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NAVORD REPORT 2195

A THEORY OF THE PROPAGATION OF SHOCKWAVES  
AND THEIR FORMATION BY EXPLOSIONS

**FC**

1 September 1951



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A THEORY OF THE PROPAGATION OF SHOCKWAVES  
AND THEIR FORMATION BY EXPLOSIONS

by

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ABSTRACT: A set of simultaneous ordinary differential equations is derived for the peak pressure and time factor of shockwaves. These equations satisfy the Rankine-Hugoniot conditions and the partial differential equations of fluid dynamics. No particular assumptions are made as to the nature of the medium, so the equations hold for any gaseous or liquid medium. For application to explosion phenomena, expressions for the initial conditions are found for the interface between the explosion products and the surrounding medium (which in this case is the place where the shockwave is formed). Relationships are derived for two cases:

- (1) that the shockwave is observed at a fixed point (applicable to the measurement of airblast waves).
- (2) that the point of observation moves with the medium (applicable to shockwave measurements in water).

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This report is part of a comprehensive project dealing with the calculation of shockwave and detonation parameters from the chemical composition of an explosive which is being carried out under Task NOL-Re2c-3-1. The report is for information only, and the opinions expressed therein are those of the authors. The authors wish to acknowledge the assistance of Miss Alys B. Russell in preparing this report.

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PAUL M. FYE  
By direction

# NAVORD Report 2195

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# A THEORY OF THE PROPAGATION OF SHOCKWAVES AND THEIR FORMATION BY EXPLOSIONS

## I INTRODUCTION

1. It has long been recognized that the disturbances emitted from explosions are shockwaves, i.e., waves having an infinitely steep front. Such shock-fronts are fully described by the well known Rankine-Hugoniot conditions, whereas for the phenomena behind the front the hydrodynamic equations of an inviscid fluid, neglecting gravitational effects, are applicable. It is the objective of this paper to combine these relations in such a way that information on the change of the pressure of the front with distance can be obtained. This can be done only if the distribution of the pressure behind the front is taken into account. In our case, the pressure distribution immediately behind the front is sufficiently described by two parameters, the time factor  $\alpha$  and the shape factor  $\alpha_t/\alpha^2$ . In this paper two different definitions for the time factor are used, namely:

$$(A) \quad \alpha = - \frac{1}{p} \left( \frac{\partial p}{\partial t} \right) \frac{a_0}{c_0}$$

and

$$(B) \quad \alpha^{(u)} = - \frac{1}{p} \left( \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial r} \right) \frac{a_0}{c_0}$$

$p$  and its derivatives refer to the pressure immediately behind the shock-front;  $t$  is the time,  $r$  the radial distance,  $u$  the velocity,  $a_0$  a reference length (in the study of explosion phenomena, the radius of the unexploded charge) and  $c_0$ , a reference velocity (for instance, the sound velocity of the undisturbed medium). These two reference magnitudes are introduced in order to obtain the time factor as a dimensionless magnitude. The time factors  $\alpha$  and  $\alpha^{(u)}$  are related to the initial decay of the pressure-time curve which, in the case of  $\alpha$

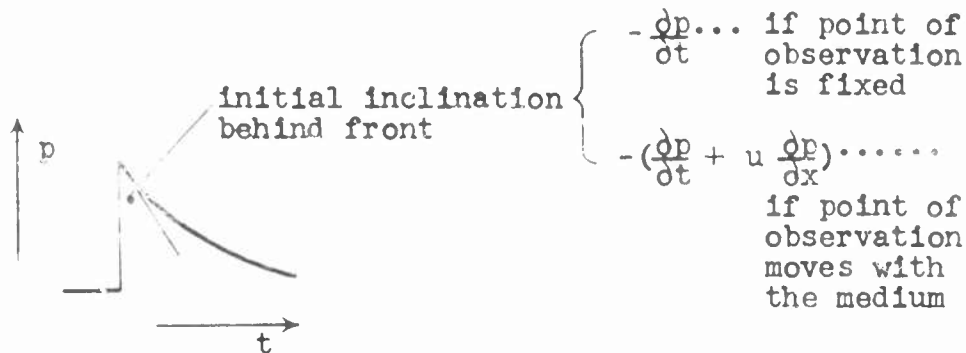


Figure 1  
Sketch of a pressure-time diagram.

is obtained by observing the pressure from a fixed point, but in the case of  $\alpha^{(u)}$ , from a point moving with the medium. These two cases have analogies in the experimental measurement of shockwaves due to explosions. In air blast measurements, the pressure recording gauges are usually so rigidly fixed that they cannot move significantly under the action of the impinging blast wave. Here  $\alpha$  applies. For the measurement of underwater shockwaves, the gauges are usually not very rigidly mounted. Considering the high pressures of the shockwave and the relatively small particle displacements in water, the case  $\alpha^{(u)}$  seems to be more appropriate for underwater explosions. However, it should be noted that only small differences are to be expected from these two treatments in the range of practical shockwave measurements.

2. The analysis presented in this paper yields two simultaneous ordinary differential equations for the peak pressure and the time factor as functions of distance. In the equations for the latter magnitude the shape factor appears as

$$\frac{\alpha_t}{\alpha^2} \quad \text{or} \quad \frac{\alpha_t^{(u)}}{\alpha^{(u)2}}$$

These magnitudes are related to the second derivative of the pressure with respect to time immediately behind the front. Again, the first of these refers to a fixed point of observation, the second, to a point moving with the medium.

3. To integrate the differential equations, the shape factor must be known as a function of distance. This magnitude depends only on the shape of the wave; no information about the pressure or the time scale is necessary for its evaluation. For instance, if the pressure-time curve (observed from either of the points of reference discussed above) is an exponential curve, the shape factor has the value one; if it is a straight line the shape factor is zero. Shockwaves from explosive sources have the important property that their shape factors are constant within an appreciable range of distance. This holds true particularly for the initial portion of an underwater shockwave which, at all distances experimentally investigated, is an exponential curve to a surprising degree of accuracy.

This fact has first been used by Kirkwood and Bethe [a]\* in the calculation of shockwave phenomena. Following their usage, we call this procedure the "peak approximation". In our analysis - which is rigorous up to this point - this amounts to an a priori assignment of a value to the shape factor.

4. No assumption of a particular shape of the wave is incorporated in the equations derived here. This, together with the rigorous character of these equations, makes it possible to study the validity of the peak approximation and to show how sensitive the solutions are to the assumed shape of the wave.

5. In order to obtain numerical solutions of the differential equations, the initial conditions must be known. In the calculation of shockwaves from explosions these conditions are provided by studying the phenomena at the interface between the reaction products and the surrounding medium, i.e., by studying the formation of the shockwave. It has often been pointed out that a high-amplitude pressure pulse must, during propagation, change its shape in such a way that a shock-front is finally built up. This phenomenon seldom occurs with shockwaves due to explosions. The rapidity with which the pressure is built up by an explosion causes the pressure wave to have a steep front from the very beginning. It is a shock-front which emerges from the surface of explosion products into the surrounding medium

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\* All such letters refer to the list of references at the end of this report.



a very short time after the detonation of the charge. Therefore, this front can provide the initial conditions for the differential equations for the shockwave peak pressure and for the time factor.

6. Several similar approaches to the problem of shockwaves due to explosions have been made previously, notably by Kirkwood and co-workers. The Kirkwood-Bethe theory [a] - though very successful in its application to underwater explosions - is based on a propagation theory which is not a rigorous solution of the hydrodynamic equations. The Kirkwood Brinkley theory [b] which is applicable to any fluid is a rigorous approach based on energy considerations and a hybrid form of the Euler and Lagrange equations of fluid dynamics. The peak approximation (or "similarity restraint" as it is called in that paper) is incorporated in the final equations by assuming an exponential wave.

7. In the present paper, an attempt is made to treat this problem in a straightforward manner, starting with the Euler equations. A similar approach has been made previously in other papers [c - f]. In these papers equation 1.6 or similar expressions are derived in various ways, but the treatment is not carried much further.

## II HYDRODYNAMIC RELATIONS

8. The fundamental equations for the fluid motion of an inviscid medium, neglecting the influence of gravity, are

$$\begin{aligned} (0.1) \quad & u_t + u u_r + \frac{p_r}{\rho} = 0 \\ (0.2) \quad & \rho_t + u \rho_r + \rho u_r + \frac{2u\rho}{r} = 0 \\ (0.3) \quad & S_t + u S_r = 0 \end{aligned}$$

The subscripts  $t$  or  $r$  denote the partial derivatives with respect to these magnitudes. In general,

$$f_y = \frac{\partial f}{\partial y}$$

Throughout this paper,  $p$  denotes the pressure in excess of the pressure of the undisturbed medium,  $p_0$ .

In the above equations  $u$  is the particle velocity,  $\rho$  the density and  $S$  the entropy.  $r$  and  $t$  are the space and time coordinates respectively. Equation (0.3) states that we are considering reversible processes, which is consistent with the assumption of an inviscid fluid. Equations (0.1) to (0.3) are not applicable across discontinuities (shock-fronts), as discussed below.

9. Equation (0.2) is written for the case of spherical symmetry. The case of cylindrical symmetry is obtained by omitting the factor 2 of the last term of (0.2), the plane wave case, by omitting the last term entirely.

10. Since

$$\rho = \rho(p + p_0, S)$$

the density can be eliminated in (0.2). Introducing the sound velocity

$$c^2 = \left( \frac{\partial (p + p_0)}{\partial \rho} \right)_S = \left( \frac{\partial p}{\partial \rho} \right)_S$$

We obtain from (0.2) and (0.3)

$$(0.4) \quad p_t + u p_r + \rho c^2 u_r + \frac{2 \rho u c^2}{r} = 0$$

11. Introducing the notation

$$f_t^{(q)} = \frac{\partial f}{\partial t} + q \frac{\partial f}{\partial r}$$

(0.1) and (0.4) can be combined to take the form

$$(0.5) \quad \frac{1}{\rho c} p_t^{(c+u)} + u_t^{(c+u)} = - \frac{2cu}{r}$$

$$(0.6) \quad \frac{1}{\rho c} p_t^{(c-u)} - u_t^{(c-u)} = - \frac{2cu}{r}$$

$p_t^{(c+u)}$  can be interpreted as the derivative of  $p$  with respect to time along a "(c+u) - characteristic", i.e., along a curve for which in the  $r$ - $t$  plane

$$\frac{dr}{dt} = c + u$$

In general,  $f_t^{(\eta)}$  denotes the derivative along a g-characteristic for which

$$\frac{dr}{dt} = \eta$$

12. If a fully isentropic flow is considered ( $S_t = 0, S_r = 0$ , therefore also  $S_t^{(c,u)} = 0$ ), we may introduce the Riemann function which is defined by

$$\left(\frac{\partial \sigma}{\partial p}\right)_s = \frac{1}{\rho}$$

(0.5) and (0.6) can then be transformed into

$$(0.7) \quad (\sigma + u)_t^{(c+u)} = - \frac{2uc}{r}$$

$$(0.8) \quad (\sigma - u)_t^{(c-u)} = - \frac{2uc}{r}$$

which is the well-known Riemann formulation of high-amplitude waves. These equations are not always applicable in the cases considered here.

13. For the shock-front, these equations are supplemented by the Rankine-Hugoniot conditions. These are three equations which permit  $u, \rho, c$  and the propagation velocity of the front,  $U$ , to be expressed in terms of  $p$ , as soon as the equation of state of the medium considered is known. Here,  $p$  denotes the amplitude of the shock-front. There are tables available which give these relations for air, sea water and fresh water [g - k]. For our purpose we can consider that  $u, \rho, c$  and  $U$  are known functions of at the shock-front.

Since we are interested in the change of the shockwave peak-pressure (i.e., the front pressure) with distance, we use the notation

$$\begin{aligned} \frac{df}{dx} &= f_t^{(U)} \\ &= \frac{\partial f}{\partial t} + U \frac{\partial f}{\partial x} \end{aligned}$$

That is, we arbitrarily define the derivative along the  $U$  - characteristic (which is equivalent to the shock-front) as the total derivative.

14. The concept of the characteristic gives us a geometrical approach to our problem. Consider the "characteristic triangle" below.

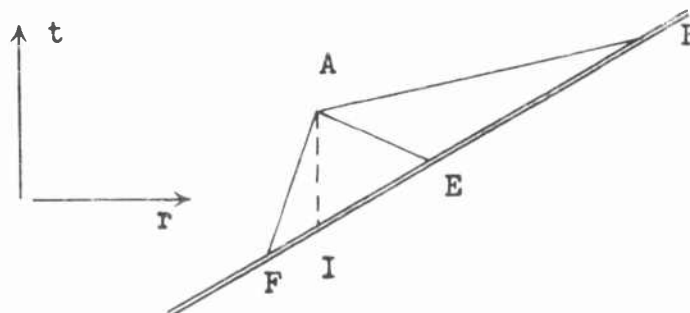


Figure 2  
Characteristic Triangle.

The lines in this figure are drawn in a  $t$ - $r$  plane in such a way that

|       |                                  |  |
|-------|----------------------------------|--|
| along |                                  |  |
| AB    | $\frac{dt}{dr} = -\frac{1}{c+u}$ |  |
| AE    | $\frac{dt}{dr} = \frac{1}{u-c}$  |  |
| AF    | $\frac{dt}{dr} = \frac{1}{u}$    |  |
| FB    | $\frac{dt}{dr} = \frac{1}{U}$    |  |

We then call

|      |                               |
|------|-------------------------------|
| AB   | the " $c+u$ - characteristic" |
| AE   | the " $c-u$ - characteristic" |
| AF   | the " $u$ - characteristic"   |
| FIEB | the " $U$ - characteristic"   |

These lines are not straight lines, since, e.g.,  $1/U$  at E differs generally from the  $1/U$  at B. This characteristic triangle is based on the fact that  $c+u > U$  which holds generally and accounts for shock phenomena.

15. In Figure 2, the line FIEB represents the shock-front. The regime below this line is that of the undisturbed medium into which the shock propagates; the regime above represents the shockwave, where there are high pressures and a particle velocity towards the front.

16. Considering the equations above, we find that (0.5) holds along the line between A and B and (0.6), between A and E. Furthermore, along FIEB the Rankine-Hugoniot conditions hold and along FA equation (0.3) holds. Thus, at A we have the same entropy as at F, whereas at B, in general, the entropy is different. This is because shock-fronts, which involve irreversible processes, cause an entropy increment which depends on the amplitude of the front.

17. As far as the thermodynamic state is concerned, we note that along FA the common adiabatic (i.e., the isentropic) holds, whereas along FB the Hugoniot adiabatic applies. Since both of these thermodynamic relationships are known, we can express the thermodynamic state along FA as well as along FB in terms of the pressure along these lines, if the state at F is known. The same holds for the particle velocity  $u$  along FB, but not along FA.

#### Case I Fixed Point of Observation

18. The Peak Pressure. Using the characteristic triangle, we will now derive the differential equation for the shock-wave peak pressure as a function of distance. In Case I, we consider the point of observation as fixed.

On integrating (0.5) and (0.6) along AB and AE respectively, we have

$$(1) \quad \frac{1}{(\rho c)_{AB}} (p_B - p_A) + (u_B - u_A) = - \left( \frac{2cu}{r} \right)_{AB} (t_B - t_A)$$

$$(2) \quad \frac{1}{(\rho c)_{AE}} (p_A - p_E) - (u_A - u_E) = - \left( \frac{2cu}{r} \right)_{AE} (t_A - t_E)$$

where the double subscript denotes the average obtained in the integration.

In this paper, we will be primarily concerned with first or second order expressions. Hence, these averages need to be correct only to the first order. For instance,  $1/\rho c_{AB}$  then becomes:

$$\frac{1}{\rho c_{AB}}(p_B - p_A) = \int_A^B \frac{dp}{\rho c} \sim \frac{1}{2} \left[ \frac{1}{(\rho c)_A} + \frac{1}{(\rho c)_B} \right] (p_B - p_A)$$

With this approximation, (1.1) and (1.2) can be combined to give

$$(1.3) \quad \frac{1}{2} \left[ \frac{1}{(\rho c)_A} + \frac{1}{(\rho c)_B} \right] (p_B - p_A) - \frac{1}{2} \left[ \frac{1}{(\rho c)_A} + \frac{1}{(\rho c)_E} \right] (p_A - p_E) + (u_B - u_E) \\ = - \left[ \left( \frac{cu}{r} \right)_A + \left( \frac{cu}{r} \right)_B \right] (t_B - t_A) + \left[ \left( \frac{cu}{r} \right)_A + \left( \frac{cu}{r} \right)_E \right] (t_A - t_E)$$

19. If we expand (1.3) by means of a Taylor's series, terms of order zero, one, two, and higher appear. Considering only first order terms, equation (1.3) becomes

$$(1.4) \quad \frac{2}{\rho c} \frac{dp}{dt} (t_B - t_I) - \frac{2}{\rho c} p_t (t_A - t_I) - \frac{1}{\rho c} \frac{dp}{dt} (t_B - t_E) \\ + \frac{du}{dp} \frac{dp}{dt} (t_B - t_E) + \frac{2cu}{r} (t_B - t_A) - \frac{2cu}{r} (t_A - t_E) = 0$$

All magnitudes without subscript refer to the point I, i.e., to  $r$  and  $t$ . For the various differences in  $t$ , we obtain the following first order relations:

$$t_B - t_I = \frac{\Delta r}{U} \\ t_B - t_A = \frac{\Delta r}{c+u} \\ t_A - t_I = \frac{c+u-U}{U(c+U)} \Delta r \\ t_B - t_E = \frac{2c \Delta r}{(c+u)(c+U-u)} \\ t_A - t_E = \frac{(c+u-U) \Delta r}{(c+u)(c+U-u)}$$

where

$$\Delta r = r_B - r_I$$

Substituting these expressions in (1.4) and dividing by  $\Delta r$ , we obtain on collecting terms:

$$(1.5) \quad \frac{1}{U} \frac{dp}{dt} \left[ c^2 + u(U-u) + \frac{du}{dp} \rho c^2 U \right] - \rho c \frac{[c^2 - (U-u)^2]}{U} + \frac{2\rho c^2 u (U-u)}{r} = 0$$

Using the notation

$$\alpha = - \frac{1}{p} \rho c \frac{a_e}{c_0}$$

and

$$\frac{r}{a_0} = x$$

(1.5) becomes

$$(1.6) \quad \frac{dp}{dx} + \frac{P_{11}}{x} + P_{12} \alpha = 0$$

where  $P_{11}$  and  $P_{12}$  are functions of the pressure  $p$  alone:

$$P_{11} = \frac{2\rho c^2 u (U-u)}{c^2 + u(U-u) + \frac{du}{dp} \rho c^2 U}$$

$$P_{12} = \frac{\rho c_0 \frac{[c^2 - (U-u)^2]}{U}}{[c^2 + u(U-u) + \frac{du}{dp} \rho c^2 U]}$$

When a double subscript appears in functions such as these, the first subscript refers to the case being discussed (I or II) and the second subscript to the number of the function. Thus  $P_{12}$  is the second P-function of Case I.

20. The differential equation just formed has been derived by other methods (see paragraph 7). For instance, equations (0.1) and (0.2) have been combined to give (1.6) directly, [d, e].

21. The Time Factor. In order to obtain a differential equation involving  $d\alpha/dx$ , we have to consider the second order terms in the expansion of (1.3). Rather than following the same lines as in the foregoing paragraphs, we will obtain the desired equation by an analytic method in which the computations are shorter. The two procedures are analogous, however.

Differentiating (0.2) and (0.4) partially with respect to  $t$ , we have

$$(2.1) \quad u_{tt} + u u_{tr} + u_t u_r + \frac{1}{\rho} p_{tr} - \frac{1}{\rho^2} \rho_t p_r = 0$$

$$(2.2) \quad p_{tt} + u p_{tr} + u_t p_r + \rho c^2 u_{tr} + u_r (\rho c^2)_t + \frac{2u}{r} (\rho c^2)_t + \frac{2\rho c^2}{r} u_t = 0$$

Differentiating (1.6) along  $U$  yields

$$(2.3) \quad \frac{d^2 p}{dx^2} = \frac{p_N}{x^2} \left( 1 + \frac{dp_N}{dp} \right) + \frac{\alpha}{x} \left( p_{12} \frac{dp_{11}}{dp} + p_{11} \frac{dp_{12}}{dp} \right) + \alpha^2 p_{12} \frac{dp_{12}}{dp} - p_{12} \frac{d\alpha}{dx}$$

In addition,

$$\begin{aligned} \frac{d^2 u}{dt^2} &= \frac{du}{dp} \frac{d^2 p}{dt^2} + \frac{d^2 u}{dp^2} \left( \frac{dp}{dt} \right)^2 \\ &= U^2 \frac{du}{dp} \frac{d^2 p}{dx^2} + U \left( \frac{dp}{dx} \right)^2 \left[ \frac{du}{dp} \frac{dU}{dp} + U' \frac{d^2 u}{dp^2} \right] \end{aligned}$$

so that we obtain

$$(2.4) \quad \begin{aligned} \frac{d^2 u}{dt^2} &= \frac{p_N}{r^2} \left[ \left( 1 + \frac{dp_{11}}{dp} \right) U^2 \frac{du}{dp} + U p_{11} \left( \frac{du}{dp} \frac{dU}{dp} + U \frac{d^2 u}{dp^2} \right) \right] \\ &+ \frac{\alpha}{a_0 r} \left[ U^2 \frac{du}{dp} \left( p_{12} \frac{dp_{11}}{dp} + p_{11} \frac{dp_{12}}{dp} \right) + 2 U p_N p_{12} \left( \frac{du}{dp} \frac{dU}{dp} + U \frac{d^2 u}{dp^2} \right) \right] \\ &+ \frac{\alpha^2}{a_0^2} \left[ U^2 p_{12} \frac{du}{dp} \frac{dp_{12}}{dp} + U p_{12}^2 \left( \frac{du}{dp} \frac{dU}{dp} + U \frac{d^2 u}{dp^2} \right) \right] - \frac{U^2 p_{12}}{a_0} \frac{dx}{dp} \frac{d\alpha}{dr} \end{aligned}$$

Moreover,

$$(2.5) \quad \frac{d}{dt} \left( \frac{du}{dt} \right) = u_{tt} + U u_{tr} + \frac{d}{dt} (U u_r)$$

Introducing  $\alpha$  as well as (1.6) into (0.4), we obtain

$$U u_r = \frac{Q_{101}}{r} + \frac{Q_{102} \alpha}{a_0}$$



where  $q_{101}$  and  $q_{102}$  are functions of  $p$  alone. Many such functions will appear. They are listed for ready reference at the end of this section. As above, each function is identified by a double subscript: the first number, 1, refers to Case I, and the other two give the sequence number of the function.

22. By differentiation,

$$\begin{aligned} \frac{d}{dt}(U u_r) &= -\frac{1}{r^2} \left( U q_{101} + U p_{11} \frac{d q_{101}}{d p} \right) \\ &- \frac{\alpha}{a_0 r} \left( U p_{12} \frac{d q_{101}}{d p} + U p_{11} \frac{d q_{102}}{d p} \right) - \frac{\alpha^2}{a_0^2} \left( U p_{12} \frac{d q_{102}}{d p} \right) + \frac{U q_{102}}{a_0} \frac{d \alpha}{d r} \end{aligned}$$

Combining this equation with (2.4) and (2.5), we obtain an expression for  $u_{tt}$  as follows:

$$u_{tt} = \frac{q_{103}}{r^2} + \frac{q_{104}}{a_0 r} \alpha + \frac{q_{105}}{a_0^2} \alpha^2 + \frac{q_{106}}{a_0} \frac{d \alpha}{d r} - U u_{tr}$$

Substituting for  $u_{tt}$  in (2.1), we may eliminate  $u_{tr}$  between this equation and (2.2) to obtain a single equation in which  $p_{tt}$  and  $p_{tr}$  appear. We eliminate  $p_{tr}$  by using the following relation:

$$p_{tt} + U p_{tr} = -\frac{U c_0}{a_0} \frac{d}{d r} (p \alpha)$$

so that we obtain the equation

$$\begin{aligned} &\frac{\rho c^2 q_{103}}{r^2} + \frac{\rho c^2 q_{104}}{a_0 r} \frac{\alpha}{r} + \frac{\rho c^2 q_{105}}{a_0^2} \alpha^2 + \frac{\rho c^2 q_{106}}{a_0} \frac{d \alpha}{d r} \\ &+ \rho c^2 u_r u_r - \frac{c^2}{r} p_t p_r + (U-u) u_t p_r + (U-u) (\rho c^2)_t u_r \\ (2.6) \quad &+ \frac{2 u (U-u)}{r} (\rho c^2)_t + \frac{2 \rho c^2 (U-u)}{r} u_t - \left[ \frac{c^2 - (U-u)^2}{U} \right] p_{tt} \\ &- \frac{c_0}{a_0} \left[ c^2 + u (U-u) \right] \frac{d(p \alpha)}{d r} = 0 \end{aligned}$$

23. Solving for  $u_r$ ,  $u_t$ ,  $p_r$ ,  $c_t$  and  $\rho_t$ , we obtain the following results:

$$\begin{aligned} u_r &= \frac{Q_{101}}{rU} + \frac{Q_{102}}{a_o U} \\ u_t &= \frac{Q_{102}}{r} + \frac{Q_{103}}{a_o} \alpha \\ p_r &= -\frac{P_{11}}{r} + \left( \frac{P_{12}}{a_o U} - \frac{P_{12}}{a_o} \right) \alpha \\ c_t^2 &= -\frac{Q_{102}}{r} - \frac{Q_{110}}{a_o} \alpha \\ \rho_t &= -\frac{Q_{111}}{r} - \frac{Q_{112}}{a_o} \alpha \end{aligned}$$

Further, we define

$$Q_i = \frac{1}{\rho} P_{oi} \left( \frac{a_o}{c_o} \right)^2$$

Substituting in (2.6) yields the following equation:

$$(2.7) \quad \frac{\phi_{11}}{a_o} \frac{d\alpha}{dr} + \frac{\phi_{12}}{r^2} + \frac{\phi_{13}\alpha}{a_o r} + \frac{\phi_{14}\alpha^2}{a_o^2} + \frac{\phi_{15}\alpha_t}{a_o^2} = 0$$

or

$$\frac{d\alpha}{dx} + \frac{P_{13}}{x^2} + \frac{P_{14}\alpha}{x} + P_{15}\alpha^2 \left( 1 + P_{16} \frac{\alpha_t}{\alpha^2} \right) = 0$$

Again, the functions denoted by  $P$  depend on the shockwave peak pressure only.

24. The functions denoted above by  $P$  and  $Q$  terms are explicitly:

$$Q_{101} = U \left( \frac{P_{11}}{\rho_o^2} - 2u \right)$$

$$Q_{102} = \frac{u U P_{12}}{\rho_o^2} + \frac{c_o \phi (U \cdot u)}{\rho_o^2}$$

$$\begin{aligned} Q_{103} &= P_{11} \left[ \left( 1 + \frac{dP_{11}}{dp} \right) U^2 \frac{du}{dp} + U P_{11} \left( \frac{du}{dp} \frac{dU}{dp} + U \frac{d^2 u}{dp^2} \right) \right] \\ &\quad + U Q_{101} + U P_{11} \frac{dQ_{101}}{dp} \end{aligned}$$

$$Q_{104} = U^2 \frac{du}{dp} \left( P_{12} \frac{dP_{11}}{dp} + P_{11} \frac{dP_{12}}{dp} \right) + 2UP_{11}P_{12} \left( \frac{du}{dp} \frac{dU}{dp} + U \frac{d^2u}{dp^2} \right) \\ + UP_{12} \frac{dQ_{101}}{dp} + UP_{11} \frac{dQ_{102}}{dp}$$

$$Q_{105} = U^2 P_{12} \frac{du}{dp} \frac{dP_{12}}{dp} + UP_{12}^2 \left( \frac{du}{dp} \frac{dU}{dp} + U \frac{d^2u}{dp^2} \right) + UP_{12} \frac{dQ_{102}}{dp}$$

$$Q_{106} = -P_{12} U^2 \frac{du}{dp} - U Q_{102}$$

$$Q_{107} = \frac{P_{11}}{\rho} - \frac{P_{11}u^2}{\rho c^2} + 2u^2$$

$$Q_{108} = \frac{P_{12}}{\rho} - \frac{P_{12}u}{\rho U} - \frac{P_{12}u^2}{\rho c^2} - \frac{P_{12}u(U-u)}{\rho c^2 U}$$

$$Q_{109} = \frac{uUP_{11}}{U-u} \left[ \left( \frac{\partial c^2}{\partial p} \right)_s - \frac{dc^2}{dp} \right]$$

$$Q_{110} = P_{12} \left( \frac{\partial c^2}{\partial p} \right)_s + \frac{uUP_{12}}{U-u} \left[ \left( \frac{\partial c^2}{\partial p} \right)_s - \frac{dc^2}{dp} \right]$$

$$Q_{111} = \frac{P_{11}uU}{U-u} \left[ \left( \frac{\partial \rho}{\partial p} \right)_s - \frac{d\rho}{dp} \right]$$

$$Q_{112} = P_{12} \left( \frac{\partial \rho}{\partial p} \right)_s + \frac{uUP_{12}}{U-u} \left[ \left( \frac{\partial \rho}{\partial p} \right)_s - \frac{d\rho}{dp} \right]$$

$$\phi_{11} = - \rho c_o \left[ c^2 + u (U-u) \right] + \rho c^2 Q_{106}$$

$$\begin{aligned} \phi_{12} = & \rho c^2 Q_{103} + \frac{\rho c_o^2}{U} Q_{101} Q_{107} - \frac{c^2}{\rho} P_n Q_{111} - (U-u) P_{11} Q_{107} \\ & - \frac{U-u}{U} \rho Q_{101} Q_{107} - \left( \frac{U-u}{U} \right) c^2 Q_{101} Q_{111} + 2 \rho c^2 (U-u) Q_{107} \\ & - 2 u \rho (U-u) Q_{107} - 2 u c^2 (U-u) Q_{111} \end{aligned}$$

$$\begin{aligned} \phi_{13} = & \rho c^2 Q_{104} + \frac{\rho c_o^2}{U} (Q_{101} Q_{108} + Q_{102} Q_{107}) - \frac{c^2}{\rho} \left[ P_n Q_{112} - Q_{111} \left( \frac{P_{12}}{U} - P_{12} \right) \right] \\ & - 2 u \rho (U-u) Q_{110} + (U-u) \left[ Q_{107} \left( \frac{P_{12}}{U} - P_{12} \right) - P_n Q_{108} \right] \\ & - \left( \frac{U-u}{U} \right) \rho (Q_{101} Q_{110} + Q_{102} Q_{107}) - 2 u c^2 (U-u) Q_{112} - \left( \frac{U-u}{U} \right) c^2 (Q_{101} Q_{112} + Q_{102} Q_{111}) \\ & + 2 \rho c^2 (U-u) Q_{108} + P_n c_o \left[ c^2 + u (U-u) \right] \end{aligned}$$

$$\begin{aligned} \phi_{14} = & \rho c^2 Q_{105} + \frac{\rho c_o^2}{U} Q_{102} Q_{108} + \frac{c^2}{\rho} \left( \frac{P_{12}}{U} - P_{12} \right) Q_{112} + (U-u) \left( \frac{P_{12}}{U} - P_{12} \right) Q_{108} \\ & - \left( \frac{U-u}{U} \right) \rho Q_{102} Q_{110} - \left( \frac{U-u}{U} \right) c^2 Q_{102} Q_{112} + \left[ c^2 + u (U-u) \right] c_o P_{12} \end{aligned}$$

$$\phi_{15} = - \frac{P_{12}^2}{U} \left[ c^2 - (U-u)^2 \right]$$

$$P_{13} = \frac{\phi_{12}}{\phi_{11}}$$

$$P_{14} = \frac{\phi_{13}}{\phi_{11}}$$

$$P_{15} = \frac{\phi_{14}}{\phi_{11}}$$

$$P_{16} = \frac{\phi_{15}}{\phi_{14}}$$

# Case II Moving Point of Observation

25. The Peak Pressure. In Case II, it is assumed that the point of observation moves with the particles of the fluid. The approach to a differential equation for the shockwave peak pressure is much the same as the one given in paragraph 19 for equation (1.6). The characteristic triangle, Figure 2, takes a slightly different form, as the line AF must now be considered instead of the line AI. The initial steps are the same, so that we arrive at equation (1.3) again. Introducing

$$p_t^{(u)} = p_t + u p_r$$

instead of  $p_t$ , we obtain on expanding (1.3)

$$(3.1) \quad \frac{2}{\rho c} \frac{dp}{dt} (t_B - t_F) - \frac{2}{\rho c} p_t^{(u)} (t_A - t_F) - \frac{1}{\rho c} \frac{dp}{dt} (t_B - t_E) + \frac{du}{dp} \frac{dp}{dt} (t_B - t_E) + \frac{2cu}{\gamma} (t_B - t_A) - \frac{2cu}{\gamma} (t_A - t_E) = 0$$

where

$$\begin{aligned} t_B - t_F &= \frac{c}{u + c - U} (t_A - t_F) \\ t_A - t_E &= \frac{U - u}{c + U - u} (t_A - t_F) \\ t_B - t_E &= \frac{2c(U - u)}{(u + c - U)(c + U - u)} (t_A - t_F) \\ t_B - t_A &= \frac{U - u}{u + c - U} (t_A - t_F) \end{aligned}$$

Making these substitutions in (3.1) and dividing through by  $t_A - t_F$  we obtain

$$(3.2) \quad \frac{dp}{dt} \left[ c^2 + \frac{du}{dp} \rho c^2 (U - u) \right] - p_t^{(u)} \left[ c^2 - (U - u)^2 \right] + \frac{2\rho u c^2 (U - u)^2}{\gamma} = 0$$

For consistency of notation, let

$$\alpha^{(u)} \equiv - \frac{1}{p} p_t^{(u)} \frac{a_s}{c_s}$$

so that (3.2) becomes

$$(3.3) \quad \frac{dp}{dx} + \frac{p_{21}}{x} + p_{22} \alpha^{(u)} = 0$$

where

$$p_{21} = \frac{2\rho u c^2 (U-u)^2}{U \left[ c^2 + \frac{du}{dp} \rho c^2 (U-u) \right]}$$

$$p_{22} = \frac{p c_s \left[ c^2 - (U-u)^2 \right]}{U \left[ c^2 + \frac{du}{dp} \rho c^2 (U-u) \right]}$$

**26. The Time Factor.** The procedure of obtaining a differential equation for the time factor parallels that for Case I.

Here we define  $\alpha_t^{(u)}$  by

$$(3.4) \quad \alpha_t^{(u)} = \frac{1}{p} \left( \frac{a_s}{c_s} \right)^2 (p_t^{(u)})_c$$

$$= \frac{1}{p} \left( \frac{a_s}{c_s} \right)^2 (p_{tt} + 2u p_{rt} + u_t p_r + u^2 p_{rr} + u u_r p_r)$$

The differential equation has the form

$$(3.5) \quad \frac{d\alpha^{(u)}}{dx} + \frac{p_{23}}{x^2} + \frac{p_{24}}{x} \alpha^{(u)} + p_{25} \alpha^{(u)2} \left( 1 + p_{26} \frac{\alpha_t^{(u)}}{\alpha^{(u)2}} \right) = 0$$

where the  $P$ 's can be expressed in terms of the functions and  $\phi$  in an analogous way to Case I:

$$p_{23} = \frac{\phi_{22}}{\phi_{21}}$$

$$p_{24} = \frac{\phi_{23}}{\phi_{21}}$$

$$p_{25} = \frac{\phi_{24}}{\phi_{21}}$$

$$p_{26} = \frac{\phi_{25}}{\phi_{24}}$$

$$\phi_{21} = \rho c^2 Q_{200} + \rho c_0 u U - \frac{\rho c^2 c_0 u U}{(U-u)^2} - \frac{\rho c^2 c_0 U}{U-u} \frac{P_{22}}{P_{12}}$$

$$\begin{aligned} \phi_{22} = & \rho c^2 Q_{201} + \frac{\rho c^2 Q_{201} Q_{207}}{U} - \frac{c^2}{\rho} Q_{211} \left[ P_{21} - \frac{\rho c_0}{U} \left( \frac{P_{21} - P_{11}}{P_{12}} \right) \right] - \left( \frac{U-u}{U} \right) \rho Q_{201} Q_{207} \\ & - \left( \frac{U-u}{U} \right) c^2 Q_{201} Q_{211} - 2u\rho(U-u)Q_{209} - 2uc^2(U-u)Q_{211} + 2\rho c^2(U-u)Q_{207} \\ & - \frac{c^2}{U-u} Q_{207} \left[ P_{21} - \frac{\rho c_0}{U} \left( \frac{P_{21} - P_{11}}{P_{12}} \right) \right] + \frac{\rho c^2 c_0 U}{U-u} \left[ \frac{P_{21} - P_{11}}{P_{12}} + \frac{P_{21}}{P_{12}} \left( \frac{dP_{21}}{dP} - \frac{dP_{11}}{dP} \right) - \frac{P_{21}(P_{21} - P_{11})}{P_{12}^2} \frac{dP_{12}}{dP} \right] \\ & + \frac{c^2 c_0 U}{U-u} P_{21} \left( \frac{P_{21} - P_{11}}{P_{12}} \right) \end{aligned}$$

$$\begin{aligned} \phi_{23} = & \rho c^2 Q_{204} + \frac{\rho c^2}{U} (Q_{201} Q_{208} + Q_{202} Q_{207}) - \frac{c^2}{\rho} \left\{ Q_{212} \left[ P_{21} - \frac{\rho c_0}{U} \left( \frac{P_{21} - P_{11}}{P_{12}} \right) \right] - Q_{211} \left( \frac{\rho c_0}{U} \frac{P_{22}}{P_{12}} - P_{22} \right) \right\} \\ & - \left( \frac{U-u}{U} \right) \rho (Q_{201} Q_{210} + Q_{202} Q_{207}) - \left( \frac{U-u}{U} \right) c^2 (Q_{201} Q_{212} + Q_{202} Q_{211}) - 2u\rho(U-u)Q_{210} \\ & - 2uc^2(U-u)Q_{212} + 2\rho c^2(U-u)Q_{208} - c_0 u U P_{21} + \frac{c^2 c_0 u U P_{21}}{(U-u)^2} \\ & - \frac{c^2}{U-u} \left\{ Q_{208} \left[ P_{21} - \frac{\rho c_0}{U} \left( \frac{P_{21} - P_{11}}{P_{12}} \right) \right] - Q_{207} \left( \frac{\rho c_0}{U} \frac{P_{22}}{P_{12}} - P_{22} \right) \right\} + \frac{c_0 c^2 U}{U-u} \left[ P_{22} \left( \frac{P_{21} - P_{11}}{P_{12}} \right) + P_{21} \frac{P_{22}}{P_{12}} \right] \\ & + \frac{\rho c_0 c^2 U}{U-u} \left[ \frac{P_{22}}{P_{12}} \left( \frac{dP_{21}}{dP} - \frac{dP_{11}}{dP} \right) - \frac{P_{22}}{P_{12}^2} (P_{21} - P_{11}) \frac{dP_{12}}{dP} + \frac{P_{21}}{P_{12}} \frac{dP_{22}}{dP} - \frac{P_{21} P_{22}}{P_{12}^2} \frac{dP_{12}}{dP} \right] \end{aligned}$$

$$\begin{aligned} \phi_{24} = & \rho c^2 Q_{205} + \frac{\rho c^2}{U} Q_{202} Q_{208} + \frac{c^2}{\rho} Q_{212} \left( \frac{\rho c_0}{U} \frac{P_{22}}{P_{12}} - P_{22} \right) - \frac{U-u}{U} \rho Q_{202} Q_{210} \\ & - \left( \frac{U-u}{U} \right) c^2 Q_{202} Q_{212} - c_0 u U P_{22} + \frac{c^2 c_0 u U}{(U-u)^2} P_{22} + \frac{c^2}{(U-u)} Q_{208} \left( \frac{\rho c_0}{U} \frac{P_{22}}{P_{12}} - P_{22} \right) \\ & + \frac{c_0 c^2 U}{U-u} \frac{P_{22}^2}{P_{12}} + \frac{\rho c_0 c^2 U}{U-u} \left[ \frac{P_{22}}{P_{12}} \frac{dP_{22}}{dP} - \frac{P_{22}^2}{P_{12}^2} \frac{dP_{12}}{dP} \right] \end{aligned}$$

$$\phi_{25} = \rho c_0^2 U - \frac{\rho c_0^2 c^2 U}{(U-u)^2}$$

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$$Q_{201} = -2uU$$

$$Q_{202} = \frac{\rho c_o U}{\rho c^2}$$

$$Q_{203} = P_{21} \left[ \left( 1 + \frac{dP_{21}}{dp} \right) U^2 \frac{du}{dp} + U P_{21} \left( \frac{du}{dp} \frac{dU}{dp} + U \frac{d^2 u}{dp^2} \right) \right] \\ + U Q_{201} + U P_{21} \frac{dQ_{201}}{dp}$$

$$Q_{204} = U^2 \frac{du}{dp} \left( P_{22} \frac{dP_{21}}{dp} + P_{21} \frac{dP_{22}}{dp} \right) + 2 U P_{21} P_{22} \left( \frac{du}{dp} \frac{dU}{dp} + U \frac{d^2 u}{dp^2} \right) \\ + U P_{22} \frac{dQ_{201}}{dp} + U P_{21} \frac{dQ_{202}}{dp}$$

$$Q_{205} = U^2 P_{22} \frac{du}{dp} \frac{dP_{22}}{dp} + U P_{22}^2 \left( \frac{du}{dp} \frac{dU}{dp} + U \frac{d^2 u}{dp^2} \right) + U P_{22} \frac{dQ_{202}}{dp}$$

$$Q_{206} = -P_{22} U^2 \frac{du}{dp} - U Q_{202}$$

$$Q_{207} = \left( \frac{P_{21}}{\rho} - \frac{u}{U} Q_{204} \right) - \frac{\rho c_o}{U \rho} \left( \frac{P_{21} - P_{11}}{P_{12}} \right)$$

$$Q_{208} = -\frac{\rho c_o}{\rho U} \frac{P_{22}}{P_{12}} + \frac{P_{22}}{\rho} - \frac{u}{U} Q_{202}$$

$$Q_{209} = \frac{u U P_{21}}{U-u} \left[ \left( \frac{\partial c^2}{\partial p} \right)_s - \frac{dc^2}{dp} \right] + \left( \frac{P_{21} - P_{11}}{P_{12}} \right) \rho c_o \left( \frac{\partial c^2}{\partial p} \right)_s$$

$$Q_{210} = \frac{P_{22}}{P_{12}} \rho c_o \left( \frac{\partial c^2}{\partial p} \right)_s + \frac{u U P_{22}}{U-u} \left[ \left( \frac{\partial c^2}{\partial p} \right)_s - \frac{dc^2}{dp} \right]$$

$$Q_{211} = \frac{u U P_{21}}{U-u} \left[ \left( \frac{\partial \rho}{\partial p} \right)_s - \frac{d\rho}{dp} \right] + \frac{P_{21} - P_{11}}{P_{12}} \rho c_o \left( \frac{\partial \rho}{\partial p} \right)_s$$

$$Q_{212} = \frac{P_{22}}{P_{12}} \rho c_o \left( \frac{\partial \rho}{\partial p} \right)_s + \frac{u U P_{22}}{U-u} \left[ \left( \frac{\partial \rho}{\partial p} \right)_s - \frac{d\rho}{dp} \right]$$



### III THE FORMATION OF SHOCKWAVES BY EXPLOSIONS

27. The initial conditions for the differential equations derived above can be obtained for the case of explosion phenomena by considering the motion of the interface between the gaseous reaction products and the surrounding medium. The conditions of continuity in pressure and particle velocity yield all the necessary equations for the initial pressure. The use of the characteristic triangle is particularly convenient for determining the initial time factor.

#### Initial Conditions for the Steady State Explosion

28. Let us consider the idealized steady state explosion.\* It is assumed that the explosive is converted to gaseous reaction products in such a way that at the very first moment the pressure is constant throughout the space previously occupied by the explosive, and all particles are at rest. For the very beginning of the expansion of the reaction products at the boundary we have the conditions

$$(5.1) \quad u_1^* = u_1, \quad p_1^* = p_1$$

$$(5.2) \quad u_1^* = -\sigma_1^*$$

The asterisk refers to the reaction products of the explosive, the symbols without asterisk, to the surrounding medium. Equation (5.1) represents the condition of continuity of pressure and velocity at the interface. Equation (5.2) determines the velocity resulting from the expansion of the initially motionless reaction products. Since these (at least in the moments we are considering here) undergo only expansions, which are necessarily isentropic - the Riemann function is used:

$$-\sigma_1^* = \int_{p_1}^{p_2} \frac{dp}{c_p}$$

\*It would be more realistic to assume that the charge is initiated at the center and that a spherical detonation wave spreads through the charge causing the explosion of the particles in concentric shells, one after another. This case can be treated approximately. It will be presented in a later publication.

Equations (5.1) and (5.2), together with the equations of state and the Rankine-Hugoniot equations determine the pressure  $p_1$  and the velocity  $u_1$ . The procedure is well known and will not be repeated here; see for instance [1]. This yields the initial conditions for the differential equations, giving the pressure versus distance, (1.6) or (2.3).

29. Case I. We will now derive an expression for the initial time factor.

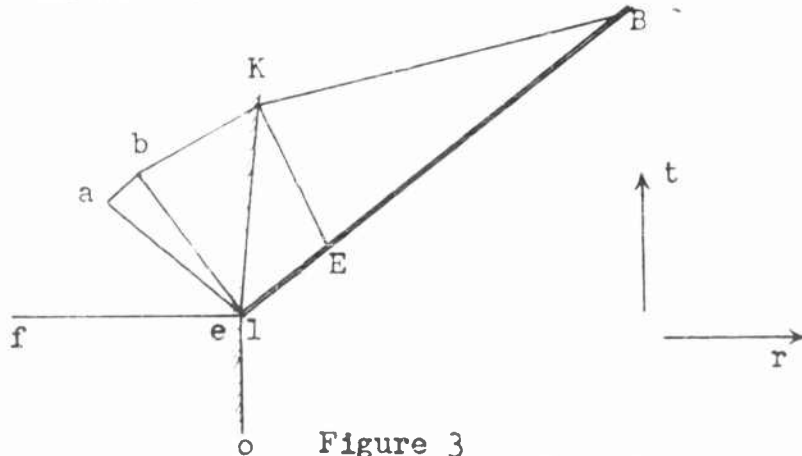


Figure 3  
Characteristic Triangles at the Interface between  
Reaction Products and Medium for the Steady  
State Explosion

In Figure 3, the line  $OelK$  represents the interface between the reaction products (left hand side) and the surrounding medium (right hand side). The regime below  $feO$  is that of the unreacted explosive. Between  $aef$  we have the reaction products at steady state conditions. The expansion takes place in the regime above  $Klea$ . The line  $lEB$  represents the front of the shock which spreads into the surrounding medium. At  $e$  the pressure of the reaction products drops instantaneously from  $p_e$  to  $p_1$ .

30. To summarize, we have

$$(5.3) \quad u_e^* = 0 \quad \sigma_e^* = 0 \quad u_a^* = 0 \quad \sigma_a^* = 0$$

$$(5.4) \quad u_1^* + \sigma_1^* = 0 = u_e^* + \sigma_e^* = u_a^* + \sigma_a^*$$

Following the same lines as above, we obtain

$$\begin{aligned}
 (5.5) \quad & \frac{1}{(\rho c)_{KE}} (p_K - p_E) - (u_K - u_E) = - \left( \frac{2uc}{r} \right)_{KE} (t_K - t_E) \\
 & (\sigma_K^* - \sigma_E^*) + (u_K^* - u_E^*) = - \left( \frac{2u^*c^*}{r} \right)_{KE} (t_K - t_E) \\
 & (\sigma_b^* - \sigma_a^*) + (u_b^* - u_a^*) = - \left( \frac{2u^*c^*}{r} \right)_{ba} (t_b - t_a)
 \end{aligned}$$

With (5.4) and  $u_K^* = u_K$ , we find by adding the quantities in (5.5)

$$\begin{aligned}
 & \left( \frac{1}{\rho c} \right)_{KE} (p_K - p_E) + (\sigma_K^* - \sigma_E^*) + u_E - u_i \\
 & = - \left( \frac{2uc}{r} \right)_{KE} (t_K - t_E) - \left( \frac{2u^*c^*}{r} \right)_{KE} (t_K - t_E) - \left( \frac{2u^*c^*}{r} \right)_{ab} (t_b - t_a)
 \end{aligned}$$

Considering the first order terms only, we obtain

$$\begin{aligned}
 (5.6) \quad & \left( \frac{1}{\rho c} + \frac{1}{\rho^* c^*} \right) p_K^{(u)} (t_K - t_i) + \left( \frac{du}{dp} - \frac{1}{\rho c} \right) \frac{dp}{dt} (t_E - t_i) \\
 & = - \frac{2uc}{r} (t_K - t_E) - \frac{2u^*c^*}{r} \left( t_K - t_b + \frac{t_b - t_a}{2} \right)
 \end{aligned}$$

All magnitudes without subscripts refer to the thermodynamic state at the point 1.

30. The differences in  $t$  can be expressed as follows:

$$\begin{aligned}
 t_E - t_i &= (t_K - t_i) \frac{c}{c + U - u} \\
 t_K - t_E &= (t_K - t_i) \frac{U - u}{c + U - u} \\
 t_K - t_b &= \frac{1}{2} (t_K - t_i) \\
 t_b - t_a &= t_K - t_i \frac{u^* - c^* - c_e^*}{3c_e^* + u^* + c^*}
 \end{aligned}$$

so that (5.6) becomes

$$(5.7) \quad \left( \frac{1}{\rho c} + \frac{1}{\rho^* c^*} \right) p_t^{(u)} + \left( \frac{du}{dp} - \frac{1}{\rho c} \right) \left( \frac{c}{c+U-u} \right) \frac{dp}{dt} + \frac{2uc}{r} \left( \frac{U-u}{c+U-u} \right) + \frac{2u^*c^*}{r} \left[ \frac{1}{2} + \frac{u^*c^* - c^2}{2(3c_e^2 + u^* + c^2)} \right] = 0$$

$p_t^{(u)}$  can be eliminated by means of

$$p_t^{(u)} = u \frac{dp}{dr} + \frac{U-u}{U} p_t$$

Further substitution for  $dp/dr$  and  $p_t$  yields

$$(5.8) \quad - \left( \frac{1}{\rho c} + \frac{1}{\rho^* c^*} \right) \left( \frac{U-u}{U} \right) \frac{p c_0}{a_0} \alpha_i - \frac{P_{11}}{r_0} \left[ u \left( \frac{1}{\rho c} + \frac{1}{\rho^* c^*} \right) + \frac{cU}{c+U-u} \left( \frac{du}{dp} - \frac{1}{\rho c} \right) \right] - \frac{P_{12} \alpha_i}{a_0} \left[ u \left( \frac{1}{\rho c} + \frac{1}{\rho^* c^*} \right) + \frac{cU}{c+U-u} \left( \frac{du}{dp} - \frac{1}{\rho c} \right) \right] + \frac{2uc}{r_0} \left( \frac{U-u}{c+U-u} \right) + \frac{2u^*c^*}{r_0} \left( \frac{2c_e^2 + u^*}{3c_e^2 + u^* + c^2} \right) = 0$$

where  $\alpha_i$  refers to the initial state of the surrounding medium at  $r_0 = a_0$ . Solving (5.8) for  $\alpha_i$ , we obtain

$$(5.9) \quad \alpha_i = G_{11} \left[ \frac{G_{12} + G_{12}^* + \frac{1}{\rho^* c^*} G_{13}}{G_{14} + \frac{1}{\rho c} + \frac{1}{\rho^* c^*}} \right]$$

where

$$\begin{aligned} G_{11} &= \frac{U}{p c_0 (U-u) + u U P_{12}} \\ G_{12} &= \frac{2uc(U-u)}{c+U-u} - \frac{P_{12}u}{\rho c} - \frac{cU P_{11}}{c+U-u} \left( \frac{du}{dp} - \frac{1}{\rho c} \right) \\ G_{12}^* &= \frac{2u^*c^*(2c_e^2 + u^*)}{3c_e^2 + u^* + c^2} \\ G_{13} &= -P_{12}u \\ G_{14} &= \frac{cU P_{12}}{c+U-u} \left( \frac{du}{dp} - \frac{1}{\rho c} \right) \left( \frac{U}{p c_0 (U-u) + u U P_{12}} \right) \end{aligned}$$

G-functions without an asterisk refer to the surrounding medium and depend only on the pressure  $p$ .  $G_{12}^*$  refers to the reaction products of the explosion.

31. It is of interest to note that if we had set up equations (5.5) using the characteristic KB instead of KE, the result would have been the same.

32. Case II. Retaining  $p_t^{(u)}$  in (5.7), we obtain

$$(6.1) \quad -\alpha_i^{(u)} \left[ \left( \frac{1}{\rho c} + \frac{1}{\rho^2 c^2} \right) \frac{p c_0}{a_0} + \frac{UP_{22}}{a_0} \left( \frac{du}{dp} - \frac{1}{\rho c} \right) \left( \frac{c}{c+U-u} \right) \right] - \frac{UP_{21}}{r_0} \left( \frac{du}{dp} - \frac{1}{\rho c} \right) \left( \frac{c}{c+U-u} \right) + \frac{2uc(U-u)}{r_0(c+U-u)} + \frac{2u^2 c^2}{r_0} \left[ \frac{1}{2} + \frac{u^2 - c^2 - c c^2}{2(3c c^2 + u^2 + c^2)} \right] = 0$$

Solving for  $\alpha_i^{(u)}$ , yields

$$(6.2) \quad \alpha_i^{(u)} = \frac{1}{r_0} \left\{ \frac{\frac{2uc(U-u)}{c+U-u} - UP_{21} \left( \frac{du}{dp} - \frac{1}{\rho c} \right) \left( \frac{c}{c+U-u} \right) + 2u^2 c^2 \left( \frac{2c c^2 + u^2}{3c c^2 + u^2 + c^2} \right)}{\left( \frac{1}{\rho c} + \frac{1}{\rho^2 c^2} \right) \frac{p c_0}{a_0} + \frac{UP_{22}}{a_0} \left( \frac{du}{dp} - \frac{1}{\rho c} \right) \left( \frac{c}{c+U-u} \right)} \right\}$$

$$(6.3) \quad \alpha_i^{(u)} = G_{21} \left[ \frac{G_{22} + G_{22}^*}{G_{22} + \frac{1}{\rho_0} + \frac{1}{\rho^2 c^2}} \right]$$

where

$$G_{21} = \frac{1}{p c_0}$$

$$G_{22} = \frac{2uc(U-u)}{c+U-u} - UP_{21} \left( \frac{du}{dp} - \frac{1}{\rho c} \right) \left( \frac{c}{c+U-u} \right)$$

$$G_{22}^* = \frac{2u^2 c^2 (2c c^2 + u^2)}{3c c^2 + u^2 + c^2}$$

$$G_{24} = UP_{22} \left( \frac{du}{dp} - \frac{1}{\rho c} \right) \left( \frac{c}{c+U-u} \right) \left( \frac{1}{p c_0} \right)$$

# IV PLANE AND CYLINDRICAL WAVES

33. It is of interest to know how the functions of the foregoing sections would look in the case of plane waves or for cylindrical waves. Once the spherical case has been worked out, it requires only very simple transformations to obtain the equations for the two cases mentioned above. All of the calculations start with equations (0.5) and (0.6). In the plane wave case, the term  $-2cu/\rho$  is zero; as a result no  $1/r$  term will appear. For the cylindrical wave, the divergence term is  $-cu/\rho$ .

Cylindrical wave:

$$(1.6a) \quad \frac{dp}{dx} + \frac{P_{11}}{2r} + \frac{P_{12}}{a_0} \alpha = 0$$

$$(2.3a) \quad \frac{dp}{dx} + \frac{P_{21}}{2r} + \frac{P_{22}}{a_0} \alpha^{(u)} = 0$$

$$(3.7a) \quad \frac{d\alpha}{dx} + \frac{P_{13}}{4r^2} + \frac{P_{14}}{2r} \alpha^{(u)} + \frac{P_{15}}{a_0^2} \alpha^2 \left(1 + \frac{\alpha_+}{\alpha^2}\right) = 0$$

$$(4.1a) \quad \frac{d\alpha^{(u)}}{dx} + \frac{P_{23}}{4r^2} + \frac{P_{24}}{2r} \alpha^{(u)} + \frac{P_{25}}{a_0^2} \alpha^{(u)2} \left(1 + \frac{\alpha_+^{(u)}}{\alpha^{(u)2}}\right) = 0$$

$$(5.9a) \quad \alpha_1 = \frac{G_{11}}{2} \left[ \frac{G_{12} + G_{12}^* + \frac{1}{\rho^2 c^2} G_{13}}{\frac{1}{\rho c} + \frac{1}{\rho^2 c^2} + G_{14}} \right]$$

$$(6.3a) \quad \alpha_1^{(u)} = \frac{G_{21}}{2} \left[ \frac{G_{22} + G_{22}^*}{\frac{1}{\rho c} + \frac{1}{\rho^2 c^2} + G_{24}} \right]$$

Plane wave:

$$(1.6b) \quad \frac{dp}{dx} + P_{12} \alpha = 0$$

$$(2.3b) \quad \frac{dp}{dx} + P_{22} \alpha^{(u)} = 0$$

$$(3.7b) \quad \frac{d\alpha}{dx} + P_{15} \alpha^2 \left(1 + P_{16} \frac{\alpha_+}{\alpha^2}\right) = 0$$

$$(4.1b) \quad \frac{d\alpha^{(u)}}{dx} + P_{25} \alpha^{(u)2} \left(1 + P_{26} \frac{\alpha_+^{(u)}}{\alpha^{(u)2}}\right) = 0$$

It is interesting to note that the functions  $P_{1...}$  and  $P_{2...}$  are the same for the spherical, cylindrical and plane wave. Hence, if they are computed for a particular medium they may be applied to any one of those types of waves.

In a plane wave  $Q_1$  is zero. That means a plane steady state explosion produces a step wave, whose pressure is constant with distance. This holds only fairly close to the charge. Later the rarefaction of the explosion products becomes effective. A rarefaction wave follows this step wave and finally overtakes the front thus causing the pressure to drop. These phenomena are well known from the theory of the shock tube.

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