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WADC TECHNICAL REPORT 55-231

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ANALYSIS OF VISCOUS INCOMPRESSIBLE AND COMPRESSIBLE FLOWS THROUGH AXIAL FLOW TURBOMACHINES WITH INFINITESIMAL AND FINITE BLADE SPACING

T. PAUL TORDA

UNIVERSITY OF ILLINOIS

MARCH 1956

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AERONAUTICAL RESEARCH LABORATORY
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FOREWORD

This Technical Report was prepared in the Aeronautical Engineering Department of the University of Illinois, Urbana, Illinois, for the Fluid Dynamics Research Branch, Aeronautical Research Laboratory, Wright Air Development Center, on Contract AF 33(616)-52, Project No. 3066, "Gas Turbine Technology," Task No. 70154, "Viscosity Effects in Subsonic Flow Compressors and Turbines," with Mr. S. Hasinger acting as task scientist. This is the sixth and final report on this project.

The work was carried out under the direction of Dr. T. P. Torda, Professor, Project Director, who is responsible for the writing of this report. Several workers cooperated on this project and it would be difficult to evaluate the contribution of each individual. Instead, the author wishes to express his sincere thanks to his co-workers for their efforts on this project. Appreciation is due to Dr. H. H. Hilton, Assistant Professor, Messrs. P. W. Born, F. C. Hall, H. A. Hassan, H. R. Russell, Research Assistants, and to the undergraduate computers who worked on the project.

ABSTRACT

A historical review of the various turbomachine theories is given. The Lorenz channel theory is developed for the analysis of viscous flows through turbomachines. Both the infinitesimally and the finitely spaced blades are considered. First, the incompressible, viscous and then the compressible, viscous flows are treated. The method of analysis consists of an iteration procedure in which the first step is the evaluation of an equivalent axially symmetric turbomachine stage with infinitesimally spaced blades. Thus, an axially symmetric flow field is used and the streamlines are determined for a desired pressure rise and an appropriately chosen velocity field in the stage. These streamlines form the infinitely thin streamline surfaces, i.e. the blades of the axially symmetric case. The results of this first step in the analysis are used as the first terms of infinite series for the velocities and pressure, when the transition to finitely spaced blades is made. These infinite series are written in powers of the tangential coordinate and the coefficients are functions of the radial as well as the axial coordinates. The fictitious force field of the analysis of infinitesimally spaced blades is eliminated and the complete Navier-Stokes equations are solved. The iteration starts from a number of frozen streamline surfaces obtained from the analysis of the axially symmetric case.

There are a number of boundary conditions which must be satisfied by the solution of the finitely spaced blade analysis. These are: 1. Along the frozen streamline surfaces, the values of the flow variables have been prescribed to be those of the infinitesimally spaced blade analysis. 2. Along the new and corrected streamline surfaces forming the actual blade surfaces in the finitely spaced blade analysis, the resultant velocity should be zero (non-slip condition). This latter boundary condition would make the already non-linear problem more non-linear and render the solution practically impossible. Therefore, these conditions have been relaxed. Since in modern turbomachines the blades are very thin, it was thought sufficiently accurate to prescribe that the non-slip conditions be enforced at the chord surfaces of the new blades and only approximately fulfilled at the actual blade surfaces. 3. The conditions of closure of the blade surfaces at the leading and trailing edges of the blades must be satisfied. Numerical illustrative examples are included. Thirty-three references are given.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:

Aldro Lingard

ALDRO LINGARD, Colonel, USAF
Chief, Aeronautical Research
Laboratory
Directorate of Research

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INTRODUCTION

Application of turbojet engines for aircraft propulsion intensified the efforts to develop a useful method for the evaluation and design of turbomachines. In the experimental field, systematic investigations of elements and also of complete engines have been carried out in order to determine the behavior of airfoil blades in cascade arrangement. The analytical methods employed at first were those used for steam and gas turbines and were of a semiempirical nature using average velocity distributions between blades in cascade arrangement. This approach may be called the momentum theory and it gives reasonably good results for the first evaluation of turbomachines. When refinements in analysis were required, two methods emerged.

The first method of approach, the direct method, attempts the evaluation of the flow between selected airfoils of known characteristics. Conformal mapping, source and sink or vortex distribution is used for the analysis of the flow fields and pressure distribution around the blades. The influence of the cascade arrangement is approximated by empirical factors accounting for the interaction of the flow and pressure fields around adjacent blades, since the evaluation of this effect by mathematical methods is practically impossible. Thus, at best, only an approximation of the

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NOTE - This technical report released for publication in November 1955.

interaction between blades is possible with this method. However, if the number of blades is small, i.e. the interaction between blades is small, the direct method can yield useful data for turbomachine analysis.

The second method of approach is the channel theory, or indirect method. The passages between blades are treated as channels and the shapes of their boundaries, i.e. the shape of the blades, are determined from the desired distribution of the flow variables (the velocity and pressure distributions) through the individual stages. It is of interest to note that the channel theory was developed first. When the circulation theory was introduced for analysis of cascades, the channel theory was forgotten, probably due to the fact that in its early form of development it was suitable only for the treatment of a turbomachine stage with infinite number of blades and only approximate corrections could be made to account for a finite number of blades.

In the following brief historical review is given the development of analytical methods for the evaluation and design of turbomachines. No attempt is made to present a complete review, but rather the early development is treated in more detail. This is done because the more recent work is well known by men working in this field. Some representative references are included which seem to have been of importance in the development of the channel theory. This list should not be considered as complete by any means.

The development of the theories of turbomachinery can be traced to analyses of water turbines and ship propellers which were the first practical applications of rotating flow-machinery. The first important step in the formulation of a theory was made by L. Euler in about 1740 with the derivation of the equations of motion for a nonviscous, homogeneous, incompressible fluid. Somewhat later he discussed the theory of water turbines proposed by Segner in 1750[1*]. He also gave a somewhat more general treatment of the subject in subsequent papers[2,3]. F. Prasil applied Euler's equations of motion to the investigation of water turbines in cylindrical coordinates in 1903[4]. Inspired by Prasil's work, H. Lorenz developed the theory of water turbines and ship propellers and published his work in several papers in 1905 and 1906.[5,6,7,8,9,10,11] The results are given in his famous book, A New Theory and Calculations of Turbine Wheels, first published in 1906[12]. Lorenz based his investigations on the analysis of an axially symmetric flow field in which the streamline surfaces correspond to an infinite number of blades, hence the axial symmetry of the flow. He introduced the concept of "forced accelerations" (Zwangsbeschleunigung), i.e. a force field per unit of mass and represented the effect of the infinite number of blades on the working fluid by these accelerations. Thus, the

*Superscripts denote References

"jump" in pressure distribution on either side of the infinitesimally spaced "blades" was represented by the force field. W. Bauersfeld showed that the force field must be perpendicular to the streamlines for a nonviscous fluid[13]. The concept of the Lorenz "forced accelerations" has been criticized severely by Prasil and von Mises, but Stodola[14], Bauersfeld[15], and Loewy[16] have successfully defended Lorenz and have shown that the attacks on his theory were based on misunderstandings. In 1909, R. von Mises published his "Theory of Water Wheels"[17] and derived equations similar to those of Lorenz. In 1912, Bauersfeld applied the Lorenz theory to the analysis and design of a Francis wheel and, thus, established its usefulness for practical applications.

In the meantime, Kutta applied the Joukowsky concept of circulation to the analysis of a straight lattice of airfoil blades[19] and in 1918, H. Foettinger introduced the use of circulation in turbine theory[20]. Many workers, among them Bauersfeld, Koenig, and Spannhake, applied the circulation theory to plane stationary lattices of airfoils. However, the momentum theory which is given in Stodola's "Steam and Gas Turbines"[21] has become the standard method for the evaluation of turbomachinery. Since this theory is based on average velocities over cross sections, a first approximation of design and output of the machines is possible only with this method.

As far as the author is aware of developments, H. J. Reissner in the late 1940's had revived the Lorenz theory to the analysis of turbomachinery and stationary lattices. Although Lorenz had already introduced corrections for a finite number of blades in his analysis, Reissner modified the method and introduced the transition from the infinitesimally spaced blades to those of finite spacing by "freezing" a certain number of streamline surfaces obtained from the axially symmetric analysis and calculating the flow between them by a perturbation method. Reissner and his collaborators published several papers[22,23,24] on the flows between finitely spaced lattices for incompressible and for compressible non-viscous flows. In 1949, the author suggested a method of application of the Lorenz-Reissner theory to the analysis of turbomachines with viscous flows[25]. This approach has been applied by Reissner and his co-workers to viscous flows between stationary lattices[26]. However, all of Reissner's work has been done with a one-step perturbation only and certain boundary conditions (the closure of the blade surfaces at the leading edge and the trailing edge) could not be satisfied.

Work at the NACA by C. Stanitz, C. H. Wu and others should also be mentioned, as well as analyses by F. E. Marble, W. D. Rannie, and others at California Institute of Technology. In nearly all work done to date the direct method of approach has been used, or a mixture of direct and indirect methods, and in most cases axial symmetry of the flow was assumed. Some numerical methods for correction for finite blade spacing exist, but a general treatment for finite blade spacing has not been given. As far as the author knows, the first complete analysis of viscous flows through turbomachines was done on this contract for the case of infinitesimally spaced blades[27] and also for the case of finitely spaced blades[28].

The analysis developed under this contract for the evaluation and design of turbomachines is based on the Lorenz-Reissner channel theory. The method consists of an iteration procedure in which the first step is the evaluation of an equivalent axially symmetric turbomachine stage with infinitesimally spaced blades. Thus, an axially symmetric flow field is used and the streamlines are determined for a desired pressure rise and an appropriately chosen velocity field in the stage. These streamlines form the infinitely thin streamline surfaces, i.e. the blades of the axially symmetric case. The results of this first step in the analysis are used as the first terms in infinite series for the velocities and pressure. These infinite series are written in powers of the tangential coordinate and the coefficients are functions of the radial as well as the axial coordinates.

The basis for the iteration is the fact that for the axially symmetric case a force field per unit of volume is assumed which represents the "jumps" of the pressure on either side of the streamline surfaces, i.e. the force field represents the action of the "blades" on the working fluid. This force field is eliminated, in the analysis of the flow through the stage with a finite number of blades, and the additional terms of the series correct the streamlines and evaluate the deviations from the axially symmetric streamlines. The corrected streamline surfaces form the upper and lower faces of the actual finitely spaced blades. Along with the correction of the streamlines, the velocity and pressure distributions of the axially symmetric case are corrected as well and the flow field in the stage with finitely spaced blades is evaluated. The method of transition from the axially symmetric to the axially non-symmetric case consists of "freezing" a certain number of streamline surfaces of the axially symmetric case and assuming that these frozen streamline surfaces will lie somewhere near the center of the channel between two blades. The freezing means that along these surfaces the flow in the finite case is prescribed to be the same as in the infinitesimal case.

In the analysis of the axially symmetric case, the Navier-Stokes momentum equations (including the terms of the Lorenz force field for a viscous fluid) and the continuity equation are used for the determination of the flow variables for prescribed inflow and exit conditions. The type of the resultant velocity profile in the channel between the infinitesimally spaced blades is assumed and the Navier-Stokes equations integrated. Then the streamline equations are integrated along the stage. The pressure and velocity distributions along the blade channels can be plotted once the intake and exit conditions are satisfied and the desired variation of the flow variables achieved by judicious choice of the arbitrary functions of integration.

The same equations are used for the analysis of finitely spaced blades, except that the Lorenz force field is eliminated from the Navier-Stokes equations and infinite series are introduced for the flow variables (the three velocity components and the pressure). By substituting the series into the Navier-Stokes and the continuity equations, recurrence formulae are obtained for the coefficients of the series. It was mentioned that the first terms of the series (u_0, v_0, w_0, and p_0) are those which were obtained from the analysis of the infinitesimally spaced blades. The series of two out of three of the velocity variables contain arbitrary functions as the second terms of the series. Thus, u_1 and either v_1 or w_1 have to be determined from boundary conditions. Once the coefficients of the series are determined, the streamline equations may be integrated. They yield the corrected streamline surfaces forming the upper and lower faces of the actual blades.

There are a number of boundary conditions which must be satisfied by the solution of the finitely spaced blade analysis. These are: 1. Along the frozen streamline surfaces, the values of the flow variables have been prescribed to be those of the infinitesimally spaced blade analysis. 2. Along the new and corrected streamline surfaces forming the actual blade surfaces in the finitely spaced blade analysis, the resultant velocity should be zero (non-slip condition). This latter boundary condition would make the already non-linear problem more non-linear and render the solution practically impossible. Therefore, these conditions have been relaxed. Since in modern turbomachines the blades are very thin, it was thought sufficiently accurate to prescribe that the non-slip conditions be enforced at the chord surfaces of the new blades and only approximately fulfilled at the actual blade surfaces. This means that the resultant velocity at the blade surfaces will not be zero but have a very small value. This relaxation of the non-slip conditions makes the boundary conditions linear. 3. The conditions of closure of the blade surfaces at the leading and trailing edges of the blades must be satisfied. The closure at the leading edges can be enforced by judicious choice of arbitrary integration constants and the closure at the trailing edges by the correct choice of the arbitrary functions u_1, and v_1 (or w_1). At first, the analysis of flows through turbomachine stages with infinitesimally spaced blades will be presented. The basic idea for such an analysis is given in Ref. 25 which made the inclusion of viscous effects possible. This method was applied first to laminar, incompressible flows[27] and somewhat generalized later along the lines suggested by author to include laminar as well as turbulent compressible or incompressible flows[29].

Numerical illustrative examples are also included.

Then, the extension of the analysis to turbomachine stages with finite number of blades is given for incompressible[28,30] and compressible[31] flows and the difficulties of considering turbulence are discussed.

ANALYSIS

I. VISCOUS FLOWS THROUGH TURBOMACHINE STAGES WITH LARGE NUMBER OF BLADES INFINITESIMAL BLADE SPACING.

I.1. METHOD OF SOLUTION.

In considering viscous flows through turbomachine stages with infinitesimally spaced blades, the Lorenz force field is generalized to include the action of shear forces. The force field, as conceived by Lorenz, accounted for the action of the "blades" on the fluid. This means that only the difference of pressure on either side of the streamline surfaces was represented by inclusion of force field terms on the momentum equations.

The assumption of infinitesimally spaced blades is equivalent to axial symmetry of the flow and it also allows the use of the average resultant velocity between these blade surfaces instead of the actual profile of the resultant velocity.

The analysis, as presented, is valid for rotating coordinate systems (rotating stage). For a stator stage, the Coriolis force terms are equal to zero.

The Navier-Stokes equations for axially symmetric flows ($\frac{\partial}{\partial \varphi} = 0$) written in cylindrical coordinates and including the Coriolis forces and the force field terms are

$$\left.\begin{aligned} u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} - 2v\omega - \omega^2 r &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + f_r \\ u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} + 2u\omega &= f_\varphi \\ u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + f_z \end{aligned}\right\} \qquad (I.1.1)$$

where

r, φ, z	are the cylindrical coordinates
u, v, w	are the velocity components in the r, φ and z directions, respectively. These velocity components are measured relative to the rotating coordinate system
p	is the pressure
ρ	is the density
ω	is the constant angular velocity
f	is the force field component per unit of mass
	subscripts r, φ and z denote directions

Fig. 1 gives the notation used.

The force field accounts for the effects of the dilatation and the shear stresses. Its components, f_r , f_φ and f_z , are inclined to the streamline surfaces and each may be written as the vector sum of two components, $g_r + h_r$ etc., such that g_r, g_φ and g_z are force components per unit of mass perpendicular to the streamline surfaces, and h_r, h_φ and h_z are force components per unit of mass tangent to the streamline surfaces.

The magnitude of the resultant velocity, relative to the rotating coordinate system, is

$$V^2 = u^2 + v^2 + w^2 \tag{I.1.2}$$

As was mentioned, instead of the actual value of the resultant velocity V , its average will be used in the analysis. This average may be determined depending on the laminar or turbulent nature of the flow. The procedure, though well known, will be explained later in the chapter on "Numerical Examples". Therefore, from here on, V will denote the average of the resultant relative velocity at any point between the streamline surfaces. The resultant of the force field due to shear forces h and its components satisfy the relationship

$$hV = h_r u + h_\varphi v + h_z w \tag{I.1.3}$$

and the g - components of f satisfy

$$g_r u + g_\varphi v + g_z w = 0 \tag{I.1.4}$$

since they are perpendicular to V .

Multiplication of the Navier-Stokes equations (I.1.1) by u , v and w, respectively, and summation of the three component equations, yields

$$\frac{D\lambda}{Dt} = \mathcal{h} V \qquad \text{(I.1.5)}$$

where $\frac{D}{Dt} = u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z}$

$$\text{and } \lambda \equiv \frac{V^2}{2} - \frac{\omega^2 r^2}{2} + P \qquad \text{(I.1.6)}$$

with P , the enthalpy function,

$$P = \int \frac{dp}{\rho} \qquad \text{(I.1.7)}$$

Eq. (I.1.5) may be re-written as

$$u\frac{\partial \lambda}{\partial r} + w\frac{\partial \lambda}{\partial z} = \mathcal{h} V \qquad \text{(I.1.8)}$$

and eq. (I.1.8) is a Lagrange equation which may be solved by finding a solution of the following system of ordinary differential equations, the subsidiary equations of (I.1.8)

$$\frac{dr}{u} = \frac{dz}{w} = \frac{d\lambda}{\mathcal{h} V} \qquad \text{(I.1.9)}$$

The complete integral of eq. (I.1.8) is

$$\psi\left[\xi(r,z,\lambda), \eta(r,z,\lambda)\right] = 0 \qquad \text{(I.1.10)}$$

where

$$\xi(r,z,\lambda) = \alpha \quad \text{and} \quad \eta(r,z,\lambda) = \beta \qquad \text{(I.1.11)}$$

are two independent solutions of eqs. (I.1.9), the characteristics of the system, α and β are constants of integration, and Ψ is an arbitrary function.

The first equation in (I.1.9), i.e.

$$\frac{dr}{u} = \frac{dz}{w} \qquad \text{(I.1.12)}$$

is the equation of the $r-z$ trace of the streamlines. Let the integral of eq. (I.1.12) be

$$\zeta(r,z) = \sigma \qquad \text{(I.1.13)}$$

where σ is a constant of integration and to each value of $\sigma = \sigma_i$ corresponds an $r-z$ trace of a streamline.

Eq. (I.1.15) (and, therefore, eq. (I.1.8)) must be integrated along streamlines, since they represent the resultant of the component momentum equations. Let in (I.1.11)

$$\xi = \lambda - \int \frac{hV}{u}\, dr = \alpha \qquad \text{(I.1.14)}$$

and

$$\eta = \lambda - \int \frac{hV}{w}\, dz = \beta \qquad \text{(I.1.15)}$$

Let

$$R(r,z) \equiv \int \frac{hV}{u}\, dr \quad \text{and} \quad Z(r,z) \equiv \int \frac{hV}{w}\, dz \qquad \text{(I.1.16)}$$

then, from (I.1.14) and (I.1.15)

$$\lambda = R(r,z) + \alpha \qquad \text{(I.1.17)}$$

and

$$\lambda = Z(r,z) + \beta \qquad \text{(I.1.18)}$$

However, through (I.1.10), (I.1.11), (I.1.12) and (I.1.13), a functional relationship exists between α, and β. Let

$$\alpha = \Theta_1(\sigma) \quad \text{and} \quad \beta = \Theta_2(\sigma) \qquad \text{(I.1.19)}$$

then, eqs. (I.1.17) and (I.1.18) may be written as

$$\lambda = R(r,z) + \Theta_1(\sigma_i) \qquad \text{(I.1.20)}$$

and

$$\lambda = Z(r,z) + \Theta_2(\sigma_i) \qquad \text{(I.1.21)}$$

since the integration must be carried out along streamlines (here, along $r-z$ traces of the streamlines).

For the solution of (I.1.18), the values of λ must be given along an initial curve, $C\,(r_o(\sigma), z_o(\sigma))$, e.g. the leading edge of the compressor blade. This means that

$$\lambda = \lambda_o(r_o, z_o) \qquad \text{(I.1.22)}$$

must be known and the only condition on $C(r_o, z_o)$ is that it should not be a streamline trace (characteristic line).

The initial value of λ defines the functions $\Theta_1(\sigma)$ and $\Theta_2(\sigma)$. If σ is eliminated between eqs. (I.1.20) and (I.1.21), the solution $\lambda = \lambda(r, z)$ is obtained which satisfies the differential equation (I.1.8) and the initial conditions (I.1.22).

The solution of eq. (I.1.8), as described, is straightforward, if u, w and $\mathcal{R} V$ are known functions of r, z and λ, and if u, w and $\mathcal{R} V$ and their first derivatives are continuous in the r - z - λ space of interest. As a matter of fact, the channel theory requires the determination of the blade shapes from desired velocity and pressure distribution through the stage.

In what follows, a somewhat more general treatment of the problem is given. Instead of prescribing $u(r, z)$, $w(r, z)$ and $p(r, z)$ (and, therefore $\lambda(r, z)$), the types of $R(r, z)$ and $Z(r, z)$, see eqs. (I.1.16), will be prescribed and these functions, together with the initial conditions, will determine the flow parameters as well as the blade shapes throughout the stage. Assumption of separable variables in R and Z greatly simplifies the solution and yields good flow parameter distribution and blade shapes.

For the complete solution of the problem, the continuity equation

$$\frac{\partial(\rho r u)}{\partial r} + \frac{\partial(\rho r w)}{\partial z} = 0 \qquad (I.1.23)$$

must be satisfied also, and either an equation of state,

$p = p(\rho, T)$, or a $p - \rho$ relationship must be given in form of a change of state equation, $p = p(\rho)$.

I.2. SPECIAL SOLUTIONS.

In the following, some special solutions are given. It will be seen that reasonably general cases are obtained even though simplifying assumptions are made.

I.2.a. ANALYSIS OF LAMINAR INCOMPRESSIBLE FLOWS WITH CONSTANT AXIAL VELOCITY COMPONENT.

It is usual to design axial flow compressors with a constant axial velocity, i.e.

$$w = w_0 \qquad \text{(I.2.a.1)}$$

For incompressible flow, the density ρ = const. Since the flow is laminar, one may assume that the velocity distribution in each channel between the blades is a non-symmetric parabola (Fig. 2) of the non-dimensional form*

$$\eta = (1 - \xi^2)/(1 - 2a\xi + a^2) \quad \text{and} \quad |a| < 1 \qquad \text{(I.2.a.2)}$$

where

$$\eta = V/V_{max} \qquad \xi = y/b$$
$$a = e/b \qquad b = d/2 \qquad \text{(I.2.a.3)}$$

and e is the eccentricity.

*The concept of parabolic velocity distribution in the channel was first introduced in Ref. 6.

It should be noted that the parabola used here (non-symmetric with respect to the resultant velocity, V) is representative only. Other, higher order polynomials may be used instead if turbulent flows are investigated and a flat velocity distribution exists between the blades. However, the choice of the type of velocity distribution is significant only for the results of the case of infinitesimal blade spacing, since this velocity distribution is corrected by the series when the finite blade spacing is treated. Therefore, only the convergence of the series will be affected by the original choice of the curve which represents the resultant velocity profile between blades.

For laminar flow the shear stress τ equals

$$\tau = \mu \left\{ \left[\partial V / \partial n \right]_{n = \frac{d}{2}} + \left[\partial V / \partial (-n) \right]_{n = -\frac{d}{2}} \right\} \qquad \text{(I.2.a.4)}$$

where n is the direction normal to the streamlines

d is the blade spacing

μ is the viscosity (taken as constant).

The blade spacing (Fig. 1) is

$$d = r \phi \qquad \text{(I.2.a.5)}$$

and ϕ is the pitch angle.

The resultant shear force per unit volume may now be calculated from eqs. (I.2.a.4) and (I.2.a.2)

$$h = \frac{F}{d} = -12\mu V_{ave}(1+a^2)/[k b^2(1-a^2)^2] \qquad \text{(I.2.a.6)}$$

where

$$V_{max} = 3V_{ave}/2k \qquad \text{(I.2.a.7)}$$

and

$$k = \frac{3}{4}\int_{-1}^{+1} \frac{(1-\xi^2)}{(1-2a\xi+a^2)}\, d\xi \qquad \text{(I.2.a.8)}$$

for $0 < |a| < 1$

$$k = \frac{3}{8}\left\{\frac{(1-a^2)^2}{2a^3}\, ln\left|\frac{1-a}{1+a}\right| + \frac{1+a^2}{a^2}\right\} \qquad \text{(I.2.a.9)}$$

and

$$\text{for} \quad a = 0 : k = 1$$
$$a = 1 : k = \frac{3}{4} \qquad \text{(I.2.a.10)}$$

The analysis which follows is valid for any prescribed velocity profile, since only the value of the constant k varies for various velocity profiles.

Eq. (I.1.3) with (I.2.a.6) becomes

$$u h_r + v h_\varphi + w h_z = hV = \frac{-12\mu V^2(1+a^2)}{k b^2(1-a^2)^2} \qquad \text{(I.2.a.11)}$$

For incompressible flow, and with the assumption of constant axial velocity through the blade, eq. (I.2.a.1), the continuity equation (I.1.23) reduces to

$$\frac{\partial(ru)}{\partial r} = 0 \qquad \text{(I.2.a.12)}$$

with the solution

$$u = Z_1(z)/r \qquad \text{(I.2.a.13)}$$

where $Z_1(z)$ is an arbitrary function of z to be determined by the boundary conditions of the problem.

Eq. (I.1.5) will be for this case

$$\frac{D\lambda}{Dt} = \frac{-12\mu V^2(1+a^2)}{\rho \kappa b^2(1-a^2)^2} \qquad \text{(I.2.a.14)}$$

If the quantity a, eq. (I.2.a.3), is taken as constant, then all the points of maximum velocity V_{max} in the channel lie along a radius at any particular z. Substitution for b^2 from eq. (I.2.a.3) and use of eqs. (I.2.a.5) and (I.2.a.3) reduces eq. (I.2.a.14) to

$$u\frac{\partial \lambda}{\partial r} + w_0\frac{\partial \lambda}{\partial z} = \frac{AV^2}{r^2} \qquad \text{(I.2.a.15)}$$

with

$$A = \frac{-48\mu(1+a^2)}{\rho \kappa \phi (1-a^2)^2} \qquad \text{(I.2.a.16)}$$

Eq. (I.2.a.15) is the Lagrange partial differential equation for this case and its subsidiary equations are

$$\frac{dr}{u} = \frac{dz}{w_0} = \frac{r^2 d\lambda}{AV^2} \qquad \text{(I.2.a.17)}$$

One integral of (I.2.a.17) is

$$\lambda = \frac{A}{w_0}\int \frac{V^2}{r^2}\, dz + C_1 \qquad \text{(I.2.a.18)}$$

where C_1 is an arbitrary constant.

Let

$$\frac{V^2}{r^2} = \frac{\partial \theta}{\partial z} \quad \text{and} \quad \theta = \theta(r,z) \qquad \text{(I.2.a.19)}$$

then eq. (I.2.a.18) becomes

$$\lambda = \frac{A\theta}{w_0} + C_1 \qquad \text{(I.2.a.20)}$$

Eliminating λ and $\frac{V^2}{r^2}$ with help of eqs. (I.2.a.19) and (.2.a.20), the differential equation (I.2.a.15) simplifies to

$$u \frac{\partial \theta}{\partial r} = 0 \qquad \text{(I.2.a.21)}$$

Since $u \neq 0$, the solution is

$$\theta = Z_2(z) \qquad \text{(I.2.a.22)}$$

and $Z_2(z)$ is an arbitrary function of z along each $r - z$ streamline trace. In general, $Z_2 = Z_2(r,z)$.

With the above results, the solution of the problem is completed. Thus

$$V^2 = r^2 \frac{\partial \theta}{\partial z} \qquad \text{(I.2.a.23)}$$

$$p = \left[-r^2 \frac{\partial \theta}{\partial z}/2 + \frac{\theta}{w_0} + \omega^2 r^2/2 + C_1 \right] \qquad \text{(I.2.a.24)}$$

$$v^2 = r^2 \frac{\partial \theta}{\partial z} - \left[\frac{Z_1(z)}{r} \right]^2 - w_0^2 \qquad \text{(I.2.a.25)}$$

The velocity and pressure fields in the compressor are completely described by the foregoing equations. The traces of the streamlines may be determined from

$$\frac{dr}{u} = \frac{r d\varphi}{v} = \frac{dz}{w} \qquad \text{(I.2.a.26)}$$

The integration of pairs of differential equations will then yield the streamline traces in any plane.

For any given stage of the compressor, the angular velocity ω, the magnitude and direction of the entrance velocity, the entrance pressure and hub and shroud radii at the entrance are prescribed. These known quantities are used to determine the arbitrary functions $Z_1(z)$ and $\theta_1(r,z)$ and the constant C_1.

It should be remembered that since these latter functions are arbitrary it is permissible to assume their type. Such assumptions in no way detract from the uniqueness of the solution, since for any given combination of Z, θ, and C_1 one and only one blade profile is found. Different velocity and pressure fields will naturally require different blade shapes.

C_1 and the initial values of $Z_1(o)$ and $\theta(o)$ are determined from the given entrance conditions. The variation of Z_1 and θ from entrance to exit has to be chosen to conform with the physical limitations on pressure distribution through the stage and the curvature of the infinitesimal "blades". Too rapid a rise of pressure corresponds to extreme curvature of the "blade" surface at the leading edge which would increase viscous losses and facilitate flow-separation. The functions Z_1 and θ also determine the total twist, $\varphi(\ell)$, of the "blade" surfaces. Excessive values of $\varphi(\ell)$ also increase the viscous losses due to the increase in "blade" surface area. There is no method available for the a priori determination of the types of functions to be used for Z_1 and θ . Since the infinitesimal "blade" surfaces will serve as chord-surfaces of the finitely spaced blades, the choice of Z_1 and θ functions may be made to obtain known efficient chord curvatures for the finitely spaced blades.

I.2.b. NUMERICAL EXAMPLE FOR LAMINAR INCOMPRESSIBLE FLOWS WITH CONSTANT AXIAL VELOCITY COMPONENT

The aim of the numerical work was to determine under which conditions a maximum pressure rise could be achieved through one rotating stage of infinitesimally spaced blades (stream-surfaces). At the same time large twist and large curvatures of these surfaces were to be avoided. A number of numerical examples were worked; however, only one is presented in this report showing trend and limitations of the design.

There are two arbitrary functions $Z_1(z)$ and $\Theta(r,z)$, and one arbitrary constant C_1 in the analysis. C_1 may be determined from eq. (I.2.a.24) with the intake conditions. Thus,

$$C_1 = \frac{p_i}{\rho} + \frac{V_i^2}{2} - \frac{\omega^2 r^2}{2} - \frac{A Z_2(0)}{w_0} \qquad \text{(I.2.b.1)}$$

where subscript i denotes values at the intake of the rotating stage.

The choice of the arbitrary functions, $Z_1(z)$ and $\Theta(r,z)$, determines the blade shape.

Eq. (I.2.a.13) shows that the variation of the radial component, u, of the resultant velocity, V, depends on the arbitrary function Z_1. Through the equation of the streamlines, eq. (I.2.a.26), Z_1 also determines the $r - z$ variation of shroud and hub. The requirement of small curvature of the stream surfaces restricts the choice of the type of functions used for Z_1.

The pressure rise through the rotating stage along each streamline is due to dynamic recovery and centrifugal action.

The pressure rise due to dynamic recovery is a maximum if the resultant velocity at the exit, V_e, is zero. (Subscript e denotes exit conditions.) Since, however, the axial component of the relative velocity, w_o , is taken as constant and since the influence of the radial velocity component is small, the tangential velocity component at the exit, v_e , was taken as zero. Thus

$$w_o^2 + \left[\frac{Z_1(\ell)}{r_e}\right]^2 = r_e^2 Z_2'(\ell) \qquad \text{(I.2.b.2)}$$

where ℓ is the value of z at exit. With the condition of v_e , the stream-surface at the exit is tangential to the $r - z$ plane.

The pressure rise due to centrifugal action is directly dependent on the ratio of the exit and intake radii, i.e., r_e / r_i , and on the angular velocity ω . Use of eq. (I.2.b.1) in (I.2.b.2) yields

$$\frac{p}{\rho} = \left[\frac{p_i}{\rho} + \frac{(v_i^2 - \omega^2 r_i^2)}{2}\right] + \frac{(\omega^2 - Z_2'^2) r^2}{2} + \frac{A Z_2}{w_o} \qquad \text{(I.2.b.3)}$$

The angular velocity is prescribed for any particular design problem. Then, the pressure rise due to centrifugal action is a function of the exit radius, r_e . If this radius is chosen, e.g., the exit hub-radius might be prescribed as an allowable maximum, then the maximum pressure is defined. The shape of each streamline in the $r - z$ plane is, thus, completely determined by prescribed values of the respective r_i and by a compromise between the value of a maximum pressure rise and a reasonable hub exit radius, and by the requirement of gentle variation from intake to exit. The choice of Z_1 yielding a straight conical shroud and slightly conical hub shape is thought to be appropriate, and this is the reason for the choice of the function

$$Z_1(z) = w_0 \left[\frac{\partial r_s}{\partial z}\right]_i \left\{ r_{si} + \left[\frac{\partial r_s}{\partial z}\right]_i z \right\} \qquad \text{(I.2.b.4)}$$

where subscript s denotes shroud. Note: For the case of straight conical shroud $\partial r_s/\partial z$ = const. = M_s .

Once the $r - z$ variation of the streamlines is given, the twist of the stream-surface is defined by θ , since $V^2/r^2 = \partial\theta/\partial z$. In order to reduce the length of the stream-surfaces, and thus, reduce the magnitude of the viscous dissipation, the twist angle should be small. This requirement and also the necessity of avoiding sharp curvatures of the stream-surfaces allow the choice of θ . From the various functions examined, a hyperbolic form was selected:

$$\frac{\partial\theta}{\partial z} = \frac{Z_1^2}{r^4} + \frac{w_0^2}{r^2} + \omega^2 \left\{ \frac{1-(z/l)}{1+(z/l)/\beta} \right\}^2 \qquad \text{(I.2.b.5)}$$

in which the parameter β allows the choice of the total twist angle.

It should be noted that the numerical work for the infinitesimal blade spacing is exploratory at best and the final design of axial flow turbomachines will be the result of the analysis involving a finite number of blades.

One more remark is necessary. The two-dimensional vortex flow may be obtained as a special case of the presented three-dimensional analysis. Then, u and Z_1 are zero. It was not thought necessary to include numerical work for the two-dimensional analysis, since the choice of w as a constant would make the flow a rather restricted one. However, the solution of the vortex flow problem with $w = w(r,z)$ may be obtained in a similar manner.

From the condition on the selected shroud shape, i.e.,

$$\frac{\partial r_s}{\partial z_s} = \text{const.} = M_s \tag{I.2.b.6}$$

the shroud radius may be obtained as

$$r_s = r_{si} + M_s z \tag{I.2.b.7}$$

Hence, $Z_1(z)$ takes the form

$$Z_1 = w_0 M_s (r_{si} + M_s z) \tag{I.2.b.8}$$

From eq. (I.2.a.13)

$$u = w_0 M_s (r_{si} + M_s z)/r \tag{I.2.b.9}$$

Eq. (I.2.a.26) yields for the $r - z$ variation of the streamlines

$$r^2 = r_i^2 + 2 M_s r_{si} z + M_s^2 z^2 \tag{I.2.b.10}$$

From eq. (I.2.b.5), the tangential component of the velocity is

$$v = r\omega\{1 - (z/l)\} / \{1 + (z/l)/\beta\} \qquad \text{(I.2.b.11)}$$

and eq. (I.2.a.26) yields for the $\varphi - z$ variation of the streamlines

$$\varphi = \left(\frac{\omega\beta}{w_0}\right)\left\{ l(\beta+1) \ln\left[(z/l\beta) + 1\right] - z \right\} \qquad \text{(I.2.b.12)}$$

At the exit

$$\varphi(l) = \left(\frac{\omega\beta l}{w_0}\right)\left\{ (\beta+1) \ln\left[(1/\beta) + 1\right] - 1 \right\} \qquad \text{(I.2.b.13)}$$

and this represents the total twist of the "blade" (Streamline surface). The arbitrary constant β allows the selection of the total twist.

The expression for the pressure distribution is obtained by substituting (I.2.b.1) into (I.2.a.24). This gives

$$p = p_i + (v_i^2 - \omega^2 r_i^2)/2 + (\omega^2 r^2 - v^2)/2 + \left(\frac{A}{w_0}\right)\int_0^z \left(\frac{v^2}{r^2}\right) dz \qquad \text{(I.2.b.14)}$$

For the case of no pre-rotation of the flow entering the rotating stage, $v_i = \omega r_i$. If the total energy is taken as constant for all entering streamlines, then

$$p_i = p_{Ti} - (w_0^2 + u_i^2)/2 \qquad \text{(I.2.b.15)}$$

where p_{Ti} denotes the ratio of the total pressure at entrance to the density. Use of (I.2.b.15) in (I.2.b.14) yields

$$p = p_{Ti} + (\omega^2 r^2 - V^2)/2 + \left(\frac{A}{w_0}\right)\int_0^z \left(\frac{v^2}{r^2}\right) dz \qquad \text{(I.2.b.16)}$$

The pressure-density ratio distribution at the entrance is

$$p_i = p_{Ti} - \frac{w_i^2}{2} - (w_0^2 M_s^2 r_{si}^2)\left(\frac{1}{2 r_i^2}\right) \qquad \text{(I.2.b.17)}$$

The power imput, torque, mass rate of flow, useful power output and the average pressure rise through the stage may be computed easily. However, such calculations are not included in this example, since they are, at best, approximations. They are given in Ref. 28.

Table I contains the numerical values used in the illustrative example. Tables II, III, IV, and V give the numerical results of the flow parameters for the numerical parameters selected in Table I. Table II gives the twist φ computed for various values of z (axial dimension) from $z = 0$ to $z = \ell$. This table is good for all streamlines. That is to say, the $\varphi - z$ variation is independent of r due to the selection of v/r as a function of z alone. Tables IIIa, IIIb, and IIIc give the variation of r , u , v , V , and p with z along the hub, a medium, and shroud streamlines respectively. Table IV lists the variation of static and total pressures at exit for various values of r_e . Table V lists the variation of static pressure at inlet for various values of r_i .

The $r-z$ variation of streamlines are shown in Fig. 3, and their $\varphi - z$ variation in Fig. 4. The radius variation with z is presented in Fig. 5 in non-dimensional form, Figs. 6 and 7 show the variation along z of the velocity components and of the resultant velocity. The pressure ratio distribution is presented in Fig. 8. Fig. 9 is a schematic drawing of the resulting streamline surface ("blade").

I.2.c. ANALYSIS OF LAMINAR OR TURBULENT INCOMPRESSIBLE OR COMPRESSIBLE FLOWS WITH VARIABLE AXIAL VELOCITY COMPONENT.

In the analysis that follows, the functions (I.1.16) were chosen as

$$R \equiv R(r) \quad \text{and} \quad Z = Z(z) \qquad \text{(I.2.c.1)}$$

Eq. (I.1.12) multiplied by $r V$ yields as the equation of the $r-z$ streamline trace (I.1.13)

$$R = Z + \sigma \qquad (\sigma = \sigma_i \text{ , a constant}) \qquad \text{(I.2.c.2)}$$

From (I.1.16) and (I.2.c.1), the following relationship is obtained

$$\frac{u}{w} = \frac{Z'}{R'} \qquad \text{(I.2.c.3)}$$

where prime denotes differentiation (with respect to r and z, respectively).

With (I.2.c.3), eq. (I.1.23) becomes

$$Z'\frac{\partial}{\partial r}(\rho r u) + R'\frac{\partial}{\partial z}(\rho r u) = (\rho r u)\, R'\frac{Z''}{Z'} \qquad \text{(I.2.c.4)}$$

which again is a Lagrange differential equation with the subsidiary equations

$$\frac{dr}{Z'} = \frac{dz}{R'} = \frac{d(\rho r u)}{(\rho r u)\, R'\frac{Z''}{Z'}} \qquad \text{(I.2.c.5)}$$

A general solution of (I.2.c.4) may be written as

$$X\left(\frac{\rho r u}{Z'},\ R - Z\right) = 0 \qquad \text{(I.2.c.6)}$$

where X is an arbitrary function.

From (I.2.c.6), u may be expressed in the form

$$u = \frac{Z'}{\rho r}\,\chi(R - Z) \qquad \text{(I.2.c.7)}$$

where χ is the inverse function of X and is determined from the initial conditions.

Eq. (I.2.c.3) yields

$$w = \frac{R'}{\rho R}\,\chi(R - Z) \qquad \text{(I.2.c.8)}$$

and $\hbar V$ may be found as

$$\hbar V = \frac{R'Z'}{\rho r}\,\chi(R - Z) \qquad \text{(I.2.c.9)}$$

In laminar and turbulent flows, the relationship between $\hbar$ and V is known

$$\mathcal{R} = \frac{1}{\rho} F(V) \qquad \text{(I.2.c.10)}$$

Then, if

$$\mathcal{R}V = \frac{V}{\rho} F(V) \qquad \text{I.2.c.11)}$$

the expression for the resultant velocity will be

$$V = G\left\{\frac{R'Z'}{r} \chi(R-Z)\right\} \qquad \text{(I.2.c.12)}$$

where G is a function defined by (I.2.c.11).

For the special case of incompressible flow, the density is constant, $\rho = \rho_0 =$ const., and the pressure is found from eqs. (1.6), (1.21) and (I.2.c.12) as

$$p = -\frac{\rho}{2}\left[G\left\{\frac{R'Z'}{r}\chi(R-Z)\right]^2 + \frac{1}{2}\omega^2 r^2 + Z + \theta_2(\sigma_i) \qquad \text{(I.2.c.13)}$$

along the $\sigma = \sigma_i$ streamline trace.

For compressible flows, if adiabatic change of state is assumed through the stage, or

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} \qquad \text{(I.2.c.14)}$$

and the density variation along any streamline trace may be found as

$$\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = \frac{\gamma-1}{\gamma}\frac{\rho_0}{p_0}\left[Z + \theta_2(\sigma) - \frac{1}{2}\left\{G\left[\frac{R'Z'}{r}\chi(R-Z)\right]\right\}^2 + \frac{1}{2}\omega^2 r^2\right] \qquad \text{(I.2.c.15)}$$

The pressure, resultant velocity and its components may then be found from the given relationships.

I.2.d. NUMERICAL EXAMPLE FOR LAMINAR COMPRESSIBLE ADIABATIC FLOWS WITH VARIABLE AXIAL VELOCITY COMPONENT.

In this example, the following numerical values were used

Length of stage $\ell = 0.25$ ft

Constant angular velocity $\omega = 1570.8$ rad/sec

Velocity components along leading edge of blade

Radial velocity $u_0 = 10$ ft/sec

Axial velocity $w_0 = 450$ ft/sec

Tangential velocity $v_0 = 1250\sqrt{r}$ ft/sec

Density $\rho_0 = 0.002378$ slugs/ft^3

Pressure $p_0 = 2116.8$ lb/ft^2

Kinematic viscosity $\nu_0 = \frac{\mu}{\rho_0} = 1.6 \times 10^{-4}$ ft^2/sec

Number of blades 48

The equation of the leading edge was chosen as

$$r_0 = z + a \quad \text{and} \quad a \leq r \leq b \tag{1.2.d.1}$$

with $a = 0.25$ ft and $b = 0.33$ ft.

Fig. 10 shows the $r - z$ projection of typical streamlines. Again, as in the example of Chapter 2.b, the parabolic distribution of the resultant velocity is used (Ref. 25), see Fig. 2. Then, $F(v)$ in eq. (I.2.c.10) becomes

$$F(V) = c_3 \frac{V}{r^2} \tag{I.2.d.2}$$

with

$$c_3 = -12 \nu_0 \left(\frac{4g}{\pi}\right)^2 \tag{I.2.d.3}$$

The functions in (I.2.c.1) will be

$$\left.\begin{aligned} R'(r) &= \frac{c_4}{r} \\ \text{and} \\ Z'(z) &= \frac{c_5}{z+a} \end{aligned}\right\} \tag{I.2.d.4}$$

with

$$c_4 = \frac{V_0^2}{r}\frac{c_3}{u_0} \quad \text{and} \quad c_5 = \frac{V_0^2 c_3}{w_0} \tag{I.2.d.5}$$

Fig. 11 shows the streamlines passing through $r = a$ (hub), $r = \frac{a+b}{2}$ (medium) and $r = b$ (shroud). Fig. 12 presents the resultant velocity, V, variation through the stage and Fig. 13 the velocity components. The static pressure variation is given in Fig. 14 and the density variation in Fig. 15. From Fig. 14 it may be seen that the pressure rise across the stage is approximately 11%.

I.2.e. NUMERICAL EXAMPLE FOR TURBULENT COMPRESSIBLE FLOW WITH VARIABLE AXIAL VELOCITY COMPONENT.

The initial values for the turbulent case are the same as those for the laminar one (Chapter 2.d), except that

$$v_0 = 674 \text{ ft/sec} \qquad \text{(I.2.e.1)}$$

Von Kármán's one-seventh power law[32] is used for the profile of the resultant velocity between blades, and Schlichting's expression for the average shear stress[33]. These give

$$F(V) = c_6 \frac{V^{4/5}}{r} \qquad \text{(I.2.e.2)}$$

with

$$c_6 = 0.072 \left(\frac{\mu}{2}\right)^{1/5} \left(\frac{8\rho_0}{7}\right)^{4/5} \qquad \text{(I.2.e.3)}$$

The average value of the resultant velocity is

$$V_{ave.} = \frac{7}{8} V_{max.} \qquad \text{(I.2.e.4)}$$

The results for this example are presented in Figs. 16-20, showing the blade shape, resultant velocity variation, velocity components, static pressure and density through the stage, respectively.

II. INCOMPRESSIBLE VISCOUS FLOWS THROUGH TURBOMACHINE STAGES WITH LARGE NUMBER OF BLADES. FINITE BLADE SPACING

II.1. METHOD OF SOLUTION. INCOMPRESSIBLE FLOWS.

In this analysis the Lorenz force-field is eliminated and the three-dimensional, incompressible continuity and Navier-Stokes equations are used. The equations are used in their general form taking the rotation of the stage into account. However, the same equations and analysis are valid for stationary stages except that the angular velocity is zero for non-rotating stages.

The following basic equations are used in cylindrical coordinates:

The continuity equation

$$\left(\frac{1}{r}\right)\frac{\partial(ru)}{\partial r} + \left(\frac{1}{r}\right)\frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial z} = 0 \qquad \text{(II.1.1)}$$

The Navier-Stokes equations

$$\left.\begin{aligned}
&\frac{Du}{Dt} - \frac{v^2}{r} - 2v\omega + \frac{\partial p}{\partial r} = \nu\left[\nabla^2 u - \frac{u}{r^2} - \left(\frac{2}{r^2}\right)\frac{\partial v}{\partial \varphi}\right] \\
&\frac{Dv}{Dt} + \frac{uv}{r} + 2u\omega + \frac{1}{r}\frac{\partial p}{\partial \varphi} = \nu[\nabla^2(v + \omega r) \\
&\qquad\qquad + \frac{2}{r^2}\frac{\partial u}{\partial \varphi} - \frac{(v - \omega r)}{r^2} \\
&\frac{Dw}{Dt} + \frac{\partial p}{\partial z} = \nu\left[\nabla^2 w\right]
\end{aligned}\right\} \qquad \text{(II.1.2)}$$

where

r , φ , z	are the cylindrical coordinates
u , v , w	are the velocity components in the r , φ , z directions, respectively
$P = \frac{p}{\rho}$	is the static pressure over the density
$\nu = \frac{\mu}{\rho}$	is the kinematic viscosity
μ	is the viscosity

$$\frac{D}{Dt} \equiv u\frac{\partial}{\partial r} + \frac{1}{r}(v+\omega r)\frac{\partial}{\partial \varphi} + w\frac{\partial}{\partial z}$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

The streamline equations for this case are

$$\frac{dr}{u} = \frac{r\,d\varphi}{v} = \frac{dz}{w} \qquad \text{(II.1.3)}$$

Infinite series in powers of the tangential coordinate, φ , are introduced for the three velocity components, u , v and w , and for the pressure variable, P . The coefficients of these series are functions of the radial and axial coordinates, r and z . These series are substituted into eqs. (II.1.1) and (II.1.2) and recurrence formulae are determined for the coefficients of the series. Conditions on the flow variable distribution along a surface between adjacent blades as well as boundary conditions are used for the evaluation of the unknown coefficients in terms of which the remainder of the coefficients of the infinite series are expressed.

The number of blades is chosen and a corresponding number of streamline surfaces are "frozen". These frozen streamline surfaces are those which have been determined in the analysis of the turbomachine stage with infinitesimally spaced blades (Chapter I.2.b). The distribution of the flow variables along these "frozen" streamline surfaces are assumed to be the same as those calculated in the "infinitesimal" analysis and are denoted by subscript zero. If the first coefficients in the four series are taken to be equal to the corresponding values of the flow variables along the frozen streamline surface, the flow variables will vary from these values toward the actual blade surfaces of the "finite" case. Therefore, the tangential variable is measured from the "frozen" streamline surface. The introduction of these "frozen" streamline surfaces and the prescribed distribution of the flow variables along them will serve to ensure a more rapid convergence of the series solution.

The boundary conditions used for the evaluation of the remaining unknown coefficients in the series are the non-slip conditions along the surfaces of the blades and the closure of the streamlines, which form the blades, at the leading and trailing edges.

The blades of modern turbomachines are thin and, therefore, it is thought to be sufficient to enforce the non-slip conditions along the chord surfaces of the blades instead of at the surfaces themselves. This relaxation of the non-slip conditions greatly simplifies the analysis and the numerical computations, since the enforcement of the non-slip conditions along the as yet to be determined blade surfaces would make the boundary conditions highly non-linear. The error introduced by this relaxation is thought to be negligible.

The closure of the blades at the leading edge is ensured by chosing the streamline surfaces forming both the upper and the lower sides of the blades so as to pass through the leading edge. This is possible by the choice of two arbitrary constants which arise from integration of the streamline equations (II.1.3). The closure of the blade surfaces at the trailing edge, however, cannot be achieved in general form, but has to be done for each design case by judicious choice of two arbitrary functions representing two of the three coefficients of the first power terms of the tangential variable in the series for u , v and w .

The series for the velocity components are taken in the form

$$\frac{a}{a_0} = \left[1 - k S - S^2\right]\left[1 + \sum_{n=1}^{\infty} a_n S^n\right] \qquad \text{(II.1.4)}$$

where

$$k = \frac{\psi^l - \psi^u}{\sqrt{\psi^l \psi^u}}$$

$$S = \frac{\psi}{\sqrt{\psi^l \psi^u}}$$

$\psi = \varphi - \varphi_0(r,z)$	is the tangential variable measured from the "frozen" streamline surface
$\varphi_0(r,z)$	is the tangential coordinate of the "frozen" streamline surface
ψ^l	is the tangential coordinate measured from φ_0 to the blade chord above φ_0
ψ^u	is the tangential coordinate measured from φ_0 to the blade chord below φ_0
a	denotes any of the velocity components u, v, or w
a_0	denotes the corresponding "infinitesimal" velocity components u_0, v_0, or w_0

a_n denotes the coefficients in the series for the velocity components, u_n, v_n, or w_n

The series for the pressure variable is chosen as

$$\frac{P}{P_0} = \left[1 + \sum_{n=1}^{\infty} P_n S^n\right] \qquad \text{(II.1.5)}$$

where

P_0 is the pressure variable along the "frozen" streamline surface

P_n is the coefficient in the series for the pressure variable

For $\psi = 0$, $S = 0$ and the expressions for the velocity components and the pressure variable, (II.1.4) and (II.1.5), satisfy the conditions that along the "frozen" streamline surface $a = a_0$ and $P = P_0$. Since $\psi = \psi^l$ and $\psi = \psi^u$ denote the two chord surfaces of the blades adjacent to the "frozen" streamline surface, it can be seen from the general expression of the velocity components (II.1.4) that for these values of ψ the velocity components will be zero. Therefore, the non-slip conditions along the chord surfaces will be satisfied by the expressions for the velocity components.

Fig. 21 shows the notation used in the $\varphi - z$ plane. From this figure it can be seen that

$$k = 2\epsilon/\sqrt{\psi^l \psi^u} \qquad \text{(II.1.6)}$$

where

$\epsilon = (\psi^l - \psi^u)/2$ is the angular distance between the mean line and the "frozen" streamline surface; the mean line lies half way between two adjacent chord surfaces

Since the blade spacing

$$\phi = \psi^l + \psi^u \qquad \text{(II.1.7)}$$

the choice of k and ϕ determines ψ^l, ψ^u and ϵ.

Eq. (II.1.4) written for the three velocity components becomes

$$\left.\begin{aligned}
\frac{u}{u_0} &= [1 - kS - S^2][1 + \sum_{n=1}^{\infty} u_n S^n] \\
\frac{v}{v_0} &= [1 - kS - S^2][1 + \sum_{n=1}^{\infty} v_n S^n] \\
\frac{w}{w_0} &= [1 - kS - S^2][1 + \sum_{n=1}^{\infty} w_n S^n]
\end{aligned}\right\} \qquad \text{(II.1.8)}$$

By substituting eqs. (II.1.5) and (II.1.8) into eqs. (II.1.1) and (II.1.2) collecting terms of equal powers of S and equating their coefficients to zero one obtains four simultaneous algebraic equations for each set of four coefficients. The solution of these equations yields the recurrence equations for the coefficients of the series in terms of u_0 , v_0 , w_0 , P_0 , u_1 and either v_1 or w_1 . The only two arbitrary functions of r and z amongst these are u_1 and either v_1 or w_1. These two arbitrary functions may be used to ensure closure of the blade surfaces at the trailing edge.

Before presenting the recurrence equations, a brief discussion of ψ and its partial derivatives should be given. After eq. (II.1.4), ψ is defined as

$$\psi = \varphi - \varphi_0(r,z) \tag{II.1.9}$$

It can be seen from eq. (II.1.9) that $\partial\psi/\partial\varphi$ equals unity, but both $\partial\psi/\partial r$ and $\partial\psi/\partial z$ are functions of r and z . If, however, the "frozen" streamline surface is made up of radial lines, i.e., if $\varphi_0 = \varphi_0(z)$ only, then $\partial\psi/\partial\varphi$ is unity, $\partial\psi/\partial r$ is zero and the $\partial\psi/\partial z$ is $-v_0/r w_0$, as can be calculated from the equations of the streamlines, eq. (II.1.3), for the "infinitesimal" case. In this somewhat restricted case, the derivatives of the tangential variable S simplify and the recurrence equations for the coefficients of the series become much more manageable. After eq. (II.1.4), S is defined as

$$S = \psi / \sqrt{\psi^{l}\psi^{u}} \qquad \text{(II.1.10)}$$

and its partial derivatives are

$$\left.\begin{aligned} \frac{\partial S}{\partial r} &= 0 \\ \frac{\partial S}{\partial \varphi} &= \frac{1}{\sqrt{\psi^{l}\psi^{u}}} \\ \frac{\partial S}{\partial z} &= \frac{-v_0}{r w_0 \sqrt{\psi^{l}\psi^{u}}} \end{aligned}\right\} \qquad \text{(II.1.11)}$$

It should be noted, however, that the restriction on the "frozen" streamline surface that it be made up of radial lines does not necessarily restrict the blade surfaces of the "finite" case to radial ones.

In the following, the recurrence equations are presented for the general case, i.e. for the case where $\varphi = \varphi(r, z)$. The recurrence equations give general expressions for the coefficients from u_8 , v_8 , w_8 and P_7 on (including these coefficients).

The recurrence equations are

$$^{\alpha}A_n u_n + {}^{\alpha}B_n v_n + {}^{\alpha}C_n w_n = -{}^{\alpha}E_n \qquad \text{(II.1.12)}$$

$$^{\beta}A_n u_n + {}^{\beta}D_n P_{n-1} = -{}^{\beta}E_n \qquad \text{(II.1.13)}$$

$$ {}^{\gamma}B_n v_n + {}^{\gamma}D_n P_{n-1} = - {}^{\gamma}E_n \qquad \text{(II.1.14)} $$

$$ {}^{\delta}C_n w_n + {}^{\delta}D_n P_{n-1} = - {}^{\delta}E_n \qquad \text{(II.1.15)} $$

In the above four equations, the expressions for u_n, v_n, w_n and P_n are determined from the following determinants. The fundamental determinant is denoted by (F.D.)

$$ (F.D.) = \begin{vmatrix} {}^{\alpha}A_n & {}^{\alpha}B_n & {}^{\alpha}C_n & 0 \\ {}^{\beta}A_n & 0 & 0 & {}^{\beta}D_n \\ 0 & {}^{\gamma}B_n & 0 & {}^{\gamma}D_n \\ 0 & 0 & {}^{\delta}C_n & {}^{\delta}D_n \end{vmatrix} $$

$$ (F.D.)u_n = - \begin{vmatrix} {}^{\alpha}E_n & {}^{\alpha}B_n & {}^{\alpha}C_n & 0 \\ {}^{\beta}E_n & 0 & 0 & {}^{\beta}D_n \\ {}^{\gamma}E_n & {}^{\gamma}B_n & 0 & {}^{\gamma}D_n \\ {}^{\delta}E_n & 0 & {}^{\delta}C_n & {}^{\delta}D_n \end{vmatrix} $$

$$(F.D.)v_n = -\begin{vmatrix} {}^{\alpha}A_n & {}^{\alpha}E_n & {}^{\alpha}C_n & 0 \\ {}^{\beta}A_n & {}^{\beta}E_n & 0 & {}^{\beta}D_n \\ 0 & {}^{\gamma}E_n & 0 & {}^{\gamma}D_n \\ 0 & {}^{\delta}E_n & {}^{\delta}C_n & {}^{\delta}D_n \end{vmatrix}$$

$$(F.D.)w_n = -\begin{vmatrix} {}^{\alpha}A_n & {}^{\alpha}B_n & {}^{\alpha}E_n & 0 \\ {}^{\beta}A_n & 0 & {}^{\beta}E_n & {}^{\beta}D_n \\ 0 & {}^{\gamma}B_n & {}^{\gamma}E_n & {}^{\gamma}D_n \\ 0 & 0 & {}^{\delta}E_n & {}^{\delta}D_n \end{vmatrix}$$

$$(F.D.)P_{n-1} = -\begin{vmatrix} {}^{\alpha}A_n & {}^{\alpha}B_n & {}^{\alpha}C_n & {}^{\alpha}E_n \\ {}^{\beta}A_n & 0 & 0 & {}^{\beta}E_n \\ 0 & {}^{\gamma}B_n & 0 & {}^{\gamma}E_n \\ 0 & 0 & {}^{\delta}C_n & {}^{\delta}E_n \end{vmatrix}$$

The series for u, v and w allow the numerical integration of the streamline equations

$$\frac{dr}{u} = \frac{r\,d\varphi}{v} = \frac{dz}{w} \qquad \text{(II.1.21)}$$

For the determination of the r and φ projections as functions of z, eq. (II.1.21) will be integrated by Taylor series expansions about consecutive points of z along the streamlines.

The expressions used in eqs. (II.1.12) to (II.1.15) are quite lengthy and are not included here. They are presented in Ref. 28.

II.2. NUMERICAL WORK

II.2.a. PRELIMINARY REMARKS

In the general analysis of laminar, incompressible flows through turbomachines with finite blade spacing and finite blade thickness (Section II.1), the velocities and the pressure-density ratio are expanded in infinite series in the tangential variable φ about points $\varphi = \varphi_0(r,z)$, where $\{\varphi_0(r,z)\}$ is the set of points on the "frozen" streamline surface (see Fig. 21). The recurrence equations determine the coefficients of the series in terms of $u_0(r,z)$, $v_0(r,z)$, $w_0(r,z)$, $P_0(r,z)$, $u_1(r,z)$, and $v_1(r,z)$ (or $w_1(r,z)$, since $v_1 = w_1$ by the first recurrence equation which is derived from the continuity equation). The functions u_0, v_0, w_0 and P_0 are known from the analysis of incompressible flows through turbomachines with infinitesimally spaced blades (Section I.1). The "frozen streamline" surface, $\varphi = \varphi_0(r,z)$, and the approximate blade shape are also known.

There remain to be chosen only the functions $u_1(r,z)$ and $v_1(r,z)$. These functions are thought to be determined by the conditions of closure of the blade surface and of convergence of the infinite series.

II.2.b. THE BLADE SHAPE - CLOSURE.

The numerical example of Section I.2.a with 16 blades has as solution

$$u_0 = w_0 \, M_s (r_{s_i} + M_s z) \frac{1}{r}$$

$$v_0 = r\,\omega \left\{ \frac{1 - \frac{z}{\ell}}{1 + \frac{z}{\beta \ell}} \right\}$$

$$w_0 = \text{constant}$$

$$P_0 = P_{T_i} + \frac{(\omega^2 r^2 - V^2)}{2} + J(r,z)$$

$$J(r,z)=\frac{A}{w_0}\Bigg\{\Bigg(\omega^2\beta^2\Big\{z-2\ell(1+\beta)\ln\Big[\frac{(\beta\ell+z)}{\beta\ell}\Big]+\frac{\ell^2(1+\beta)^2 z}{\beta\ell(\beta\ell+z)}\Big\}$$

$$-\frac{1}{2}w_0^2 M_s^2\left[\frac{r^2-(r_{s_i}+M_s z)(2r_{s_i}+M_s z)}{(r^2+r_{s_i}^2)-(r_{s_i}+M_s z)^2}\right]\left(\frac{z}{r^2}\right) \qquad \text{(II.2.b.1)}$$

$$-\frac{1}{4}\left[\frac{w_0^2(M_s^2+2)}{M_s}\right]\left[(r_{s_i}+M_s z)^2-r^2\right]^{-\frac{1}{2}}\ln\epsilon$$

$$\epsilon=\frac{r^2-M_s z(r_{s_i}+M_s z)-M_s z\left[(r_{s_i}+M_s z)-r^2\right]^{\frac{1}{2}}}{r^2-M_s z(r_{s_i}+M_s z)+M_s z\left[(r_{s_i}+M_s z)-r^2\right]^{\frac{1}{2}}}$$

For this solution, the velocity profile was chosen as a parabola with zero eccentricity. Having chosen this symmetric profile means that the "frozen streamline", $\varphi=\varphi_0(r,z)$, may be fixed in the middle of the blade channel, i.e., $\psi^u\equiv\psi^\ell$, so that $k=0$ in eqs. (II.1.4).

The problem may be stated as the determination of the as yet unknown functions $u_1(r,z)$ and $v_1(r,z)$ such that the infinite series are convergent in the region between the blades and that the blade of finite thickness, determined by the velocity series, closes at the leading and trailing edges. The form of the series assures that the components u, v, w and p will reduce to u_0, v_0, w_0 and p_0 on the "frozen streamline" surface, $\varphi=\varphi_0(r,z)$. The requirement that the velocity is a maximum on this surface may be met by enforcing

$$\left.\frac{\partial u}{\partial s}\right]_{s=0} = 0$$

$$\left.\frac{\partial v}{\partial s}\right]_{s=0} = 0 \qquad \text{(II.2.b.2)}$$

and

$$\left.\frac{\partial w}{\partial s}\right]_{s=0} = 0$$

which, upon substituting the velocity series, eq. (II.1.4), clearly implies that the coefficients $u_1 = v_1 = w_1 = 0$ and also $k = 0$ for all values of r and z. It is not immediately clear, however, that this choice of the arbitrary functions u_1 and v_1 will lead to closure of the blade surface and convergence of the series.

There are several ways to treat the condition of closure. One of the most natural of these is to specify the leading edge of the blade, determine the functions u_1 and v_1 from other considerations, as for example, the maximal condition of the preceding paragraph, and allow the resulting blade to close where it will, if at all. If the resulting blade does not close, it is necessary to reconsider the functions u_1 and v_1, alter them in some way, rework the numerical example and examine the resulting blade for closure.

This procedure must be repeated until closure is attained. However, to simplify the design of the stator stage, it is desirable to formulate the closure condition so that both the leading and trailing edges of the blade of the rotor are specified. This may be done as follows:

The upper blade surface, $\varphi = \varphi_u(r,z)$, is generated as the solution of the equation

$$u \frac{\partial \varphi_u}{\partial r} + w \frac{\partial \varphi_u}{\partial z} = \frac{v}{r} \qquad \text{(II.2.b.3)}$$

such that $\varphi_u(r,z)$ meets the conditions that where $z = 0$, $\varphi_u(r,0) = \varphi_o(r,0) + \psi^u$ and where $z = \ell$, $\varphi_u(r,\ell) = \varphi_o(r,\ell) + \psi^u$. Similarly, the lower blade surface, $\varphi = \varphi_\ell(r,z)$, is generated as the solution of

$$u \frac{\partial \varphi_\ell}{\partial r} + w \frac{\partial \varphi_\ell}{\partial z} = \frac{v}{r} \qquad \text{(II.2.b.4)}$$

such that where $z=0$, $\varphi_\ell(r,0) = \varphi_o(r,0) - \psi^\ell$ and where $z=\ell$, $\varphi_\ell(r,\ell) = \varphi_o(r,\ell) - \psi^\ell$ It should be noted that the subsidiary equations of (II.2.b.3) and (II.2.b.4) are the streamline equations, i.e.

$$\frac{dr}{u} = \frac{dz}{w} = \frac{r\,d\varphi}{v} \qquad \text{(II.2.b.5)}$$

Here, the leading and trailing edges are specified to be radial lines with the same "twist" as the blade of the infinitesimal case. This statement of the problem may be written in terms of line integrals along the characteristics (streamlines) of eqs. (II.2.b.3) and (II.2.b.4). The appropriate integral representation is

$$\int_{\Gamma_1} \frac{v}{rw}\, dz = \beta_0 \qquad \text{(II.2.b.6)}$$

where Γ_1 is a streamline passing through any initial point $z=0$, $r=r_i$, on the initial line $\varphi_u(r,0)=\varphi_0(r,0)+\psi^u$ (or $\varphi_l(r,0)=\varphi_0(r,0)-\psi^l$), terminating in the plane $z=l$. β_0 is the total angle of twist of the "infinitesimal" blade.

From I.2.a and I.2.b,

$$\beta_0 = \int_{\Gamma_2} \frac{v_0}{rw_0}\, dz \qquad \text{(II.2.b.7)}$$

where Γ_2 is a streamline passing through any initial point $z=0$, $r=r_i$ and terminating in the plane $z=l$. Since, from eqs. (II.2.b.1), the ratio $\frac{v_0}{rw_0}$ is independent of r and φ, the "infinitesimal" blade is a surface generated by radial lines and β_0 is a constant (independent of r). It is also to be noticed that, since the ratio $\frac{v_0}{rw_0}$ is independent of r and φ, the value β_0 is independent of the path of integration between the planes $z=0$ and $z=l$. So the streamline, Γ_2, may be replaced by any path extending from the entrance plane, $z=0$, to the exit plane $z=l$. In particular, it may be replaced by the curve Γ_1.

Combining (II.2.b.6) and (II.2.b.7), one sees that

$$\int_{\Gamma_1} \frac{v}{rw}\, dz = \beta_0 = \int_{\Gamma_1} \frac{v_0}{rw_0}\, dz$$

or

$$\int_{\Gamma_1} \left\{ \frac{v}{w} - \frac{v_0}{w_0} \right\} \frac{dz}{r} = 0 \qquad \text{(II.2.b.8)}$$

is a statement of the condition of closure.

Substitution from (II.1.4) into (II.2.b.7) yields

$$\int_{\Gamma_1} \left\{ \frac{1 + \Sigma v_n S^n}{1 + \Sigma w_n S^n} - 1 \right\} \frac{v_0}{w_0} \frac{dz}{r} = 0 \qquad \text{(II.2.b.9)}$$

This condition is difficult to manipulate in its present form. If, however, it is desired to enforce the closure condition to only the second order approximation of the velocity series, so that (II.2.b.9) may be written

$$\int_{\Gamma_1} \left\{ \frac{1 + v_1 S + v_2 S^2}{1 + w_1 S + w_2 S^2} - 1 \right\} \frac{v_0}{w_0} \frac{dz}{r} = 0 \qquad \text{(II.2.b.10)}$$

it becomes possible to find elementary conditions on u_1 and v_1 (or w_1) such that closure occurs to this approximation.

Clearing fractions in the integrand yields

$$\int_{\Gamma_1} \frac{(v_1 - w_1)S + (v_2 - w_2)S^2}{1 + w_1 S + w_2 S_2} \frac{v_0}{w_0} \frac{dz}{r} = 0 \qquad \text{(II.2.b.11)}$$

From the first continuity recurrence equation and the functions in (II.2.b.1) it is seen that

$$v_1(r,z) \equiv w_1(r,z) \tag{II.2.b.12}$$

so that

$$\int_{\Gamma_1} \frac{(v_2 - w_2)\, S^2}{1 + w_1 S + w_2 S^2} \frac{v_0}{w_0} \frac{dz}{r} = 0 \tag{II.2.b.13}$$

From the recurrence equations of Section II.1,

$$v_2(r,z) = \frac{{}^{\alpha}E_2\, {}^{\gamma}D_2\, {}^{\delta}C_2 + {}^{\alpha}B_2\left({}^{\delta}E_2\, {}^{\gamma}D_2 - {}^{\gamma}E_2\, {}^{\delta}D_2\right)}{{}^{\alpha}B_2\left({}^{\gamma}B_2\, {}^{\delta}D_2 - {}^{\gamma}D_2\, {}^{\delta}C_2\right)} \tag{II.2.b.14}$$

and

$$w_2(r,z) = \frac{{}^{\alpha}E_2\, {}^{\gamma}B_2\, {}^{\delta}D_2 + {}^{\alpha}B_2\left({}^{\delta}E_2\, {}^{\gamma}D_2 - {}^{\gamma}E_2\, {}^{\delta}D_2\right)}{{}^{\alpha}B_2\left({}^{\gamma}B_2\, {}^{\delta}D_2 - {}^{\gamma}D_2\, {}^{\delta}C_2\right)} \tag{II.2.b.15}$$

so that $v_2 \equiv w_2$ if ${}^{\alpha}E_2 = 0$.

Using the special forms (II.2.b.1), ${}^{\alpha}E_2 = 0$ reduces to

$$u_0 \frac{\partial u_1}{\partial r} + w_0 \frac{\partial w_1}{\partial z} = 0 \tag{II.2.b.16}$$

The integral in (II.2.b.13) is zero if $v_2 = w_2$, and the closure condition may be met by choosing u_1 and v_1 $(= w_1)$ such that (II.2.b.16) is satisfied.

As was pointed out in the discussion of eq. (II.2.b.2), the condition that a maximum of the velocity occurs on the surface $\varphi = \varphi_0(r,z)$, may be met by choosing the functions u_1 and v_1 zero. Such functions trivially satisfy eq. (II.2.b.16), (since $v_1 = w_1$) and, therefore, were tried for the numerical example. Unfortunately, the resulting velocity series appear to be divergent. Several possible reasons for this will be discussed in Section II.2.c. But now, it appears proper to discuss eqs. (II.2.b.13) and (II.2.b.16) in slightly greater detail, and show that eq. (II.2.b.16) leads to a thin airfoil shape.

If the integrand of eq. (II.2.b.13) is identically zero ($v_2 \equiv w_2$), then, to the second order approximation, the "blade" must close everywhere, i.e., the integral (II.2.b.13) becomes a function identically zero in r , z and S . This means that the functions resulting from the integration of (II.2.b.3) and (II.2.b.4), φ_u and φ_ℓ (where the series coefficients in these equations have been terminated at the term in S^2) must differ only by the angular distance between successive "frozen" stream surfaces, i.e.

$$\varphi_u - \varphi_\ell = \psi^u - \psi^\ell \qquad \text{(II.2.b.17)}$$

This condition corresponds to a blade of zero thickness. Further, the only admissible blade of zero thickness is the original "chord surface", $\varphi = \varphi_c(r,z) + \psi^u$ or $\varphi = \varphi_c(r,z) - \psi^\ell$. For, suppose there were an admissible blade of zero thickness different from this blade, then such a blade must have points above or below the original chord surface upon which the velocities are zero. And, since the blade is generated as a family of streamlines, the flow must somewhere have passed through the original chord surface. But flow with zero velocity is stationary and so could not pass through this surface. A contradiction.

It is seen that if the closure condition (II.2.b.13) is enforced identically by making ${}^{\beta}E_{2} = 0$, then any deviation of the final blade shape from that of the original chord line airfoil must be due to the influence of terms in the velocity series of order S^3 and higher. Such an airfoil must then be a thin airfoil.

II.2.c. THE NUMERICAL EXAMPLES

Before proceeding to a detailed consideration of the numerical examples, it should be pointed out that derivatives of preceding coefficients up to u_{n-1}, v_{n-1}, w_{n-1}, and ρ_{n-1} occur in the recurrence equations for u_n, v_n, w_n and ρ_n. The calculation of these derivatives is a laborious process which was carried out only for the specific examples considered. Because of the special nature of these rather lengthy calculations, they are not included in this report, but are placed on file in the departmental office in Urbana and are available there to interested persons.

Following eq. (II.2.b.16) in the discussion on closure, it was mentioned that an example has been worked for the solution of the case with infinitesimal blade spacing, and for the choice of the arbitrary functions, $u_1 = v_1 = w_1 = 0$ and also $\psi = 0$. Also noted there was the fact that this choice of the arbitrary functions leads to closure in the second order approximation and to a velocity profile between the blades which has a relative maximum (or a minimum) on the frozen streamline surface. To guarantee the relative maximum, the slope of the tangent must be proceeding from positive values to negative values in the neighborhood of the points on this surface. This is the usual condition of negativeness on the second derivative, which in this case is seen to be satisfied if

$$a_2 < 1 \qquad \text{(II.2.c.1)}$$

where a_2 represents each of u_2 , v_2 and w_2 .

More important was the fact that the first three terms of the velocity series were seen to be rapidly divergent in the region where $S \approx \pm \frac{1}{2}$. Since approximation formulae were used for the term P_0 , as discussed in Section I.2.b, involving the assumption that $\lambda = P_{T_i}$, i.e. the stagnation enthalpy is constant throughout the stage, and since the coefficients a_n depend on the derivatives $P_{0,r}$, $P_{0,z}$, $P_{0,rr}$, $P_{0,rz}$, $P_{0,zz}$, etc., it is thought that the divergence of the series is traceable to errors introduced by this approximation. Another possibility, however, is that choosing $u_1 = v_1 = 0$ might be inconsistent with the series expansion and the differential equations. To investigate this possibility, a new function u_1 was chosen satisfying two conditions; first, $u_1 = u_1(z)$ so that eq. (II.2.b.16) is satisfied, and second, $u_1(z)$ makes $u_2(r,z) = 0$ at approximately the center of the stage (r = 0.35). This choice is accomplished by solving the equation $^{\beta}E_2$ (r = 0.35, z) = 0, since it can be seen from Section II.1 that, if $^{\beta}E_2$ = 0, then u_2= 0. A solution, $u_1(z)$, for this equation (with the approximations on $\rho_{0,r}$ retained) is displayed in Fig. 22.

With this new choice of u_1, a second numerical calculation was then performed. Here again the first three terms of the velocity series were noticed to be rapidly divergent. Figs. 23-29 illustrate this second case.

To get an idea of how the approximation on ρ_0 for the second case $(u_1 \neq 0)$, affects the values of the derivatives $\rho_{0,r}$, $\rho_{0,z}$, $\rho_{0,rr}$, $\rho_{0,rz}$, etc., ρ_0, $\rho_{0,r}$, and $\rho_{0,z}$ were programmed for Illiac computation utilizing the complete solution $\lambda = \lambda(r,z)$ eq. (II.2.b.1). A cursory examination of the formula for $\frac{\partial \lambda}{\partial z}$ indicates that this value will be very much larger than the approximated values near the shroud and near the exit of the compressor stage. The exact calculations show that the approximated values at z = 0.2 of $\rho_{0,r}$ are off by about 20 per cent, and those of $\rho_{0,z}$ by about 50 per cent. The values of these first derivatives at smaller values of z appear to be less in error. One naturally expects higher order derivatives to be more in error than these first order derivatives. However, the exact numerical values for the higher derivatives, e.g., $\rho_{0,zz}$, $\rho_{0,rr}$, $\rho_{0,rz}$, were not computed with the Illiac, although the formulae for their calculation have been prepared.

However, it is not known whether convergence of the series will be uniformly effected by this refinement in calculation. It was thought that a re-examination of the physical principles underlying the choice of the arbitrary functions u_1 and v_1 would be helpful. The gross results of the numerical examples already calculated are useful for this re-examination.

It is evident from the example calculated for $v_1 \equiv 0$ and $u_1 = u_1(z)$ (such that u_2 (0.35, z) = 0) that where u_2 is small, i.e., in the neighborhood of r = 0.35, the other coefficients are small also (c.f. Fig. 24). This leads one to expect that convergence of the series may be obtained by choosing $u_1(r,z)$ and $v_1(r,z)$ such that $u_2 \equiv 0$ and such that closure occurs to the second order. It is actually possible to choose $u_1 = v_1 = 0$ at the entrance of the first stage. This means that the axially symmetric flow leaving the entrance diffuser is undistrubed to the first order.

The condition of closure is given in (II.2.b.16) and the condition $u_2 \equiv 0$ may be met by taking ${}^{\beta}E_2 = 0$ or by

$$A \frac{\partial u_1}{\partial z} + B u_1 + C v_1 + D = 0 \qquad \text{(II.2.c.2)}$$

where A, B, C, and D are functions of r and z known from the zero order terms. Dividing (II.2.c.2) by C, differentiating with respect to z and using (II.2.b.16), one obtains a linear, parabolic, second order partial differential equation in u_1

$$\frac{A}{C} \frac{\partial^2 u_1}{\partial z^2} + \left[\frac{\partial}{\partial z} \left(\frac{A}{C} \right) + \frac{B}{C} \right] \frac{\partial u_1}{\partial z} - \frac{u_0}{w_0} \frac{\partial u_1}{\partial r}$$
$$+ \frac{\partial \left(\frac{B}{C} \right)}{\partial z} u_1 + \frac{\partial \left(\frac{D}{C} \right)}{\partial z} = 0 \qquad \text{(II.2.c.3)}$$

The characteristics of this equation are lines parallel to the z-axis. Therefore, physically, a solution is expected to exist reducing to arbitrary values of u_1 and its normal derivative on the initial value curve $z = 0$. Eq. (II.2.c.2), with specification e.g., $u_1 = v_1 = 0$ on the curve $z = 0$, leads to a solution of (II.2.c.3) for u_1 throughout the compressor. Knowledge of u_1 and again use of eq. (II.2.c.2) yields v_1 throughout the compressor stage. Therefore, it is thought that the above procedure in solving the problem may yield convergent series for the flow variables.

III. VISCOUS COMPRESSIBLE FLOWS THROUGH TURBOMACHINE STAGES WITH LARGE NUMBER OF BLADES. FINITE BLADE SPACING.

III.1. LAMINAR FLOWS.

The basic equations used are the Navier-Stokes equations together with the continuity equation and the energy equation in their most general form. Infinite series in powers of the tangential coordinate (with the coefficients being functions of the radial and axial coordinates) are introduced for the dependent variables. The Navier-Stokes equations and the energy equation are of second order and, therefore, the coefficients of the series are expressed in terms of the coefficients of the zeroth and first order terms of the tangential coordinate.

The analysis of axially symmetric compressible flow through a rotating turbomachine stage with infinitesimal blades, Section I, gives the zeroth order terms. The first order terms are specified from the boundary conditions which are the non-slip condition on the surface of the blade and the closure condition.

The method of solving the problem using the analysis for the infinitesimally spaced blades and the boundary conditions is explained below.

By using cylindrical polar coordinates, the basic equation can be written as:

The continuity equation

$$\rho\Theta + D\rho/Dt = 0 \qquad \text{(III.1.1)}$$

The Navier-Stokes equations

$$\left.\begin{aligned}
&\rho\left\{\frac{Du}{Dt} - \frac{v^2}{r} - 2\omega v - \omega^2 r\right\} = \frac{\partial p}{\partial r} + \frac{\mu}{3}\frac{\partial \Theta}{\partial r} \\
&+ \mu\left[\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \varphi}\right] - \frac{2}{3}\Theta\frac{\partial \mu}{\partial r} \\
&+ \frac{1}{r}\frac{\partial \mu}{\partial \varphi}\frac{\partial (v+\omega r)}{\partial r} + \frac{\partial w}{\partial r}\frac{\partial \mu}{\partial z} \\
&- \left[\frac{1}{r^2}\left(\frac{\partial \mu}{\partial \varphi}\right)\left(\frac{\partial u}{\partial \varphi} - v - \omega r\right) + \left(\frac{\partial \mu}{\partial z}\right)\left(\frac{\partial u}{\partial z}\right)\right]
\end{aligned}\right\} \qquad \text{(III.1.2)}$$

$$\left.\begin{aligned}
&\rho\left\{\frac{Dv}{Dt} + \frac{uv}{r} + 2\omega u\right\} = -\frac{1}{r}\frac{\partial p}{\partial \varphi} + \frac{\mu}{3r}\frac{\partial \Theta}{\partial \varphi} \\
&+ \mu\left[\nabla^2 v - \frac{v^2}{r} + \frac{2}{r^2}\frac{\partial u}{\partial \varphi}\right] \\
&- \frac{2}{3r}\Theta\frac{\partial \mu}{\partial \varphi} + \frac{1}{r}\left[\left(\frac{\partial u}{\partial \varphi}\right)\left(\frac{\partial \mu}{\partial r}\right) + \left(\frac{\partial w}{\partial \varphi}\right)\left(\frac{\partial \mu}{\partial z}\right)\right] \\
&- \left[\frac{\partial \mu}{\partial r}\frac{\partial (v+\omega r)}{\partial r} + \frac{u}{r^2}\frac{\partial \mu}{\partial \varphi} + \left(\frac{\partial \mu}{\partial z}\right)\left(\frac{\partial v}{\partial z}\right)\right]
\end{aligned}\right\} \qquad \text{(III.1.3)}$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\mu}{3}\left(\frac{\partial \Theta}{\partial z}\right) + \mu \nabla^2 w - \frac{2}{3}\Theta\left(\frac{\partial \mu}{\partial z}\right)$$

$$+\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial \mu}{\partial r}\right) + \frac{1}{r}\left(\frac{\partial v}{\partial z}\right)\left(\frac{\partial \mu}{\partial \varphi}\right) - \left[\left(\frac{\partial \mu}{\partial r}\right)\left(\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial \mu}{\partial \varphi}\right)\left(\frac{\partial w}{\partial \varphi}\right)\right] \qquad \text{(III.1.4)}$$

The energy equation

$$\rho C_p \frac{DT}{Dt} = \frac{DP}{Dt} + \frac{1}{r}\frac{\partial\left[r\lambda\left(\frac{\partial T}{\partial r}\right)\right]}{\partial r}$$
$$+ \frac{1}{r^2}\frac{\partial\left[\lambda\left(\frac{\partial T}{\partial \varphi}\right)\right]}{\partial \varphi} + \frac{\partial\left[\lambda\left(\frac{\partial T}{\partial z}\right)\right]}{\partial z}$$
$$+ \mu\left[-\frac{2}{3}\Theta^2 + 2\left(\frac{\partial u}{\partial r}\right)^2 + \frac{2}{r^2}\left(\frac{\partial v}{\partial \varphi} + u\right)^2\right.$$
$$+ 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \varphi}\right)^2$$
$$\left. + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \varphi}\right)^2\right] \qquad \text{(III.1.5)}$$

The equation of state

$$P = R\rho T \qquad \text{(III.1.6)}$$

where

r, φ, z are the cylindrical coordinates

u, v, w	are the velocity components in the r , φ , z directions respectively
p	is the static pressure
ρ	is the density
T	is the temperature
μ	is the coefficient of viscosity
λ	is the coefficient of heat conduction
R	is the gas constant
c_p	is the specific heat at constant pressure
ω	is the angular velocity of the rotating stage

$$\Theta = (v r) \frac{\partial (r u)}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial z}$$

$$\frac{D}{Dt} = u\left(\frac{\partial}{\partial r}\right) + \frac{1}{r}(v + \omega r)\frac{\partial}{\partial \varphi} + w\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

The quantities μ and λ are given by

$$\mu = CT , \quad \lambda = \alpha T \qquad (C , \alpha \text{ constants}) \qquad (III.1.7)$$

and C_p is constant.

The streamline equations for this case are

$$\frac{dr}{u} = \frac{r\,d\varphi}{v} = \frac{dz}{w} \qquad \text{(III.1.8)}$$

The pressure P is eliminated from eqs. (III.1.2 to III.1.5) by using eq. (III.1.6). Infinite series in powers of the tangential coordinate (φ) are introduced for the three velocity components (u , v , w), for the density ρ and for the temperature T . The coefficients of these series are functions of the radial and axial coordinates (r , z). These series are substituted into eqs. (III.1.1 to III.1.5) and recurrence formulae are determined for the coefficients of the series. Conditions on the flow variable distributions along a surface between adjacent blades as well as the boundary conditions are used for the evaluation of the unknown coefficients in terms of which the remainder of the coefficients of the infinite series are expressed.

The number of blades is chosen and a corresponding number of streamline surfaces are "frozen". These frozen streamline surfaces are those which have been determined in the analysis of the turbomachine stage with infinitesimally spaced blades. The distribution of the flow variables along these frozen streamline surfaces are assumed to be the same as those calculated in the "infinitesimal" analysis and are denoted by the subscript zero. If the first coefficients in the five series are taken to be equal to the corresponding values of the flow variables along the streamline surface, the flow variables will vary from these values toward the actual blade surfaces of "finite" spacing. Therefore, the tangential variable is measured from the frozen streamline surface.

The boundary conditions used for the evaluation of the remaining unknown coefficients in the series are the non-slip conditions along the surfaces of the blades and the closure of the streamlines which form the blades at the leading and trailing edges.

The blades of modern turbomachines are thin and, therefore, it is thought to be sufficient to enforce the non-slip conditions along the chord surfaces of the blades instead of at the surfaces themselves. This relaxation of the non-slip conditions greatly simplifies the analysis and the numerical computation since the enforcement of the non-slip conditions along the as yet to be determined blade surfaces would make the boundary conditions highly non-linear. This same "relaxing" of the boundary conditions was used in previous Sections.

The series for the velocity components are taken in the form

$$\frac{u}{u_o} = \left[1 - KS - S^2\right]\left[1 + \sum_{n=1}^{\infty} u_n S^n\right]$$

$$\frac{v}{v_0} = \left[1 - KS - S^2\right]\left[1 + \sum_{n=1}^{\infty} v_n S^n\right]$$

$$\frac{w}{w_0} = \left[1 - KS - S^2\right]\left[1 + \sum_{n=1}^{\infty} w_n S^n\right] \tag{III.1.9}$$

and, the series for the temperature and the density are chosen as

$$\frac{T}{T_0} = 1 + \sum_{n=1}^{\infty} T_n S^n$$

$$\frac{\rho}{\rho_0} = 1 + \sum_{n=1}^{\infty} \rho_n S^n \tag{III.1.10}$$

where

$$K = \frac{(\psi^{l} - \psi^{u})}{\sqrt{\psi^{l}\,\psi^{u}}}$$

$$S = \frac{\psi}{\sqrt{\psi^{l}\,\psi^{u}}}$$

$\psi = \varphi - \varphi_0(r,z)$ is the tangential variable measured from the "frozen" streamline surface

$\varphi_0(r,z)$ is the tangential coordinate of the "frozen" streamline surface

ψ^u is the tangential coordinate measured from φ_o to the blade chord above φ_o

ψ^l is the tangential coordinate measured from φ_o to the blade chord below φ_o

For $\psi = 0$, $S = 0$ and the expressions for the velocity components, temperature and density satisfy the conditions that along the frozen streamline surface $u = u_o$, $v = v_o$, etc. Since $\psi = -\psi^l$ and $\psi = \psi^u$ denote the two chord surfaces of the blades adjacent to the frozen streamline surface, from eq. (III.1.9), it can be seen that for these values of ψ the velocity components will be zero. Therefore, the non-slip conditions along the chord surfaces will be satisfied by the expressions for the velocity components.

Fig. 21 shows the notation used in φ, z plane. From this figure it can be seen that $K = 2\epsilon/\sqrt{\psi^l\psi^u}$, where $\epsilon = (\psi^l - \psi^u)/2$ is the angular distance between the mean line and the frozen streamline surface, the mean line lies half-way between two adjacent chord surfaces. Since the blade spacing is $\Phi = \psi^l + \psi^u$ the choice of K and Φ determines ψ^l, ψ^u and ϵ.

By substituting eqs. (III.1.9) and (III.1.10) into eqs. (III.1.1 to III.1.5) after eliminating P, μ, λ by using eqs. III.1.6) and (III.1.7). Collecting terms of equal powers of S and equating their coefficients to zero,, one obtains five simultaneous algebraic equations for each set of five coefficients. The coefficients of S up to the S^{5} in eqs. (III.1.1) to (III.1.5) were calculated and also a general recurrence formula for S^{n} with $n \geq 6$ The solutions of these equations give the unknown coefficients in terms of u_0, v_0, w_0, ρ_0, T_0, u_1, T_1 and either v_1 or w_1. The three arbitrary functions u_1, T_1, v_1 or w_1 may be used to ensure closure of the blade surfaces at the trailing edge.

The recurrence equations for the series are not included here because they are too lengthy. They are available to interested workers at the Aeronautical Engineering Department, University of Illinois, Urbana, Illinois.

III.2. TURBULENT FLOWS

The general Navier-Stokes equations were used with the assumption that the flow is composed of a mean flow and of turbulent fluctuations of the flow variables. This gave rise to double and triple correlations between these variables.

To our knowledge, no valid theory of compressible turbulent flows exists which would allow the evaluation of these correlation functions. Therefore, several groups in this country were contacted who are working in the field of turbulence. However, it was pointed out by these workers that the problem cannot be solved at present analytically. Therefore, no further work was carried out on this phase.

IV. CONCLUSIONS

It is thought that the report presents the first comprehensive treatment of turbomachine analysis including viscous effects. The channel theory of Lorenz has been developed to a general and, it is hoped, useful stage. Both compressible and incompressible flows are treated which may be laminar or turbulent (except that the general treatment of compressible turbulent flows is not possible at present).

The analyses are valid for axial flow, as well as radial and mixed flow turbomachines (compressors and turbines), although the numerical work included treats axial flow compressor stages only. This is so because the original aim of the work was the development of a theory for axial flow turbomachines.

It is realized that much numerical work has to be done to render the analyses useful for engineers. Only after such work has been carried out will it be possible to evaluate the usefulness of the theory presented.

TABLE I. DESIGN PARAMETERS FOR NUMERICAL EXAMPLE

Length of stage	$(\ell) = 0.250$ ft
Intake radii	
Hub	$(r_{H_i}) = 0.250$ ft
Intermediate	$(r_{m_i}) = 0.293$ ft
Shroud	$(r_{s_i}) = 0.333$ ft
Slope of shroud (constant)	$(M_s) = 0.5$
Number of blades	16
Parameter in eq. (I.2.b.5)	$(\beta) = 0.043$
Pitch angle	$(\Phi) = \pi/8$ radians
Axial velocity component	$(w_o) = 100$ ft/sec
Intake radial velocity component at shroud	$(u_{s_i}) = 50$ ft/sec
Angular velocity	$(\omega) = 15{,}000$ rpm
Eccentricity of velocity parabola	$(a) = 0$
Intake pressure-density ratios	
Hub	$(P_{H_i}) = 882{,}770$ ft^2/sec^2
Intermediate	$(P_{m_i}) = 883{,}377$ ft^2/sec^2
Shroud	$(P_{s_i}) = 883{,}742$ ft^2/sec^2
Density	$(\rho) = 0.002378$ slugs/ft^3

TABLE II. BLADE TWIST VARIATION ALONG AXIS

$\frac{z}{\ell}$	ϕ
0	0
.05	.1274
.10	.1947
.20	.2712
.30	.3150
.40	.3386
.50	.3622
.60	.3751
.70	.3836
.80	.3890
.90	.3918
1.00	.3927

TABLES III. VARIATION OF RADIUS, VELOCITIES, AND PRESSURE-DENSITY RATIOS ALONG AXIS

IIIa. Hub

z/ℓ	r(ft)	u(ft/sec)	v(ft/sec)	V(ft/sec)	$P(ft^2/sec^2)$
0	.2500	66.667	392.700	410.68	882,770
.05	.2583	65.741	178.190	214.64	949,240
.10	.2664	64.899	113.250	164.43	964,040
.20	.2845	63.428	62.805	134.04	979,430
.30	.2982	62.185	41.096	124.72	991,880
.40	.3135	61.122	29.445	120.84	1,003,990
.50	.3287	60.204	20.444	118.50	1,016,280
.60	.3437	59.404	13.286	117.07	1,028,850
.70	.3585	58.701	8.623	116.28	1,041,730
.80	.3731	58.080	5.977	115.80	1,054,950
.90	.3875	57.527	2.775	115.40	1,068,560
1.00	.4018	57.032	0	115.12	1,082,530

IIIb. Intermediate

z/ℓ	r(ft)	u(ft/sec)	v(ft/sec)	V(ft/sec)	$P(ft^2/sec^2)$
0	.2933	56.833	460.650	474.79	883,380
.05	.3003	56.533	207.210	236.92	973,210
.10	.3074	56.253	130.660	173.89	991,440
.20	.3214	55.747	71.458	134.96	1,008,300
.30	.3353	55.304	46.209	123.22	1,021,060
.40	.3491	54.912	32.775	118.70	1,033,240
.50	.3627	54.563	22.557	116.13	1,045,550
.60	.3763	54.252	15.811	114.86	1,058,100
.70	.3899	53.973	10.631	114.13	1,070,970
.80	.4033	53.721	6.462	113.70	1,084,190
.90	.4167	53.493	2.984	113.45	1,097,780
1.00	.4301	53.286	0	113.31	1,111,730

IIIc. Shroud

z/ℓ	r(ft)	u(ft/sec)	v(ft/sec)	V(ft/sec)	$P(ft^2/sec^2)$
0	.3333	50	523.600	535.40	883,740
.05	.3396	50	234.280	259.59	998,560
.10	.3458	50	146.999	184.69	1,020,480
.20	.3583	50	79.672	137.29	1,038,970
.30	.3708	50	51.111	122.93	1,052,080
.40	.3833	50	35.994	117.46	1,064,360
.50	.3958	50	24.616	114.48	1,076,720
.60	.4083	50	17.155	113.11	1,089,280
.70	.4208	50	11.475	112.39	1,102,150
.80	.4333	50	6.943	112.02	1,115,360
.90	.4458	50	3.123	111.85	1,128,940
1.00	.4583	50	0	111.80	1,142,880

TABLE IV. VARIATION OF STATIC AND TOTAL PRESSURE-DENSITY RATIOS AT EXIT

	r_e(ft)	P_e(ft^2/sec^2)	P_{T_e}(ft^2/sec^2)
Hub	.4018	1,082,530	1,288,338
Intermediate	.4301	1,111,730	1,346,347
Shroud	.4583	1,142,880	1,408,287

TABLE V. VARIATION OF STATIC PRESSURE-DENSITY RATIO AT INTAKE

	r_i(ft)	P_i(ft^2/sec^2)
Hub	.2500	882,770
Intermediate	.2933	883,380
Shroud	.3333	883,740

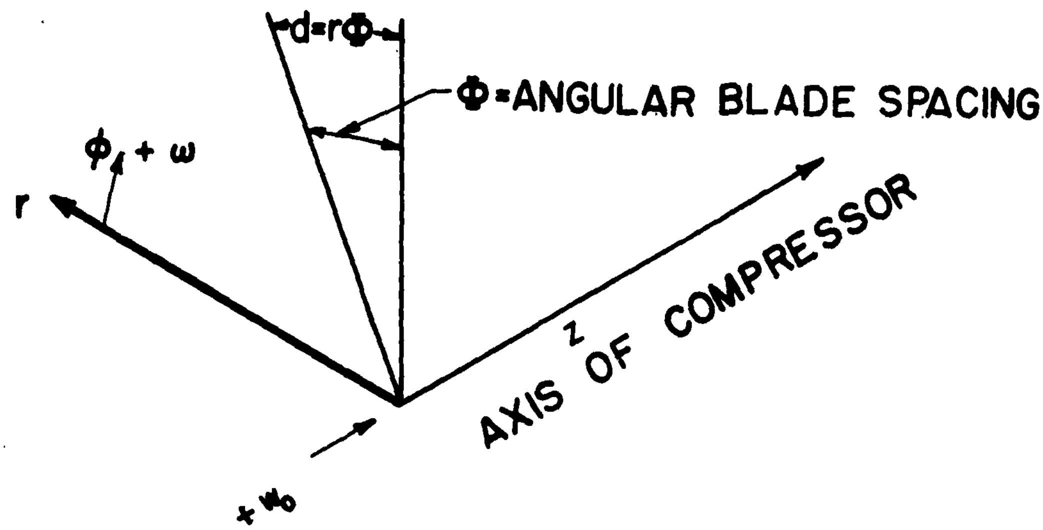


FIG. 1
NOTATION FOR INFINITESIMAL BLADE SPACING

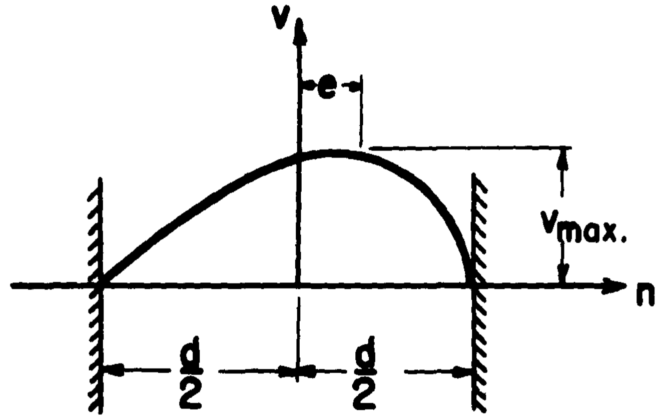


FIG. 2
RESULTANT VELOCITY DISTRIBUTION
INFINITESIMAL BLADE SPACING

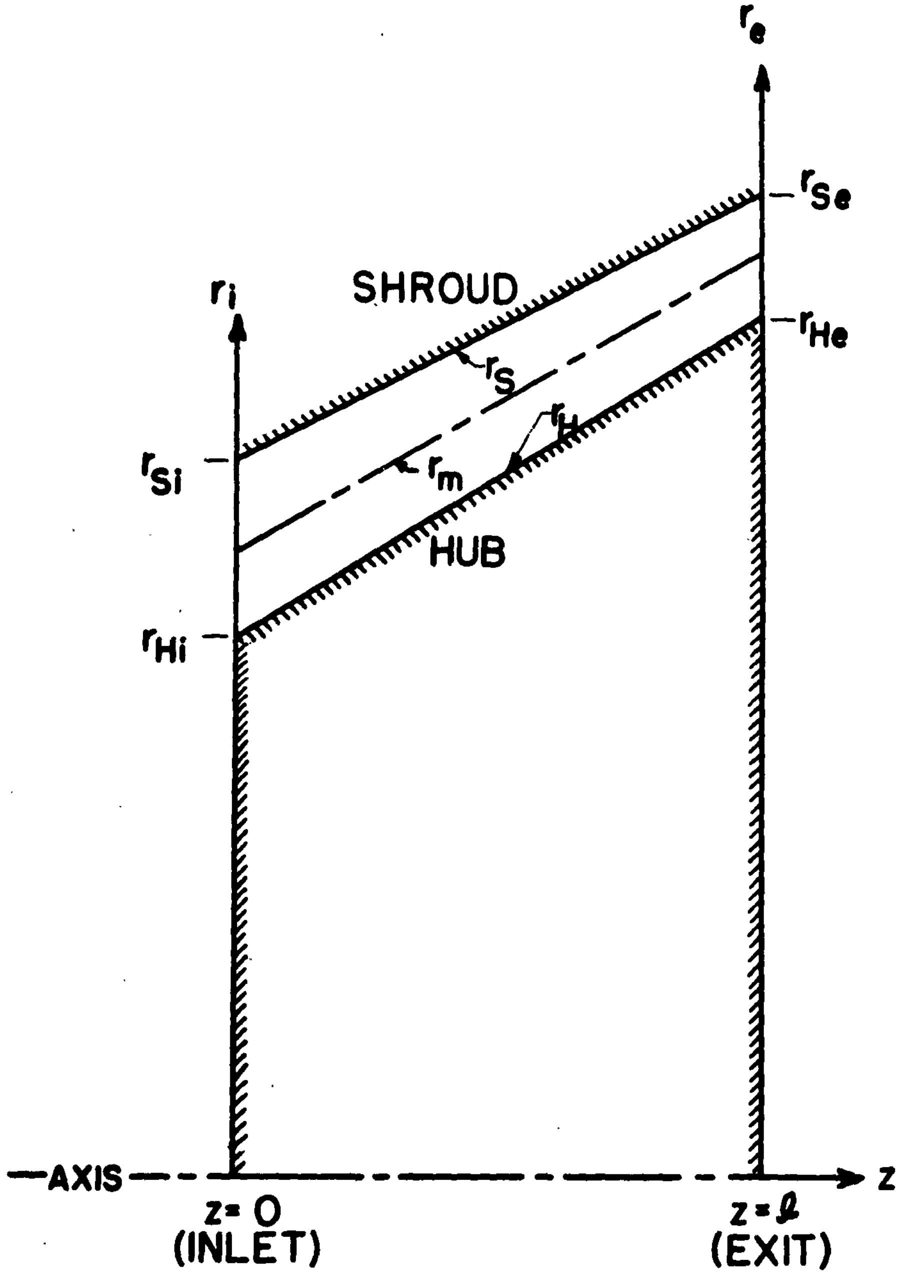


FIG. 3

r, z PROJECTION (SECTION WITH AXIAL PLANE)

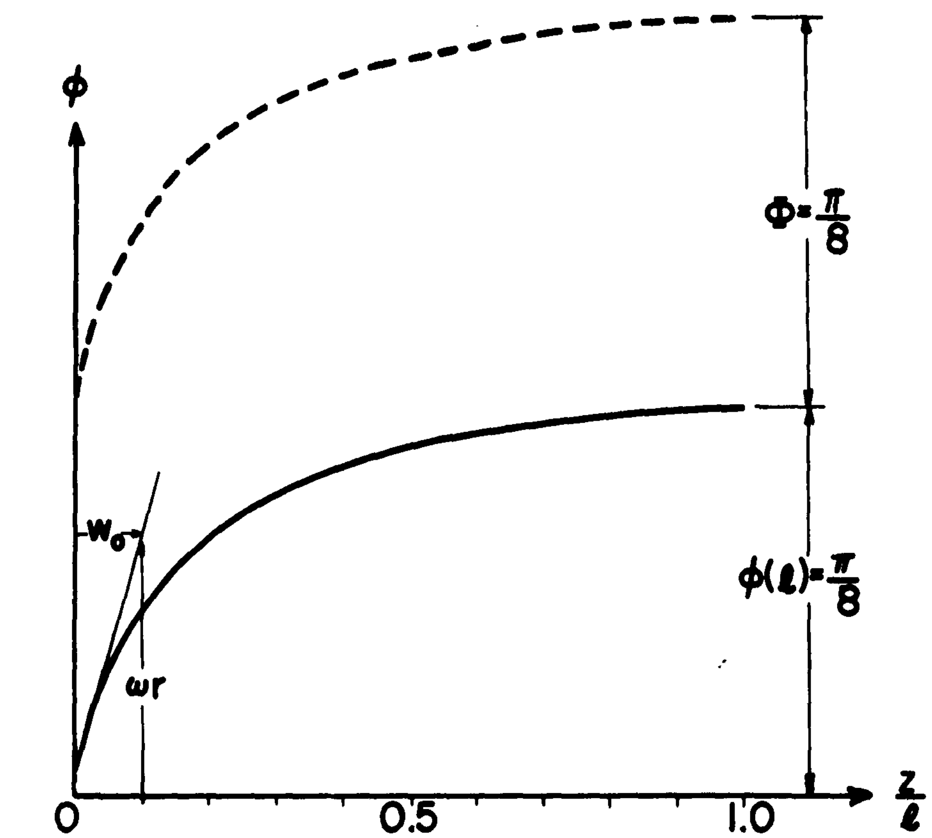


FIG. 4 ϕ, z PROJECTION OF STREAMLINES

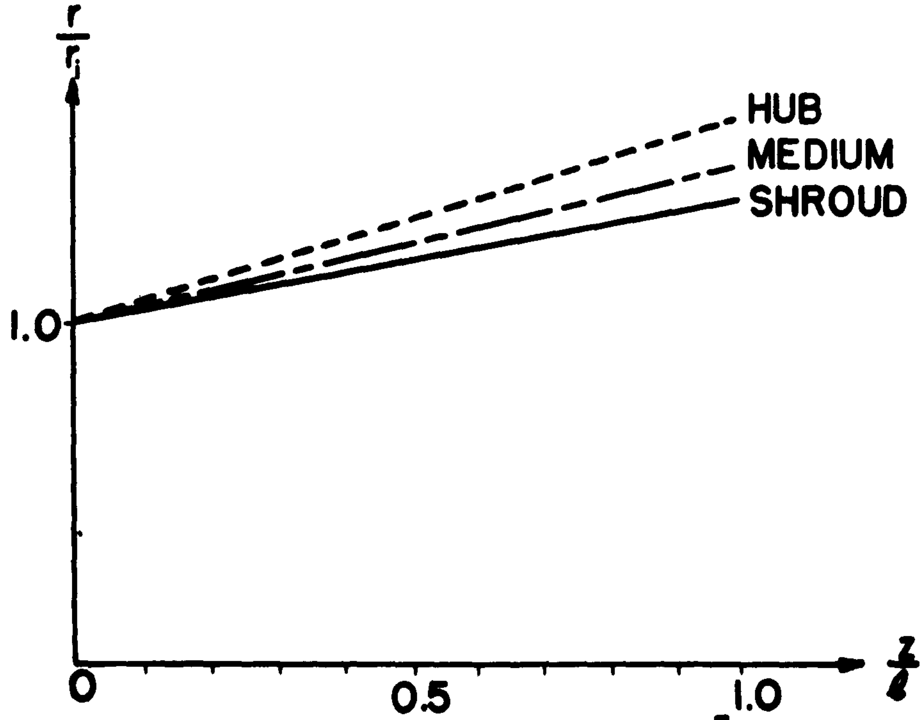


FIG. 5 RADIUS RATIO VERSUS $\frac{z}{\ell}$

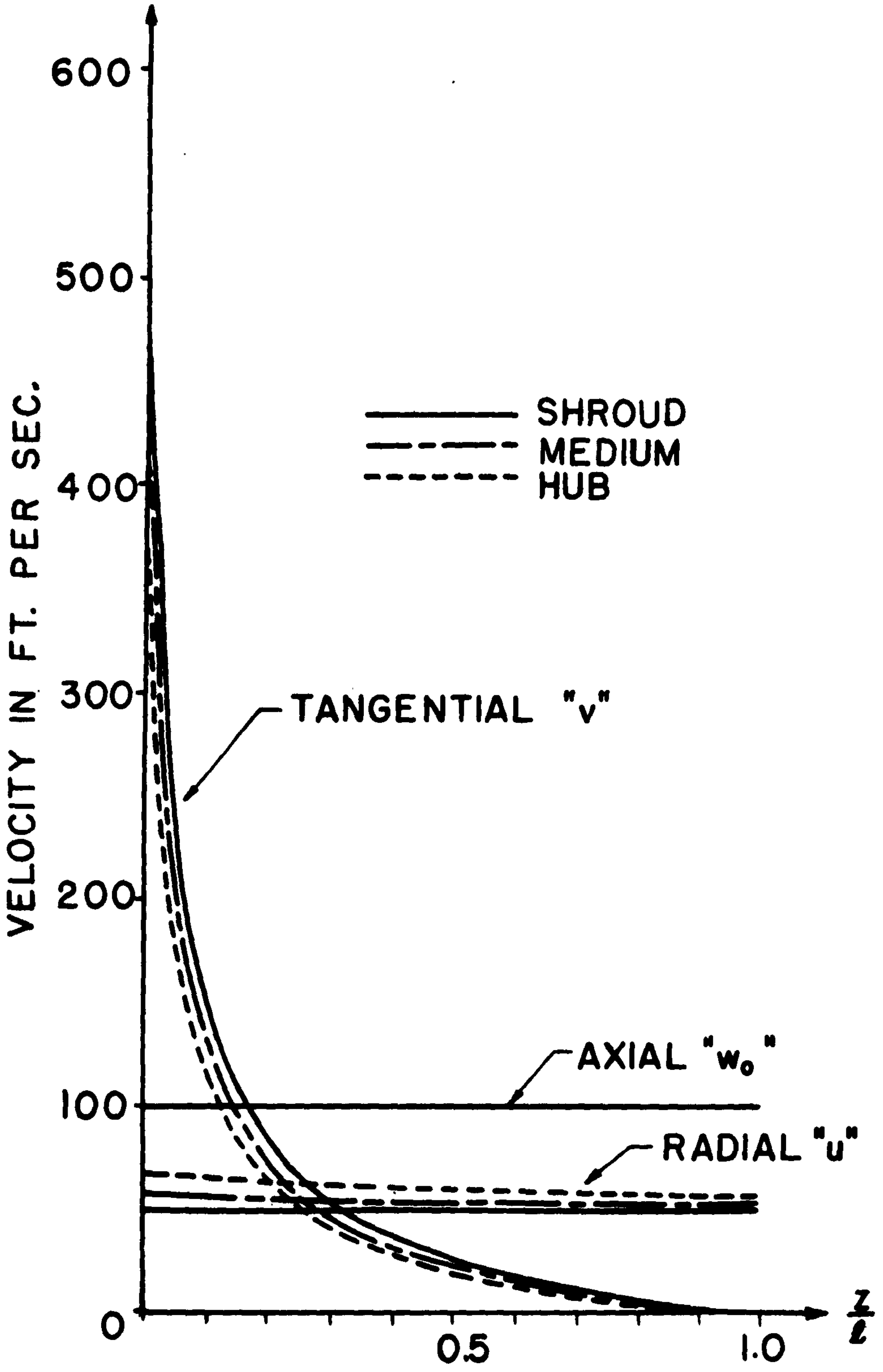


FIG. 6 VELOCITY COMPONENTS VERSUS $\frac{z}{l}$

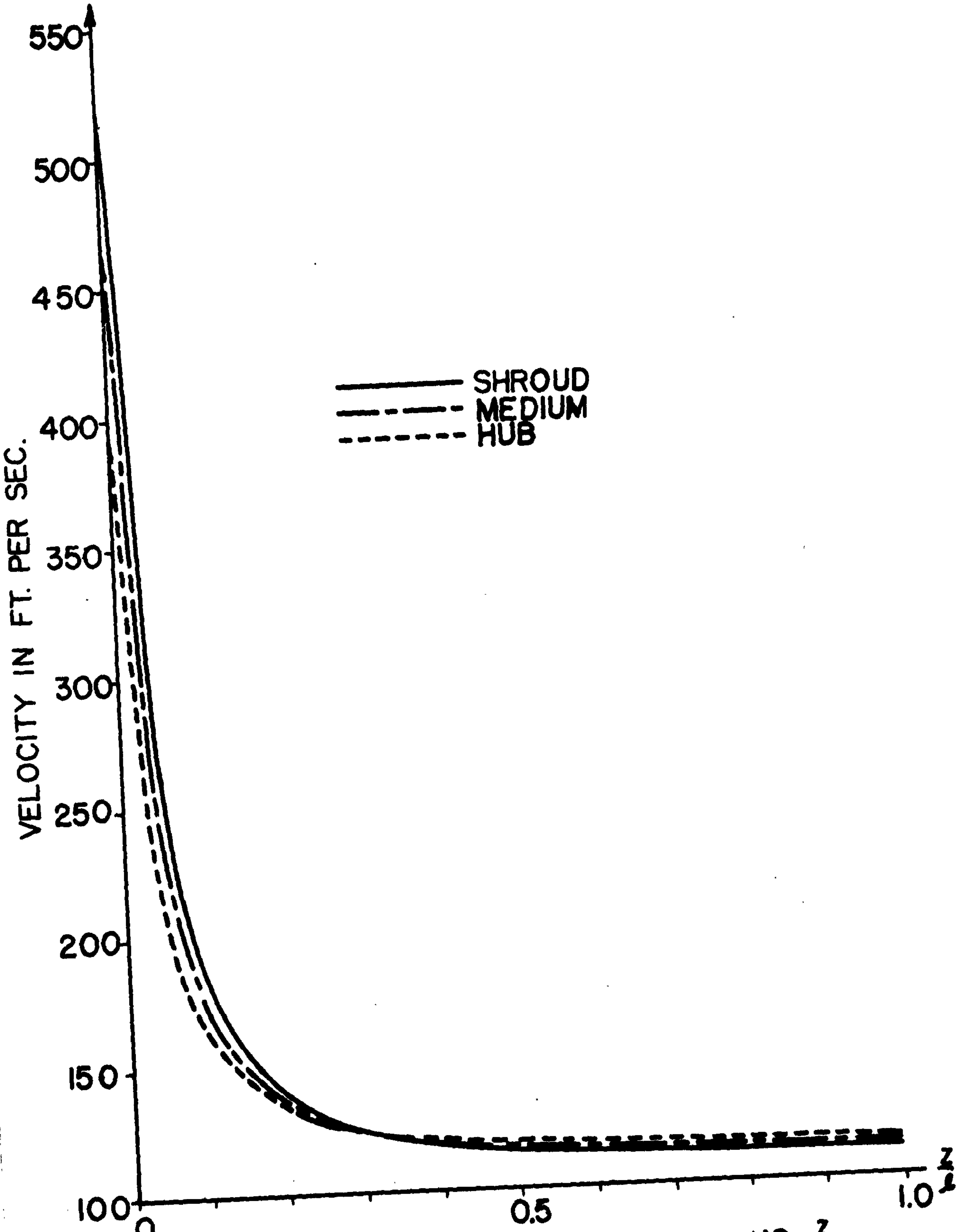


FIG. 7 TOTAL RELATIVE VELOCITY VERSUS $\frac{z}{l}$

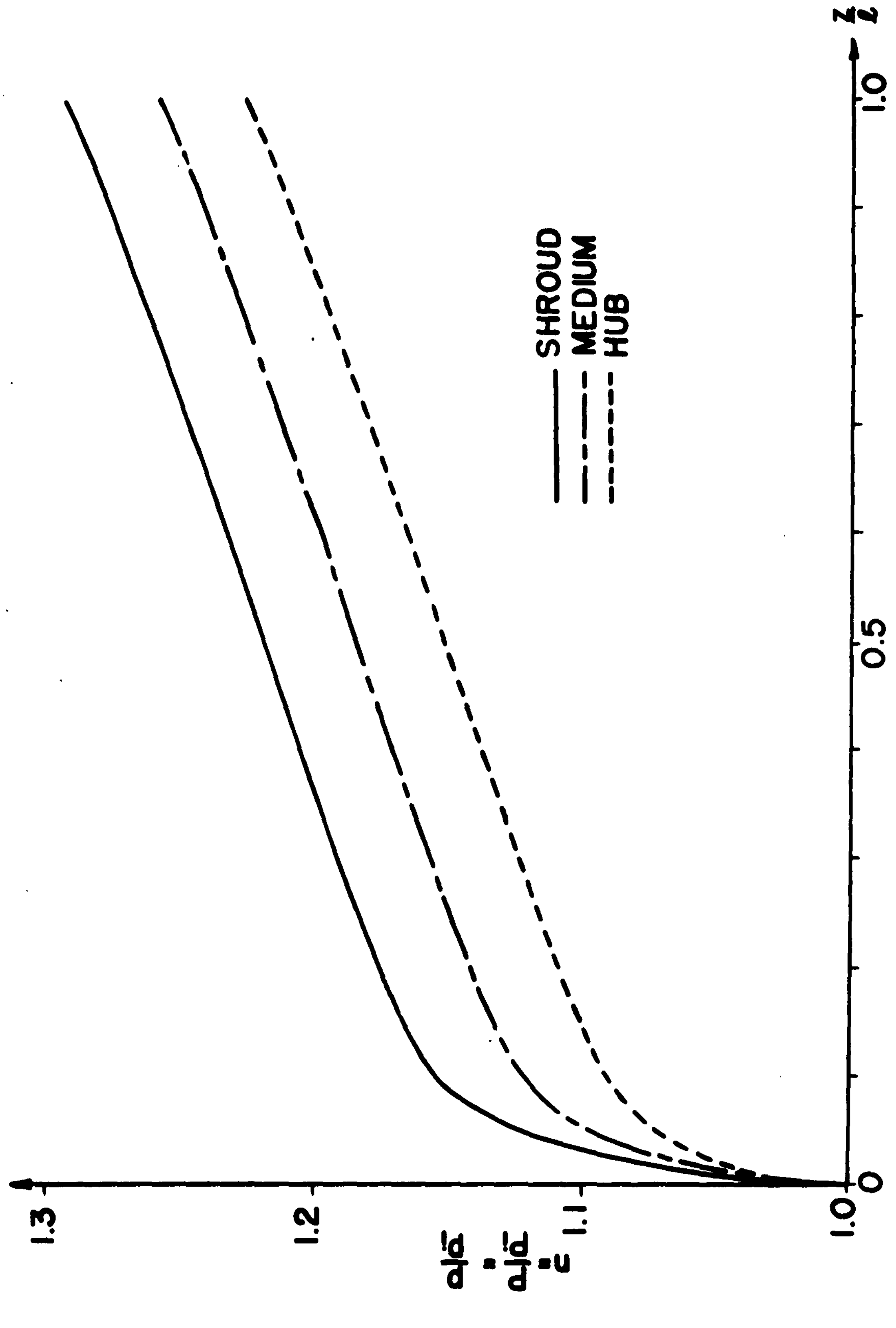


FIG. 8 PRESSURE RATIO VERSUS $\frac{z}{\ell}$

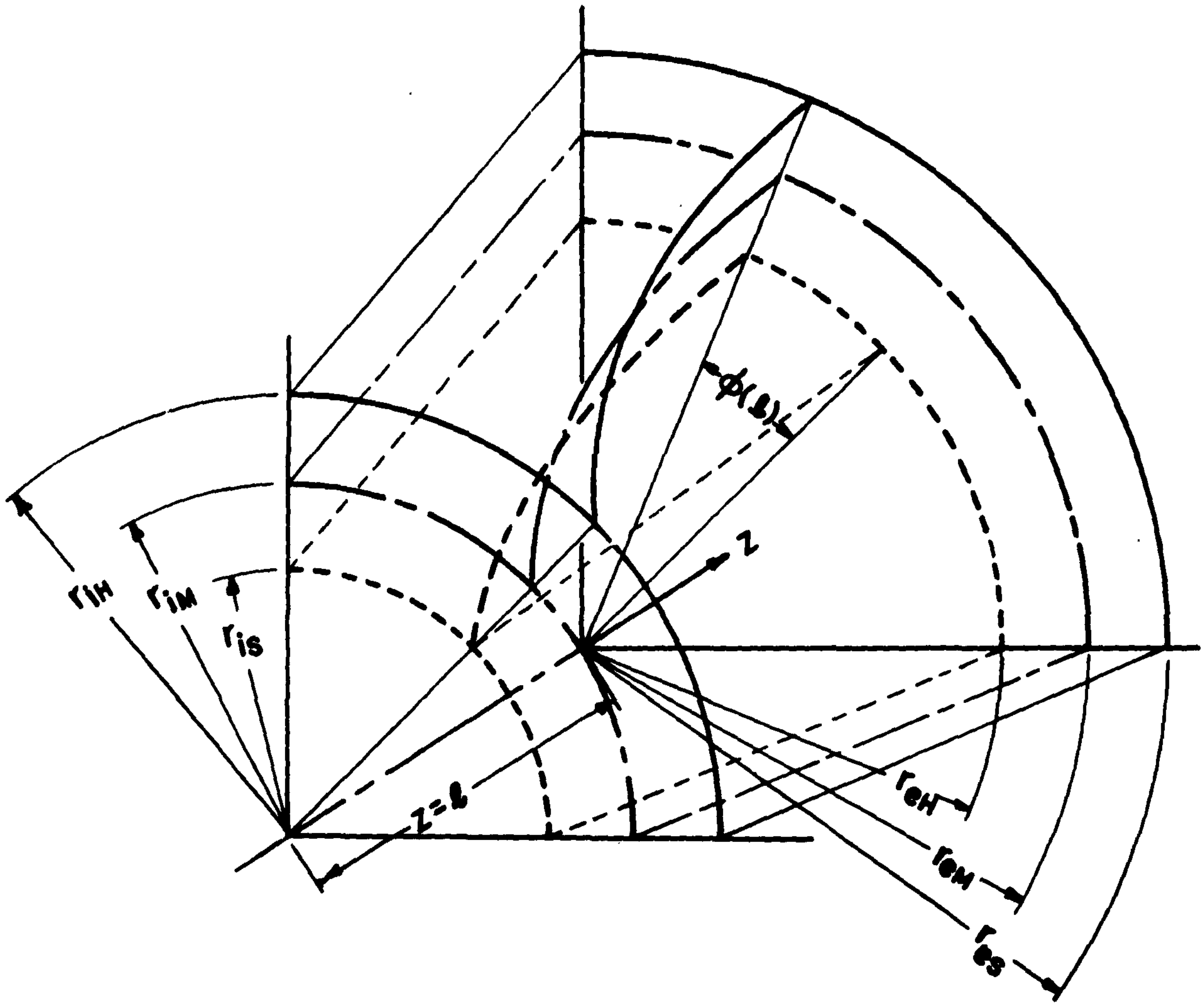


FIG. 9 HUB AND INFINITESIMAL BLADE SURFACE THREE — DIMENSIONAL VIEW

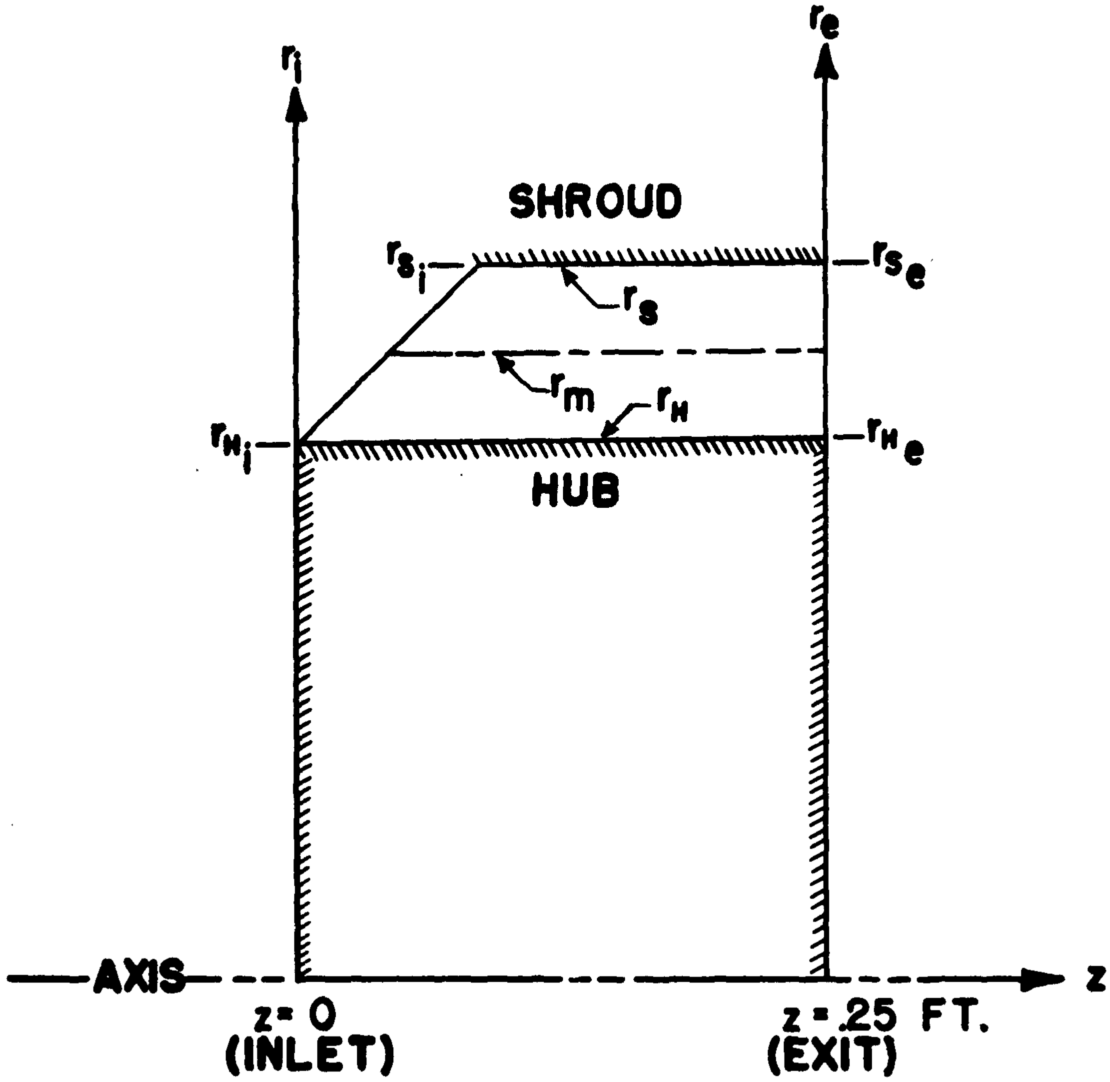


FIG. 10 r, z PROJECTION OF TYPICAL STREAMLINES

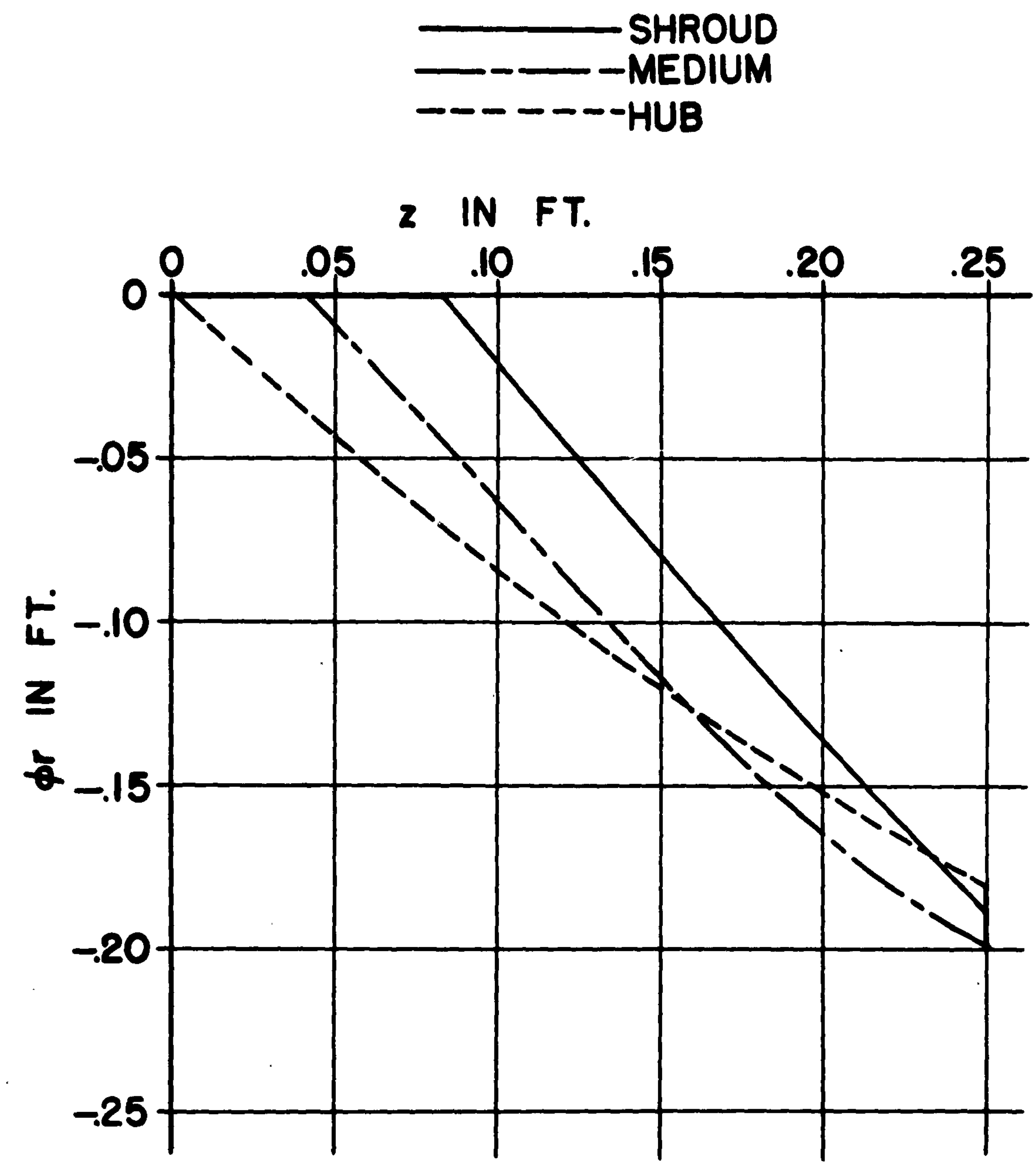


FIG. 11 TYPICAL STREAMLINES IN THE DEVELOPED HUB, MEDIUM, AND SHROUD SURFACES (BLADE SHAPE)

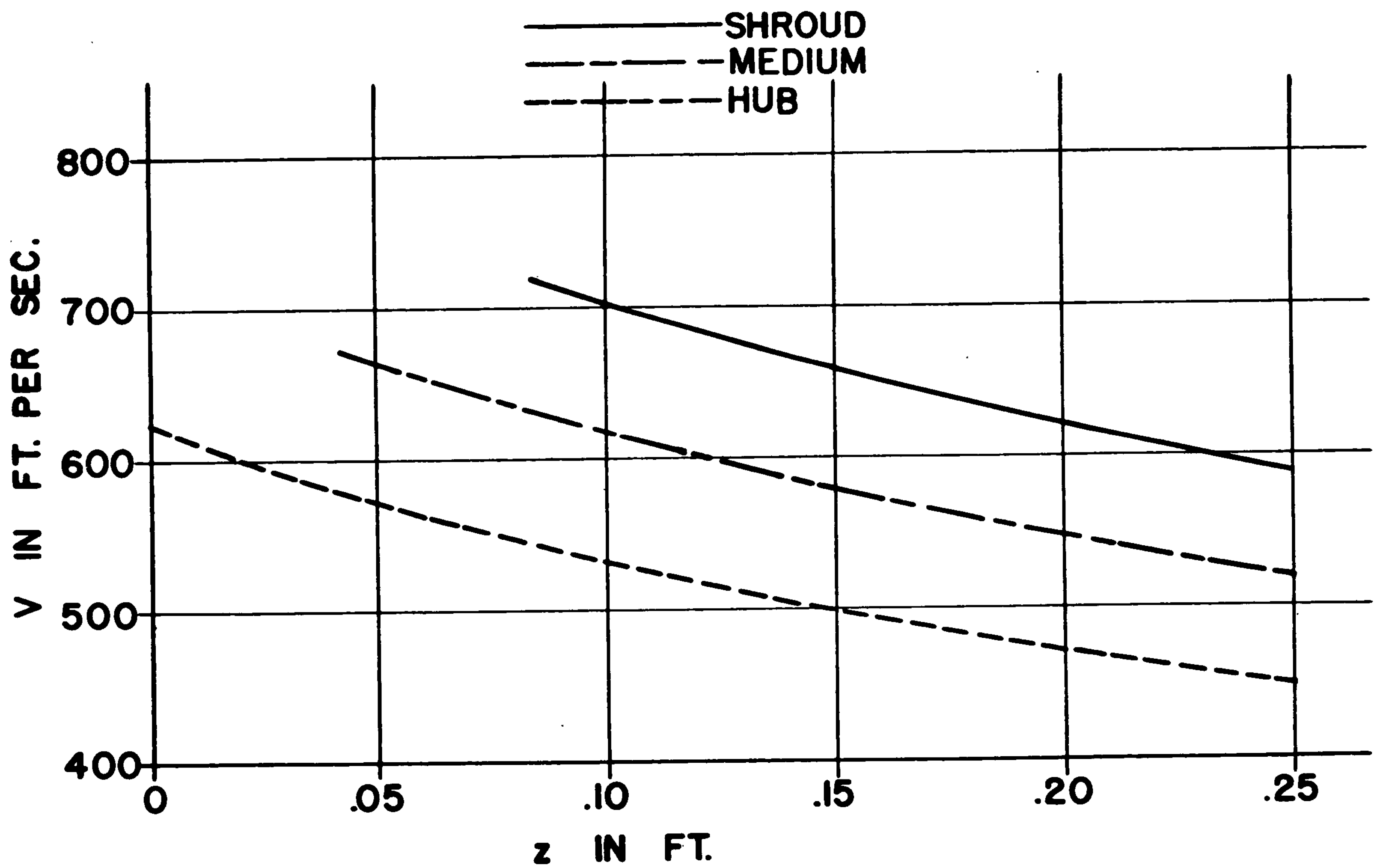


FIG. 12 AXIAL VARIATION OF RESULTANT VELOCITY IN THE HUB, MEDIUM, AND SHROUD SURFACES

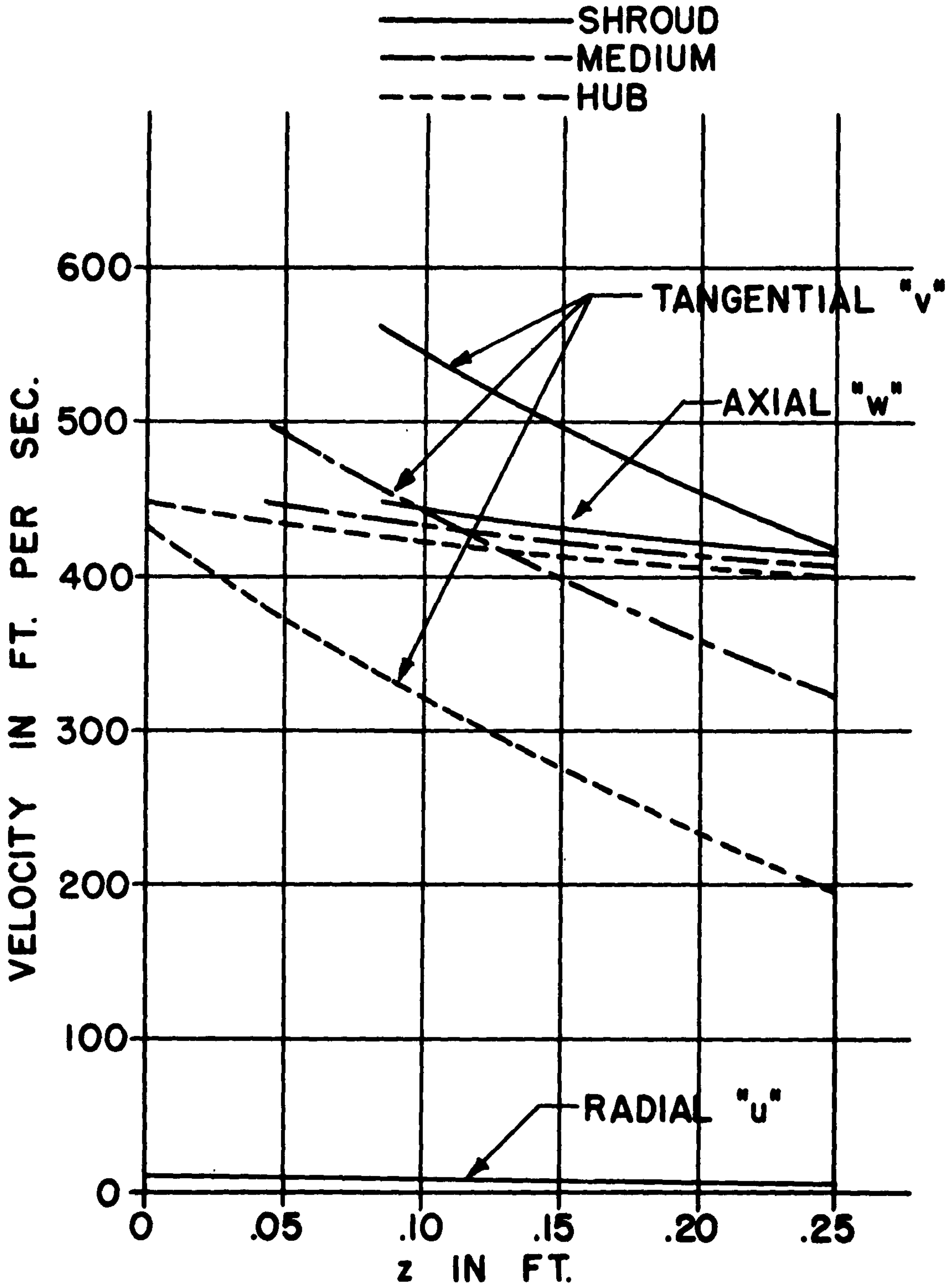


FIG. 13 AXIAL VARIATION OF VELOCITY COMPONENTS IN THE HUB, MEDIUM, AND SHROUD SURFACES

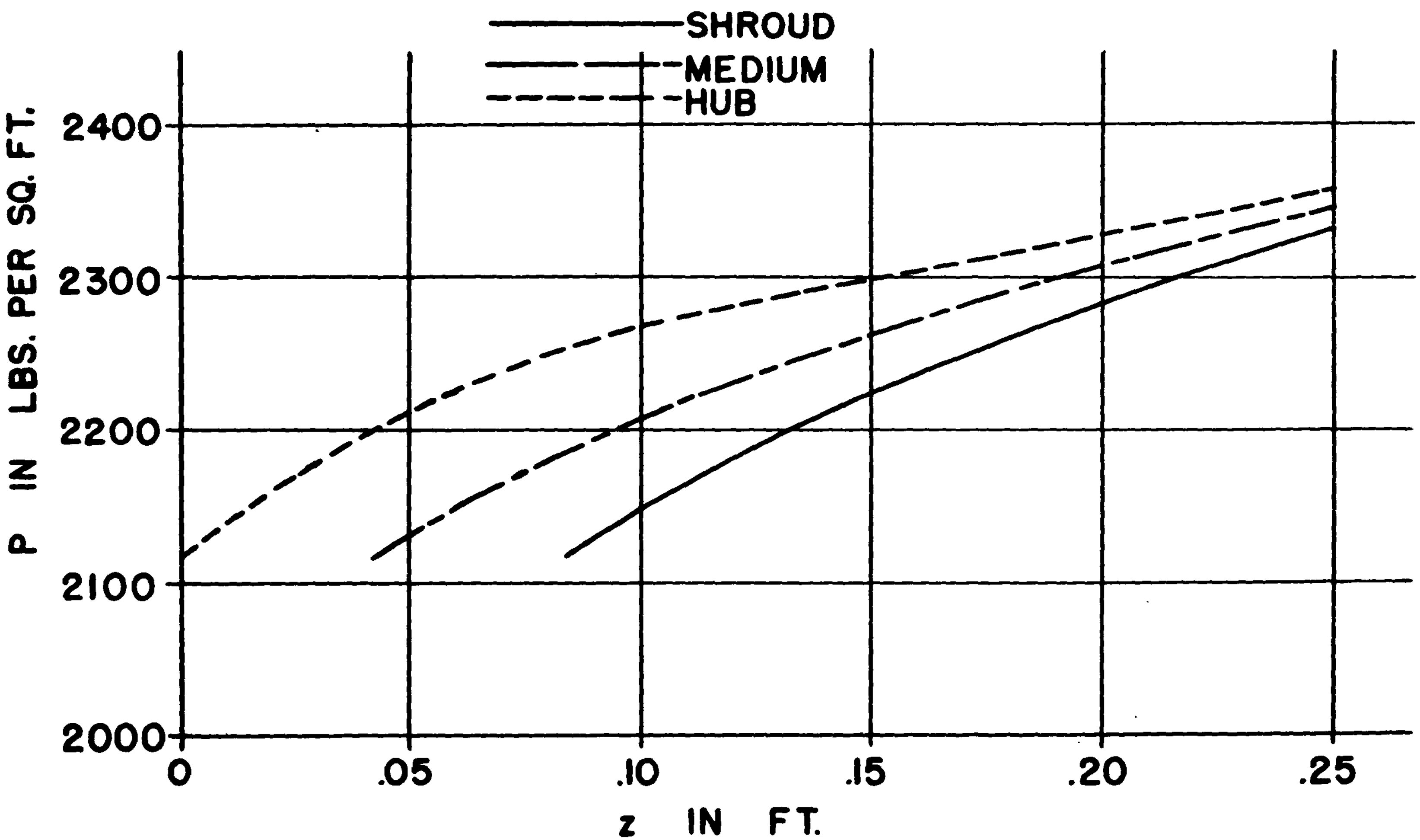


FIG. 14 AXIAL VARIATION OF STATIC PRESSURE IN THE HUB, MEDIUM, AND SHROUD SURFACES

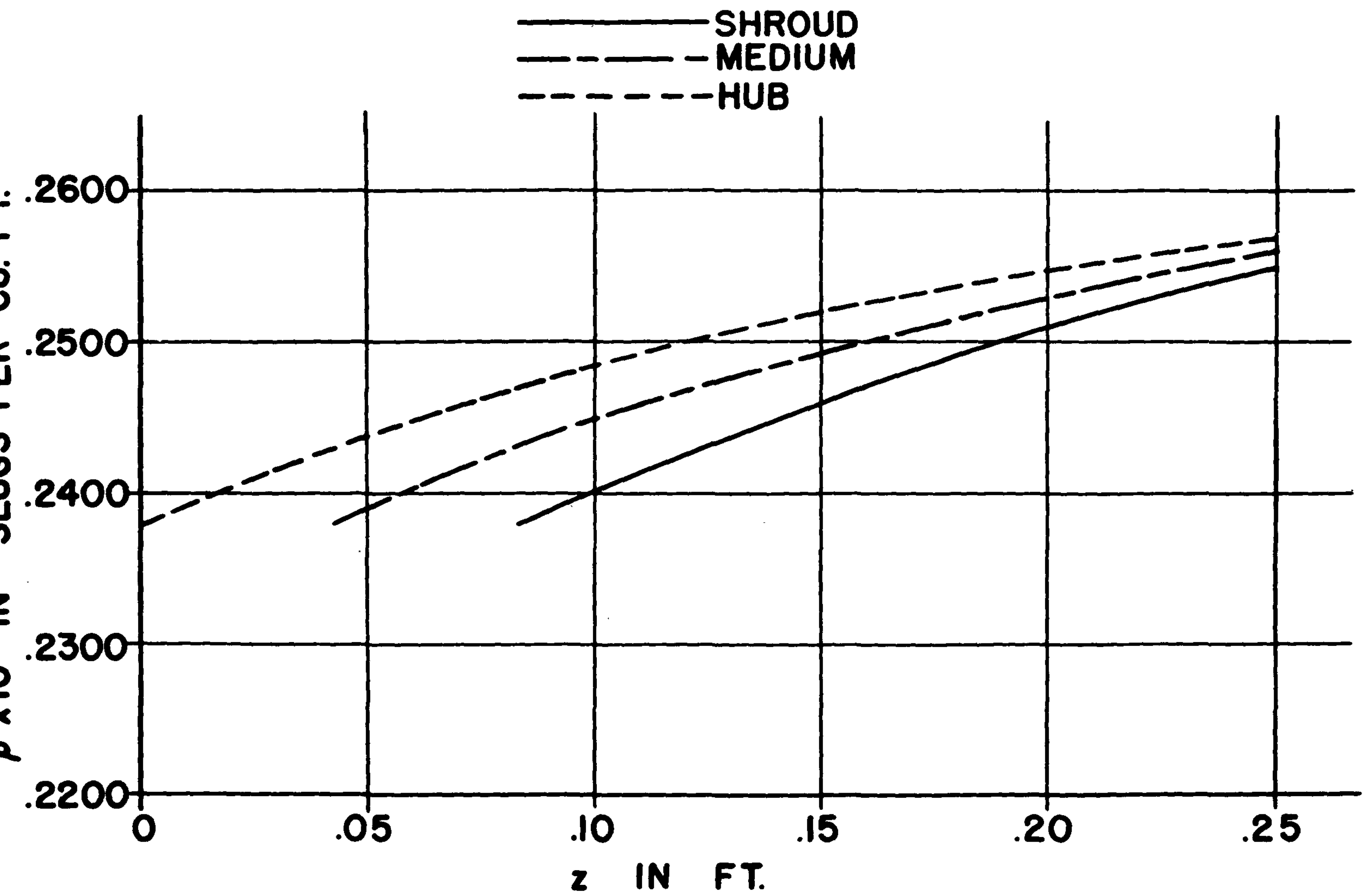


FIG. 15 AXIAL VARIATION OF DENSITY IN THE HUB, MEDIUM, AND SHROUD SURFACES

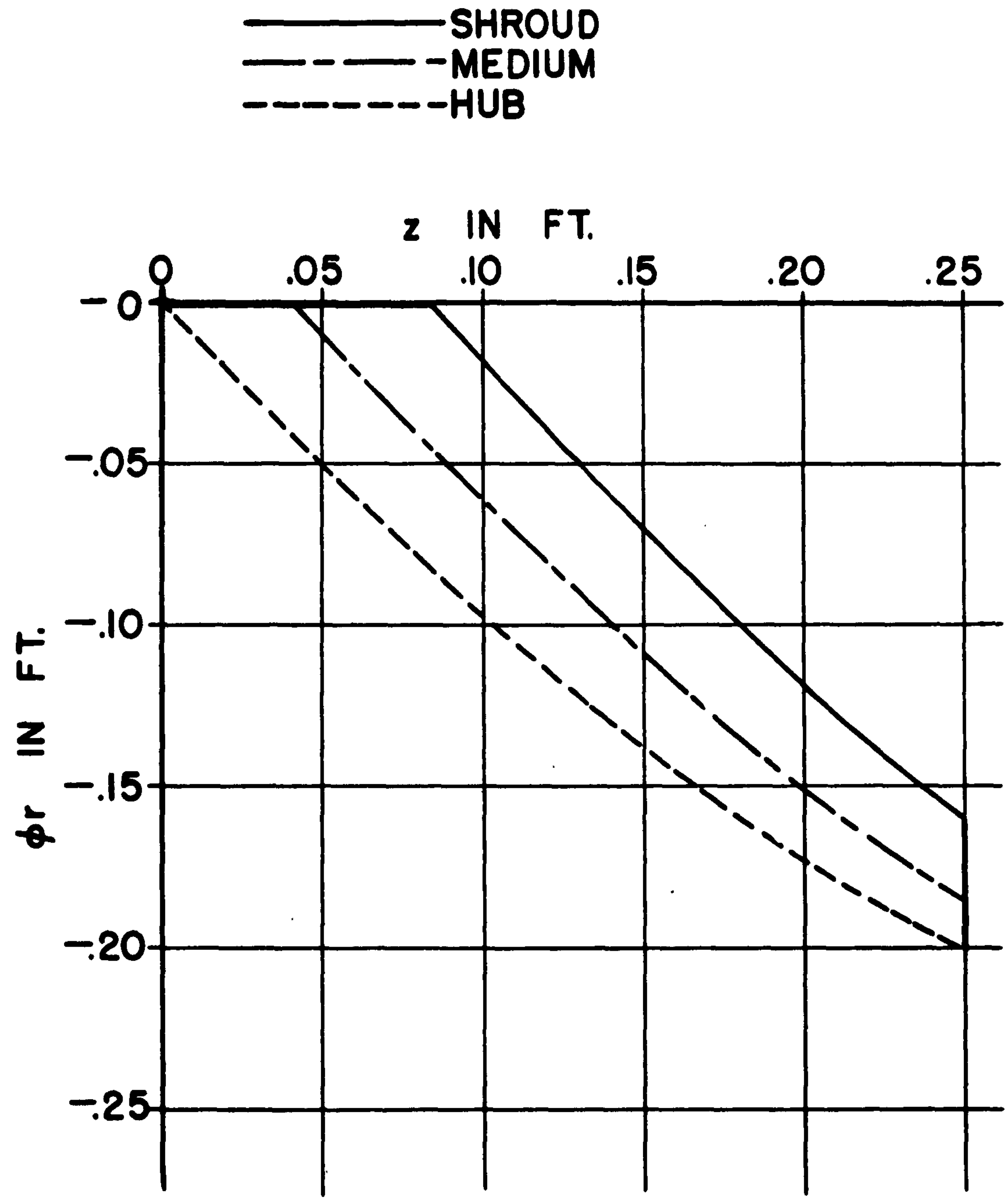


FIG. 16 TYPICAL STREAMLINES IN THE DEVELOPED HUB, MEDIUM, AND SHROUD SURFACES (BLADE SHAPE)

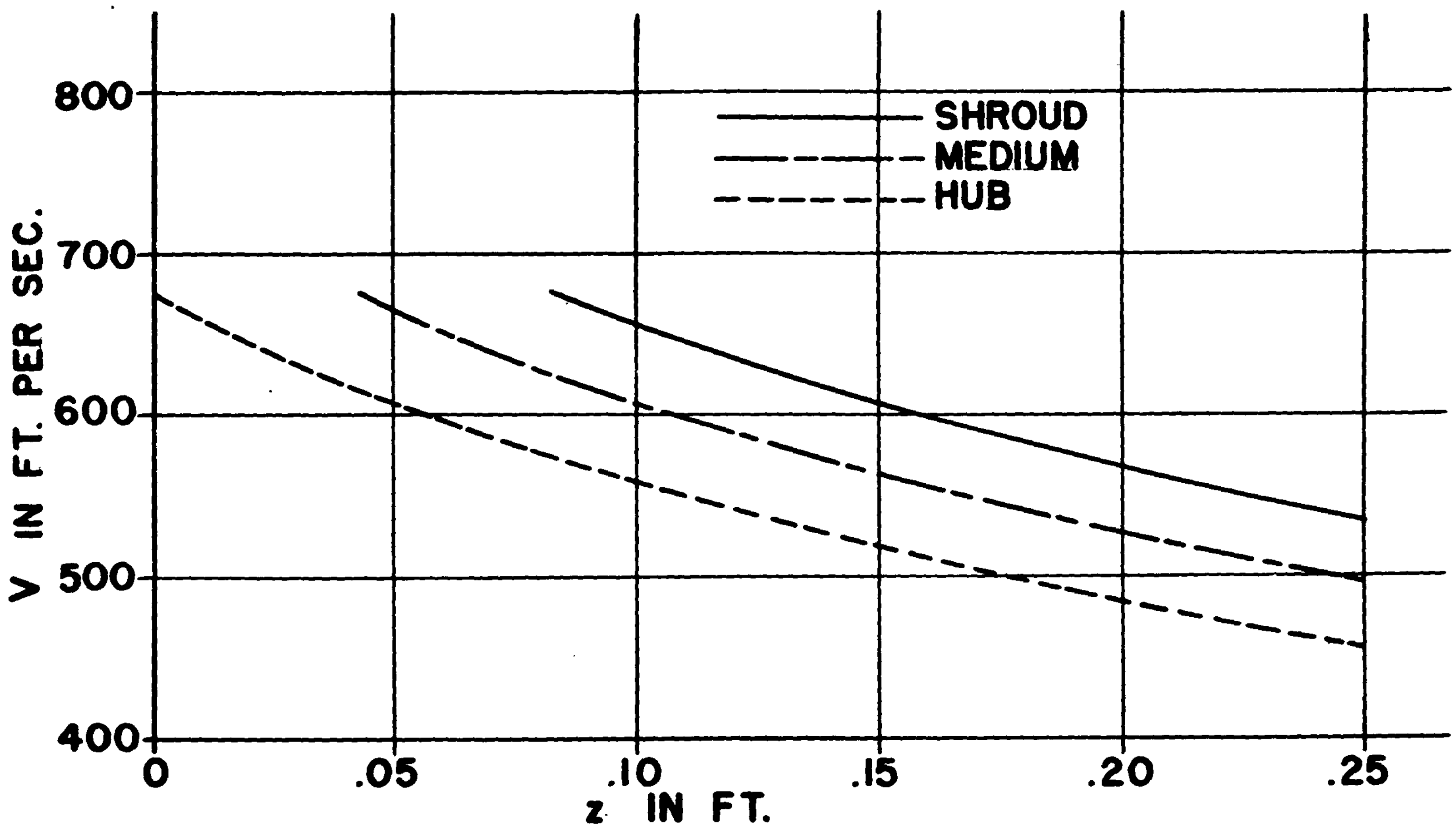


FIG. 17 AXIAL VARIATION OF RESULTANT VELOCITY IN THE HUB, MEDIUM AND SHROUD SURFACES

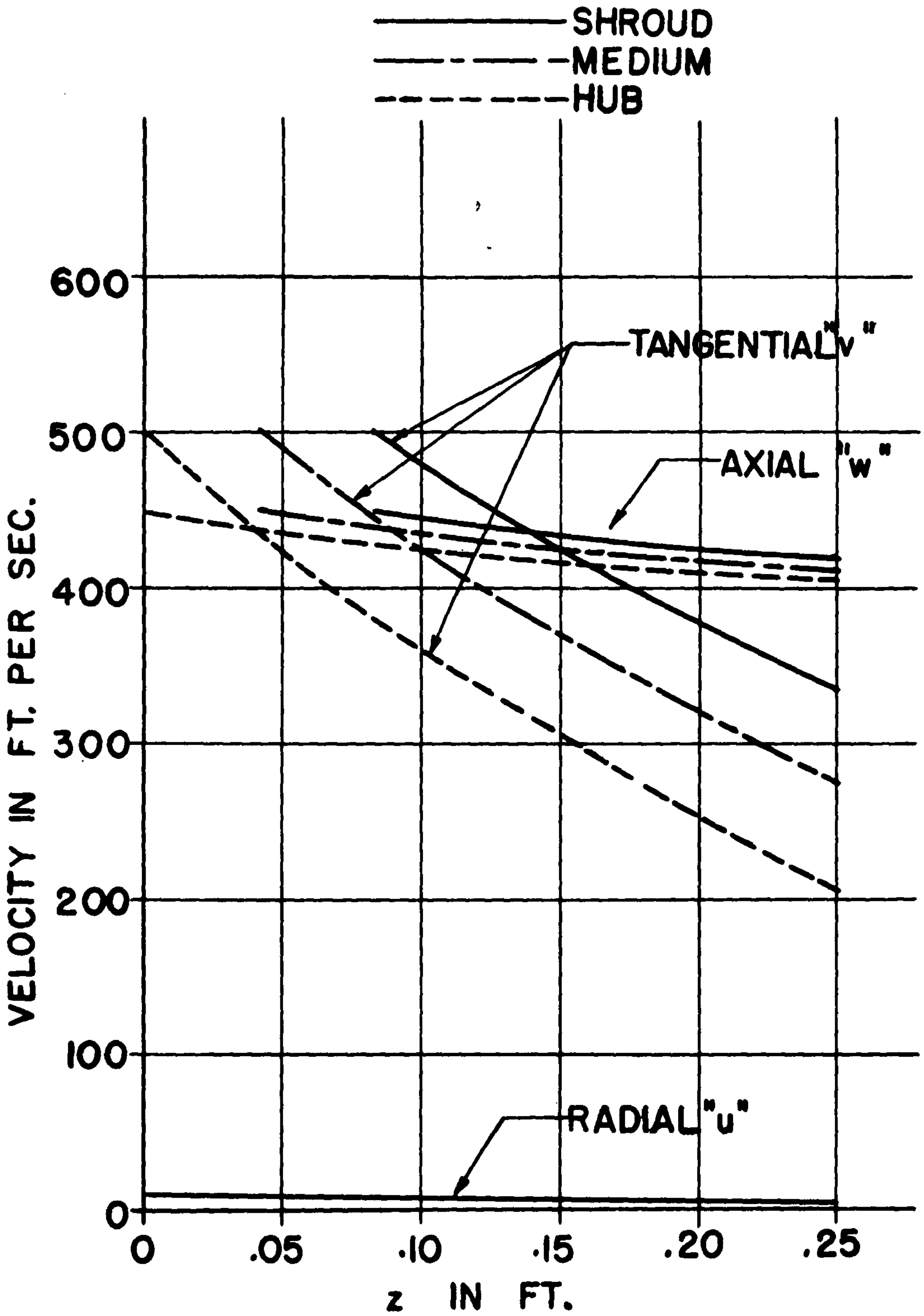


FIG. 18 AXIAL VARIATION OF VELOCITY COMPONENTS IN THE HUB, MEDIUM, AND SHROUD SURFACES

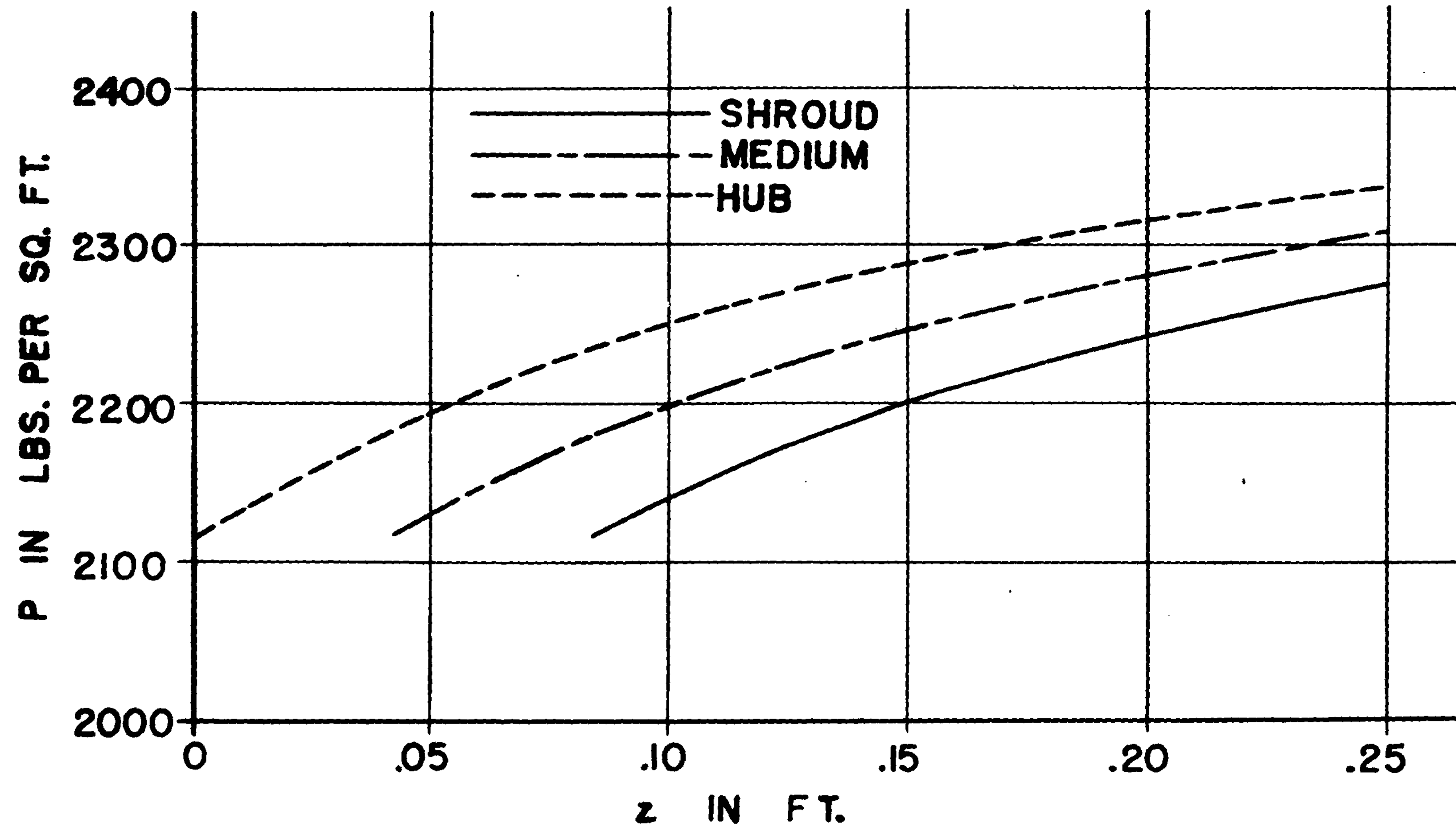


FIG. 19 AXIAL VARIATION OF STATIC PRESSURE IN THE HUB, MEDIUM, AND SHROUD SURFACES

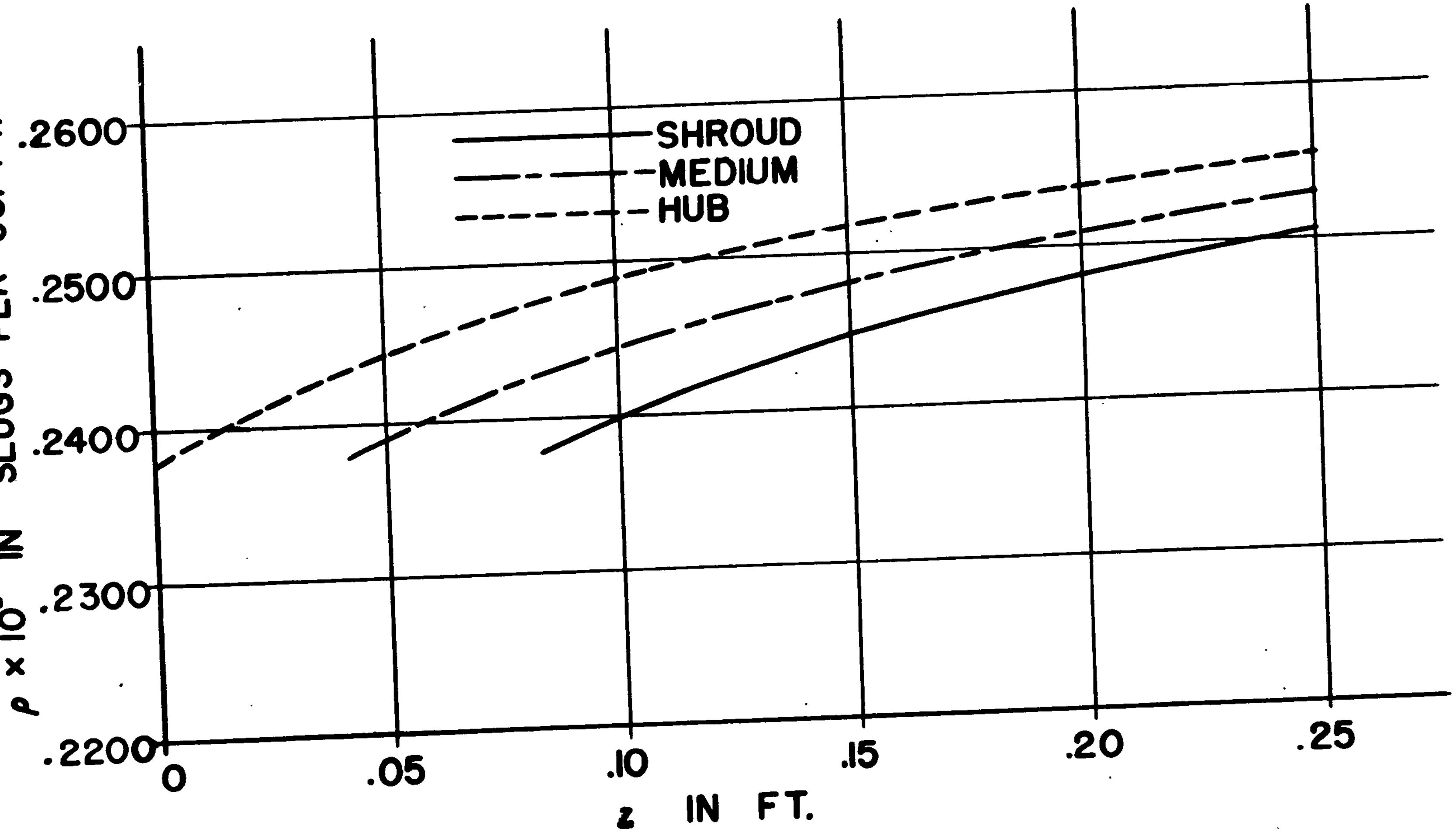
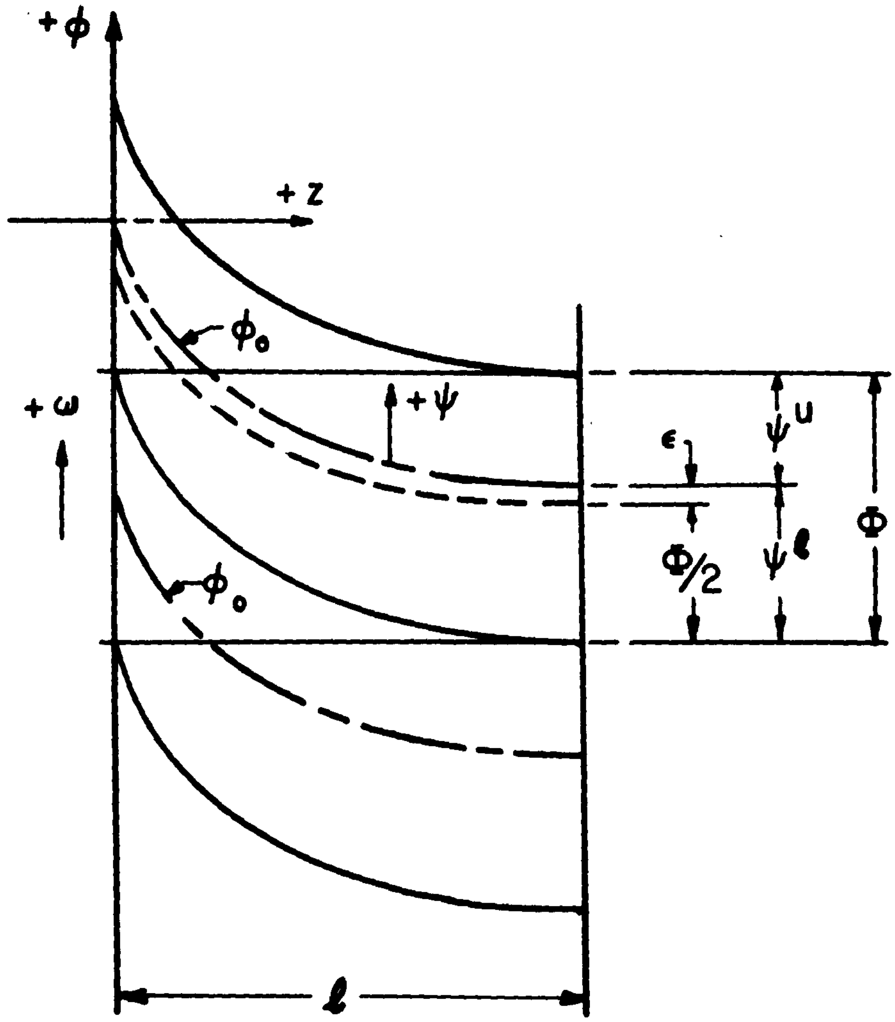


FIG. 20 AXIAL VARIATION OF DENSITY IN THE HUB, MEDIUM, AND SHROUD SURFACES



ψ^{u} :UPPER CHORD SURFACE

ψ^{ℓ} :LOWER CHORD SURFACE

ϕ_{o} :FROZEN STREAMLINE SURFACE

$\psi = \phi - \phi_{o}(r, z)$

FIG. 21 GEOMETRY OF BLADE CHANNEL

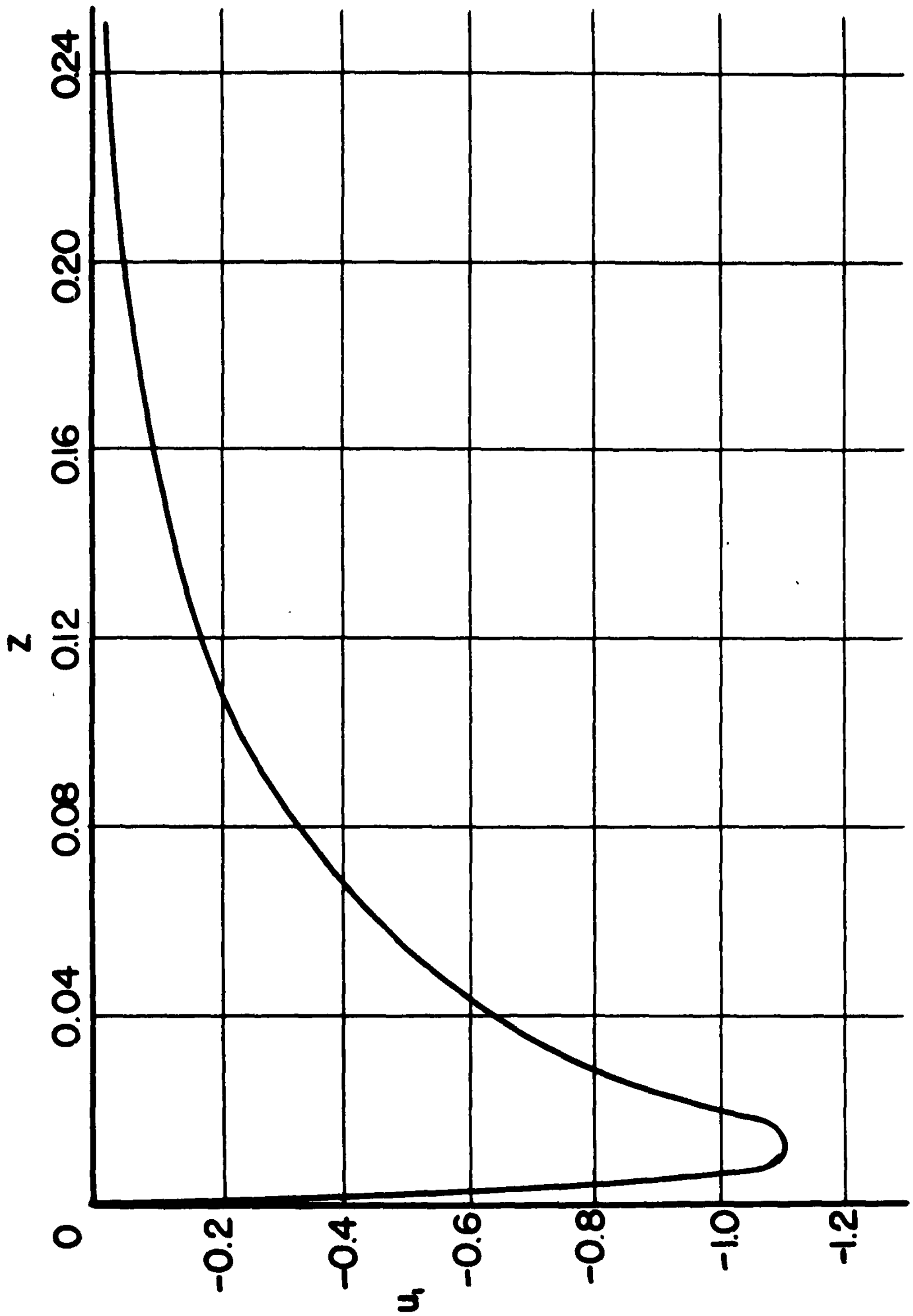


FIG. 22 u_1 VERSUS Z

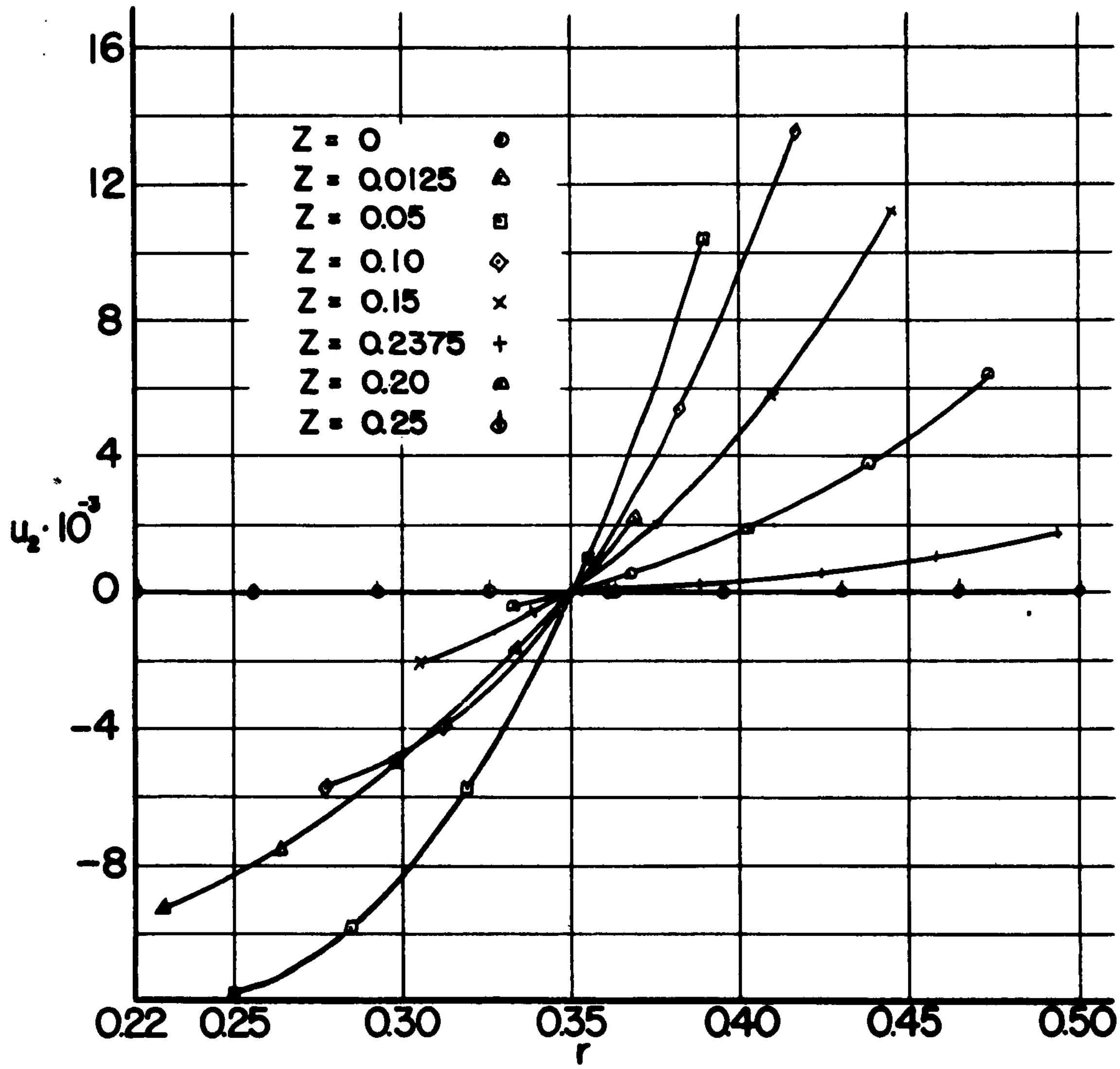


FIG.23 u_2 VERSUS r

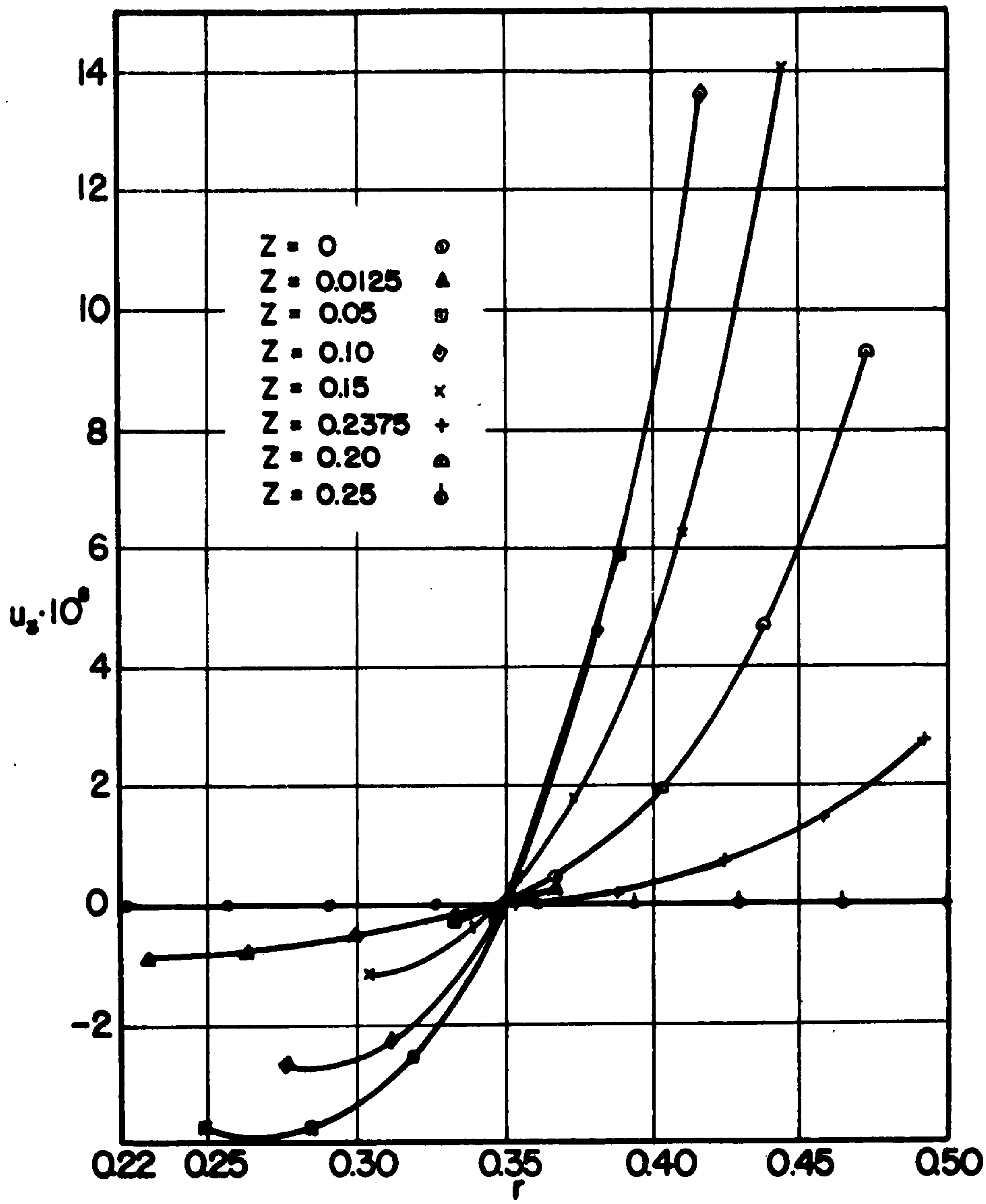


FIG. 24 u_s VERSUS r

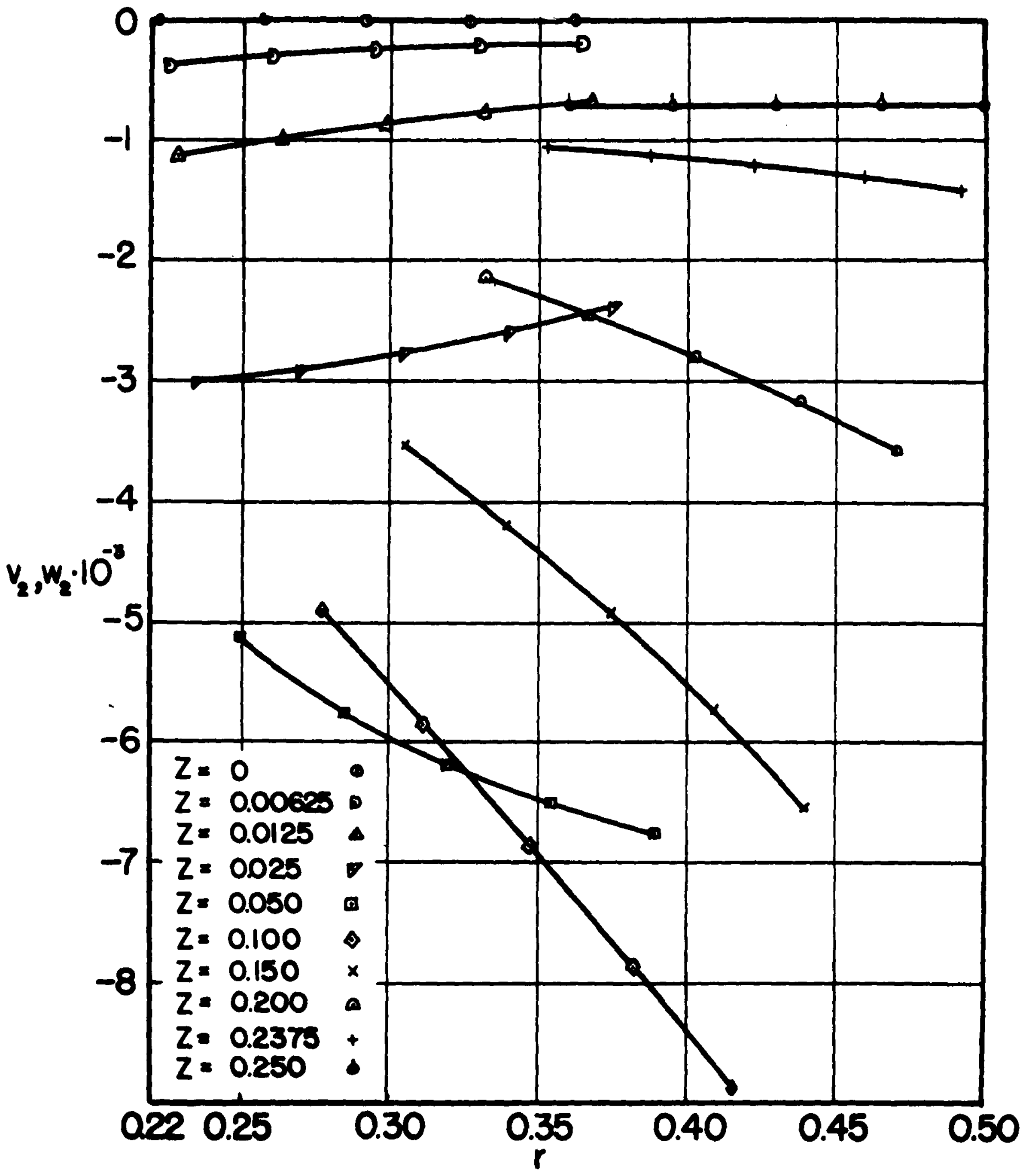


FIG. 25 V_2, W_2 VERSUS r ($V_2 = W_2$)

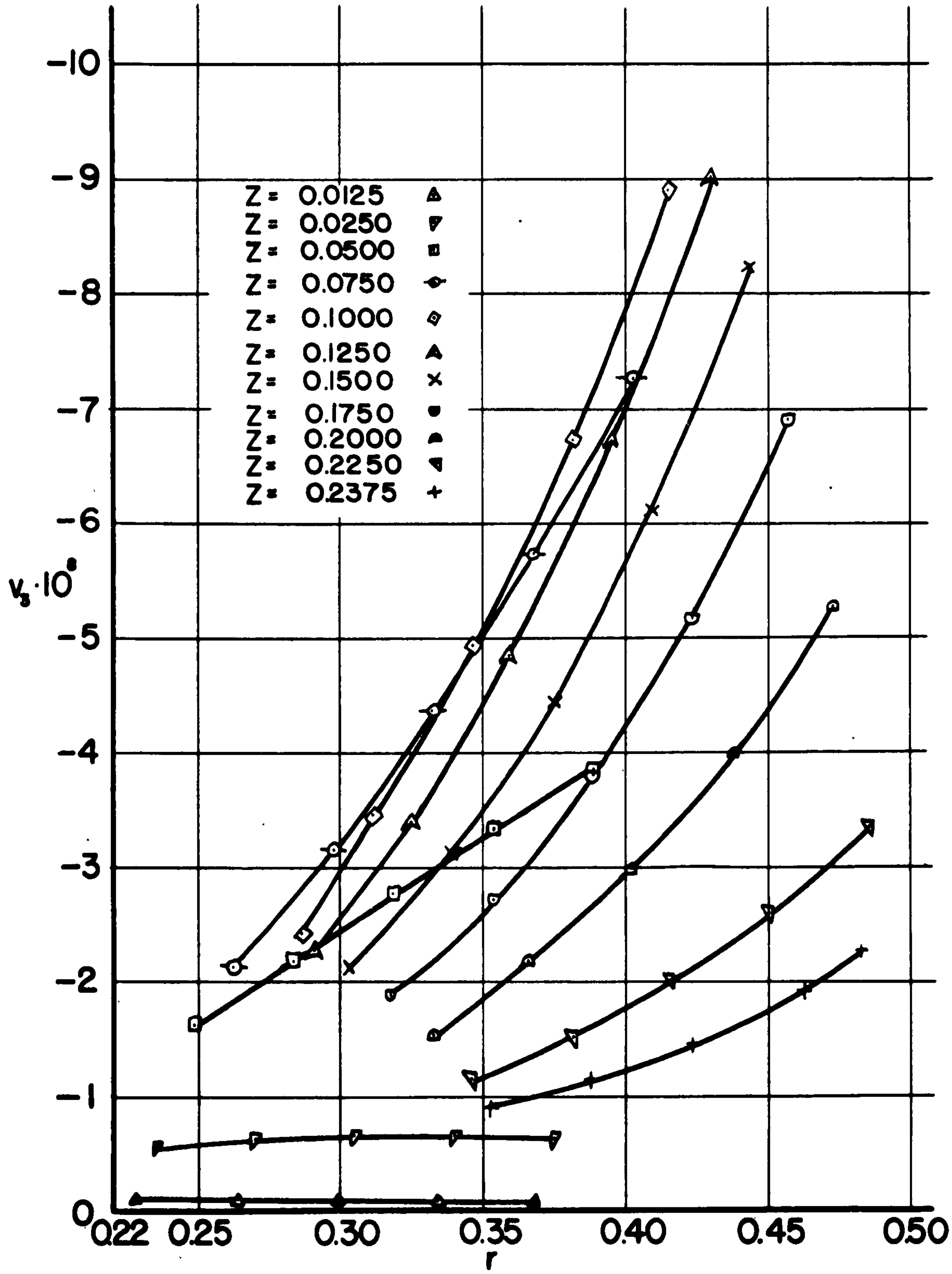


FIG. 26 v_3 VERSUS r

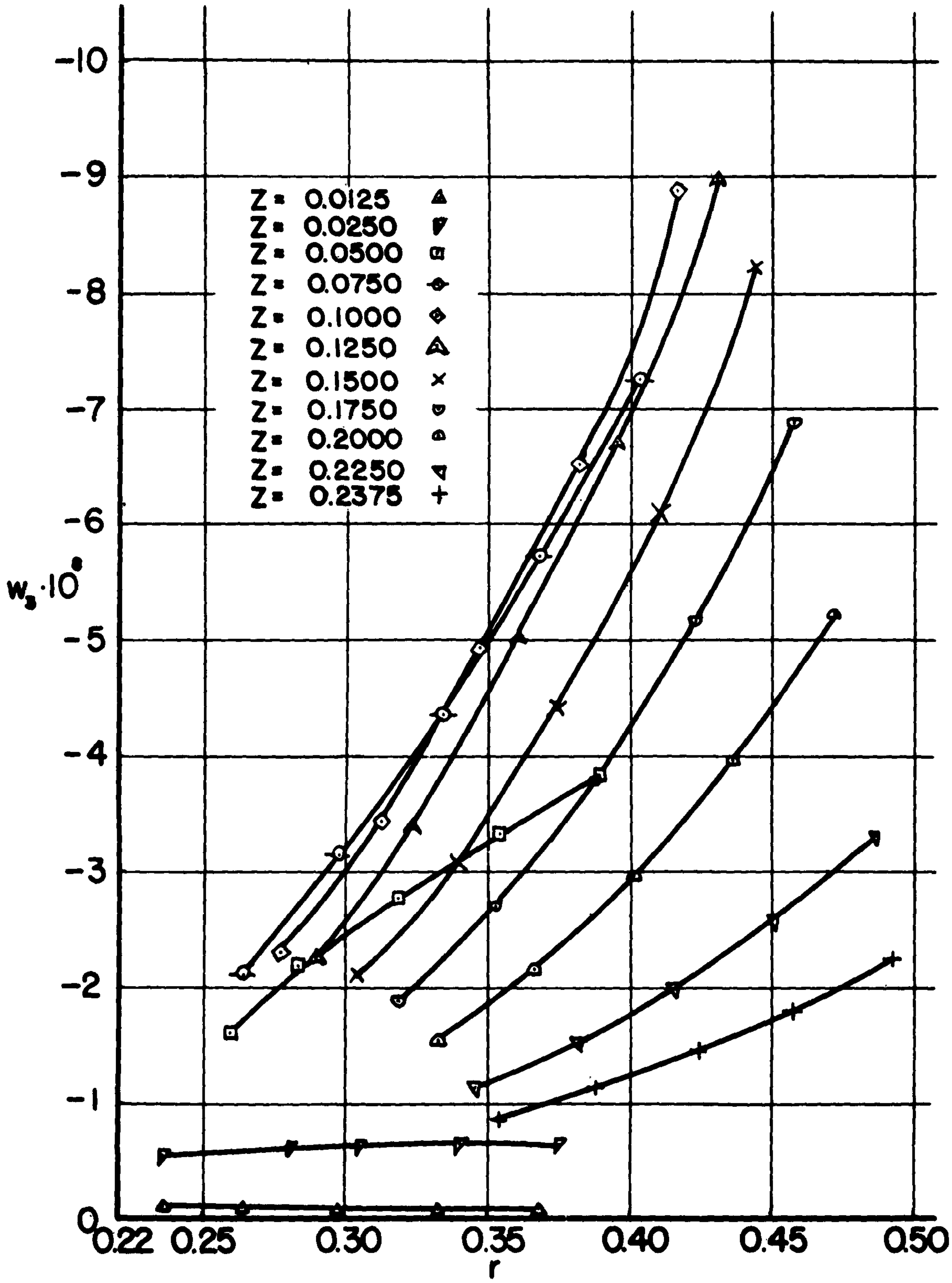


FIG. 27 w_s VERSUS r

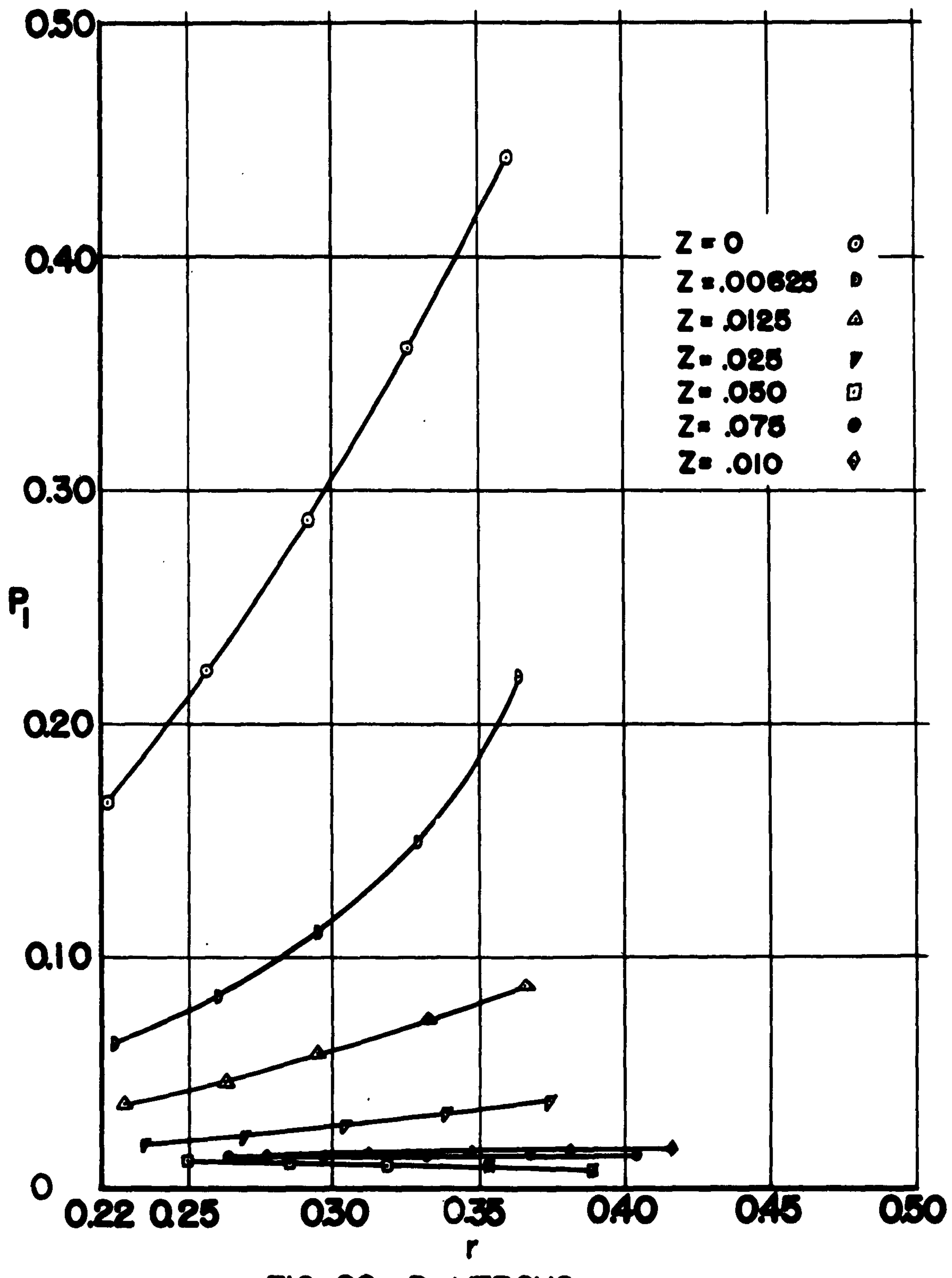


FIG. 28 P_1 VERSUS r

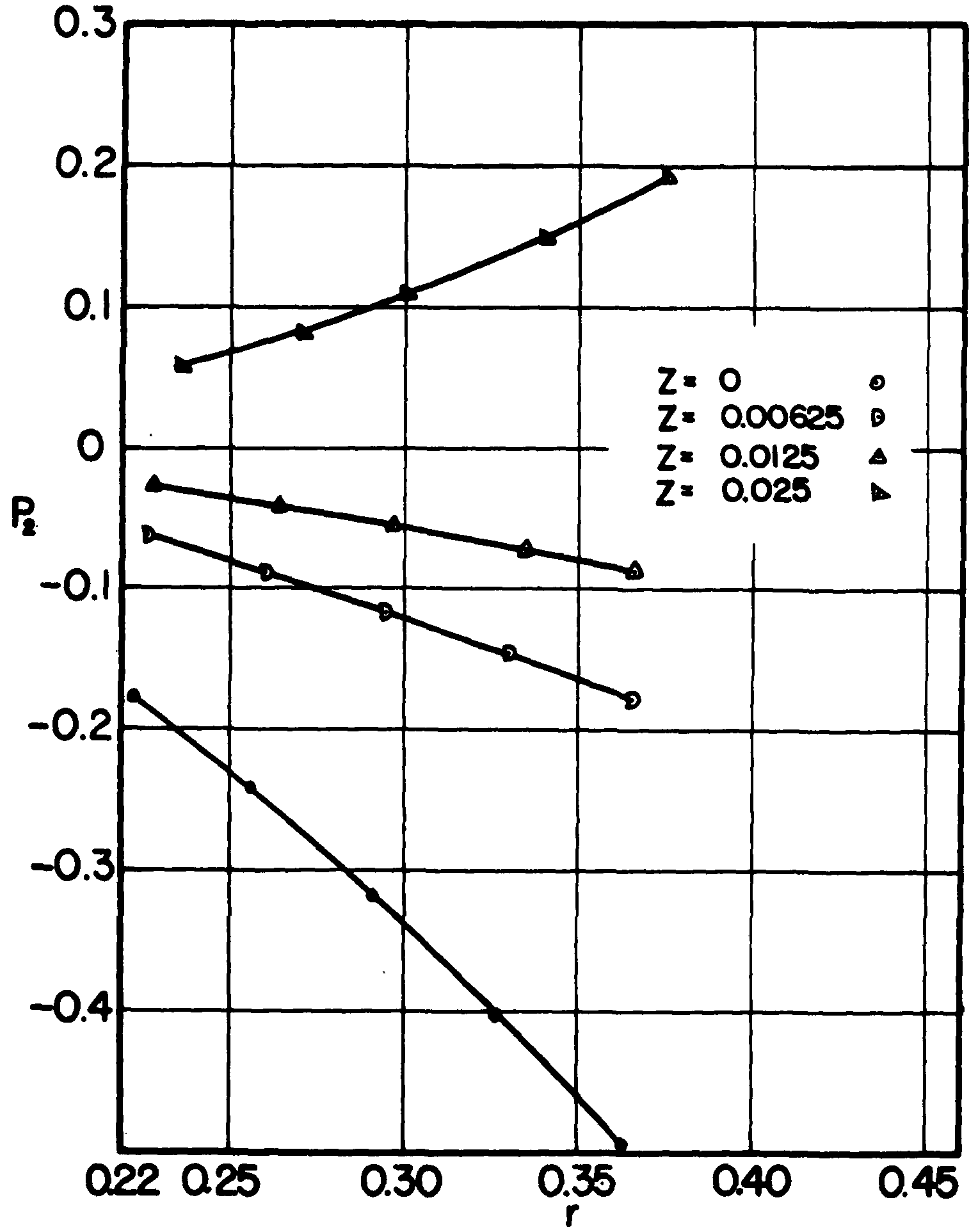


FIG. 29 P_2 VERSUS r

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