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REPORT NO. RM-824-P-5

Technical Report

DISCHARGE OF CORONA POINT CURRENT FROM AIRCRAFT

Contract Nonr - 904(00)

1 December 1954

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CORNELL AERONAUTICAL LABORATORY, INC.

BUFFALO, N. Y.

REPORT NO. RM-824-P-5

Technical Report

DISCHARGE OF CORONA POINT CURRENT FROM AIRCRAFT

Contract Nonr - 904(00)

Prepared by: Seville Chapman
Seville Chapman, Head
Physics Department

Date: 1 December 1954

Table of Contents

Abstract

In a system for maintaining zero electrostatic charge on aircraft, where electric field meters on the surfaces of the wings control a high-voltage corona discharge point behind the tail of the aircraft, the primary factors influencing the magnitude of blow-away discharge current i are point potential V, aircraft speed v, point geometry, and space charge from the already discharged current. V must be large enough to create an electric field vector toward the rear to drive the discharge current into the space charge behind the point, but since this same V creates a field vector forward toward the aircraft skin, the point must be disposed so that the wind past the point prevents return current to the aircraft. The problem is approached mathematically from several points of view, to evaluate constants F, G, and H in $i = \epsilon_0(FkV^2/\ell + GvV + H\ell v^2/k)$, where k is the mobility and ℓ a length. The current should vary with the first powers of the speed and point potential, and be independent of mobility, and hence of polarity and altitude. For an aircraft speed of 100 meter/sec and a point potential of 100 kilovolts, the current should lie in the range 200-250 microamperes although calculations based on different approximations vary from 175-395 microamperes. Experiments in the laboratory with wind simulated by an electric field yield 225 microamperes. Limited experimental and in-flight data imply current of about 200 microamperes.

FORM 145-A

CORNELL AERONAUTICAL LABORATORY, INC.

BUFFALO, N. Y.

PREPARED BY Seville Chapman

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DISCHARGE OF CORONA POINT CURRENT FROM AIRCRAFT

1. Introduction

When aircraft fly through precipitation, they frequently become charged electrostatically at a rate of 100 microamperes or more. Since the capacitance of an aircraft such as an R6D (or DC6) is approximately 800 picafarads (1 picafarad or "puff" = 1 micromicrofarad = 10^{-12} farad), it is clear that the rate of change of potential may be 80 000 volts/second. After a few seconds many undesirable phenomena occur, especially those which interfere with electrostatic measurements or with communications. Even in fair weather engine exhausts can generate charging currents of several microamperes depending on speed, mixture, and the idiosyncrasies of the engine.

The first major consideration of problems of electrostatic charging of aircraft occurred during World War II when precipitation static became a very serious communication problem. During the war and since, numerous investigators [1, 2, 3, 4, 5, 6, 7, 8] have reported in the scientific literature various investigations pertaining to measurements of electrostatics and atmospheric electricity from aircraft.

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1. Gunn et al, "Army-Navy Precipitation-Static Project," Proc. Inst. Radio Eng. 34, 156-177 (April 1946), and 34, 234-254 (May 1946).

2. Pelton et al, Cornell Aeronautical Laboratory "Final Report RA-766-P-10. Investigation of Means to Maintain Zero Electrostatic Charge on Aircraft." October 31, 1953, Contract AF 19(122)475.

3. Byers, editor, "Thunderstorm Electricity," University of Chicago Press, 1953, containing 16 articles. See also Byers and Braham, "The Thunderstorm," Superintendent of Documents, 1949.

4. Malone, editor, "Compendium of Meteorology," American Meteorological Society, 1951. Six articles on atmospheric electricity.

5. Proceedings of the Wentworth Conference on Atmospheric Electricity (in press) Articles by Koenigsfeld, Stergis, Sagalyn, Schilling, Schaefer, Holzer, Chalmers, and others.

6. Meteorological Abstracts and Bibliography, issued monthly by the American Meteorological Society, regularly contains a section Electrical Phenomena

7. Seville Chapman, "Influence of Space Charge on Magnitude of Corona Discharge Current from Aircraft," Wentworth Conference on Atmospheric Electricity, May 1954
   Seville Chapman, "Corona Discharge Current from Aircraft," Physical Review 96, 831 (1954)
   Seville Chapman, "Effects of Wind and Space Charge on Corona Point Discharge, Particularly from Aircraft," Cornell Aeronautical Laboratory Report RA-766-P-11, Contract No. AF19(122)475, October 31, 1954

8. Chalmers, J. Alan, "Atmospheric Electricity," Oxford University Press, London, 1949
   Chalmers, J. Alan, "Atmospheric Electricity," Reports on Progress in Physics 17, 101-134 (1954)

The water-drop discharge system[1] was cumbersome, so that Cornell Aeronautical Laboratory developed[2] an active corona discharge system to maintain zero electrostatic charge on aircraft in flight.

Reasons for maintaining zero electrostatic charge on aircraft include alleviating effects of precipitation static[1], avoiding explosions during refueling operations in flight, or avoiding distortions of electrostatic conditions while in-flight measurements are being made of such quantities as the electrical conductivity of the air[2].

In order that effects of electrostatic charge on the aircraft itself not interfere with the precision measurement of electrostatic quantities in the atmosphere, the aircraft[2] is equipped with a Zero-Charge System to maintain the electrostatic charge at a non-fluctuating value, usually chosen to be zero. This report is directed toward one phase of the Zero-Charge System, namely, the magnitude of corona discharge current.* The system is described briefly in the next section and also in project reports[2].

*A preliminary version of a fraction of the material in this report was presented at the Wentworth Conference on Atmospheric Electricity in May 1954 which was sponsored jointly by the Navy and Air Force. A shortened ten-minute version of the Wentworth talk was presented to the American Physical Society in Minneapolis, June 1954 (See reference 7). At Wentworth and at Minneapolis the entire discussion of hardware and data was confined to that given in reference 2 of this report. Since the Wentworth Conference, the amount of material has been very considerably expanded (perhaps quadrupled), and made more rigorous. It should be remarked that most of the time spent on this paper has been out-of-hours. The Proceedings of the Wentworth Conference are not likely to be available for many months. The paper submitted for publication in the Proceedings, though more detailed than the talk, contains much less information than is given here.

While corona points have been used to discharge aircraft as just described, and for investigating atmospheric electricity from ground stations[8], most calibrations have been done in still air in the laboratory. In flight, and generally at a ground station, there is wind past the corona point. Since conditions with wind are quite different from those without, the effects of wind receive primary attention in this discussion. We shall see that the usual type of calibration done in still air is completely inapplicable to the case of an isolated point in wind.

2. The Zero-Charge Instrumentation System

The Zero Charge Instrumentation System senses the electrostatic charge on the aircraft and uses this information to control a high-voltage corona discharge point behind the tail of the aircraft, so that excess charge may be discharged by the corona point and blown away by the slip-stream.

The electrostatic charge on the aircraft may be determined by measuring the surface electrostatic field with generating electric field meters (sometimes called generating voltmeters or field mills). Almost invariably there is an electrostatic field in the atmosphere; hence the electric field at the surface of the aircraft consists of two components, one due to the ambient field (which is vertical in fair weather), and the other due to self-charge. Clearly a minimum of two meters is required to distinguish between these two fields. There are several sets of two points on the aircraft where two meters could be placed so that their readings could be combined to yield self-charge. Most of these points, however, are in awkward positions, such as where the fuselage is highly curved, and where slight inaccuracies of positioning the meters would give false indications.

The ambient electric field may have horizontal components as well as vertical components, (for example in the vicinity of thunderclouds). All components may be determined in addition to the self-charge of the aircraft. A convenient arrangement is to place two meters far out on each wing well away from the fuselage and engine, but not close to the wing tips. In these regions the wings are fairly flat, and one may calculate the "form factors" associated with any given location. Meters should be placed along the electrical center lines of the wing so as to be insensitive to longitudinal horizontal fields (see Figure 1).

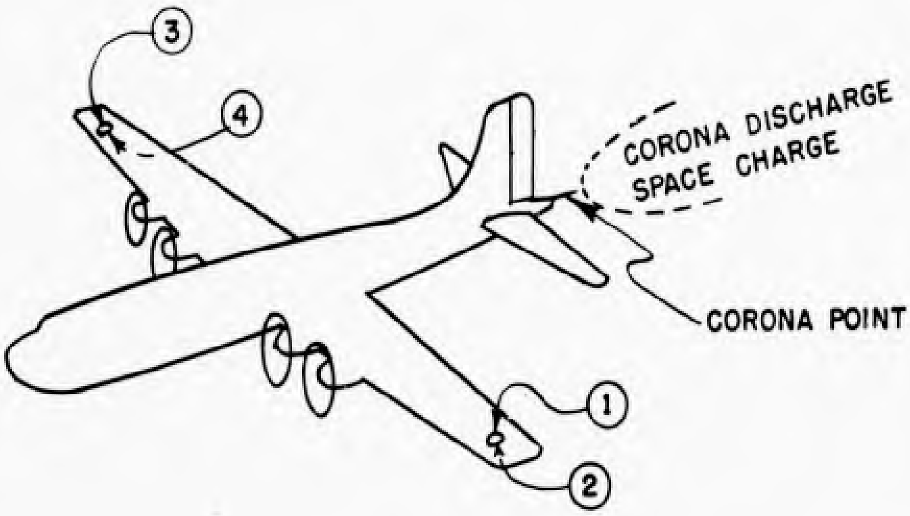

FIGURE 1

LOCATION OF FIELD-METERS AND CORONA POINT
IN THE ZECA SYSTEM

When due account is taken of the form factors it is not difficult to see that

$$E_1 - E_2 + E_3 - E_4 \propto E_z \tag{2.1}$$

$$E_1 + E_2 - E_3 - E_4 \propto E_y \tag{2.2}$$

$$E_1 + E_2 + E_3 + E_4 \propto E_q \tag{2.3}$$

where E_1, E_2, E_3, and E_4 are the form-factor corrected readings of meters 1, 2, 3, 4; E_z is the vertical component of the ambient electric field, E_y is the transverse horizontal component, and E_q is the field at the meters due to electrostatic charge on the aircraft. Actually there is some redundancy since four meters are used to give three quantities. In fact, if one of the meters becomes inoperative in flight, it is possible to switch the meters so that only three are needed. For convenience, however, and from symmetry, it seems best to install four meters as shown.

Actually the subject of form-factors is quite complicated.

In principle the reading of each meter is affected by each component of the ambient field, and the self-charge. If there are four meters (the minimum number for a complete measurement) then sixteen coefficients must be evaluated. By proper placement of the meters, many coefficients can be made zero; and from symmetry, the number of unknown coefficients may be reduced further. Electrolytic tank measurements or extensive calculations may yield values for others. Finally certain experiments can be carried out in the air to yield the remaining coefficients, for example banking the aircraft at a known angle, or discharging a known quantity of charge from the aircraft (measured current times measured time to yield a measured change in self-charge field).

The readings of the several meters are combined to control a 100-kilovolt dc voltage supply[2]. One side of the voltage supply is connected to the airframe, while the other is connected to a corona point insulated from the skin of the aircraft. The tip of the corona point is about 40 cm behind the tail of the aircraft, where in principle it is subjected to the full force of the slip-stream past the aircraft.

Or one may use a group of points. In the laboratory five points spaced at the corners and center of a square whose diagonal was 3.75 inches yield a little less than twice the current of a single point. Thus at 100 kilovolts when a plate was placed 7.5 inches from the point or points, five points gave 225 microamperes while one gave 130. At 4.5 inches, 520 microamperes was obtained from the five points. As shown later in the report, Sections 4 and 14, theory implies little basis for supposing five closely spaced points in flight will give much more current than one.

Obviously it is important to estimate the magnitude of corona current which can be obtained, in terms of the potential applied to the point, the aircraft speed, ion mobility, and geometrical quantities such as the distance of the point (or points) behind the aircraft. Such an expression is required so that design information can be obtained, performance data can be estimated, and results can be analyzed properly. Assertions have been made that current blown away from the airplane by unipolar ions created in a gaseous discharge will be limited to rather small values of discharge current of about 15 microamperes. While we have measured currents greater than this, in order to understand the mechanism of discharge of the aircraft (as distinct from the detailed mechanism of corona discharge itself[9]), it seems worthwhile to endeavor to calculate the currents which may be obtained. For the sake of comparison, results are normalized to conditions with a point potential of 100 000 volts and with an aircraft flying with a speed of 100 meters/second at an altitude of about 18 000 feet, where the pressure is half of sea-level pressure.

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9) Loeb, "Recent Developments in Analysis of Mechanisms of Positive and Negative Coronas in Air," J. Applied Physics 19, 882-897 (1948).

3. Basic Mathematical Relations

a. The fundamental relations governing the motion of space charge created by the corona point are:

$$\nabla^2 V = -\rho/\varepsilon_0 \quad (3.1)$$

$$E = -\text{grad } V \quad (3.2)$$

$$u = kE + v \quad (3.3)$$

$$\text{div}(\rho u) = 0 \text{ except at the point and at electrodes} \quad (3.4)$$

$$i = \rho u S \quad (3.5)$$

where

V = the electric potential

ρ = the volume space charge density

ε_0 = the permittivity of empty space = 8.854×10^{-12} farad/meter

E = the electric field

u = the ion speed

k = the ion mobility ($+1.6 \times 10^{-4}$ or -2.2×10^{-4} m^2/sec volt at sea-level or about 4×10^{-4} m^2/sec volt at 18 000 feet altitude)

v = the airspeed of the airplane

i = the current

S = an appropriate area

Equations (3.1) and (3.2) are the fundamental equations of electrostatics, equation (3.3) essentially defines mobility (since in still air, ion speed = kE), equation (3.4) is the equation of continuity, and equation (3.5) relates current to space charge density.

Other useful equations can be derived from equations (3.1) to (3.5). Complete derivations are given in the appendices and the results only are given below.

b. Sometimes it is convenient to consider the effect of wind blowing ions along as equivalent to an electric field E_w. From equation (3.3), E_w is seen to be

$$E_w = v/k \quad (3.6)$$

c. In the absence of space charge, at a distance r from a point charge q, the electrostatic equations can be written

$$E = -q/4\pi \varepsilon_o r^2 \tag{3.7}$$

$$E = -V/r \tag{3.8}$$

from which it is readily seen that $V = q/4\pi \varepsilon_o r$.

d. Consider a spherical system without wind having a small grounded inner electrode and an outer electrode at a fixed potential, with total space charge current i between electrodes. At any point at radius r within the system or at the outer electrode, to a very close approximation:

$$i = 6\pi \varepsilon_o k r E^2 \tag{3.9}$$

$$i = \frac{3\pi \varepsilon_o}{2} \frac{kV^2}{r} \tag{3.10}$$

$$E = -V/2r \tag{3.11}$$

e. Consider a semi-infinite cylinder of space charge of uniform density ρ, whose radius is r_3, and whose axis lies along the X-axis. The cylinder extends to infinity from a plane normal to the X-axis at x_o. It may be imagined that the space charge was emitted at a rate $i = d\rho/dt$ by a source of area πr_3^2 traveling with a speed v along the axis. The space charge is considered to remain fixed and not to spread out radially. It is apparent that the

$$\text{linear charge density} = i/v = \pi r_3^2 \rho. \tag{3.12}$$

The electric field E_c at the origin of coordinates is directed along the axis, its value is

$$E_c = -\frac{i}{2\pi \varepsilon_o r_3 v} \left[\sqrt{1 + (x_o/r_3)^2} - x_o/r_3\right] \tag{3.13}$$

If the end of the cylinder is at the origin, then $x_o = 0$ and for the field E_{co} equation (3.13) simplifies to

$$E_{co} = -i/2\pi \varepsilon_o r_3 v \tag{3.14}$$

From equations (3.13) and (3.14) we see that most of the field at the end of the cylinder is due to space charge close to the end. Space charge between the end and $x = 0.75r_3$ contributes half of the field, while space charge beyond $x = 5r_3$ contributes only one-tenth of the field.

f. Consider a paraboloid of space charge whose axis lies along the X-axis. The paraboloid has a radius R_0 at the origin and extends to infinity along the positive direction of the axis. Because of the self-repulsive nature of space charge, it can be shown that the space charge will spread out in the form of a paraboloid whose radius R at an abscissa x is given by (see Appendix C)

$$R^2 = R_0^{\ 2} + \frac{ki}{\pi \epsilon_0 v^2} x \tag{3.15}$$

where i = current or rate of emission of space charge

v = speed of source

k = mobility of the ions comprising the space charge.

g. If the paraboloid is truncated at an abscissa x and extends to infinity beyond x, then the electric field at the origin is given by

$$E_p = -\frac{v}{2k} \ln \frac{\left(\sqrt{R_0^2+bx+x^2}+x+\frac{b}{2}\right)\left(b^2-4R_0^2\right)}{2\left(R_0^2 b+b^2 x-2R_0^2 x-2R_0^2\sqrt{R_0^2+bx+x^2}\right)} \tag{3.16}$$

where

$$b = ki/\pi \epsilon_0 v^2 \tag{3.17}$$

If the paraboloid is truncated at the origin $x = 0$ where the radius of truncation is R_0, the field at the origin E_{po} becomes

$$E_{po} = -\frac{v}{k} \ln \left(1 + \frac{k\,i}{2\pi \epsilon_0 v^2 R_0}\right) \tag{3.18}$$

h. To find the field on the X-axis at a point $x = -2r_2$ of the paraboloid truncated at the origin (equation 3.15), it is convenient to translate the truncated paraboloid of equation (3.15) by a distance $2r_2$, so that it is now truncated at $x = 2r_2$. Then we can find the field E_p at a distance $2r_2$ from a truncated paraboloid whose radius of truncation is R_o, this field being formed by a current source i having a speed v. We find from equations (3.15) and (3.16) that

$$E_p = -\frac{v}{2k} \ln \frac{\left(\left\{\sqrt{R_o^2+4r_2^2}+2r_2\right\}+\frac{b}{2}\right)\left(\frac{b^2}{4}-\left\{R_o^2-2br_2\right\}\right)}{\left(R_o^2\frac{b}{2}-\left\{R_o^2-2br_2\right\}\left\{\sqrt{R_o^2+4r_2^2}+2r_2\right\}\right)} \qquad (3.19)$$

where b is given by equation (3.17).

4. Dimensional Analysis

We can learn some facts about corona discharge from dimensional analysis. In electrical phenomena we may choose four fundamental quantities, for example, length, mass, time, and charge. Actually it is simpler to take charge, potential, length, and time. By the usual type of analysis, the relevant parameters can be combined to yield the following result for the blow-away current i: (See Appendix D)

$$i = \varepsilon_o \left(FkV_1^2/\ell + GvV_1 + H\ell v^2/k\right) \qquad (4.1)$$

where

ε_o = the permittivity of empty space = 8.854×10^{-12} farads per meter

F,G,H, are non-dimensional coefficients undetermined by the dimensional analysis

k = the mobility of the ions

V_1 = the potenti.. of the discharge point relative to the aircraft skin

ℓ = some geometrical length or combination of lengths

v = aircraft speed.

Three special cases are of interest.

First, suppose the point is at rest in the air ($v = 0$) as in a laboratory test. Two of the terms vanish, and the corona current is proportional to V_1^2, and also to k. Since the ratio of positive to negative mobilities is 1.6/2.2, the current at a given potential would be expected to vary in the same ratio as polarity is changed. Further, mobility varies inversely with density. The isothermal density of the atmosphere varies exponentially inversely with altitude, the density at 18 000 feet being about half that at sea-level, and the density at 36 000 feet being about one-quarter that at sea-level. Therefore one would expect current to vary accordingly in altitude chamber tests. All of these effects are observed quite rigorously. For example, see Figure 9 of reference 2.

Second, suppose that no potential is applied to the discharger. Again two terms vanish in equation (4.1). This is virtually the case in the Philco system[10], where the outer cylinder is at the same potential as the airplane skin, and the high voltage points are shielded by the surrounding cylinders. Although their wind tunnel tests were hardly definitive, the current did vary with a power of the wind speed. Theoretically one would expect the current to vary with the square of the wind speed.

Third, suppose that there is no significant geometrical distance in the experiment. This would be the case if the corona point were isolated far enough behind the aircraft to prevent return current from flowing back to the aircraft skin. Once again two terms in equation (4.1) vanish, and now the current is proportional to the aircraft velocity and to the point potential. Further, the current will be independent of mobility, and hence independent of polarity (although $k+$ is not equal to $k-$), as well as independent of altitude (although k varies with altitude). In wind, an independence of polarity and a linear dependence on potential has been reported[11].

10) Green and Laurent, Philco Corporation, Research Division, Final Engineering Report on Precipitation Static Reduction, Contract USAF, No. W33-038-ac-20763, February 9, 1950.

11) Langmuir, "Final Report on Investigation of Fundamental Phenomena of Precipitation-Static," Contract W-33-106-sc-65, General Electric Co., October 1943-May 1945.

Any theory of the magnitude of corona discharge current must reduce dimensionally to these cases.

5. Physical Relations

When an aircraft becomes highly charged in flight, it is because an excess of current of one polarity is being delivered to it, generally by impact or "frictional" electrification all over its frontal area. Most regions in the atmosphere are electrically neutral, or at least neutral in comparison with considerations involving currents of many microamperes; hence the airplane will acquire one sign of charge while it leaves behind it a broad region of space charge of opposite polarity. If the electrostatic charge on the aircraft is maintained at zero by an active corona point in the tail of the aircraft, as in the system of Section 2, then there will be a narrow dense region of space charge discharged by the corona point directly behind the point. In later Sections, this narrow dense region of space charge is assumed to have different mathematical forms, such as a cylinder, or paraboloid.

At large distances behind the aircraft, the two space charges of opposite polarity will intermingle and exert no influence on the discharge. Close to the point, however, the broad space charge will have a minor influence, so that we shall be concerned mainly with the narrow dense region of space charge just behind the point.

Clearly, very close to the isolated corona point, there must be spherical symmetry (or almost spherical symmetry), since virtually all of the effects will be associated with the point potential and wind will be unimportant. It has already been mentioned (see equation 3.15) that far behind the point the space charge will assume the form of a paraboloid. We thus have boundary conditions defined in general terms. It is the purpose of the mathematical discussions which follow to try to fit a solution to these conditions.

As a method of procedure we may assume that out to a certain distance from the point, spherical conditions apply; beyond that distance, the situation may be described in terms of a cylinder or truncated paraboloid of space charge (see Figure 2). Point potential

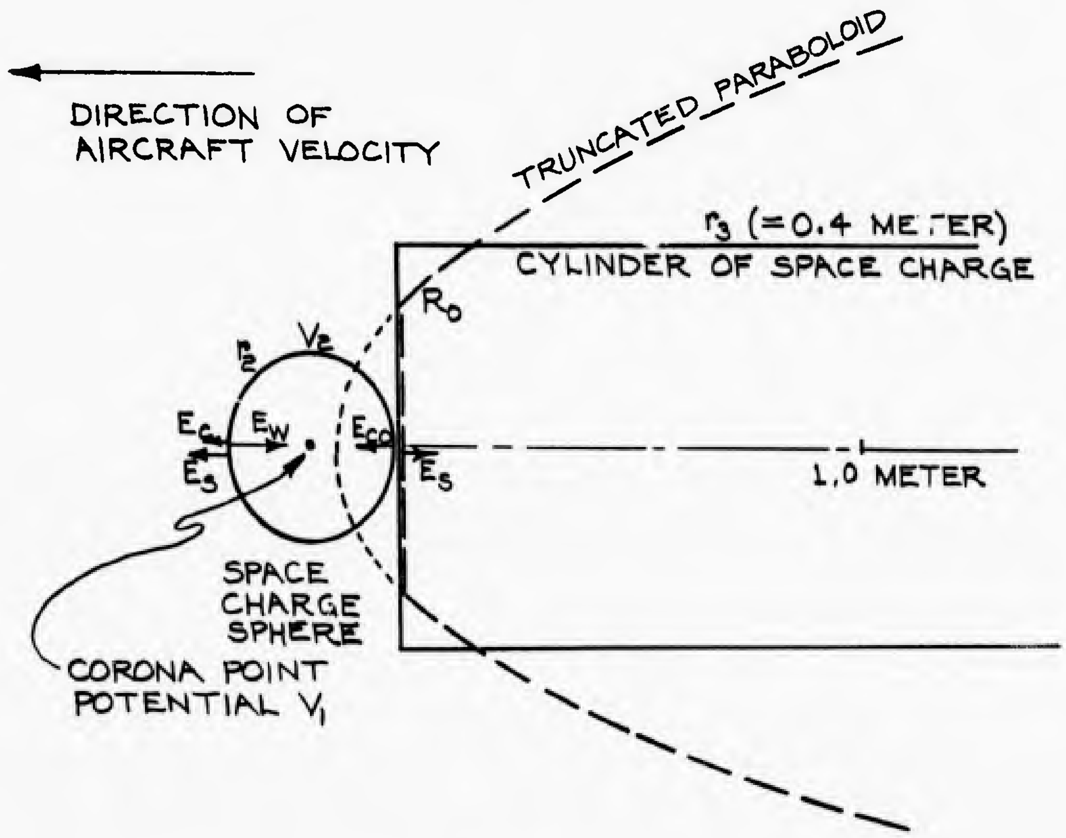

FIGURE 2

Geometry of the Methods of the Space Charge Sphere and Cylinder (solid line) or Truncated Paraboloid (broken line)

V_1, ion mobility k, aircraft speed v may be assumed to be known. There are four main unknown parameters: <u>discharge current</u> i; <u>radius</u> r_2 of the sphere about the corona point in which spherical conditions apply; <u>radius</u> r_3 of the cylinder, or the radius R_0 of truncation of the paraboloid (depending on which model is adopted for calculation);

and the potential V_2 of the sphere of radius r_2. To solve for these four unknowns we must have four mathematical conditions for which we can write appropriate equations. After we consider some physical relations influencing the phenomena, we will formulate the mathematical conditions. Physical reasoning suggests that there are five kinds of relations that might be important.

1. Space Charge Sphere. In the space charge sphere where electrical forces predominate over wind force, equation 3.10 must apply. We understand by V the potential difference between the point and the surface of the sphere, that is, $V_1 - V_2$.

2. Surface Potential of the Sphere. The surface potential V_2 of the sphere must be related in some way to the point potential V_1. In the next section the relation $V_2 = V_1/3$ is derived; in Section 10, a different relation is suggested.

3. Balance of electrical forces between sphere and cylinder or paraboloid of space charge. Directly behind the point the electric field vector toward the rear associated with the point potential must balance the electric field vector forward associated with the narrow dense region of already-discharged space charge. In other words, the point potential must be great enough to force the ions into the opposing already-discharged space charge.

4. Balance of electrical forces and wind. Forward of the point toward the aircraft skin, the electric field vector associated with the space charge sphere (and perhaps in addition the electric field vectors of the cylinder or paraboloid of already discharged space charge) must be balanced by the effective field vector of the wind. In other words, the point potential tends to create an electric field that drives ions back to the airframe; hence there must be a strong enough wind past the point to blow the ions away from the airplane and thus prevent return current to the aircraft. If discharge current returns to the aircraft, it does not influence the net charge on the airplane, and therefore serves no function.

5. Geometry of the discharge paths. The relative dimensions of the space charge sphere and cylinder of paraboloid must be appropriate in view of the fact that the electric charge which leaves the sphere is supposed to form the cylinder or paraboloid.

From these several types of physical relations we shall now formulate corresponding mathematical relations. Already we can see that we have more than enough kinds of relations to formulate the necessary mathematical relations to solve for the four unknowns. Below we list nine mathematical conditions appropriate to these five types of physical relations, although in fact, even more conditions may be formulated. From the nine we explore the solutions for six general types of solutions in Sections 7 to 12. A quite different approach is given in Section 13, and a comparison of the solutions is given in Section 16.

Condition I From relation 1) we see that we have

$$i = \frac{3\pi \varepsilon_0}{2} \frac{k(V_1-V_2)^2}{r_2} \qquad (5.1)$$

Condition II From relation 2) we have

$$V_2 = fV_1 \qquad (5.2)$$

As stated in the next section we shall take $f = 0.33$ unless specified otherwise.

Condition IIIc From relation 3), for the cylinder we have

$$E_s = E_{co} \qquad (5.3)$$

Condition IIIp From relation 3), for the paraboloid we have

$$E_s = E_{po} \qquad (5.4)$$

Condition IV From relation 4), we may choose to set the wind field equal to the field from the space charge sphere, thus

$$E_w = E_s \tag{5.5}$$

Condition IVc On the other hand, from relation 4), we may choose to set the wind field equal to the sum of the fields from the space charge sphere and the cylinder, thus

$$E_w = E_s + E_c \tag{5.6}$$

Condition IVp Still again, from relation 4), we may choose to set the wind field equal to the sum of the fields from the space charge sphere and the paraboloid, thus

$$E_w = E_s + E_p \tag{5.7}$$

Condition Vc From relation 5) we may consider it appropriate for the space charge sphere to have the same radius as the cylinder of space charge, thus

$$r_2 = r_3 \tag{5.8}$$

or we might choose to have r_3 larger than r_2, for example as shown in Figure 2.

Condition Vp From relation 5) we may choose to set the radius of the space charge sphere to have the same radius as the radius of truncation of the paraboloid, thus

$$r_2 = R_o \tag{5.9}$$

or of course we might prefer to have a different relation between r_2 and R_o.

In these equations, the principal symbols are

i = current

k = ion mobility

V_1 = point potential

V_2 = space charge sphere surface potential

r_2 = radius of the space charge sphere

f = a fraction

E_s = surface field of the space charge sphere

E_{co}, E_c, E_{po}, E_p, E_w, r_3, and R_o are fields from the cylinder, paraboloid, and wind; and radii of the cylinder and paraboloid as defined in equations 3.14, 3.13, 3.18, 3.16, 3.15, and 3.6.

6. The V/3 Rule; $V_2 = V_1/3$.

Consider a spherical region of radius r_2 around the corona point. Within this region assume that electrical and space charge forces on the ions predominate over the force due to the wind (see equation 3.6), so that the wind is assumed not to exist. Within the sphere equations 3.9, 3.10, and 3.11 apply. Outside of the sphere, suppose that the wind force predominates over the space charge forces, so that we may assume that the free space equations 3.7 and 3.8 apply. (This last assumption neglects the effect of the narrow dense region of space charge behind the point, but we shall take this into account in Section 10.)

The potential of the corona point relative to infinity (or relative to the aircraft skin, providing the aircraft is zero-charged) is V_1. The potential on the surface of the space charge sphere is V_2. Just within the surface of the space charge sphere, from equation 3.11 the radial electric field is

$$E_{in} = -(V_1 - V_2)/2r_2 \qquad (6.1)$$

Just outside the surface of the space charge sphere, from equation 3.8 the radial electric field is

$$E_{out} = -V_2/r_2 \tag{6.2}$$

Since there is no finite surface charge on the boundary, $E_{out} = E_{in}$ and hence

$$V_1 - V_2 = 2V_2 \tag{6.3}$$

or

$$V_2 = V_1/3 \tag{6.4}$$

Thus, if a point at infinity is considered to have zero potential, the potential on the surface of the space charge sphere is one-third the potential of the point. Equation 6.4 will be referred to as Condition II.

The conclusion of equation 6.4 may be reached also by a more elegant argument[11].

7. Method of the Simple Space Charge Sphere

The following analysis is a simple one, and is not rigorous since it neglects the inhibiting influence of the electric field of the already-discharged space charge. It is included because most of the material in this Section is required for the two following Sections, which are more rigorous, and do consider the influence of the already-discharged space-charge.

As in Section 6, let us suppose[11] that all of space may be divided into two regions. The inner region about the corona point of potential V_1 is a sphere of radius r_2 having a surface potential V_2. Within this region space charge equations 3.9, 3.10, and 3.11 apply. The outer region comprises all the rest of space. In the outer region space charge forces may be neglected and only wind forces (equation 3.6) apply.

In this situation we have seen that Condition II is

$$V_2 = V_1/3. \tag{7.1}$$

In equation 3.10, the potential difference between the corona point and the sphere is $V_1 - V_2$; thus Condition I is

$$i = \frac{3\pi\varepsilon_0}{2} \frac{k(V_1-V_2)^2}{r_2} \tag{7.2}$$

If we substitute equation 7.1 we obtain

$$i = 2\pi \varepsilon_0 k V_1^2/3r_2 \tag{7.3}$$

Since we consider that space charge effects may be ignored outside the sphere of radius r_2, from equations 3.8 and 7.1 for the radial electric field E_s just outside the surface of the sphere we have

$$E_s = V_1/3r_2 \tag{7.4}$$

Now let us apply a variant of Condition IV, and take the field at the front surface of the sphere E_s to be just equal to the wind field $E_w = v/k$ (equation 3.6). Thus

$$V_1/3r_2 = v/k \tag{7.5}$$

When equations 7.5 and 7.3 are combined we have the results

$$i = 2\pi \varepsilon_0 v V_1 \tag{7.6}$$

$$r_2 = kV_1/3v \tag{7.7}$$

If we substitute our normalized values $v = 100$ meter/sec, $V_1 = 100\ 000$ volts, and $k = 4\times10^{-4} m^2$/sec volt,

$$i = 555\times10^{-6} \text{ amperes or 555 microamperes} \tag{7.8}$$

$$r_2 = 0.133 \text{ meter} \tag{7.9}$$

It is to be noticed that the blow-away current is proportional to the first power of the speed and the first power of the point potential. The current is independent of mobility and any distances. Equation 7.6 is consistent with the dimensional analysis of equation 4.1.

8. Method of the Space Charge Sphere and Cylinder

In this Section we consider that there is a space charge sphere about the corona point in which wind forces are neglected, and behind this sphere there is a non-expanding cylinder of radius r_3 of already-discharged space charge, as shown in Figure 2.

We make use of Conditions I and II as in Section 7 so that equations 7.3 and 7.4 are applicable to this Section too. We shall also use Conditions IIIc and IVc.

Condition IIIc specifies that the axial field E_{co} at the end of the cylinder of already-discharged space charge, given by equation 3.14, is equal to the radial field E_s at the surface of the space charge sphere given by equation 7.4.

Thus

$$E_s = E_{co} \qquad (8.2)$$

or

$$V_1/3r_2 = i/2\pi\, \epsilon_o r_3 v \qquad (8.3)$$

Condition IVc relates E_c, E_s, and E_w.

where E_c is the field at the forward surface of the sphere, from the cylinder of already-discharged space charge, and is given by equation 3.14,

E_s is the field at the surface of the space charge sphere given by equation 7.4, and

E_w is the wind field given by equation 3.6.

Thus

$$E_w = E_s + E_c \qquad (8.4)$$

or

$$\frac{v}{k} = \frac{V_1}{3r_2} + \frac{i}{2\pi\, \epsilon_o r_3 v}\left[\sqrt{1 + \left(\frac{2r_2}{r_3}\right)^2} - \frac{2r_2}{r_3}\right] \qquad (8.5)$$

The solution of equations 7.3, 8.3, and 8.5 is[12]

$$r_3 = kV_1/v \quad (8.6)$$

$$r_2 = 10kV_1/21v \quad (8.7)$$

$$i = 1.40\pi \, \epsilon_o v V_1 \quad (8.8)$$

and of course

$$V_2 = V_1/3. \quad (8.9)$$

Substituting normalized values $v = 100$ m/sec, $V_1 = 10^5$ volts, and $k = 4 \times 10^{-4} m^2$/sec volt, we have

cylinder radius r_3	=	0.40 meter	(8.10)
sphere radius r_2	=	0.191 meter	(8.11)
current i	=	388×10^{-6} amperes	(8.12)
sphere potential V_2	=	33 000 volts	(8.13)

The form of equation 8.8 is the same as for equation 7.6, the only difference being the numerical coefficient. Equations 8.6, 8.7, 8.8, and 6.4 are the solution for the four unknowns of Section 5, as determined by the four conditions of Section 5, applied to the non-expanding cylinder of space charge for the narrow dense region of space charge behind the corona point.

9. Method of the Space Charge Sphere and Paraboloid

If we replace the non-expanding cylinder of space charge of Section 8 by the truncated paraboloid referred to by equations 3.16, 3.17, and 3.18, and shown in Figure 2, then in the analysis of Section 8 we should replace E_{co} by E_{po} and E_c by E_p. Thus we use Conditions I, II, IIIp, and IVp. The algebra is extremely tedious and solutions must be obtained in part by trial and error. The results are (Appendix C)

radius of truncated paraboloid	R_o	= 0.274 meter	(9.1)
radius of space charge sphere	r_2	= 0.188 meter	(9.2)
current	i	= 394×10^{-6} amperes	(9.3)
sphere potential	V_2	= 33 000 volts	(9.4)

It is seen that the results of Section 9 are not very different from those of Section 8.

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10. The Method of Variation of f; $V_2 = fV_1$

The method of Section 8 may be repeated utilizing Conditions I, IIIc, and IVc, without using the result of Section 6 that the sphere potential V_2 is $V_1/3$, where V_1 is the point potential. For Condition II, we may place

$$V_2 = fV_1, \tag{10.1}$$

where the fraction f is a parameter whose value, as we shall see, lies somewhere in the range from about 0.3 to 0.5. Results for a few arbitrary values of f are given in the table.

Parameter	f		0.333	0.40	0.442	0.48	0.49	0.50
Current	i	µamps	388	292	245	197	188	179
Sphere	r_2	m	0.191	0.205	0.212	0.228	0.229	0.231
Cylinder	r_3	m	0.40	0.27	0.212	0.169	0.160	0.150
Field	E_s	v/m	175,000	195,000	199,000	210,000	212,000	215,000
Wind Field	$E_w = v/k$	v/m	250,000	250,000	250,000	250,000	250,000	250,000

The problem now is to determine a physical basis for selecting the proper value of f. The results of Section 8 constitute the first column of figures. In Section 8, the presence of the narrow dense region of space charge behind the space charge sphere was ignored. While we must not be too rigid in our thinking regarding the physical existence of a mathematically defined space charge sphere, it does seem true that the presence of the narrow dense region of space charge so near to the sphere ought to give it a surface potential greater than would be the case if the narrow dense region of space charge were not there.

The capacitance of an isolated object is defined as the ratio of its charge to its potential. It is well known that if C is the capacitance of an isolated conducting sphere, then the capacitance of two equal touching conducting spheres is

$2C \ln 2$, or 69.3 per cent of the sum of the capacitances of two isolated spheres. Thus with a given charge on a sphere, its potential is increased in the ratio 1/0.693 by the presence of a second equally charged sphere.

We may consider the second sphere as an approximation to the narrow dense region of space charge behind the space charge sphere. Applying this argument, we might expect the V/3 rule, Condition II, to be modified to V/(3x0.693), or in equation 10.1 we might expect

$$f = 0.333/0.693 = 0.480. \qquad (10.2)$$

Hence the fourth column of figures in the table would be the correct one; that is, the current would be 197 microamperes.

Alternatively we might select a value of f intermediate between 0.333 and 0.480. Since ions do move away from the sphere about the point, there is the presumption that the second sphere should be chosen to have a smaller potential than the first, an effect that would yield a value for f of less than 0.48. The solution for $f = 0.48$ is somewhat disturbing also, since r_3 is less than r_2, and such a geometry does not seem particularly appropriate. Again in place of the condition of equation 10.2 we might select the Condition Vc $r_2 = r_3$ or $f = 0.442$. In any case 0.333 seems too small for f, and probably the value lies between 0.4 and 0.5.

11. The Method of Integration from Infinity

If one attempts to evaluate the potential of the surface of the space charge sphere, by integrating outward from the known potential of the point, for example by the use of equation 3.9, the expression clearly involves the current, which is an unknown. If one attempts to integrate from the known zero potential at infinity, by using any appropriate equations for the cylinder or paraboloid (see appendices B and C), the result invariably is infinite, a result which is neither enlightening nor physically correct.

As remarked at the beginning of Section 5, in a steady state of, say, precipitation charging, just behind the corona point there is a narrow dense region of space charge discharged by the point. Outside of this region there is a broad region of space charge of opposite polarity. The two regions must contain the same linear charge density, for if they do not, the potential of the aircraft ultimately would become infinite.

Since the two linear charge densities of opposite polarities have equal magnitudes, the field at large distances will decrease with distance at least as fast as an inverse square law, and hence an integration will yield a finite potential. As an approximation, we may consider two coaxial cylinders of charge, the inner one being the narrow dense region of the corona discharge, and the outer one being the broad region of space charge. We assume the radius of the inner cylinder r_3 to be equal to the radius r_2 of the space charge sphere and that both are equal to 0.2 meter. The radius of the outer cylinder should be the "radius" of the aircraft as viewed nose on. As an average radius r_4, we may take a rough geometric mean between the semi-wing-span and the half-height of the fuselage. We adopt $r_4 = 6$ meters, although granting that it is a very rough value. The space charge in the outer cylinder is assumed to be distributed evenly. The distribution in the inner cylinder does not influence the integration.

We integrate radially in toward the corona point so that the radial field is of the form λ/r, where λ is the linear space charge density, and r is the radius. Since the cylinders go to infinity in only one direction, the exact expression for the radial field is

$$E = \lambda/4\pi \epsilon_0 r \qquad (11.1)$$

For the case where r_4 is much greater than r_2, we may neglect r_2^2 in comparison with r_4^2. Since

$$\lambda = i/v \qquad (11.2)$$

where i is current and v is speed, the net linear charge within any radius r is

$$\frac{i}{v}\left(1 - \frac{r^2}{r_4^2}\right) \quad (11.3)$$

and the field is

$$E = \frac{i}{v}\left(1 - \frac{r^2}{r_4^2}\right)\frac{1}{4\pi\varepsilon_0 r} \quad (11.4)$$

The potential at radius r_2 is the integral of the field from infinity, or what is the same thing, from r_4, since in this case from the cylindrical symmetry, the field is zero beyond r_4. (See Figure 3). The potential at radius r_2 is

$$V_2 = -\int_{r_4}^{r_2} E\,dr \quad (11.5)$$

$$V_2 = \frac{i}{4\pi\varepsilon_0 v}\left[\left(\ln\frac{r_4}{r_2}\right) + \frac{r_2^2 - r_4^2}{2r_4^2}\right] \quad (11.6)$$

Substituting the numerical values for r_2 and r_4, we have

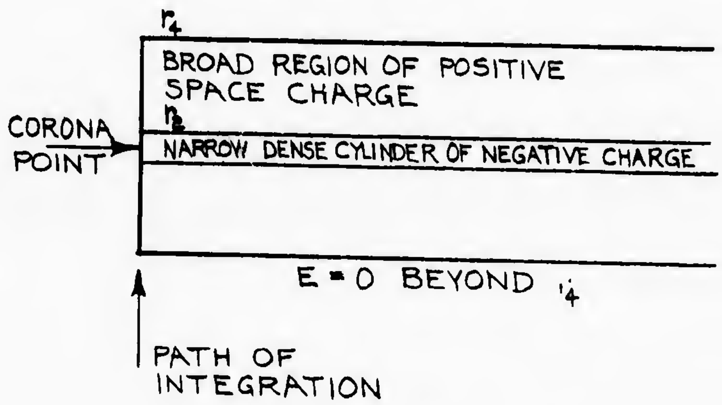

FIGURE 3
Geometry of the Method of Integration from Infinity

$$V_2 = 2.9\ i/4\pi\varepsilon_0 v \quad (11.7)$$

We know from previous analyses, for example from equation 8.8, that the current is of the form

$$i = K\ 2\pi\varepsilon_0 v V_1 \quad (11.8)$$

where K is a coefficient. In equation 8.8, K would be 0.7. We see that

$$K = i/555 \text{ microamperes.} \quad (11.9)$$

Equation 11.8 may be rewritten as

$$V_1 = i/K2\pi \varepsilon_0 v \tag{11.10}$$

Dividing equation 11.7 by equation 11.10, we have

$$\frac{V_2}{V_1} = 1.45K \tag{11.11}$$

or

$$f = 1.45\ i/555 \text{ microamperes} \tag{11.12}$$

Equation 11.12 gives a relation between f and i. We may inspect the table of Section 10 to obtain the solution

$$i = 188 \text{ microamperes} \tag{11.13}$$

$$f = 0.49 \tag{11.14}$$

We see that equation 11.6 is not very sensitive to changes in ratio of radii, and a smaller ratio would yield somewhat larger current. On the other hand, if the second term in the bracket is ignored, as would be proper if the broad region of space charge were not circular in cross-section but were much wider than it is high, the current would be reduced to about 170 microamperes, with $f = 0.51$.

It seems clear that the method of this section yields currents of about 180 microamperes with f about 0.50.

12. <u>The Method of</u> $r_2 = r_3$

In this method we apply conditions I, IIIc, IV, and Vc. The procedure is similar to that of the second method of Section 10, mentioned in the last paragraph of that section, except that here we use the simpler condition IV in place of condition IVc. Since we use condition IV rather than IVc, we must expect that ions from the point will reach a position somewhat closer to the airplane skin than the distance r_2 from the point.

From the four conditions respectively we have

I $$i = \frac{3\pi \epsilon_0}{2} \frac{kV_1^2}{r_2} (1 - f)^2 \qquad (12.1)$$

IIIc $$fV_1/r_2 = i/2\pi\epsilon_0 r_3 v \qquad (12.2)$$

IV $$fV_1/r_2 = v/k \qquad (12.3)$$

Vc $$r_2 = r_3 \qquad (12.4)$$

From conditions I and IV we obtain

$$i = 2\pi\epsilon_0 vV_1 \, 3 \, (1 - f)^2/4f \qquad (12.5)$$

and from conditions IIIc and Vc we get

$$i = 2\pi\epsilon_0 vV_1 f \qquad (12.6)$$

From equations 12.5 and 12.6, the solution for f is obtained easily, and from the other equations the complete solution, which is

$$f = 0.465$$
$$i = 258 \text{ microamperes}$$
$$r_2 = r_3 = 0.186 \text{ meter}$$

This seems far enough to carry the mathematical analyses, although other methods come to mind.

13. Still Air Simulation

Consider a laboratory set-up in still air, the set-up consisting of two horizontal parallel plates separated a distance d, as shown in Figure 4. A potential V is applied to the upper plate, and a corona point

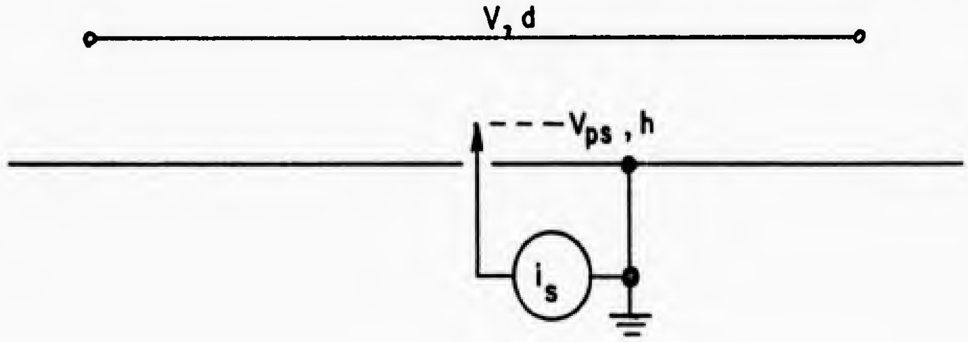

FIGURE 4
The experimental arrangement for corona discharge measurements using an electric field to simulate the effect of wind.

is placed through a hole in the lower plate so that the point is at a height h above the lower plate. At the same height h, but at a place removed from the point, the potential in the space between the planes is V_{ps}, so that the potential difference between this region and the point is V_{ps}.

Since the electric field between the plates is V/d, ions of mobility k will move in the region between the plates at a speed kV/d. Thus the arrangement of Figure 4 simulates a condition where ions are in a simulated wind of speed

$$v_s = kV/d \qquad (13.1)$$

and the point is at a potential

$$V_{ps} = Vh/d \qquad (13.2)$$

relative to its surroundings.

From the analysis of Section 8, we expect corona current from aircraft in flight to be proportional to the first power of the point potential and the wind speed. Thus the true current i_t from a corona point is

$$i_t = \frac{V_{pt}}{V_{ps}} \frac{v_t}{v_s} i_s \tag{13.3}$$

where

V_{pt} = point potential on the aircraft
V_{ps} = simulated point potential, equation 13.2
v_t = true airspeed
v_s = simulated wind speed

If we note that for actual corona points there is a starting potential V_o of 3 or 4 kilovolts, the equation 13.3 becomes

$$i_t = \frac{V_{pt} - V_o}{\frac{Vh}{d} - V_o} \frac{v_t}{\frac{kV}{d}} i_s \tag{13.4}$$

My colleague, Mr. Roland Pilie, has made careful measurements with an experimental arrangement like Figure 4. The lower plate was 0.81 meter in diameter; the upper plate was 0.61 meter in diameter and its circumferential edge was protected by a corona shield consisting of a bicycle tube painted with a conducting layer of colloidal graphite. The point was a Recoton Superchrome Phoneedle (a standard 0.006-centimeter radius, 78 revolution per minute phonograph needle) placed at various heights h from 0.04 to 0.12 meter above the lower plate. Plate separation d was 0.30 meter.

Curves followed the usual quadratic form above a starting potential, as would be expected. For example if the analysis of equation 13.4 is correct, one would expect

$$i_s \propto hk(V-V_o') \; V/d^2 \tag{13.5}$$

where

$$V_o' = V_o \frac{d}{h} \qquad (13.6)$$

Thus current would be quadratic with V above a starting potential V_o'.

Some representative data taken at 20°C and 74cm Hg pressure, with h = 0.10 meter and d = 0.30 meter, were

i_s = ±22.7x10^{-6} amperes, V = +64,500 volts or -76,500 volts; and

i_s = zero at onset, V_o' = +9,000 volts or -12,000 volts.

Taking values of mobility as k+ = 1.6x10^{-4} m^2/sec volt, and k- = -2.2x10^{-4} m^2/sec volt, we can reduce the data to 0°C and 76cm Hg, and then normalize to V_{pt} = 100,000 volts and v = 100 m/sec, we have for the positive point (negative upper plate)

$$i_{t+} = +225 \times 10^{-6} \text{ amperes} \qquad (13.7)$$

and for the negative point (positive upper plate)

$$i_{t-} = -227 \times 10^{-6} \text{ amperes} \qquad (13.8)$$

In view of the fact that the simulated speed of equation 13.1 depends on mobility and on potential, it is interesting that the measured data give essentially equal currents. This independence of current upon mobility was referred to in Sections 7 and 8. The simulated wind-speeds from equation 13.1 may be readily calculated to be 45 m/sec for the positive point, and 52 m/sec for the negative point.

Data of the type given here taken by various observers are not always in quantitative agreement. In one experiment[11] data were given for the negative point only where d was 0.05 meter. Using that data, the negative point current would be about twice the current given here. It may be that in the smaller apparatus, all the electrons from the discharge may not have attached to form negative ions. The near equality of the positive and negative currents using the data given here suggest that in these experiments ions rather than free electrons were involved.

If the simulation could be done with a strictly isolated point, it is the opinion of the author that the method would be quite good, in spite of possible electrical mirror image effects in the plate. The point, however, is supported mechanically with an electrical conductor, which to some extent distorts the field lower than the point. It is difficult to say whether this effect decreases the current by reducing the simulated wind behind (or lower than) the point, or whether it increases it by yielding a correspondingly greater field ahead of (or higher than) the point. I should think both effects would be fairly small.

14. Miscellaneous Comments

a. All the preceding analyses have involved approximations, and obviously the results must not be considered quantitatively precise. For example, we have considered a space charge sphere from which wind is excluded, with the wind having full speed elsewhere. Clearly this model can be only approximate, especially since the "sphere" probably will be skew.

b. If the airplane is not aerodynamically clean with sharp trailing edges, there will be image forces in the airplane. A flat plate transverse to the wind and transparent to it (if such can be imagined) would reduce the current by a factor of about 2.

c. If there is a conducting plasma around the point, there will be little voltage drop for a short distance. Hence V_1 will apply to a surface having a radius r_1 of a few centimeters. The effect on the current is the same as increasing the point potential. This phenomenon was perhaps responsible for the fact that altitude chamber measurements of current[2] quantitatively were greater than theory would call for on the basis of the chamber radius. A ratio of plasma radius to chamber radius of one to nine increases the current 2.25 times over that for a point.

d. It is worth noting that the starting potential for a high voltage flame is zero, but for a metal point it is a few thousand volts.

e. With an ordinary point, the discharge is very unsteady until the current reaches a value at least as great as about half a microampere. If radioactivity is applied to the corona point, then a finite stable discharge is obtained with a very low voltage. The corona point attached to the author's garage is 32 feet high, and has 320 micrograms of radium on it. In fair weather a current of 0.01 to 0.05 microampere commonly is obtained. With thunderstorms or snow squalls in the vicinity, the current commonly is 1 to 5 microamperes. Peaks as great as 20 microamperes of either polarity have been observed. The radioactivity is of negligible significance (other than in stabilizing the current) when the current exceeds about 0.3 microampere since the total ionization created by the radioactivity is of that order of magnitude.

f. When discharge currents are being measured in fair weather, it is important to note that performance data should be taken when the aircraft is zero-charged. Otherwise the potential of the aircraft may influence the discharge, so as to give misleading results. To obtain maximum current under zero-charge conditions, the aircraft should be charged (by manual over-ride on the controls) with one polarity. Then the discharge polarity should be changed, and current recorded as the aircraft passes through zero-charge.

g. Clearly, if several well-separated points are used, the current should be proportional to the number of points. If points are not spaced considerably farther apart than twice the space charge sphere radius ($2r_2$ = about 40 cm at 18 000 feet altitude or about 20 cm at sea level) then interference effects between points would be expected to reduce the current per point. The point must be placed behind the airframe skin a distance distinctly greater than r_2 if return current to the aircraft is to be avoided.

15. Data

Unfortunately, the data of reference 2 are very meager; in fact there is only one really useful current measurement at +23 kilovolts at a true airspeed of 273 miles/hour. The result is somewhat questionable and should not be considered as definitive. Adopting a 4-kilovolt starting potential, and extrapolating to the normalized values of 100 kilovolts at 100 m/sec, the current would have been 154 microamperes. Aerodynamicists at the Laboratory[2] consider that the true airspeed in the vicinity of the point was between 0.6 and 0.8 of the true aircraft speed, since the special fairing at the tail surface built to accommodate the point somewhat shielded the point aerodynamically, and hence probably reduced the wind speed there. If we adopt an average value of 0.7 for the speed reduction factor, then at the full speed of 100 m/sec the current would have been 220 microamperes.

The data taken 28 April 1954 were taken by Dr. Pelton with a voltage supply whose maximum potential was about 13 kilovolts. Data were taken at two altitudes, 3000 and 8000 feet, at 75, 90, and 105 knots. At that time the field-meters suffered from considerable drift. The commercial recorders also were subject to some drift. Consequently there was some doubt about the zero-charge reference, and about the absolute stability from one run to the next.

Over the voltage range available, from a starting potential of about 3 to 4 kilovolts, the currents were quite closely linear with point potential, although it must be admitted that there was considerable scatter to the points. There is no evidence, however, of any curvature.

The speed in different runs varied by 33 per cent of the average value. The current variation with speed apparently was not exactly linear with speed, varying about 20 per cent of the average value. The experimental spread in values was quite large, perhaps 20 per cent, so that these speed data cannot be said to be inconsistent with the principle of current being proportional to speed. On the other hand, they do not support the principle very strongly either.

There was no consistent variation with polarity, or with altitude.

Extrapolating the 90-knot data to the normalized speed and potential of 100 meters/sec and 100 000 volts, the currents at 3000 feet altitude were +137 microamperes and -158 microamperes, and at 8000 feet altitude were +186 microamperes and -144 microamperes. The average of the four magnitudes is 156 microamperes.

Mr. Schwartz subsequently made another single measurement when the drift problems had been eliminated. His "best" figure for a group of five points spaced about 2 inches apart, showed a starting potential of 4 kilovolts, with -20 microamperes at -30 kilovolts at 130 knots at 8000 feet. This datum when normalized yields -114 microamperes, the smallest value observed.

Mr. Breeden made a ground test on 28 June 1954, using the aircraft engines to create a wind past the tail. A velocity meter was used to measure the speed at the five points. Speed was adjusted to be 3000 feet/minute in one run, and 4000 feet/minute in another. Starting potential was 4000 volts, and the four curves of current plotted against voltage (two speeds, both polarities) were accurately straight lines. Maximum current was about 15 microamperes at 4000 feet/minute at 40 000 volts. The currents were almost identically the same magnitude for positive and for negative point potential. The current ratio for 4000 feet/minute and 3000 feet/minute, instead of being 1.33 as might be expected, was quite closely 1.20. It is unfortunate that more velocity data could not be taken. Extrapolating the data to the normalized speed of 100 m/sec and normalized potential of 100 kilovolts, the currents for 3000 feet/minute are +203 microamperes and -209 microamperes, and for 4000 feet/minute +180 microamperes and -183 microamperes. The average of the four magnitudes is 194 microamperes.

16. Conclusions

Form of the equations. It seems to be established theoretically that for this type of discharger, the current is proportional to the first powers of the point potential and velocity, and is independent of mobility, and hence of polarity and altitude.

The experimental verification of the dependence of current on the first power of the potential seems established. The relation with speed is less certain. Current varies approximately with the first power of speed. If the variation is not quite with the first power, certainly it is rather close to the first power. There appears to be no dependence on polarity. In fact, the independence of current with polarity means that current is independent of mobility. This independence of mobility is good evidence that the current should vary with the first power of the speed, in accordance with the analysis of Section 4. It is believed that this is the first report to show both theoretical and experimental evidence for the proportionality of current to the first powers of potential and speed, together with an independence of polarity.

A summary is given below of magnitude of current values normalized to the speed of 100 m/sec and a point potential of 100 kilovolts, and for mobility $4x10^{-4}$ m^2/sec volt.

	Current in Microamperes
a. Method of the Simple Space Charge Sphere, Section 7, Condition I, f = 0.33, $E_s = E_w$ (r_2 = 0.13m)	555
b. Method of the Space Charge Sphere and Cylinder, Section 8, Condition I, f = 0.33, $E_s = E_{co}$, $E_s + E_c = E_w$ (r_2 = 0.191m, r_3 = 0.40m)	388
c. Method of the Space Charge Sphere and Paraboloid Section 9, Condition I, f = 0.33, $E_s = E_{po}$, $E_s + E_p = E_w$ (r_2 = 0.188m, R_o = 0.27m)	394

d. Method of Variation of f, Section 10
Condition I, $f = 0.48$, $E_s = E_{co}$,
$E_s + E_c = E_w$
($r_2 = 0.23$m, $r_3 = 0.17$m) 197

e. Method of Variation of f, Section 10
Condition I, $r_2 = r_3 \therefore f = 0.442$,
$E_s = E_{co}$, $E_s + E_c = E_w$
($r_2 = r_3 = 0.212$m) 245

f. Method of Integration from Infinity, Section 11
Condition I, $\therefore f \cong 0.50$, $E_s = E_{co}$, $E_s + E_c = E_w$
($r_2 = 0.16$m) ~180

g. Method of $r_2 = r_3$, Section 12
Condition I, $E_s = E_{co}$, $E_s = E_w$, $r_2 = r_3$ ($f = 0.465$,
$r_2 = r_3 = 0.19$m) 258

h. Observation by Simulation (data of Pilié), Section 13 225

i. Reference 2, B-29, by measurement 154
by correcting for speed 220

j. 28 April 1954, average 156

k. Schwartz "best" figure 114

l. 28 June 1954, on ground, average 194

As mentioned in Section 7, the result shown as item a) should be rejected from consideration. Values of methods b) and c) probably are distinctly too large. Values of methods d) and f) probably are somewhat small, and methods e) and g) are probably somewhat too large. The conflict among these four methods is whether to take the larger values of f (about 0.49), which seems right, and obtain as a result that r_2 is larger than r_3, which seems wrong; or to set $r_2 = r_3$, which seems right, and obtain as a result the smaller value of f (about 0.45). Probably the distinction is not important enough to worry about. The average of the four methods is 220 microamperes. The value of method h) should be reliable. Not too much confidence should be placed in any one value of the measurements i), j), k). Measurement k) is disturbing, but it is a single datum, not a group of measurements. Measurement l) should be reliable.

For V = 100 000, v = 100 meter/sec, k = 4×10^{-4} m^2/sec volt (18 000 feet altitude), the author suggests the value 225 ± 25 microamperes is the best value for an isolated corona point discharger based on the investigation of this report, including theory, simulation, and various measurements. The radius of the space charge sphere is about 0.20 meter at 18 000 feet or 0.10 meter at sea-level, and the potential at the surface of the sphere is about 48 000 volts. Current is proportional to speed and potential, and is independent of mobility, and hence of altitude, and polarity.

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APPENDIX A:
Space Charge Relations for Spherical, Planar, and Cylindrical Symmetry

Spherical Symmetry

To derive the space-charge current-field-voltage relationships, equations 3.9, 3.10, and 3.11, in a stationary spherical system consisting of inner and outer spherical conductors from equations 3.1 to 3.5, we proceed as follows. The inner electrode is considered to be small.

By the definition of mobility k,

$$u = kE \tag{A-1}$$

From the equation of continuity $\operatorname{div}(\rho u) = 0$

$$\frac{1}{r^2}\frac{d}{dr}(r^2 \rho u) = 0 \tag{A-2}$$

From Poisson's equation for electrostatics $\nabla^2 V = -\rho/\epsilon_0$

$$\frac{1}{r^2}\frac{d}{dr}(r^2 E) = \frac{\rho}{\epsilon_0} \tag{A-3}$$

From (A-2) it is evident that

$$r^2 \rho u = K_1 = \text{a constant}$$

or

$$\rho = K_1/r^2 kE \tag{A-4}$$

Then substituting for ρ, (A-3) may be written

$$r^2 E \, d(r^2 E) = \frac{K_1}{\epsilon_0 k} r^2 dr$$

which is separable and integrable.

$$\frac{(r^2 E)^2}{2} = \frac{K_1}{3\epsilon_0 k} r^3 + K_2$$

$$E^2 = \frac{2K_1}{3\epsilon_0 kr} + \frac{2K_2}{r^4}$$

If the current is emitted uniformly in all directions, then

$$i = 4\pi r^2 \rho u = 4\pi r^2 \rho k E$$

Whence from (A-4)

$$K_1 = i/4\pi$$

(and incidentally $\rho = i/4\pi r^2 kE$)

Therefore

$$E^2 = \frac{i}{6\pi \epsilon_0 kr} + \frac{2K_2}{r^4}$$

Now when $i = 0$, or i is just at onset, $E_0{}^2 = 2K_2/r_0{}^4$, so that

$$E^2 = \frac{i}{6\pi \epsilon_0 kr} + E_0{}^2 \frac{r_0{}^4}{r^4} \qquad \text{(A-5)}$$

where the subscript "o" on E_0 and r_0 refers either to the inner or outer spherical electrode for both E_0 and r_0 when i is at onset. For a flame onset occurs for zero initial field E_0. In any case the second term on the right member generally is small, and we will neglect it for large values of E or i. Thus

$$E^2 = i/6\pi \epsilon_0 kr$$

$$i = 6\pi \epsilon_0 kr E^2. \qquad \text{(A-6)}$$

(and incidentally $\rho^2 = \frac{3 \epsilon_0 i}{8\pi kr^3}$ so that ρ decreases as r increases)

But

$$E = -\frac{dV}{dr} = \sqrt{\frac{i}{6\pi \epsilon_0 kr}}$$

Therefore

$$V = -2 \sqrt{\frac{i\, r}{6\pi \epsilon_0 k}} + V_1$$

where V_1 is not a starting potential, but is the potential of the inner electrode relative to ground. Sometimes the inner electrode is considered grounded, in which case $V_1 = 0$.

It is worth noting that if the outer electrode is at an infinite radius, an infinite voltage is needed for a finite current.

It is clear on rearranging that

$$i = \frac{3\pi\varepsilon_0 k}{2} \frac{(V-V_1)^2}{r} \qquad \text{(A-7)}$$

(If we had allowed the inner electrode to have a radius r_1 and a potential V_1, and the outer electrode to have a radius r_2 and a potential V_2, we would have obtained

$$i = \frac{3\pi\varepsilon_0 K (V_2 - V_1)^2}{2r_2 \left[1-\sqrt{r_1/r_2}\right]^2} \qquad \text{(A-7a)}$$

which implies a current greater than given by equation A-7.)

Combining A-6 and A-7

$$V - V_1 = -2Er \qquad \text{(A-8)}$$

Equations A-6, A-7, and A-8 which correspond to equations 3.9, 3.10, and 3.11 are the fundamental equations we need. They express the relationships among current, field, and potential at any radius within the space charge region. For example, if the inner electrode is grounded ($V_1 = 0$), and the radius of the outer electrode is 1 meter, and its potential is 10^5 volts, the field just within the outer electrode is 5×10^4 volts/meter, and for ions of mobility 4×10^{-4} m^2/sec volt, the current is 167 microamperes.

Parallel Planes

For reference it is worth having the corresponding equations for the parallel plane case, where x is the plate separation, and S is the area. Thus:

$$dE/dx = \rho/\epsilon_0$$

$$d(\rho u)/dx = 0$$

$$u = kE$$

Therefore

$$\rho u = \text{constant} = i/S$$

$$dE/dx = i/S\,\epsilon_0\,kE$$

$$EdE = idx/S\epsilon_0$$

$$E^2 = \frac{2i}{S\,\epsilon_0 k}\,x + E_0^{\,2}$$

If when $i = 0$, the starting field is zero as for a flame, then $E_0 = 0$. Note that the field is proportional to $\sqrt{x}$, not constant as is the case in absence of space charge. Hence ρ is proportional to $1/\sqrt{x}$. Thus

$$i/S = \epsilon_0 k\,E^2/2x \qquad \text{(A-9)}$$

We see that

$$-\,dV/dx = \sqrt{2i\,x\,/\,S\,\epsilon_0 k}$$

$$V = -\frac{2}{3}\sqrt{\frac{2i}{S\,\epsilon_0 k}}\;x^{3/2} \quad \text{if } V = 0 \text{ when } x = 0.$$

$$V^2 = \frac{8}{9}\;\frac{i}{S\,\epsilon_0 k}\;x^3$$

Note again that if the high voltage plate is at infinity, an infinite voltage is needed to get a finite current.

$$\frac{i}{S} = \frac{9}{8}\;\frac{\epsilon_0 kV^2}{x^3} \qquad \text{(A-10)}$$

$$V = -2\,E\,x/3 \qquad \text{(A-11)}$$

A-9, A-10, and A-11 are the relations for the parallel plane case.

Cylindrical Symmetry

In cylindrical coordinates, for reference, we have

$$\frac{1}{r}\frac{d}{dr}(rE) = \rho/\epsilon_0$$

and we get

$$i/L = 2\pi\epsilon_0 kE^2 \quad \text{(A-12)}$$

$$i/L = 2\pi\epsilon_0 kV^2/r^2 \quad \text{(A-13)}$$

$$V = -E\,r \quad \text{(A-14)}$$

APPENDIX B: Equations for the Cylinder of Space Charge

To obtain equations 3.12, 3.13, and 3.14 for the cylinder of space charge, we proceed as follows. We require the axial field E_c at the origin of coordinates (x, r, θ) from the one-directional or semi-infinite cylinder whose axis lies along the X-axis and whose uniform space charge density is ρ. Applying Coulomb's law for electrostatics to a ring element, and taking the axial component of the field

$$E_c = -\int_{x=x_o}^{\infty} \int_{r=0}^{r_3} \frac{\rho 2\pi r}{4\pi \varepsilon_o (x^2+r^2)} \frac{x}{\sqrt{x^2+r^2}} \, dr\, dx \qquad \text{(B-1)}$$

Hence

$$E_c = -\frac{\rho}{2\varepsilon_o} \int_{x_o}^{\infty} \int_{0}^{r_3} \frac{xr}{\sqrt{(r^2+x^2)^3}} \, dr\, dx$$

$$= -\frac{\rho}{2\varepsilon_o} \int_{x_o}^{\infty} x\,dx \left[\frac{-1}{\sqrt{r^2+x^2}}\right]_0^{r_3}$$

$$= +\frac{\rho}{2\varepsilon_o} \int_{x_o}^{\infty} \frac{x\,dx}{\sqrt{r_3^2+x^2}} - \frac{\rho}{2\varepsilon_o} \int_{x_o}^{\infty} \frac{x\,dx}{\sqrt{x^2}}$$

$$= \frac{\rho}{2\varepsilon_o} \left[\sqrt{r_3^2+x^2}\right]_{x_o}^{\infty} - \frac{\rho}{2\varepsilon_o} \left[x\right]_{x_o}^{\infty}$$

To evaluate these brackets, we should actually replace the upper limit by a large number, say A, and take the limit as A goes to infinity.

$$E_c = \frac{\rho}{2\varepsilon_o} \left[\sqrt{r_3^2+A^2} - \sqrt{r_3^2+x_o^2} - A + x_o\right]$$

$$E_c = -\frac{\rho r_3}{2\varepsilon_o} \left[\sqrt{1+(x_o/r_3)^2} - x_o/r_3\right]$$

(B-2)

We have

$$\text{linear charge density} = i/v = \pi r_3^2 \rho \qquad \text{(B-3)}$$

where i is current, and v is speed.
Substituting (B-3) in (B-2) we obtain

$$E_c = \frac{-i}{2\pi \varepsilon_0 r_3 v}\left[\sqrt{1 + (x_0/r_3)^2} - x_0/r_3\right] \qquad \text{(B-4)}$$

When $x_0 = 0$, equation (B-4) becomes

$$E_{co} = -i/2\pi \varepsilon_0 r_3 v \qquad \text{(B-5)}$$

Equations (B-3), (B-4), and (B-5) are equations (3.12), (3.13), and (3.14).

APPENDIX C: The Equations of the Paraboloid of Space Charge

In this section we derive equations (3.15) to (3.19).

To derive equation (3.15) we proceed as follows. Consider a corona point or other source of current moving in a straight line with a speed v emitting space charge. The problem is to find the shape assumed by the space charge. Some distance back from the point we cannot be far wrong in assuming that all motion of the space charge is radial in planes normal to the axis along which the source moves with no axial motion of the space charge. The radial field E_{rad} at a distance r from the axis of a cylinder of space charge or a line charge of linear density λ is

$$E_{rad} = \frac{\lambda}{2\pi \epsilon_o r} \qquad \text{(C-1)}$$

From the equation for mobility k, (drift speed) = (mobility) times (field), the outermost ions move with a radial speed dr/dt

$$dr/dt = k E_{rad}$$

$$= \frac{k \lambda}{2\pi \epsilon_o r}$$

Thus

$$r dr = \frac{k \lambda}{2\pi \epsilon_o} dt$$

and since the ions are presumed to start at a radius R_o at time $t = 0$.

$$\frac{R^2}{2} - \frac{R_o^2}{2} = \frac{k \lambda}{2\pi \epsilon_o} t$$

where R is the radius of the paraboloid. But the time $t = x/v$ where x is the distance the airplane moves in time t and v is its velocity. Thus

$$R^2 = R_o^2 + \frac{k\lambda}{\pi \epsilon_o v} x \qquad \text{(C-2)}$$

or since current $i = \lambda v$, therefore

$$R^2 = R_o^2 + \frac{k\,i}{\pi\,\varepsilon_o\,v^2}\,x \qquad \text{(C-3)}$$

which is a paraboloid whose vertex is not at the origin, but at the point

$$x = -\pi\,\varepsilon_o\,v^2\,R_o^2/ki$$

Equation (C-3) is equation (3.15).

Momentarily consider $R_o = 0$, then at a distance $x = 1.0$ meter behind the corona point, for $k = 4\text{x}10^{-4}\ m^2/\text{sec volt}$, $i = 100$ microamperes, and $v = 100$ m/sec, then $R = 0.38$ meter. We can differentiate equation (C-3) for the paraboloid and find that when $dR/dx = 1$, then $R = 0.072$ meter and $x = 0.036$ meter, or in this region the radial ion speed equals the axial aircraft speed. For a current of 10 microamperes at $x = 1.0$ meter, then $R = 0.12$ meter.

To derive equations (3.16), (3.17), and (3.18) we proceed as follows. To calculate the axial field E_p at the origin for the paraboloid of equation (C-3), truncated in a plane normal to the axis at a distance x from the origin, we proceed:

$$E_p = -\int_x^\infty \int_{r=0}^R \frac{\rho\,2\pi r}{4\pi\,\varepsilon_o(x^2+r^2)}\ \frac{x}{\sqrt{x^2+r^2}}\,dr\,dx$$

But space charge density $\rho = \lambda/\pi R^2$ where R is the (outer) radius of the paraboloid, and $\lambda = i/v$, so that $\rho = i/\pi R^2 v$

$$E_p = -\int_x^\infty \int_{r=0}^R \frac{i\,x}{2\pi\,\varepsilon_o v R^2}\ \frac{r\,dr}{\sqrt{(x^2+r^2)^3}}\,dx$$

$$E_p = -\int_x^\infty \int_{r=0}^R \frac{i\,x}{2\pi\,\varepsilon_o v R^2}\left[\frac{-1}{\sqrt{x^2+r^2}}\right]_0^R dx$$

$$E_p = -\int_x^\infty \frac{i\,x}{2\pi\,\varepsilon_o v R^2}\left[\frac{-1}{\sqrt{R^2+x^2}} + \frac{1}{x}\right] dx$$

$$E_p = -\frac{i}{2\pi \varepsilon_o V}\int_x^{\infty}\left[\frac{-x}{R^2\sqrt{R^2+x^2}} + \frac{1}{R^2}\right]dx$$

We rewrite equation (C-3) as

$$R^2 = a + b\,x \qquad \text{(C-4)}$$

where $a = R_o^2$ (C-5)

and $b = ki/\pi \cdot \varepsilon_o V^2.$ (C-6)

Equation (C-6) is equation (3.17).

We have

$$E_p = -\frac{i}{2\pi \varepsilon_o V}\int_x^{\infty}\left[\frac{-x}{(a+bx)\sqrt{a+bx+x^2}} + \frac{1}{a+bx}\right]dx \qquad \text{(C-7)}$$

The method of solution for this equation is obvious*.

Replace the upper limit of integration by A where A is finite, and take $\lim_{A\to\infty} E_{pA}$ where

$$E_{pA} = -\frac{i}{2\pi \varepsilon_o V}\int_x^{A}\left[\frac{1}{a+bx} - \frac{x}{(a+bx)\sqrt{a+bx+x^2}}\right]dx$$

To fit the integrals 160 and 195 in Peirce's tables, manipulate the integrand

*Not to me, but to my colleague, Dr. Leonard Geller.

$$E_{pA} = \frac{-i}{2\pi\varepsilon_0 V}\left[\int_x^A \frac{dx}{a+bx} - \frac{1}{b}\int_x^A \frac{dx}{\sqrt{a+bx+x^2}} + \frac{a}{b}\int_x^A \frac{dx}{(a+bx)\sqrt{a+bx+x^2}}\right]$$

$$E_{pA} = \frac{-i}{2\pi\varepsilon_0 V}\left[\frac{1}{b}\ln\frac{a+bA}{a+bx} - \frac{1}{b}\ln\frac{\sqrt{a+ba+A^2}+A+\frac{b}{2}}{\sqrt{a+bx+x^2}+x+\frac{b}{2}}\right.$$
$$\left. + \frac{a}{b}\cdot\frac{1}{a}\ln\frac{(a+bx)\left[2a^2+(b^2-2a)(a+bA)-2ba\sqrt{a+bA+A^2}\right]}{(a+bA)\left[2a^2+(b^2-2a)(a+bx)-2ba\sqrt{a+bx+x^2}\right]}\right]$$

$$= \frac{-i}{2\pi\varepsilon_0 Vb}\ln\frac{\left(\sqrt{a+bx+x^2}+x+\frac{b}{2}\right)\left(ab+b^2A-2aA-2a\sqrt{a+bA+A^2}\right)}{\left(\sqrt{a+bA+A^2}+A+\frac{b}{2}\right)\left(ab+b^2x-2ax-2a\sqrt{a+bx+x^2}\right)}$$

$$\lim_{A\to\infty} E_{pA} = \frac{-i}{2\pi\varepsilon_0 Vb}\ln\frac{\left(\sqrt{a+bx+x^2}+x+\frac{b}{2}\right)A(b^2-4a)}{2A\left(ab+b^2x-2ax-2a\sqrt{a+bx+x^2}\right)}$$

$$E_p = -\frac{i}{2\pi\varepsilon_0 Vb}\ln\frac{\left(\sqrt{a+bx+x^2}+x+\frac{b}{2}\right)(b^2-4a)}{2\left(ab+b^2x-2ax-2a\sqrt{a+bx+x^2}\right)} \qquad \text{(C-8)}$$

This is the field at the origin for the paraboloid of equation (C-3) truncated at an abscissa x. Substituting in part for b

$$E_p = -\frac{V}{2k}\ln\frac{\left(\sqrt{a+bx+x^2}+x+\frac{b}{2}\right)(b^2-4a)}{2\left(ab+b^2x-2ax-2a\sqrt{a+bx+x^2}\right)} \qquad \text{(C-9)}$$

Substitution of the identity $a = R_0^2$ (equation C-5) in equation (C-9) yields equation (3.16).

As a special case equation (C-8) may be simplified by taking $x = 0$, that is, truncating the paraboloid at the origin.

$$E_{po} = -\frac{V}{2k}\ln\frac{\left(\sqrt{a}+\frac{b}{2}\right)(b^2-4a)}{2(ab-2a\sqrt{a})}$$

$$E_{po} = -\frac{V}{2k}\ln\frac{\left(\sqrt{a}+\frac{b}{2}\right)(b+2\sqrt{a})(b-2\sqrt{a})}{2a(b-2\sqrt{a})}$$

$$E_{po} = -\frac{V}{2k}\ln\left(\frac{b+2\sqrt{a}}{2\sqrt{a}}\right)^2$$

$$E_{po} = -\frac{V}{k}\ln\left(1+\frac{b}{2R_0}\right) \qquad \text{(C-10)}$$

or

$$E_{po} = - \frac{v}{k} \ln\left(1 + \frac{ki}{2\pi \varepsilon_o v^2 R_o}\right) \qquad \text{(C-11)}$$

Equation (C-11) is equation (3.18) for the axial field at the origin of the paraboloid truncated at the origin.

To derive equation (3.19) we proceed as follows.

Equation (C-3) for the paraboloid

$$R^2 = R_o^2 + (ki/\pi \varepsilon_o v^2)\, x \qquad \text{(C-12)}$$

may be abbreviated as

$$R^2 = R_o^2 + b\,x \qquad \text{(C-13)}$$

where

$$b = ki/\pi \varepsilon_o v^2 \qquad \text{(C-14)}$$

as has been shown in equations (C-4), (C-5), (C-6).
The resulting equations (C-9) and (C-11) are the expressions for the axial field at the origin of the paraboloid truncated at an abscissa x and having a radius R_o at the origin. When $x = 0$, (C-9) simplifies to (C-11).

Now suppose we require the axial field at the origin for a paraboloid translated along the X-axis so that the paraboloid is truncated at $x = x_o$ where its radius is R_o. The equation (C-13) becomes

$$R^2 = R_o^2 + b\,(x - x_o) \qquad \text{(C-15)}$$

which may be replaced by

$$R^2 = (R_o')^2 + b\,x \qquad \text{(C-16)}$$

where

$$(R_o')^2 = R_o^2 - b\,x_o = a - b\,x_o \qquad \text{(C-17)}$$

using the notation of equation (C-5).

Thus if we wish to find the axial field E_p' at a distance $2r_2$ from a truncated paraboloid whose radius at its plane of truncation is R_o, in equation (C-9) we will

$$\text{replace } a \text{ by } a - bx_o \qquad \text{(C-18)}$$

and set

$$x = x_o = 2r_2. \qquad \text{(C-19)}$$

Thus (C-9) becomes

$$E_p = -\frac{v}{2k} \ln \frac{\left(\sqrt{a-bx_o+bx_o+x_o^2}+x_o+\frac{b}{2}\right)(b^2-4a+4bx_o)}{2(ab-b^2x_o+b^2x_o-2ax_o+2bx_o^2-2a\sqrt{a+x_o^2}+2bx_o\sqrt{a+x_o^2})} \qquad \text{(C-20)}$$

$$E_p = -\frac{v}{2k} \ln \frac{\left(\sqrt{R_o^2+x_o^2}+x_o+\frac{b}{2}\right)(b^2-4R_o^2+4bx_o)}{2\left[R_o^2 b-(2R_o^2-2bx_o)\left(x_o+\sqrt{R_o^2+x_o^2}\right)\right]} \qquad \text{(C-21)}$$

By substituting equation (C-19) in equation (C-21) and dividing numerator and denominator of the logarithm term by 4, we obtain equation (3.19). For purposes of calculation it is convenient to proceed further.

Since $b = ki/\pi \epsilon_o v^2$ if we may substitute equation (7.3) for i, we get

$$b = \frac{2k^2 V_1^2}{3v^2 r_2} \qquad \text{(C-22)}$$

If we substitute the usual values for k V_1 and v we have

$$b = \frac{2}{3}\frac{(4\times10^{-4})^2\,(10^5)^2}{(100)^2\;r_2} = 0.1067/r_2 \qquad \text{(C-23)}$$

Further $\quad x = x_o = 2r_2 \qquad$ (C-24)

Thus dividing a factor 4 out of numerator and denominator under the logarithm

$$E_p = -\frac{v}{2k} \ln \frac{\left(\left\{\sqrt{R_o^2+4r_2^2}+2r_2\right\}+\left\{\frac{.0533}{r_2}\right\}\right)\left(\left\{\frac{.0533}{r_2}\right\}^2-\left\{R_o^2-.2133\right\}\right)}{\left(R_o^2\left\{\frac{.0533}{r_2}\right\}-\left\{R_o^2-.2133\right\}\left\{\sqrt{R_o^2+4r_2^2}+2r_2\right\}\right)} \qquad \text{(C-25)}$$

If we let

$$A = \left\{\sqrt{R_o^2 + 4\,r_2^2} + 2\,r_2\right\} \qquad (C\text{-}26)$$

$$B = \left\{.0533/r_2\right\} \qquad (C\text{-}27)$$

$$C = \left\{R_o^2 - .2133\right\} \qquad (C\text{-}28)$$

then

$$E_p = -\frac{v}{2k}\,\ln\frac{(A+B)(B^2-C)}{(R_o^2B-AC)} \qquad (C\text{-}29)$$

Equations (C-26) to (C-29) give expressions for the axial field of a truncated paraboloid at a distance $2r_2$ from the truncation. These equations are used in the calculation of Section 9.

APPENDIX D: Dimensional Analysis Applied to the Corona Discharge Problem

In Section 4 the result of applying dimensional analysis to the corona discharge problem is stated. This appendix displays the derivation of this result given as equation 4.1 of Section 4.

The Buckingham Pi Theorem is outlined in any standard text on fluid mechanics, such as that by R. C. Binder. It states that in a physical situation characterized by n physical quantities, if m fundamental dimensions are involved in these quantities, then a single functional relationship can be written involving (n-m) independent non-dimensional parameters.

$$\text{Given: } F(x_1, x_2, x_3, \ldots\ldots\ldots x_n) = 0, \quad (1)$$

in which the n x's involve m primary dimensions, then (1) can be written alternately as

$$\phi(\pi_1, \pi_2 \ldots\ldots\ldots \pi_{n-m}) = 0. \quad (2)$$

Any method which finds n-m independent non-dimensional parameters is useable. The one used below is convenient.

The symbols used below are defined in Section 4.

In the corona discharge problem, the physical quantities involved are: current i, permittivity ϵ_0, point potential V_1, speed v, mobility k, and a length ℓ. Therefore n = 6.

There are four fundamental dimensions involved in the above. Therefore m = 4. A satisfactory set of four are Q, V, L, and T, standing for charge, potential, length, and time, respectively. It is therefore clear from the above and Buckingham's Theorem that a relationship of the form $\phi(\pi_1, \pi_2) = 0$ can be written, where there are two non-dimensional combinations. An alternate form is

$$\pi_1 = f(\pi_2) \quad (3)$$

Proceeding as in Binder, Chapter 6, we set

$$\pi_1 = i\, \varepsilon_0^{\,a}\, v^b\, V_1^{\,c}\, \ell^d \qquad (4)$$

where a, b, c, and d are exponents whose values are to be determined such that π_1 is dimensionless. Inserting the dimensions of the variables in (4) we obtain:

$$0 = \frac{Q}{T}\left(\frac{Q}{VL}\right)^a \left(\frac{L}{T}\right)^b (V)^c (L)^d \qquad (5)$$

which leads to four simultaneous equations in the exponents.

$$\begin{aligned} 1+a &= 0 \\ -a+b+d &= 0 \\ -a+c &= 0 \\ -1-b &= 0 \end{aligned} \qquad (6)$$

The solution of equations (6) is:

$$\begin{aligned} a &= -1 \\ b &= -1 \\ c &= -1 \\ d &= 0 \end{aligned} \qquad (7)$$

Substituting (7) in (4) yields:

$$\pi_1 = \frac{i}{\varepsilon_0\, v\, V_1} \qquad (8)$$

Proceeding similarly we let

$$\pi_2 = k\, \varepsilon_0^{\,e}\, v^f\, V_1^{\,g}\, \ell^h \qquad (9)$$

From which we can find that

$$\pi_2 = \frac{v\ell}{kV_1} \qquad (10)$$

Substituting (8) and (10) in (3), we obtain:

$$\frac{i}{\varepsilon_0\, v\, V_1} = f\left(\frac{v\,\ell}{k\,V_1}\right) \qquad (11)$$

The normal course of a dimensional analysis stops with equation (6). Additional information is gathered from experiment, and using the two non-dimensional coordinates indicated in (11), the form of the function f is determined.

In this case, however, we can get more information about the function from the necessary conditions that the expression be finite for zero voltage and zero velocity. We proceed as follows:

Assume that the function in (6) can be expanded as a Laurent series about the origin. The expression obtained is

$$\frac{i}{\varepsilon_0 v V_1} = \sum_{n=0}^{\infty} a_n \left(\frac{v l}{k V_1}\right)^n + \sum_{m=1}^{\infty} b_m \left(\frac{k V_1}{v l}\right)^m \tag{12}$$

or:

$$\frac{i}{\varepsilon_0} = \sum_{n=0}^{\infty} a_n v V_1 \left(\frac{v l}{k V_1}\right)^n + \sum_{m=1}^{\infty} b_m v V_1 \left(\frac{k V_1}{v l}\right)^m \tag{13}$$

Multiplying and rearranging gives

$$\frac{i}{\varepsilon_0} = \sum_{n=0}^{\infty} a_n \left(\frac{l}{k}\right)^n \frac{v^{n+1}}{V_1^{n-1}} + \sum_{m=1}^{\infty} b_m \left(\frac{k}{l}\right)^m \frac{V_1^{m+1}}{v^{m-1}} \tag{14}$$

Writing out the first few terms of each series

$$\frac{i}{\varepsilon_0} = a_0 v V_1 + a_1\left(\frac{l}{k}\right)v^2 + a_2\left(\frac{l}{k}\right)^2 \frac{v^3}{V_1} + \text{higher order terms}$$

$$+ b_1\left(\frac{k}{l}\right)V_1^2 + b_2\left(\frac{k}{l}\right)^2 \frac{V_1^3}{v} \quad + \text{higher order terms}$$

(15)

It is obvious from this expression that if the current is to be finite when V_1 is zero, a_2 and all higher order a's must be zero. Similarly, if the current is to be bounded when v is zero, b_2 and all higher order b's must be zero. Making these substitutions, the current becomes

$$\frac{i}{\varepsilon_0} = a_0 v V_1 + a_1 \frac{\ell}{k} v^2 + b_1 \frac{k}{\ell} V_1^2 \tag{16}$$

Rearranging and supplying new constants, we obtain equation 4.1 of Section 4:

$$i = \varepsilon_0 \left(FkV_1^2/\ell + GvV_1 + H\ell v^2/k\right) \tag{17}$$