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Underwater Explosion Bubbles III. The Effects Of The Surface And The Bottom On The Shape And Motion Of The Bubble.

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UNDERWATER EXPLOSION BUBBLES III. THE EFFECTS OF THE SURFACE AND THE BOTTOM ON THE SHAPE AND MOTION OF THE BUBBLE.

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CONTENTS

UNDERWATER EXPLOSION BUBBLES III: THE EFFECT OF THE SURFACE AND THE BOTTOM ON THE SHAPE AND MOTION OF THE BUBBLE.

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I. Introduction.

In the absence of gravity and in an unbounded liquid, an initially spherical underwater explosion bubble would remain spherical at all times. It would perform radial oscillations, while the center of the bubble would remain at rest, and if the liquid were incompressible these oscillations would be periodic and undamped. In our first report, "Underwater Explosion Bubbles I", [1], we considered a compressible liquid and showed that the oscillations would then be damped. In our second report, "Underwater Explosion Bubbles II", [2], we considered gravity and showed how it would effect the shape of the bubble and also cause it to rise. In the present report we consider the effects of the water surface and bottom in addition to gravity, and show how they effect the shape and motion of the bubble.

Of course all of these effects have been considered to some extent by other authors. However we have attempted in each case to present a more complete and more systematic analysis than has previously been given. Thus in our first report we showed that radiation of energy from the bubble can account for most of observed energy losses. In our second report we showed that the observed changes of shape could be accounted for, at least qualitatively, by considering gravity, and that these changes also diminished the rate of rise and modified the bubble period in the observed manner.

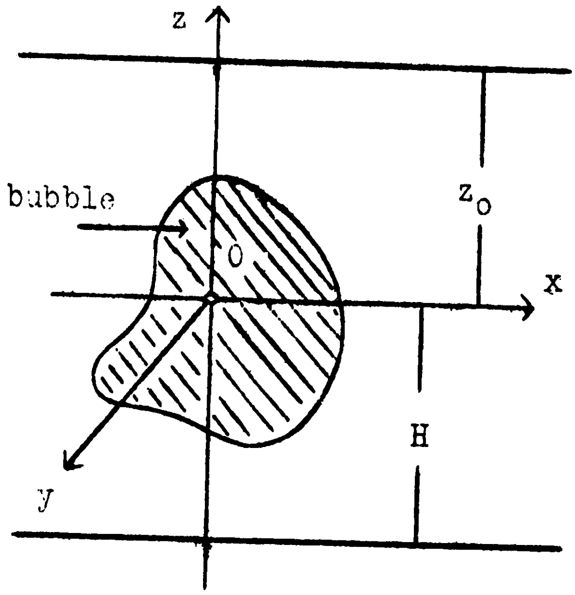
In the present report in considering the effects of the top surface, the bottom, and gravity, we do not constrain the bubble to be spherical at all times, as was done in the previous work on these effects by Shiffman and Friedman [3], and Friedman [4]. We find that certain effects, which we call effects of lowest order, agree with those calculated by the above cited authors, but higher order effects also occur which were not predicted by them. In particular the change of shape and its consequences were obviously excluded from their work. Insofar as our results coincide with the previous results, we believe that they constitute a proof of the correctness of the previous results, since our results are based upon a more accurate theory. Of course in addition we obtain new results involving the change of shape.

The present results are based upon the linearized boundary condition at the top surface of the liquid, rather than upon the exact nonlinear boundary condition. This linearization is employed because it simplifies the calculation, and because it does not affect the bubble shape or motion to the order of approximation considered in this report. Nevertheless we are at present considering this problem, using the exact boundary condition. The results of this analysis will be presented in a future report. They should verify the present results to the order considered here, and differ in higher order.

A discussion of results will be found in Section VII. This discussion is qualitative to a large extent. Numerical results are published in a companion report, "Underwater Explosion Bubbles IV - Summary and Numerical Results", IMM-NYU Report No. 233.

II. Formulation.

We assume that an incompressible, inviscid fluid bounded below by the rigid surface $z = -H$ and above by the free surface $z = z_o$ contains a gas bubble within it. See Figure 1. The velocity $\vec{u}(x,y,z,t)$ in the fluid is assumed derivable from a velocity potential $\phi(x,y,z,t)$ which then satisfies the Laplace's equation



(2.1) $\Delta \phi = 0 \; , \quad \vec{u} = \nabla \phi \; .$

The pressure $p(x,y,z,t)$ in the water is then given by the Bernouilli equation,

(2.2) $p = P_o - \rho[\phi_t + \frac{1}{2}(\nabla \phi)^2] - \rho g z$

Here, $P_o = p_o + \rho g z_o$, where p_o is the pressure of air above the free surface, ρ is the density of the fluid (water), and g is the acceleration of gravity. Thus, P_o is the static pressure at the level $z = 0$.

The equation of the bubble's surface is assumed to be of the form

$$(2.3) \qquad F(x,y,z,t) = 0 \; .$$

On this surface, the kinematic and the dynamic conditions must be satisfied, namely

$$\left.\begin{aligned} &(2.4) \qquad \nabla F \cdot \nabla \phi + F_t = 0 \\ &(2.5) \qquad p = P_o - \rho[\phi_t + \tfrac{1}{2}(\nabla \phi)^2] - \rho g z = KV^{-\gamma} \end{aligned}\right\} \text{ on } F = 0 \; .$$

The last condition follows from the assumption that within the bubble the pressure is uniform and is related to the bubble volume V by the adiabatic relation, pressure = $KV^{-\gamma}$, with K and γ constants.

At the bottom we have no normal flow, so

$$\phi_z(x,y,-H,t) = 0 . \tag{2.6}$$

In a precise formulation, the top surface must be also considered as unknown, and two conditions, kinematic and dynamic, must be imposed there. For simplicity, however, we assume that on the free surface, $(\nabla\phi)^2$ is small compared with other terms appearing in (2), and that this surface is approximately $z = z_0$ at all times. Since the pressure there is p_0, the two conditions are now replaced by a single linearized condition, $\phi_t(x,y,z_0,t) = 0$, which together with the assumption on $(\nabla\phi)^2$ implies that $\phi(x,y,z_0,t)$ = constant, and the constant may be taken as zero. Thus

$$\phi(x,y,z_0,t) = 0 , \quad \text{for } z = z_0 . \tag{2.7}$$

This approximation was discussed by Friedman [4], and is commented upon in the introduction above.

The mathematical problem which we consider is that of finding ϕ and F satisfying equations (1) through (7), given $F(x,y,z,0)$ and $F_t(x,y,z,0)$ (i.e., given the initial surface and velocity of the bubble). Once ϕ is known, the pressure field can be computed from equation (2). We will assume that the bubble is initially a sphere of radius A and that its initial velocity is radial and equal to $\dot{A}$, a constant. If equations (6) and (7) are omitted (or if z_0 and H become infinite) the present problem reduces to that considered in [2], provided that ϕ is required to be regular at infinity. Here, as in the case of the reduced problem, the assumption of very special initial conditions on the bubble surface is essential for applicability of our method of solution.

In the sequel, it will be convenient to employ the energy balance equation, which is a direct consequence of the preceding equations. Since in the linearized problem there is actual mass flow across the linearized free surface (since, $\phi_z(x,y,z_0,t) \not\equiv 0$), it is not at all obvious that the total energy of the system is conserved. This is, nevertheless, true, and we now proceed to show it.

Denote by B, S_1, S_2 the surfaces of the bubble, of the free surface, and of the rigid surface, respectively, by $\bar{V}$, the volume of the water. Let $d\tau$ be the volume element, $d\vec{s}$ - the surface element measured along the outward normal. Multiply equation (5) by $\vec{u} \cdot d\vec{s}$ and integrate over B, obtaining

$$(2.8) \qquad \int_B (\rho[\phi_t + \tfrac{1}{2}(\nabla\phi)^2] - P_0 + \rho g z + KV^{-\gamma})\vec{u} \cdot d\vec{s} = 0 .$$

This is, as we shall show by properly identifying various terms, the desired energy equation.

Let T be the kinetic energy of water,

$$(2.9) \qquad T = \tfrac{1}{2}\rho \int_{\bar{V}} (\nabla\phi)^2 d\tau = \tfrac{1}{2}\rho \int_S \phi\nabla\phi \cdot d\vec{s} = \tfrac{1}{2}\rho \int_B \phi\nabla\phi \cdot d\vec{s} .$$

Here $S = B + S_1 + S_2$. The first equality follows from the Green's theorem for potential functions, while the second follows from the fact that on S_1, $\phi \equiv 0$, while on S_2, the normal components of $\nabla\phi$ is $\phi_n = \phi_z = 0$. Now

$$\frac{d}{dt} T = \frac{1}{2}\rho\frac{d}{dt}\int(\nabla\phi)^2 d\tau = \frac{1}{2}\rho[2\int_V (\nabla\phi \cdot \nabla\phi_t)d\tau + \int_B (\nabla\phi)^2\vec{v} \cdot d\vec{s}]$$

where $\vec{v}$ is the velocity of the bubble surface. We observe that $(\nabla\phi \cdot \nabla\phi_t) = \nabla \cdot (\phi_t\nabla\phi) - \phi_t\Delta\phi = \nabla \cdot (\phi_t\nabla\phi)$. Hence, by Gauss' Theorem, the first integral above is $2\int_S \phi_t\nabla\phi \cdot d\vec{s}$, which reduces to $2\int_B \phi_t\nabla\phi \cdot d\vec{s}$, since on $S_1, \phi_t \equiv 0$, while on S_2 , $\phi_n \equiv 0$. As to the second term, we observe that on the bubble $v_n = u_n$, so that, finally,

$$\text{(2.10)} \qquad \rho\int_B [\phi_t + \frac{1}{2}(\nabla\phi)^2]\vec{u} \cdot d\vec{s} = \frac{d}{dt} T = \frac{1}{2}\rho\frac{d}{dt}(\int_B \phi\vec{u} \cdot d\vec{s}) ,$$

identifying the first term in (8).

Let $d\vec{s}' = - d\vec{s}$, the surface element measured along the outward normal to the bubble. We have, since $u_n = v_n$

$$-\int_B \vec{u} \cdot d\vec{s} = \int \vec{v} \cdot d\vec{s}' = \frac{d}{dt} V \quad .$$

Hence

$$\text{(2.11)} \qquad - P_o\int_B \vec{u} \cdot d\vec{s} = \frac{d}{dt}(P_o V)$$

$$\text{(2.12)} \qquad KV^{-\gamma}\int_B \vec{u} \cdot d\vec{s} = - KV^{-\gamma}\frac{d}{dt} V = \frac{d}{dt}(\frac{K}{\gamma-1} V^{1-\gamma}) \ .$$

Lastly

$$\text{(2.13)} \qquad + \int_B z\vec{u} \cdot d\vec{s} = - \int_B z\vec{v} \cdot d\vec{s}' = - \frac{d}{dt}\int_V z d\tau = - \frac{d}{dt}(BV) ,$$

where B is the location of the center of gravity of the bubble. Substituting (10) through (13) in (8) and integrating with respect to time, we now get

$$T + (P_0 - \rho g B)V + \frac{K}{\gamma-1} V^{1-\gamma} = \text{constant} = E_0 \quad . \tag{2.14}$$

The energy equation (14) is identical in form with that obtained for the bubble placed in infinite water; see [2], equation (20). The total energy E_0 is, however, not necessarily the same as in case of infinite water. We write

$$E_0 = E + \overline{E} \quad , \tag{2.15}$$

where E is the energy that system would possess if the water boundaries were thrown to infinity, while the bubble has the same initial data. As is well known,

$$E = \frac{4\pi}{3} A^3 [\tfrac{3}{2} \rho \dot{A}^2 + P_0 + \frac{K}{\delta-1} (\tfrac{4\pi}{3} A^3)^{-\gamma}] \quad . \tag{2.16}$$

III. Moving Coordinates and Dimensionless Variables.

The method of solution of the problem posed in Section II will naturally be similar to that used previously in [2]. In particular, we introduce new coordinates, ξ, η, ζ, as previously, by

$$\xi = x, \quad \eta = y \ , \quad \zeta = z - B(t) \ . \tag{3.1}$$

$B(t)$ has been defined in Section II as the position of the center of gravity of the bubble, and measures, in a certain sense, the migration of the bubble. The moving coordinate system defined by (1) is, therefore, such that its origin will remain at the center of gravity of the bubble at all times. We also introduce the polar coordinates r, θ, ω, relative to the new origin. Since the problem still presents rotational symmetry, the results will be independent of ω, and we can assume this at the outset. If we write the equation of the bubble surface in the form

$$(3.2) \qquad F(x,y,z,t) \equiv r - R(\theta,t) = 0 , \quad \text{or} \quad r = R(\theta,t) ,$$

equations (2.4) through (2.7) become, using $\dot{} = \frac{d}{dt}$

$$(3.3) \qquad \phi_r - \frac{R_\theta}{r^2}\phi_\theta = R_t + \dot{B}(\cos\theta + \frac{\sin\theta}{r}R_\theta) , \quad \text{for} \quad r = R(\theta,t)$$

$$(3.4) \qquad P_o - \rho[g(r\cos\theta + B) + \phi_t - \dot{B}(\cos\theta\,\phi_r - \frac{\sin\theta}{r}\phi_\theta) + \frac{1}{2}(\phi_r^2 + \frac{1}{r^2}\phi_r^2)] = K(\frac{2\pi}{3}\int_0^\pi R^3 \sin\theta\, d\theta)^{-\gamma} , \quad \text{for} \quad r = R(\theta,t)$$

$$(3.5) \qquad \cos\theta\,\phi_r - \frac{\sin\theta}{r}\phi_\theta = 0 , \quad \text{for} \quad r\cos\theta = -(H + B)$$

$$(3.6) \qquad \phi = 0 , \quad \text{for} \quad r\cos\theta = z_o - B .$$

In addition to those the initial conditions on the bubble are,

$$(3.7) \qquad R(\theta,0) = A , \quad R_t(\theta,0) = \dot{A} , \quad B(0) = 0 , \quad \dot{B}(0) = 0 ,$$

while the condition that the moving origin be at the center of gravity of the bubble is

$$(3.8)\qquad \int_0^\pi R^4 \cos\theta \sin\theta \, d\theta = 0 .$$

The energy equation is, explicitly,

$$(3.9)\qquad -\pi\rho \int_0^\pi R^2\left(R_t + \dot{B}\left[\cos\theta + \frac{\sin\theta}{r}R_\theta\right]\right)\phi(R,\theta,t)\sin\theta \, d\theta$$
$$+ \frac{2\pi}{3}(P_o - \rho g B)\int_0^\pi R^3 \sin\theta \, d\theta + \frac{K}{\gamma-1}\left(\frac{2\pi}{3}\int_0^\pi R^3 \sin\theta \, d\theta\right)^{1-\gamma} = E + \bar{E} .$$

Finally, we define the dimensionless functions, variables, and constants, $\bar{r}$, $\bar{t}$, $\bar{\phi}$, λ, b, a_o, $\dot{a}_o$, σ, k, $\hat{k}$, $\hat{a}$, by introducing a unit of length L and a unit of time T. These quantities are defined as previously, by

$$(3.10)\quad \begin{cases} r = L\bar{r} , \quad t = T\bar{t} \\ R(\theta,t) = L\lambda(\theta,\bar{t}), \quad A = La_o, \quad \dot{A} = LT^{-1}\dot{a}_o, \quad B(t) = Lb(\bar{t}) \\ \phi(r,\theta,t) = L^2T^{-1}\bar{\phi}(\bar{r},\theta,\bar{t}) \\ \sigma = L\rho g P_o^{-1} , \quad k = \dfrac{KP_o^{\gamma-1}E^{-\gamma}}{\gamma-1} , \quad \hat{k} = \max k = \dfrac{1}{\gamma-1}\left(\dfrac{\gamma-1}{\gamma}\right)^\gamma \\ \hat{a} = \left(\dfrac{\gamma-1}{\gamma}\right)^{1/3} . \end{cases}$$

For the units we choose, as before,

$$(3.11) \qquad L = \left[\frac{3(\gamma-1)E}{4\pi\gamma P_o}\right]^{1/3} , \qquad T = L\sqrt{\rho P_o^{-1}} \quad . *$$

In addition to these, we define two new dimensionless ratios, ν and μ , by

$$(3.12) \qquad \nu = H\rho g P_o^{-1} = \frac{H}{L}\sigma , \qquad \mu = z_o \rho g P_o^{-1} = \frac{z_o}{L}\sigma .$$

Clearly, $\mu \leq 1$, and equality holds only when the pressure at the free surface vanishes. The ratios ν and μ measure the distance of the bottom and top, respectively, from $z = 0$ where the bubble is initially located. Thus, for fixed σ, the larger μ and ν , the less important are the boundary effects compared to gravity effects.

In terms of all these new variables, the previous conditions become, omitting all bars,

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* These units are different from those introduced by Taylor, Friedman and Shiffman, and frequently used in literature on underwater bubble motion. If we denote by L' and T' the units defined in [4], their connection with the present is,

$$L = \hat{\alpha} L' , \qquad T = \sqrt{\frac{3}{2}}\, \hat{\alpha} T' .$$

The reason for breaking with the custom is that L represents exactly the equilibrium radius of the bubble, and thus has a greater physical significance that L' which gives only an upper bound for the maximum radius and is close to the actual maximum radius only for high bubble energies. The new unit of time was chosen for convenience.

(3.13) $\Delta \phi = 0$, for $r > R(\theta,t)$ and $-(\frac{\nu}{\sigma} + b) < r \cos\theta < \frac{\mu}{\sigma} - b$

(3.14) $g(\theta,t) \equiv \phi_r - \lambda_t - \dot{b}\cos\theta - \frac{\lambda\theta}{\lambda}(\frac{\phi_\theta}{r} + \dot{b}\sin\theta) = 0$, $r = \lambda(\theta,t)$,

(3.15)
$$h(\theta,t) \equiv \phi_t + \frac{1}{2}\{[1 + (\frac{\lambda_\theta}{\lambda})^2](\frac{1}{r}\phi_\theta + \dot{b}\sin\theta)^2 + 2\lambda_t\frac{\lambda_\theta}{\lambda}(\frac{1}{r}\phi_\theta + \dot{b}\sin\theta) + \lambda_t^2 - \dot{b}^2\} + \frac{k}{\hat{k}}(\frac{1}{2}\int_0^\pi \lambda^3 \sin\theta\, d\theta)^{-\gamma} + \sigma(\lambda\cos\theta + b) - 1 = 0 ,$$

$r = \lambda(\theta,t)$,

(3.16) $\phi_\zeta = \cos\theta\,\phi_r - \frac{\sin\theta}{r}\phi_\theta = 0$, on $r\cos\theta = -(\frac{\nu}{\sigma} + b)$,

(3.17) $\phi = 0$, on $r\cos\theta = \frac{\mu}{\sigma} - b$,

(3.18) $\int \lambda^4 \cos\theta \sin\theta\, d\theta = 0$,

whereas the dimensionless energy equation is

(3.19)
$$-\frac{3}{4}\int_0^\pi \lambda^2(\lambda_t + \dot{b}[\cos\theta + \frac{\sin\theta}{\lambda}\lambda_\theta])\phi(\lambda,\theta,t)\sin\theta\, d\theta + \frac{1}{2}(1 - \sigma b)\int_0^\pi \lambda^3 \sin\theta\, d\theta + \hat{a}^{-3\delta}k(\frac{1}{2}\int_0^\pi \lambda^3\sin\theta\, d\theta)^{1-\gamma} = \hat{a}^{-3}(1 + \frac{\overline{E}}{E}) .$$

The functions g and h in (14), (15) are defined by those equations. Equation (15) is obtained from (4) by using (3) to eliminate ϕ_r .

Our problem is now to find ϕ, λ, and b satisfying (13-18), subject to the initial conditions

$$\lambda(\theta,0) = a_o \ , \quad \lambda_t(\theta,0) = \dot{a}_o \ , \quad b(0) = \dot{b}(0) = 0 \ . \tag{3.20}$$

a_o and $\dot{a}_o$ cannot be prescribed independently, since they are related by equation (2.16), which in terms of the new variables becomes,

$$[\tfrac{3}{2}\,\dot{a}_o^2 + 1 + k(\hat{\alpha} a_o)^{-3\gamma}]a_o^3 = \hat{\alpha}^{-3} \quad . \tag{3.21}$$

IV. Method of Solution.

In order to find ϕ, λ, b, we assume that they can be expresses as power series in σ (convergent or asymptotic) in the following form:

$$\phi(r,\theta,t) = \sum_{n=o}^{\infty} \sigma^n \sum_{m=o}^{\infty} c_{nm}(t)(r^{-(m+1)}P_m(\cos\theta) + \phi_m^*(r,\theta)) \tag{4.1}$$

$$\equiv \sum_{n=o}^{\infty} \sigma^n \sum_{m=o}^{\infty} c_{nm}(t)\Phi^m$$

$$\lambda(\theta,t) = \sum_{n=o}^{\infty} \sigma^n \sum_{m=o}^{\infty} a_{nm}(t)P_m(\cos\theta) \tag{4.2}$$

$$b(t) = \sum_{n=o}^{\infty} b_n(t) \ . \tag{4.3}$$

Here, $c_{nm}(t)$, $a_{nm}(t)$, $b_n(t)$ are functions of time to be determined. In view of (3.20), they must satisfy the following initial conditions:

$$(4.4)\quad \begin{cases} a_{oo}(0) = a_o\ , \quad \dot{a}_{oo}(0) = \dot{a}_o \\ a_{nm}(0) = \dot{a}_{nm}(0) = 0 \quad \text{for} \quad n+m \geq 1 \\ b_n(0) = \dot{b}_n(0) = 0 \quad \text{for all} \quad n\ . \end{cases}$$

We require that the ϕ_m^* be regular harmonic functions in the strip $-(\frac{\nu}{\sigma} + b) < \zeta < \frac{\mu}{\sigma} - b$, such that $\Phi^m = r^{-(m+1)} P_m(\cos\theta) + \phi_m^*$ satisfies:

$$(4.5)\quad \begin{cases} \Phi^m = 0\ , \quad \text{for} \quad \zeta = \frac{\mu}{\sigma} - b \\ \Phi^m_\zeta = 0\ , \quad \text{for} \quad \zeta = -(\frac{\nu}{\sigma} + b)\ . \end{cases}$$

Then (1), which certainly satisfies (3.13) will also satisfy the boundary conditions (3.16, 3.17), no matter what the c_{nm} turn out to be.

The ϕ_m^* are determined in the Appendix A . Obviously ϕ_m^* depends on σ and t (through b) since, as equations (5) show, their boundary values depend on σ and t . Since ϕ_m^* is regular at $r = 0$ and has rotational symmetry, it can be represented as a linear superposition of axially symmetric spherical harmonics regular at $r = 0$; hence

$$(4.6)\quad \phi_m^* = \sum_{k=0}^{\infty} d_{mk}\, r^k P_k(\cos\theta)\ .$$

The coefficients d_{km}, (see Appendix B), depend on σ and t , and as is shown in Appendix C , they can be expanded in power series in σ

(4.7) $$d_{mk} = \sum_{\ell=0}^{\infty} d_{mk\ell}(t)\sigma^{\ell} \quad .$$

On the other hand, since the problem reduces to that of the spherical bubble theory when $\sigma = 0$, we have

(4.8) $$a_{om} \equiv c_{om} \equiv b_o \equiv 0 \ , \quad \text{for} \quad m > 0 \ ,$$

whereas

(4.9) $$\begin{cases} a_{oo}(t) \equiv a(t) > 0 \\ c_{oo}(t) = - a^2\dot{a} \ , \end{cases}$$

where $a(t)$ is the well known zero order solution, discussed, in particular, in [2].

In view of (7), (8) and (9), ϕ, ϕ_r, ϕ_θ, ϕ_t for $r = \lambda(\theta,t)$ can be expressed as power series in σ. Likewise the volume of the bubble can be expressed as power series in σ. To find the b_n, a_{nm} and c_{nm} for $n \geq 1$ we substitute the power series thus obtained in (3.14) and (3.15), and equate the coefficients of $\sigma^n P_m(\cos\theta)$, for every n and m. This way we get a sequence of equations involving the coefficients and their derivatives.

To obtain these equations explicitly, it is convenient, as was done previously in [2], to introduce the functions $\bar{g}(\theta,t)$ and $\bar{h}(\theta,t)$ defined by

(4.10) $$\bar{g}(\theta,t) \equiv g(\theta,t) + \sum_{n=1}^{\infty} \sigma^n (\dot{b}_n P_1 + \sum_{m=o}^{\infty} [\dot{a}_{mn} + 2\frac{\dot{a}}{a}\, a_{nm} + (m+1)a^{-(m+2)} c_{nm}]P_m) \equiv \sum_{n=o}^{\infty} \sigma^n \sum_{m=o}^{\infty} g_{nm}P_m$$

$$(4.11)\quad \bar{h}(\theta,t) \equiv h(\theta,t) - \sum_{n=1}^{\infty} \sigma^n\Big(-3\gamma\frac{k}{\hat{k}}\, a^{-3\gamma-1} a_{no} + \sum_{m=o}^{\infty}\Big[\frac{(a^2\dot{a})^{\cdot}}{a^2}\, a_{nm} + a\ddot{a}_{nm} + a^{-(m+1)}\dot{c}_{nm}\Big]P_m\Big) \equiv \sum_{n=o}^{\infty} \sigma^n \sum_{m=o}^{\infty} h_{nm}P_m \,.$$

Now, inserting (1) through (3) in (3.14), we obtain, for $n \geq 1$

$$(4.12)\quad \begin{cases} c_{nm} = -\dfrac{1}{m+1}\, a^{m+2}\Big[\dot{a}_{nm} + 2\dfrac{\dot{a}}{a}\, a_{nm} - g_{nm}\Big] , & m \neq 1 \\[2ex] c_{n1} = -\dfrac{1}{2}\, a^3\Big[\dot{b}_n + \dot{a}_{n1} + 2\dfrac{\dot{a}}{a}\, a_{n1} - g_{n1}\Big] , & m = 1 \,. \end{cases}$$

Similarly, inserting (1) through (3) in (3.15) and making use of (12) we get, after some manipulations,

$$(4.13)\quad \begin{cases} a\ddot{a}_{no} + 3\dot{a}\dot{a}_{no} + \Big(\ddot{a} + 3\gamma\dfrac{k}{\hat{k}}\, a^{-3\gamma-1}\Big)a_{no} = a\dot{g}_{no} + 2\dot{a}g_{no} + h_{no} , & m = \\[1ex] (a^3\dot{b}_n)^{\cdot} = (a^3[g_{n1} - \dot{a}_{n1}])^{\cdot} + 2a^2h_{n1} , & m = 1 \\[1ex] a\ddot{a}_{nm} + 3\dot{a}\dot{a}_{nm} + (1-m)\ddot{a}a_{nm} = a\dot{g}_{nm} + (m+2)\dot{a}g_{nm} + (m+1)h_{nm}, & m \geq 2 \end{cases}$$

So far, $\bar{g}$ and $\bar{h}$ are general formal power series in σ with coefficients depending on a_{nm}, c_{nm}, and b_n. If g_{nm} and h_{nm} were independent of a_{nm} for $m \neq 1$, and of b_n for $m = 1$, equations (13) together with the initial conditions (4) would determine the a_{nm} and b_n uniquely, provided a_{n1} were also known. Even so, in general we would have to solve for each n an infinite set of equations, and therefore we would not be able to get beyond the terms of first order in σ. Fortunately, the following theorem can be proved:

1. for each n , $g_{nm} \equiv h_{nm} \equiv 0$ for $m > n$.
2. for each n , the g_{nm}, h_{nm} depend only on the a_{km}, b_k, c_{km} with $k < n$.

A similar theorem was proved in [2], and since the method of proof is the same, we omit it here. (The theorem proved in [2] was somewhat stronger in that we managed to show the $g_{nm} \equiv h_{nm} \equiv 0$ also whenever $n + m$ = odd integer. This is not true in the present case.)

From the first part of the theorem it follows then that the equations (13) are linear and homogeneous for $m > n$, and, since all the unknowns are subject to homogeneous initial conditions, that

$$a_{nm} \equiv 0 \quad \text{for} \quad m > n . \tag{4.14}$$

Hence for each n , we have only a finite number (exactly $n+1$) of equations to solve. Also using (14) in (12), one gets

$$c_{nm} \equiv 0 \quad \text{for} \quad m > n . \tag{4.15}$$

Thus, the expansions (1) and (3) simplify to

$$\phi(r,\theta,t) = \sum_{n=0}^{\infty} \sigma^n \sum_{m=0}^{n} c_{nm}(t)(r^{-(m+1)}P_m(\cos\theta) + \phi_m^*) \tag{4.16}$$

$$\lambda(\theta,t) = \sum_{n=0}^{\infty} \sigma^n \sum_{m=0}^{n} a_{nm}(t)P_m(\cos\theta) . \tag{4.17}$$

In view of the second part of our theorem, the equations (13) for a fixed n and $m \leq n$, $m \neq 1$, can be solved provided that we know a_{km}, b_k, c_{km}, with $k < n$. Also, since by (3.18) a_{n1} can be determined in terms of a_{km} with $k < m$, the equation for b_n can be likewise solved. Hence, the system (13) can be solved in succession.

V. Bubble Coefficients up to Terms in σ^3.

Up to terms in σ^3 the expansions for ϕ, λ, and b are

$$(5.1)\quad \phi = c_{oo}(r^{-1} + d_{oo} + d_{o1}rP_1 + d_{o2}r^2P_2) + [c_{1o}(r^{-1} + d_{oo} + d_{o1}rP_1) + c_{11}(r^{-2}P_1 + d_{1o})]\sigma + [c_{2o}(r^{-1} + d_{oo}) + c_{21}r^{-2}P_1 + c_{22}r^{-3}P_2]\sigma^2 + [c_{3o}r^{-1} + c_{31}r^{-2}P_1 + c_{32}r^{-3}P_2 + c_{33}r^{-4}P_3]\sigma^3$$

$$(5.2)\quad \lambda = a + a_{1o}\sigma + (a_{2o} + a_{22}P_2)\sigma^2 + (a_{3o} + a_{32}P_2 + a_{33}P_3)\sigma^3 + \ldots$$

$$(5.3)\quad b = b_1\sigma + b_2\sigma^2 + b_3\sigma^3 .$$

In the expansion for λ terms involving a_{n1} were omitted since, as follows from (3.18) they are zero for $n \leq 4$. (This assertion is not true for all n, though. As may be verified, $a_{51} = -\frac{27}{70}a^{-1}a_{22}a_{33} \neq 0$.)

In this section we consider the equations for a_{nm} with $m \neq 0$. The equations for bubble coefficients with $m = 0$ are derived in Section VI in a simpler way. Since, however, the present equations will depend on the a_{no}, some of the results of Section VI, namely those for a_{1o} and a_{2o}, are already used here.

Let $\bar{a}_{nm}$, $\bar{b}_n$ denote the bubble coefficients when the boundaries are removed. The equations for these were found in [2], and will be omitted here. Let

$$\text{(5.4)} \qquad \begin{cases} a_{nm} = \bar{a}_{nm} + \tilde{a}_{nm} \\ b_n = \bar{b}_n + \tilde{b}_n \end{cases} .$$

Thus the $\tilde{a}_{nm}$ and $\tilde{b}_n$ are corrections to the bubble coefficients due to existence of boundaries. It was previously established that $\bar{a}_{1o} = \bar{b}_2 = \bar{a}_{3o} = \bar{a}_{32} = 0$.

For $n = 1$, the only coefficient (except for a_{1o}) to be determined is b_1 . We find, $g_{11} = 0$, $h_{11} = a$, and consequently

$$\text{(5.5)} \qquad \tilde{b}_1 = 0 , \quad c_{11} = - \frac{1}{2} a^3 \dot{b}_1 .$$

Hence $b_1 = \bar{b}_1$. It was previously shown that the formula for $\bar{b}_1$ is equivalent to the Herring rise formula.

For $n = 2$, the only coefficients (except for a_{2o}) to be determined are b_2 , and a_{22} . We find,

$$\text{(5.6)} \qquad \begin{cases} g_{21} = - [3\dot{b}_1 a^{-1} a_{1o} + a^2 \dot{a}\, d_{oo1}] \\ g_{22} = 0 \\ h_{21} = 3a_{1o} - a(a^2 \dot{a})^{\dot{}}\, d_{oo1} \\ h_{22} = - \frac{3}{4} \dot{b}_1^2 \end{cases} ,$$

and, on substituting in (4.13), obtain

(5.7) $$b_2 = \tilde{b}_2 = \beta_2 d_{oo1} + \beta_2' d_{o12} ,$$

where β_2 and β_2' are determined by ,

(5.8) $$\begin{cases} (a^3\dot{\beta}_2)^{\cdot} = \frac{3}{2}\dot{b}_1 a(\dot{a}a^2 + [\ddot{a}a - \dot{a}^2]\int_0^t a(\tau)d\tau) , & \beta_2(0)=\dot{\beta}_2(0)=0 \\ (a^3\dot{\beta}_2')^{\cdot} = -3a^2(a^3\dot{a})^{\cdot} , & \beta_2'(0) = \dot{\beta}_2'(0) = 0 , \end{cases}$$

while,

(5.9) $$\tilde{a}_{22} \equiv 0 .$$

In terms of b_2 and a_{22}, we get from (6) and (4.12) ,

(5.10) $$\begin{cases} c_{21} = -\frac{1}{2}[a^3\dot{b}_2 + 3\dot{b}_1 a^2 a_{1o} + a^5\dot{a}\, d_{o12}] \\ c_{22} = -\frac{1}{3} a^2(a^2 a_{22})^{\cdot} . \end{cases}$$

In establishing equation (8), we used the result $a_{1o} = \tilde{a}_{1o} = -\frac{1}{2} d_{oo1}\dot{a}\int_0^t a(\tau)d\tau$, (see equation (6.12)) .

For $n = 3$, the only coefficients (except for a_{3o}) to be determined are b_3, a_{32}, and a_{33} . We find

(5.11) $$\begin{cases} g_{31} = -a^{-3}[3\dot{b}_2 a^2 a_{1o} + 3\dot{b}_1 a a_{1o}^2 + 3\dot{b}_1 a^2(a_{2o} - \frac{1}{5}a_{22}) + (a^5 a_{1o})^{\cdot} d_{o1} \\ g_{32} = -a^{-4}[2(a^3 a_{1o}a_{22})^{\cdot} + 2a^3 a_{1o}\dot{a}_{22} + 2a^7\dot{a}d_{o23}] \\ g_{33} = -\frac{18}{5}\dot{b}_1 a^{-1}a_{22} \\ h_{31} = \frac{3}{5}a^{-2}(\dot{b}_1 a^2 a_{22})^{\cdot} + 3a_{2o} + 3a^{-1}a_{1o}^2 - [(a^3)^{\cdot\cdot}a_{1o} + a(a^2 a_{1o})^{\cdot\cdot}]d_{o1} \end{cases}$$

$$h_{32} = -\frac{3}{2}\dot{b}_1(\dot{b}_2 + a^2\dot{a}d_{o12}) - a^2(a^2\dot{a})^{\cdot}d_{o23} + a_{1o}\ddot{a}_{22} + (\dot{a}_{1o} + 4a^{-1}\dot{a}a_{1o})\dot{a}_{22}$$
$$+ (\ddot{a}_{1o} + 4a^{-1}\dot{a}\dot{a}_{1o} + 2a^{-1}\ddot{a}a_{1o})a_{22}$$

$$h_{33} = -\frac{3}{5}a\dot{b}_1(a^{-1}a_{22})^{\cdot} + \frac{9}{5}a_{22}\ .$$

Substitution in (4.13) now yields, using α_{1o}, α_{2o} , defined by $a_{1o} = \alpha_{1o}d_{oo1}$, $\tilde{a}_{2o} = \alpha_{2o}d_{oo1}^2$, (see equations (6.12) and (6.14),

(5.12) $$\tilde{b}_3 = \beta_3 d_{oo1}^2 + \beta_3' d_{oo1}d_{o12}$$

(5.13) $$a_{32} = \tilde{a}_{32} = \alpha_{32}d_{oo1} + \alpha_{32}'d_{o12} + \alpha_{32}''\ d_{o23}$$

(5.14) $$\tilde{a}_{33} = 0\ ,$$

where β_3 , β_3' , α_{32}, α_{32}' , and α_{32}'' satisfy the following equations:

(5.15) $$\begin{cases} (a^3\dot{\beta}_3)^{\cdot} = -3(\dot{\beta}_2 a^2\alpha_{1o} + \dot{b}_1 a\alpha_{1o}^2 + \dot{b}_1 a^2\alpha_{2o})^{\cdot} \\ \qquad + 6(a^2\alpha_{2o} + a\alpha_{1o}^2)\ , \quad \beta_3(0) = \dot{\beta}_3(0) = 0 \\ (a^3\dot{\beta}_3')^{\cdot} = -3(\dot{\beta}_2' a^2\alpha_{1o} + (a^5\alpha_{1o})^{\cdot})^{\cdot} \\ \qquad + 12a^2\dot{a}(a^2\alpha_{1o})^{\cdot}\ , \quad \beta_3'(0) = \dot{\beta}_3'(0) = 0 \end{cases}$$

(5.16) $$\begin{cases} a\ddot{\alpha}_{32} + 3\dot{a}\dot{\alpha}_{32} - \ddot{a}\alpha_{32} = -\frac{9}{2}\dot{b}_1\dot{\beta}_2 - (\alpha_{1o}\ddot{a}_{22} + 3\dot{\alpha}_{1o}\dot{a}_{22} - \ddot{\alpha}_{1o}a_{22} \\ \qquad \alpha_{32}(0) = \dot{\alpha}_{32}(0) = 0 \\ a\ddot{\alpha}_{32}' + 3\dot{a}\dot{\alpha}_{32}' - \ddot{a}\alpha_{32}' = -\frac{9}{2}\dot{b}_1(\dot{\beta}_2' + a^2\dot{a})\ , \quad \alpha_{32}'(0) = \dot{\alpha}_{32}'(0) = \\ a\ddot{\alpha}_{32}'' + 3\dot{a}\dot{\alpha}_{32}'' - \ddot{a}\alpha_{32}'' = -5(a^4\dot{a})^{\cdot}\ , \quad \alpha_{32}''(0) = \dot{\alpha}_{32}''(0) = 0\ . \end{cases}$$

Formulae for c_{31}, c_{32}, and c_{33} can be obtained in terms of b_{31} a_{32} and a_{33} using (6) and (4.12). However, c_{32} and c_{33} are of no interest, and the expression for c_{31} is

$$(5.17)\quad c_{31} = -\tfrac{1}{2}[a^3\dot{b}_3 + 3\dot{b}_2a^2a_{1o} + 3\dot{b}_1aa_{1o}^2 + 3\dot{b}_1a^2(\bar{a}_{2o} - \tfrac{1}{5}a_{22}) + (a^5a_{1o})^{\cdot}d_{o12}] \quad .$$

For the equilibrium bubble $(a \equiv 1)$ discussed at length in [2], we find $\alpha_{1o} \equiv 0$, $\alpha_{2o} \equiv 0$, and consequently all $\tilde{a}_{nm}$, $\tilde{b}_n$ equal identically to zero except for $\tilde{a}_{3o}$. Thus, the only case which can be solved explicitly is uninteresting from the point of view of effect of boundaries. To obtain quantitative results in the general case, equations (8), (15) and (16) must be integrated numerically.

VI. Use of the Energy Equation - Bubble Coefficients with $m = 0$

The bubble coefficients a_{no} may be computed by solving equations (4.13) with $m = 0$, namely

$$(6.1)\quad a\ddot{a}_{no} + 3\dot{a}\dot{a}_{no} + (\ddot{a} + 3\gamma\frac{k}{\hat{k}}\,a^{-(3\gamma+1)})a_{no} = a\dot{g}_{no} + 2\dot{a}g_{no} + h_{no} ,$$

$$a_{no}(0) = \dot{a}_{no}(0) = 0 .$$

Using the equation defining $a(t)$, namely

$$(6.2)\qquad \tfrac{3}{2}\,a^3\dot{a}^2 + a^3 + \frac{k}{\hat{k}(\gamma-1)}\,a^{-3(\gamma-1)} = \hat{a}^{-3} \quad ,$$

(cf. [2] or earlier works on spherical bubbles), one verifies that $\dot{a}(t)$ is a particular solution of the homogeneous equation associated with (1). Formally then, the problem of solving (1) may be reduced to two quadratures, since there exists a formula involving two integrations which will produce the general solution of a second order linear differential equation once a particular solution of the associated homogeneous equation is known. As is shown in [5], this formula involves, in the case when this particular solution vanishes at some points - which, indeed, is the case with $\dot{a}(t)$ - the computation of finite parts of divergent integrals, unless, of course, the integrations can be carried out explicitly. We shall show here that in part these integrations can be carried out, but in general it is preferable to solve (1) numerically rather than to use formulae.

In any case, equation (1) can be reduced to a first order equation which is,

$$(6.3) \qquad \dot{a}\dot{a}_{no} - \ddot{a}a_{no} = k_{no} = a^{-3}\int_0^t (a\dot{g}_{no} + 2\dot{a}g_{no} + h_{no})a^2\dot{a}dt, \quad a_{no}(0)=0$$

To proceed further we must, of course, evaluate g_{no} and h_{no} in terms of the $a_{k\ell}$, b_μ with $k < n$, and the actual expressions for these are a great deal more involved than the formulae for g_{nm} and h_{nm} with $m > 0$ encountered so far (equations 5.6, 11).

A simplification is, fortunately, possible. In Section II we derived the energy equation which in dimensionless form is equation (3.19). The energy equation is a first integral of the problem. Hence on substitution in it of our expansions for ϕ, λ, and b, and on identification of coefficients of various powers of σ, one ought to get a sequence of <u>first order</u> differential equations for the bubble coefficients. Now, since the equations for a_{nm} with $m > 0$ cannot be reduced to first order equations, we expect to get from the energy equation the equations for a_{no}, namely the equations (3). Moreover, since the integrations intended in (3.19) are with respect to θ and not with respect to t, and since the expansions used are algebraic forms in the bubble coefficients and their derivatives, the equations so obtained will involve the bubble coefficients algebraically. Thus we shall not only avoid the computation of the g_{no}, h_{no}, but will also obtain the integrals k_{no} in (3) as algebraic expressions in the bubble coefficients. (From the form (3) of k_{no} it is of course not at all obvious that these integrals can be evaluated explicitly.)

We now proceed to express equation (3.19) in terms of bubble coefficients up to order 4 in σ. Let

(6.4) $$a_m = \sum_{n=o}^{\infty} \sigma^n a_{nm} , \qquad c_m = \sum_{n=o}^{\infty} \sigma^n c_{nm} .$$

In view of the theorem of Section IV, $a_m = O(\sigma^m)$, $c_m = O(\sigma^m)$. In place of a_o it is convenient to use the average radius of the bubble, a,

(6.5) $$a = (\tfrac{1}{2} \int \lambda^3 \sin\theta \, d\theta)^{1/3}$$

in terms of which a_o is, up to terms in σ^4, (6.6) $$\alpha_o = \alpha - \frac{1}{5}\alpha^{-1}\alpha_2^2 \ .$$

Using the above notation, and substituting (4.16,17) in (3.19), we now get, on carrying out the intended integrations,

(6.7) $$- \frac{3}{2}[c_o(\alpha\dot{\alpha} - \frac{1}{5}\alpha_2\dot{\alpha}_2 + d_{oo}\alpha^2\dot{\alpha}) + c_1 d_{1o}\alpha^2\dot{\alpha} + \frac{1}{5}c_2(\alpha^{-1}\alpha_2)^{\cdot} + \dot{b}(\frac{1}{3}c_o d_{o1}\alpha^3 + \frac{1}{3}c_1 - \frac{2}{5}c_1\alpha^{-1}\alpha_2)] + (1 - \sigma b)\alpha^3 + k\hat{\alpha}^{-3\gamma}\alpha^{-3(\gamma-1)} = \hat{\alpha}^{-3}(1 + \frac{\overline{E}}{E}) \ .$$

We still need to express c_o, c_1, and c_2 in terms of bubble coefficients, of which c_o is needed to order 4, c_1 to order 3, and c_2 to order 2 in σ, if (7) is to be correct to order 4 in σ. c_o can be evalueated exactly (to any order) by evaluating the surface integral of the normal derivative of ϕ over the surface of the bubble. Since ϕ_m^* (see equation (A.9)) is a regular potential function inside the bubble, its contribution to the surface integral is zero, by a well known theorem. Hence

$$Q = \int_{\text{Bubble}} \phi_n ds = \int_{\text{Bubble}} \frac{\partial}{\partial n}(\sum_m c_m r^{-(m+1)} P_m) ds$$

$$= - 2\pi \sum_m c_m \int_0^\pi [(m+1)P_m + \lambda^{-1}\lambda_\theta P_m']^{-m}\lambda \sin\theta \, d\theta \ .$$

The integrand above is $\frac{1}{m}\frac{d}{d\theta}(\lambda^{-m}P_m' \sin\theta)$, if $m \neq 0$, while for $m = 0$ it is equal to $\sin\theta$. On integrating one gets then,

$$Q = - 4\pi c_o \ .$$

On the other hand, Q can be computed by using the boundary condition at the bubble, equation (3.14), namely

$$Q = 2\pi \int_0^\pi [\lambda_t + \dot{b}(\cos\theta + \frac{\lambda\theta}{\lambda}\sin\theta)]\lambda^2 \sin\theta \, d\theta =$$

$$= 2\pi \int_0^\pi \frac{1}{3}(\lambda^3)^{\cdot}\sin\theta + \frac{1}{2}\dot{b}\frac{d}{d\theta}(\lambda^2\sin^2\theta)d\theta = \frac{4\pi}{3}\frac{d}{dt}(\frac{1}{2}\int_0^\pi \lambda^3 \sin\theta d\theta)$$

$$= 4\pi a^2\dot{a} \ .$$

Equating the two expressions for Q, one gets

$$c_0 = - a^2\dot{a} \ . \tag{6.8}$$

The expressions for c_1 and c_2 to the desired order can be obtained by combining results (5.5,10). One gets thus,

$$c_1 = -\frac{1}{2}\dot{b}(a^3 - \frac{3}{5}a^2a_2) - \frac{1}{2}a^5\dot{a}d_{01} \ , \quad \text{to terms in } \delta^3 \tag{6.9}$$

$$c_2 = -\frac{1}{3}a^2(a^2a_2)^{\cdot} \ , \quad \text{to terms in } \delta^2 \ . \tag{6.10}$$

Substituting (8), (9) and (10), in (7), we get finally

$$\frac{3}{2}\{\dot{a}^2(a^3 + d_{00}a^4 + \frac{1}{2}d_{01}d_{10}a^7 - \frac{2}{15}aa_2^2) + \dot{b}^2(\frac{1}{6}a^3 - \frac{3}{10}a^2a_2)$$
$$+ \frac{1}{15}a^3\dot{a}_2^2 + \frac{1}{2}\dot{a}\dot{b}(d_{10} + d_{01})a^5 - \frac{2}{15}a^2\dot{a}a_2\dot{a}_2\}$$
$$+ (1 - \sigma b)a^3 + k\hat{a}^{-3\gamma}a^{-3(\gamma-1)} = \hat{a}^{-3}(1 + \frac{\bar{E}}{E})$$
$$= \hat{a}^{-3} + \frac{3}{2}a_0^4\dot{a}_0^2(d_{00} + \frac{1}{2}d_{01}d_{10}a_0^3). \tag{6.11}$$

The last part of this equality was obtained by evaluating the energy at $t = 0$ and using (3.21). The equations for the a_{no} are now obtained by replacing α, α_2, b by their expansions in (4) and identifying the coefficients of various powers of σ. Results are as follows.

1. For $a_{oo}(t) = a(t)$ we get (as expected) equation (2)
2. For a_{1o} we get the equation

$$\ddot{a}a_{1o} - \ddot{a}a_{1o} = \frac{1}{2} d_{oo1}\left(\frac{a_o^4\dot{a}_o^2}{a^3} - a\dot{a}^2\right) .$$

This equation can be solved explicitly if $\dot{a}_o = 0$, while if $\dot{a}_o \neq 0$, the solution is obtained in terms of finite parts of divergent integrals. We shall assume, here and subsequently, that $\dot{a}_o = 0$.*
One gets then,

(6.12) $$a_{1o} = \alpha_{1o}d_{oo1} , \qquad \alpha_{1o} = -\frac{1}{2}\dot{a}\int_0^t a d\tau .$$

3. For a_{2o} we get the equation

$$\ddot{a}a_{2o} - \ddot{a}a_{2o} = \frac{1}{3} b_1 - \frac{1}{12}\dot{b}_1^2 + \frac{1}{2}\frac{\dddot{a}}{a} a_{1o}^2 - \frac{1}{2}\dot{a}_{1o}^2 - \dot{a}\dot{a}a_{1o}d_{oo1}$$
$$- \frac{1}{2}\dot{a}^2 a_{1o}d_{oo1} .$$

---

* The significance of this assumption was discussed in [2] .

Using the formula (12) for a_{10} , one now finds that

(6.13) $$a_{20} = \bar{a}_{20} + \alpha_{20} d_{001}^2 ,$$

where $\bar{a}_{20}$ is the part of a_{20} found in connection with the bubble in infinite water, (see equation (54) in [2]), while α_{20} is given by

(6.14) $$\alpha_{20} = \frac{1}{8} \{ \ddot{a} (\int_0^t a d\tau)^2 + 2a\dot{a} \int_0^t a d\tau + \dot{a} \int_0^t a^2 d\tau \} .$$

4. For a_{30} we get the equation

$$\begin{aligned} a\ddot{a}_{30} - \ddot{a}a_{30} = {} & \frac{\dddot{a}}{\dot{a}} a_{10} a_{20} + \frac{1}{6\dot{a}} \left(\frac{\dddot{a}}{a} \right)^{\cdot} a_{10}^3 - \dot{a}_{10} \dot{a}_{20} \\ & - \frac{1}{2} a (\dot{a}_{10}^2 + 2a\ddot{a}_{20}) d_{001} \\ & - a\ddot{a}_{10} a_{10} d_{001} - \frac{1}{2} \dot{a}^2 (2ab_1 d_{012} + a_{20} d_{001}) - \frac{1}{6} \dot{b}_1 \dot{b}_2 \\ & - \frac{1}{2} a^2 \ddot{a} b_1 d_{012} + \frac{1}{3} b_2 . \end{aligned}$$

Using previous results for a_{10}, a_{20}, and b_2 , one now finds that

(6.15) $$a_{30} = \alpha_{30} d_{001} + \alpha_{30}' d_{012} + \alpha_{30}'' d_{001}^3 ,$$

where α_{30} and α_{30}' satisfy the equations

(6.16) $$\begin{aligned} a\ddot{\alpha}_{30} - \ddot{a}\alpha_{30} = {} & \frac{1}{2} (\ddot{a} \int_0^t a d\tau - a\dot{a}) \dot{\bar{a}}_{20} - \frac{1}{2} (\ddot{a} \int_0^t a d\tau + \dot{a}^2) \bar{a}_{20} \\ & - \frac{1}{6} \dot{b}_1 \dot{\beta}_2 + \frac{1}{3} \beta_2 , \end{aligned}$$

$$(6.17)\qquad \dddot{a}\alpha'_{30} - \ddot{a}\alpha'_{30} = -(\dot{a}^2 a b_1 + \dot{a}a^2\dot{b}_1 + \tfrac{1}{6}\,\dot{b}_1\dot{\beta}'_2 - \tfrac{1}{3}\beta'_2) \quad ,$$

with the initial conditions $\alpha_{30}(0) = 0$, $\alpha'_{30}(0) = 0$. (For the definition of β_2 and β'_2 see equations (5.8).) The equation that α''_{30} satisfies can be integrated explicitly and one gets

$$(6.18)\qquad \alpha''_{30} = -\frac{1}{16}\Big\{\frac{1}{3}\dddot{a}\Big(\int_0^t a\,d\tau\Big)^3 + \ddot{a}\Big(\int_0^t a\,d\tau\Big)\Big(\int_0^t a^2 d\tau\Big)$$

$$+ (2a\ddot{a} + \dot{a}^2)\Big(\int_0^t a\,d\tau\Big)^2 + 3a^2\dot{a}\int_0^t a\,d\tau + a\dot{a}\int_0^t a^2 d\tau + \dot{a}\int_0^t a^3 d\tau\Big\} \ .$$

The equations (16, 17) are singular, since the coefficient $\dot{a}$ of the leading term vanishes at some points. Yet, as we know already, they can be solved, although it is not convenient to use them in the given form for numerical solution. We know, however, that the a_{no} satisfy the equations (4.13) with $m = 0$, which are regular, and therefore α_{30} and α'_{30} satisfy equations with the same principal part and finite non-homogeneous term. Actually, equations (4.13) can be obtained from (16, 17) by multiplying them by a^3 , differentiating, and then dividing by $a^2\dot{a}$. One gets, however, simpler <u>regular</u> second order equations by differentiation of (16, 17) without prior multiplication by a^3 . The result is, on dividing by $\dot{a}$,

$$(6.16)'\quad \ddot{\alpha}_{30} - \Big(\frac{\dddot{a}}{\dot{a}}\Big)\alpha_{30} = -\frac{1}{2}\Big\{\Big(\frac{\dddot{a}}{\dot{a}}\Big)^{\bullet}\int_0^t a\,d\tau + 2a\Big(\frac{\dddot{a}}{\dot{a}}\Big) + 4\ddot{a}\Big\}\bar{a}_{20} - \frac{1}{3}\,b_1$$

$$+ \frac{1}{4}\,\dot{b}_1^2\Big(a^{-2}\dot{a}\int_0^t a\,d\tau - \frac{5}{3}\Big) + a^{-1}\dot{b}_1\dot{\beta}_2,\quad \alpha_{30}(0) = \alpha'_{30}(0) = 0$$

$$(6.17)' \quad \ddot{\alpha}'_{30} - \left(\frac{\dddot{a}}{\dot{a}}\right)\dot{\alpha}'_{30} = -(2a\ddot{a} + \dot{a}^2)b_1 + a\dot{a}\dot{b}_1 - a^2 - a\dot{b}_1\dot{\beta}'_2 \,,$$

$$\alpha'_{30}(0) = \dot{\alpha}'_{30}(0) = 0 \,.$$

The right hand terms in $(16)'$ and $(17)'$ were arranged so as to exhibit their finiteness. The only doubtful parts are those involving $\frac{\dddot{a}}{\dot{a}}$ and $\left(\frac{\dddot{a}}{\dot{a}}\right)^{\cdot}$. From equation (2) one finds,

$$\frac{\dddot{a}}{\dot{a}} = 4\hat{a}^{-3}a^{-5}\left(1 - \frac{\gamma}{4}(3\gamma+1)k[\hat{\alpha}a]^{-3(\gamma-1)}\right)$$

$$\left(\frac{\dddot{a}}{\dot{a}}\right)^{\cdot} = -20\hat{a}^{-3}a^{-6}\left(1 - \frac{\gamma}{20}(3\gamma+1)(3\gamma+2)k[\hat{\alpha}a]^{-3(\gamma-1)}\right)\dot{a} \,,$$

which are both finite for all t since $a > 0$.

VII. Discussion of Results.

Formulae for the bubble and migration coefficients are too involved to allow for a complete discussion of our results. In the companion report, Underwater Explosion Bubbles IV, we use these formulae for computations in a special case. Here we shall carry out only a qualitative discussion. In particular, we shall show that certain effects predicted by other authors are easily deducible from the present study.

We recall that to the third order in σ

$$(7.1) \quad \begin{cases} \lambda = a + a_{10}\sigma + (a_{20} + a_{22}P_2)\sigma^2 + (a_{30} + a_{32}P_2 + a_{33}P_3)\sigma^3 \\ b = b_1\sigma + b_2\sigma^2 + b_3\sigma^3 \,, \end{cases}$$

where

$$
(7.2)\quad \begin{cases}
a_{1o} = \alpha_{1o} d_{oo1} \\
a_{2o} = \bar{a}_{2o} + \alpha_{2o} d_{oo1}^2 \\
a_{22} = \bar{a}_{22} \\
a_{3o} = \alpha_{3o} d_{oo1} + \alpha'_{3o} d_{o12} + \alpha''_{3o} d_{oo1}^3 \\
a_{32} = \alpha_{32} d_{oo1} + \alpha'_{32} d_{o12} + \alpha''_{32} d_{o23} \\
a_{33} = \bar{a}_{33} \\
b_1 = \bar{b}_1 \\
b_2 = \beta_2 d_{oo1} + \beta'_2 d_{o12} \\
b_3 = \bar{b}_3 + \beta_3 d_{oo1}^2 + \beta'_3 d_{oo1} d_{o12} \quad .
\end{cases}
$$

The barred quantities are those that would be obtained if the boundaries were removed to infinity.

1. <u>Theory of bubble periods</u>.

In view of formula (6.6), the average radius of the bubble α is to the third order in σ given by α_o ,

$$
(7.3)\qquad \alpha = \alpha_o = a + a_{1o}\sigma + a_{2o}\sigma^2 + a_{3o}\sigma^3 \quad .
$$

In general α is not a periodic function, but may be assumed approximately periodic for small σ, since $a(t)$ is periodic. Let $\bar{\tau}$ be the half period of $a(t)$. The half period $\tau(\sigma)$ of α can be defined as the time elapsed until the first maximum is reached. That is, $\tau(\sigma)$ is defined by the equation

(7.4) $$\dot{a}(\tau(\sigma)) = 0 \quad .$$

Assume that

(7.5) $$\tau(\sigma) = \bar{\tau} + \tau_1\sigma + \ldots \quad .$$

Substituting in (4), expanding and keeping terms up to the first order, we get

(7.6) $$\dot{a}(\bar{\tau}) + \sigma(\tau_1\ddot{a}(\bar{\tau}) + \dot{a}_{1o}(\bar{\tau})) + \ldots = 0 \ .$$

Solving for τ_1, and using (6.12), we get

(7.7) $$\tau_1 = \frac{1}{2}d_{oo1}\left(\int_0^{\bar{\tau}} a(\tau)d\tau\right) = \frac{1}{2}d_{oo1}\left(\int_{a_{min}}^{a_{max}} \frac{a}{\dot{a}}\, da\right) \quad .$$

This formula is identical with that obtained in [4], p. 181, formula (1.15).

The formula for d_{oo1} which is derived in Appendix C is

(7.8) $$d_{oo1} = -\frac{1}{\mu+\nu}(f(x) + \log 2) \ ,$$

where

(7.9) $$f(x) = 2x\sum_{s=o}^{\infty} \frac{(-1)^s}{(2s+1)^2 - x^2} \ , \qquad x = \frac{\nu-\mu}{\nu+\mu} \quad .$$

The function $f(x)$ has been tabulated in [4], p. 190. d_{oo1} is negative for $x > -\frac{1}{3}$, and positive for $x < -\frac{1}{3}$. Thus the bubble period is smaller or larger than that of a corresponding bubble in infinite water, depending on whether $\mu < 2\nu$ or $\mu > 2\nu$.

2. Theory of migration.

To the first order, the migration of the bubble is given by the Herring formula,

$$(7.10) \qquad b = \sigma b_1 = 2\sigma \int_0^t \frac{d\tau}{a^3(\tau)} \int_0^\tau a^3(\sigma)\,d\sigma \ .$$

For derivation, see, e.g. [2]. Experimental data indicate that Herring's formula predicts a too large migration, and the discrepancy is of the order of 50°/o for moderate explosions (300 lb. TNT). For bubbles in infinite water, $b_2 \equiv 0$, and $b_3 < 0$. This shows that our results are in qualitative agreement with experimental data.

In presence of boundaries, $b_2 \neq 0$, and it dominates the third order correction. We shall estimate its effect by computing $\dot{b}_2$ at the time of the first minimum. In this estimate, we assume that $\dot{b}_1$ is appreciable only during a short time interval near the first minimum of the bubble, and that the bubble radius, is during most of its motion, near its maximum. Then formulae (5.8) yield, approximately, at $t = 2\tau$

$$(7.11) \qquad \begin{cases} \dot{\beta}_2 \sim 3\tau\left(\frac{a_M}{a_m}\right)\ddot{a}_m\, b_1(2\tau) \\ \dot{\beta}_2' \sim 12 a_m^{-3}\tau A \end{cases} \ .$$

Here, M refers to data at the maximum, m, to data at the minimum of the bubble, and A is the average value of $a^4\dot{a}^2$. Both

$\dot{\beta}_2$ and $\dot{\beta}_2'$ are positive. Using the above estimate, we now get

$$(7.12)\begin{cases} \dot{b}(2\tau) \sim \dot{\bar{b}}_1\sigma + \dot{\tilde{b}}_2\sigma^2 \sim 4(\frac{a_M}{a_m})^3\sigma\tau + 3(\frac{a_M}{a_m})b_1(2\tau)\ddot{a}_m\sigma^2\tau d_{oo1} \\ \qquad + 12\, a_m^{-3}\, A\sigma^2\tau d_{o12} \quad . \end{cases}$$

For $x > -\frac{1}{3}$, i.e. for $\mu < 2\nu$, both d_{oo1} and d_{o12} are negative showing that the effect of boundaries is to retard the upward migration of the bubble. For $x < -\frac{1}{3}$, i.e. for $\mu > 2\nu$, d_{oo1} becomes positive, while d_{o12} remains negative. Both $|d_{oo1}|$ and $|d_{o12}|$ tend to infinity as $x \rightarrow -1$. Hence, although it is not impossible that $\dot{\tilde{b}}_2$ remains negative for all x, it will certainly tend to $-\infty$ for $x \rightarrow -1$, that is for $\nu \rightarrow 0$, since $|d_{o12}| \rightarrow \infty$ faster than $|d_{oo1}|$. This can be seen from the table for $f'(x)$ found in [6], Fig. 15. We can conclude therefore that there exists a critical value of ν for which $\dot{b}(2\tau) \sim 0$, and for ν smaller than this critical value the bubble will migrate downwards.

3. Shape of the bubble.

We have shown in [2] that in the absence of boundaries the bubble would flatten in the first approximation and would become kidney shaped in the third approximation. This is because $\bar{a}_{22} \leq 0$ and $\bar{a}_{33} \geq 0$. The effect of boundaries on the shape is felt only in the third approximation and they affect only the value of a_{32}, (see equations (7.2)). By inspecting equations (5.16) it may be verified that $\tilde{a}_{32} \geq 0$ for sufficiently small ν. Hence α_2

$= \sigma^2 a_{22} + \sigma^3 a_{32} + \ldots$ may become positive, resulting in an elongated bubble. This possibility admits of a very simple explanation. The rigid bottom attracts the bubble downwards while a free surface repels the bubble, also downwards. On the other hand, due to gravity, the bubble has the tendency to rise. When the bubble is near the bottom, the upward pull due to gravity is larger than the repelling action of the free surface. As a result, the upper part of the bubble will move upward, and the lower part - downward. This may result is an elongation of the bubble, and even in a tear into two bubbles, one below the other.

Our deductions have been made by considering only the most significant terms in (7.2). Thus in discussing periods, we considered only effects due to first order terms, and neglected to review the effect of second order and third order terms. In studying the migration, we also neglected the third order terms. All these neglected terms may play a significant role. However, our conclusions are fully confirmed by a numerical example reported on in "Underwater Explosion Bubbles IV."

APPENDIX

A. Determination of the ϕ_m^*

Let $\phi_m(r,\theta) = r^{-(m+1)}P_m(\cos\theta)$. ϕ_m^* was defined in Section III as a regular harmonic function in the strip $-u < \zeta < v$, satisfying the boundary conditions

$$\text{(A1)} \qquad \begin{cases} \phi_m + \phi_m^* = 0, & \text{for} \quad \zeta = v, \\ (\phi_m + \phi_m^*)_\zeta = 0, & \text{for} \quad \zeta = -u, \end{cases}$$

where,

$$\text{(A2)} \qquad u = \frac{\nu}{\sigma} + b, \qquad v = \frac{\mu}{\sigma} - b.$$

ϕ_m^* may be constructed by using the method of images. Denote by ζ_k the ζ coordinates of images of the origin in the planes $\zeta = v$, and $\zeta = -u$, lying below the plane $z = -u$, and by $\bar{\zeta}_k$ the ζ coordinates of images lying above the plane $z = v$. We have,

$$\text{(A3)} \qquad \begin{cases} \xi_k = \begin{cases} -2sw, & \text{if} \quad k = 2s \\ -2sw - 2u, & \text{if} \quad k = 2s+1 \end{cases} \\ \bar{\xi}_k = \begin{cases} 2sw, & \text{if} \quad k = 2s \\ 2sw + 2v, & \text{if} \quad k = 2s+1 \end{cases} \end{cases}, \qquad s = 0, 1 \ldots$$

where,

(A4) $$w = u + v = \frac{\mu+\nu}{\sigma} \; .$$

It is easily verified that (A3) yields the ζ coordinates of all the images if k runs from 1 on, whereas for $k = 0$, $\zeta_o = \overline{\zeta}_o = 0$.

Our construction is based on the following proposition: If $g(\xi,\eta,\zeta) = g(\zeta)$ is a regular harmonic function everywhere except at the origin, then

(A5) $$G(\xi,\eta,\zeta) = - 2g(\zeta) + \sum_{n=o}^{\infty} (-1)^n [g(\zeta_{2n+1} - \zeta) + g(\zeta - \zeta_{2n}) + g(\zeta - \overline{\zeta}_{2n}) - g(\overline{\zeta}_{2n+1} - \zeta)]$$

is a regular harmonic function in the strip $-u < \zeta < v$, and $G(\xi,\eta,v) = - g(v)$, $G_\zeta(\xi,\eta,u) = g_\zeta(-u)$. We specialize g to $\phi_m(r,\theta) = \phi_m(\xi,\eta,\zeta)$, obtaining

(A6) $$\phi_m^* = \sum_{n=o}^{\infty}{}' (-1)^n [\phi_m(\zeta_{2n+1} - \zeta) + \phi_m(\zeta - \zeta_{2n}) + \phi_m(\zeta - \overline{\zeta}_{2n}) - \phi_m(\overline{\zeta}_{2n+1} - \zeta)] \; .$$

Here the ($'$) indicates that the terms involving $\zeta_o = \overline{\zeta}_o = 0$ should be omitted from the series.

Defining r_k, θ_k, $\overline{r}_k$, $\overline{\theta}_k$ by

$$\text{(A7)}\quad \begin{cases} r_k^2 = \xi^2 + \eta^2 + (\zeta - \zeta_k)^2 = r^2 + \zeta_k^2 - 2r\zeta_k\cos\theta \\ \bar{r}_k^2 = \xi^2 + \eta^2 + (\zeta - \bar{\zeta}_k)^2 = r^2 + \bar{\zeta}_k^2 - 2r\bar{\zeta}_k\cos\theta \\ r_k\cos\theta_k = (-1)^k(\zeta - \zeta_k) = (-1)^k(r\cos\theta - \zeta_k) \\ \bar{r}_k\cos\theta_k = (-1)^k(\zeta - \bar{\zeta}_k) = (-1)^k(r\cos\theta - \bar{\zeta}_k) \end{cases},$$

we now get

$$\text{(A8)}\quad \begin{cases} \phi_m(\xi,\eta,(-1)^n(\zeta - \zeta_k)) = \phi_m(r_k, \theta_k) = r_k^{-(m+1)}P_m(\cos\theta_k) \\ \phi_m(\xi,\eta,(-1)^n(\zeta - \bar{\zeta}_k)) = \phi_m(\bar{r}_k, \bar{\theta}_k) = \bar{r}_k^{-(m+1)}P_m(\cos\bar{\theta}_k) \end{cases}.$$

Substituting in (A6), we finally obtain

$$\text{(A9)}\quad \phi_m^*(r,\theta) = \sum_{n=0}^{\infty} (-1)^n[r_{2n+1}^{-(m+1)}P_m(\cos\theta_{2n+1}) + r_{2n}^{-(m+1)}P_m(\cos\theta_{2n}) + \bar{r}_{2n}^{-(m+1)}P_m(\cos\bar{\theta}_{2n}) - \bar{r}_{2n+1}^{-(m+1)}P_m(\cos\bar{\theta}_{2n+1})] .$$

B. Expansion of ϕ_m^* near $r = 0$.

In order to evaluate $\phi_m^*(\lambda,\theta)$ as a power series in σ, an expansion of ϕ_m^* in powers of r near $r = 0$ is needed.

By definition of Legendre Polynomials,

$$\frac{1}{\sqrt{1-2x\cos\theta_n+x^2}} = \sum_{k=0}^{\infty} x^k P_k(\cos\theta_n) , \qquad |x| < 1$$

Hence, on replacing x by $\frac{x}{r_n}$, (B1) $$\frac{1}{\sqrt{r_n^2-2xr_n\cos\theta_n+x^2}} = \sum_{k=0}^{\infty} x^k r_n^{-(k+1)} P_k(\cos\theta_n) , \quad |x| < r_n$$

from which we conclude that

$$\phi_m(r_n,\theta_n) = \frac{1}{m!}\left(\frac{\partial}{\partial x}\right)^m \frac{1}{\sqrt{r_n^2-2xr_n\cos\theta_n+x^2}}\Bigg|_{x=0} .$$

Since, however, by (A7)

$$r_n^2 - 2xr_n\cos\theta_n + x^2 = r^2 - 2r(\zeta_n + (-1)^n x)\cos\theta + (\zeta_n + (-1)^n x)^2 ,$$

we get

$$\phi_m(r_n,\theta_n) = (-1)^{nm}\frac{1}{m!}\left(\frac{\partial}{\partial\zeta_n}\right)^m \frac{1}{\sqrt{r^2-2r\zeta_n\cos\theta+\zeta_n^2}} .$$

Hence, using (B1) with r replacing x, and ζ_n replacing r_n, we get, on carrying out the differentiation,

(B2) $$\phi_m(r_n,\theta_n) = |\zeta_n|^{-1}\sum_{k=0}^{\infty}(-1)^{m(n+1)}\frac{(k+m)!}{k!m!}\zeta_n^{-(k+m)} r^k P_k(\cos\theta) ,$$

$$r \leq |\zeta_n| .$$

Similarly,

(B3) $$\phi_m(\bar{r}_n,\bar{\theta}_n) = |\bar{\zeta}_n|^{-1}\sum_{k=0}^{\infty}(-1)^{m(n+1)}\frac{(k+m)!}{k!m!}\bar{\zeta}_n^{-(k+m)} r^k P_k(\cos\theta) ,$$

$$r \leq |\bar{\zeta}_n| .$$

Substitution in (A9) now yields the desired expansion,

(B4) $$\phi_m^*(r,\theta) = \sum_{k=o}^{\infty} d_{mk} r^k P_k(\cos\theta) ,$$

where

$$d_{mk} = \frac{(k+m)!}{k!m!} \sum_{n=o}^{\infty}{}' [(-1)^n \frac{\zeta_{2n+1}^{-(k+m)}}{|\zeta_{2n+1}|} + (-1)^{n+m} \frac{\zeta_{2n}^{-(k+m)}}{|\zeta_{2n}|}$$
$$+ (-1)^{n+m} \frac{\bar{\zeta}_{2n}^{-(k+m)}}{|\bar{\zeta}_{2n}|} - (-1)^n \frac{\bar{\zeta}_{2n+1}^{-(k+m)}}{|\bar{\zeta}_{2n+1}|}] .$$

Using the definitions of ζ_n , $\bar{\zeta}_n$, we get,

(B5) $$d_{mk} = (\frac{\sigma}{2(\nu+\mu)})^{k+m+1} \frac{(k+m)!}{k!m!} \Big\{ \sum_{s=1}^{\infty} (-1)^{s+m}(1 + (-1)^{k+m}) s^{-(k+m+1)}$$
$$+ \sum_{s=o}^{\infty} (-1)^s [(-1)^{k+m}(s + \frac{\nu+\sigma b}{\nu+\mu})^{-(k+m+1)} - (s + \frac{\mu-\sigma b}{\nu+\mu})^{-(k+m+1)}]$$

C. <u>Expansion of</u> d_{mk} <u>in power of</u> σ.

Since $\nu + \sigma b > 0$, $\mu - \sigma b > 0$, each term in the brace of (B5) may be expanded in powers of σ. In the text we considered the solution for λ, ϕ, b up to terms in σ^3 . Since the expansion of d_{mk} starts with a term in σ^{k+m+1} , and since, in view of the theorem of Section III, each ϕ_m^* is multiplied in the expression for ϕ by σ to at least m'th power, for our purposes it will be sufficient to consider the expansion of the brace in (B5) up to the power $3 - (2m+k+1)$. Thus we only need to consider the cases $m = 0$, $k = 0$, 1, or 2, and $m = 1$, $k = 0$.

Writing

(C1) $$d_{mk} = \sum_{\ell=0}^{\infty} d_{mk\ell}\,\sigma^{\ell} \quad .$$

we have

(C2) $$\begin{cases} d_{oo} = d_{oo1}\sigma + d_{oo3}\sigma^3 + \dots \\ d_{o1} = d_{o12}\sigma^2 + \dots \\ d_{o2} = d_{o23}\sigma^3 + \dots \\ d_{1o} = d_{1o2}\sigma^2 + \dots \quad . \end{cases}$$

Here, the terms $d_{oo2}\sigma^2$ and $d_{o13}\sigma^3$ were omitted, since, as may be verified they are zero. We now get,

(C3) $$\begin{cases} d_{oo1} = \frac{1}{2(\mu+\nu)}\left\{2\sum_{s=1}^{\infty}\frac{(-1)^s}{s} + \sum_{s=o}^{\infty}(-1)^s\left[(s + \frac{\nu}{\mu+\nu})^{-1} - (s + \frac{\mu}{\mu+\nu})^{-1}\right]\right\} \\ d_{o12} = -\frac{1}{4(\mu+\nu)^2}\sum_{s=o}^{\infty}(-1)^s\left[(s + \frac{\nu}{\mu+\nu})^{-2} + (s + \frac{\mu}{\mu+\nu})^{-2}\right] \\ d_{o23} = \frac{1}{8(\mu+\nu)^3}\left\{2\sum_{s=1}^{\infty}\frac{(-1)^s}{s^3} + \sum_{s=o}^{\infty}(-1)^s\left[(s+ \frac{\nu}{\mu+\nu})^{-3} - (s + \frac{\mu}{\mu+\nu})^{-3}\right]\right\} \\ d_{1o2} = d_{o12} \\ d_{oo3} = 2b_1 d_{o12} \quad . \end{cases}$$

The series in (C3) can be expressed in terms of tabulated functions of

(C4) $$x = \frac{\nu-\mu}{\nu+\mu} = \frac{H-z_o}{H+z_o} ,$$

namely

(C5) $$f(x) = \sum_{s=o}^{\infty}(-1)^s[(2s+1-x)^{-1}-(2s+1+x)^{-1}]$$

$$= 2\frac{d}{dx}[\log\Gamma(\frac{1+x}{4}) + \log\Gamma(\frac{1-x}{4})] - \frac{\pi}{2}\tan\frac{\pi x}{2}$$

and its derivatives. We find

(C6) $$\begin{cases} d_{oo1} = -(f(x) + \log 2)(\mu+\nu)^{-1} \\ d_{o12} = -f'(x)(\mu+\nu)^{-2} \\ d_{o23} = -(\frac{1}{2}f''(x) + \frac{3}{16}\zeta(3))(\mu+\nu)^{-3} . \end{cases}$$

Here ζ denotes the ζ-Riemann function.

In special cases $\mu = \infty$, or $\nu = \infty$, formulae (C6) simplify to:

(C7) $$\begin{cases} d_{oo1} = \frac{1}{2\nu} \\ d_{o12} = -\frac{1}{4\nu^2} \\ d_{o23} = \frac{1}{8\nu^3} \end{cases} , \quad \text{for } \mu = \infty ,$$

(C8) $$\begin{cases} d_{oo1} = -\frac{1}{2\mu} \\ d_{o12} = -\frac{1}{4\mu^2} \\ d_{o23} = -\frac{1}{8\mu^3} \end{cases} , \quad \text{for } \nu = \infty .$$

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