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OCTOBER 1956

Prediction Of The Motion Of Missiles Acted On By Non-Linear Forces And Moments

CHARLES H. MURPHY

DEPARTMENT OF THE ARMY PROJECT No. 5B03-03-001
ORDNANCE RESEARCH AND DEVELOPMENT PROJECT No. TB3-0108

BALLISTIC RESEARCH LABORATORIES

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A B E R D E E N   P R O V I N G   G R O U N D,   M A R Y L A N D

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ABSTRACT

The application of the usual techniques of non-linear mechanics to fourth order non-linear systems is severely handicapped by algebraic complexities. It is shown that for an important subset of the set of fourth order systems, the use of the complex variable allows the quick derivation of the required results. This technique is applied in some detail to the prediction of missile yawing motion. In this discussion the useful concept of an amplitude plane is introduced. Comparison of the theory with the results of exact computations indicate the value of the equivalent linearization approach.

The effects of gravity-induced yaw of repose and of small aerodynamic asymmetries on the general non-linear problem are discussed in an appendix.

TABLE OF SYMBOLS*

A,B,C,D	complex constants in Eq. (2)
a,b,c,d	real constants in Eq. (58)

$$A = \frac{\bar{\nu}\, T'(K^2)}{\tilde{\phi}_1' - \tilde{\phi}_2'}$$

a_1, a_2	real coefficients in Eq. (16)
b_1, b_2	real coefficients in Eq. (16)
c_i	real coefficients in Eqs. (1)

$$D = (b - c)^2 + 4\,ad$$

d	diameter

$$G = \gamma' - \left[\frac{\rho d^3}{m}\,(K_D - k_2^{-2} K_H) - J_g + i\bar{\nu}\right]\gamma$$

g_T	trajectory component of the gravitational acceleration
g_2, g_3	components of the gravitational acceleration perpendicular to the missile's axis

$$H = \frac{\rho d^3}{m}\left[\ell K_L - K_D + k_2^{-2}\,(K_H - \ell K_{MA})\right]$$

$$J_i = \frac{\rho d^3}{m}\,K_i$$

$$J_g = \frac{g_T\, d}{u^2}$$

$$K_i = \sum_{k=0}^{n} K_{i_{\delta^{2k}}}\,\delta^{2k} \qquad i = H, L, M, MA, T$$

*Only those symbols which appear in the body of this report are listed here. Symbols which are introduced in the appendices appear close to their definitions.

$K_i = K_{i0}\, e^{-\alpha_i p}$; $i = 1, 2$ — amplitude of i - th frequency

K_H — moment coefficient due to cross angular velocity

K_L — lift force coefficient due to yaw

K_M — moment coefficient due to yaw (static moment coefficient)

K_{MA} — moment coefficient due to cross acceleration

K_T — Magnus moment coefficient

$K_i(p)$ — parametric function for damping correction to first approximation

k_1 — axial radius of gyration in calibers

k_2 — transverse radius of gyration in calibers

ℓ — cosine of yaw angle

$M = \frac{\rho d^3}{m} \left[\ell\, k_2^{-2}\, K_M\right]$

m — mass

$m = \left| m_1 \right|$

m_1, m_2 — complex constants in Eq. (3)

$P(x,y)$ — function in Eq. (58)

p — independent variable

p — arclength along the trajectory in calibers

$Q(x,y)$ — function in Eq. (58)

$s = \frac{m_2}{m_1}$

$T = \frac{\rho d^3}{m}\, \ell \left[K_L - k_1^{-2}\, K_T\right]$

u — magnitude of velocity

x,y — real dependent variables in Eqs. (1)

α_i exponential damping coefficient of i-th frequency

$$\mathscr{V} = \frac{(g_2 + ig_3)d}{u^2} - J_g \lambda$$

$\delta = \sqrt{\lambda\bar{\lambda}}$ (sine of yaw angle)

$$\left[\delta^{2k}\right]_{e1} = \frac{1}{2\pi} \int_0^{2\pi} \delta^{2k} \left[1 + \frac{K_2}{K_1} \cos \phi\right] d\phi$$

$$\left[\delta^{2k}\right]_{e2} = \frac{1}{2\pi} \int_0^{2\pi} \delta^{2k} \left[1 + \frac{K_1}{K_2} \cos \phi\right] d\phi$$

θ arg m_1

λ complex dependent variable in Eq. (3) or complex yaw

$\bar{\nu}$ $\dfrac{k_1^2}{k_2^2} \dfrac{\omega_1 d}{u}$ (gyroscopic spin)

ζ complex dependent variable in Eq. (2)

ρ air density

σ arg C

$\phi_i = \phi_{i0} + \phi_i' p$ phase angle of i-th frequency

$\phi = \tilde{\phi}_1 - \tilde{\phi}_2$

$\psi(p)$ parametric function for frequency correction of first approximation

ω arg D

ω_1 axial spin

$\sim$ tilde superscript denotes quantities appearing in the epicyclic first approximation of the non-linear equation

1. INTRODUCTION

One of the most challenging current problems in exterior ballistics is the prediction of missile motion for missiles flying at large yaw angles.* Highberg[1] and Zaroodny[2] have treated the special case of circular yawing motion while the case of almost circular motion has been rather elegantly analysed by Davis, Follin, and Blitzer.[3] The problem of more general motion for cubic static and Magnus moments has been attacked by Lietmann[4] but this treatment is obscured by lengthy algebraic manipulations and does not give its results in a convenient form for the exterior ballistician.

There are two difficulties in extending the usual methods of non-linear mechanics to the problem of missile yawing motion. The first difficulty is primarily algebraic and the second conceptual. Since the yawing motion has two degrees of freedom, the problem requires the solution of a fourth order system of equations and, hence, the application of the Kryloff-Bogoliuboff[5] techniques can require rather complicated algebraic operations. Secondly, vibrations of this fourth order system can take on two frequencies so that the averaging step of the K - B method would require an averaging of two different frequencies. Although this is mathematically possible, the precise meaning of such an average is vague.

In this report it will be shown that for an important subset of the set of fourth order differential systems, the fourth order equation in a real dependent variable can be replaced by a second order analytic equation in a complex variable. This use of the complex variable introduces important simplifications into the non-linear problem. In Reference 6 it is shown that with the proper selection of geometrical

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*The angle between the missile's axis and the tangent to its trajectory is called the yaw angle. The yawing motion is usually described by two variables. Two common choices are the Eulerian angles locating the missile axis in wind-fixed coordinates and the direction cosines of the velocity vector with respect to missile-fixed coordinates.

variables, the system of differential equations describing the yawing motion can be a member of this subset. In this report it will be shown that for a large class of non-linearities the averaging step of the K-B method need only be done for a single frequency and that most important aerodynamic non-linearities lie in this class.

By means of an "amplitude plane" the results of the analysis are presented in a simple and revealing form. It is shown that the whole character of the motion can be described when the location and nature of a small number of singularities are determined. The effect of a cubic Magnus moment is discussed in some detail. Finally, predictions for limit motion of two different cases of quintic Magnus moments are compared with exact numerical integrations made at the Naval Proving Ground.[7]

Although only the homogeneous part of the complex yaw equation is treated in the report proper, the effect of gravity-induced "yaw of repose" and the effect of small asymmetries is considered in an appendix. The direct substitution method of Reference 6 is used and the results are compared with additional Naval Proving Ground calculations.[8]

2. EPICYCLIC SUBSET

The most general linear homogeneous fourth order system of differential equations with constant coefficients may be written in the form

$$x'' = c_1 x' + c_2 x + c_3 y' + c_4 y \tag{1a}$$

$$y'' = c_5 x' + c_6 x + c_7 y' + c_8 y \tag{1b}$$

where c_i are constants and primes indicate derivatives with respect to the independent variable, p. If Eq. (1b) is multiplied by i and added to Eq. (1a) and the complex variable $\xi = x + iy$ is introduced, the following complex equation may be written

$$\xi'' + A \xi' + B \xi + C \bar{\xi}' + D \bar{\xi} = 0 \tag{2}$$

where the complex coefficients, A, B, C, D, are linear combinations of the c_i. When C and D vanish, Eq. (2) becomes an analytic differential equation in the complex variable, ξ, and can be easily solved. Because of this simplicity it is of some interest to determine the conditions under which it is possible to transform Eq. (2) to this special form by means of a reversible linear transformation. For a reason which we will give later in this section this subset of the set of all differential systems of the fourth order with constant coefficients will be called the epicyclic subset.

The most general linear transformation can be written in the form

$$\lambda = m_1 \xi + m_2 \bar{\xi} \tag{3}$$

where m_1 and m_2 are complex constants. The inverse equation can be obtained by eliminating $\bar{\xi}$ between Eq. (3) and the conjugate of Eq. (3).

$$\therefore \quad \xi = \frac{\bar{m}_1 \lambda - m_2 \bar{\lambda}}{m_1 \bar{m}_1 - m_2 \bar{m}_2} \tag{4}$$

Thus the restriction to reversible transformations is equivalent to the relation

$$m_1 \bar{m}_1 - m_2 \bar{m}_2 \neq 0 \tag{5}$$

Without loss of generality we can assume that m_1 is non-zero*, and, hence, relations (3) and (5) become

$$\lambda = m e^{i\theta} (\xi + s\bar{\xi}) \tag{6a}$$

$$m^2 (1 - s\bar{s}) \neq 0 \tag{6b}$$

where $m e^{i\theta} = m_1$,

$$s = \frac{m_2}{m_1}, \text{ and}$$

m is real.

$$\therefore \quad \xi = \frac{e^{-i\theta}\lambda - s e^{i\theta} \bar{\lambda}}{m(1 - s\bar{s})} \tag{7}$$

*If m_1 were zero, we would consider the conjugates of Eq. (2) and (3).

Substituting Eq. (7) in Eq. (2),

$$\lambda'' + (A - C\bar{s})\lambda' + (B - D\bar{s})\lambda$$

$$- e^{2i\theta} \left[s\,\bar{\lambda}'' + (As - C)\bar{\lambda}' + (Bs - D)\bar{\lambda} \right] = 0 \qquad (8)$$

If $\bar{\lambda}''$ is eliminated between Eq. (8) and its conjugate,

$$(1 - s\bar{s})\lambda'' + (A - C\bar{s} + \bar{C}s - \bar{A}\,s\bar{s})\lambda'$$

$$+ (B - D\bar{s} + \bar{D}s - \bar{B}\,s\bar{s})\lambda - e^{2i\theta} \left\{ \left[\bar{C}\,s^2 + (A - \bar{A})s - C \right] \bar{\lambda}' \qquad (9)$$

$$+ \left[\bar{D}\,s^2 + (B - \bar{B})s - D \right] \bar{\lambda} \right\} = 0$$

where $(1 - s\bar{s})$ is non zero by equation (6b).

The requirement that Eq. (2) belong to the epicyclic subset means that it must be possible to select s so that the coefficients of $\bar{\lambda}'$ and $\bar{\lambda}$ vanish. From Eq. (9) it can be seen that m and θ cannot make the coefficients of $\bar{\lambda}'$ and $\bar{\lambda}$ vanish and so they may be arbitrarily selected. If we make the definitions $A = A_1 + iA_2$, $B = B_1 + iB_2$, $C = |C|e^{i\sigma}$, and $D = |D|\,e^{i\omega}$, then we have the following pair of quadratic equations for s.

$$|C|\,s^2 + 2iA_2 s e^{i\sigma} - |C|\,e^{2i\sigma} = 0 \qquad (10)$$

$$|D|\,s^2 + 2iB_2 s e^{i\omega} - |D|\,e^{2i\omega} = 0 \qquad (11)$$

If neither C nor D vanish, the solutions of Eqs. (10 - 11) are:

$$s = \left[\frac{A_2}{|C|} \pm \sqrt{\left|\frac{A_2}{C}\right|^2 - 1} \right] e^{i(\sigma - \frac{\pi}{2})} \qquad (12)$$

$$s = \left[\frac{B_2}{|D|} \pm \sqrt{\left|\frac{B_2}{D}\right|^2 - 1} \right] e^{i(\omega - \frac{\pi}{2})} \qquad (13)$$

Since s is unity when either $\left|\frac{A_2}{C}\right|$ or $\left|\frac{B_2}{D}\right|$ is less than or equal to one and this is contrary to relation (6b), we have the restrictions:

$$\left|\frac{A_2}{C}\right| > 1, \quad \left|\frac{B_2}{D}\right| > 1 \qquad (14)$$

Under restriction (14), s can satisfy both Eq. (12) and Eq. (13) only if $\frac{A_2}{|C|} = \frac{B_2}{|D|}$ and $\sigma = \omega$ or $\frac{A_2}{|C|} = \frac{-B_2}{|D|}$ and $\sigma = \omega + \pi$. This can be given in the single equation

$$\frac{A_2}{C} = \frac{B_2}{D} \tag{15}$$

By means of Eqs. (10 - 15) we can now state the following theorem:

Theorem Eq. (2) is a member of the epicyclic subset if one of the following conditions is satisfied:

(1) $C = D = 0$

(2) $C = 0,\ A_2 = 0,\ \left|\frac{B_2}{D}\right| > 1$

(3) $D = 0,\ B_2 = 0,\ \left|\frac{A_2}{C}\right| > 1$

(4) $\frac{A_2}{C} = \frac{B_2}{D},\ \left|\frac{A_2}{C}\right| > 1.$

For the trivial case (1), s is zero; for the other cases it is fixed by either Eq. (12) or Eq. (13).

By means of such a linear transformation all equations belonging to the epicyclic subset may be written in the form

$$\lambda'' + (a_1 + ia_2)\lambda' + (b_1 + ib_2)\lambda = 0 \tag{16}$$

where a_1, a_2, b_1, and b_2 are constants. Substituting $\lambda = e^{(\int \alpha + i\phi')p}$ in Eq. (16) and separating real and imaginary parts, we see that ϕ' and α must satisfy the following equations:

$$\phi'^2 + a_2\,\phi' - b_1 + \alpha(a_1 - \alpha) = 0 \tag{17}$$

$$\alpha = \frac{a_1\,\phi' + b_2}{2\,\phi' + a_2} \tag{18}$$

Under the usual assumption of small damping during a cycle, the α term in Eq. (17) can be neglected and it reduces to

$$\phi'_j = \frac{-a_2 \pm \sqrt{a_2^2 + 4 b_1}}{2}, \quad j = 1,2. \tag{19}$$

$$\therefore \quad a_2^2 + 4 b_1 > 0$$

If the above inequality is not satisfied or the damping is not small over a cycle, Eqs. (17 - 18) lead to the need for the solution of a fourth order equation. This difficulty can be avoided by not separating Eq. (16) into real and imaginary parts after making the substitution.

All the cases treated by the methods of this report will be restricted to those with small damping over a cycle and Eq. (19) will apply.

Since $a_2 = -(\phi_1' + \phi_2')$, symmetric forms of the equation for the damping exponents can be obtained from Eq. (18).

$$\alpha_1 = \frac{a_1 \phi_1' + b_2}{\phi_1' - \phi_2'} \tag{20}$$

$$\alpha_2 = \frac{a_1 \phi_2' + b_2}{\phi_2' - \phi_1'} \tag{21}$$

The equation for the general solution to Eq. (16), therefore, is

$$\lambda = K_1 e^{i(\phi_{10} + \phi_1' p)} + K_2 e^{i(\phi_{20} + \phi_2' p)} \tag{22}$$

where $K_j = K_{j0} e^{-\alpha_j p}$; ϕ_j' and α_j are given by Eqs. (19 - 21); and ϕ_{j0} and K_{j0} are constants. This solution is a linear combination of two complex vectors which are rotating at certain fixed frequencies and are exponentially damped. Since the curve swept out in the complex plane is called an epicycle, the reason for the name of this subset is now clear.

The solution for ξ when Eq. (2) belongs to the epicyclic subset can be obtained by use of Eq. (4), s is determined to be a solution of Eqs. (10 - 11), m and θ are arbitrary and for convenience m will be made unity and θ will be fixed at zero.

$$\therefore \quad \xi = \frac{\lambda - s\bar{\lambda}}{1 - s\bar{s}}$$

$$= \frac{K_1 e^{i\phi_1} + K_2 e^{i\phi_2} + s K_1 e^{-i\phi_1} + s K_2 e^{-i\phi_2}}{1 - s\bar{s}} \tag{23}$$

where $\phi_j = \phi_{j0} + \phi_j' p$.

3. SOLUTION OF THE NON-LINEAR EPICYCLIC EQUATION

Although the treatment of non-linear fourth order systems is usually quite laborious, the procedure for systems which may be linearized to members of the epicyclic subset is much simpler. In this section the approximate solution to equations of the form of Eq. (16) but with coefficients a_1, a_2, b_1, b_2, which are functions of λ, $\bar{\lambda}$, λ', and $\bar{\lambda}'$, will be considered. The method of solution which will be employed will be essentially that of Kryloff and Bogoliuboff.[5]

This method is based on a pertubation of the solution of the linear equation with no damping. If this solution is identified by $\tilde{\lambda}$ then

$$\tilde{\lambda} = K_{10} e^{i\tilde{\phi}_1} + K_{20} e^{i\tilde{\phi}_2} \tag{24}$$

where

$$\tilde{\phi}_j = \tilde{\phi}_{j0} + \tilde{\phi}_j' p$$

$$\tilde{\phi}_j' = \frac{-\tilde{a}_2 \pm \sqrt{\tilde{a}_2^2 + 4\tilde{b}_1}}{2}$$

$$\tilde{a}_2^2 + 4\tilde{b}_1 > 0$$

$\tilde{a}_2$, $\tilde{b}_1$ constant parts of a_2 and b_1 (their values when $\lambda' = \lambda = 0$.); and

K_{j0}, $\tilde{\phi}_{j0}$ are constants.

The complete non-linear differential equation can be written in the form:

$$\lambda'' + i\tilde{a}_2 \lambda' + \tilde{b}_1 \lambda = - \left[a_1 + i(a_2 - \tilde{a}_2)\right] \lambda' - \left[(b_1 - \tilde{b}_1) + ib_2\right] \lambda$$

$$= f(\lambda, \bar{\lambda}, \lambda', \bar{\lambda}'). \qquad (25)$$

where a_1, a_2, b_1, b_2 are functions of λ, $\bar{\lambda}$, λ', $\bar{\lambda}'$.

The pertubation process is quite similar to the well-known method of variation of parameters. First the solution is assumed to be that expressed by Eq. (24) with the constants K_{j0}, $\tilde{\phi}_{10}$ replaced by unknown functions of p, $K_j(p)$ and $\psi_j(p)$.

$$\therefore \lambda = K_1 e^{i(\psi_1 + \tilde{\phi}_1)} + K_2 e^{i(\psi_2 + \tilde{\phi}_2)} \qquad (26)$$

This equation is then differentiated to yield:

$$\lambda' = i\tilde{\phi}_1' K_1 e^{i(\psi_1 + \tilde{\phi}_1)} + i\tilde{\phi}_2' K_2 e^{i(\psi_2 + \tilde{\phi}_2)}$$

$$+ (K_1' + i\psi_1' K_1)e^{i(\psi_1 + \tilde{\phi}_1)} + (K_2' + i\psi_2' K_2)e^{i(\psi_2 + \tilde{\phi}_2)} \qquad (27)$$

The last two expressions in Eq. (27) are set equal to zero so that

$$(K_1' + i\psi_1' K_1)e^{i(\psi_1 + \tilde{\phi}_1)} + (K_2' + i\psi_2' K_2)e^{i(\psi_2 + \tilde{\phi}_2)} = 0 \qquad (28)$$

and Eq. (27) then simplifies to

$$\lambda' = i\tilde{\phi}_1' K_1 e^{i(\psi_1 + \tilde{\phi}_1)} + i\, \tilde{\phi}_2' K_2 e^{i(\psi_2 + \tilde{\phi}_2)} \qquad (29)$$

Differentiating again we see that

$$\lambda'' = - \tilde{\phi}_1'^2 K_1 e^{i(\psi_1 + \tilde{\phi}_1)} - \tilde{\phi}_2'^2 K_2 e^{i(\psi_2 + \tilde{\phi}_2)} \qquad (30)$$

$$+ i\, \tilde{\phi}_1'(K_1' + i\psi_1' K_1)e^{i(\psi_1 + \tilde{\phi}_1)} + i\, \tilde{\phi}_2'(K_2' + i\psi_2' K_2)e^{i(\psi_2 + \tilde{\phi}_2)}$$

If Eqs. (26, 29, 30) are substituted in Eq. (25), the terms without derivatives of the parameters must satisfy the homogeneous equation and therefore cancel.

$$\therefore \quad i\tilde{\phi}_1'(K_1' + i\psi_1'K_1)e^{i(\psi_1+\tilde{\phi}_1)} + i\tilde{\phi}_2'(K_2' + i\psi_2'K_2)e^{i(\psi_2+\tilde{\phi}_2)} = f(p) \qquad (31)$$

$(K_2' + i\psi_2'K_2)$ can be eliminated between Eqs. (28) and (31).

$$\therefore \quad \frac{K_1'}{K_1} + i\psi_1' = \frac{f(p)\, e^{-i(\psi_1 + \tilde{\phi}_1)}}{i(\tilde{\phi}_1' - \tilde{\phi}_2')K_1} \qquad (32)$$

$$= -(\tilde{\phi}_1' - \tilde{\phi}_2')^{-1}\left\{-\left[a_1 + i(a_2 - \tilde{a}_2)\right]\left[\tilde{\phi}_1' + \tilde{\phi}_2'\frac{K_2}{K_1}e^{i\phi}\right] - \left[b_2 - i(b_1 - \tilde{b}_1)\right]\left[1 + \frac{K_2}{K_1}e^{i\phi}\right]\right\}$$

where $\phi = \tilde{\phi}_2 + \psi_2 - \tilde{\phi}_1 - \psi_1$.

This simple derivation of Eq. (32) shows the considerable reduction in algebraic complication which the use of the complex variable and the restriction to the epicyclic subset have introduced.

At this point in the K-B method, due to the difficulty in solving the differential equations for the parametric functions, averages of the right sides of the parametric equations are taken. In general, the right hand side of Eq. (32) has two basic frequencies and the necessary averages have a rather vague physical meaning. If only the difference frequency (ϕ) is allowed in the functional dependence of a_1, a_2, b_1, and b_2 on λ, $\bar{\lambda}$, λ', and $\bar{\lambda}'$, a single average would be needed and this average would have a simple meaning. This means that only those non-linearities which can be associated with a body of revolution are considered.[15] For example, under this restriction there are four possible quadratic combinations of λ, $\bar{\lambda}$, λ' and $\bar{\lambda}'$:*

* The effect of damping has been neglected in the calculation of λ' and $\bar{\lambda}'$.

$$\delta^2 = \lambda\bar{\lambda} = K_1{}^2 + K_2{}^2 + 2K_1K_2 \cos(\phi) \tag{33}$$

$$\lambda'\bar{\lambda}' = \phi_1'^2 K_1{}^2 + \phi_2'^2 K_2^2 + 2\,\phi_1'\phi_2' K_1K_2 \cos(\phi) \tag{34}$$

$$\lambda\bar{\lambda}' = -i\left[\phi_1' K_1{}^2 + \phi_2' K_2{}^2 + K_1K_2(\phi_1' e^{i\phi} + \phi_2' e^{-i\phi})\right] \tag{35}$$

$$\lambda'\bar{\lambda} = i\left[\phi_1' K_1{}^2 + \phi_2' K_2{}^2 + K_1K_2(\phi_1' e^{-i\phi} + \phi_2' e^{i\phi})\right] \tag{36}$$

From Eq. (33 - 34) we see that both the magnitude of λ and the magnitude of its derivative have the frequency of ϕ and, hence, the averaging will be made over a period of these amplitudes.

It is now necessary to assume that the modal amplitudes, K_j, change slowly over a period of the difference frequency. With this assumption in mind we can average the right side of Eq. (32) over a period of ϕ and separate the result into real and imaginary parts.

$$\therefore\ \frac{K_1'}{K_1} = \frac{-1}{2\pi(\tilde{\phi}_1'-\tilde{\phi}_2')} \int_0^{2\pi} \left\{ a_1\left[\tilde{\phi}_1' + \tilde{\phi}_2' \frac{K_2}{K_1}\cos\phi\right] + b_2\left[1 + \frac{K_2}{K_1}\cos\phi\right] \right.$$

$$\left. + \left[(a_2 - \tilde{a}_2)\tilde{\phi}_2' - (b_1 - \tilde{b}_1)\right] \frac{K_2}{K_1}\sin\phi \right\} d\phi = -\alpha_1 (K_1,K_2) \tag{37}$$

$$\psi_1' = \frac{-1}{2\pi(\tilde{\phi}_1'-\tilde{\phi}_2')} \int_0^{2\pi} \left\{ (a_2 - \tilde{a}_2)\left[\tilde{\phi}_1' + \tilde{\phi}_2' \frac{K_2}{K_1}\cos\phi\right] \right.$$

$$\left. - (b_1 - \tilde{b}_1)\left[1 + \frac{K_2}{K_1}\cos\phi\right] - \left[a_1\tilde{\phi}_2' + b_2\right]\frac{K_2}{K_1}\sin\phi \right\} d\phi\ . \tag{38}$$

From symmetry, the equations for the second mode of oscillation are

$$\frac{K_2'}{K_2} = \frac{-1}{2\pi(\phi_2'-\phi_1')} \int_0^{2\pi} \left\{ a_1 \left[\tilde{\phi}_2' + \tilde{\phi}_1' \frac{K_1}{K_2} \cos \phi\right] + b_2 \left[1 + \frac{K_1}{K_2} \cos \phi\right] - \left[(a_2 - \tilde{a}_2)\tilde{\phi}_1 - (b_1 - \tilde{b}_1)\right] \frac{K_1}{K_2} \sin \phi \right\} d\phi = -\alpha_2 (K_1,K_2) \qquad (39)$$

$$\psi_2' = \frac{-1}{2\pi\ (\tilde{\phi}_2' - \tilde{\phi}_1')} \int_0^{2\pi} \left\{ (a_2 - \tilde{a}_2) \left[\tilde{\phi}_2' + \tilde{\phi}_1' \frac{K_1}{K_2} \cos \phi\right] - (b_1 - \tilde{b}_1) \left[1 + \frac{K_1}{K_2} \cos \phi\right] + \left[a_1\tilde{\phi}_1' + b_2\right] \frac{K_1}{K_2} \sin \phi \right\} d\phi \qquad (40)$$

Thus for each cycle of ϕ, the K-B approximation is a damped epicycle with damping exponents, α_j, given by Eqs. (37) and (39) and with frequencies, ϕ_j', equal to $\tilde{\phi}_j' + \psi_j'$ where the ψ_j''s are given by Eqs. (38) and (40). The damping exponents predicted by Eqs. (37) and (39) for the linear case are precisely those given by Eqs. (20 - 21).

If $\tilde{a}_2$ and $\tilde{b}_1$ had been chosen to be the average values of a_2 and b_1 instead of their values for zero amplitude motion, the expressions for ψ_j' would have been simplified. The implications of such a choice are described in Appendix C. In most cases, however, the advantages of this choice do not outweigh the algebraic complexities which are introduced.

4. APPLICATION TO MISSILE YAWING MOTION

With the aid of the preceding two sections on the approximate solution of non-linear equations of the epicyclic type, the problem of non-linear yawing motion of a symmetric missile in free flight can be easily handled. For this case the complex plane is the plane perpendicular to the missile's axis. The complex vector, λ, lies on the intersection of this plane with the plane of the yaw angle with magnitude equal to the sine of that angle. In Reference 6 it is shown that the general equation of yawing motion in a non-rotating coordinate system is

$$\lambda'' + \left[H + J_g - \frac{\ell'}{\ell} - i\bar{\nu} \right] \lambda' + \left[- M + \ell J_L' - i\bar{\nu}\, T \right] \lambda = G - \frac{\ell'}{\ell} \gamma, \quad (41)$$

where $H = \frac{\rho d^3}{m} \left[\ell K_L - K_D + k_2^{-2}(K_H - \ell K_{MA}) \right]$

$\ell = \sqrt{1 - \delta^2}$ (cosine of yaw angle)

$\delta = \sqrt{\lambda\bar{\lambda}}$ (magnitude of sine of yaw angle)

$$\bar{\nu} = \frac{k_1^{\,2}}{k_2^{\,2}} \quad \frac{\omega_1 d}{u}$$

$$M = \frac{\rho d^3}{m} \left[\ell k_2^{-2} K_M \right]$$

$$T = \frac{\rho d^3}{m} \ell \left[K_L - k_1^{-2} K_T \right]$$

$$J_L' = \frac{\rho d^3}{m} \left[\frac{dK_L(\delta^2)}{d\delta^2} \quad (\delta^2)' \right]$$

$$G = \gamma' - \left[\frac{\rho d^3}{m} (K_D - k_2^{-2} K_H) - J_g + i\bar{\nu} \right] \gamma$$

$$J_g = \frac{g_T d}{u^2}$$

$$\gamma = \frac{(g_2 + ig_3)d}{u^2} - J_g\lambda$$

k_1 axial radius of gyration in calibers

k_2 transverse radius of gyration in calibers

ρ density of air

d diameter

m mass

ω_1 axial spin

u magnitude of velocity

g_T trajectory component of the gravitational acceleration

g_2, g_3 components of gravitational acceleration perpendicular to missile's axis and the K_i's are aerodynamic coefficients which are defined in the Table of Symbols. Since only the homogeneous equation has been considered in Sections 2 and 3, the discussion of the inhomogeneous part of Eq. (41) will be deferred to Appendix A.

The most common non-linear forms of Eq. (41) arise from a dependence of the aerodynamic coefficients on δ^2. Since any function of δ^2 is an even periodic function of $\phi_1 - \phi_2$, its Fourier series expansion is a cosine series and, hence, most of the sine terms in Eqs. (37 - 40) vanish.* Eqs. (37 - 40) can be written as

$$\psi_1' = \frac{-1}{2\pi(\phi_1' - \phi_2')} \int_0^{2\pi} \left\{ \left[M(\delta^2) - M(0)\right] \left[1 + \frac{K_2}{K_1} \cos \phi\right] + \frac{\ell'}{\ell}\, \phi_2'\, \frac{K_2}{K_1} \sin \phi \right\} d\phi \qquad (42)$$

$$\psi_2' = \frac{-1}{2\pi(\phi_2' - \phi_1')} \int_0^{2\pi} \left\{ \left[M(\delta^2) - M(0)\right] \left[1 + \frac{K_1}{K_2} \cos \phi\right] - \frac{\ell'}{\ell}\, \phi_1'\, \frac{K_1}{K_2} \sin \phi \right\} d\phi \qquad (43)$$

---

*Since ℓ' and J_L' are the derivatives of even functions, they are odd functions and so have sine expansions. For these quantities only the sine terms have non-zero contributions.

$$\frac{K_1'}{K_1} = \frac{-1}{2\pi(\tilde{\phi}_1' - \tilde{\phi}_2')} \int_0^{2\pi} \left\{ \left[H(\delta^2) + J_g\right] \left[\tilde{\phi}_1' + \tilde{\phi}_2' \frac{K_2}{K_1} \cos\phi\right] - \bar{\nu}\, T(\delta^2) \left[1 + \frac{K_2}{K_1}\cos\phi\right]\right.$$

$$\left. - \ell J_L' \frac{K_2}{K_1} \sin\phi \right\} d\phi = -\alpha_1(K_1, K_2) \qquad (44)$$

$$\frac{K_2'}{K_2} = \frac{-1}{2\pi(\tilde{\phi}_2' - \tilde{\phi}_1')} \int_0^{2\pi} \left\{ \left[H(\delta^2) + J_g\right] \left[\tilde{\phi}_2' + \tilde{\phi}_1' \frac{K_1}{K_2} \cos\phi\right] - \bar{\nu}\, T(\delta^2) \left[1 + \frac{K_1}{K_2}\cos\phi\right]\right.$$

$$\left. + \ell J_L' \frac{K_1}{K_2} \sin\phi \right\} d\phi = -\alpha_2(K_1, K_2) \qquad (45)$$

where

$$\tilde{\phi}_j' = \frac{1}{2}\left[\bar{\nu} \pm \sqrt{\bar{\nu}^2 - 4M}\right] \text{ and*}$$

$$\bar{\nu}^2 - 4M > 0$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\ell'}{\ell} \sin\phi\, d\phi = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-\delta^2)'}{2(1-\delta^2)} \sin\phi\, d\phi$$

$$\doteq \frac{1}{2\pi}\int_0^{2\pi} K_1 K_2 \sin^2\phi\, (\tilde{\phi}_1' - \tilde{\phi}_2')(1 + \delta^2)\, d\phi$$

$$= \frac{(\tilde{\phi}_1' - \tilde{\phi}_2')\, K_1 K_2 \,(1 + K_1^2 + K_2^2)}{2}, \text{ and}$$

---

*The inequality $\bar{\nu}^2 - 4M > 0$ is equivalent to the requirement that the missile be gyroscopically stable.[9] For positive spin, ϕ_1' is the larger frequency and is usually called the nutational frequency while ϕ_2', the smaller frequency, is called the precessional frequency. On page 19 of Ref. 9, it is shown that this use of the terms "precession" and "nutation" is not compatible with their use in the theory of the top.

$$\phi_j' = \tilde{\phi}_j' + \psi_j'$$

then

$$\phi_1' \doteq \tilde{\phi}_1' - \frac{\tilde{\phi}_2' K_2^2 (1+K_1^2 + K_2^2)}{2} - \frac{1}{2\pi(\tilde{\phi}_1'-\tilde{\phi}_2')}\int_0^{2\pi} \left[M(\delta^2)-M(0)\right] (1 + \frac{K_2}{K_1}\cos\phi)d\phi \tag{47}$$

$$\phi_2' \doteq \tilde{\phi}_2' - \frac{\tilde{\phi}_1' K_1^2 (1+K_1^2+K_2^2)}{2} + \frac{1}{2\pi(\tilde{\phi}_1' - \tilde{\phi}_2')} \int_0^{2\pi} \left[M(\delta^2)-M(0)\right] (1 + \frac{K_1}{K_2}\cos\phi)d\phi \tag{48}$$

Although Eqs. (47 - 48) for the equivalent linear frequencies* are of some interest, the primary concern of a designer is the behavior of the amplitude of the motion. For the type of non-linearities which are usually encountered, it will be convenient to describe this motion in terms of the squared amplitudes of each mode. In the more detailed analysis that follows we will make the convenient but not necessary assumptions of linear damping moment** and linear lift force (H = constant, $J_L' = 0$). Eqs. (44 -45) then reduce to

$$\frac{(K_1^2)'}{K_1^2} = - 2\alpha_1 (K_1^2, K_2^2) \tag{49}$$

$$\frac{(K_2^2)'}{K_2^2} = -2\alpha_2 (K_1^2, K_2^2) \tag{50}$$

$$\alpha_1 = \frac{(H + J_g)\tilde{\phi}_1' - \frac{1}{2\pi} \int_0^{2\pi} \bar{\nu}\, T(\delta^2) \left[1 + \frac{K_2}{K_1} \cos \phi\right] d\phi}{\tilde{\phi}_1' - \tilde{\phi}_2'} \tag{51}$$

$$\alpha_2 = \frac{(H + J_g)\tilde{\phi}_2' - \frac{1}{2\pi} \int_0^{2\pi} \bar{\nu}\, T(\delta^2) \left[1 + \frac{K_1}{K_2} \cos \phi\right] d\phi}{\tilde{\phi}_2' - \tilde{\phi}_1'} \tag{52}$$

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* These frequencies are measured in the non-rotating coordinate system. The relations between the non-rotating coordinate system and the fixed plane coordinates are given in Reference 6.

** In Appendix D the interesting special case of a non-linear damping moment and zero spin is considered.

where

$$\delta^2 = K_1^{\ 2} + K_2^{\ 2} + 2K_1K_2 \cos \phi.$$

5. THE AMPLITUDE PLANE

In Reference 6 the analysis of non-linear spark range data was developed in some detail. The yawing motion over the relatively short flat trajectory under observation was assumed to be epicyclic with constant frequencies and damping exponents which were related to the appropriate effective magnitudes of yaw. For the cubic Magnus and static moments, which were assumed, Eqs. (47 - 48) and (51 - 52) reduced to*

$$\phi_1' \doteq \tilde{\phi}_1' - \frac{\tilde{\phi}_2' K_2^{\ 2}}{2} - \frac{M_2 (K_1^{\ 2} + 2K_2^{\ 2})}{\tilde{\phi}_1' - \tilde{\phi}_2'} \tag{53}$$

$$\phi_2' \doteq \tilde{\phi}_2' - \frac{\tilde{\phi}_1' K_1^{\ 2}}{2} - \frac{M_2 (K_2^{\ 2} + 2K_1^{\ 2})}{\tilde{\phi}_2' - \tilde{\phi}_1'} \tag{54}$$

$$\begin{aligned} \alpha_1 &= \alpha_{10} + \alpha_{12} (K_1^{\ 2} + 2K_2^{\ 2}) \\ &= \frac{H \tilde{\phi}_1' - \bar{\nu} \left[T_0 + T_2(K_1^{\ 2} + 2K_2^{\ 2})\right]}{\tilde{\phi}_1' - \tilde{\phi}_2'} \end{aligned} \tag{55}$$

$$\begin{aligned} \alpha_2 &= \alpha_{20} + \alpha_{22} (K_2^{\ 2} + 2K_1^{\ 2}) \\ &= \frac{H \tilde{\phi}_2' - \bar{\nu} \left[T_0 + T_2 (K_2^{\ 2} + 2K_1^{\ 2})\right]}{\tilde{\phi}_2' - \tilde{\phi}_1'} \end{aligned} \tag{56}$$

---

*The quartic terms in Eqs. (47 - 48) are omitted for $\delta < \sin 15^\circ$.

where

$$M = \sum_{k=0}^{n} \ell \, k_2^{-2} \, J_{M_{2k}} \, \delta^{2k} \doteq M_0 + M_2 \, \delta^2$$

$$T = \sum_{k=0}^{n} \ell \left[J_{L_{2k}} - k_1^{-2} \, J_{T_{2k}} \right] \delta^{2k} \doteq T_0 + T_2 \, \delta^2 .$$

$$J_{\ddot{1}} = \frac{\rho d^3}{m} \quad K_1 .$$

Using Eqs. (53 - 56) a large number of spark range firings were analysed with outstanding success. The swerving motion was treated in a similar fashion and cubic lift force and cubic Magnus force coefficients were obtained which showed excellent consistency with their corresponding moment coefficients.

Although this work on spark range firings was so successful, it suffered from its limitations to cubic non-linearities and short portions of trajectories. In Reference 6, the influence of polynomial expansion of M and T was calculated up to 14th degree polynomials* and Eqs. (47 - 48, 51 - 52) will apply when M and T are arbitrary functions** of δ^2. Thus the basic limitation is the restriction to short trajectories.

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* The effective yaws of that report are related to the integrals of this report by the definitions

$$\left[\delta^{2k}\right]_{e1} = \frac{1}{2\pi} \int_0^{2\pi} \delta^{2k} \left[1 + \frac{K_2}{K_1} \cos \phi \right] d\phi \quad \text{and}$$

$$\left[\delta^{2k}\right]_{e2} = \frac{1}{2\pi} \int_0^{2\pi} \delta^{2k} \left[1 + \frac{K_1}{K_2} \cos \phi \right] d\phi .$$

** In Appendix B effective values of negative powers of δ are computed and a possible use indicated.

This restriction lies in the assumption that the yawing motion is epicyclic with constant frequencies and constant damping exponents. (These constants depend on the effective amplitude of the motion.) A study of Eqs. (49 - 50) shows how this restriction can be relaxed. The theory requires that these quantities be constant over a period of δ^2 but they may vary over longer intervals. Dividing Eq. (50) by Eq. (49), we can write a single first order non-linear equation for this variation

$$\frac{(K_2^2)'}{(K_1^2)'} = \frac{d(K_2^2)}{d(K_1^2)} = \frac{K_2^2 \alpha_2 (K_1^2, K_2^2)}{K_1^2 \alpha_1 (K_1^2, K_2^2)} \tag{57}$$

Equation 57 describes the character of the yawing motion by means of the movement of a point in the K_1^2, K_2^2 plane which we will call the amplitude plane. For any point in this plane the equivalent linear frequencies and damping exponents can be calculated and thus, except for the phase angles, ϕ_{j0}, the motion is completely determined. As will be seen, this amplitude plane will have a number of similarities with the phase plane associated with the one degree of freedom problem. Although the usual four dimensional phase space associated with two degrees of freedom reduces to three essential dimensions for the epicyclic subset, it will not in general reduce to two dimensions and, hence, the amplitude plane is definitely not a phase plane.

Differential equations of the form of Eq. (57) have been treated in some detail by Poincare[10] who showed that the essential properties of the solution curves are fixed by the location and type of their singularities. Singularities are points for which both the numerator and the denominator of the right side of Eq. (57) vanish. They can vanish in four different ways. These ways correspond to the following points:

(1) the origin;

(2) the K_1^2 intercepts of the $\alpha_1 = 0$ curve;

(3) the K_2^2 intercepts of the $\alpha_2 = 0$ curve; and

(4) the intersections of the zero damping curves $\alpha_1 = 0$ and $\alpha_2 = 0$.

According to Poincare's classification there are four possible types of first order singularities; nodes, saddles, spirals, and centers. In Fig. 1, a node and a saddle are shown, while a spiral appears in Fig. 3b and a center in Fig. 4. The particular curve along which the actual yawing motion moves is determined by the initial modal amplitudes as specified by the initial conditions. If the coordinates are translated so that the singularity is at the origin, Eq. (57) has the form:

$$\frac{dy}{dx} = \frac{ax + by + P(x, y)}{cx + dy + Q(x, y)} \tag{58}$$

where* $ad - bc \neq 0$ and P and Q vanish to at least the second order at the origin. The criteria for the type of the singularity can now be stated in terms of the coefficients a, b, c, d and their discriminant $D = (b - c)^2 + 4\,ad$. (See page 44 of Ref. 11.)

I. The singularity is a node if (1) $D > 0$ and $ad - bc < 0$ or (2) $D = 0$. (Note that if $ad = 0$, it is a node if $bc > 0$.)

II. The singularity is a saddle if $D > 0$ and $ad - bc > 0$.

III. The singularity is a spiral if $D < 0$ and $b + c \neq 0$.

IV. The singularity may be a center if $D < 0$ and $b + c = 0$; otherwise it is a spiral. The higher order terms P and Q must be considered for a final determination.[14]

For Eq. (57), the character of the singularity at the origin can be easily determined since a and d both vanish and b and c are the linear values of the damping exponents α_{20}, α_{10}. The origin is a node when these damping exponents are of the same sign $(\alpha_{10}\alpha_{20} > 0)$ and a saddle when they are of opposite sign $(\alpha_{10}\alpha_{20} < 0)$. In Fig. 1a, the first possibility is shown for $|\alpha_{10}| > |\alpha_{20}|$ and in Fig. 1b, the second for $|\alpha_{10}| < |\alpha_{20}|$ is shown. The direction of motion is not shown since this depends on the actual sign of the damping. If, for example, both exponents are positive, the origin is a stable node and small amplitude motion must damp to zero.

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*If $ad - bc = 0$ and the origin is a singularity, then the singularity is at least second order and the P and Q functions must be considered.

The singularities on the K_1^2 or K_2^2 axes can be located by consideration of pure mode motion. They are, however, very important for "almost" pure modes as well. For pure nutational motion, i.e., $(K_2^2 = 0)$, Eq. (51) reduces to

$$\alpha_1(K_1^2, 0) = \frac{(H+J_g)\tilde{\phi}_1' - \bar{\nu}\, T(K_1^2)}{\tilde{\phi}_1' - \tilde{\phi}_2'} \tag{59}$$

Similarly for pure precessional motion,

$$\alpha_2(0, K_2^2) = \frac{(H+J_g)\tilde{\phi}_2' - \bar{\nu}\, T(K_2^2)}{\tilde{\phi}_2' - \tilde{\phi}_1'} \tag{60}$$

Eqs. (59 - 60) could have been obtained by taking the expressions for linear damping and replacing the constant Magnus moment coefficient by its actual non-linear dependence on δ^2. This is the process used in Refs. 1 and 2 in their treatment of pure mode motions.

In order to consider motion near a pure mode of amplitude K, $T(\delta^2)$ must be expanded in Taylor series about this amplitude.

$$\therefore \quad T(\delta^2) = T(K^2) + T'(K^2)(\delta^2 - K^2) \tag{61}$$

Substituting Eq. (61) in Eqs. (51 - 52),

$$\alpha_1(K_1^2, K_2^2) = \frac{(H+J_g)\tilde{\phi}_1' - \bar{\nu}\, T(K^2) - \bar{\nu} T'(K^2)\left[K_1^2 + 2K_2^2 - K^2\right]}{\tilde{\phi}_1' - \tilde{\phi}_2'} \tag{62}$$

$$\alpha_2(K_1^2, K_2^2) = \frac{(H+J_g)\tilde{\phi}_2' - \bar{\nu} T(K^2) - \bar{\nu} T'(K^2)\left[K_2^2 + 2K_1^2 - K^2\right]}{\tilde{\phi}_2' - \tilde{\phi}_1'} \tag{63}$$

where the point (K_1^2, K_2^2) is near either $(K^2, 0)$ or $(0, K^2)$. If singularities on the nutational axis $(K_2^2 = 0)$ are considered, their location may be obtained by setting Eq. (59) equal to zero. Identifying such a singular point by the coordinates $(K^2, 0)$, Eq. (59) becomes

$$\frac{(H+J_g)\tilde{\phi}_1' - \bar{\nu}\, T(K^2)}{\tilde{\phi}_1' - \tilde{\phi}_2'} = 0 \tag{64}$$

If Eq. (64) is substituted in Eqs. (62 - 63),

$$\alpha_1\ (K_1^{\,2}, K_2^{\,2}) = -A\left[(K_1^{\,2} - K^2) + 2K_2^{\,2}\right] \tag{65}$$

$$\alpha_2\ (K_1^{\,2}, K_2^{\,2}) = \left[H+J_g + AK^2\right] + A\left[K_2^{\,2} + 2(K_1^{\,2} - K^2)\right] \tag{66}$$

where $A = \dfrac{\bar{\nu}\, T'(K^2)}{\tilde{\phi}_1' - \tilde{\phi}_2'}$. Substituting Eqs. (65 - 66) in Eq. (57) and shifting the origin to the singular point by the translation $x = K_1^{\,2} - K^2$, $y = K_2^{\,2}$, we see that

$$\frac{dy}{dx} = \frac{y\left[(H+J_g + AK^2) + A(y + 2x)\right]}{(x + K^2)\left[-A(x + 2y)\right]} \tag{67}$$

$$\begin{aligned} &\therefore\ a = 0 && c = -AK^2 \\ &b = H+J_g + AK^2 && d = -2AK^2, \end{aligned} \tag{68}$$

and

$$D = (H + J_g + 2AK^2)^2 \geq 0 \tag{69}$$

According to relation (69) the singularity must be either a node or a saddle. It will be a node if $A(H+J_g + AK^2)$ is negative and a saddle otherwise. Conversely, if the singularity is on the precessional axis, it can be located by setting Eq. (60) equal to zero. Since Eqs. (59 -60) are symmetric in the subscripts 1 and 2, the test for type of singularity can be obtained by interchanging subscripts in the above discussion. Therefore, the precessional singularity will be a node if $-A\ (H+J_g + AK^2)$ is negative and a saddle otherwise.

The situation for singularities which are not on the coordinate axes is much more complex. In order to treat these singularities it is necessary to know much more than the value of T and T' at a point. To illustrate the proper procedure for handling these singularities the possible amplitude planes for a cubic Magnus moment will be discussed in some detail.

According to Eqs. (55 - 56) the curves of zero damping are lines described by the following equations:

$$\alpha_{10} + \alpha_{12} (K_1^2 + 2K_2^2) = 0 \tag{70}$$

$$\alpha_{20} - \alpha_{12} (K_2^2 + 2K_1^2) = 0 \tag{71}$$

where $\alpha_{10} = \dfrac{H \tilde{\phi}_1' - \bar{\nu}\, T_0}{\tilde{\phi}_1' - \tilde{\phi}_2'}$

$$\alpha_{20} = \frac{H \tilde{\phi}_2' - \bar{\nu}\, T_0}{\tilde{\phi}_2' - \tilde{\phi}_1'}$$

$$\alpha_{12} = - \alpha_{22} = \frac{- \bar{\nu}\, T_2}{\tilde{\phi}_1' - \tilde{\phi}_2'} .$$

The intersection of the lines is a singularity and it has coordinates $\left(\dfrac{\alpha_{10} + 2\alpha_{20}}{3\alpha_{12}}, \dfrac{\alpha_{20} + 2\alpha_{10}}{-3\alpha_{12}}\right)$. In addition to this there are singularities at the origin and the points $\left(- \dfrac{\alpha_{10}}{\alpha_{12}}, 0\right)$, $\left(0, \dfrac{\alpha_{20}}{\alpha_{12}}\right)$. These three singularities can, however, be treated by methods already developed.

Since only the first quadrant of the amplitude plane has physical meaning, the intersection of the zero damping lines has importance only when it lies in the first quadrant.

$$\therefore \quad \frac{\alpha_{10} + 2\alpha_{20}}{3\, \alpha_{12}} > 0, \quad \frac{\alpha_{20} + 2\alpha_{10}}{- 3\, \alpha_{12}} > 0 \tag{72}$$

If these are multiplied together and divided by $-\dfrac{9\,\alpha_{12}^{\;2}}{2\,\alpha_{20}^{\;2}}$, then

$$\left(\frac{\alpha_{10}}{\alpha_{20}} + 2\right)\left(\frac{1}{2} + \frac{\alpha_{10}}{\alpha_{20}}\right) < 0 \tag{73}$$

$$\therefore\; -2 < \frac{\alpha_{10}}{\alpha_{20}} < -\frac{1}{2} \tag{74}$$

Comparing inequality (74) with the original inequalities (72), we see that the intersection will be in the first quadrant when $\dfrac{\alpha_{10}}{\alpha_{20}}$ lies between -2 and $-\frac{1}{2}$ and α_{20} has the same sign as α_{12}.

In order to classify this singularity Eqs. (55 - 56) are now placed in Eq. (57) and the origin is translated to the point of intersection by the translation $x = K_1^{\;2} - \dfrac{\alpha_{10} + 2\,\alpha_{20}}{3\,\alpha_{12}}$, $y = K_2^{\;2} + \dfrac{\alpha_{20} + 2\,\alpha_{10}}{3\,\alpha_{12}}$.

$$\therefore\; \frac{dy}{dx} = \frac{(\alpha_{20} + 2\,\alpha_{10})(y + 2x) - 3\,\alpha_{12}\,(y + 2x)y}{(\alpha_{10} + 2\,\alpha_{20})(x + 2y) + 3\,\alpha_{12}\,(x + 2y)x} \tag{75}$$

The coefficients of Eq. (58) can now be obtained from Eq. (75).

$$a = 2(\alpha_{20} + 2\,\alpha_{10}) \qquad c = \alpha_{10} + 2\,\alpha_{20}$$
$$b = \alpha_{20} + 2\,\alpha_{10} \qquad d = 2\,(\alpha_{10} + 2\,\alpha_{20}) \tag{76}$$

$$\therefore\; D = 33\,\alpha_{10}^{\;2} + 78\,\alpha_{10}\alpha_{20} + 33\,\alpha_{20}^{\;2}$$
$$= \frac{33}{\alpha_{20}^2}\left[\frac{\alpha_{10}}{\alpha_{20}} + 1.812\right]\left[\frac{\alpha_{10}}{\alpha_{20}} + .552\right] \tag{77}$$

$$a\,d - bc = 3(\alpha_{10} + 2\,\alpha_{20})(\alpha_{20} + 2\,\alpha_{10}) \tag{78}$$

$$b + c = 3(\alpha_{10} + \alpha_{20}) \tag{79}$$

If the point of intersection lies in the first quadrant, bc - ad must be always positive. Applying the criteria for singularities to relations (72, 75 - 77), we can make the following statements for the singularity at the point of intersection.

(1) It is a node if $-2 < \frac{\alpha_{10}}{\alpha_{20}} \leqq -1.812$ or $-.552 \leqq \frac{\alpha_{10}}{\alpha_{20}} < -.5$.

(2) It is a spiral if $-1.812 < \frac{\alpha_{10}}{\alpha_{20}} < -.552$ and $\frac{\alpha_{10}}{\alpha_{20}} \neq -1$.

(3) It may be a center if $\frac{\alpha_{10}}{\alpha_{20}} = -1$. (An application of the criteria of Ref. 14 shows that it is a center.)

In Figures 1 - 4*, all but one of the possible different amplitude planes for a cubic Magnus moment have been drawn by the EBL Analog Computer. If $\alpha_{10}\alpha_{20} > 0$, the origin is a node and only one of the lines of zero damping falls in the first quadrant. Since the intercept of this line has to be a saddle point, the amplitude plane is that shown in Fig. 2. This figure shows a characteristic property of non-linear equations, namely a dependence of the form of the solution on initial conditions. If the origin is a stable node, the motion is down toward the K_1^2 - axis and, hence, for certain initial conditions, the motion goes to zero while for others it goes to large values of K_1^2.

If $\alpha_{10}\alpha_{20} < 0$, the origin is a saddle and either both lines go through the first quadrant or neither does**. When both lines go through the first quadrant but do not intersect there (Fig. 3a), our criteria reveal that the singularity on the line closer to the origin is a node while the other singularity is a saddle point. Thus, for a stable node, the yawing motion can approach a limit cycle of circular yawing motion or diverge to large values of K_1^2. This limit motion is a second property of non-linear equations.

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* Although Fig. 1 was drawn for the linear equation, the amplitude planes for non-linear equations whose only singularity is located at the origin are quite similar to those shown.

** If neither line goes through the first quadrant, the amplitude plane is similar to that shown in Fig. 1b.

If the zero damping lines intersect in the first quadrant and $\alpha_{10}+\alpha_{20} \neq 0$, the intersection must be either a node or a spiral. For this case the three singularities on the axes are saddles as shown in Fig. (3b). When the intersection is a node, it lies close to either axis and so this case is not shown. The limit cycle predicted for a stable node or spiral is now a limit epicycle. In other words, for a wide range of initial conditions the motion should eventually be epicyclic with certain fixed values of amplitudes and no damping. Because of symmetry considerations, the special case of $\alpha_{10}+\alpha_{20} = 0$ has to be a center and this is shown in Fig. 4.

6. COMPARISON WITH NUMERICAL INTEGRATIONS

After developing the technique for treating non-linear fourth order systems of a certain type, it is natural to wonder how well it would predict actual motion. Fortunately, the exact fourth order equations of motion for a missile acted on by non-linear forces and moments have already been programmed for the NORC computer at the Naval Proving Ground. In Reference 8 there are described a number of calculations which were made to investigate the effect of initial conditions on the character of the yawing motion. Since these calculations involved rather large yarying yaw of repose and a non-constant $\bar{\nu}$, the comparison of theory with these calculations will be deferred to the appendix.

Dr. Cohen and Mr. Hubbard of NPG offered to make a number of special runs as a critical test of the equivalent linearization theory[7]. It was felt that the prediction of limit epicycles was an essentially new prediction and that this prediction should be checked. Two cases of ten NORC runs each were considered. For both cases the Magnus moment was assumed to be a quintic function of δ and all other forces and moments were assumed to be linear.

$$\therefore T = T_0 + T_2 \delta^2 + T_4 \delta^4. \qquad (80)$$

In Reference 6 it is shown that the effective values of δ^4 are

$$\left[\delta^4\right]_{e1} = K_1^{\,4} + 6K_1^{\,2}K_2^{\,2} + 3K_2^{\,4} \tag{81}$$

$$\left[\delta^4\right]_{e2} = K_2^{\,4} + 6K_1^{\,2}K_2^{\,2} + 3K_1^{\,4}. \tag{82}$$

$$\therefore \alpha_1 = \alpha_{10} + \alpha_{12}(K_1^{\,2} + 2K_2^{\,2}) + \alpha_{14}(K_1^{\,4} + 6K_1^{\,2}K_2^{\,2} + 3K_2^{\,4}) \tag{83}$$

$$\alpha_2 = \alpha_{20} + \alpha_{22}(K_2^{\,2} + 2K_1^{\,2}) + \alpha_{24}(K_2^{\,4} + 6K_1^{\,2}K_2^{\,2} + 3K_1^{\,4}) \tag{84}$$

where $\alpha_{14} = -\alpha_{24} = \dfrac{-\bar{\nu}\, T_4}{\phi_1' - \phi_2'}$.

Thus the loci of zero damping are now hyperbolas.

The two sets of aerodynamic coefficients and spin were selected so that the left branches of the zero damping hyperbolas fell in about the same location as the zero damping lines in Figures 3a and 3b. The other branches were far enough to the right that they could be neglected for small yaw conditions. The exact conditions of these two cases are given in Table 1 and the location and type of their amplitude plane singularities are given in Table 2. (Although the large yaw singularities are listed in Table 2, these points are not important in this discussion.)

The initial conditions for Case 1 were selected to lie to the left of the dashed curve in Figure 3a. It was, therefore, expected that the motion would quickly become a pure precessional mode with amplitude..0818. In Table 3,initial values of $K_j^{\,2}$ are tabulated with their values* after 18,000 calibers of travel. From Table 3, we see that the final values are close to the predicted point. In Figure 5, the final values at 18,000 calibers are plotted on a greatly enlarged scale to show how the motion is slowly approaching the pure mode.

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* The amplitude of the modes may be easily calculated from pairs of successive stationary values of δ. ($\delta_{max} = K_1 + K_2$, $\delta_{min} = |K_1 - K_2|$).

TABLE 1

Parametric Values for NORC Runs

Both Cases

$\bar{\nu} = .0055$

$10^3 H = .400$

$J_g = 0$

$10^3 T_0 = -1.30$

$10^3 T_2 = 156$

$10^3 T_4 = -985$

	Case 1	Case 2
$10^5 M =$	- 4.16	- 2.08
$\frac{2\pi}{\tilde{\phi}_1' - \tilde{\phi}_2'} =$	448 calibers	590 calibers

TABLE 2

Location and Type of Singular Points

Case 1

K_1	K_2	K_1^2	K_2^2	Type
0	0	0	0	saddle
.119	0	.0141	0	saddle
0	.0818	0	.006686	stable node
.381	0	.1449	0	saddle
0	.390	0	.1523	saddle
.257	.162	.0660	.0261	stable spiral

Case 2

K_1	K_2	K_1^2	K_2^2	Type
0	0	0	0	saddle
.115	0	.0132	0	saddle
0	.088	0	.0075	saddle
.0268	.0784	.00072	.00615	stable spiral
.382	0	.1458	0	saddle
0	.389	0	.1514	saddle
.250	.169	.0625	.0286	stable spiral

TABLE 3

NORC Results for Case 1

Runs	Initial $10^2K_1^2$	Initial $10^2K_2^2$	Final $10^2K_1^2$	Final $10^2K_2^2$
1	.0392	.5083	.0069	.6512
2	.0400	.5069	.0046	.6561
3	.4096	.1823	.0031	.6577
4	.0655	1.2343	.0079	.6464
5	.0610	.5520	.0058	.6529
6	.0132	.4761	.0021	.6577
7	.0004	.4900	.0002	.6642
8	.2601	.1109	.0015	.6610
9	.0071	1.2144	.0061	.6529
10	.0036	.5256	.0010	.6626

From Table 2 we see that the small yaw limit cycle for Case 2 is a zero-damped epicycle with amplitudes $K_1 = .0268$, $K_2 = .0784$. Five pairs of initial conditions were used. Although both members of a pair had the same initial modal amplitudes, the first member was initially at the maximum yaw while the second was at minimum yaw. The equivalent linear theory makes no distinction between these initial conditions and the NORC computations seemed to verify this characteristic. The actual initial amplitudes are given in Table 4.

TABLE 4

Initial Squared Amplitudes for Case 2

Runs	K_1	K_2	$10^2K_1^2$	$10^2K_2^2$
11, 16	.0268	.0784	.072	.615
12, 17	.0141	.0784	.020	.615
13, 18	.0700	.0500	.490	.250
14, 19	.0100	.1380	.010	1.904
15, 20	.0194	.0825	.038	.681

Once again computer runs of 18,000 caliber duration (30-1/2 maxima of yaw) were made. In all cases the motion became an epicycle with zero damping and the amplitudes of both modes agreed with the theory to four decimal places. Since runs 11, 12, 15, 16, 17, and 20 had initial amplitudes near the limit amplitudes, they attained their limit values rather rapidly, i.e., in less than 5,000 calibers. The remaining runs required the full distance of 18,000 calibers.

In Figure 6, the complete amplitude plane for Case 2 is presented. The small yaw position together with the initial points for runs (13, 18) and (14, 19) are shown in Figure 7. The time history of the amplitudes for runs 13 or 18 were calculated from Eqs. (49 - 50) by the Exterior Ballistics Analog Computer. The values of the two modal amplitudes between successive maxima and minima were computed from the exact NORC calculations for Run 13 and are compared with the analog computer results in Figure 8. The agreement is quite good.

As a final check of the theory it was decided to verify the predicted location of the separatrix for Case 2. (This is the dashed curve in Fig. 7 which separates the large yaw trajectories from the small yaw trajectories.) For a value of K_2^2, values of K_1^2 were chosen which bracketed the predicted K_1^2 of the separatrix. This was done for two values of K_2^2. Although the theory predicts that the results are independent of relative phase of the modes, runs were made for modes both initially in phase and initially out of phase. The values of K_1^2 which were selected are given in Table 5.

Exact six degree of freedom calculations were then made on the NORC for these initial conditions. The results for 3000-caliber long trajectories fell into two groups. For one group the yawing motion decreased while for the other it grew. For one run it was not possible, on the basis of the first 3000 calibers, to decide to which group it belonged. These three possibilities are denoted by S (stable), U(unstable), and N (neutral) in Table 5. As can be seen from that table, the agreement with the theory is excellent.

TABLE 5

Trajectories to Locate Separatrix

K_2^2	K_1^2	In Phase	Out of Phase
.006	.00800	S	S
.006	.00825	S	S
.006	.00850	S	S
.006	.00875	S	S
.006	.00900	N	U
.006	.00925	U	U

Predicted K_1^2: .00895

K_2^2	K_1^2	In Phase	Out of Phase
.012	.00450	S	S
.012	.00475	S	S
.012	.00500	S	S
.012	.00525	S	S
.012	.00550	S	U
.012	.00575	U	U
.012	.00600	U	U

Predicted K_1^2: .00555

7. SUMMARY

1. A subset of the set of fourth order linear systems with constant coefficients has been identified and its important property of an epicyclic solution described.

2. An approximate solution has been obtained for a class of non-linear equations which can be linearized to members of this subset.

3. This solution was applied to the equations of yawing motion of a symmetric missile and the very useful concept of an amplitude plane introduced.

4. The predictions of the approximate theory have been compared with exact results and excellent agreement has been obtained.

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APPENDIX A: DISCUSSION OF THE INHOMOGENEOUS EQUATION

In this appendix the stability characteristics of the inhomogeneous equation will be considered. Since for constant spin a small aerodynamic asymmetry[12] has a form similar to the gravity term in Eq. (41), the treatment will be general enough to cover both cases. More precisely, we will consider the following differential equation.*

$$\lambda'' + (H + J_g - i\bar{\nu})\lambda' - (M + i\bar{\nu}\ \ T)\lambda = J\lambda_\epsilon\, e^{i\psi} \qquad \text{(A1)}$$

where $J\lambda_\epsilon\, e^{i\psi}$ is either G or $J_\epsilon \lambda_\epsilon\, e^{i(\nu p + \psi_0)}$

$$\nu = \frac{\omega_1 d}{u}$$

$$G \doteq \bar{\nu}\ \left(\frac{gd}{u^2}\right)\ \cos\theta$$

$$J_\epsilon = \frac{\rho d^3}{m}\left[i\nu\ \left(1 - \frac{A}{B}\right)\ K_{N_\epsilon} - k_2^{-2}\, K_{M_\epsilon}\right]$$

K_{N_ϵ} is the asymmetric force coefficient

K_{M_ϵ} is the asymmetric moment coefficient

λ_ϵ is asymmetry angle

ψ_0 is initial orientation of the asymmetric force

θ is the inclination of the trajectory

A is axial moment of inertia and B is transverse moment of inertia

The particular solution for the linearized form of Eq. (A1) with a constant ψ' has the form $K_3\, e^{i\phi_3}$ where $\phi_3{}' = \psi'$ and K_3 is a real constant. (For the yaw of repose case $\psi' = 0$; for aerodynamic asymmetry $\psi' = \nu$.)

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* The small terms $\frac{\ell'}{\ell}\ \lambda'$ and $\ell J_L{}'\lambda$ have been omitted for simplicity.

If H, M, T, and J are assumed to be functions of δ^2 and the pure mode steady state solution is sought, the problem can be quickly solved. For this case,

$$\lambda = K_3 e^{i\phi_3} \tag{A2}$$

$$\delta^2 = K_3^2 \tag{A3}$$

where $\phi_3 = \phi_{30} + \psi' p$.

Substituting Eqs. (A2 - A3) in Eq. (A1),

$$K_3 e^{i\phi_{30}} = \frac{-\left[J(K_3^2)\right] \lambda_\epsilon e^{i\psi_0}}{\psi'(\psi' - \bar{\nu}) + M(K_3^2) + i\left[\bar{\nu} T(K_3^2) - \psi' \left[H(K_3^2) + J_g\right]\right]} \tag{A4}$$

For the inhomogeneous gravity term, Eq. (A4) reduces to

$$\lambda_R = \delta_R e^{i\phi_R} = \frac{-\nu \left(\frac{gd}{u^2}\right) \cos\Theta}{M(\delta_R^2) + i\bar{\nu} T(\delta_R^2)} \tag{A5}$$

where λ_R is the yaw of repose, $\delta_R = K_3$, $\phi_R = \phi_{30}$. On the other hand, for the small asymmetry term it becomes

$$K_3 e^{i\phi_{30}} = \frac{-\left[J_\epsilon(K_3^2)\right] \lambda_\epsilon e^{i\psi_0}}{\nu^2(1 - \frac{A}{B}) + M(K_3^2) + i\nu\left[\frac{A}{B} T(K_3^2) - H(K_3^2) - J_g\right]} \tag{A6}$$

An important difference between Eqs. (A5 - A6) and their linearized versions is the fact that more than one solution of the non-linear expressions is possible. In usual practice it is difficult to construct an important example of a multivalued yaw of repose. For missiles possessing small aerodynamic asymmetries, a multivalued trim angle is quite possible. As is shown in Figure 9, which is based on Eq. (A6) for a common non-linear static moment curve, the well known linear resonance curve is quite distorted and for certain spins three values of K_3 are possible.* This non-linear response curve possesses the characteristic non-linear property of jumps between roots where the curve possesses a vertical tangent.

In order to treat the more general case of mixed oscillations, we will make use of the direct substitution method of Reference 6. If Eq. (A1) is written for cubic non-linearities in M, T, and J,

$$\lambda'' + (H+J_g-i\bar{\nu})\lambda' - \left[M_0 + M_2 \ \delta^2 + i\bar{\nu}\ (T_0 + T_2 \ \delta^2)\right] \lambda = \left[J_0 + J_2 \ \delta^2\right] \lambda_\epsilon \ e^{i(\psi' p + \psi_0)} \tag{A7}$$

$$\lambda = K_1 \ e^{i\phi_1} + K_2 \ e^{i\phi_2} + K_3 \ e^{i\phi_3} \tag{A8}$$

$$\delta^2 = \lambda\bar{\lambda} = K_1^{\ 2} + K_2^{\ 2} + K_3^{\ 2} + K_1K_2 \left[e^{i(\phi_1-\phi_2)} + e^{i(\phi_2-\phi_1)}\right] + K_2K_3 \left[e^{i(\phi_2-\phi_3)} + e^{i(\phi_3-\phi_2)}\right] + K_3K_1 \left[e^{i(\phi_3-\phi_1)} + e^{i(\phi_1-\phi_3)}\right] \tag{A9}$$

where
$$\left.\begin{aligned} K_j &= e^{-\alpha_j p} \\ \phi_j &= \phi_{j0} + \phi_j' p \end{aligned}\right\} \quad j = 1,\ 2$$

K_3 = constant and

$\phi_3 = \phi_{30} + \psi' p.$

---

*It is quite likely that the middle value of K_3 is unstable. This is the case for the single degree of freedom case. (See Refs. 11 and 13.)

If Eqs. (A8 - A9) are substituted in Eq. (A7) and terms of the same frequency are collected,

$$K_1 e^{i\phi_1} \Big[(-\alpha_1 + i\phi_1')^2 + (H + J_g - i\bar{\nu})(-\alpha_1 + i\phi_1') - (M_0 + i\bar{\nu}\, T_0)$$

$$- (M_2 + i\bar{\nu}\, T_2) \left[\delta^2\right]_{e1} - J_2 \left[\delta^2\right]_{e31} e^{i(\psi_0 - \phi_{30})} \Big]$$

$$+ K_2 e^{i\phi_2} \Big[(-\alpha_2 + i\phi_2')^2 + (H+J_g - i\bar{\nu})(-\alpha_2 + i\phi_2') - (M_0 + i\bar{\nu}\, T_0)$$

$$- (M_2 + i\bar{\nu}\, T_2) \left[\delta^2\right]_{e2} - J_2 \left[\delta^2\right]_{e32} e^{i(\psi_0 - \phi_{30})} \Big]$$

$$+ e^{i\phi_3} \Big\{ \Big[-\psi'^2 + i\psi' (H + J_g - i\bar{\nu}) - (M_0 + i\bar{\nu}\, T_0)$$

$$- (M_2 + i\bar{\nu}\, T_2) \left[\delta^2\right]_{e3} \Big] K_3 - \Big[J_0 + J_2 \left[\delta^2\right]_{e33} \Big] \lambda_\epsilon e^{i(\psi_0 - \phi_{30})} \Big\}$$

$$- E = 0 \tag{A10}$$

where $\left[\delta^2\right]_{e1} = K_1^{\,2} + 2K_2^{\,2} + 2K_3^{\,2}$,

$\left[\delta^2\right]_{e2} = K_2^{\,2} + 2K_3^{\,2} + 2K_1^{\,2}$,

$\left[\delta^2\right]_{e3} = K_3^{\,2} + 2K_1^{\,2} + 2K_2^{\,2}$,

$\left[\delta^2\right]_{e31} = \left[\delta^2\right]_{e32} = K_3\lambda_\epsilon$,

$\left[\delta^2\right]_{e33} = K_1^{\,2} + K_2^{\,2} + K_3^{\,2}$, and

$$E = \left[M_2 + i\bar{\nu}\, T_2\right] \left[K_1K_2^2\, e^{i(2\phi_2-\phi_1)} + K_1^2K_2\, e^{i(2\phi_1-\phi_2)} \right.$$

$$+ K_2K_3^2\, e^{i(2\phi_3-\phi_2)} + K_2^2K_3\, e^{i(2\phi_2-\phi_3)}$$

$$+ K_3K_1^2\, e^{i(2\phi_1-\phi_3)} + K_3^2K_1\, e^{i(2\phi_3-\phi_1)}$$

$$\left. + 2K_1K_2K_3\, (e^{i(\phi_1+\phi_2-\phi_3)} + e^{i(\phi_2+\phi_3-\phi_1)} + e^{i(\phi_3+\phi_1-\phi_2)})\right]$$

$$+ J_2\lambda_\epsilon\, e^{i(\psi_0 + \psi' p)} \left[K_2K_3\, e^{i(\phi_3-\phi_2)} + K_1K_3\, e^{i(\phi_3-\phi_1)} \right.$$

$$\left. + K_1K_2\, (e^{i(\phi_1-\phi_2)} + e^{i(\phi_2-\phi_1)})\right] .$$

If certain isolated sets of frequencies are avoided*, the error term in Eq. (A10) will contain frequencies which are different from any of the "primary" frequencies ϕ_j'. Making our usual assumption that the solution to the non-linear equation has the same form as the solution to the linearized equation, we neglect the error term E in Eq. (A10) and replace the local amplitudes, K_j, in the coefficients of cubic terms by their average values.

$$\therefore \quad (-\alpha_j + i\phi_j')^2 + (H + J_g - i\bar{\nu})(-\alpha_j + i\phi_j')^2 - (M_0 + i\bar{\nu}\, T_0)$$

$$- (M_2 + i\bar{\nu}\, T_2) \left[\delta^2\right]_{ej} - J_2 \left[\delta^2\right]_{e3j} e^{i(\psi_0-\phi_{30})} = 0 \quad \text{(A11)}$$

$$(j = 1,2).$$

$$K_3\, e^{i\phi_{30}} = \frac{-\left[J_0 + J_2 \left[\delta^2\right]_{e33}\right] \lambda_\epsilon\, e^{i\psi_0}}{\psi'(\psi'-\bar{\nu}) + M_0 + M_2\left[\delta^2\right]_{e3} + i\left[\bar{\nu}(T_0+T_2 \left[\delta^2\right]_{e3}) - \psi'(H+J_g)\right]} \quad \text{(A12)}$$

---

* The cases which are to be avoided include stability factor equal to one for which $\phi_1' = \phi_2'$ and resonance for which either $\phi_3' = \phi_1'$ or $\phi_3' = \phi_2'$.

Eqs. (A11 - A12) may be extended to handle more general non-linearities by the use of higher order polynomials in δ^2. In Table 6, values of $\left[\delta^{2k}\right]_{e1}$ and $\left[\delta^{2k}\right]_{e31}$, which were calculated by an algorithm similar to that used in Ref. 6, are tabulated for k = 1, 2, 3. $\left[\delta^{2k}\right]_{e2}$, $\left[\delta^{2k}\right]_{e3}$, and $\left[\delta^{2k}\right]_{e32}$ can be obtained from $\left[\delta^{2k}\right]_{e1}$ and $\left[\delta^{2k}\right]_{e31}$ by replacing 1 by 2 or 3 in the subscripts while $\left[\delta^{2k}\right]_{e33}$ is the symmetric part* of $\left[\delta^{2k}\right]_{e1}$.

An important feature of this three frequency problem can be seen from Eq. (A9). According to this equation the non-linearities are functions of three difference frequencies. These difference frequencies are linearly dependent, and, therefore, this treatment is essentially an average over two frequencies and suffers from the hazy physical meaning of such a process.

As an application of Eqs. (A11 - A12) we will consider the amplitude plane for a quintic Magnus moment as modified by a constant yaw of repose. If Eqs. (A11) are modified for a quintic Magnus moment, it follows from their imaginary parts that

$$\alpha_j = \alpha_{j0} + \alpha_{j2}\left[\delta^2\right]_{ej} + \alpha_{j4}\left\{\delta^4\right\}_{ej} \qquad \text{(A13)}$$

where $\alpha_{j0} = \dfrac{(H + J_g)\phi_j' - \bar{\nu}\, T_0}{2\phi_j' - \bar{\nu}}$

$$\alpha_{j2} = \frac{-\bar{\nu} T_2}{2\phi_j' - \bar{\nu}}$$

$$\alpha_{j4} = \frac{-\bar{\nu}\, T_4}{2\phi_j' - \bar{\nu}}$$

---

* This symmetric part is identified by braces in Table 6.

TABLE 6

$$\left[\delta^2\right]_{e1} = \left\{K_1^{\,2} + K_2^{\,2} + K_3^{\,2}\right\} + (K_2^{\,2} + K_3^{\,2}) = K_1^{\,2} + 2(K_2^{\,2} + K_3^{\,2})$$

$$\left[\delta^4\right]_{e1} = \left\{(K_1^{\,2} + K_2^{\,2} + K_3^{\,2})^2 + 2(K_1^{\,2}K_2^{\,2} + K_2^{\,2}K_3^{\,2} + K_3^{\,2}K_1^{\,2})\right\} + 2(K_1^{\,2} + K_2^{\,2} + K_3^{\,2})(K_2^{\,2} + K_3^{\,2}) + 4K_2^{\,2}K_3^{\,2}$$

$$= (K_1^{\,2} + K_2^{\,2} + K_3^{\,2})(K_1^{\,2} + 3K_2^{\,2} + 3K_3^{\,2}) + 2(K_1^{\,2}K_2^{\,2} + 3K_2^{\,2}K_3^{\,2} + K_3^{\,2}K_1^{\,2})$$

$$\left[\delta^6\right]_{e1} = \left\{(K_1^{\,2} + K_2^{\,2} + K_3^{\,2})^3 + 6(K_1^{\,2} + K_2^{\,2} + K_3^{\,2})(K_1^{\,2}K_2^{\,2} + K_2^{\,2}K_3^{\,2} + K_3^{\,2}K_1^{\,2}) + 12K_1^{\,2}K_2^{\,2}K_3^{\,2}\right\}$$

$$+ 3(K_1^{\,2} + K_2^{\,2} + K_3^{\,2})^2 (K_2^{\,2} + K_3^{\,2}) + 18(K_1^{\,2} + K_2^{\,2} + K_3^{\,2})K_2^{\,2}K_3^{\,2} + 3(K_2^{\,2} + K_3^{\,2})^2 K_1^{\,2}$$

$$\left[\delta^2\right]_{e31} = K_3\lambda_\epsilon$$

$$\left[\delta^4\right]_{e31} = 2K_3\lambda_\epsilon\,(K_1^{\,2} + 2K_2^{\,2} + K_3^{\,2})$$

$$\left[\delta^6\right]_{e31} = 3K_3\lambda_\epsilon\left[(K_1^{\,2} + K_2^{\,2} + K_3^{\,2})(K_1^{\,2} + 3K_2^{\,2} + K_3^{\,2}) + 2K_2^{\,2}(K_1^{\,2} + K_3^{\,2}) + K_1^{\,2}K_3^{\,2}\right]$$

$$\left[\delta^2\right]_{e1} = K_1^{\,2} + 2K_2^{\,2} + 2\delta_R^{\,2}$$

$$\left[\delta^4\right]_{e1} = (K_1^{\,4} + 6K_1^{\,2}K_2^{\,2} + 3K_2^{\,4}) + 6\delta_R^{\,2}(K_1^{\,2} + 2K_2^{\,2}) + 3\,\delta_R^{\,4}$$

$$\left[\delta^2\right]_{e2} = K_2^{\,2} + 2K_1^{\,2} + 2\,\delta_R^{\,2} \text{ and}$$

$$\left[\delta^4\right]_{e2} = (K_2^{\,4} + 6K_1^{\,2}K_2^{\,2} + 3K_1^{\,4}) + 6\,\delta_R^{\,2}\,(K_2^{\,2} + 2K_1^{\,2}) + 3\,\delta_R^{\,4}.$$

In most cases the effect of T on δ_R is small. From Eq. (A5),

$$\delta_R \doteq \left| \frac{\bar{\nu}\ gd\ \cos\theta}{u^2\ M} \right| . \qquad \text{(A14)}$$

Eqs. (A13 - A14) can now be used to interpret the results of Ref. 8. The aerodynamic coefficients which were used in that reference are listed in Table 7. Although a number of spins were considered, we will limit our attention to $\bar{\nu}$ = .0055. The quintic expansion of the Magnus moment was good up to 14° and so only the small yaw singularities are listed.*

TABLE 7

Parameters Used in Reference 8

$10^3 H$ = .928	$10^3 T_0$ = - 2.83
$10^5 M_o$ = -4.10	$10^3 T_2$ = 313
$10^5 M_2$ = -6	$10^3 T_4$ = -1976
$\bar{\nu}$ = .0055	$\dfrac{2\pi}{\phi_1{}' - \phi_2{}'}$ = 452 calibers
$\dfrac{2\pi}{\phi_1{}'}$ = 643 calibers	$\dfrac{2\pi}{\phi_2{}'}$ = 1496 calibers

* Since the frequencies are not important to this discussion, the indicated cubic nature of the static moment will not affect our results.

Small Yaw Singularities for
Initial Yaw of Repose of .016

K_1	K_2	K_1^2	K_2^2	Type
0	0	0	0	saddle
.120	0	.0144	0	saddle
0	.082	0	.0067	stable node

The actual variation of the yaw of repose as a function of p is given in Figure 10. Since the summit value of the yaw of repose is about five times its initial value of .016, its square magnitude grows to twenty-five times its initial value. In Figure 11 the small yaw amplitude plane for a constant yaw of repose of .016 is shown. The corresponding amplitude plane for summit conditions, yaw of repose of .100, is given in Fig. 12. Thus it can be seen that the growth in δ_R has the effect of moving pure mode singularities towards the origin and past it.

This variation in δ_R means that G in Eq. (A1) is a function* of p. If G and therefore δ_R changes slowly during an appropriate period of the motion, it is reasonable to expect that our technique will still apply. According to Table 7 the shortest period associated with any of the three difference frequencies is 452 calibers. From Figure 10 it can be seen that for the first thousand calibers the yaw of repose varies reasonably slowly in comparison with this period. Since the results of Ref. 8 indicate that the motion over this distance is sufficient to determine the character of the yawing motion, it seems reasonable to compare the predictions of the equivalent-linear theory with those of that reference.

In the NORC calculations of that report, certain physical parameters were varied and a large number of trajectories were calculated for a variety of initial conditions. It was found that the computed yawing motion either became quite large or damped out. The critical initial conditions which separated these two types of yawing motion were determined.

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* Under the assumptions of Reference 8, $\bar{v}$ and J_g are also functions of p. Their variation, however, is overwhelmed by the variation in δ_R. In any event the remarks made about δ_R will apply to these terms as well.

The initial modal amplitudes associated with six of these critical sets of initial conditions are listed in Table 8 and plotted in Figure 11. According to the theory these points should fall near the separatrix associated with the nutational saddle. The qualitative description of this saddle point's motion to the left also explains why the point for Case 581 lies farther to the left of the separatrix than the other points. In all events, the agreement is quite good.

TABLE 8

NORC Run No.	K_1	K_2	K_1^2	K_2^2
Standard Case	.103	.060	.0106	.0036
446	.106	.054	.0113	.0029
479	.103	.064	.0107	.0041
504	.101	.062	.0103	.0039
531	.096	.073	.0092	.0053
581	.079	.105	.0062	.0111

APPENDIX B: CALCULATION OF THE AMPLITUDE PLANE FOR NON-POLYNOMIAL NON-LINEARITIES

Although the results of this report have been stated in terms of general non-linearities, the applications have been for polynomial functions. These functions possess the handicap of growing rapidly for large angles. Most non-linear moments either level off or tend to zero with increasing angle. Since constant moments imply moment coefficients which are proportional to δ^{-1}, and moments tending to zero can be described by higher negative powers of δ, effective values of negative powers of δ are of some interest.

The definitions of the effective values of δ^m are

$$\left[\delta^m\right]_{e1} = \frac{1}{2\pi} \int_0^{2\pi} \delta^m \left[1 + \frac{K_2}{K_1} \cos \phi\right] d\phi \tag{B1}$$

$$\left[\delta^m\right]_{e2} = \frac{1}{2\pi} \int_0^{2\pi} \delta^m \left[1 + \frac{K_1}{K_2} \cos \phi\right] d\phi \tag{B2}$$

where $\delta^2 = K_1^2 + K_2^2 + 2K_1K_2 \cos \phi$. If m is a negative even integer, the integrals may be computed in closed form by means of Pierce formulas Nos. 304-306. This has been done for $m = -2, -4, -6, -8$, and the results for $\left[\delta^m\right]_{e1}$ listed in Table 9. The results for $\left[\delta^m\right]_{e2}$ may be obtained by interchanging the subscripts 1 and 2.

The important case of $\left[\delta^{-1}\right]_{e1}$, which arises from a constant moment, requires the use of elliptic integrals. After some algebraic manipulation we have the result:

$$\left[\delta^{-1}\right]_{e1} = \frac{1}{\pi K_1^2} \left[(K_1 - K_2)\, E_1(k) + (K_1 + K_2)\, E_2(k)\right] \tag{B3}$$

$$k^2 = \frac{4\, K_1K_2}{(K_1 + K_2)^2} \tag{B4}$$

where E_1 is the complete elliptic integral of the first kind usually denoted by K and E_2 is the complete elliptic integral of the second kind usually denoted by E.

TABLE 9

$$\left[\delta^{-2}\right]_{el} = \frac{1}{2K_1^2}\left[1 + \frac{\left|K_1^2 - K_2^2\right|}{K_1^2 - K_2^2}\right] = \begin{cases} \frac{1}{K_1^2}, & K_1 > K_2 \\ 0, & K_1 < K_2 \end{cases}$$

$$\left[\delta^{-4}\right]_{el} = \frac{K_1^2 - K_2^2}{\left|K_1^2 - K_2^2\right|^3}$$

$$\left[\delta^{-6}\right]_{el} = \frac{K_1^4 + K_1^2K_2^2 - 2K_2^4}{\left|K_1^2 - K_2^2\right|^5}$$

$$\left[\delta^{-8}\right]_{el} = \frac{K_1^6 + 5\,K_1^4K_2^2 - 3\,K_1^2K_2^4 - 3\,K_2^6}{\left|K_1^2 - K_2^2\right|^7}$$

Since the inverse powers of δ^2 can not be used to describe aerodynamic moments for small angles, the actual moment curve would have to be approximated by at least two analytic segments. For example the Magnus moment could be represented by a cubic function of δ for $0 \leqq \delta \leqq \delta_a$, and a constant for $\delta_a \leqq \delta \leqq \delta_b$. The Magnus moment coefficient for this case would be determined by the following conditions:

$$K_T = K_{T_0} + K_{T_{\delta^2}}\,\delta^2, \qquad 0 \leqq \delta \leqq \delta_a \tag{B5}$$

$$K_T = K_{T_{\delta^{-1}}}\,\delta^{-1}, \qquad \delta_a \leqq \delta \leqq \delta_b \tag{B6}$$

where $K_{T_{\delta^{-1}}}\,\delta_a^{-1} = K_{T_0} + K_{T_{\delta^2}}\,\delta_a^2$.

As a consequence of this two segment approximation of the Magnus moment curve, three different types of yawing motion have to be considered:

(1) δ lying in the interval $(0, \delta_a)$ for which Eq. (B5) applies and the effective values of $\langle\delta^2\rangle$ can be used to describe the motion;

(2) δ lying in the interval (δ_a, δ_b) for which Eq. (B6) applies and the effective values of $\langle\delta^{-1}\rangle$ are used; and

(3) δ lying in both intervals for which both equations apply and the motion calculated by means of Eqs. (51 - 52).

The boundaries of the corresponding regions in the amplitude plane are

$$\delta_{max} = K_1 + K_2 = \delta_a$$

$$\delta_{min} = \left|K_1 - K_2\right| = \delta_a \qquad \text{(B7)}$$

$$\delta_{max} = K_1 + K_2 = \delta_b$$

These regions are shown in Figure 13.

It should be noted that the amplitude plane for arbitrary non-linear moments can be computed by numerical means from Eqs. (44 - 45). This integration would be simpler than that of the complete fourth order non-linear equation and the resulting amplitude plane would display simple relationships between initial conditions and the type of yawing motion.

APPENDIX C: AN IMPROVED FIRST APPROXIMATION

An important feature of the K-B equivalent linearization method which was developed in the report proper was the use of an epicycle with zero damping as the first approximation. This was defined by the equation

$$\tilde{\lambda} = K_{10}\, e^{i\tilde{\phi}_1} + K_{20}\, e^{i\tilde{\phi}_2} \qquad \text{(C1)}$$

where $\tilde{\phi}_j = \phi_{j0} + \tilde{\phi}_j{}'p$

$$\tilde{\phi}_j{}' = \frac{-\tilde{a}_2 \pm \sqrt{\tilde{a}_2^{\,2} + 4\tilde{b}_1}}{2}$$

$\tilde{a}_2^{\,2} + 4\tilde{b}_1 > 0$ and $\tilde{a}_2$ and $\tilde{b}_1$ are the values of a_2 and b_1 when λ and λ' are zero.

This selection of a first approximation suffers from the handicap that it can not be near the actual motion for amplitudes which make the non-linear a_2 and b_1 very different from their linear values. In particular, if the small amplitude values of a_2 and b_1 should not satisfy the inequality $\tilde{a}_2^{\,2} + 4\tilde{b}_1 > 0$, small amplitude epicyclic motion would be impossible and the first approximation would not exist. Yet it is quite possible that for certain non-linearities large amplitude epicyclic motion could occur.

For this reason we will consider, in this section, a better choice for the first approximation. From an examination of Eqs. (38) and (40) it can be seen that these equations could be considerably simplified if $\tilde{a}_2$ and $\tilde{b}_1$ had been selected to be the average values of a_2 and b_1. More precisely, they would be defined by the relations

$$\tilde{a}_2 = \frac{1}{2\pi} \int_0^{2\pi} a_2 \, d\phi \qquad \text{(C2)}$$

$$\tilde{b}_1 = \frac{1}{2\pi} \int_0^{2\pi} b_1 \, d\phi \qquad \text{(C3)}$$

Under these definitions Eqs. (38) and (40) would reduce to

$$\psi_1' = \frac{-K_2}{2\pi(\tilde{\phi}_1' - \tilde{\phi}_2')K_1} \int_0^{2\pi} \left\{ \left[(a_2 - \tilde{a}_2)\tilde{\phi}_2' - (b_1 - \tilde{b}_1) \right] \cos \phi - (a_1 \tilde{\phi}_2' + b_2) \sin \phi \right\} d\phi \tag{C4}$$

$$\psi_2' = \frac{-K_1}{2\pi(\tilde{\phi}_2' - \tilde{\phi}_1')K_2} \int_0^{2\pi} \left\{ \left[(a_2 - \tilde{a}_2)\tilde{\phi}_1' - (b_1 - \tilde{b}_1) \right] \cos \phi + (a_1 \tilde{\phi}_1' + b_2) \sin \phi \right\} d\phi \tag{C5}$$

This use of an improved first approximation which is based on definitions (C2) and (C3) would clearly overcome the problem of non-existent small amplitude epicyclic motion*. The remainder of the analysis would be identical with that of the text. The calculation of the amplitude planes would, however, be made more complicated by the need to consider the variation of $\tilde{\phi}_j'$ with amplitude.

---

* This definition also possesses the advantage of yielding the exact frequency of constant amplitude pure mode motion. This can be seen from the fact that if damping is neglected, Eq. (25) is linear for pure mode motion and the linear frequency formula for the pure mode is correct.

APPENDIX D: AMPLITUDE PLANES FOR MISSILES WITH NON-LINEAR DAMPING MOMENTS AND ZERO SPIN

A rather interesting special case of non-linear damping moment is that for zero spin. Since $\emptyset_1' = -\emptyset_2'$, Eqs. (44 - 45) assume the simple form*:

$$\frac{K_1'}{K_1} = -\frac{1}{4\pi} \int_0^{2\pi} H(\delta^2) \left[1 - \frac{K_2}{K_1} \cos \emptyset\right] d\emptyset = -\alpha_1(K_1^2, K_2^2) \quad (D1)$$

$$\frac{K_2'}{K_2} = -\frac{1}{4\pi} \int_0^{2\pi} H(\delta^2) \left[1 - \frac{K_1}{K_2} \cos \emptyset\right] d\emptyset = -\alpha_2(K_1^2, K_2^2) \quad (D2)$$

If H is assumed to be represented by a power series in δ^2, Eqs. (D1 - D2) become

$$\frac{K_1'}{K_1'} = -\frac{1}{2} \sum_{k=0}^{n} H_{2k} \left[\delta^{2k}\right]_{e1} \quad (D3)$$

$$\frac{K_2'}{K_2} = -\frac{1}{2} \sum_{k=0}^{n} H_{2k} \left[\delta^{2k}\right]_{e2} \quad (D4)$$

where $H = \sum_{k=0}^{n} H_{2k} \delta^{2k}$

$$\left[\delta^{2k}\right]_{e1} = \frac{1}{2\pi} \int_0^{2\pi} \delta^{2k} \left(1 - \frac{K_2}{K_1} \cos \emptyset\right) d\emptyset$$

$$\left[\delta^{2k}\right]_{e2} = \frac{1}{2\pi} \int_0^{2\pi} \delta^{2k} \left(1 - \frac{K_1}{K_2} \cos \emptyset\right) d\emptyset$$

---

* The small gravity term and small J_L' term have been neglected for simplicity.

The definitions of $\left[\delta^{2k}\right]_{\underline{e}j}$ should have a simple relationship with the $\left[\delta^{2k}\right]_{ej}$ listed in Table I of Reference 6. If the sign in front of the coefficient of the brackets which appear in that table is changed from plus to minus, this new effective yaw can be computed. For example

$$\left[\delta^{2}\right]_{\underline{e}1} = K_1^{\,2} + K_2^{\,2} - \left[K_2^{\,2}\right] = K_1^{\,2}, \tag{D5}$$

$$\begin{aligned}\left[\delta^{4}\right]_{\underline{e}1} &= (K_1^{\,2} + K_2^{\,2})^2 + 2K_1^{\,2}K_2^{\,2} - 2(K_1^{\,2} + K_2^{\,2}) \left[K_2^{\,2}\right] \\ &= K_1^{\,4} + 2K_1^{\,2}K_2^{\,2} - K_2^{\,4}.\end{aligned} \tag{D6}$$

From symmetry,

$$\left[\delta^{2}\right]_{\underline{e}2} = K_2^{\,2}, \tag{D7}$$

$$\left[\delta^{4}\right]_{\underline{e}2} = K_2^{\,4} + 2K_1^{\,2}K_2^{\,2} - K_1^{\,4}. \tag{D8}$$

Note. If H had been assumed to be a polynomial function of $\left|\lambda'\right|^2 = \lambda'\bar{\lambda}'$, an interesting simplification would follow. For $\tilde{\phi}_1' = -\tilde{\phi}_2'$, Eq. (34) becomes

$$\lambda'\bar{\lambda}' = \tilde{\phi}_1'^{\,2} \left[K_1^{\,2} + K_2^{\,2} - 2K_1K_2 \cos (\tilde{\phi}_1 - \tilde{\phi}_2)\right]. \tag{D9}$$

$$\therefore \quad \left[(\lambda'\bar{\lambda}')^k\right]_{\underline{e}1} = \frac{\tilde{\phi}^{2k}}{2\pi} \int_0^{2\pi} (K_1^{\,2} + K_2^{\,2} - 2K_1K_2 \cos \phi)^k \left(1 - \frac{K_2}{K_1} \cos \phi\right) d\phi. \tag{D10}$$

Replacing ϕ by $\phi + \pi$ we, therefore, see that

$$\left[(\lambda'\bar{\lambda}')^k\right]_{\underline{e}j} = \tilde{\phi}^{2k} \left[\delta^{2k}\right]_{ej} . \tag{D11}$$

Returning to the assumption that H is a function of δ^2 we now consider the properties of amplitude planes associated with Eqs. (D3 - D4). Since these relations are symmetric in the two modes, we expect all amplitude planes to be symmetric with respect to the line $K_1^2 = K_2^2$. The first consequence of this symmetry lies in the fact that the origin must be a node.

In order to consider other singularities on the axes we expand H in a Taylor expansion about the amplitude K^2.

$$H = H(K^2) + H'(K^2) \left[\delta^2 - K^2\right] \tag{D12}$$

$$\therefore \alpha_1 = \frac{1}{2} \left\{ H(K^2) + H'(K^2) \left[K_1^2 - K^2\right] \right\} \tag{D13}$$

$$\alpha_2 = \frac{1}{2} \left\{ H(K^2) + H'(K^2) \left[K_2^2 - K^2\right] \right\} \tag{D14}$$

If the singularity is on the one axis at $(K^2, 0)$, α_1 vanishes and from Eq. (D13) it can be seen that

$$H(K^2) = 0. \tag{D15}$$

Shifting the origin to the singular point by the translation $x = K_1^2 - K^2$, $y = K_2^2$, the differential equation for the amplitude plane becomes

$$\frac{dy}{dx} = \frac{-y\ (y - K^2)}{x\ (x + K^2)} \tag{D16}$$

The usual tests show that the singularity must be a saddle. Therefore, all pure mode singularities with the exception of the origin must be saddles.

The case for a quadratic H is quite simple. The damping curves are given by Eqs. (D13 - D14) for $K^2 = 0$, $H_0 = H(0)$ and $H_2 = H'(0)$. If H_0 and H_2 have the same sign, the origin is the only singularity. (Fig. 14a). If they are of unlike sign, three more singularities appear in the first quadrant. The two pure mode singularities at $(-\frac{H_0}{H_2}, 0)$ and $(0, -\frac{H_0}{H_2})$ have to be saddles while it can be shown that the singularity at $(-\frac{H_0}{H_2}, -\frac{H_0}{H_2})$ is a node. (Fig. 14b).

For negative H_0, this second node is a stable node and the limit motion is planar yawing motion with amplitude, $K_1 + K_2 = 2\sqrt{-\frac{H_0}{H_2}}$. For planar yawing motion, the yaw equation reduces to a real equation in the magnitude of yaw, δ.

$$\delta'' + (H_0 + H_2\,\delta^2)\delta' - M\,\delta = 0, \quad M < 0 \tag{D17}$$

Eq. (D17) is the well-known van der Pol Equation and our predicted amplitude agrees with the amplitude obtained by other methods.*

If a quartic H is considered, singularities can appear off the axes in three different ways. In Table 10 the off-axis-singularities for these three cases are listed and the amplitude planes for cases B and C are given in Figs. 15 and 16. (Since the amplitude plane for case A is essentially a distortion of Fig. 14b, it is not shown). The limit motion for case C is particularly interesting. It is an ellipse with semi-major axis $r_4 + r_3$ and semi-minor axis $r_4 - r_3$.

* The interesting character of the amplitude plane for no spin, quadratic H, and $H_0 H_2 < 0$ was pointed out to the author by H. L. Reed.

TABLE 10

OFF-AXIS SINGULARITIES FOR $H = H_0 + H_2\,\delta^2 + H_4\,\delta^4$

Case A

$$\frac{H_0}{H_4} < 0$$

$(r_2^{\,2}, r_2^{\,2})$ node

Case B

$$0 < \frac{H_0}{H_4} < \frac{1}{8}\left[\frac{H_2}{H_4}\right]^2; \qquad \frac{H_2}{H_4} < 0$$

$(r_1^{\,2}, r_1^{\,2})$ node

$(r_2^{\,2}, r_2^{\,2})$ node

Case C

$$\frac{1}{8}\left[\frac{H_2}{H_4}\right]^2 < \frac{H_0}{H_4} < \frac{1}{4}\left[\frac{H_2}{H_4}\right]^2; \qquad \frac{H_2}{H_4} < 0$$

$(r_3^{\,2}, r_4^{\,2})$ center*

$(r_4^{\,2}, r_3^{\,2})$ center*

where
$$r_1^{\,2} = -\frac{1}{4}\left\{\left[\frac{H_2}{H_4}\right] + \sqrt{\left[\frac{H_2}{H_4}\right]^2 - 8\left[\frac{H_0}{H_4}\right]}\right\}$$

$$r_2^{\,2} = -\frac{1}{4}\left\{\frac{H_2}{H_4} - \sqrt{\left[\frac{H_2}{H_4}\right]^2 - 8\left[\frac{H_0}{H_4}\right]}\right\}$$

$$r_3^{\,2} = -\frac{1}{4}\left\{\frac{H_2}{H_4} + \sqrt{8\left[\frac{H_0}{H_4}\right] - \left[\frac{H_2}{H_4}\right]^2}\right\}$$

$$r_4^{\,2} = -\frac{1}{4}\left\{\frac{H_2}{H_4} - \sqrt{8\left[\frac{H_0}{H_4}\right] - \left[\frac{H_2}{H_4}\right]^2}\right\}$$

* According to the criteria of Ref. 14, this point is a center if the cubic terms which appear in the numerator and denominator do not affect the character of the singularity. Fig. 16 is based on this assumption.

AMPLITUDE PLANE FOR LINEAR EQUATION

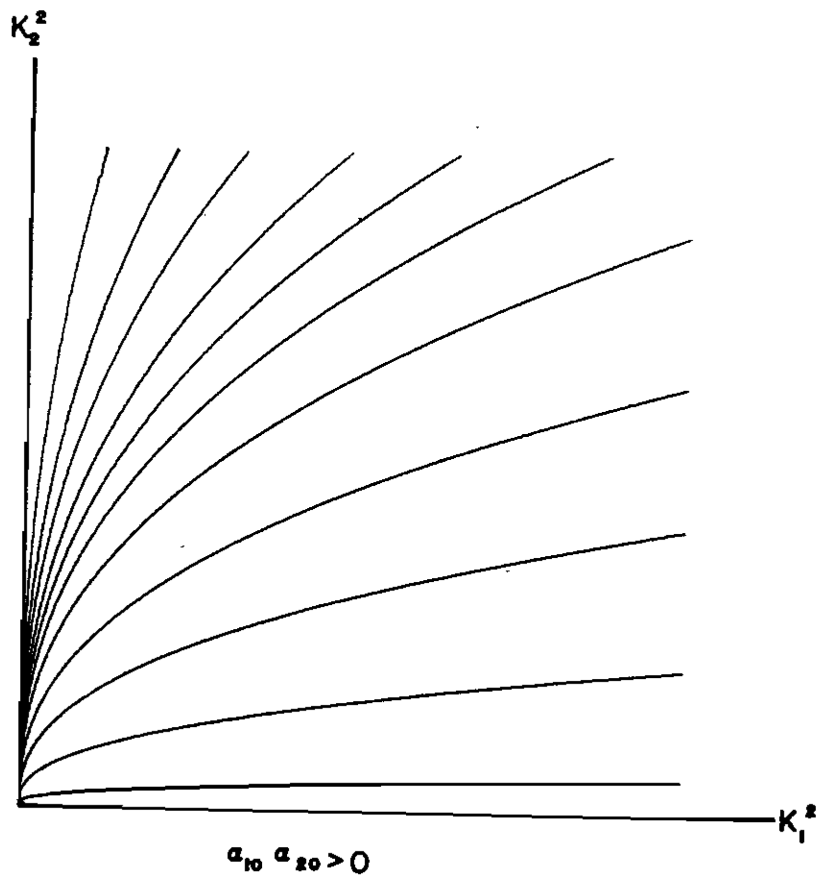


FIG. 1a

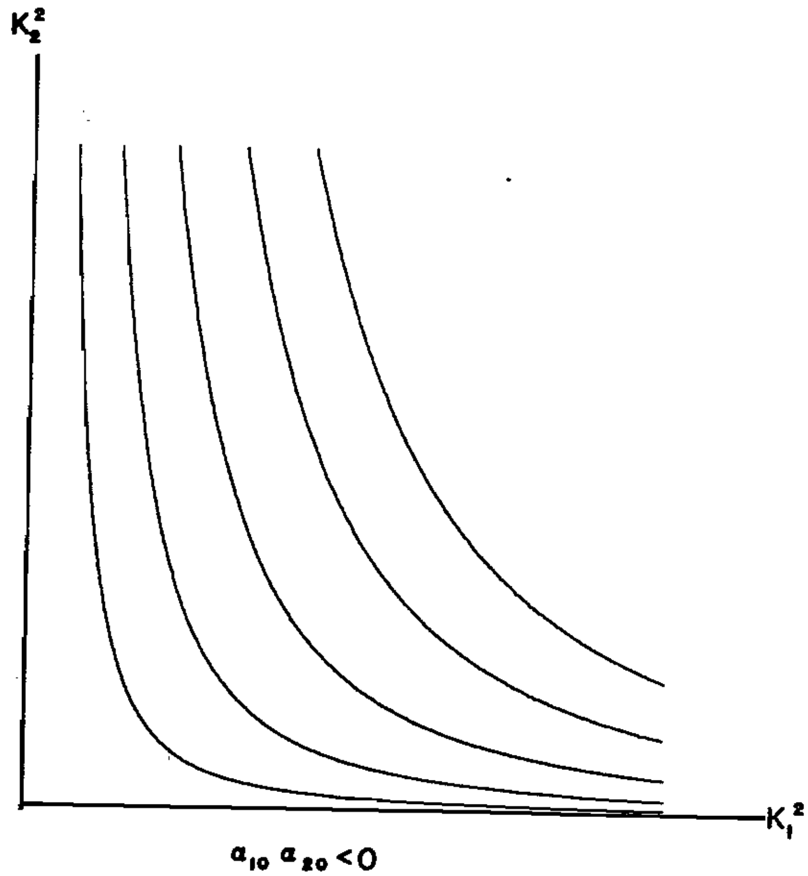


FIG. 1b

AMPLITUDE PLANE FOR CUBIC MAGNUS MOMENT ($\alpha_{10}\alpha_{20} > 0$)

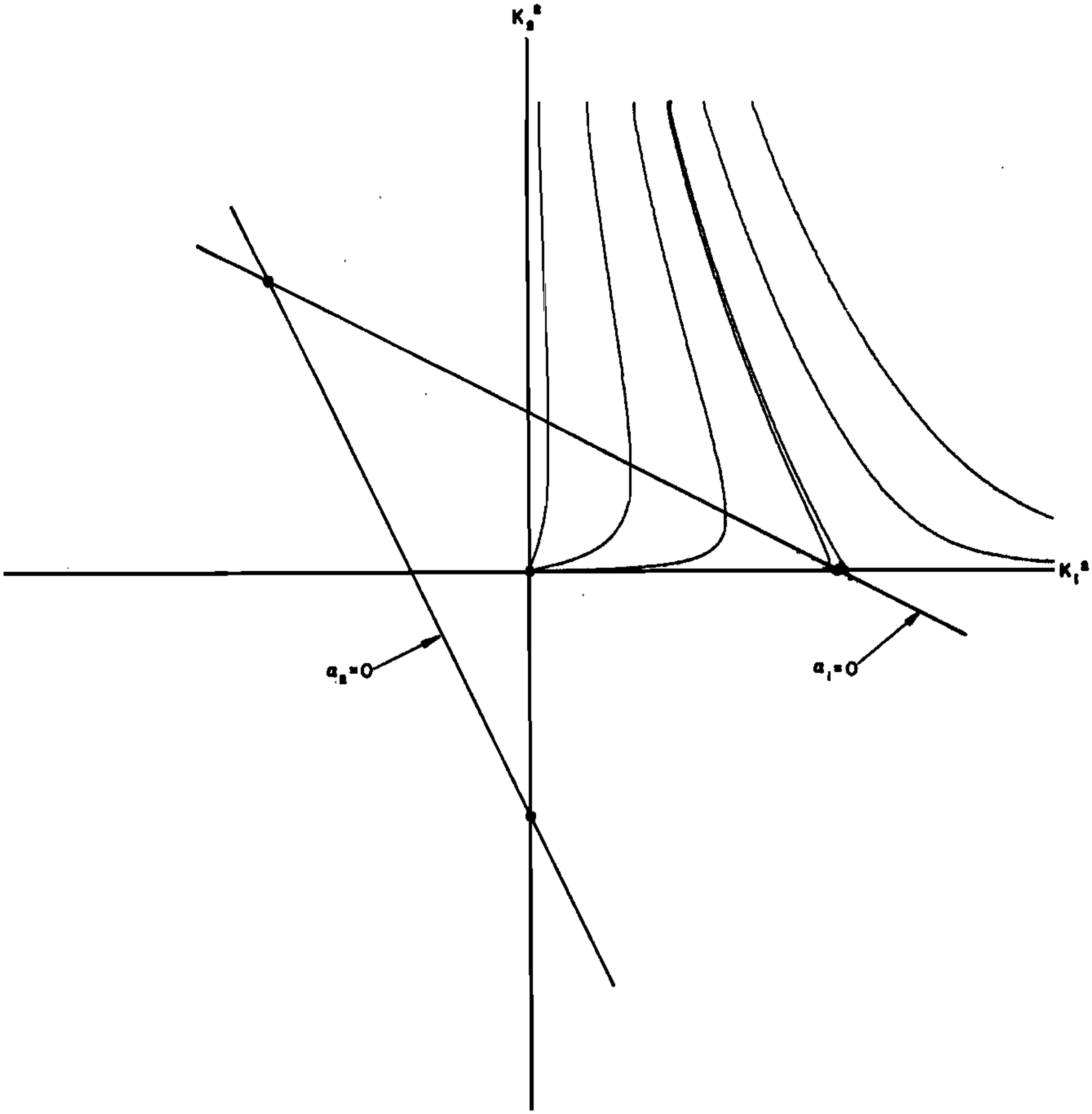


FIG. 2

AMPLITUDE PLANE FOR CUBIC MAGNUS MOMENT ($\alpha_{10}\,\alpha_{20} < 0$)

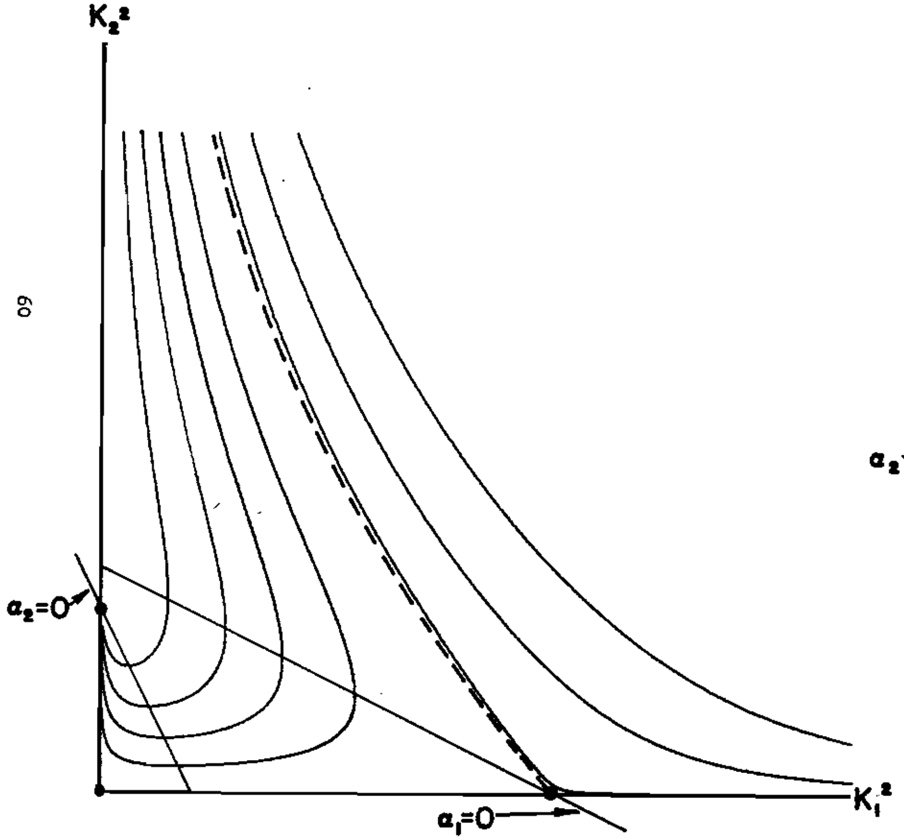


FIG. 3a

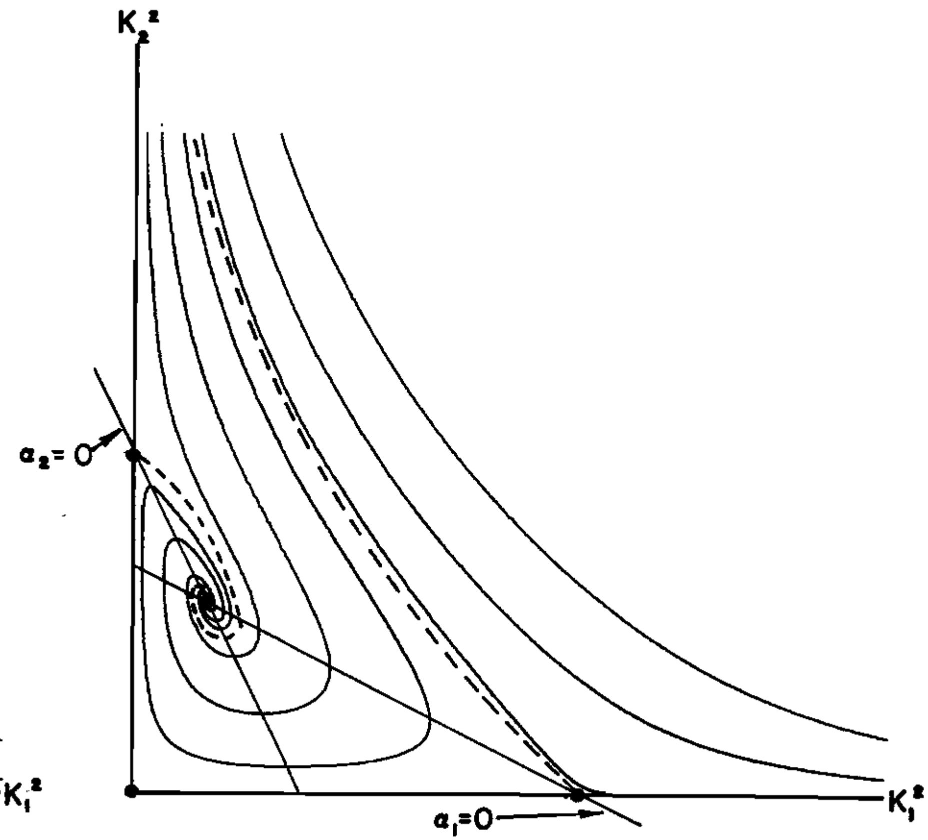


FIG. 3b

AMPLITUDE PLANE FOR CUBIC MAGNUS MOMENT ($\alpha_{10} = -\alpha_{20}$)

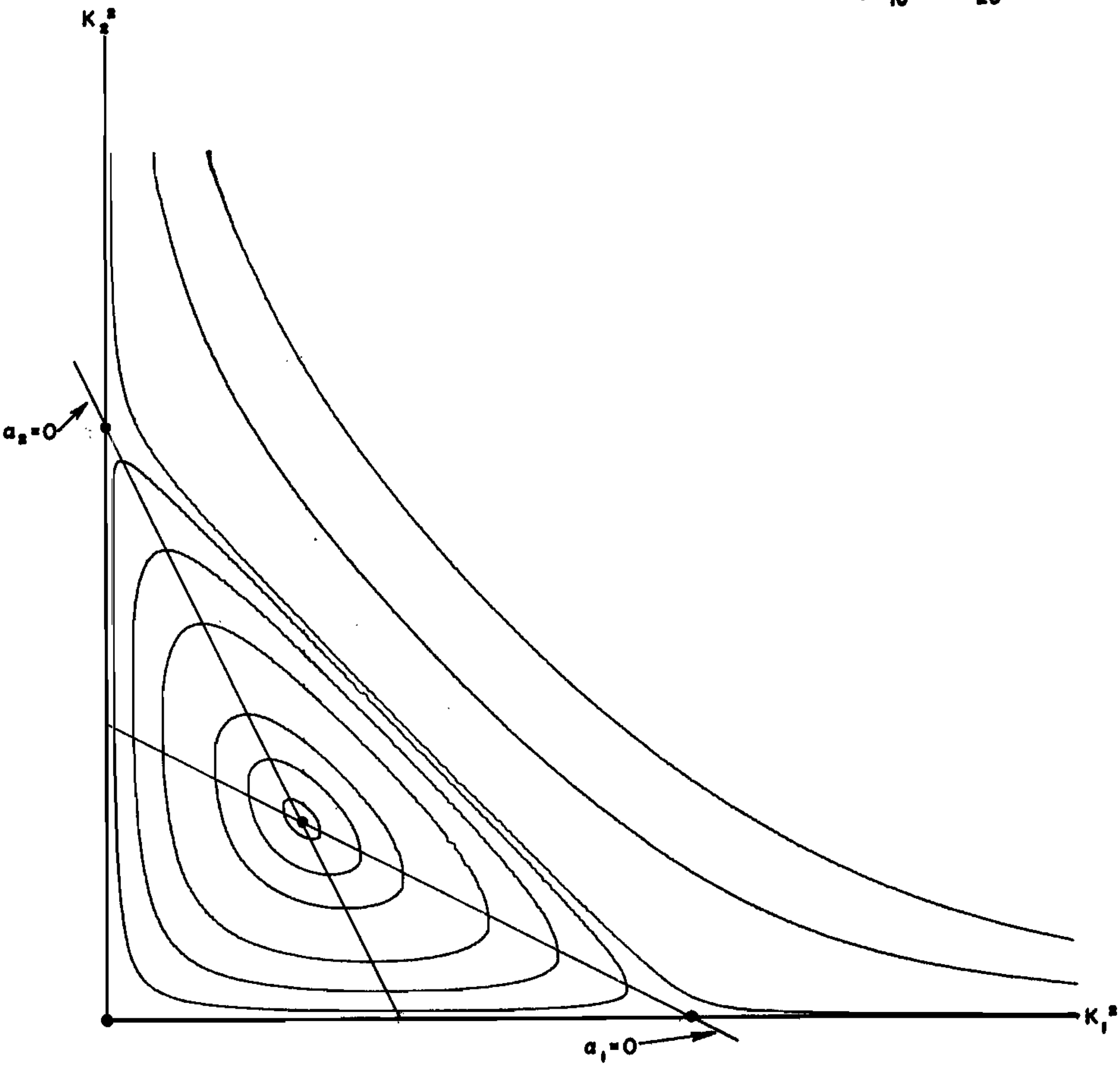


FIG. 4

PORTION OF AMPLITUDE PLANE FOR CASE I

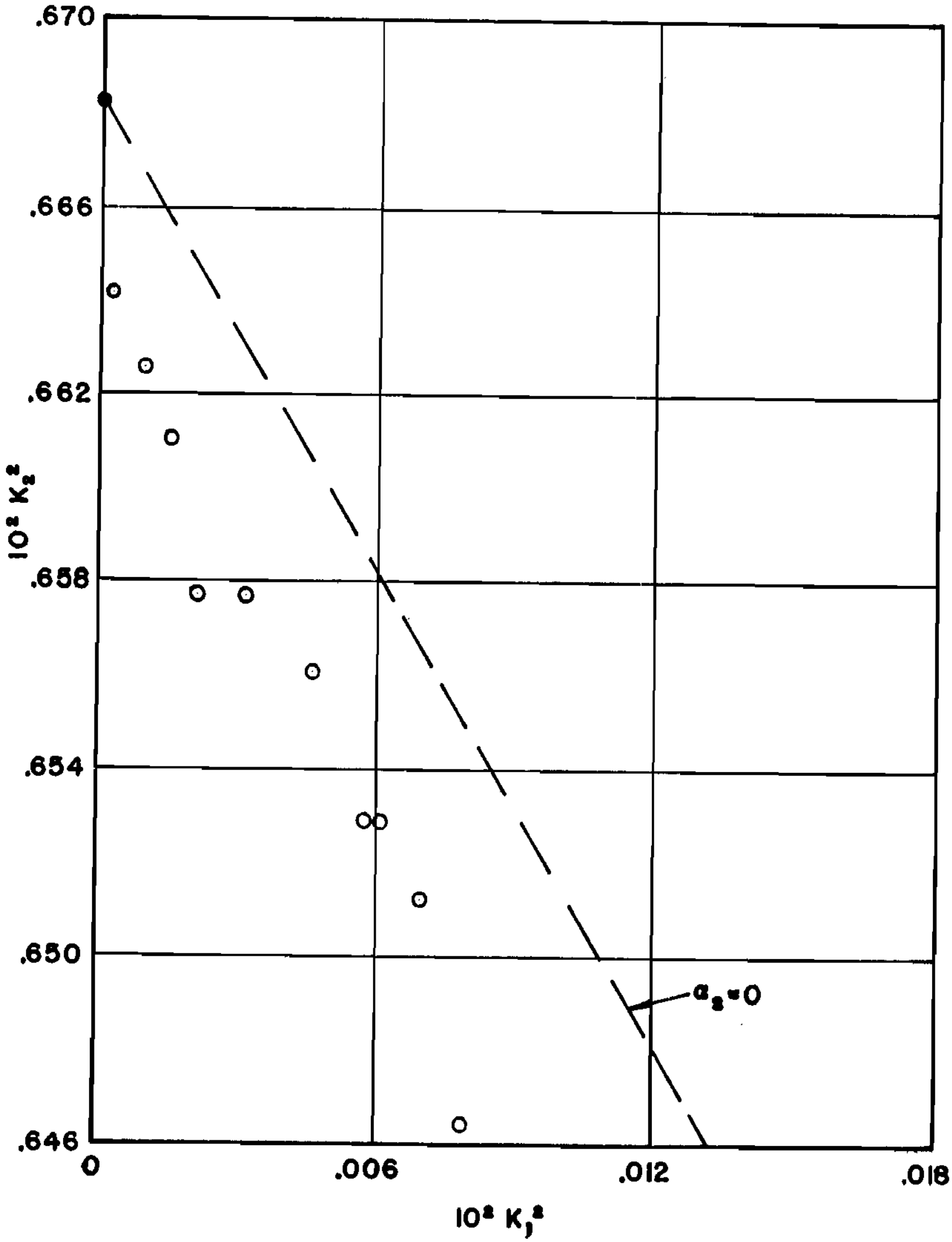


FIG. 5

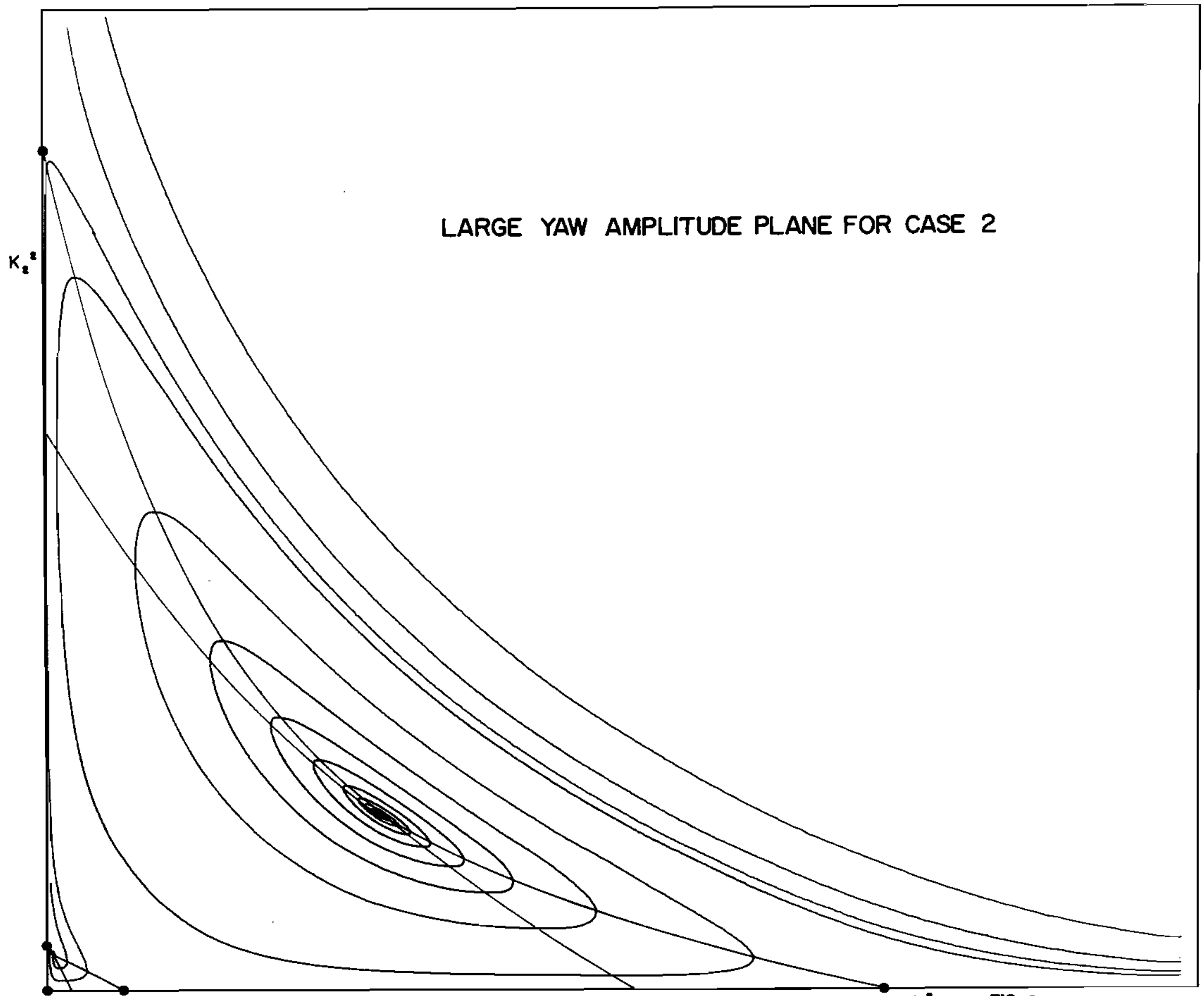


FIG. 6

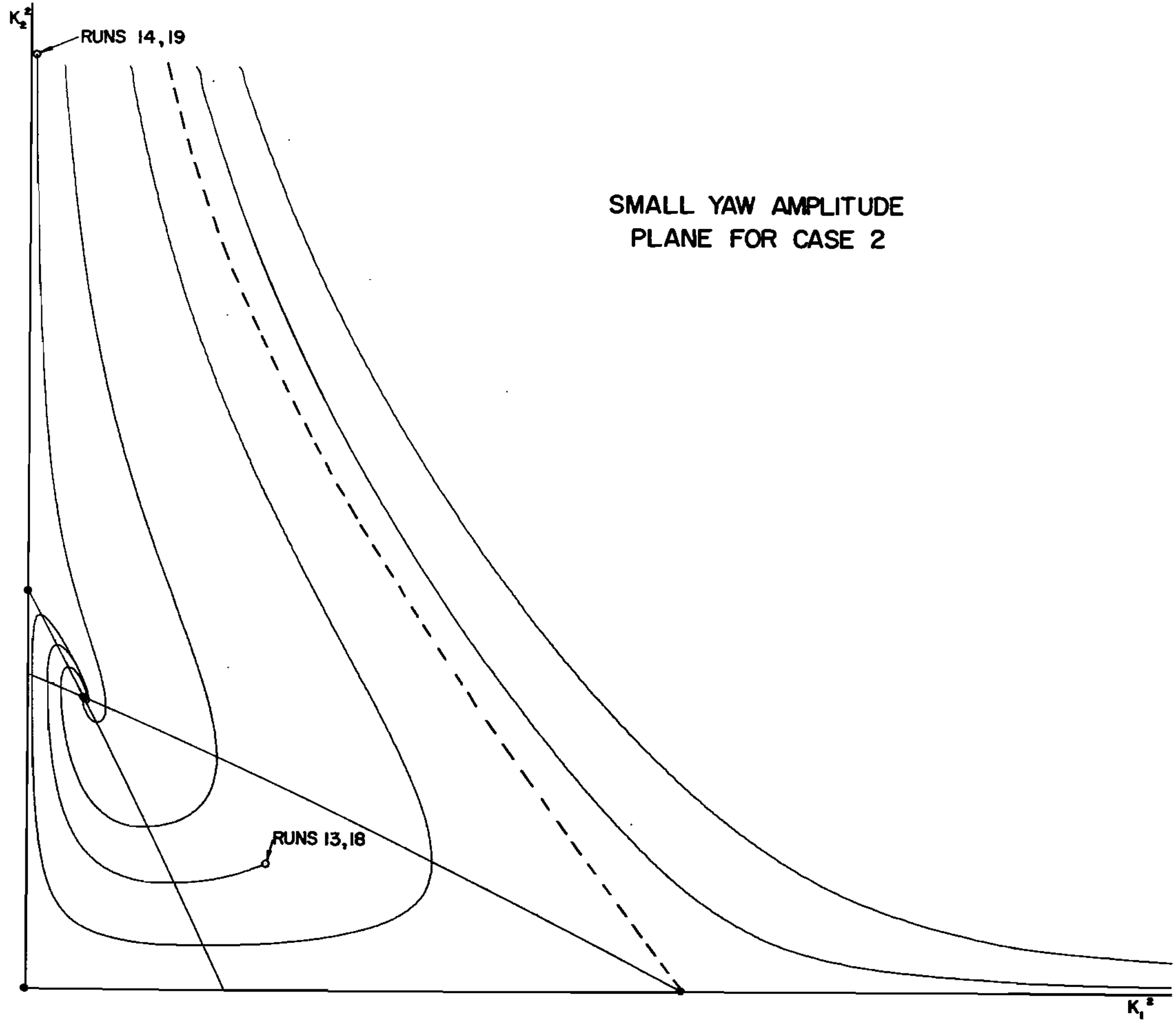

SMALL YAW AMPLITUDE
PLANE FOR CASE 2
K_2^2
RUNS 14, 19
RUNS 13, 18
K_1^2

FIG. 7

K_1^2 AND K_2^2 vs. PERIODS OF δ^2

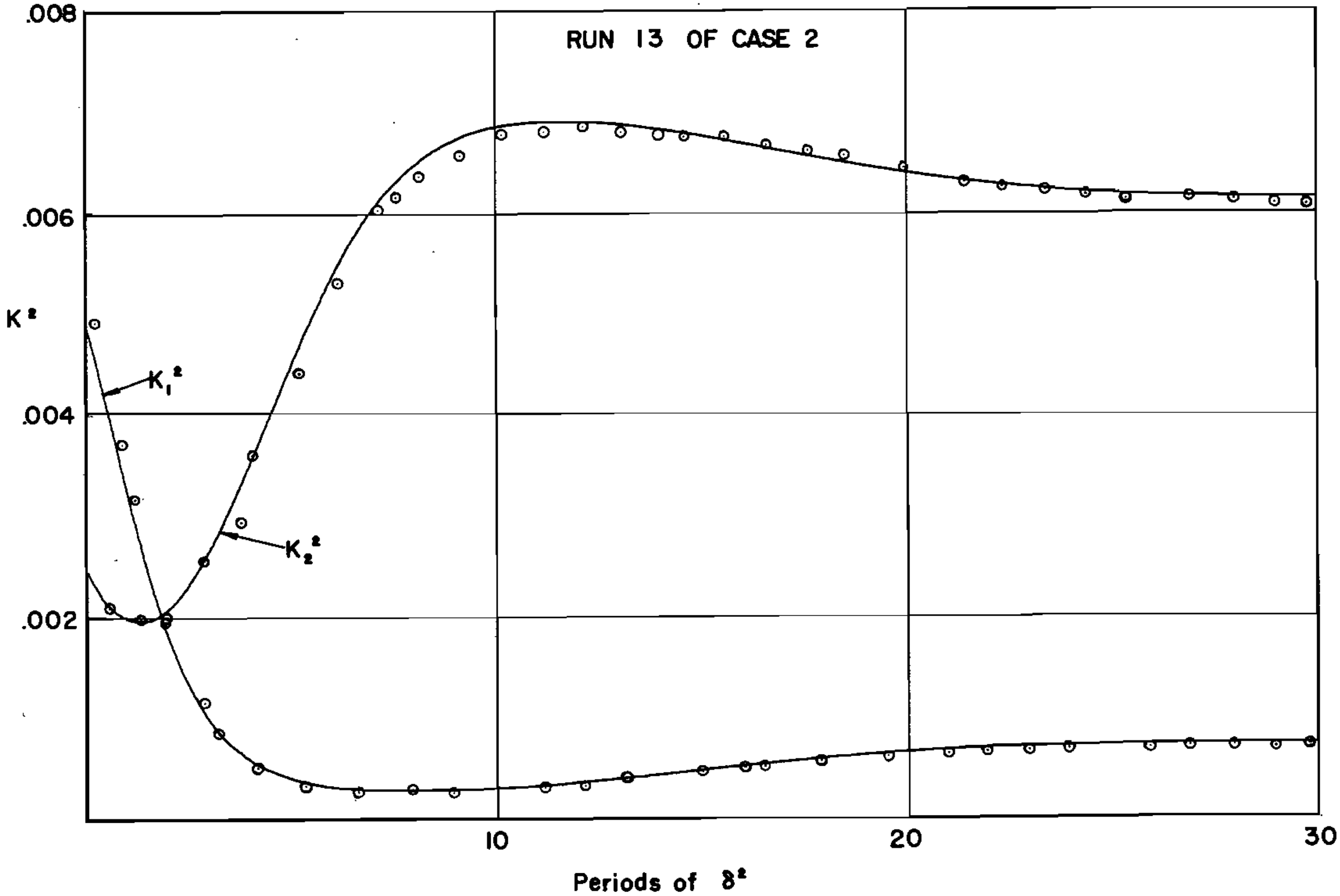


FIG. 8

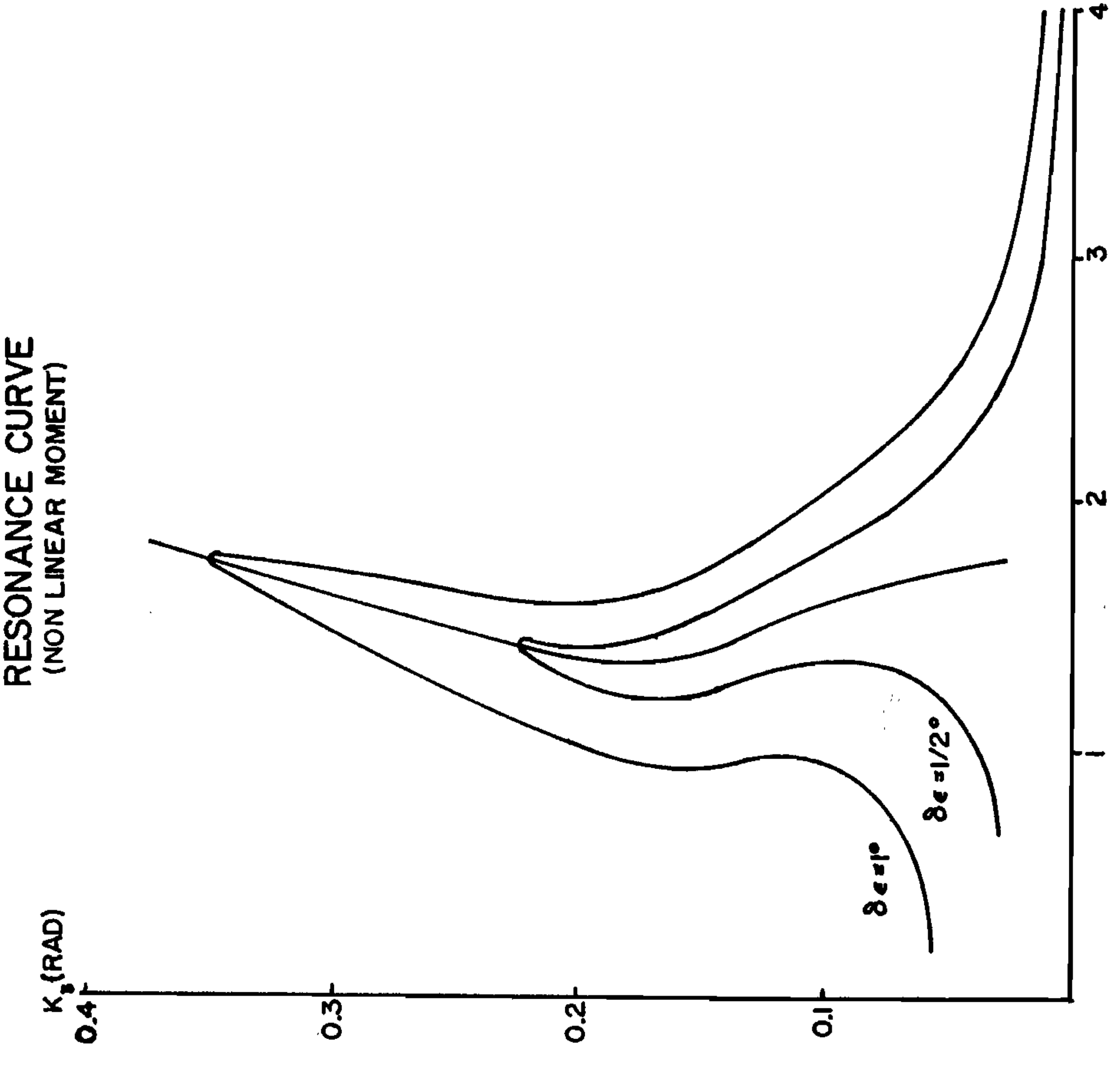

RESONANCE CURVE
(NON LINEAR MOMENT)
K_s (RAD)
0.4
0.3
0.2
0.1
1
2
3
4
$10^2 \nu$
$\delta_e = 1°$
$\delta_e = 1/2°$

FIG. 9

YAW OF REPOSE vs DISTANCE

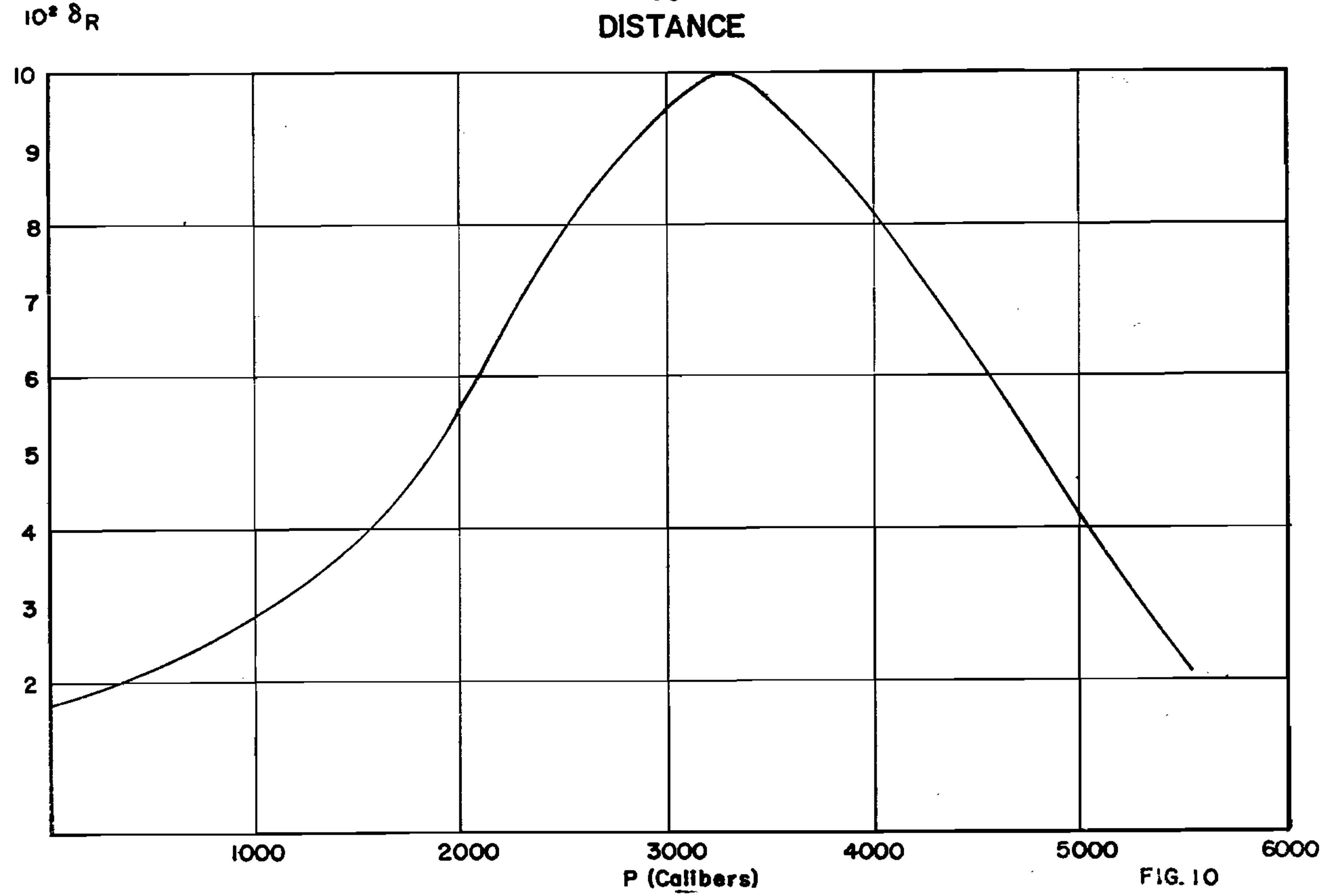


FIG. 10

INITIAL AMPLITUDE PLANE (REF. 8)

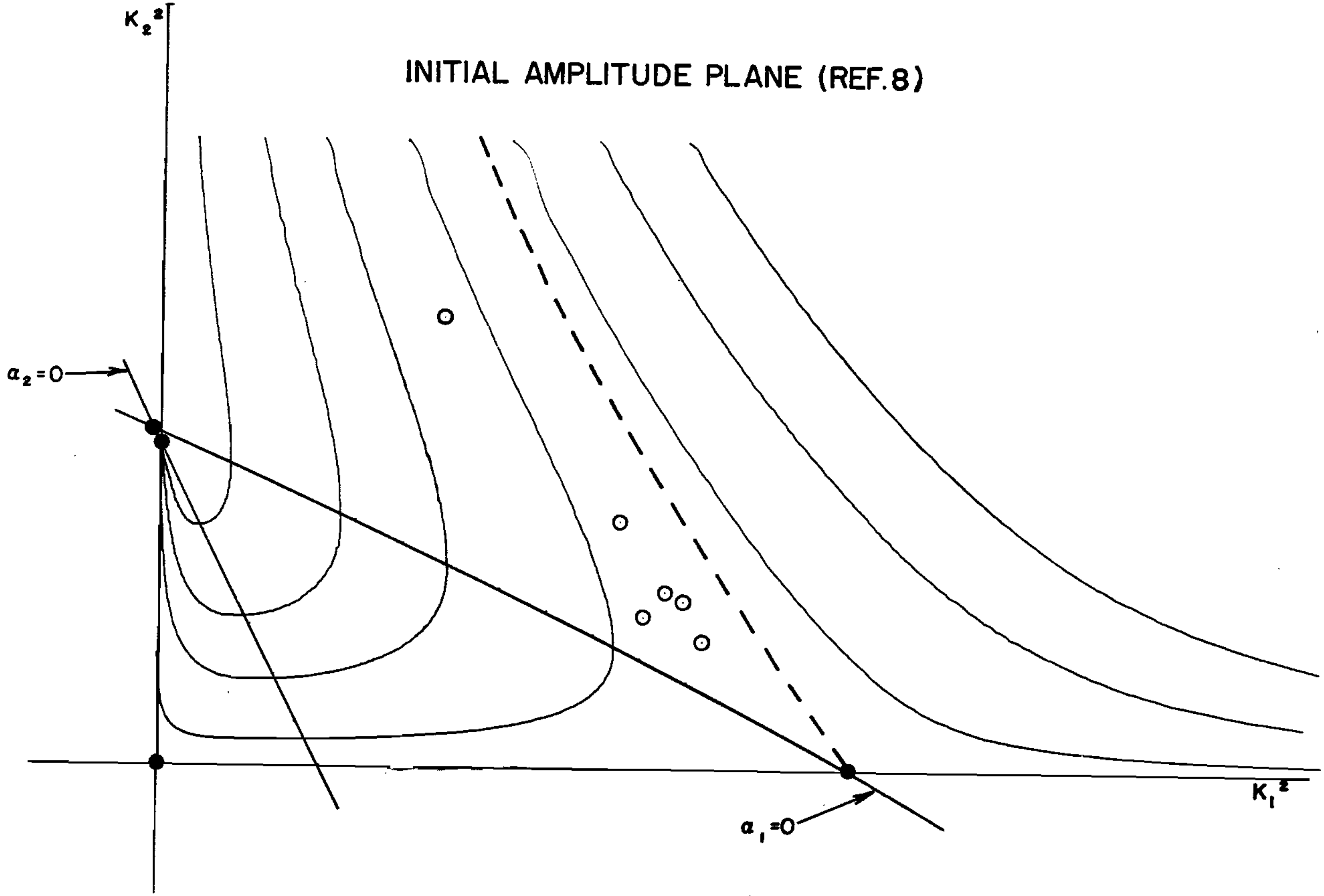


FIG. 11

SUMMITAL AMPLITUDE PLANE
(REF. 8)

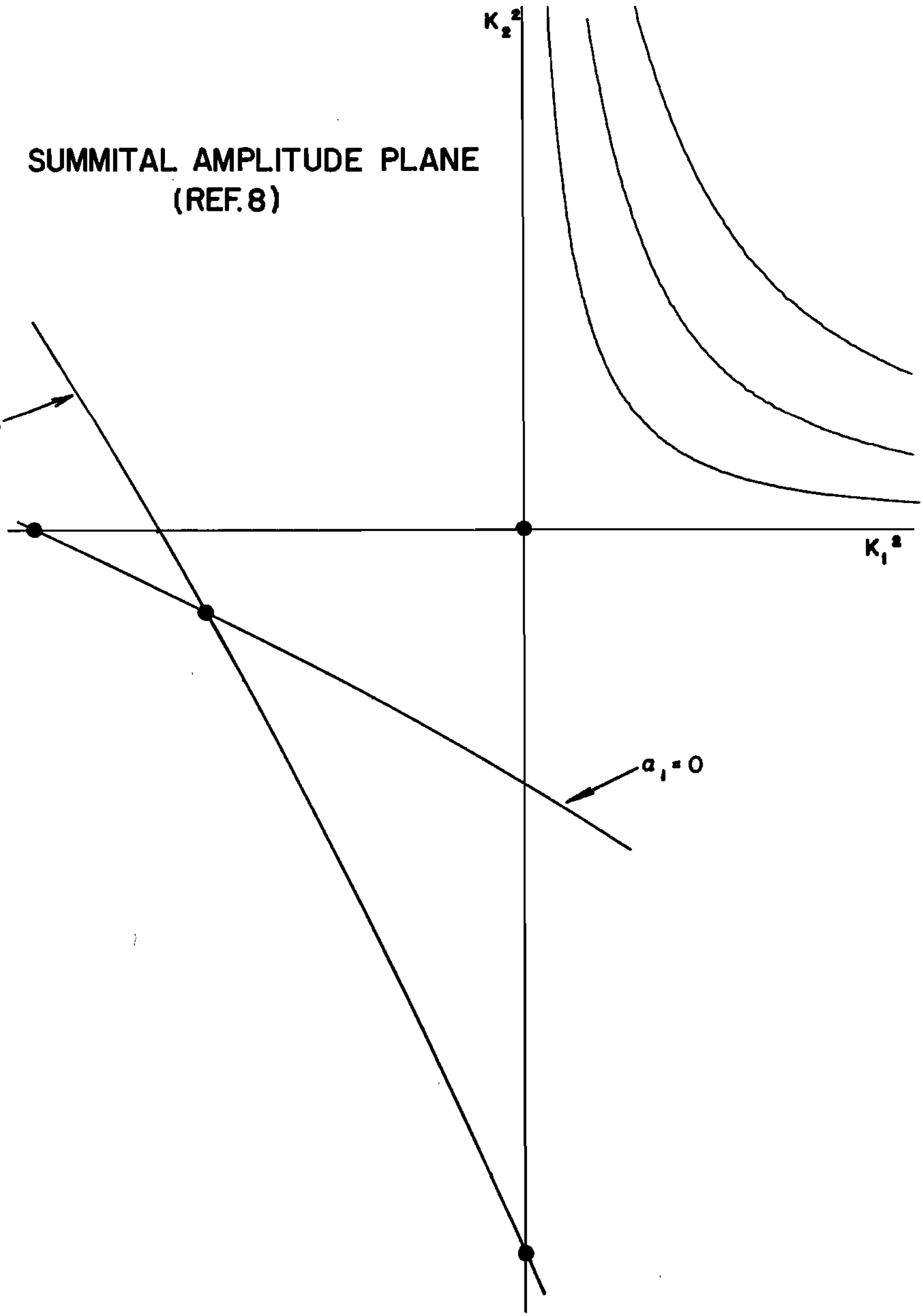


FIG. 12

AMPLITUDE PLANE FOR TWO SEGMENTS

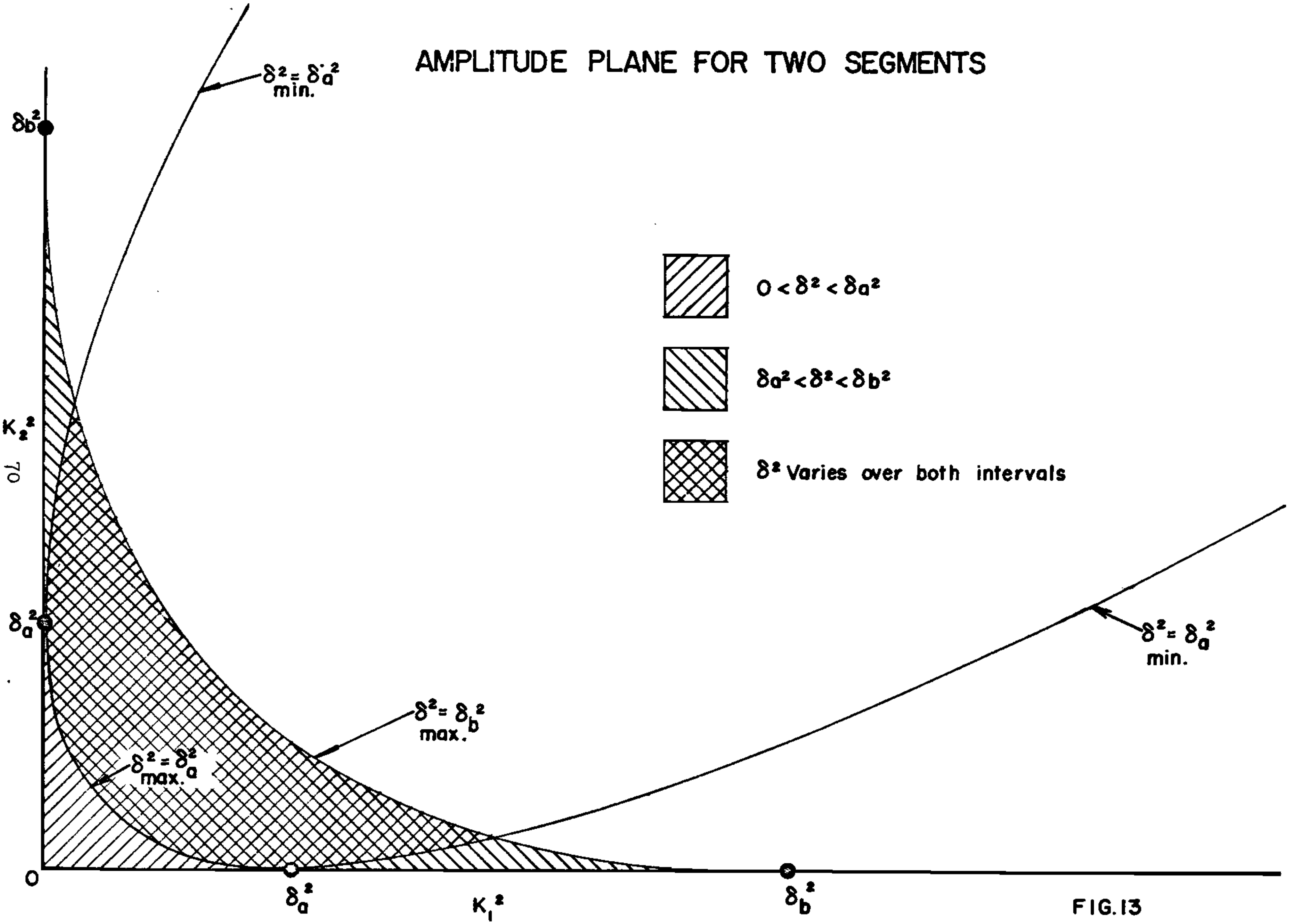
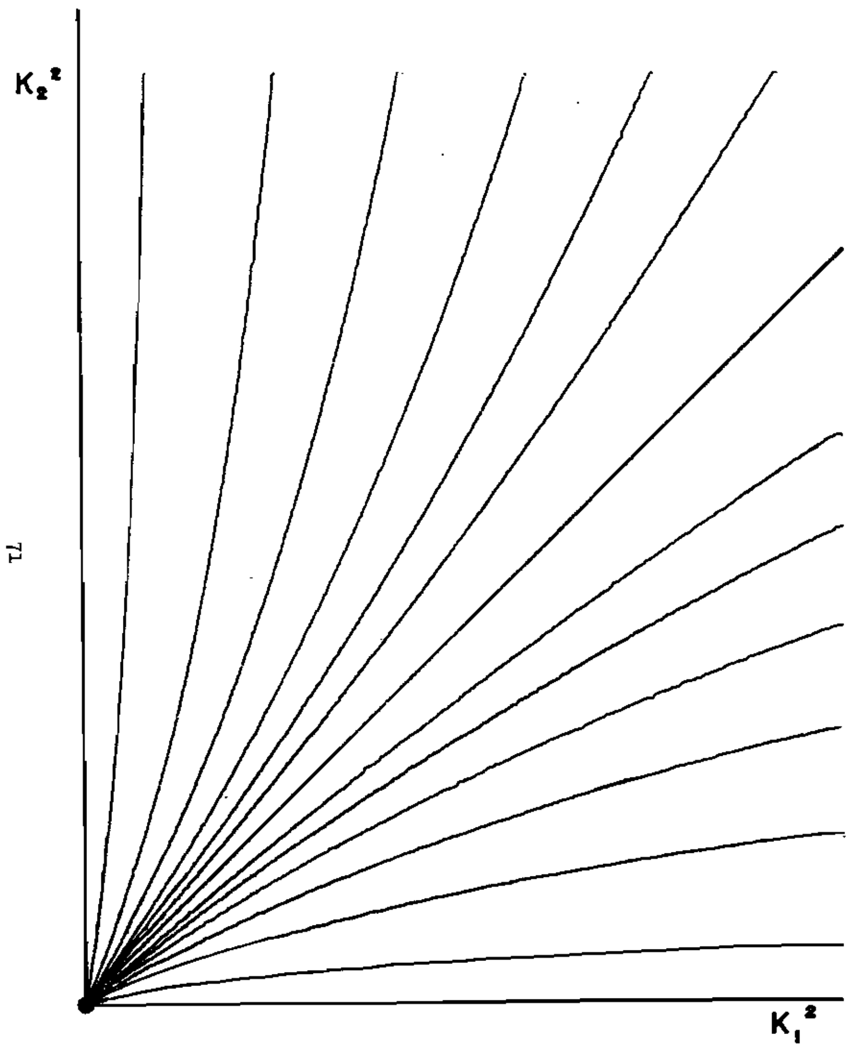
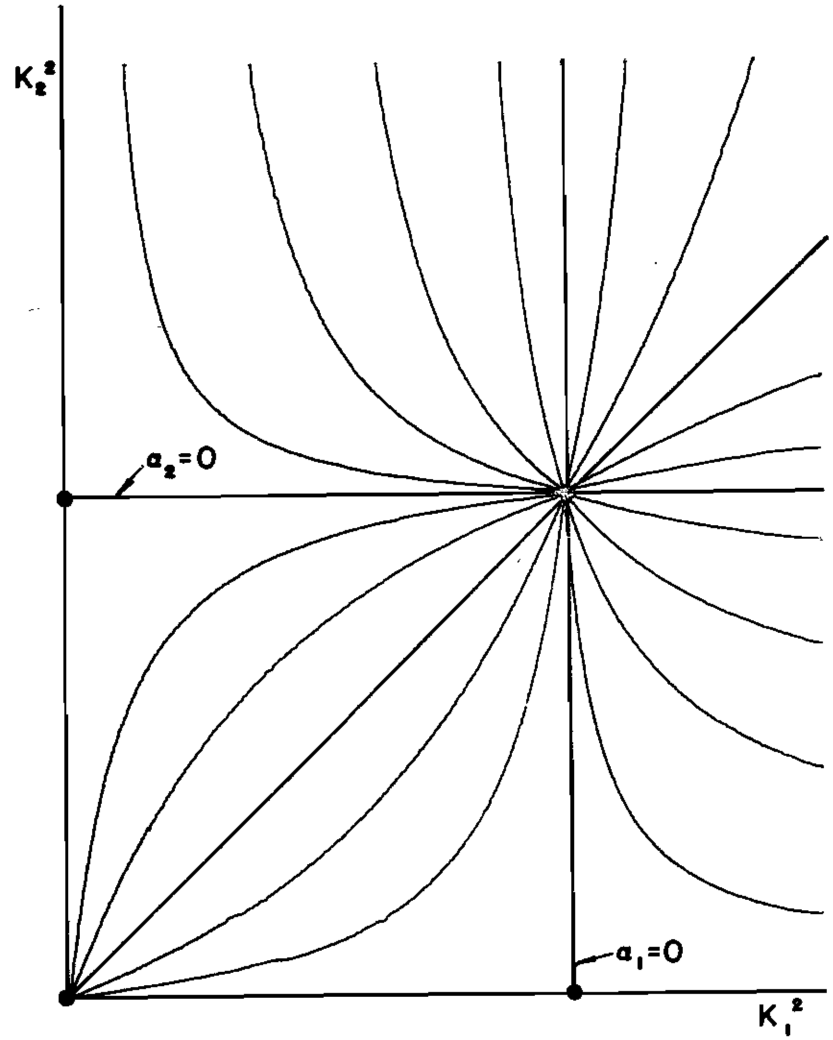


FIG. 13



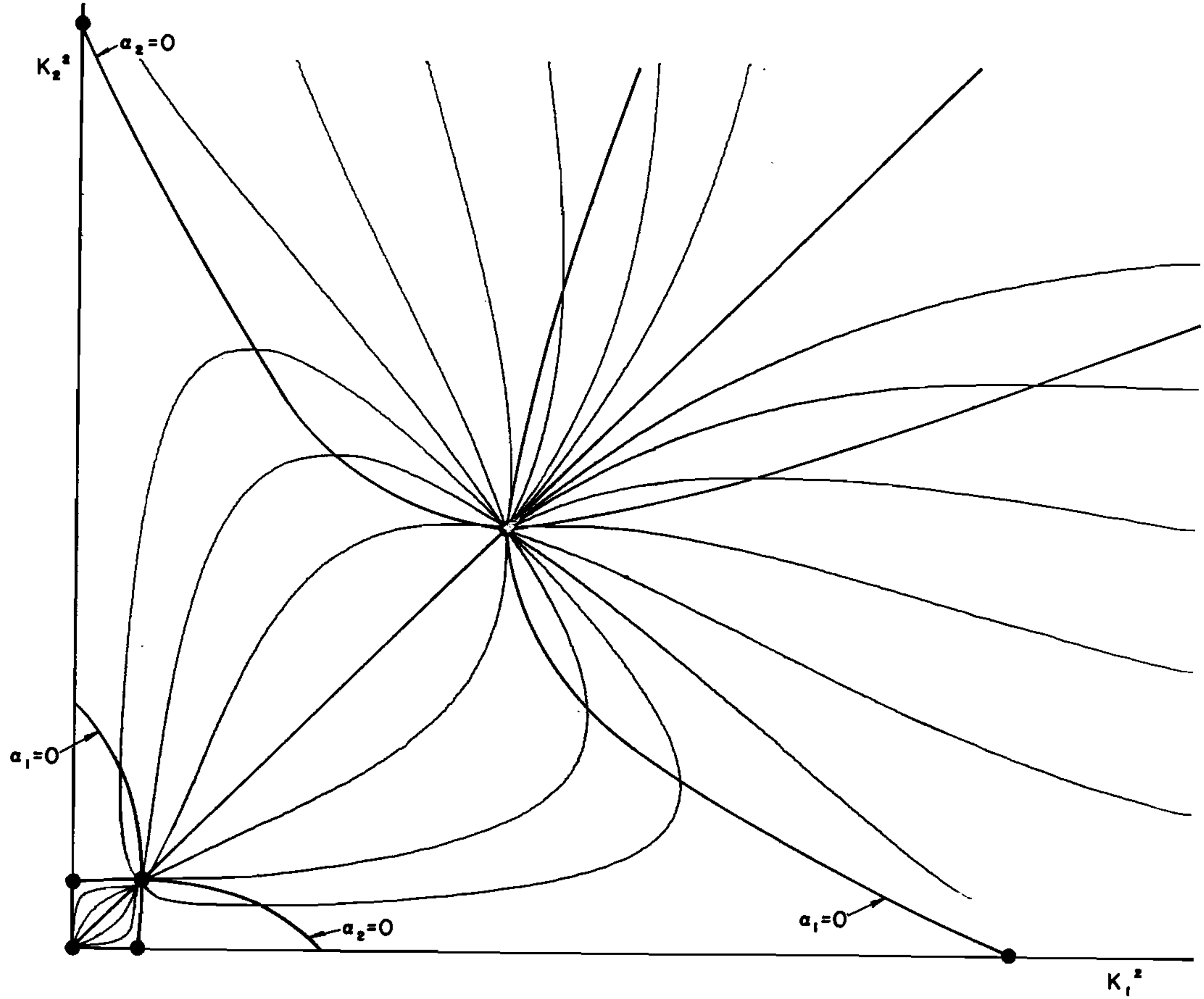
CONSTANT H

FIG. 14a



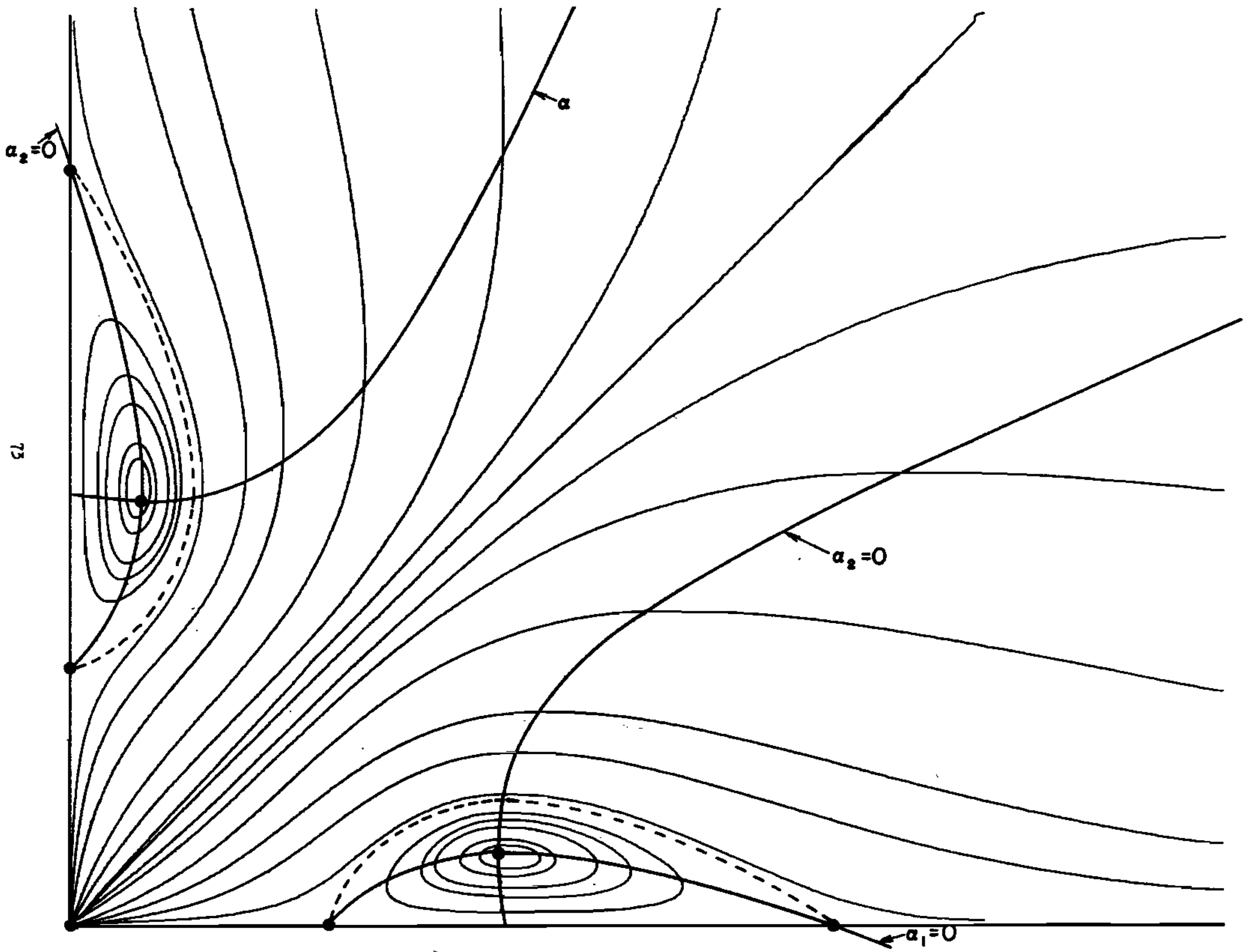
QUADRATIC $H, (H_0H_2<0)$

FIG. 14b



AMPLITUDE PLANE FOR CASE B

FIG. 15



AMPLITUDE PLANE FOR CASE C

FIG.

REFERENCES

1. Highberg, I., NOTS TN 5036-94 (July 1951).

2. Zaroodny, S.J., Spiral Yawing Motions of 81-mm M56 Shell - A Study In Non-Linear Theory, BRLM 682 (May 1953).

3. Davis, L., Follin, J.W., Blitzer, L., Exterior Ballistics of Rockets, Office of Naval Research, P-1002 (July 1953)

4. Lietmann, G., Nonlinear Equations of Motion of Spin-Stabilized Missile Solved by a Perturbation Method, NAVORD 3364 (Aug. 1954).

5. Kryloff, N., and Bogoliuboff, N.B., Introduction to Non-Linear Mechanics, translated by S. Lefschetz, Princeton University Press (1947).

6. Murphy, C.H., The Measurement of Non-Linear Forces and Moments by Means of Free Flight Tests, BRL Report 974 (Feb. 1956).

7. Cohen, C.J., and Hubbard, E.C., Unpublished NORC Computations, (May 1956).

8. Cohen, C.J., and Hubbard, E.C., Predictions of the Motion of a Slowly Spinning Research Rocket, NPG Report 1421 (Nov. 1955)

9. Murphy, C.H., On Stability Criteria of the Kelley-McShane Linearized Theory of Yawing Motion, BRL Report 853 (April 1953).

10. Poincare, H., Sur les Courber Definies par une Equation Differentielle, Oeuvies, Gauthier-Villon, Paris, Vol. I (1892).

11. Stoker, J.J., Nonlinear Vibrations in Mechanical and Electrical Systems, Interscience Publishers, New York (1950).

12. Nicolaides, J.D., On the Free Flight Motion of Missiles Having Slight Configurational Asymmetries, BRL Report 858 (June 1953).

13. Minorski, N., Introduction to Non-Linear Mechanics, J. W. Edwards, Ann Arbor (1947).

14. Frommer, M., Uber das Auftreten von Wirbeln und Strudeln (geschlossener und spiraliger Integralkurven) in der Umgebung rationaler Unbestimmtheitsstellen, Math. Ann. 109 (1934) 395 - 424.

15. Maple, C. G., Synge, J. L., Aerodynamic Symmetry of Projectiles, Q.A.M. Vol. IV, No. 4, Jan. 1949.

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