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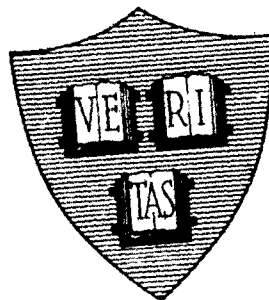
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AN EXPANSION OF THE KUMMER FUNCTION
 ${}_1F_1\left(\alpha; \frac{1}{2}; \beta x^2\right)$

FC



By
Gordon Kent

March 5, 1956

Technical Report No. 234

Cruft Laboratory
Harvard University
Cambridge, Massachusetts

Office of Naval Research

Contract N5ori-76

Task Order No. 1

NR-372-012

Technical Report

on

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Technical Report No. 234

Cruft Laboratory

Harvard University

Cambridge, Massachusetts

An Expansion of the Kummer Function ${}_1F_1(a; \frac{1}{2}; \beta x^2)$

by

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In recent years the confluent hypergeometric or Kummer functions have become useful in the treatment of a vast variety of problems in mathematical physics. The Kummer function expansion presented here was developed by the author for the solution of an electron beam problem.¹ Although its form is not the most convenient for this and for similar applications, it has the virtue that the terms of the expansion are easily calculated and are expressible in elementary functions.

The Kummer function ${}_1F_1(a; \frac{1}{2}; \beta x^2)$ is defined by the series²

$${}_1F_1(a; \frac{1}{2}; \beta x^2) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{(4\beta x^2)^n}{(2n)!} \quad (1)$$

The behavior of the function under the conditions

$$|a| \rightarrow \infty$$

$$|\beta| \rightarrow 0 \quad (2)$$

$$|a\beta| \text{ is finite}$$

is to be determined. Evidently, the parameter β is only a scale factor and could be eliminated; but it is more convenient to keep the parameter so that the conditions (2) do not have to be imposed on the variable, which may be

permitted to be of the order of unity without invalidating the results.

Inspection of the n 'th term of the Kummer series, (1), reveals that it contains a finite sum of terms of descending orders of magnitude, specifically of orders $(4a\beta)^n$, $4\beta(4a\beta)^{n-1}$, $\dots(4\beta)^{n-1}(4a\beta)$. If the terms of the Kummer series, which is absolutely and uniformly convergent, can be rearranged, then these orders of magnitude can be separated. To accomplish this rearrangement, let

$$\frac{\Gamma(a+n)}{\Gamma(a)} = \sum_{k=0}^{k=n} a_k^{(n)} a^k \quad (3)$$

and substitute (3) into (1). When the order of summation is reversed, there results the expression

$${}_1F_1(a; \frac{1}{2}; \beta x^2) = \sum_{m=0}^{\infty} b_m(x) (4\beta)^m, \quad (4)$$

where

$$b_m = \sum_{n=0}^{\infty} \frac{(4a\beta)^{n-m} a_{n-m}^{(n)} x^{2n}}{(2n)!} \quad (5)$$

It can be seen by examination of (3) that

$$\begin{aligned} a_n^{(n)} &= 1 \\ a_{n-m}^{(n)} &= 0, \text{ for } m > n \\ a_p^{(q)} &= 0, p > q \end{aligned} \quad (6)$$

and the general expression for $a_{n-m}^{(n)}$ is

$$a_{n-m}^{(n)} = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{k_1-1} \sum_{k_3=0}^{k_2-1} \dots \sum_{k_m=0}^{k_{m-1}-1} k_1 k_2 k_3 \dots k_m. \quad (7)$$

With the use of the functional equation for $\Gamma(a+n)$ and (6) the validity of

(7) can be proved by induction.

For computational purposes, it is more convenient to write (7) in the form

$$a_{n-m}(n) = \sum_{k=m}^{n-1} k a_{k+1-m}(k), \quad m=1, 2, 3, \dots \quad (8)$$

Since $a_n(n)=1$, the expressions for $a_{n-m}(n)$ for subsequent values of m can be calculated with the aid of the following two relations from combinatorial analysis:

$$\sum_{\ell=0}^{n-1} \binom{\ell}{k} = \binom{n}{k+1} \quad (9)$$

and

$$n \binom{n}{m} = (m+1) \binom{n}{m+1} + m \binom{n}{m} \quad (10)$$

The first four of these coefficients are

$$\left. \begin{aligned} a_n(n) &= 1, \\ a_{n-1}(n) &= \binom{n}{2}, \\ a_{n-2}(n) &= 3\binom{n}{4} + 2\binom{n}{3}, \\ a_{n-3}(n) &= 15\binom{n}{6} + 20\binom{n}{5} + 6\binom{n}{4}. \end{aligned} \right\} \quad (11)$$

The first four coefficients of $(4\beta)^m$ in (4) may now be written

$$\left. \begin{aligned} b_0 &= \sum_{n=0}^{\infty} \frac{q^n}{(2n)!}, \\ b_1 &= \frac{1}{4\alpha\beta} \sum_{n=0}^{\infty} \frac{q^n}{(2n)!} \binom{n}{2}, \end{aligned} \right\} \quad (12)$$

$$\left. \begin{aligned} b_2 &= \left(\frac{1}{4\alpha\beta}\right)^2 \sum_{n=0}^{\infty} \frac{q^n}{(2n)!} [3\binom{n}{4} + 2\binom{n}{3}], \\ b_3 &= \left(\frac{1}{4\alpha\beta}\right)^3 \sum_{n=0}^{\infty} \frac{q^n}{(2n)!} [15\binom{n}{6} + 20\binom{n}{5} + 6\binom{n}{4}], \end{aligned} \right\} \quad (12)$$

in which the new variable $q = 4\alpha\beta x^2$. These coefficients may be expressed in closed form in terms of $b_0(q)$ and its derivatives. The expressions are

$$\left. \begin{aligned} b_0 &= \cosh \sqrt{q}, \\ b_1 &= \frac{(4\alpha\beta x^4)}{2!} b_0^{(2)}, \\ b_2 &= \frac{3(4\alpha\beta)^2 x^8}{4!} b_0^{(4)} + \frac{2(4\alpha\beta)x^6}{3!} b_0^{(3)}, \\ b_3 &= \frac{15(4\alpha\beta)^3 x^{12}}{6!} b_0^{(6)} + \frac{20(4\alpha\beta)^2 x^{10}}{5!} b_0^{(5)} + \frac{6(4\alpha\beta)x^8}{4!} b_0^{(4)} \end{aligned} \right\} \quad (13)$$

where $b_0^{(k)}$ indicates the k 'th derivative of $b_0(q)$ with respect to q .

It is apparent that the procedure by which (13) was developed can be applied to as many terms as desired, and that all the coefficients will be expressible as a linear combination of derivatives of $b_0(q)$.

The first four terms of the expansion of the Kummer function obtained by completing the operations indicated in (13) are

$$\begin{aligned} {}_1F_1\left(a; \frac{1}{2}; \beta x^2\right) &= \cosh kx + \left(\frac{\beta x^2}{2}\right) \left\{ \cosh kx - \frac{\sinh kx}{kx} \right\} \\ &+ \frac{1}{2!} \left(\frac{\beta x^2}{2}\right)^2 \left\{ \left[1 - \frac{1}{(kx)^2}\right] \cosh kx - \left[\frac{2}{3} - \frac{1}{(kx)^2}\right] \frac{\sinh kx}{kx} \right\} \\ &+ \frac{1}{3!} \left(\frac{\beta x^2}{2}\right)^3 \left\{ \left[1 - \frac{7}{(kx)^2} - \frac{15}{(kx)^4}\right] \cosh kx + \left[1 + \frac{12}{(kx)^2} + \frac{15}{(kx)^4}\right] \frac{\sinh kx}{kx} \right\} \\ &+ O(\beta^4), \end{aligned} \quad (14)$$

where for convenience $k^2 = 4\alpha\beta$. Analysis shows that the terms in the $\{ \}$ brackets vanish uniformly as $x \rightarrow 0$. For small values of x , these terms, in the order given in (14), approach the forms

$$\left. \begin{aligned} \frac{2}{3} (kx)^2 & , \\ \frac{7}{18} (kx)^2 & , \\ \frac{107}{120} (kx)^2 & , \end{aligned} \right\} \quad (15)$$

Thus it appears that in the neighborhood of the origin the approximation is extraordinarily good.

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