A GENERAL LEAST SQUARES SOLUTION FOR SUCCESSIVE INTERVALS

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Abstract

A general least squares solution for successive intervals is presented, along with iterative procedures for obtaining stimulus scale values, discriminal dispersions, and category boundaries. Because provisions for weighting were incorporated into the derivation, the solution may be applied without loss of rigor to the typical experimental matrix of incomplete data, i.e., to a data matrix with missing entries, as well as to the rarely occurring matrix of complete data. The use of weights also permits adjustments for variations in the reliability of estimates obtained from the data. The computational steps involved in the solution are enumerated, the amount of labor required comparing favorably with other procedures. A quick, yet accurate, graphical approximation suggested by the least squares derivation is also described.

A GENERAL LEAST SQUARES SOLUTION FOR SUCCESSIVE INTERVALS

Since Thurstone first developed the scaling method of successive intervals, it has appeared in several essentially identical forms under various names, such as absolute scaling (15), equal discriminability scaling (6), and graded dichotomies (1). The procedure was first published as a psychological scaling method by Saffir (14), the basic rationale having been previously presented by Thurstone in his absolute scaling of psychological tests (15, 18).

Experimental Procedures for Successive Intervals

The experimental procedure for the method of successive intervals requires n stimuli to be sorted into $(k + 1)$ categories on some attribute continuum. The categories are usually ranked so that a stimulus placed in category g is judged to have a higher psychological scale value and, in some sense, "more of" the attribute in question than a stimulus placed in category $(g - 1)$. Analogously to Cases I and II of the Law of Comparative Judgment (17), the sorting procedure may be repeated N times by the same person or performed once by each of N different people.

This procedure yields a frequency distribution for each stimulus over several of the categories on the attribute continuum, i.e., it yields the number of times, f_{ig}, that the ith stimulus was placed in the gth category. These frequencies provide the raw material for the analytical procedure of successive intervals, which is an attempt to account for the data by a single psychological scale. The basic consideration is whether or not these frequency distributions can be simultaneously converted to a common distribution, allowing unequal means and variances, on the same base line. The means of the converted distributions would then correspond to stimulus scale values and the standard deviations to what Thurstone has called "discriminal dispersions" (16). Scale values for category boundaries are also obtained from the method of successive intervals, thus permitting estimates of the size of categories rather than assuming them to be equal as in the method of equal-appearing intervals (19).

Solutions to the Scaling Problem

Successive intervals solutions for the $\underline{n}$ stimulus values have been suggested by Saffir (14), Guilford (9), Mosier (12), Bishop (2), Attneave (1), Garner and Hake (6), Edwards (3), Gulliksen (10), and Rimoldi (13). Some of these articles also offer solutions for the $\underline{n}$ discriminal dispersions and the $\underline{k}$ category boundaries. The procedures vary in computational routine and with respect to certain restricting assumptions, but they are essentially equivalent. These procedures involve obtaining the proportion, p_{ig}, of times a stimulus $\underline{i}$ was placed in category $\underline{g}$. The p_{ig} values are then cumulated to give the proportion of times stimulus $\underline{i}$ was placed below the $\underline{g}$th category boundary, t_g, and these cumulative proportions are usually converted into normal deviate values, z_{ig}. Various successive intervals solutions presented in the literature have used normal curve transformations or other similar transformations such as the logistic. Any similar function giving a one-to-one correspondence between p_{ig} and z_{ig} could be used (see 10).

The normal deviate value obtained from such a transformation may be expressed as follows:

$$z_{ig} = \frac{t_g - m_i}{s_i} \quad , \tag{1}$$

where z_{ig} = the normal deviate value corresponding to a cumulated proportion,
t_g = the upper boundary of the gth category,
m_i = the scale value of stimulus i , and
s_i = the discriminal dispersion for stimulus i .

The category boundary can then be expressed as

$$t_g = m_i + s_i z_{ig} \quad . \tag{2}$$

This equation is what Torgerson calls a special case of the Law of Categorical Judgment (20). Algebraic solutions for m_i , s_i , and t_g can then be obtained from this relationship by arbitrarily choosing one of the s_i values as a unit and one of the m_i values or their average as an origin.

Gulliksen (10) derived an explicit least squares solution for m_i , s_i , and t_g by minimizing the following error term:

$$E = \frac{1}{b^2} \sum_{i=1}^{n} \sum_{g=1}^{k} (m_i + s_i z_{ig} - t_g)^2 \quad , \tag{3}$$

where b is an arbitrary scale factor.

The following restrictions, which fix an origin and a unit, were attached to the function to be minimized by Lagrange multipliers:

$$\sum_{g=1}^{k} t_g = ka \qquad (4)$$

and

$$\sum_{g=1}^{k} t_g^2 = k(a^2 + b^2) \quad , \qquad (5)$$

where $\underline{k}$ is the number of category boundaries, i.e., one less than the number of categories. These restrictions place the mean scale value of the category boundaries, t_g , at $\underline{a}$ and their standard deviation at $\underline{b}$.

A General Least Squares Solution

This paper is concerned with a general least squares solution for successive intervals, which is an attempt to express the variables m_i , s_i , and t_g in terms of each other, instead of solving for them explicitly. Once such relationships have been obtained, an iterative procedure can be utilized to converge upon m_i , s_i , and t_g . In an attempt to obtain a least squares solution that would apply equally well to data with either complete or incomplete overlap, a weighting system was incorporated into the derivation. Thus the "incomplete overlap" situation could be easily handled without loss of rigor by assigning weights of zero to the missing entries. As it turned out, the present iterative solution also involves a computational routine that is not excessively laborious.

The general strategy, then, of the present least squares derivation is to solve implicitly for m_i, s_i, and t_g in terms of each other and then to set up rules for iterating to convergence values. The derivation proceeds as follows:

Let $\hat{t}_{ig}$ represent an estimate of t_g that would be obtained from stimulus i if m_i and s_i were known:

$$\hat{t}_{ig} \equiv m_i + s_i z_{ig} \quad . \tag{6}$$

Let e_{ig} represent the error made by estimating t_g in this fashion:

$$e_{ig} \equiv \hat{t}_{ig} - t_g = m_i + s_i z_{ig} - t_g \quad . \tag{7}$$

Since the error term to be minimized must contain provisions for weighting, let it be written as

$$E \equiv \frac{1}{b^2} \sum_{i=1}^{n} \sum_{g=1}^{k} w_{ig} e_{ig}^2 \quad , \tag{8}$$

where w_{ig} is a weight that may be chosen in any fashion as long as the following restriction is met:

$$\text{For } z_{ig} = \pm\infty, \quad \text{i.e., when } \begin{cases} p = 0 \\ p = 1 \end{cases}, \text{ then } w_{ig} = 0 \text{ and } w_{ig}z_{ig} = 0 .$$

An arbitrary origin, a, and unit, b, for the t-scale may now be specified by

$$\sum_{g=1}^{k} t_g \sum_{i}^{n} w_{ig} = Wa \quad , \tag{9}$$

$$\sum_{g=1}^{k} t_g^2 \sum_{i}^{n} w_{ig} = W(a^2 + b^2) \quad , \tag{10}$$

where W is the constant defined as

$$W \equiv \sum_{gi}^{kn}\sum w_{ig} \quad .$$

Thus, the weighted mean scale value of the category boundaries is set at a and their weighted standard deviation at b, as can be readily seen if equations 9 and 10 are rewritten as

$$\sum_{g}^{k}(t_g - a) \sum_{i}^{n} w_{ig} = 0 \tag{11}$$

and

$$\sum_{g}^{k}(t_g - a)^2 \sum_{i}^{n} w_{ig} = Wb^2 \quad . \tag{12}$$

These restrictions generalize for the weighted case the definitions used by Gulliksen in the unweighted case (see equations 4 and 5). Since the t-scale is determined only within a linear transformation, a and b may be set at any values desired, e.g., a convenient possibility for the origin, a , might be zero and for the scale unit, b , which must be positive, might be unity.

Using two Lagrange multipliers γ and 2λ , the restrictions setting origin and unit may be included in the error term as follows:

$$Q = \frac{1}{b^2} \sum_{i=1}^{n} \sum_{g=1}^{k} w_{ig} (m_i + s_i z_{ig} - t_g)^2 + 2\lambda \left(\sum_{g=1}^{k} t_g \sum_{i=1}^{n} w_{ig} - Wa \right)$$

$$- \gamma \left[\sum_{g=1}^{k} t_g^2 \sum_{i}^{n} w_{ig} - W(a^2 + b^2) \right] \quad . \qquad (13)$$

Except for the weights, w_{ig} , equation 13 is identical to the term minimized by Gulliksen (see equations 3, 4, and 5). His solution, then, is the special case of the present one in which all of the weights are equal to unity and $\sum_{g}^{k} w_{ig} = k$ and $\sum_{i}^{n} w_{ig} = n$. Because of this restriction to unit weights, only data with complete overlap can be considered. Equation 13 is also similar to the term minimized by Tucker (22) in developing a least squares solution to the normal ogive model for categorical data, which is formally equivalent to the successive intervals situation (see 21, Chapter 13). Tucker's solution, like the present one, involves weights and is iterative, but instead of minimizing the sum of squared differences between theoretical and estimated t-values, he minimized the sum of squared differences between theoretical and observed z-values.

The differentiation of Q with respect to each of the m_i in turn yields the n equations

$$\frac{1}{2}\frac{\partial Q}{\partial m_i} = \frac{1}{b^2}\sum_{g=1}^{k} w_{ig}(m_i + s_i z_{ig} - t_g)\ (+1) \qquad (i=1\ldots n) \quad . \tag{14}$$

Expanding and setting the partial derivative equal to zero,

$$m_i \sum_g^k w_{ig} + s_i \sum_g^k w_{ig} z_{ig} - \sum_g^k w_{ig} t_g = 0 \quad . \tag{15}$$

The solution for m_i can now be written as

$$m_i = \frac{\sum_g^k w_{ig} t_g - s_i \sum_g^k w_{ig} z_{ig}}{\sum_g^k w_{ig}} \quad . \tag{16}$$

Q is now differentiated with respect to each s_i in turn to yield the n equations

$$\frac{1}{2}\frac{\partial Q}{\partial s_i} = \frac{1}{b^2}\sum_g^k w_{ig}(m_i + s_i z_{ig} - t_g)(z_{ig}) \qquad (i=1\ldots n) \quad . \tag{17}$$

Expanding and setting the partial derivative equal to zero,

$$m_i \sum_g^k w_{ig} z_{ig} + s_i \sum_g^k w_{ig} z_{ig}^2 - \sum_g^k w_{ig} t_g z_{ig} = 0 \quad . \tag{18}$$

Substituting the value of m_i from equation 16 we have

$$\frac{(\sum_g^k w_{ig} z_{ig})(\sum_g^k w_{ig} t_g)}{\sum_g^k w_{ig}} - \frac{s_i(\sum_g^k w_{ig} z_{ig})^2}{\sum_g^k w_{ig}} + s_i \sum_g^k w_{ig} z_{ig}^2 - \sum_g^k w_{ig} t_g z_{ig} = 0 \quad . \tag{19}$$

Solving for s_i ,

$$s_i = \frac{(\sum_g^k w_{ig} t_g z_{ig})(\sum_g^k w_{ig}) - (\sum_g^k w_{ig} z_{ig})(\sum_g^k w_{ig} t_g)}{(\sum_g^k w_{ig} z_{ig}^2)(\sum_g^k w_{ig}) - (\sum_g^k w_{ig} z_{ig})^2} \quad . \tag{20}$$

For the purpose of parenthetical comment upon the form of s_i , equation 20 can be rewritten in the following manner:

$$s_i = \frac{\sum_g^k w_{ig}(t_g - \bar{t}_i)(z_{ig} - \bar{z}_i)}{\sum_g^k w_{ig}(z_{ig} - \bar{z}_i)^2} \quad , \tag{21}$$

where

$$\bar{z}_i = \frac{\sum_g^k w_{ig} z_{ig}}{\sum_g^k w_{ig}}$$

and

$$\bar{t}_i = \frac{\sum_g^k w_{ig} t_g}{\sum_g^k w_{ig}} \quad .$$

It is apparent from the form of equation 21 that s_i is the slope of a regression line. It is the coefficient for the regression of $\underline{t}$ on $\underline{z}$. This immediately suggests a graphical representation of the data, which will be presented in a later section. It is also interesting to note that

Tucker's solution (22), which minimized the sum of squared errors in the $\underline{z}$ direction instead of in the $\underline{t}$ direction, involves the other regression--the regression of $\underline{z}$ on $\underline{t}$.

Utilizing the value of s_i obtained in equation 20, it is also possible to rewrite the formula for m_i in terms of t_g as follows:

$$m_i = \frac{(\sum_g^k w_{ig}t_g)(\sum_g^k w_{ig}z_{ig}^2) - (\sum_g^k w_{ig}t_g z_{ig})(\sum_g^k w_{ig}z_{ig})}{(\sum_g^k w_{ig}z_{ig}^2)(\sum_g^k w_{ig}) - (\sum_g^k w_{ig}z_{ig})^2} \quad . \tag{22}$$

As will be seen in a later section, this is the form of m_i which it will be convenient to use in computational routines.

Differentiating Q with respect to each t_g in turn yields the $\underline{k}$ equations

$$\frac{1}{2}\frac{\partial Q}{\partial t_g} = \frac{1}{b^2}\sum_i^n w_{ig}(m_i + s_i z_{ig} - t_g)(-1) + \lambda \sum_i^n w_{ig} - \gamma t_g \sum_i^n w_{ig}$$

$$(g=1\ldots k) \quad . \tag{23}$$

Expanding and setting the partial derivative equal to zero,

$$t_g \sum_i^n w_{ig} - \sum_i^n w_{ig}(m_i + s_i z_{ig}) + b^2\lambda \sum_i^n w_{ig} - b^2\gamma t_g \sum_i^n w_{ig} = 0 \quad . \tag{24}$$

Define

$$v_g \equiv \frac{\sum_i^n w_{ig}(m_i + s_i z_{ig})}{\sum_i^n w_{ig}} \quad . \tag{25}$$

Rearranging the terms of equation 24, dividing both sides by $\sum_i^n w_{ig}$, and substituting the definition of v_g from equation 25, we may write

$$(1 - b^2\gamma)t_g = v_g - b^2\lambda \quad . \tag{26}$$

The solution for t_g can now be written as

$$t_g = \frac{v_g - b^2\lambda}{1 - b^2\gamma} \quad . \tag{27}$$

Summing equation 24 over $\underline{g}$ and utilizing the definition of equation 9, we may write

$$Wa - \sum_{gi}^{kn}\sum w_{ig}(m_i + s_i z_{ig}) + Wb^2\lambda - Wab^2\gamma = 0 \quad . \tag{28}$$

Consider the term $\sum_{gi}^{kn}\sum w_{ig}(m_i + s_i z_{ig})$. By interchanging the order of summation and by inserting the values of s_i and m_i given in equations 20 and 22, respectively, this term may be written as

$$\sum_i^n(m_i \sum_g^k w_{ig} + s_i \sum_g^k w_{ig}z_{ig}) = \sum_i^n \left[\frac{(\sum_g^k w_{ig}t_g)[(\sum_g^k w_{ig}z_{ig}^2)(\sum_g^k w_{ig}) - (\sum_g^k w_{ig}z_{ig})^2]}{(\sum_g^k w_{ig}z_{ig}^2)(\sum_g^k w_{ig}) - (\sum_g^k w_{ig}z_{ig})^2}\right]$$

$$= \sum_{ig}^{nk}\sum w_{ig}t_g \quad . \tag{29}$$

Utilizing the definition of an origin given in equation 9, the term can be further simplified to

$$\sum_{ig}^{nk}\sum w_{ig}(m_i + s_i z_{ig}) = \sum_{ig}^{nk}\sum w_{ig} t_g = Wa \quad . \tag{30}$$

Now, equation 28 can be written as

$$Wb^2\lambda = Wab^2\gamma \quad ,$$

from which

$$\lambda = a\gamma \quad . \tag{31}$$

It should be noted in passing that equation 30, in terms of the v_g of equation 25, indicates that the weighted mean of v_g is

$$\bar{v} = \frac{\sum_{g}^{k} v_g \sum_{i}^{n} w_{ig}}{\sum_{gi}\sum w_{ig}} = a \quad . \tag{32}$$

If the above value of λ is substituted into equation 26 and $(1 - b^2\gamma)a$ is subtracted from both sides of the equation, then equation 26 becomes

$$(1 - b^2\gamma)(t_g - a) = v_g - a \quad . \tag{33}$$

Squaring both sides of equation 33, multiplying through by $\sum_{i}^{n} w_{ig}$, and summing over g , we may write

$$(1 - b^2\gamma)^2 \sum_{g}^{k}(t_g - a)^2 \sum_{i}^{n} w_{ig} = \sum_{g}^{k}(v_g - a)^2 \sum_{i}^{n} w_{ig} \quad . \tag{34}$$

Utilizing the definition of a scale unit in the form given in equation 12, we may write equation 34 as

$$(1 - b^2\gamma)^2 Wb^2 = \sum_g^k (v_g - a)^2 \sum_i^n w_{ig} \quad , \tag{35}$$

from which

$$(1 - b^2\gamma) = \sqrt{\frac{\sum_g^k (v_g - a)^2 \sum_i^n w_{ig}}{Wb^2}} \quad . \tag{36}$$

Using the value of λ found in equation 31, the solution for t_g given in equation 27 may be written as

$$t_g = \frac{v_g - ab^2\gamma}{1 - b^2\gamma} = \frac{v_g - a}{1 - b^2\gamma} + a \quad . \tag{37}$$

Substituting the value of $(1 - b^2\gamma)$ from equation 36, the solution for t_g becomes

$$t_g = \frac{v_g - a}{\sqrt{\frac{\sum_g^k (v_g - a)^2 \sum_i^n w_{ig}}{Wb^2}}} + a \quad . \tag{38}$$

γ may also be represented in terms of the sum of squares of errors as follows: Equation 8 may be written as

$$b^2E = \sum_{ig}^{nk}\sum w_{ig}(m_i + s_iz_{ig} - t_g)^2 = \sum_{ig}^{nk}\sum w_{ig}(m_i + s_iz_{ig})^2$$

$$- 2\sum_{g}^{k} t_g \sum_{i}^{n} w_{ig}(m_i + s_iz_{ig}) + \sum_{g}^{k} t_g^2 \sum_{i}^{n} w_{ig} \quad . \tag{39}$$

Consider the first term of the above expression:

$$\sum_{ig}^{nk}\sum w_{ig}(m_i + s_iz_{ig})^2 = \sum_{ig}^{nk}\sum w_{ig}(m_i + s_iz_{ig})(m_i + s_iz_{ig}) = \sum_{i}^{n} m_i \sum_{g}^{k} w_{ig}(m_i + s_iz_{ig})$$

$$+ \sum_{i}^{n} s_i \sum_{g}^{k} w_{ig}z_{ig}(m_i + s_iz_{ig}) \quad . \tag{40}$$

Substituting into the last term of equation 40 the values of s_i and m_i from equations 20 and 22, respectively, and simplifying, and remembering that $\sum_{g}^{k} w_{ig}(m_i + s_iz_{ig})$ was found to be equal to $\sum_{g}^{k} w_{ig}t_g$ in equation 29, we may rewrite equation 40 as

$$\sum_{ig}^{nk}\sum w_{ig}(m_i + s_iz_{ig})^2 = \sum_{i}^{n} m_i \sum_{g}^{k} w_{ig}t_g + \sum_{i}^{n} s_i \sum_{g}^{k} w_{ig}t_gz_{ig} = \sum_{g}^{k} t_g \sum_{i}^{n} w_{ig}(m_i + s_iz_{ig}) \quad . \tag{41}$$

Equation 39 may now be written as

$$b^2E = - \sum_{g}^{k} t_g \sum_{i}^{n} w_{ig}(m_i + s_iz_{ig}) + \sum_{g}^{k} t_g^2 \sum_{i}^{n} w_{ig} \quad . \tag{42}$$

Utilizing the definition of a scale unit from equation 10, equation 42 becomes

$$b^2E = W(a^2 + b^2) - \sum_{g}^{k} t_g \sum_{i}^{n} w_{ig}(m_i + s_iz_{ig}) \quad . \tag{43}$$

The size of the error term at the solution may now be evaluated by substituting for t_g in equation 43 its value from equation 37, and by noting that $\sum_i^n w_{ig}(m_i + s_i z_{ig}) = v_g \sum_i^n w_{ig}$ from equation 25. Equation 43 then becomes

$$b^2E = W(a^2 + b^2) - \frac{1}{1 - b^2\gamma}\left[\sum_g^k v_g^2 \sum_i^n w_{ig} - a\sum_g^k v_g \sum_i^n w_{ig}\right] - a\sum_g^k v_g \sum_i^n w_{ig} \quad . \tag{44}$$

In equation 32 it was seen that $\sum_g^k v_g \sum_i^n w_{ig} = Wa$, so equation 44 may be written as

$$b^2E = Wa^2 + Wb^2 - \frac{1}{1 - b^2\gamma}\left[\sum_g^k v_g^2 \sum_i^n w_{ig} - Wa^2\right] - Wa^2 \quad . \tag{45}$$

Rewriting $\left[\sum_g^k v_g^2 \sum_i^n w_{ig} - Wa^2\right]$ as $\sum_g^k (v_g - a)^2 \sum_i^n w_{ig}$, and noting the value of this term in equation 35, we may write equation 45 as

$$b^2E = Wb^2 - (1 - b^2\gamma)Wb^2 \quad . \tag{46}$$

The error term may now be expressed as

$$E = Wb^2\gamma \quad . \tag{47}$$

The Iterative Procedure

Since the t-scale of successive intervals is determined only with a linear transformation, the origin, a , and the scale unit, b , may be set at any values desired. The values most convenient for computational routines are $a = 0$ and $b = 1$. With such a placement of origin and unit, the restrictions given in equations 9 and 10 may be restated as

$$\sum_{g}^{k} t_g \sum_{i}^{n} w_{ig} = 0 \qquad \text{and} \tag{48}$$

$$\sum_{g}^{k} t_g^2 \sum_{i}^{n} w_{ig} = W \qquad \text{, respectively.} \tag{49}$$

The formula for category boundaries given in equation 38 may also be rewritten in terms of this origin and unit as follows:

$$t_g = \frac{v_g}{\sqrt{\frac{1}{W} \sum_{g}^{k} v_g^2 \sum_{i}^{n} w_{ig}}} \quad . \tag{50}$$

Now, equations 20, 22, 25, and 50 may be used to set up an iterative procedure to obtain convergence values for s_i , m_i , and t_g . If some initial estimate, t_{g1} , of the category boundaries, t_g , were available, equation 20 could be solved to obtain initial estimates, s_{i1} , of the discriminal dispersions, s_i . The initial _t_-estimates, t_{g1} , would, of course, have to be converted to meet the restrictions of equations 48 and 49.

In other words, if some set of _k_ numbers, v_{g1} , is available as possible estimates of the category boundaries, before they are used in the above solution, they must first be converted to meet the restrictions of equations 48 and 49. This can be done as follows:

$$t_{g1} = \frac{v_{g1} - \bar{v}_1}{\sigma_{v_1}} \qquad , \text{ where}$$

$$\bar{v}_1 = \frac{1}{W} \sum_g^k v_{g1} \sum_i^n w_{ig} \qquad \text{and}$$

$$\sigma_{v_1} = \sqrt{\frac{1}{W} \sum_g^k (v_{g1} - \bar{v}_1)^2 \sum_i^n w_{ig}} \quad . \tag{51}$$

Initial estimates of s_i can then be obtained from equation 20 as follows:

$$s_{i1} = \frac{(\sum_g^k w_{ig} t_{g1} z_{ig})(\sum_g^k w_{ig}) - (\sum_g^k w_{ig} z_{ig})(\sum_g^k w_{ig} t_{g1})}{(\sum_g^k w_{ig} z_{ig}^2)(\sum_g^k w_{ig}) - (\sum_g^k w_{ig} z_{ig})^2} \quad . \tag{52}$$

If a subscript $\underline{\alpha}$ is introduced to indicate the $\underline{\alpha}$th cycle in the iterative procedure, equation 52 can be rewritten as a formula for the $\underline{\alpha}$th estimate of s_i :

$$s_{i\alpha} = \frac{(\sum_g^k w_{ig} t_{g\alpha} z_{ig})(\sum_g^k w_{ig}) - (\sum_g^k w_{ig} z_{ig})(\sum_g^k w_{ig} t_{g\alpha})}{(\sum_g^k w_{ig} z_{ig}^2)(\sum_g^k w_{ig}) - (\sum_g^k w_{ig} z_{ig})^2} \quad . \tag{53}$$

In order to set up systematic computational routines, it would seem desirable at this point to define the following coefficients:

$$A_i \equiv \frac{\sum_g^k w_{ig}}{(\sum_g^k w_{ig}z_{ig}^2)(\sum_g^k w_{ig}) - (\sum_g^k w_{ig}z_{ig})^2} \quad , \tag{54}$$

$$B_i \equiv \frac{\sum_g^k w_{ig}z_{ig}}{(\sum_g^k w_{ig}z_{ig}^2)(\sum_g^k w_{ig}) - (\sum_g^k w_{ig}z_{ig})^2} \quad , \tag{55}$$

$$C_i \equiv \frac{\sum_g^k w_{ig}z_{ig}^2}{(\sum_g^k w_{ig}z_{ig}^2)(\sum_g^k w_{ig}) - (\sum_g^k w_{ig}z_{ig})^2} \quad . \tag{56}$$

Since the components of the above coefficients are obtainable directly from the data, they are the same for all cycles of iteration and need be computed only once for the entire procedure.

Equation 53 for the $\underline{\alpha}$th estimate of the discriminal dispersions, s_i , may now be written as

$$s_{i\alpha} = A_i \sum_g^k w_{ig}t_{g\alpha}z_{ig} - B_i \sum_g^k w_{ig}t_{g\alpha} \quad . \tag{57}$$

Having found estimates for t_g and s_i , the $\underline{\alpha}$th estimate of the scale values, m_i , may now be obtained from equations 22, 55, and 56 as follows:

$$m_{i\alpha} = C_i \sum_g^k w_{ig}t_{g\alpha} - B_i \sum_g^k w_{ig}t_{g\alpha}z_{ig} \quad . \tag{58}$$

Now, new estimates of v_g may be obtained from a formula analogous to equation 25:

$$v_{g(\alpha+1)} = \frac{\sum_i^n w_{ig} m_{i\alpha} + \sum_i^n w_{ig} s_{i\alpha} z_{ig}}{\sum_i^n w_{ig}} \quad . \tag{59}$$

A new estimate of the t-scale may now be found by using equation 50 as follows:

$$t_{g(\alpha+1)} = \frac{v_{g(\alpha+1)}}{\sqrt{\frac{1}{W} \sum_g^k v^2_{g(\alpha+1)} \sum_i^n w_{ig}}} \quad . \tag{60}$$

The above procedure may be repeated by inserting this value of $t_{g(\alpha+1)}$ into equation 57 to obtain $s_{i(\alpha+1)}$, which in turn may be used to obtain $m_{i(\alpha+1)}$ and, subsequently, $t_{g(\alpha+2)}$. This cycle may then be iterated until two successive estimates of t_g are as similar as desired, *i.e.*, until $[\, t_{g(\alpha+1)} - t_{g\alpha}]$ is negligible.

The amount of scaling error at any given cycle of iteration may be evaluated from equation 43, with a set equal to zero and b to unity, as follows:

$$E_\alpha = W - \sum_g^k t_{g\alpha} \sum_i^n w_{ig} (m_{i\alpha} + s_{i\alpha} z_{ig}) \quad . \tag{61}$$

The error should decrease as iteration proceeds until at convergence

$$E = W\gamma = W(1 - \sqrt{\frac{1}{W} \sum_g^k v^2_g \sum_i^n w_{ig}}) \quad . \tag{62}$$

The one step remaining to be considered before the above iterative procedure may be applied in practice is the initial estimation of the t-scale. One obvious starting point might be a set of equally spaced numbers, such as the integers from 1 to k, to which the conversion of equation 51 had been applied. By using such equally spaced t_{g1} values, a set of "equal-appearing" intervals is used as the starting point for iteration to successive intervals. It may be possible, also, to increase the rate of convergence in the iterative procedure by doubling or tripling the difference between successive t-estimates, i.e., instead of using $t_{g(\alpha+1)}$ on the $(\alpha + 1)$ trial, use $t'_{g(\alpha+1)} = t_{g\alpha} + 2[t_{g(\alpha+1)} - t_{g\alpha}]$.

A cycle or two, and in some cases perhaps several cycles, may be eliminated from the iterative procedure by using a computationally simple linear solution for t_g as a first estimate. One of the simplest methods for estimating t_g has been suggested by Garner and Hake (6) and by Edwards (3) and involves averages of successive differences in z_{ig} values. Such averages are estimates of $(t_g - t_{g-1})$ provided the discriminal dispersions may be assumed equal. Torgerson (21) also gives a simple algebraic ratio solution for t_g which does not require equal s_i . Any of these algebraic solutions may be used to obtain initial estimates for the iterative procedure, but the labor involved might turn out to be as great as that in the cycle or two eliminated.

Some Weighting Systems Appropriate for Successive Intervals Data

Some comment is necessary at this point concerning the weights, w_{ig} , involved in the above least squares solution. It will be recalled that the only restriction placed upon the choice of these weights was that

$$w_{1g} = 0 \quad \text{and} \quad w_{1g} z_{1g} = 0 \quad \text{when} \quad \begin{cases} p = 0 \\ p = 1 \end{cases}$$

If p equals neither zero nor one, the weights may be set at any values desired, e.g., w_{1g} may be set equal to unity for $0 < p < 1$.

However, as Urban (23) has pointed out, it is a principle of the method of least squares that more reliable observations should have greater importance than less reliable ones, so it would seem reasonable to choose the weights in such a way as to take account of differences in the reliability of the z_{1g} values. Since the reliability of a proportion is inversely related to its variance, the weights used for such a purpose would be proportional to $\frac{1}{pq}$. These weights would also be directly proportional to the information available from the observations, since the reciprocal of the variance may be identified with quantity of information (5).

Another point must also be considered in selecting weights for successive intervals data. As reported by Guilford (8, p. 175), "Müller argued that the proportions near .50 should be weighted more than the proportions deviating in either direction from .50...an error in p near .50, where p is changing at its maximum rate, as compared with the change in (z), is not nearly so serious as at the extremes where a slight error in p is reflected in a large error in (z)." Müller, then, was advocating a system which would weight directly in proportion to the rate of change of p with respect to z . These weights turn out to be the ordinates, X , of the normal distribution corresponding to proportions, p ; a recent derivation of these weights is given by

Finney (4). In practice both weighting conditions are usually combined to produce the Müller-Urban weights, $\frac{X^2}{pq}$.

It can also be shown that within an approximation the Müller-Urban weights are the proper values for weighting normally transformed scores inversely to their variance. Consider the following approximate relationship stated by Kendall (11, p. 206):

$$\text{var } p = \left(\frac{dp}{dz}\right)^2 \text{var } z \quad , \tag{63}$$

where var is an abbreviation for variance. If p and z are related by the unit normal distribution function, as is the case for successive intervals, $\frac{dp}{dz}$ is equal to X , the ordinate of the unit normal curve corresponding to a proportion, p . Therefore,

$$\text{var } p = X^2 \text{ var } z \qquad \text{or}$$

$$\text{var } z = \frac{pq}{NX^2} \quad . \tag{64}$$

Thus, in weighting inversely to the variance of z , the weight would be proportional to $\frac{X^2}{pq}$, since N is a constant in the successive intervals situations.

However, it is possible to use a simpler set of weights than X^2/pq , without sacrificing completely the differentiation between reliable and unreliable z_{ig} values. For instance, one possible rule for weighting would be to assign zero if the corresponding proportion contained less than some specified fraction, $\frac{1}{r}$, of the maximum possible information (corresponding to $p = .5$) and unity if it contained more than $\frac{1}{r}$ th the maximum information. Or, all $|z_{ig}| > c$ could be weighted zero, and all $|z_{ig}| \leq c$ could be weighted unity; such a rule with a value of $c = 2$ has been found to be convenient in practice (7, 8).

The use of a simple set of unit and zero weights also simplifies some of the procedures involved in the above iterative solution. For instance, if

$$w_{ig} = \begin{cases} 0 & \text{for } |z_{ig}| > c \\ 1 & \text{for } |z_{ig}| \leq c \end{cases} ,$$

or alternatively stated, if

$$w_{ig} = \begin{cases} 0 & \text{for } F_{ig} < c_1 \text{ or } F_{ig} > c_2 \\ 1 & \text{for } c_1 \leq F_{ig} \leq c_2 \end{cases} , \quad c_1 < c_2 ,$$

the weights can be applied simultaneously with the conversion to z_{ig} , i.e., $F_{ig} < c_1$ or $F_{ig} > c_2$ can be converted to zero, while $c_1 \leq F_{ig} \leq c_2$ are converted to the corresponding normal deviates, z_{ig} . With this kind of procedure, the deviate values available for manipulation have already had the weights applied; this would greatly simplify subsequent computations.

Summary and Illustration of Analytical Procedures

The analytical procedures involved in the above least squares solution for successive intervals will now be summarized, and an errorless numerical example will be used to illustrate the computational routine.

1. The experimental method of successive intervals yields the category (1 to k+1) into which each of <u>n</u> stimuli was placed by each of N individuals. These data may be summarized into an $n\ X(k+1)$ table, the cell entries of which, f_{ig}, represent the number of times the <u>i</u>th stimulus was placed in the <u>g</u>th category. By cumulating the frequencies in each row of this table so that each entry now represents the number of times the <u>i</u>th stimulus appeared below the <u>g</u>th category boundary, t_g, a set of cumulated frequencies, F_{ig}, is obtained, which can be considered to be the starting point for successive intervals analysis.
2. The cumulated frequencies are then converted into proportions, P_{ig}, and then to normal deviate values, z_{ig}. For the purpose of illustrating computational procedures, consider the set of z_{ig} values presented in Table 1. The four scale values, m_i, four discriminal dispersions, s_i, and three category boundaries, t_g, which exactly fit these z_{ig} values under the restrictions of equations 48 and 49 are also given in Table 1. Knowing a "true" set of scale values and category boundaries, the convergence of the above iterative solution may be illustrated.
3. A set of weights, w_{ig}, and an initial estimate, t_{g1}, of the category boundaries must now be determined. For the present example, it was decided to use the weights given in Table 1; they were assigned so that

$$w_{ig} = \begin{cases} 0 \text{ for } |z_{ig}| > 3.0 \\ 1 \text{ for } 3.0 \geq |z_{ig}| \geq 2.0 \\ 2 \text{ for } |z_{ig}| < 2.0 \end{cases} .$$

It was also decided to use an equally-spaced scale as a first estimate of t_g . Accordingly, using the integers 1, 2, and 3 as v_{g1} , the conversion of equation 51 produced as t_{g1} the values -1.16422, .15523, 1.47469. It should be noted that if the "true" t_g values given in Table 1 were used as first estimates in the present iterative procedure, they would be exactly reproduced at the end of one cycle.

4. Now, the coefficients A_i , B_i , and C_i may be computed according to equations 54, 55, and 56, respectively; these values are presented in Table 2, along with the values of $\sum_g^k w_{ig} t_{g1}$ and $\sum_g^k w_{ig} t_{g1} z_{ig}$.
5. Sufficient information is now available to solve for first estimates of the discriminal dispersions, s_{i1} , using equation 57 (see Table 2).
6. Now, first estimates of the scale values, m_{i1} , can be obtained, using equation 58 (see Table 2).
7. New estimates, t_{g2} , of the category boundaries may now be found from equations 59 and 60. It will be noted that t_{g2} given in Table 2 is closer to the "true" t-scale than t_{g1} was.
8. In order to iterate this solution, new values of $\sum_g^k w_{ig} t_{g2}$ and $\sum_g^k w_{ig} t_{g2} z_{ig}$ must be computed. Using these values,

equations 57 and 58, respectively, may be solved to obtain s_{i2} and m_{i2} . Then, equations 59 and 60 may be used to obtain t_{g3} , the third estimate of the t-scale (see Table 3). It should again be noted that the estimates of t_g at each successive cycle are approaching closer and closer the "true" t_g scale. This procedure may now be repeated until two successive t-estimates are as similar as desired.

A Graphical Successive Intervals Scaling Procedure

It was seen from equation 21 that s_i is the regression coefficient for the regression of t on z . This suggests a graphical solution to successive intervals, which will be summarized below; the procedures to be presented bear some similarity to the graphical methods of Mosier (12) and Garner and Hake (6).

1-3. Steps 1 through 3 of the graphical procedure are identical to the corresponding steps of the above analytical procedure. In order to utilize the graphical method, a first estimate, t_{g1} , of the category boundaries must be available, along with normal deviate values and their corresponding weights.

4. The estimated t_{g1} values are then marked off as the ordinate of a graph with z values used as the abscissa. The t_{g1} values are horizontal lines that hold for all stimuli, so several plots can be made on one graph (see Figure 1). For each stimulus, the z_{ig} values are plotted at the appropriate t_{g1} points, i.e., for stimulus 2 in the above numerical example, points would be plotted at ($t_{g1} = -1.164$, $z = -.5$), ($t_{g1} = .155$, $z = .5$), and ($t_{g1} = 1.475$, $z = 2.0$), as illustrated in Figure 1. Weights can be applied in the graphical

procedure by clustering around each point a number of dots proportional to the corresponding weight. A straight line can now be fitted to the points by eye, giving more emphasis to those points with bigger dot clusters in determining the slope of the line.

5. The equation of each of these lines can be written as

$$t_{g1} = m_{i1} + s_{i1} z_{ig}$$

or, more generally, as

$$t_{g\alpha} = m_{i\alpha} + s_{i\alpha} z_{ig} \quad . \tag{65}$$

The slope, $s_{i\alpha}$, of each line is the $\underline{\alpha}$th estimate of the discriminal dispersion, and the intercept, $m_{i\alpha}$, when $z = 0$ is the $\underline{\alpha}$th estimate of the scale value. These intercepts and slopes can be read directly from the graph, but they need not be recorded until the final iteration.

6. In practice, the straight lines fitted to the plotted values will rarely cross every point; each point will usually deviate from the line by some amount, the amount of this deviation in the vertical direction representing a scaling error (see Figure 1). The vertical projection of a plotted point on the fitted line produces another point, $\hat{t}_{ig}$, which, since it lies directly on the line, represents a theoretical or fitted estimate of t_g (see Figure 1). For a given category boundary, there are $\underline{n}$ fitted estimates $\hat{t}_{ig}$, one for each stimulus. If the ordinates of these $\hat{t}_{ig}$ values are recorded, weighted averages of the

ordinates can be used to obtain a new estimate of t_g as follows:

$$\frac{\sum_i^n w_{ig}\hat{t}_{ig\alpha}}{\sum_i^n w_{ig}} = v_{g(\alpha+1)} \quad , \tag{66}$$

$$t_{g(\alpha+1)} = \frac{v_{g(\alpha+1)} - \bar{v}_{(\alpha+1)}}{\sigma_{v(\alpha+1)}} \quad , \tag{67}$$

where

$$\bar{v}_{(\alpha+1)} = \frac{1}{W}\sum_g^k v_{g(\alpha+1)} \sum_i^n w_{ig}$$

and

$$\sigma_{v(\alpha+1)} = \sqrt{\frac{1}{W}\sum_g^k \left[v_{g(\alpha+1)} - \bar{v}_{(\alpha+1)}\right]^2 \sum_i^n w_{ig}} \quad .$$

The only value, then, that need be read from the graphs in going from one iterative cycle to another is $\hat{t}_{ig\alpha}$. The slopes and intercepts corresponding to discriminal dispersions and scale values do not need to be recorded until the final iteration.

7. This new estimate of t_g may now be plotted as the ordinate of a graph with $\underline{z}$ values marked off as the abscissa. The cycle may then be repeated beginning at step 4 until two successive $\underline{t}$-estimates are as similar as desired.

Summary

A general least squares solution is developed for the method of successive intervals when weights are assigned to each error term. These weights may reflect the relative amount of information contained

in the corresponding observations. By use of zero weights, this method is rigorously applicable to matrices involving incomplete data. An iterative computational procedure is presented.

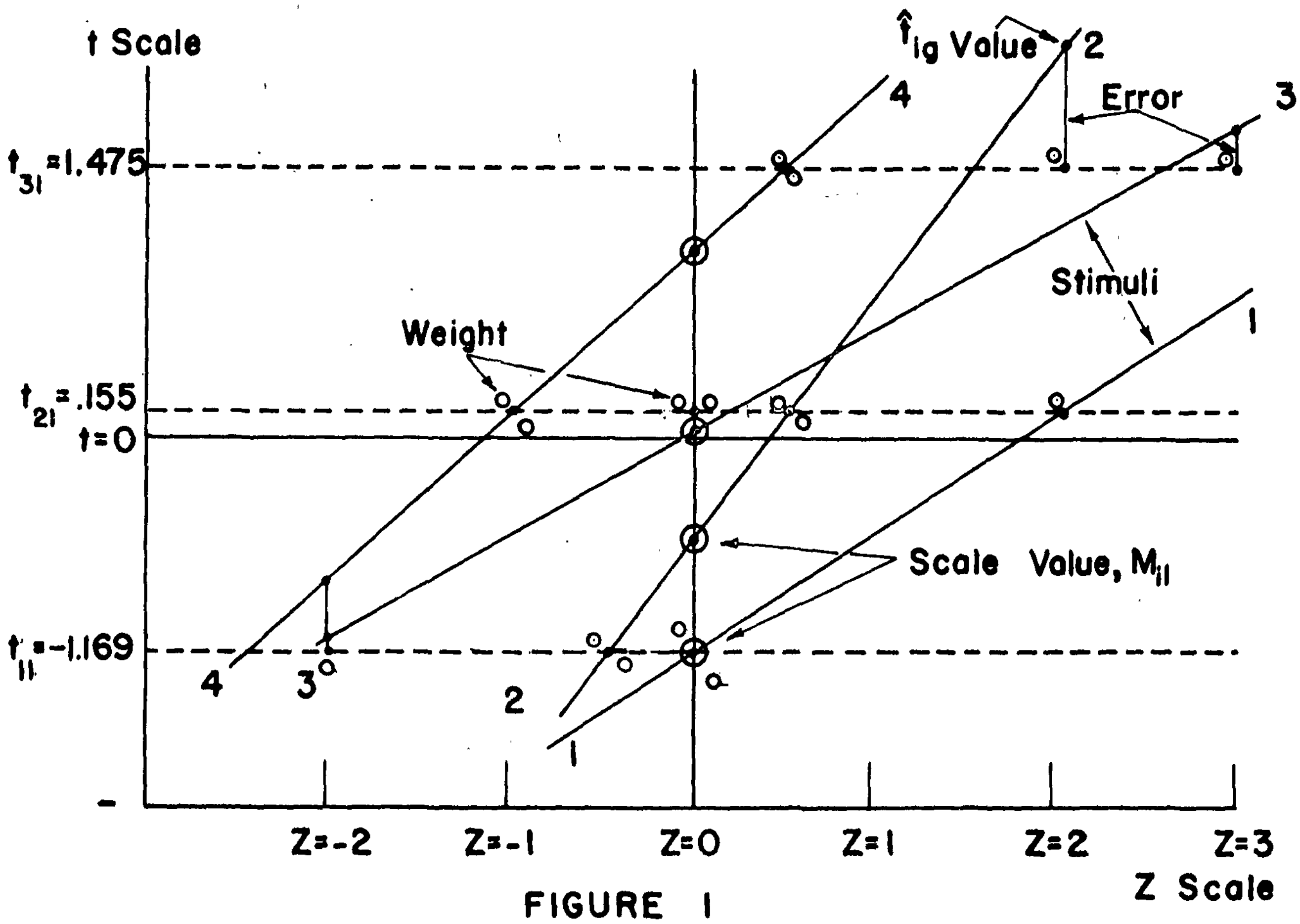

FIGURE 1

Graphical Solution for Numerical Example

Beginning with Equally Spaced t_{g1}

TABLE 1

Data for an Errorless Numerical Example, along with "True" Scale Values and Category Boundaries

Stimulus	z_{ig} Category 1	2	3	w_{ig}			m_i	s_i	t_g
1	0.0	2.0	5.0	2	1	0	-1.06458	.53229	$t_1 = -1.06458$
2	-0.5	0.5	2.0	2	2	1	-.53229	1.06458	$t_2 = 0$
3	-2.0	0.0	3.0	1	2	1	0	.53229	$t_3 = 1.59687$
4	-2.0	-1.0	0.5	1	2	2	1.06458	1.06458	

TABLE 2

First Iteration in the Numerical Example, beginning with Equally Spaced t_{g1}

Stimulus	t_{g1}	A_i	B_i	C_i	$\sum_g w_{ig} t_{g1}$	$\sum_g w_{ig} t_{g1} z_{ig}$	s_{i1}	m_{i1}	t_{g2}
1	$t_{11} = -1.16422$	.3750	.2500	.5000	-2.17321	.31046	.65972	-1.16422	$t_{12} = -1.08759$
2	$t_{21} = .15523$	.2381	.0952	.2381	-.54329	4.26883	1.06813	-.53575	$t_{22} = .03360$
3	$t_{31} = 1.47469$	.0784	.0196	.2549	.62093	6.75251	.51723	.02593	$t_{32} = 1.57282$
4		.2128	-.1277	.2766	2.09562	3.49267	1.01085	1.02566	

TABLE 3

Second Iteration in the Numerical Example

Stimulus	$\sum_g w_{ig} t_{g2}$	$\sum_g w_{ig} t_{g2} z_{ig}$	s_{i2}	m_{i2}	t_{g3}
1	-2.14158	.06720	.56060	-1.08759	$t_{13} = -1.06948$
2	-.53516	4.26683	1.06688	-.53362	$t_{23} = .00730$
3	.55243	6.89364	.52963	.00570	$t_{33} = 1.59193$
4	2.12525	3.68080	1.05467	1.05788	

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