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PART 4
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DYNAMIC SYSTEM STUDIES: TECHNICAL STAFF REQUIREMENTS

WILLIAM R. ALLEN
UNIVERSITY OF CHICAGO

MARY C. WEISS
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SEPTEMBER 1956

WRIGHT AIR DEVELOPMENT CENTER

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AERONAUTICAL RESEARCH LABORATORY
PROJECT 7060
ADVISORY BOARD ON SIMULATION
CONTRACT No. AF 33(038)-15068, SUPPLEMENTS 2 AND 11

WRIGHT AIR DEVELOPMENT CENTER
AIR RESEARCH AND DEVELOPMENT COMMAND
UNITED STATES AIR FORCE
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

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FOREWORD

The Advisory Board on Simulation has concluded a three-year research program in air weapon system dynamics sponsored by Wright Air Development Center, with P. W. Nosker/WCRR as project engineer. This volume is one of the following 16 comprising the final report, WADC TR 54-250, entitled Dynamic System Studies:

Part No.	Subtitle	Editing Agency
1	Conclusions and Recommendations	University of Chicago
2	The Design of a Facility	" " "
3	The Mission of a Facility (Confidential)	" " "
4	Technical Staff Requirements	" " "
5	Analog Computation	Naval Ord. Lab., Corona
6	Operation & Maintenance Procedures for Analog Computers	University of Chicago
7	Digital Computers	" " "
8	Recorders	" " "
9	Flight Tables (Confidential)	" " "
10	Performance Requirements for Flight Tables	Mass. Inst. of Tech.
11	Load Simulators (Confidential)	Cook Research Lab.
12	Guidance Simulation (Secret)	Naval Ord. Lab., Corona
13	Error Studies	University of Chicago
14	Error Analysis for Differential Analyzers (written by F. J. Murray, Columbia U., and K. S. Miller, N.Y.U.)	" " "
15	Air Vehicle Characteristics (Secret)	" " "
16	Aerodynamic Studies (written by M. Z. Krzywoblocki, U. of Ill.)	" " "

The history of the project and a complete bibliography may be found in Part 1. All reports may be obtained through the project engineer.

This report represents the culmination of the assignment to determine the proper mission, equipmentation, operating procedures, and personnel for an engineering facility in the field of air weapon systems dynamics. The subdivisions of the report correspond to these four basic objectives and the subsidiary work in their support, and reflect the role of simulation as a dominant technique. The functions of each part and the relations among them are indicated in the technical summary, Part 2.

The following organizations have participated directly in the program:

Organization	Contract No.	Time of Performance
University of Chicago	AF33(038)-15068 Supplements 2 and 11	1 Feb. '51-31 Aug. '54
J. B. Rea Company	AF33(038)-15068 Subcontract 2	1 Feb. '51-31 Oct. '52
Cook Research Laboratories	AF33(038)-15068 Subcontracts 3 and 9	1 Feb. '51-31 May '54
RCA Laboratories	AF33(038)-15068 Subcontract 4	1 Feb. '51-1 Mar. '53
Armour Res. Foundation of Ill. Inst. of Technology	AF33(038)-15068 Subcontract 5	1 Feb. '52-30 Nov. '52
Northwestern University, Aerial Meas. Lab.	AF33(038)-15068 Subcontract 8	17 July '52-22 Aug. '52
Mass. Inst. of Technology, Flight Control Lab.	AF33(038)-15068 Purchase Order A2086	20 Apr. '54-31 Aug. '54
Mass. Inst. of Technology, Dynamic Analysis & Control Laboratory	AF33(038)-15068 Purchase Order A23883	22 July '53-30 Nov. '53
Mass. Inst. of Tech., D.A.C.L.	AF33(616)-2263 Task Statement 2	1 Dec. '53-30 Sept. '54
Nat. Bur. of Standards Corona, which became	AF33(038)-51-4345-E	25 Feb. '51-Sept. '53
Naval Ordnance Lab., Corona	MIPR(33-616)54-154	20 Nov. '53-31 Dec. '55

This is a record of formal participation only; the program was aided immeasurably by the splendid cooperation of all governmental, industrial and educational organizations (particularly the simulation laboratories) contacted. Although it is impractical to mention them all here, the extent of their assistance is evident throughout the reports and is hereby gratefully acknowledged. Details of these affiliations, including statements of work, may be found throughout the 21 Bimonthly Progress Reports issued by the University of Chicago during the course of the work. (All formal participation in the program is recorded above; missing supplement and subcontract numbers do not pertain to this project.)

The University of Chicago was assigned prime responsibility for integration of the program. This has been effected by a full time staff at the University, and by aperiodic meetings of the following advisory committee, selected by the Air Force:

Dean Walter Bartky, Chairman	University of Chicago	1 Feb. '51-31 Aug. '54
Prof. C. S. Draper	Mass. Inst. of Tech.	1 Feb. '51-28 Feb. '53
Mr. Donald McDonald	Cook Research Lab.	1 Feb. '51-31 Aug. '54
Prof. F. J. Murray	Columbia University	1 Apr. '52-31 Aug. '54
Dr. J. B. Rea	J. B. Rea Company	1 Feb. '51-28 Feb. '53
Prof. R. C. Seamans, Jr.	Mass. Inst. of Tech.	1 Sept. '53-31 Aug. '54
Mr. R. J. Shank	Hughes Aircraft Co.	1 July '51-31 Aug. '54
Dr. H. K. Skramstad	NBS-NOLC	1 Feb. '51-31 Aug. '54
Mr. A. W. Vance	RCA Laboratories	1 Feb. '51-31 Aug. '54
ex officio:		
Mr. P. W. Nosker, Project Eng.	WADC	1 Feb. '51-31 Aug. '54
Dr. B. E. Howard, Secretary	University of Chicago	1 Feb.'51-31 Aug. '54

The meetings have been recorded in the Bimonthly Progress Reports previously mentioned. Except for Dr. Skramstad, who has participated through direct arrangement between NBS-NOLC and WADC, members of the advisory committee who are not connected directly with the University have participated in the program through consulting agreements with the University of Chicago. In addition, similar consulting agreements with the University have provided for the participation of:

Dr. R. R. Bennett	Hughes Aircraft Co.	1 Jan. '52-31 Jan. '54
Mr. J. P. Corbett	Libertyville, Ill. (formerly with the University)	11 May '54-31 Aug. '54
Mr. G. L. Landsman	Motorola, Inc.	1 May '54-31 Aug. '54
Dr. Thornton Page	Johns Hopkins Univ. (formerly with the University, and Secretary to the Board until 1 Aug. '51)	7 Aug. '51-1 Mar. '53
Prof. M.Z. Krzywoblocki	Univ. of Illinois	15 Jan. '52-31 Aug. '54
Prof. K. S. Miller	New York Univ.	2 Nov. '53-31 Aug. '54
Dr. J. Winson	Riverside, N.Y. (formerly consultant to Project Cyclone)	1 Mar. '53-30 June '54

Many others have contributed significantly to the progress of the work. Among those from other organizations in regular attendance at most of the meetings of the committee have been Mr. Charles F. West, Air Force Missile Test Center; Prof. L. L. Rauch, University of Michigan, representing Arnold Engineering Development Center; Col. A. I. Lingard, WADC; and Dr. F. W. Bubb, WADC.

Coordination of the program and administration of the prime contract at the University of Chicago has been under the charge of Dr. Walter Bartky, Dean of the Division of Physical Sciences and Director of the Institute for Air Weapons Research; Dr. B. E. Howard, Assistant to the Director; and Messrs. William R. Allen and William J. Riordan, Group Leaders. The work at the cooperating institutions has been directed by the appropriate member of the advisory committee and his assistants: Dr. H. K. Skramstad and Mr. Gerald L. Landsman at the National Bureau of Standards-Naval Ordnance Laboratory, Corona; Messrs. Donald McDonald and Jay Warshawsky at Cook Research Laboratories; Messrs. A. W. Vance, J. Lehman and Dr. E. C. Hutter at RCA Laboratories; Dr. J. B. Rea at J. B. Rea Company; Prof. R. C. Seamans at the Flight Control Laboratory and Dr. W. W. Seifert and Mr. H. E. Blanton at the Dynamic Analysis and Control Laboratory, Mass. Inst. of Technology. V. H. Disney, S. Hori, and G. F. Warnke at Armour Research Foundation and J. C. MacAnulty and George Goelz at Northwestern University, Aerial Measurements Lab., have directed the contributory studies at their respective organizations. More explicit credit is found in appropriate places throughout the reports; biographical sketches are in Part 1. Space does not allow full credit that is due to all the workers on the combined project, but special mention is certainly due the project engineer for his conception of the project and for his cooperation during its execution.

This document was edited by William R. Allen and Mary C. Weiss, and was prepared for publication by E. R. Spangler at the University of Chicago. The editors assume responsibility for the selection of the material; where not otherwise indicated Mr. Allen is the author of all sections in the text. The editors are indebted to many people and organizations for advice and information helpful in the preparation of this report. In particular, acknowledgement is due the following organizations for their assistance in the survey of procedures and personnel at simulation laboratories:

Aerial Measurements Laboratory, Northwestern University
Applied Physics Laboratory, Johns Hopkins University
Askania Regulator Corporation
Bell Telephone Laboratories
Boeing Airplane Company
Consolidated Vultee Aircraft Corporation, Pomona and San Diego divisions
Douglas Aircraft Company
Dynamic Analysis and Control Laboratory, Mass. Inst. of Tech.
Flight Control Laboratory, Mass. Inst. of Tech.
Glenn L. Martin Company
Goodyear Aircraft Company
Hughes Aircraft Company
Naval Air Development Center
Naval Ordnance Laboratory, Corona, Calif.
Project Cyclone, Reeves Instrument Company
Radio Corporation of America, Laboratories Division
Rand Corporation

The quotation from Cybernetics, by Norbert Weiner, in Chapter 3 is reprinted with the permission of its publisher, John Wiley and Sons, Inc.

ABSTRACT

A system dynamics laboratory requires a diversity of specialized personnel to accomplish its tasks. This technical staff can be organized conveniently into three branches:

(1) System Analysis Branch
(2) Simulation Equipment Branch
(3) Technical Services Branch

The System Analysis Branch has responsibility for formulating and performing studies connected with aerial weapons systems. The Simulation Equipment Branch is responsible for the operation, maintenance, and design of the simulation equipment used in the studies. The Technical Services Branch furnishes information and advice to the system engineering team on such things as component response aerodynamics, thermodynamics, structures, military requirements, statistical evaluation of data, programming of digital check problems, and the like.

PUBLICATION REVIEW

FOR THE COMMANDER:

Aldro Lingard

ALDRO LINGARD
Colonel, USAF
Chief, Aeronautical Research Laboratory
Directorate of Research

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INTRODUCTION

In the operation of a system dynamics laboratory devoted to the study of unitary aerial weapon systems, the complexity of the problems virtually forces the adoption of the practice of using an integrated team composed of specialists at whose disposal is a considerable array of simulation equipment. This part of the final report concerns itself with setting requirements for the technical personnel and laboratory organization which will tend to make a laboratory both efficient and effective.

In order to set these requirements we have studied:

(1) the purpose and field of operation of a laboratory
(2) the equipment that a laboratory needs
(3) the operation of similar laboratories throughout the country
(4) mathematical models of various mechanisms requiring service.

Information concerning the first and second studies is contained in other parts of this report. Information concerning the third was obtained from two surveys of dynamic analysis and simulation laboratories, a special report to the Advisory Board on Simulation by the Dynamic Analysis and Control Laboratory, M.I.T., the pertinent part of which is included as Appendix 1, and a literature search of the reports dealing with similar facilities. Information concerning the fourth study is contained in appendices to this volume.

The size of the staff given in this volume is adequate for a four-team one-shift operation of a facility that does not perform its own digital check problems. The organizational structure considered here takes into account only the technical portion of the laboratory; this report makes no recommendations concerning such necessary adjuncts as administrative staff, clerical staff, draftsmen, fabrication shop, or library. Neither does it concern itself with how the technical groups are to be administered, although it is appropriate to quote here the following unanimous recommendation of the Advisory Board on Simulation: "The staff of a system dynamics laboratory should be administered in accordance with accepted procedures at a scientific project, of the kind now operated by the federal government, e.g., the laboratories of the National Bureau of Standards."

The problem of determining the number of people required to maintain a piece of equipment has given rise to a study of "waiting lines." Since most of the cases ordinarily encountered consider either a single type of customer or a single server, the theoretical study described in Appendix 2 was necessary in order to provide a sound basis for making estimates of the number of maintenance personnel needed. Appendix 3 is an application of a well-known "waiting line" technique for estimating the number of maintenance men needed if the only failures are in amplifiers and power supplies. The failure rates are based on two studies of machine failures, at the Aerial Measurements Laboratory, Northwestern, and at Convair, Pomona. In addition, this study has given us useful information for planning computer maintenance.

1. ORGANIZATION OF A SYSTEM DYNAMICS LABORATORY

In discussing the organization of a system dynamics laboratory and its technical staffing, it is helpful to examine the main functions of the laboratory and to see what tools and methods are used to perform these functions. As indicated in Part 3, the chief work of this laboratory is to

(1) study the feasibility of proposed unitary aerial weapons systems,

(2) aid in the design of unitary aerial weapons systems,

(3) aid in the evaluation of current aerial weapons systems and aerial weapons systems projects.

(4) study modifications for the improvement of current aerial weapons systems.

A staff capable of carrying out such a program must include people with a knowledge of

(1) system design, particularly as it applies to flight control, guidance, and fire control systems,

(2) system environment, with particular reference to aircraft structure, aerodynamics, thermodynamics, military targets, countermeasures, and meteorological and topographical effects on guidance and control systems,

(3) methods of representing the system with mathematical and physical models usable for scientific study.

One of the main techniques of the laboratory is flight simulation. The equipment proposed for this includes electronic computers, hydraulic and electromechanically driven flight tables, hydraulic and mechanical force generators (load simulators), optical guidance simulators, radar simulators, and infrared simulators. The laboratory needs engineers and technicians capable of operating and maintaining such equipment.

To perform reasonable systems studies and flight simulation it is necessary that dynamic data be obtained on subsystems and components; as a result there must be staff members able to obtain such data by literature search, survey of manufacturers, or by having tests of their own devising executed. In addition, the experience of other organizations indicates that the laboratory can continue as an effective engineering and scientific group only if its staff continually tries to improve old and to develop new techniques and equipment

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Manuscript released by the author in July 1954 for publication as a WADC Technical Report.

for studying systems.

Analysis of the work of the laboratory indicates that it breaks down into three main categories: system analysis, simulation equipment, and technical services. With these categories in mind, the laboratory can be organized as follows:

I. System Analysis Branch
    A. System engineering section
    B. System research section
II. Simulation Equipment Branch
    A. Electronic equipment section
        1. Operation
        2. Maintenance
        3. Design
    B. Mechanical Equipment Section
        1. Operation
        2. Maintenance
        3. Design
    C. Hydraulic Equipment Section
        1. Operation
        2. Maintenance
        3. Design
III. Technical Services Branch
    A. Component Dynamic Response Section
        1. Electrical components
        2. Mechanical components
        3. Hydraulic components
    B. Technical Consulting Section

A diagram of this organizational setup is given in Figure 1.

The team working on a given problem requires engineers, physicists, and mathematicians from the system engineering section and simulator operating personnel from the various sections of the simulation equipment branch. The team would receive help as required on problems in aerodynamics, structures, components, military requirements, statistics, numerical analysis, etc. from personnel in the technical services branch. Maintenance of simulation equipment used in the problem and fabrication of special equipment needed for the problem are responsibilities of the appropriate sections of the simulation equipment branch.

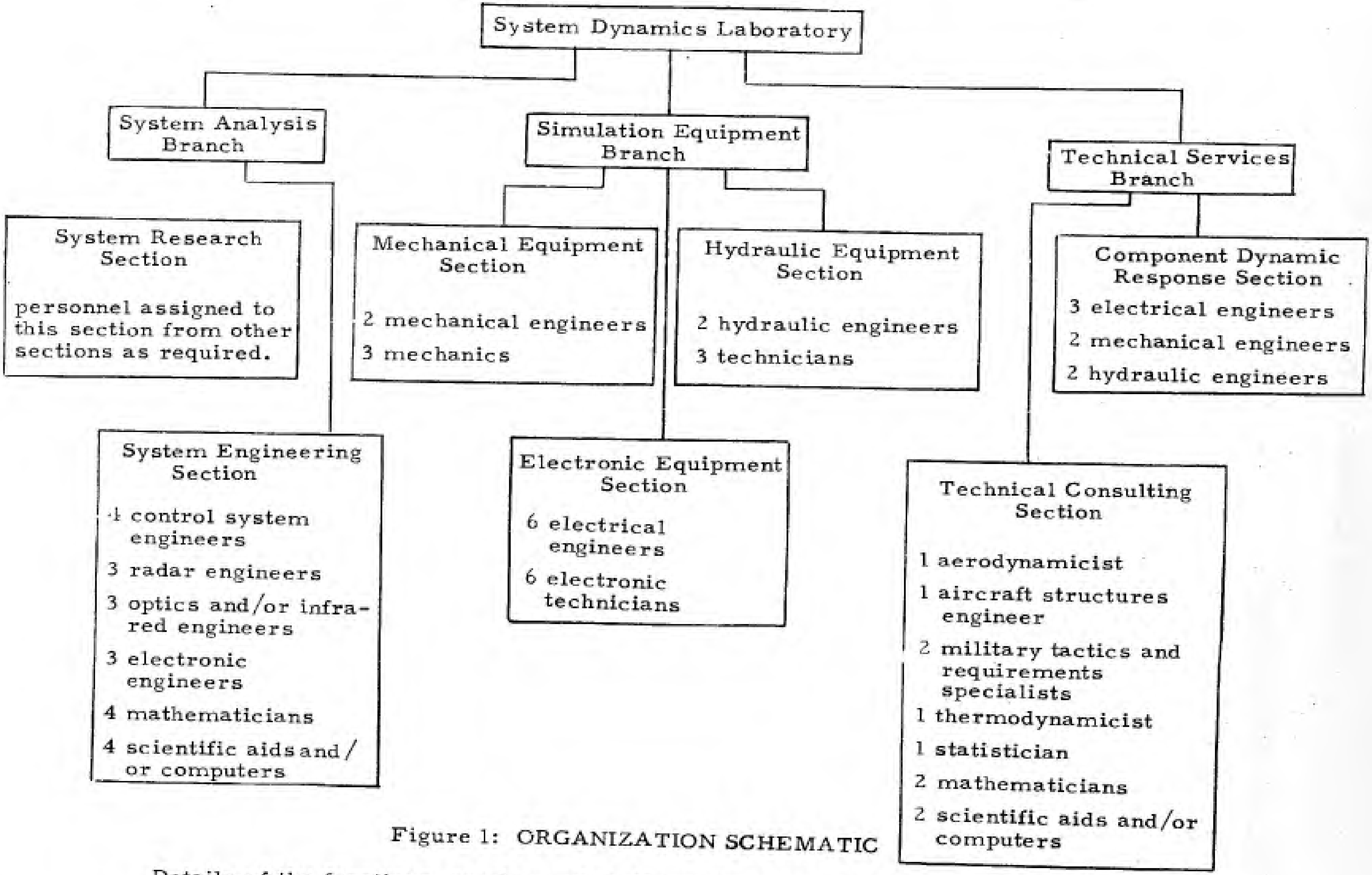

Figure 1: ORGANIZATION SCHEMATIC

Details of the functions, numbers of staff members required, staff qualifications, and organization of the problem team are given in succeeding chapters.

2. STAFF REQUIREMENTS

2.1 The System Analysis Branch

2.1.1 System Engineering Section. The system engineering section of the system analysis branch has primary responsibility for formulating and performing studies connected with the dynamics of aerial weapons systems. Most of the expected problems involve the unified operation of flight control, guidance, and fire control systems. Most of the mathematical models in current use have their representation formulated as algebraic, differential, integral, or mixed systems of equations and/or inequalities; many current models also involve stochastic processes, e.g., noise problems, random walk problems. The system engineering staff should consist of engineers, physicists, and mathematicians capable of handling both engineering problems connected with the systems and mathematical problems connected with their representations.

On the basis of the D.A.C.L. report* and our own studies of organizations performing system dynamics problems, we have drawn up the following staff requirements for the system engineering section. At the risk of being blacklisted forever by the physical society and the engineering societies, we shall lump engineers and physicists together, describing the job by the type of material to be studied with the title of engineer attached. It is felt that the number of personnel indicated is adequate for three or four teams working on system problems and allows three to five people freedom to pursue some basic problems in the system research section. The personnel types and numbers are as follows:

control system engineers:	4
radar engineers:	3
optics and/or infrared engineers :	3
electronic engineers:	3
mathematicians:	4
scientific aids and/or computers:	4

The control system engineers should be either electrical or aeronautical engineers with experience and/or training in servomechanism work. All of the engineers should have formal training at least equivalent to the bachelor's

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*See Appendix 1.

degree in engineering, and at least one of these should have approximated five years of responsible professional experience in design and evaluation of control systems or in a closely related field. The other three should have a minimum of two years' professional experience in control systems or related fields.

The radar specialists should be either electrical engineers or physicists with a knowledge of the physics of electromagnetic waves and with training and/or experience in the design of radar. At least one of the specialists should have five years of responsible experience in the design of radar or in research on radar wave patterns. The other two should have a minimum of two years' professional design or research experience. At least one of the men should be familiar with available laboratory testing and simulation techniques for radar seekers.

The optics specialists should be physicists or engineers with training and/or experience in the design of infrared or other optical seekers and with a knowledge of the physics of light waves. At least one should have about five years of responsible professional experience in the design of such seeker systems and in the propagation patterns of light waves. The other two should have a minimum of two years' professional design or research experience in this field.

The electronics specialists should be electrical engineers or physicists. At least one should have about five years of responsible experience in the design of such things as airborne computers for field use, communications systems, etc. The other two electrical engineers should have at least two years of experience in the design and/or operation of electronic gear.

Two of the mathematicians should have Ph.D.'s in mathematics or the equivalent, and in addition one of these should have about five years' experience in applied mathematics, should have extensive knowledge of ordinary and partial differential equations, particularly as they apply to mechanics, electronics, and aerodynamics, and should have a good general background in analysis. The second should have at least two years' experience in applying mathematics to physical problems and a good general background in analysis. In addition, each of the following fields should be of more than passing familiarity to at least one of these two mathematicians: integral equations, stochastic processes, and approximating polynomials. The other two mathematicians should have the master's degree or the equivalent in mathematics, with competence in analysis, particularly differential equations; and at least one should have sufficient knowledge of numerical methods to act as an advisor to the computers. All four mathematicians should have some knowledge of the operation and capabilities of electronic differential analyzers.

The four computers should have at least a bachelor's degree in one of the following: engineering, physics, mathematics, or statistics. Training in mathematics must include trigonometry, analytic geometry, differential calculus, integral calculus, differential equations, and in addition any three of the following: advanced calculus, theory of equations, determinants and matrices, introduction to modern algebra, statistics, mechanics, vector analysis, probability theory, numerical methods.

2.1.2 System Research Section. It is contemplated that the scientists in the system engineering section, technical services branch, and simulation equipment branch during the course of problems will find basic situations in system design, simulation techniques, physics, engineering, or mathematics that bear further investigation. If a staff member demonstrates that he has a reasonable line of investigation for one of these basic situations, he may be assigned to the research section for a period adequate to perform his studies. Further, if the administrator feels some basic situation is sufficiently important to warrant investigation he may assign one or more scientists to the basic research section to do this work.

No permanent technical staff is assigned to this section; staff scientists are assigned on a temporary basis as their interests and the technical needs of the facility warrant. Probably no more than 15°/o of the total scientific staff are assigned to this group at any one time. This group should also establish some arrangements for bringing in outside specialists to spend a summer, a school term, or a year to work on various basic problems, in much the same manner as the Institute for Numerical Analysis of the Bureau of Standards has done. This procedure tends to introduce the resident group to new ideas and aids in establishing effective lines of communication with other groups in the same or related fields. It has the added advantage of being a potential source of permanent employees.

2.2 Simulation Equipment Branch

The simulation equipment branch is responsible for the operation and maintenance of all laboratory equipment as well as for the design and fabrication of special equipment needed for specific problems. When a problem is being prepared for simulation, either mathematical or physical, an engineer from the appropriate simulation equipment section works with the team to help insure that the proposed simulation setup is suitable for the equipment to be used. It is felt that this procedure is of material assistance in cutting down the initial trouble shooting time, tends to reduce the amount of equipment used for a problem, and gives additional assurance of obtaining meaningful results from the simulation.

2.2.1 Electronic Equipment Section. The electronic section is responsible for the operation and maintenance of the analog computing equipment available to the facility. In a complete facility this equipment should consist of (1) a dynamic systems synthesizer with a size equivalent to approximately 900 operational amplifiers and (2) about 15 racks of standard commercial analog equipment. In addition, the equipment may include fast time computers, passive network analyzers, and other specialized apparatus as required. This section has the responsibility for maintaining the electrical components of the physical simulation equipment as well as the maintenance and operation of any electrical components of actual weapons hardware under test. Design and fabrication of special analog apparatus needed for problems and necessary or desirable modifications of the existing simulation equipment are also part of this section's task.

Our survey and studies* indicate that this section should have six electrical engineers (or physicists with electronics training) and six electronic technicians. Two of the electrical engineers should have professional training at least equivalent to the master's degree and a minimum of four years of experience, at least two of which should have been connected with design, operation, and/or maintenance of analog computing equipment. The other four should have professional training at least equivalent to the bachelor's degree and a minimum of two years' experience. All of these engineers should have sufficient mathematical training to preclude extreme fright at the mention of systems of algebraic, differential, and integral equations or inequalities. The electronic technicians should be familiar with the maintenance, repair, and fabrication of relatively complex electrical apparatus, and at least one of the technicians should be familiar with the maintenance and repair of radar and infrared equipment.

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*See Appendix 3.

2.2.2 Mechanical Equipment Section. The mechanical equipment section is responsible for the maintenance and operation of mechanical equipment such as electromechanically driven flight tables or mechanical load simulators and the mechanical components of such items as hydraulically driven flight tables. In addition, the section is responsible for operating and maintaining any mechanical components of weapon systems included in the simulation loop.

A staff adequate to handle the mechanical equipment should consist of two mechanical engineers and three mechanics. Both of the engineers should have professional training equivalent to the master's degree in mechanical engineering and a minimum of two years' professional experience. One of the engineers should have had at least one year's experience working with or designing electromechanical systems incorporating such items as electromagnetic or induction clutches. The other engineer should have at least one year's experience working with or designing gear drive systems.

At least one of the mechanics should have had some experience in the maintenance and repair of electromechanical systems such as magnetic or induction clutches. The others should be familiar with the maintenance and repair of such things as gear drive systems.

2.2.3 Hydraulic Equipment Section. The hydraulic equipment section is responsible for the operation and maintenance of the hydraulically driven equipment such as load simulators and flight tables. In addition, it is responsible for operating and maintaining any hydraulic components of weapon systems included in the simulation loop.

A staff of two engineers and three technicians or mechanics should be adequate. Both engineers should have professional training equivalent to the master's degree in engineering and a minimum of two years' professional experience. Both the engineers should have at least one year's experience working with hydraulic valves or hydraulic motors. At least one should have some experience in instrumentation for measuring temperature, pressure flow, fluid cleanliness, etc. in valves and motors.

The technicians should have some experience in operation and maintenance of hydraulic valves and motors. One of the technicians should also meet the requirements of a journeyman plumber.

2.3 Technical Services Branch

During various stages in a system study the system engineers need data on component response, aerodynamics, thermodynamics, structures, and military requirements. In addition, they may need aid in programming a digital check problem or in planning a simulation program to obtain data useful for a statistical evaluation of the weapon's performance. To furnish this information and advice, the technical services branch has two sections: the component dynamic response section and the technical consulting section.

2.3.1 Component Dynamic Response Section. This section is responsible for furnishing component data, primarily on dynamic response, to the system engineering teams. This group obtains this data in whatever manner is most efficient in the individual instance: contacting the manufacturers, making a library search, or devising tests to obtain the information. For convenience the section subdivides into three subsections: electrical, mechanical, and hydraulic, corresponding to the type of components considered.

The electrical group should consist of three electrical engineers. At least one of these should have professional training equivalent to a master's degree in electrical engineering and a minimum of three years' professional experience, one year of which should have been spent testing equipment. The other two engineers should have professional training equivalent to the bachelor's degree in electrical engineering and a minimum of two years' professional experience, six months of which should have been spent on testing equipment.

The mechanical components subsection requires two mechanical or aeronautical engineers. One of these should have professional training equivalent to the master's degree in machanical or aeronautical engineering and a minimum of three years' professional experience, at least one year of which should have been in the field of testing components for aerial weapon systems. The other should have professional training equivalent to the bachelor's degree in mechanical or aeronautical engineering and a minimum of two years' professional experience. At least one of these engineers should have had some experience with designing and/or testing gyroscopic devices.

The hydraulic components subsection needs two hydraulic or mechanical engineers. One of these should have professional training equivalent to the master's degree in hydraulic or mechanical engineering and a minimum of three years' professional experience, one year of which should have been spent on designing or testing hydraulic valves and rams of the types used in aerial weapons systems. The other should have professional training equivalent to the bachelor's degree in hydraulic or mechanical engineering and a minimum of two years' professional experience, at least six months of which should have been connected with designing or testing hydraulic systems of a type similar to those used in aerial weapons.

2.3.2 Technical Consulting Section. The technical section has the responsibility of furnishing (1) information on systems environment such as aerodynamics, thermodynamics, and air vehicle structure, (2) information on military requirements and tactics, (3) technical aid on statistical problems and coding problems for digital checks. Our studies indicate that this group should have at least the following personnel:

I. One aerodynamicist with professional training equivalent to a Ph.D. in aerodynamics and a minimum of five years' experience and research in both the practical and theoretical aspects of the effects of aerodynamic flow on air vehicle control.

II. One aircraft structures engineer with professional training equivalent to a Ph.D. in aeronautical engineering and a minimum of three years' experience and research in both practical and theoretical aspects of air vehicle structure requirements and structural effects on air vehicle control.

III. Two specialists in military requirements and tactics. It is difficult to give specifications on such men other than to say that they should have a sound scientific background and three to five years' experience in military phases of operations research.

IV. One thermodynamicist with professional training equivalent to a Ph.D. in physics with a minimum of four years' experience and research in theoretical and practical aspects of the thermodynamics of high speed flight.

V. One statistician with professional training equivalent to the Ph.D. in statistics and a minimum of four years' experience and research. His experience should include some practical experience in the design and analysis of experiments, estimation, and quality control.

VI. Two mathematicians with training equivalent to the Ph.D. in mathematics and three to five years' experience in formulating and coding problems for digital computation.

VII. Two computers or scientific aids. Their training should be equal and similar to that required for the computers in the system engineering section.

3. ORGANIZATION OF THE SYSTEM ENGINEERING TEAM

Examination of weapon system problems shows that techniques and knowledge from many scientific disciplines are required for adequate treatment of these problems. Since it is an exceedingly rare person who combines sufficient competence in all the required disciplines, a team of specialists appears necessary. This need is not unique to weapon system studies; it occurs today in many fields. In the "Introduction" to Cybernetics, Norbert Weiner writes, "There are fields of scientific work... which have been explored from the different sides of pure mathematics, statistics, electrical engineering, and neurophysiology; in which every single notion receives a separate name from each group; and in which important work has been triplicated or quadruplicated; while still other important work is delayed by the unavailability in one field of results that may have already become classical in the next field.

"It is these boundary regions of science which offer the richest opportunities to the qualified investigator. They are at the same time the most refractory to the accepted techniques of mass attack and the division of labor. If the difficulty of a physiological problem is mathematical in essence, ten physiologists ignorant of mathematics will get precisely as far as one physiologist ignorant of mathematics, and no further. If a physiologist, who knows no mathematics, works together with a mathematician who knows no physiology, the one will be unable to state his problem in terms that the other can manipulate, and the second will be unable to put the answers in any form that the first can understand. . . . a proper exploration of these blank spaces on the map of science could only be made by a team of scientists, each a specialist in his own field, but each possessing a thoroughly sound and trained acquaintance with the fields of his neighbors; all in the habit of working together, of knowing one another's intellectual customs, and of recognizing the significance of a colleague's new suggestion before it has taken on a full formal expression." *

This ability to work cooperatively in a scientific or engineering research team must be considered an additional qualification for most of the technical personnel in the laboratory. In order to understand how the team procedure

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*Norbert Weiner, Cybernetics (N.Y., 1948), pp. 8-9.

might work, we will describe how the various people in a laboratory could be called upon to participate in various phases of a problem. For the sake of concreteness we will suppose that the problem is to design and evaluate at least through the breadboard stage the guidance and control system of an air-to-air guided missile with, say, active radar guidance.

To start such a job a nuclear team of one radar specialist, one control system engineer, and one mathematician appears reasonable. They consult with the military applications specialists to discover under what tactical conditions and against what types of targets this weapon will operate. Knowledge of these military conditions helps to establish the requirements the guidance and control system must meet. Next, the team attempts a representation of the system, consulting aerodynamicists, structural engineers, and perhaps a thermodynamicist to see what environmental effects can safely be neglected in a preliminary study to determine the form of, say, the guidance equation. When the form of representation is settled and the aims of this phase of the study are established, an analog computer engineer joins the team to help plan the instrumentation of the representation for simulation. The computer engineer remains with the team until the problem is off the computer. This planning for simulation on a computer is fairly critical, since it must be determined that the analog representation is in fact a representation of the physical system. In this phase it is also advisable to consult the statistician on a plan for the analog runs in order to minimize the number of runs and, if desired, to get data suitable for statistical analysis.

With all of this settled, the problem is set up and run on this computer. It is probably wise to have one or more scientific aids to help with the evaluation of the data as it comes from the computer. When the computer phase is over, the results are analyzed and engineering judgments made concerning the meaning of the guidance equations. When this last is determined, a less simplified version of the problem is prepared with the aid of an aerodynamicist and a structural engineer to see if the effects neglected in the first computation will change things significantly. At the same time the instrumentation for the type of guidance obtained and its connection with the control loop are studied; since a considerable amount of the instrumentation is electronic, an electronic engineer may join the team.

With the aid of a computer engineer an analog of the more complicated problem is prepared. The statistician is consulted about plans for the runs, the problem is coded for digital check, and it is run and analyzed as before.

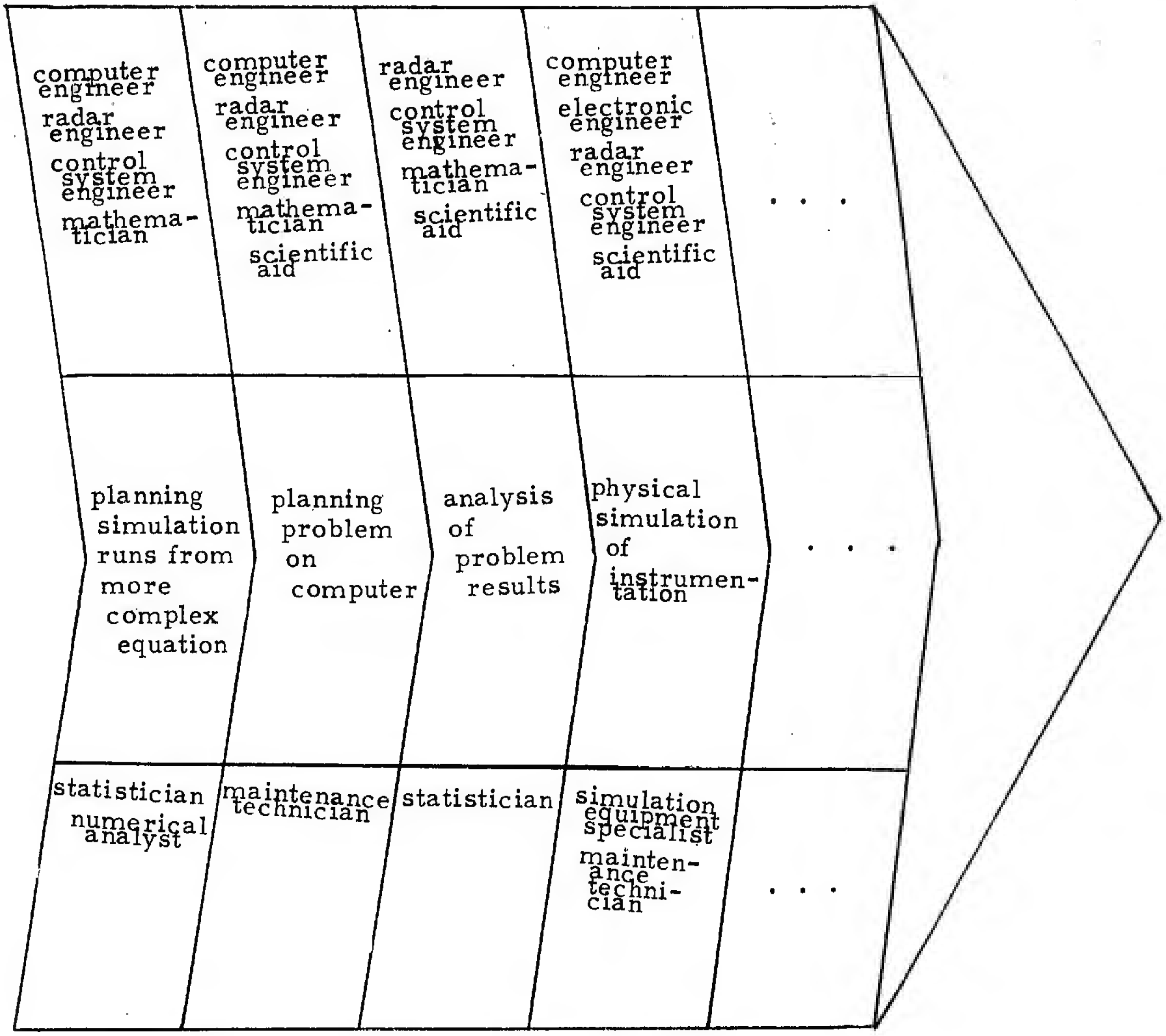
Those studying the instrumentation of the combined guidance and control loop need some hardware dynamics, and these they obtain from the component dynamic response section. They also want to make analog tests, perhaps with actual components in the loop. If there is a large amount of special electronics, an electronic engineer again joins the team. For analog simulation the procedure is the same as before. If actual hardware is incorporated in the loop, some one from the proper simulation equipment section assists in preparing the simulation. When a subsystem has been synthesized in this manner, more elaborate studies will be made of the total system in essentially the same way the subsystem was studied.

Note that all of the branches are involved in this work at various times during the study. A flow chart of the problem and the staff involved in its stages is given in Figure 2.

The procedure given here is of course simplified and because of the nature of our example somewhat specialized, but it illustrates in a concrete manner how this concept of the system engineering team applies. The team's composition is flexible; the number of people connected with the problem at any time depends only upon what needs to be done at that time.

SYSTEM ENGINEERING TEAM	radar engineer control systems engineer mathematician	computer engineer radar engineer control system engineer mathematician scientific aid	radar engineer electrical system engineer mathematician scientific aid	radar engineer electrical system engineer mathematician scientific aid	electronics engineer radar engineer control system engineer mathematician
STUDY PROGRAM PHASE	formulation of simplified guidance equations	planning simulation runs	running problem on computer	analysis of problem results	formulation of less simplified version and instrumentation of guidance loop
CONSULTANT AND SERVICE PERSONNEL	military applications specialist structural engineer thermodynamicist	statistician numerical analyst	maintenance technician	statistician	aerodynamicist structural engineer thermodynamicist component dynamic response engineer

Figure 2:


computer engineer
radar engineer
control system engineer
mathematician
computer engineer
radar engineer
control system engineer
mathematician
scientific aid
radar engineer
control system engineer
mathematician
scientific aid
computer engineer
electronic engineer
radar engineer
control system engineer
scientific aid
. . .
planning simulation runs from more complex equation
planning problem on computer
analysis of problem results
physical simulation of instrumentation
. . .
statistician
numerical analyst
maintenance technician
statistician
simulation equipment specialist
maintenance technician
. . .

FLOW CHART

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Appendix 1*

THE STAFF AND ORGANIZATION OF A LARGE ANALOG-COMPUTING CENTER

by

H. Mori and C. W. Steeg, Jr.

1. Introduction

The primary goal of this appendix is to report the way the Dynamic Analysis and Control Laboratory (D.A.C.L.) has organized and staffed the M.I.T. Flight Simulator facility and to discuss the procedure which the D.A.C.L. recommends for planning an organization capable of handling a large variety of problems on an analog computer. Examination of these topics emphasizes the particular difficulties that are likely to be encountered in the operation of a large computing center. A method for handling problems on a computer is outlined in sufficient detail to furnish the background necessary for the consideration of the organization required. A survey is made of the activities essential for efficient operation of a facility. The advantages of supplying adequate maintenance and development of equipment, consulting advice and assistance, and basic research facilities also are examined.

The organization of a computing center varies with the kind of services performed. The basic objective of a large analog-computing facility should be to provide the services and equipment adequate to solve a variety of extensive, complicated problems. The facility should also be capable of handling smaller problems as time and equipment are available.

Because of the unique position that a large computing center occupies, it can provide additional services beyond those of a small facility, possibly including the following:

(1) The computing center should maintain a constant effort to further the science of machine computation through the development of new methods for performing computations, improvement of analog-computing equipment,

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*From Dynamic Analysis and Control Laboratory, M.I.T., Research Memorandum 6463-3, 30 Nov. 1953.

and application of analog computers to new fields of study.

(2) The computing center should issue sufficiently detailed reports on each problem to allow independent interpretation of the analog data.

(3) The computing center can be a central agency for transmission of information among groups interested in similar or allied problems.

(4) The computing center can contribute more information than mere presentation of results if sufficient data are given to enable independent, theoretical analysis by extrapolating available data to predict probable performance under different conditions, and by using experience gained from previous programs.

The two general types of problems customarily undertaken in an analog-computing facility are mathematical studies and equipment analyses, sometimes referred to as mathematical and physical simulations, respectively. An equipment analysis is characterized by the use of physical apparatus in a portion of the computation and becomes necessary when some of the characteristics of a physical system cannot be expressed accurately for simulation. The correlation between equipment and simulated studies determines the validity of simplifications made in investigations involving only simulation. Problems that do not involve actual hardware may be designated as mathematical simulation.

A wide variety of problems could be handled at an installation such as the proposed dynamic systems laboratory. Possible fields of study might include the automatic control of dynamic systems such as aircraft or engines, the trajectory performance of missile and aircraft control systems, and the solution of diverse mathematical problems involving either ordinary or partial differential equations.

The number of persons needed for a program in any of these fields depends on the size of the problem. Some problems that require short preparation times and a small portion of the available equipment may be handled by a few people, and in order to obtain maximum usefulness from the computer several such problems can be run simultaneously. If a problem is so computer that a large portion of the computrr is occupied, not only is a long period of preparation before the actual machine operation required but also the most advantageous use of the computer is ensured by multiple-shift operation.

2. The D.A.C.L. Organization

The research program of the D.A.C.L. is shared between the Simulator Group and the Research and Development Group. The chart of Figure 3 shows the organization of the Simulator Group, which is divided into three divisions: Simulator Operation, Analysis and Evaluation, and Simulator Development. The present organization represents the evolution of the activities and interests of the staff members and is a result of the laboratory history more than of any preconceived master plan. At the inception of the laboratory, activities were largely confined to the development of computing equipment and particularly a three-axis flight table. During this period, the effort in analysis was directed principally to maintaining an awareness of progress in the fields where computing equipment would be most useful, but actually resulted not only in the pioneering of these areas but also in the decision to expand development to include the generalized computer. As equipment became available for the solution of specific problems, the technique of team research, which had proved so effective for the development of the computer, was adopted for the operation of the equipment and for the evaluation of the data.

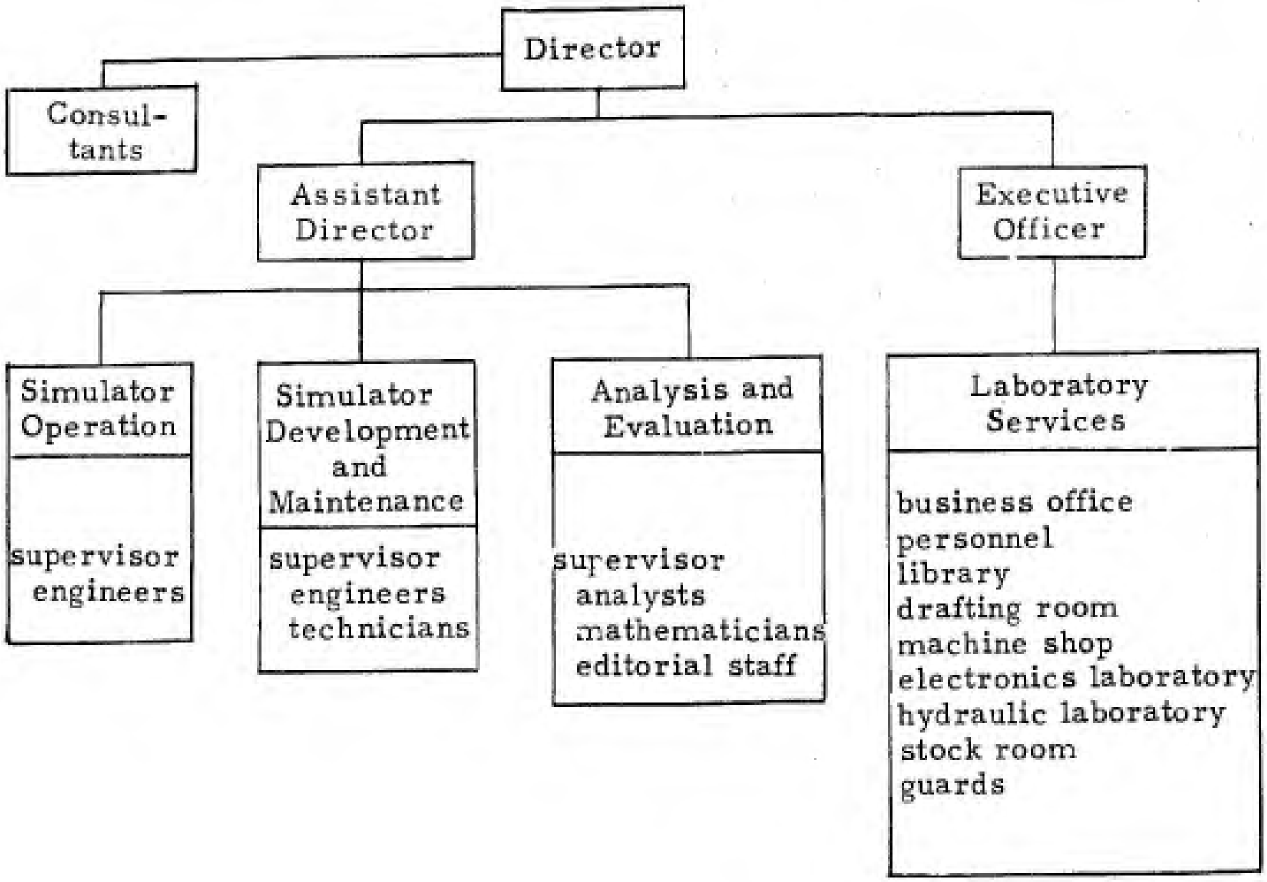

Figure 3: D.A.C.L. ORGANIZATION FOR THE SIMULATOR GROUP

A team of engineers and operating technicians has charge of each problem. Such a procedure allows scientists with various specialties, such as electrical engineering, mechanical engineering, mathematics, and physics, to work together on the same project. The individual having a particular interest closest to the major problem is named as team director, or problem engineer, acting more as a coordinator than as a supervisor. The approximate equality of team members assures each an equal opportunity for expression on all phases of the problem. The mathematician obtains a clearer concept of the electrical engineer's problem, for example, and has an opportunity to offer suggestions as well as to contribute theoretical concepts. Thus, research scientists can be specialists without being isolated from other branches of science. The success or failure of a computing program depends largely upon the initiative of the team. The operating teams are members of the Simulator Operation Division.

The contribution made by the other divisions can be explained best by reviewing the steps involved in the solution of a typical complex problem. Usually the initial contact with a problem is made by another team comprised of members from the supervisory level and including the Assistant Director for the Simulator. This team determines not only the capability and availability of the equipment and the staff to handle the problem but also the suitability of the problem to add to the background of basic research. Furthermore, at M.I.T. the Division of Industrial Cooperation imposes certain specifications that must be met before acceptance of a research project. Once a problem is selected, the nucleus of the operating team is presented all information for formulation and preliminary analysis of the equations.

In the operation of any automatic device, the errors caused by human intervention must be reduced. Elaborate methods for checking the equipment have been devised in order to minimize human errors. In addition to various static and dynamic checks of computer setup, a numerical solution using typical parameter values is prepared. The latter work represents part of the activities of the Analysis and Evaluation Division. Specifications for any special computing devices are prepared by the operating team and submitted to the Simulator Development Division for design and construction. During the period of machine operation, a member of the Analysis and Evaluation Division, which is continually apprised of simulator results, attempts to correlate computer data with analytical studies in order that some extrapolation can be made concerning the way a system may perform under conditions other than those studied.

After the machine period is finished, the simulator data are reduced to appropriate curves, graphs, charts, and tables. The final phase in the program is the preparation of a comprehensive report by the problem engineer with the cooperation of all personnel associated with the problem. Because this report represents a permanent record of the program, all information necessary for a detailed understanding of the program and its results is presented. Editing and publication of the report is the responsibility of the Analysis and Evaluation Division.

The Simulator Development Division, in addition to its previously mentioned duties, performs research on components for possible application in the computer and supervises maintenance of the Simulator.

A Division of Laboratory Services under the jurisdiction of the laboratory Executive Officer maintains a drafting room, a machine shop, and an electronics shop and performs the personnel, security, procurement, and business-office functions for both the Simulator Group and the Research and Development Group.

The Simulator Group has a staff of approximately 70 with almost half this number of professional grade. Because the D.A.C.L. is a part of M.I.T., many of the laboratory personnel are graduate students who spend a portion of their time attending classes while working for advanced degrees under the M.I.T. Research Assistantship Program. The remainder of the professional staff members have advanced degrees, or extensive post-graduate training, and at least three years of training in electronics, systems analysis, mathematics, or related scientific fields.

3. Suggested Personnel and Organization

A possible organization for a dynamic systems facility has been derived on the basis of the experience gained at the D.A.C.L. in the operation of a computing installation. The suggestions for the organization of personnel are given in this section and summarized in Figure 4. Although the organization of D.A.C.L. is an outgrowth of the type of problem and equipment, as well as of the interests of the staff, many features of the Simulator Group appear to be necessary and valuable for the full utilization of a large-scale analog computer. If the facility is partially separate from any other installation, then essentially the same organization as used at the D.A.C.L. may be applicable. If other units at Wright Air Development Center share the same administration, a significant reduction in such services as security, library, and shop facilities may be possible.

3.1. Personnel. The minimum requirement for the professional staff members of a computing facility is a bachelor's degree in engineering or in an allied science. The specific academic background of the personnel frequently is less significant than their training in the scientific discipline and method. The variety of problems to be undertaken dictates a further requirement for breadth of experience, obtained either in industry or in work for an advanced degree.

Because a computer usually is an electronic contrivance, electrical engineering education and experience are necessary in most instances for staff members working directly with the computer. A large majority of the problems solved on present-day analog computers utilizes the principles of servomechanisms and the general theory of feedback control; consequently, a knowledge of these fields greatly facilitates the understanding and analysis of problems. Traditionally these subjects have been regarded as belonging to the province of the electrical engineer; however, the same techniques have been adopted and used successfully in other fields of engineering. Because an analog computer usually is involved with the simulation of actual mechanical equipment, a background of mechanical engineering is helpful in order to explain and interpret the analogy. Furthermore, because the common bases for all computing problems are mathematical equations, workers with extensive experience in physics and mathematics are necessary. For the study of specific problems, such other specialists as aerodynamicists, radar engineers, or thermodynamicists may be required.

Inasmuch as some of the staff will not be familiar with analog computation or computers, a nucleus of experienced members must be obtained to initiate a program for the training of subsequent additions to the staff. If a training program is not feasible, careful consideration should be given to such incentives for attracting high-caliber engineers as providing adequate opportunity for the pursuit of advanced degrees or otherwise encouraging professional recognition.

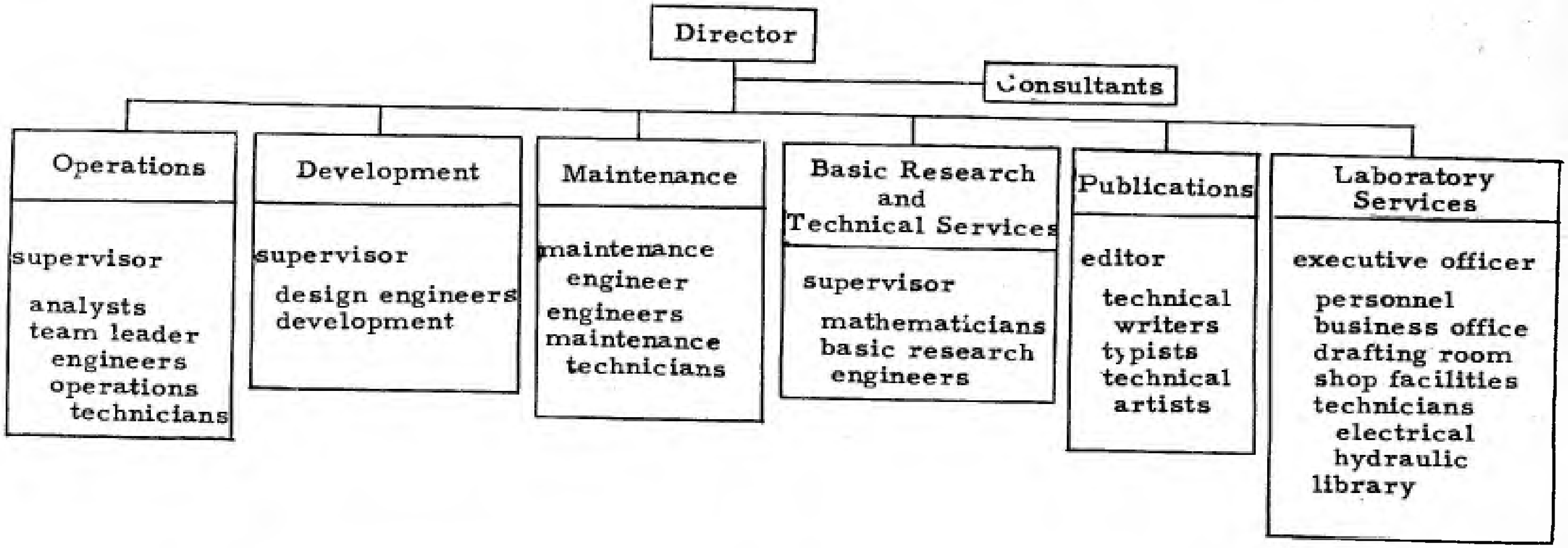

Figure 4: ORGANIZATION CHART FOR AN ANALOG-COMPUTING FACILITY

3.2. Operations. The size of the operations team and the qualifications of its members are dependent on the problems being solved at the computing center. The number of teams required is determined by such factors as the complexity of the problem and the capacity of the equipment for multiple-problem operation. The complexity of a problem dictates the amount of preliminary analysis, the duration of machine operation, and the effort devoted to data reduction and preparation of the report. The maximum benefits from team organization are derived when all of the above-mentioned phases are handled by the same group. Consequently, the composition of a team should include, in addition to the problem engineer, the following types of personnel: an analyst to complement computer results with theoretical analyses, two or more operations engineers to prescribe computer setup and operating procedures, and two operations technicians to inspect and check the computing equipment. For multiple-shift operation of the computer, the team may require an additional engineer and technician for each extra shift.

3.3. Development and Maintenance. A computer-development program should be sustained to prevent obsolescence of the equipment. New components for computer application and new operational equipment must be developed continually to preserve the effective and efficient operation of the computer. Furthermore, the development group must attend to equipment problems brought to its attention by the maintenance and operations groups and requests for special equipment for particular programs.

To insure the continuous and efficient operation of the computer, a practical maintenance program should be initiated. The determination of a suitable program, whether preventive, breakdown, or a combination of these, should be a matter of fundamental policy at the computer center. Even such an apparently extraneous decision as the selection of a maintenance program can determine the entire form of the organization. The D.A.C.L. has devoted extensive study* to the determination of the length of the maintenance cycle for a preventive-type program.

Personnel of the development group will be engaged in development of equipment, and the maintenance personnel will be concerned primarily with setting up maintenance procedures and improving existing equipment for increased reliability and efficiency. Although the background requirements for both groups depend upon the type of computer at the facility, an understanding

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*Bibliography, item 5.

of electrical and mechanical engineering would be a definite asset.

3.4. Basic Research and Technical Services. The tasks of preparing numerical solutions, performing mathematical computations, and carrying on theoretical investigations of problems being studied on the computer should be conducted by a group for basic research and technical services. As the interests, initiative, and ingenuity of the staff may dictate, basic research in machine computation and in problems arising out of the work at the center also can be performed.

In addition to several people to operate hand calculators, this group should supply mathematicians with backgrounds in numerical analysis. Because many calculations demand the use of a digital computer, at least one person with a background in digital computation would be helpful. In fact, if very many numerical computations are needed, the group should operate a small digital computer.

3.5. Publications. The publications staff should include such personnel as an editor, technical writers, and technical artists, as well as several typists. The members of the operations team assigned to the program should be able to write comprehensive reports on the program. The use of technical writers depends upon the qualifications of the team members. The time required to prepare and edit a report might be used more advantageously by the operations team in analyzing and studying additional problems. However, maximum benefit from a computer study can be gained only by requiring those team members most directly associated with the problem to prepare the report. Reports are edited, illustrated, typed, and published by the publications group.

3.6. Consultants. Because in a large computing facility many highly technical problems arise that are beyond the knowledge of the staff, a group of specialists should be available for oral consultation without interrupting the progress of the problem.

3.7. Laboratory Services. The nontechnical functions necessary for the autonomous operation of the proposed center are the responsibility of the laboratory services group. Such services should include supply shop, library, stock-room, and drafting facilities, as well as supervision of the nonprofessional staff of technicians, typists, and secretaries.

Appendix 2

ON A PROBLEM OF QUEUES WITH MANY SERVERS AND DIFFERENT TYPES OF CUSTOMERS*

by

Mary C. Weiss

The problem we wish to solve is the following: Hypothesis: There exist k types of customers who have distinct exponential arrival distributions with parameters β_i and distinct exponential service time distributions with parameters α_i. There are s servers and the queue-discipline is first come, first served. We wish the following information: at time t what is the probability that there are n people in the queue? What is the expected waiting time of an arbitrary customer? What is the limiting distribution for the queue size?

1. Formal Solution

First we consider an approach which will yield us a formal solution.

Let U be the probability that $n+\omega$ $(\omega \geq 1)$ people arrive in time t, the ith person to arrive being of type z_i. This probability is found by letting m_j denote the number of j's among $(z_1,\ldots,z_{n+\omega})$ and considering

$\exp\left(-\sum_{i=1}^{k}\beta_i t\right)\dfrac{t^{n+\omega}\beta_1^{m_1}\ldots\beta_k^{m_k}}{m_1!\ldots m_k!}$, the probability that $n+\omega$ people, m_j of type j, arrive. If this probability is multiplied by $\dfrac{m_1!\ldots m_k!}{(n+\omega)!}$, the probability that their types are in the order $z_1,\ldots,z_{n+\omega}$, then the result

$\exp\left(-\sum_{i=1}^{k}\beta_i t\right)\dfrac{t^{n+\omega}}{(n+\omega)!}\cdot\beta_{z_1}\ldots\beta_{z_{n+\omega}}$ is the probability U in question.

If $0 \leqq x_1 \leqq \ldots \leqq x_{n+\omega} \leqq t$, the probability V that the ith person arrives during the interval $(x_i, x_i + dx_i)$ is

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*From Advisory Board on Simulation Technical Note 42, 9 April 1954.

$V = U \cdot \left[\frac{(n+\omega)!}{t^{n+\omega}} \; dx_1 \ldots dx_{n+\omega} \right] \quad = \exp(-\sum_{i=1}^{k} \beta_i t) \, \beta_{z_1} \ldots \beta_{z_{n+\omega}} \; dx_1 \ldots dx_{n+\omega}$, since the measure of the set $0 \leqq x_1 \leqq \ldots \leqq x_{n+\omega} \leqq t$ is $\frac{t^{n+\omega}}{(n+\omega)!}$ and since all orderings of the x's are equally likely.

Of $n+\omega$ people arriving during the time interval (0,t) such that the ith person is of type z_i and arrives between x_i and $x_i + dx_i$, the probability that exactly ω are waiting at time t is

$$\alpha_1^{m_1} \ldots \alpha_k^{m_k} \underbrace{\int_0^{\infty} \int_0^{\infty} \ldots \int_0^{\infty}}_{n+\omega} f(t,s,\omega,x_1,\ldots,x_{n+\omega},y_1,\ldots,y_{n+\omega}) \exp(-\sum_{i=1}^{n+\omega} \alpha_{z_i} y_i) \, dy_1 \ldots dy_{n+\omega}$$

where $f(t,s,\omega,x_1,\ldots,x_{n+\omega},y_1,\ldots,y_{n+\omega}) = 1$ if of $n+\omega$ people arriving at times $x_1,\ldots,x_{n+\omega}$ and taking times $y_1,\ldots,y_{n+\omega}$ respectively to be served, ω are waiting at time t. Now the probability that exactly $n+\omega$ people arrive before time t, the ith person to arrive being of type z_i, and that there are exactly ω people waiting at time t is

$$\exp(-\sum_{i=1}^{k} \beta_i t) \, \beta_{z_1} \ldots \beta_{z_{n+\omega}} \; \alpha_{z_1} \ldots \alpha_{z_{n+\omega}} \iint\limits_{x_1 \leqq x_2 \leqq \ldots \leqq x_{n+\omega} \leqq t} \!\!\ldots \int \; dx_1 \, dx_2 \ldots dx_{n+\omega}$$

$$\left[\int_0^{\infty} \int_0^{\infty} \ldots \int_0^{\infty} f(t,s,\omega,x_1,\ldots,x_{n+\omega},y_1,\ldots y_{n+\omega}) \exp(-\sum_{i=1}^{n+\omega} \alpha_{z_i} y_i) \, d_{y_1} \ldots d_{y_{n+\omega}} \right]$$

Summing over all $(z_1,\ldots,z_{n+\omega})$ and all n we obtain the desired probability.

$$\exp(-\sum_{i=1}^{k} \beta_i t) \sum_{n=s} \sum_{(z_1,\ldots,z_{n+\omega})} \beta_{z_1} \ldots \beta_{z_{n+\omega}} \; \alpha_{z_1} \ldots \alpha_{z_{n+\omega}} \iint\limits_{x_1 \leqq x_2 \leqq \ldots \leqq x_{n+\omega} \leqq t} \!\!\ldots \int dx_1 \ldots dx_{n+\omega}$$

$$\left[\int_0^{\infty} \int_0^{\infty} \ldots \int_0^{\infty} f(t,s,\omega,x_1,\ldots,x_{n+\omega} \; y_1,\ldots,y_{n+\omega}) \exp(-\sum_{i=1}^{n+\omega} \alpha_{z_i} y_i) \, dy_1 \ldots dy_{n+\omega} \right]$$

Now it remains to determine the function $f(t,s,\omega,x_1,\ldots,x_{n+\omega},y_1,\ldots,y_{n+\omega})$. Let t_i be the waiting time of the ith person to arrive. Then $x_i+t_i+y_i$ is the time at which the service of the ith person is completed.

$t_i = 0$ for $i \leqq s$

$$t_{s+1} = (\min^1(x_1+y_1,x_2+y_2,\ldots,x_s+y_s) - x_{s+1})\, S(\min^1(x_1+y_1,x_2+y_2,\ldots,x_s+y_s) - x_{s+1})$$

$$t_{s+k} = (\min^k(x_1+y_1,x_2+y_2,\ldots,x_s+y_s,x_{s+1}+t_{s+1}+y_{s+1},\ldots,x_{s+k-1}+t_{s+k-1}+y_{s+k-1})-x_{s+k})$$
$$\cdot\, S(\min^k(x_1+y_1,x_2+y_2,\ldots,x_s+y_s,x_{s+1}+t_{s+1}+y_{s+1},\ldots,x_{s+k-1}+t_{s+k-1}+y_{s+k-1})-x_{s+k})$$

where $\min^k(a_1,\ldots,a_n)$ is the k element when the a_i's are ordered so that

$a_i^j \leqq a_{i+1}$ and $S(a) = 1$ if $a \geqq 0$; $S(a) = 0$ otherwise

$$\min^k(a_1,\ldots,a_n) = \min_{i_1+i_2+\ldots i_k}\left\{a_{i_1}+a_{i_2}+\ldots a_{i_k}\right\} - (\min^{k-1}(a_1,\ldots a_n) + \min^{k-2}(a_1,\ldots,a_n) + \ldots + \min(a_1+a_n))$$

$$\min^1(a_1,a_2,\ldots,a_n) = \min(a_n, \min(a_1,a_2,\ldots,a_{n-1}));\ \min(a_1,a_2) = \frac{a_1+a_2-|a_1-a_2|}{2}$$

$$S\min^k(a_1,\ldots,a_n) = \sum_{i=o}^{k-1}\ \sum_{i'\ldots i'} S(-a_{1'})\ldots S(a_{i'})\, S(a_{(i+1)'})\ldots S(a_n)$$

$$S\min^k(a_1,\ldots,a_n) = \sum_{i=o}^{k-1}\ \sum_{1'\ldots i'} S(-a_{1'})\ldots S(a_{i'}) S(a_{(i+1)'})\ldots S(a_{n'})$$

Let $\lambda_1 = S\min^{n-s}(x_1+y_1,\ldots,x_s+y_s,x_{s+1}+y_{s+1}+t_{s+1},\ldots,x_{n+\omega}+t_{n+\omega}+y_{n+\omega})$

$\lambda_2 = S\min^{n+1-s}(x_1+y_1,\ldots,x_s+y_s,\ldots\qquad,x_{n+\omega}+t_{n+\omega}+y_{n+\omega})$

Then $f(t,s,\omega,x_1,\ldots,x_{n+\omega},y_1,\ldots,y_{n+\omega}) = \lambda_1 + \lambda_2 - 2\lambda_1\lambda_2$

2. Limit of the Probability

2.1 k Types of Customers. Instead of considering the probability that at time t there are n people in line, we consider the limit of this probability. When these probabilities have limits that will form a distribution is a question that will be considered in part 2.2. For the present we assume that $\lim_{t \to \infty} F(t;\, n,r)$ exists and $\sum_{r=1}^{k} \sum_{n=o}^{\infty} F(t;\, n,r) = 1$. Here we initially consider the single server case and let $F(t;\, n,r)$ denote the probability that at time t there are n people in line and a person of type r is being served. Then the following differential equations apply.

$$\text{I} \quad \frac{dF(t;\, n,r)}{dt} = -\,(\beta+\alpha_r)F(t;n,r) + \beta F(t;n-1,r) + \gamma_r \sum_s \alpha_s F(t;n+1,s) \qquad \text{for } n \geqq 1$$

$$\text{II} \quad \frac{dF(t;\, 0,r)}{dt} = -\,(\beta+\alpha_r)F(t;0,r) + \beta_r F(t;0,0) + \gamma_r \sum_s \alpha_s F(t;1,s)$$

$$\text{III} \quad \frac{dF(t;\, 0,0)}{dt} = -\,\beta F(t;0,0) + \sum_s \alpha_s F(t,0,s)$$

Now as $t \to \infty$, $\frac{dF(t;n,r)}{dt} \longrightarrow 0$. Denoting $\lim_{t \to \infty} F(t;n,r)$ by $F(n,r)$ we have the following limiting equations from I, II, III

$$\text{I}' \quad -(\beta+\alpha_r)F(n,r) + \beta F(n-1,r) + \gamma_r \sum_s \alpha_s F(n+1,s) = 0$$

$$\text{II}' \quad -(\beta+\alpha_r)F(0,r) + \beta_r F(0,0) + \gamma_r \sum_s \alpha_s F(1,s) = 0$$

$$\text{III}' \quad -\beta F(0,0) + \sum_s \alpha_s F(0,s) = 0$$

Summing I' and II' over r and using the notation $\sum_r F(n,r) = F(n)$ $\sum_r \alpha_r F(n,r) = H_n$, $F(0,0) = F(-1)$ we obtain

$$\text{I}'' \quad \beta(F(n) - F(n-1)\,) = H(n+1) - H(n)$$

$$\text{II}'' \quad \beta(F(0) - F(-1)\,) = H(1) - H(0)$$

$$\text{III}'' \quad \beta F(-1) = H(0)$$

Thus we have for all n

$$\text{IV} \quad \beta F(n) = H(n+1).$$

Substituting this equation in equations I' II' III' we obtain

$$\text{I}''' \quad -(\beta+a_r)F(n,r) + \beta F(n-1,r) + \gamma_r \beta \sum_s F(n,s) = 0$$

$$\text{II}''' \quad -(\beta+a_r)F(0,r) + \beta_r F(0,0) + \gamma_r \beta \sum_s F(0,s) = 0$$

Equation III' to $\beta F(0,0) = \beta F(0,0)$

Thus we have a set of equations for $F(n,r)$ which involves only $F(k,s)$ for $k \leq n$. These equations can be solved recursively as follows:

$$\text{Let } A = \begin{bmatrix} -(\beta+a_1)+\beta_1 & \beta_1 & \beta_1 \cdots \beta_1 \\ \beta_2 & -(\beta+a_2)+\beta_2 & \beta_2 \cdots \beta_2 \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \beta_k & \beta_k & \beta_k \cdots -(\beta+a_k)+\beta_k \end{bmatrix}$$

$$\phi = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_k \end{bmatrix} \qquad \psi_n = \begin{bmatrix} F(n,1) \\ F(n,2) \\ \cdot \\ \cdot \\ \cdot \\ F(n,k) \end{bmatrix}$$

Then equation II' can be expressed:

$$A \begin{bmatrix} F(0,1) \\ F(0,2) \\ F(0,k) \end{bmatrix} = \phi F(0,0)$$

Equation I':

$$A \psi_n = -\beta \psi_{n-1}$$

Case I: A is nonsingular

$$\psi_o = A^{-1} \quad \phi F(0,0)$$

$$\psi_n = -\beta A^{-1} \psi_{n-1} = (-\beta)^n \quad (A^{-1})^{n+1} F(0,0)\, \phi$$

Then using $F(0,0) + \sum_{n=o}^{\infty} \sum_{r=1}^{k} F(n,r) = 1$ we can evaluate $F(n,r)$

Case II: A is singular

$$\text{Let } A = \begin{bmatrix} -(\beta+x_1)+\beta & \beta_1 & \beta_1 \cdots \beta_1 \\ \beta_2 & -(\beta+x_2)+\beta_2 & \beta_2 \cdots \beta_2 \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \beta_k & & \beta_k \cdots -(\beta+x_k)+\beta_k \end{bmatrix}$$

Then $d(A')$ is a polynomial p in E_k for which $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a root. Any neighborhood of $(\alpha_1, \ldots, \alpha_k)$ contains a point for which $p \neq 0$ and hence for which the determinant of our matrix is nonzero. Therefore we can replace our matrix A by a nonsingular matrix B.

$$B = \begin{bmatrix} -(\beta+\alpha_1 + \epsilon_r) + \beta_1 & & \beta_1 \cdots \beta_1 \\ \beta_2 & -(\beta+\alpha_2\alpha + \epsilon_2) \cdots\cdots & \beta_2 \\ \cdot & & \\ \cdot & & \\ \beta_k \cdots\cdots \beta_k \cdots\cdots & & -(\beta+\alpha_k + \epsilon_k)+\beta_k \end{bmatrix}$$

where $|\epsilon_i|$ can be chosen arbitrarily small.

The case $s > 1$.

In order to simplify notation we will consider $s = 2$. The problems for larger s are the same. Let $F(n,r,r') = \lim_{t \to \infty} F(t;n,r,r')$ where $F(t;\, n,r,r')$ is the probability that at time t there are n people in the waiting line, that the first server (we have ordered the servers) is serving a person of type r and the second server is serving a person of type r'. We make the assumption that if a person enters and finds no line and niether of the servers busy the probability that he goes to the ith server is $\frac{1}{2}$ $(i = 1, 2)$.

We obtain, as in the single server case, the following equations.

$$\hat{\text{I}} \quad F(n,r,r')(-\alpha_r-\alpha_{r'}-\beta)+F(n-1,r,r')\beta+\gamma_r\sum_s F(n+1,s,r')\alpha_s+\gamma_{r'}\sum_s F(n+1,r,s)\alpha_s = 0$$

$$\hat{\text{II}} \quad F(0,r,r')(-\alpha_r-\alpha_r-\beta)+F(0,r,0)\beta_{r'}+F(0,0,r')\beta_r+\gamma_r\sum_s F(1,s,r')+\gamma_{r'}\sum_s F(1,r,s)\alpha_s=0$$

$$\hat{\text{III}} \quad F(0,0,r)(-\alpha_r-\beta)+\frac{1}{2}F(0,0,0)\beta_r+\sum_s F(0,s,r)\alpha_s=0$$

$$F(0,r,0)(-\alpha_r-\beta)+\frac{1}{2}F(0,0,0)\beta_r+\sum_s F(0,r,s)\alpha_s=0$$

$$\hat{\text{IV}} \quad F(0,0,0)(-\beta)+\sum_s F(0,0,s)\alpha_s+\sum_s F(0,s,0)\alpha_s=0$$

If we observe equations I', II' and III' we see that II' and III' form a system of k+1 nonhomogeneous equations in the k+1 unknowns $F(0,r)$ and $\sum_s \alpha_s F(1,s)$. Therefore equations II' and III' can be solved uniquely for $F(0,r)$ and $\sum_s \alpha_s F(1,s)$ as functions of $F(0,0)$ and the β_i and α_i. Using this solution for $\sum_s \alpha_s F(1,s) = g(F(0,0),\alpha_i,\beta_i)$ together with equations I' for n=1 we again have k+1 equations in k+1 unknowns. Thus our system of equations has a unique solution. Now if we consider equations IV and III as nonhomogeneous equations in the unknown $F(0,0,s)$ and $\sum_s F(0,s,r)\alpha_s$ (we do not consider $F(0,r,0)$ or $\sum_s F(0,r,s)\alpha_s$ since $F(n,r,r') = F(n,r',r)$) we have k+1 equations and 2k unknowns. Hence if k>l equations IV and III have many solutions for $F(0,0,r)$ and $\sum_s F(0,s,r)\alpha_s$. Once one set of solutions has been determined upon, however, the solutions from then on are uniquely determined for the k equations $g_r(F(0,0),\alpha_i\beta_i)$; and the $\frac{k^2-k}{2}$ + k equations from II in the $\frac{k^2-k}{2}$ + k unknowns $F(0,r,r')^2$ $(r \leqq r')$ from a system of $\frac{k^2-k}{2}$ + 2k equations in the $\frac{k^2-k}{2}$ + k unknowns $F(0,r,r')$ $(r \leqq r')$ and $\sum_s F(1_1 s,r)\,\alpha_s$. Thus the equations $\hat{\text{I}}$ $\hat{\text{II}}$ $\hat{\text{III}}$ $\hat{\text{IV}}$ are insufficient to determine a unique solution for the $F(n,r,r')$'s.

In many applications of this problem the information we want is the value of F(0; 0,0). A very good approximation to it can be obtained in the following way. We can solve for F(n; r,r') in terms of the F(0; 0,s)'s and F(0; 0,0) for all (r,r') as indicated above. Then using the k+1 equations

$$2F(0;\,0,r) + 2\sum_{r'=1}^{k}\sum_{n=o}^{\infty} F(n;\,r,r') = \beta_r/\alpha_r$$

$$\sum_{(r,r')}\sum_{n=o}^{\infty} F(n;\,r,r') + 2\sum_{r} F(0;\,0,r) + F(0;\,0,0) = 1$$

which follow from the ergodic properties of the system, we can solve for the F(0; 0,s)'s and F(0; 0,0) in terms of the parameters β_i and α_i. In practice we cannot easily find a formal expression for F(n; r,r') as a function of the F(0; 0,s)'s and F(0; 0,0). However, we can find F(n; r,r') through F(N; r,r') in terms of the F(0; 0,s)'s and F(0; 0,0) for small N without too much difficulty. We can then approximate the true solution by using the k+1 almost correct equations.

$$2F(0;\,0,r) + 2\sum_{r'=1}^{k}\sum_{n=o}^{N} F(n;\,r,r') = \beta_r/\alpha_r$$

$$\sum_{(r,r')}\sum_{n=o}^{N} F(n;\,r,r') + 2\sum_{r} F(0;\,0,r) + F(0;\,0,0) = 1$$

Values of N which I estimate give an error of less than .1% , .5% , and 1% in F(0; 0,0) for values of $\sum_{i=1}^{k} \beta_i/\alpha_i$ = .3, .4, .5, .75, 1.0, 1.25, and 1.5 are given in the table.

These estimates are based on the two-server case for one type of customer.

Table of Values of N

$\sum_{i=1}^{k} \beta_i/\alpha_i$	°/o Error		
	.1°/o	.5°/o	1°/o
.3	.3	2	2
.4	4	3	3
.5	4	3	3
.75	8	3	3
1.0	12	7	6
1.25	18	12	10
1.50	28	20	16

Another method, very similar to the above, is obtained by Markov chain considerations. We look at the system when someone leaves. Let $P_i(n,r)$ be the probability that the ith man to leave was of type r and was one of n+1 people in line or being served. $q_m(r)$ = probability that m people arrive in the time it takes to serve a man of type r. Using the theorem on a periodic irreducible Markov chains,* we assume $P_i(n,r) \underset{i \to \infty}{\longrightarrow} P(n,r)$ and also that $\sum_{n,r} P(n,r) = 1$.

$$P_i(n,r) = \gamma_r \sum_{i=1}^{k} \sum_{j=o}^{n+1} P_{i-1}(j,s)\, q_{n-j+1}(r)$$

$$P(n,r) = \gamma_r \sum_{s=1}^{k} \sum_{j=o}^{n+1} P(j,s)\, q_{n-j+1}(r)$$

$$\text{Let } Q(n) = \sum_{s=1}^{k} P(n,s)$$

$$\text{then } P(n,r) = \gamma_r \sum_{j=o}^{n+1} Q(j)\, q_{n-j+1}(r)$$

Summing the above equation over all r we obtain

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*See chapter xv in Feller (Bibliography, item 4).

$$Q(n) = \sum_{j=o}^{n+1} P_{n=j+1} \, Q(j) \quad \text{where } P = \sum_{r=1}^{k} \gamma_r \, q_\lambda \, (r) \, \lambda$$

This equation can be solved for $Q(n+1)$ in terms of Q's of lower n and hence in terms of $Q(0)$. Using $\sum_{n=o} Q(n) = 1$ we can obtain $Q(0)$ and hence $Q(n)$. Generalizing this method to the case $S > 1$ we can obtain equations similar to those above, but as with our other method there are too few of them to yield a unique solution.

2.2 Single Type of Customer. Instead of considering k different types of customers each with Poissonian arrival distribution and exponential service time distributions, we consider rather a single type of customer with a Poissonian arrival distribution (since the sum of Poissonianly distributed random variables has a Poissonian distribution) and a service time distribution given by $F(v) = \sum_{i=1}^{k} i\,(1 - \exp(-\alpha_i v)) = 1 - \sum_{i} \exp(-\alpha_i v)$. Thus we have a special case of M/C/S. This system was discussed by Pollaczek in 1934,* and appears to involve simply a formal solution. The M/G/1 system was elegantly and thoroughly covered by D.G. Kendall in 1951.**

Letting β_i, α_i, and γ_i have the same meaning as earlier in the discussion then $\sum_{i=1}^{k} \beta_i$ is the expected number of arrivals in unit time and

$$\sum_{i=1}^{k} \gamma_i \alpha_i \int_{o} v \exp(-\alpha_i v)\, dv = \sum \frac{\gamma_i}{\alpha_i}$$ is the expected service time of an arbitrary customer.

Following Kendall we find

$$E\,[q] = E\,[r] + \frac{E\,[r(r-1)]}{\lambda(1 - E\,[r])}$$ where q is the limit as $i \rightarrow \infty$, α_i being the random variable which assigns to each point the size of the queue the ith person leaves behind him as his service is completed, and r_i is the random variable which is the number of people to arrive while the ith person is being served. (Since the r_i have the same distribution we use r.) In our case:

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*Bibliography, item 9.

**Item 7.

$$E\left[r\right] = \sum_{n=1} n \int_0^{\infty} \sum a_i \gamma_i - \exp(-a_i x) \frac{\exp(-\beta x)}{n!} (\beta x)^n \, dx$$

$$= \sum_{i=1}^{k} \beta \frac{\gamma_i}{a_i} = \sum_{i=1}^{k} \beta_i / a_i$$

$$E\left[r^2\right] = \sum_{n=1}^{\infty} n^2 \int_0^{\infty} \sum a_i \gamma_i \exp(-a_i x)\Big] \frac{\exp(-\beta x)}{n!} (\beta x)^n \, dx = \beta \sum \frac{\gamma_i}{a_i} + 2\beta^2 \sum (\gamma_i / a_1^2) .$$

Therefore $E\left[q\right] = \sum_{i=1}^{k} (\beta_i / a_i) + \dfrac{\beta \sum_{i=1}^{k} (\beta_i / a_i^2)}{1 - \sum_{i=1}^{k} (\beta_i / a_i)}$

If ω_i is the waiting time of the ith customer $\omega = \lim_i \omega_i$.*

$$E\left[\omega\right] = \frac{1}{\beta} E\left[q\right] - E\left[v\right] = \sum_{i=1}^{k} \frac{\gamma_i}{a_i} + \frac{\sum (\beta_i / a_i^2)}{1 - \sum_{i=1}^{k} \frac{\beta_i}{a_i}} - \sum \frac{\gamma_i}{a_i} = \frac{\sum_{i=1}^{k} (\beta_i / a_i^2)}{1 - \sum_{i=1}^{k} (\beta_i / a_i)}$$

The generating function of the queue distribution is

$$\frac{1 - \sum (\beta_i / a_i)(1-z)}{1 - z/k(z)} = H(z)$$

Where $K(z) = \int_0^{\infty} \exp\left[- \left\{(1-z)\beta\right\} v\right] \left(\sum_{i=1}^{k} \gamma_i a_i \exp(-a_i v)\right) dv$

$$= \sum_{i=1}^{k} a_i \gamma_i \int_0^{\infty} \exp\left[- \left\{(1-z)\beta + a_i\right\} v\right] dv = \sum_{i=1} \frac{a_i \gamma_i}{(1-z)\beta + a_i} .$$

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*Actually $\lim_{i \to \infty} q_i(x)$ and $\lim_{i \to \infty} {}_i(x)$ do not necessarily exist even a.e.

However, $\lim_{i \to \infty} p_r(q_i = n)$ does exist and $\sum_n \lim_{i \to \infty} p_r(q_i = n) = 1$. Hence q_i and ω_i have limiting distributions.

In all the discussions of the single server case we have assumed that the queue distribution approaches a limiting distribution. Kendall demonstrates that the condition for this to be true is that $\rho < 1$ where ρ is the expected number of arrivals in unit time multiplied by the expected service time of an arbitrary customer. The expected number of arrivals in unit time is $\sum_{i=1}^{k} \beta_i = \beta$ and the expected service time is $\sum_{i=1}^{k} (\gamma_i/a_i)$ and hence ρ is $\sum_{i=1}^{k} (\beta_i/a_i)$. Hence if $\sum_{i=1}^{k} (\beta_i/a_i) < 1$ the above formulas apply.

Appendix 3

ON PROBLEMS OF OPTIMUM SERVERS

by

Mary C. Weiss

Problem: The machine has two types of failures, amplifier failures and power supply failures. We would like to know the number of men needed to repair these failures in order to keep the machine in working order a maximum amount of time. On the other hand, we want to maintain a minimum number of repairmen idle. In order to decide what this best number is we have to consider the expected fraction of unit time that the machine is in working order, and the expected fraction of unit time that a server (repairman) is idle. Since we can repair the machine in two ways, we have two different problems to consider. The first way, problem I, is to repair the broken amplifier when it is discovered. In this case the machine cannot operate until the amplifier is repaired. In the second case, problem II, the broken amplifier is removed from the machine, replaced with a good one, and repaired in the server's spare time. In this case the machine will be working as soon as the broken amplifier is replaced with a good one. We must, in this case, be sure that the servers have enough time left over from search for and replacement of the amplifiers and repair of the power failures to repair the broken amplifiers away from the machine.

In the solutions to problems I and II, β_1, β_2, α_1, α_2, d, and E are quantities which we have determined by using records kept of breakdown and repair times. β_1 is the expected number of amplifier breakdowns in one week, β_2 is the expected number of power supply failures in one week, $\frac{1}{\alpha_1}$ is the expected fraction of a work week's time that it takes to repair an amplifier (thus if 1.5 hours is average repair time for an amplifier, $\frac{1}{\alpha_1} = \frac{1.5}{40} = \frac{3}{80}$), $\frac{1}{\alpha_2}$ is the expected fraction of time that it takes to repair a power supply failure, $\frac{1}{d}$ is the expected fraction of time required to find a broken amplifier, and $\frac{1}{E}$ is the expected fraction of time required to remove and replace a broken amplifier. Our calculations consider 100, 150, and 900 amplifiers, and expected search times of 30 and 40 minutes. p_o is the probability that the machine is working and is equal to the expected amount of unit time that the machine is working.

The assumptions we have made about the failure distributions and service time distributions are: (1) the interfailure times of an amplifier are exponentially distributed and all the amplifiers have identical distributions; (2) the interfailure times of the power supply failures are exponentially distributed; (3) the amplifier repair times, power supply failure repair times, search times for broken amplifiers, and amplifier removal and replacement times are all exponentially distributed. In problem I, given these assumptions, we can see that the combined time for search and service has a non-exponential distribution which we are unable to handle except in the single server case. We therefore obtain an approximate solution by replacing this combined distribution with an exponential one having the same mean. We then make another approximation by considering only one kind instead of two kinds of failure, with an exponential service time distribution having the same mean as the true distribution, and with the combined Poisson arrival distribution. In problem II we combine search and removal and replacement time distribution just as in problem I we combine the search and repair time distributions. We then combine the two types of failures as in problem I. This approximation is very good; it becomes better as the discrepancy between the expected amount of unit time to be spent servicing one type of failure and the expected amount of unit time to be spent servicing the other grows larger.

I: One Server

$$p_o = 1 - (\beta_1 + \beta_2)\left(\frac{\gamma_1}{\eta} + \frac{\gamma_2}{\alpha_2}\right) \text{ where } \eta = \frac{d\alpha_1}{\alpha_1 + d}$$

The expected free time of the server is p_o

Two Servers

$$p_o = \frac{1 - \frac{1}{2}(\beta_1+\beta_2)(\frac{\gamma_1}{\eta} + \frac{\gamma_2}{\alpha_2}}{1 + \frac{1}{2}(\beta_1+\beta_2)(\frac{\gamma_1}{\eta} + \frac{\gamma_2}{\alpha_2})}$$

The expected free time of a server is $p_o + \frac{p_1}{2}$

when $p_1 = (\beta_1 + \beta_2)(\frac{\gamma_1}{\eta} + \frac{\gamma_2}{\alpha_2})\, p_o$

Three Servers

$$p_o = \frac{1}{1+(\beta_1+\beta_2)(\frac{\gamma_1}{\eta}+\frac{\gamma_2}{\alpha_2})+\frac{1}{2}(\beta_1+\beta_2)^2(\frac{\gamma_1}{\eta}+\frac{\gamma_2}{\alpha_2})^2+(\beta_1+\beta_2)^3(\frac{\gamma_1}{\eta}+\frac{\gamma_2}{\alpha_2})^3\,\frac{1}{3!}\cdot\frac{1}{1-\frac{1}{3}(\beta_1+\beta_2)(\frac{\gamma_1}{\eta}+\frac{\gamma_2}{\alpha_2})}}$$

The amount of free time a server has is .

$p_o + \frac{2}{3}p_1 + \frac{1}{3}p_2$, where $p_2 = \frac{1}{2}(\beta_1 + \beta_2)^2(\frac{\gamma_1}{\ } + \frac{\gamma_2}{\alpha_2})^2 p_o$

II: Let $\omega = \frac{Ed}{E+d}$

One Server

$$p_o = 1 - (\beta_1 + \beta_2)(\frac{\gamma_1}{\omega} + \frac{\gamma_2}{\alpha_2}),$$

In order that one server be sufficient to handle the amplifier repair, we must, in addition, have

$$p_o > \frac{\beta_1}{\alpha_1}$$

Two Servers

$$p_o = \frac{1 - \frac{1}{2}(\beta_1+\beta_2)(\frac{\gamma_1}{\omega}+\frac{\gamma_2}{\alpha_2})}{1 + \frac{1}{2}(\beta_1+\beta_2)(\frac{\gamma_1}{\omega}+\frac{\gamma_2}{\alpha_2})}$$

In order that two servers be sufficient to handle traffic and repairs we must have

$$p_o + \frac{1}{2}p_1 > \frac{1}{2}\,\frac{\beta_1}{\alpha_1}$$

The expected free time of a server is

$$p_o + \frac{1}{2} p_1 - \frac{1}{2} \frac{\beta_1}{\alpha_1} \quad \text{where } p_1 = (\beta_1+\beta_2)\left(\frac{\gamma_1}{\omega} + \frac{\gamma_2}{\alpha_2}\right) p_o$$

Three Servers

$$p_o = \frac{1}{1+(\beta_1+\beta_2)\left(\frac{\gamma_1}{\omega}+\frac{\gamma_2}{\alpha_2}\right)+\frac{1}{2}(\beta_1+\beta_2)^2\left(\frac{\gamma_1}{\omega}+\frac{\gamma_2}{\alpha_2}\right)^2+(\beta_1+\beta_2)^3\left(\frac{\gamma_1}{\omega}+\frac{\gamma_2}{\alpha_2}\right)^3\frac{1}{3!}\;\dfrac{1}{1-\frac{1}{3}(\beta_1+\beta_2)\left(\frac{\gamma_1}{\omega}+\frac{\gamma_2}{\alpha_2}\right)}}$$

In order that three servers be sufficient to handle traffic and repairs we must have

$p_o + \frac{2}{3} p_1 + \frac{1}{3} p_2 > \frac{1}{3} \frac{\beta_1}{\alpha_1}$ where p_1 and p_2 have the same respective values given above.

The expected free time of a server is

$$p_o + \frac{2}{3} p_1 + \frac{1}{3} p_2 - \frac{1}{3} \frac{\beta_1}{\alpha_1}$$

Some Special Cases: 900 amplifiers

When $\beta_1 = 21.8$

$1/\alpha_1 = .028$

$\beta_2 = .275$

$1/\alpha_2 = .048$

$1/d = .0125$

$1/E = .004$

results for problem I are

One Server

$p_o = .101$ = expected free time of server

Two Servers

$p_o = .379$

expected free time of a server = .549

Three Servers

$p_o = .401$

expected free time of a server = .698

and results for problem II are

One Server

$p_o = .622$

Since .622 > .610 one server can handle traffic and repair.

expected free time of a server = .012

Two Servers

$p_o = .682$

expected free time of a server = .505

Three Servers

$p_o = .686$

expected free time of a server = .672

When $\beta_1 = 21.8$

$1/\alpha_1 = .028$

$\beta_2 = 2.75$

$1/\alpha_2 = .048$

$1/d = .0166$

$1/E = .0042$

results of problem I are

One Server

$p_o = .0022$ = expected free time of server

Two Servers

$p_o = .3343$, $p_1 = .3336$

expected free time of a server = .5011

Three Servers

$p_o = .3645$, $p_1 = .3637$, $p_2 = .1815$

expected free time of a server = .6675

and results of problem II are

One Server

$p_c = .5342$

Since $p_o = .5342 < .610 = \frac{\beta_1}{\alpha_1}$, one server is not sufficient to handle traffic repairs.

Two Servers

$p_o = .6222$, $p_1 = .2898$

Since $p_o + \frac{1}{2} p_1 = .7671 > .305 = \frac{1}{2} \frac{\beta_1}{\alpha_1}$, two servers are sufficient to handle traffic repairs.

expected free time of a server = .462

Three Servers

$p_o = .6279$, $p_1 = .2925$, $p_2 = .0681$

expected free time of a server = .643

Some Special Cases: 100 amplifiers

When $\beta_1 = 2.43$

$1/\alpha_1 = .028$

$\beta_2 = .275$

$1/\alpha_2 = .048$

$1/d = .0125$

$1/E = .0042$

results for problem I are

One Server

$p_o = .8883$ = expected free time of server

Two Servers

$p_o = .8941, p_1 = .0999$

expected free time of server = .9441

Three Servers

$p_o = .8944, p_1 = .0999, p_2 = .0056$

expected free time of a server = .9629

and results for problem II are

One Server

$p_o = .9462$

Since $.9462 > .0678 = \frac{\beta_1}{\alpha_1}$, one server is sufficient for traffic and repairs.

expected free time of the server = .8784

Two Servers

$p_o = .9476, p_1 = .0510$

expected free time of a server = .9392

Some Special Cases: 150 amplifiers

When $\beta_1 = 3.645$

$1/\alpha_1 = .028$

$\beta_2 = .275$

$1/\alpha_2 = .048$

$1/d = .0125$

$1/E = .0042$

results of problem I are

One Server

$p_o = .8389$ = expected free time of server

Two Servers

$p_o = .8510$, $p_1 = .1371$

expected free time of a server = .9195

Three Servers

$p_o = .8512$, $p_1 = .1371$, $p_2 = .011$

expected free time of a server = .9462

and results of problem II are

One Server

$$p_o = .9259 > \frac{\beta_1}{\alpha_1} = .1021$$

expected free time of a server = .8238

Two Servers

$p_o = 9286$, $p_1 = .0688$

expected free time of a server = .9110

Some Special Cases: 100 amplifiers

When $\beta_1 = 2.43$

$1/\alpha_1 = 0.028$

$\beta_2 = .275$

$1/\alpha_2 = .048$

$1/d = .0166$

1/Æ = .0042

results for problem I are

One Server

$p_o = .878$ = expected free time of a server,

Two Servers

$p_o = .885$, $p_1 = .108$

expected free time of a server = .939

Three Servers

$p_o = .886$, $p_1 = .108$, $p_2 = .008$

expected free time of a server = .960

and results for problem II are

One Server

$p_o = .936$

Since $.936 > .0678 = \frac{\beta_1}{\alpha_1}$, one server is sufficient for traffic and repairs.

expected free time of a server = .868

Two Servers

$p_o = .938$, $p_1 = .060$

expected free time of a server = .934

Some Special Cases: 150 amplifiers

When $\beta_1 = 3.645$

$1/\alpha_1 = .028$

$\beta_2 = .275$

$1/\alpha_2 = .048$

$1/d = .0166$

$1/E = .0042$

results for problem I are

One Server

$p_o = .824$ = expected free time of a server

Two Servers

$p_o = .838$, $p_1 = .147$

expected free time of a server = .912

Three Servers

$p_o = .839$, $p_1 = .147$, $p_2 = .013$

expected free time of a server = .941

and results for problem II are

One Server

p_o = .911

Since .911 > .102 = $\frac{\beta_1}{\alpha_1}$, one server is sufficient for traffic and repairs.

expected free time of a server = .809

Two Servers

p_o = .915, p_1 = .081

expected free time of a server = .843