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NEW YORK UNIVERSITY

College of Engineering
RESEARCH DIVISION
University Heights, New York 53, N. Y.
Department of Mathematics

ON THE JOINT ESTIMATION OF THE SPECTRA, COSPECTRUM AND QUADRATURE SPECTRUM OF A TWO-DIMENSIONAL STATIONARY GAUSSIAN PROCESS

by

N.R. Goodman

Scientific Paper No. 10
Engineering Statistics Laboratory

Prepared for BuShips
Contract Nobs-72018 (1734 F)
And
David Taylor Model Basin
Contract Nonr-285 (17)

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Abstract

The probability structure of a real two-dimensional stationary (zero mean) Gaussian vector process $[x(t), y(t)]$, $-\infty < t < \infty$, is specified (in the absolutely continuous case) by $f_{xx}(\lambda)$, $f_{yy}(\lambda)$ the spectral densities of the $x(t)$ and $y(t)$ processes respectively, $c(\lambda)$ the cospectral density, and $q(\lambda)$ the quadrature spectral density $(-\infty < \lambda < \infty)$. The dissertation treats the problem of jointly estimating (in a suitable sense) $f_{xx}(\lambda)$, $f_{yy}(\lambda)$, $c(\lambda)$, $q(\lambda)$ from a finite part of a sample function of the $[x(t), y(t)]$, $-\infty < t < \infty$, process. An approximation to the joint sampling distribution of the estimators for $f_{xx}(\lambda)$, $f_{yy}(\lambda)$, $c(\lambda)$, $q(\lambda)$ is obtained. This approximate sampling distribution termed a Complex Wishart distribution serves as the starting point in the derivation of approximate sampling distributions of estimators for functions of $f_{xx}(\lambda)$, $f_{yy}(\lambda)$, $c(\lambda)$, $q(\lambda)$.

The dissertation was motivated by the need of experimenters in fields such as micrometeorology, oceanography, electrical engineering, and aeronautical engineering to statistically estimate "parameters" characterizing their particular physical systems and to treat the sampling variability of estimators for the "parameters". In a number of cases the "parameters" to be estimated are functions of the densities $f_{xx}(\lambda)$, $f_{yy}(\lambda)$, $c(\lambda)$, $q(\lambda)$ of a real two-dimensional stationary (zero mean) Gaussian vector process.

Chapter 1

Statement of Problem and Results

1.0 Introduction

This chapter states the estimation problem studied, the motivation for the problem, and the results achieved.

1.1 General Statement of Problem and Results

The probability structure of a real two-dimensional stationary (zero mean) Gaussian vector process $[x(t), y(t)]$ $-\infty < t < \infty$, is specified (in the absolutely continuous case) by the spectral density matrix

$$(1.1) \quad f(\lambda) = \begin{bmatrix} f_{xx}(\lambda) & f_{xy}(\lambda) \\ f_{yx}(\lambda) & f_{yy}(\lambda) \end{bmatrix} \quad (-\infty < \lambda < \infty)$$

where

$$f(\lambda) \text{ continuous, } f_{xx}(-\lambda) = f_{xx}(\lambda),\ f_{yy}(-\lambda) = f_{yy}(\lambda),$$

$$f_{xy}(\lambda) = \overline{f_{xy}(-\lambda)},\ f_{yx}(\lambda) = \overline{f_{xy}(\lambda)}, \text{ and}$$

$$f_{xx}(\lambda) \geq 0,\ f_{yy}(\lambda) \geq 0,\ f_{xx}(\lambda)f_{yy}(\lambda) - |f_{xy}(\lambda)|^2 \geq 0.$$

In (1.1)

$f_{xx}(\lambda)$ = spectral density of $x(t)$ process,

$f_{yy}(\lambda)$ = spectral density of $y(t)$ process,

$f_{xy}(\lambda) = c(\lambda) + iq(\lambda)$ = cross spectral density of the $x(t)$ process with the $y(t)$ process.

$c(\lambda)$ = cospectral density = Real part $f_{xy}(\lambda)$,

$q(\lambda)$ = quadrature spectral density

= Imaginary part $f_{xy}(\lambda)$.

The dissertation treats the problem of jointly estimating $f_{xx}(\lambda)$, $f_{yy}(\lambda)$, $c(\lambda)$, $q(\lambda)$ from a finite part of a sample function of the $[x(t), y(t)]$ process. To be more precise, what is estimated is not the "spot" densities $f_{xx}(\lambda)$, $f_{yy}(\lambda)$, $c(\lambda)$, $q(\lambda)$ but weighted integrals of these densities where the weights are such as to concentrate most of their "mass" in narrow frequency bands. Emphasis is placed on the study of the sampling variability of the estimators. An approximation to the joint sampling distribution of the estimators for $f_{xx}(\lambda)$, $f_{yy}(\lambda)$, $c(\lambda)$, $q(\lambda)$ is obtained. This approximate sampling distribution is termed a Complex Wishart distribution. Certain distributions derived from the Complex Wishart distribution are approximate sampling distributions of estimators for functions of $f_{xx}(\lambda)$, $f_{yy}(\lambda)$, $c(\lambda)$, $q(\lambda)$ such as

$$(1.2)\quad \frac{c^2+q^2}{f_{xx}f_{yy}},\ \operatorname{Arg}(c+iq),\ \frac{\sqrt{c^2+q^2}}{f_{xx}},\ \frac{c}{f_{xx}},\ \frac{q}{f_{xx}},$$

and such distributions are extensively discussed.

1.2 Motivation and Orientation

The dissertation was motivated by the need of experimenters in fields such as micrometeorology, oceanography, electrical engineering, and aeronautical engineering to statistically estimate "parameters" characterizing their particular physical systems and to treat the sampling variability of the estimators for the "parameters". In a number of cases the "parameters" to be estimated can be regarded as functions of the densities $f_{xx}(\lambda)$, $f_{yy}(\lambda)$, $c(\lambda)$, $q(\lambda)$ of a two-dimensional stationary (zero mean) Gaussian vector process. Examples of such functions of physical significance are given by (1.2). To illustrate this the following (somewhat artifical and idealized) example from the theory of gravity waves is given to show how a statistical estimate of the function $\mathrm{Arg}_{\psi}(c+iq)$ of a suitably defined process could serve as a basis for determining the physical constant g.

Consider the small oscillations of the free surface of an incompressible infinitely deep fluid in the case when the motion is such that the free surface at any instant of time has parallel straight ridges and furrows. A simple harmonic progressive wave of frequency ω is then a wave such that the height of the free surface at position L and time t is given by

$$(1.3) \quad \eta_{\omega}(L,t) = a \sin(\omega t - \frac{\omega^2 L}{g} + \phi_{\omega}).$$

The wave $\eta_{\omega}(L,t)$ progresses in the positive L direction. For a superposition of such progressive waves the free surface is given by

$$(1.4) \quad \eta(L,t) = \int_0^\infty \sin(\omega t - \frac{\omega^2 L}{g} + \phi_\omega) dS(\omega).$$

For a random (independent phases) superposition of such progressive waves the free surface is given by

$$(1.5) \quad \eta(L,t) = \int_0^\infty [\cos(\omega t - \frac{\omega^2 L}{g}) dU(\omega) + \sin(\omega t - \frac{\omega^2 L}{g}) dV(\omega)],$$

where

$U(\omega)$ and $V(\omega)$ are real Gaussian processes,

$E\, dU(\omega) = E\, dV(\omega) = 0,$

$E\, dU(\omega) dV(\omega') = 0,$

$E\, dU(\omega) dU(\omega') = 0 = E\, dV(\omega) dV(\omega')$ if

$E\, [dU(\omega)]^2 = E[dV(\omega)]^2 = s(\omega) d\omega.$

The representation (1.5) serves as a mathematical model for what is termed in oceanography a Gaussian infinitely long crested confused sea. Consider now the two-dimensional vector process $[x^*(t), y^*(t)]$ obtained by measuring the height of the free surface at two points which are a distance L apart. If there are no errors of measurement, $[x^*(t), y^*(t)]$ is a two-dimensional stationary (zero mean) Gaussian process, and the spectral density matrix of this process is

$$(1.6) \qquad f(\omega) = \begin{bmatrix} s(\omega) & e^{\frac{-i\omega^2 L}{g}} s(\omega) \\ e^{\frac{i\omega^2 L}{g}} s(\omega) & s(\omega) \end{bmatrix}.$$

Thus

$$(1.7)\qquad \begin{aligned} f_{xx}(\omega) &= s(\omega) \\ f_{yy}(\omega) &= s(\omega) \\ c(\omega) + iq(\omega) &= e^{\frac{-i\omega^2 L}{g}} s(\omega) \end{aligned}$$

If, however, there are errors $e_x(t)$ and $e_y(t)$ in measuring the heights $x(t)$ and $y(t)$ respectively, and these errors are sample functions of stationary (zero mean) Gaussian processes incoherent with each other and with the random heights $x(t)$, $y(t)$, then the spectral density matrix of the process $[x^*(t), y^*(t)]$ is

$$(1.8)\qquad f^*(\omega) = \begin{bmatrix} s(\omega) + s_{ex}(\omega) & e^{\frac{-i\omega^2 L}{g}} s(\omega) \\ e^{\frac{i\omega^2 L}{g}} s(\omega) & s(\omega) + s_{ey}(\omega) \end{bmatrix}$$

From (1.8)

$$(1.9)\qquad \begin{aligned} f^*_{xx}(\omega) &= s(\omega) + s_{ex}(\omega), \\ f^*_{yy}(\omega) &= s(\omega) + s_{ey}(\omega), \\ c^*(\omega) + iq^*(\omega) &= e^{\frac{-i\omega^2 L}{g}} s(\omega) = s(\omega)\cos\frac{\omega^2 L}{g} - is(\omega)\sin\frac{\omega^2 L}{g} . \end{aligned}$$

Thus,

$$(1.10)\qquad \operatorname{Arg}\,[c^*(\omega) + iq^*(\omega)] = -\frac{\omega^2 L}{g} ,$$

and the physical constant g is given by

$$(1.11) \qquad g = \frac{-\omega^2 L}{\operatorname{Arg}[c^*(\omega) + iq^*(\omega)]} \, .$$

If $\operatorname{Arg}[c^*(\omega) + iq^*(\omega)]$ is estimated from a finite length of measured record $[x^*(t), y^*(t)]$, $0 \leq t \leq T$, an estimate of g can be obtained by substituting this estimate for $\operatorname{Arg}[c^*(\omega) + iq^*(\omega)]$ in (1.11). From (1.7) and (1.9) it is seen that $c^*(\omega) + iq^*(\omega) = c(\omega) + iq(\omega)$, so that the errors $e_x(t)$, $e_y(t)$ do not affect the theoretical cross-spectrum. The errors do not however affect the variability of estimators for the cross-spectrum. In general, as $s_{ex}(\omega)$, $s_{ey}(\omega)$ increase, the length of record required to estimate the cross-spectrum (with fixed confidence) also increases. The dissertation provides the machinery for quantitatively studying how the variability of the estimators for the spectra and cross-spectrum and, what is of particular interest in the situation just treated, how the variability of the estimators for $\operatorname{Arg}[c^*(\omega) + iq^*(\omega)]$ vary as the length of sample record increases.

Spectral densities are essentially variances and cospectral and quadrature spectral densities are essentially covariances. One can thus rightly think of the dissertation as dealing with a special topic in the theory of "measuring" variances and covariances, and with relation of such "measurements" to parameters of physical significance.

The idea that physical parameters can be determined by "measuring" variances or mean squares is not new. One obtains the correct orientation by considering the following classical example from the kinetic theory of gases where the molecular origin of Brownian motion was confirmed by the quantitative agreement of the measured displacements of Brownian motion particles in a given time with the predicted value of such displacement based on the theory of Einstein and von Smoluchowski. According to the theory of Einstein and von Smoluchowski

$$\text{(1.12)} \qquad \text{mean } X^2(t) = \frac{R_A T t}{3\pi N_A \eta a}$$

where

$X(t)$ = the x component of displacement of a Brownian motion particle in time t, and

R_A = the gas constant,

T = the absolute temperature,

N_A = the Avogadro number,

η = the coefficient of viscosity of the liquid,

a = the radius of the particle.

J. Perrin made measurements of the x-component of displacement of a particle in time t and computed an average $X^2(t)$ from these measurements which was in agreement with the value given by (1.12) with the constants N_A, R_A, T, a, η

determined from other experiments. Such measurements also served as a basis for determining the Avogadro number N_A when the constants R_A, T, a, η were known from other sources.

1.3 Comments on the Contents of Chapters 2-4

Chapter 2 contains an exposition of the probability theory on which the estimation problem is based. The principal references for this material are [2], [3], [15], [16].

In Chapter 3 estimators for the spectral, cospectral, and quadrature spectral densities are specified and studied. The material of this chapter extends the results of Tukey [17] to the two-dimensional case and uses many of the results and ideas of Tukey [17].

The material on the Complex-Wishart distribution and related distributions which comprises Chapter 4 is believed new. The Complex-Wishart distribution and related distributions are the distributions of the sample second moments of complex Gaussian random variables and the distributions of certain functions of these sample moments respectively. The distributions are used to study the sampling variability of estimators for the spectral, cospectral, and quadrature spectral densities, and the sampling variability of the estimators for functions of these densities. Other applications of these distributions may conceivably arise, and for this reason Chapter 4 is so written that it may be read without referring to the other chapters.

The first section of Chapter 5 contains a discussion on how the Complex-Wishart Distribution and related distributions are to be used to study the sampling variability of estimators for the spectral, cospectral, and quadrature spectral densities and functions of these densities. The second section indicates how by using the estimation theory developed earlier the frequency response function of a linear time invariant system can be statistically estimated.

Chapter 2

Probability Background

2.0 Introduction

The probability generalities and details which constitute the background essential for an understanding of the estimation problem treated in Chapter 3 are outlined in this chapter. For the most part, proofs are omitted.

2.1 Stochastic Processes

A one-dimensional random variable $X=X(\omega)$ is defined as a real-valued measurable function on a space $\Omega, \omega \varepsilon \Omega$ on which a probability measure P is defined. The space Ω is called the sample space and $\omega \varepsilon \Omega$ is called a sample point. At times it is convenient to consider complex-valued random variables $Z(\omega)$, and such random variables are defined as $Z(\omega)=X(\omega)+iY(\omega)$ where $X(\omega)$ and $Y(\omega)$ are real-valued measurable functions. A real k-dimensional random variable is defined as a k-tuple of real one-dimensional random variables, and similarly a complex k-dimensional random variable is defined as a k-tuple of complex one-dimensional random variables. The notion random variable now includes any of the specific types of random variables described above.

A *stochastic process* $X_t = X_t(\omega), t \in T$ is an indexed collection of random variables defined on the space Ω. (As is customary with random variables explicit reference to the sample space Ω is usually not made). If the random variables X_t are real (complex) the process is said to be real (complex); if the random variables X_t are k-dimensional, the process is said to be k-dimensional. A k-dimensional process $(k \geq 2)$ is referred to as a *vector process*. In this paper consideration will be limited to primarily one and two-dimensional processes. The index set T is usually assumed to be infinite. The index set may, for example, be the continuous real line $[t; -\infty < t < \infty]$. In such a case the process is said to be a *continuous parameter process.* The index set may on the other hand be a discrete set of points such as the set $[k\Delta t; k=\ldots,-1,0,1,2,\ldots]$. In such a case the process is said to be a *discrete parameter process*. The index set T may be quite general and in physically relevant situations index sets T consisting of n-dimensional vector spaces (usually for n=1,2,3,4) have frequently been employed. For the purpose of the present paper it will suffice to restrict T to be the real line $[t; -\infty < t < \infty]$ or the equispaced discrete subset $[k\Delta t; k=\ldots,-1,0,1,\ldots]$ of the real line. The parameter t in many physical contexts represents time and it has become customary to refer to stochastic processes indexed by time as *Time Series*.

2.2 Stationary Processes

Consider a k-dimensional complex-valued process $X_t = X_t(\omega), t \in T$, where T now denotes either the continuous group $[t; -\infty < t < \infty]$ or the discrete group $[k\Delta t; k=\ldots,-1,0,1,\ldots]$ where in both cases addition is the group operation. Let A denote any measurable set $A \subseteq \Omega$ defined by a constraint on the random variables $X_t, t \in T$, i.e. $A = C[X_t, t \in T]$. Let A_τ denote the set $C[X_{t+\tau}, t \in T]$ where $\tau \in T$. The process $X_t, t \in T$ is said to be strictly stationary or strongly stationary if $P(A) = P(A_\tau)$ for all $\tau \in T$ and all measurable sets $A \subseteq \Omega$. The meaning of the last statement is that the probability structure of a continuous (discrete) strictly stationary process is invariant under the group of continuous (discrete) translations.

Assume furthermore that the process $X_t, t \in T$ is such that the mean values

$$(2.1) \quad E\, X_t = \mu_t$$

and covariances

$$(2.2) \quad E\, (X_s - \mu_s)' (\overline{X_t - \mu_t}) = \text{Cov}\, (X_s, X_t) = R_{s,t}$$

exist and are finite for every $s, t \in T$. Here X_t (for fixed t) denotes a k-dimensional row vector of complex random variables and $\overline{X}_t'$ denotes the conjugate transpose of X_t. The function $R_{s,t}$ is $(k \times k)$ matrix valued and is called the autocovariance function of the process. The process $X_t, t \in T$ is called weakly stationary (or stationary in the wide sense) if

$$(2.3) \qquad \mu_t = \mu$$

and

$$(2.4) \qquad R_{s,t} = R_{s-t} \; .$$

It is clear that a strongly stationary process possessing second moments is weakly stationary.

2.3 Gaussian Processes

It will suffice to here restrict $X_t, t \in T$ to be a real stochastic process. A function X_t obtained by fixing ω in $X_t(\omega)$ and letting t vary is called a sample function or realization of the process. (If T is a discrete set, the sample functions are sample sequences). One can consider a stochastic process as a space of functions $[X_t]$ (the sample functions) on which a probability measure P is defined. Let $t_1, t_2, \ldots t_N$ be any finite set of parameter values t. The multivariate distribution of the random variables $X_{t_1}, X_{t_2}, \ldots, X_{t_N}$ i.e. Prob $[X_{t_1} \leq x_1, X_{t_2} \leq x_2, \ldots, X_{t_N} \leq x_N]$ is termed a finite dimensional distribution of the process and is denoted by $P_N(x_1, x_2, \ldots, x_N; t_1, t_2, \ldots, t_N)$. The probability measure P

on $[X_t]$ can be introduced by prescribing a mutually consistent collection of finite dimensional distributions.*
The consistency conditions called the consistency conditions of Kolmogorov are:

(2.5.1) every F_k is symmetric in all pairs (x_ν, t_ν), and

(2.5.2) $F_k(x_1,\ldots,x_j,\infty,\ldots,\infty;t_1,\ldots,t_k)$
$$= F_j(x_1,\ldots,x_j;t_1,\ldots,t_j) \quad \text{for} \quad j < k.$$

The finite dimensional distributions are thus basic distributions of a process and processes are frequently classified according to their finite dimensional distributions. If a process is strongly stationary

(2.6) $F_k(x_1,x_2,\ldots,x_k;t_1,t_2,\ldots,t_k)$
$$= F_k(x_1,x_2,\ldots,x_k;t_1+\tau,t_2+\tau,\ldots,t_k+\tau)$$
for every τ, $k = 1,2,3,\ldots$.

A stochastic process is called Gaussian if the joint distribution of every finite set of X_t's is multivariate Gaussian, that is, for every finite set of $t_1,t_2,\ldots,t_N \in T$ the joint distribution of $X_{t_1},X_{t_2},\ldots,X_{t_N}$ has the characteristic function

* See Chapters I,II of [3] for the technical details of constructing the probability measure P.

(2-7) $\phi(\vec{r}_{t_m}) = E \exp[i \vec{r}_{t_m} \vec{X}_{t_m}']$

$$= \exp[-\tfrac{1}{2} \vec{r}_{t_m} \| R(t_m,t_n) \| \vec{r}_{t_m}' + i \vec{r}_{t_m} \vec{\mu}'(t_m)]$$

where

(1) in the one-dimensional case

$$\vec{X}_{t_m} = [X_{t_1},\ldots\ldots,X_{t_N}] ,$$

$$E\, X_t = \mu(t), \qquad \vec{\mu}(t_m) = [\mu(t_1),\mu(t_2),\ldots,\mu(t_N)] ,$$

$$\vec{r}_{t_m} = [r_{t_1},r_{t_2},\ldots\ldots,r_{t_N}]$$

and $\qquad R(t_m,t_n) = \mathrm{Cov}(X_{t_m},X_{t_n})$;

and (2) in the two-dimensional case

$$X_t = [x_t,\ y_t], \qquad E\, X_t = [\mu_x(t),\ \mu_y(t)] ,$$

$$\vec{X}_{t_m} = [x_{t_1},y_{t_1},x_{t_2},y_{t_2},\ldots\ldots,x_{t_N},y_{t_N}] ,$$

$$\vec{\mu}(t_m) = [\mu_x(t_1),\mu_y(t_1),\ldots\ldots,\mu_x(t_N),\mu_y(t_N)] ,$$

$$\vec{r}_{t_m} = [r_{xt_1},r_{yt_1},\ldots\ldots,r_{xt_N},r_{yt_N}] ,$$

and

$$R(t_m,t_n) = \mathrm{Cov}(X_{t_m},X_{t_n}) = \begin{bmatrix} R_{xx}(t_m,t_n) & R_{xy}(t_m,t_n) \\ & \\ R_{yx}(t_m,t_n) & R_{yy}(t_m,t_n) \end{bmatrix} .$$

In either case the matrix $\|R(t_m,t_n)\|$ is symmetric and non-negative definite. If the matrix $\|R(t_m,t_n)\|$ is non-singular, the joint distribution of $X_{t_1}-\mu(t_1),\ldots,X_{t_N}-\mu(t_N)$ has the probability density function

$$(2.8) \qquad \frac{|R(t_m,t_n)|^{-\frac{1}{2}}}{(2\pi)^{\frac{1}{2}kN}} \exp\left[-\tfrac{1}{2}\,\vec{u}_{t_m} \,\|R(t_m,t_n)\|^{-1}\, \vec{u}_{t_m}'\right]$$

where $|R(t_m,t_n)|$ denotes the determinant of $\|R(t_m,t_n)\|$ and

(1) in the one-dimensional case $(k=1)$

$$\vec{u}_{t_m} = (u_{t_1},u_{t_2},\ldots,u_{t_N})$$

and (2) in the two-dimensional case $(k=2)$

$$\vec{u}_{t_m} = (u_{xt_1},u_{yt_1},u_{xt_2},u_{yt_2},\ldots,u_{xt_N},u_{yt_N}) .$$

2.4 Weakly Stationary Processes

2.4.1 The continuous parameter real one-dimensional process.

The process $X_t, t \in T$ is assumed to have mean $\mu = 0$. The covariance function $R_\tau = E\,X_{t+\tau}X_t$ is assumed continuous at $\tau = 0$, so that it is then bounded and everywhere continuous. The function R_τ is even and positive definite and hence the Fourier cosine transform of a real bounded monotone non-decreasing function $F(\lambda)$ called the spectral function of the process.

$$(2.9) \quad R_\tau = \int_0^\infty \cos \tau\lambda \, dF(\lambda)$$

The spectral function $F(\lambda)$ car. . expressed as the sum of three monotone non-decreasing functions

$$(2.10) \quad F(\lambda) = F_{(a)}(\lambda) + F_{(d)}(\lambda) + F_{(s)}(\lambda)$$

Here

(a) $F_a(\lambda)$ is the absolutely continuous component;

$$F_{(a)}(\lambda) = \int_0^\lambda f(x)dx \quad \text{where} \quad f(x) \geq 0 .$$

(d) $F_d(\lambda)$ is the discontinuous component;

$$F_{(d)}(\lambda) = \sum_{\lambda_r \leq \lambda} p_r$$

where λ_r denotes the, at most, denumerable number of discontinuities of $F(\lambda)$ and p_r denotes the saltus of $F(\lambda)$ at $\lambda = \lambda_r$.

(s) $F_s(\lambda)$ is the singular component;

$F_{(s)}(\lambda)$ is everywhere continuous and has a derivative equal to zero almost everywhere.

The case of physical interest is that in which $F(\lambda)$ is absolutely continuous with a continuous derivative. Unless otherwise specified it is assumed that

$$F(\lambda) = \int_0^\lambda f(x)dx \quad \text{where} \quad f \geq 0 \text{ and continuous.}$$

The function $f(\lambda)$ is called the spectral density of the process. From (2.9),

$$E X_t^2 = R(0) = \int_0^\infty f(\lambda)d\lambda .$$

The process $X_t, t \in T$ can be expressed in the canonical form (spectral representation of the process)

$$(2.11) \qquad X_t = \int_0^\infty \cos t\lambda dU(\lambda) + \int_0^\infty \sin t\lambda dV(\lambda)$$

where $U(\lambda)$ and $V(\lambda)$ are real processes with orthogonal increments and the processes $U(\lambda)$ and $V(\lambda)$ are orthogonal to each other; that is,

$$(2.12) \qquad \begin{aligned} &E\, dU(\lambda)dU(\lambda') = 0 = E\, dV(\lambda)dV(\lambda') \text{ if } \lambda \neq \lambda', \\ &E\, dU(\lambda)dV(\lambda') = 0 \quad \text{for all } \lambda, \lambda'. \\ &E\, dU(\lambda) = E\, dV(\lambda) = 0 \end{aligned}$$

Furthermore,

$$(2.13) \qquad E\,[dU(\lambda)]^2 = E\,[dV(\lambda)]^2 = dF(\lambda) = f(\lambda)d\lambda.$$

The integrals appearing in (2.11) are stochastic integrals. An integral such as

$$(2.14) \qquad I = \int_a^b g(\lambda)dU(\lambda) \qquad (g(\lambda) \text{ real and continuous})$$

is introduced as follows:

Consider a sequence of random variables $I_1, I_2, \ldots\ldots$ defined by

$$(2.15) \qquad I_n = \sum_{\nu=1}^{n} g(\lambda_\nu^{(n)})[U(\lambda_\nu^{(n)}) - U(\lambda_{\nu-1}^{(n)})]$$

where $a = X_0^{(n)} < t_1^{(n)} < \ldots < t_{n-1}^{(n)} < t_n^{(n)} = b$. Suppose that

$$(2.16) \quad \lim_{n \to \infty} \; \max_{\nu=1,2,\ldots,n} \; (t_\nu^{(n)} - t_{\nu-1}^{(n)}) = 0 \;.$$

It then follows (see [2]) that I_n converges (l.i.m.)* to a random variable I. Furthermore, if I_n^* is another sequence formed with the same $U(\lambda)$ and $g(\lambda)$ but with another system of points $t_\nu^{(n)}$ satisfying (2.16) and converging (l.i.m.) to the limit I^*, then I^* and I are equivalent. Thus

$$(2.17) \quad I = \int_a^b g(\lambda)\,dU(\lambda) = \underset{n \to \infty}{\text{l.i.m.}}\; I_n \;.$$

From the orthogonality-increment property (2.12) and (2.13) of $U(\lambda)$,

$$(2.18) \quad E\, I_n^2 = \sum_{\nu=0}^{n} g^2(\lambda_\nu^{(n)}) \, [F(\lambda_\nu^{(n)}) - F(\lambda_{\nu-1}^{(n)})]$$

so that

$$(2.19) \quad E\, I^2 = \int_0^\infty g^2(\lambda) f(\lambda)\,d\lambda \;.$$

---

* $\underset{n \to \infty}{\text{l.i.m.}}\; X_n = X$ if $\lim_{n \to \infty} E\,|X_n - X|^2 = 0$. Further if X', X'' are random variables such that

$$\text{l.i.m. } X_n = X' \quad \text{and} \quad \text{l.i.m. } X_n = X'',$$

then X' and X'' are equivalent random variables, i.e. $P\{\omega \,|\, X'(\omega) = X''(\omega)\} = 1$.

The spectral representation (2.11) exhibits the process $X_t, t \in T$ as a random superposition of trigonometric functions. The increments $dU(\lambda)$ and $dV(\lambda)$ are the random amplitudes of $\cos t\lambda$ and $\sin t\lambda$ respectively. The spectral density $f(\lambda)d\lambda$ gives the variance of the random amplitudes $dU(\lambda)$ and $dV(\lambda)$.

2.4.2 The continuous parameter real two-dimensional process.

Here $X_t = [x_t, y_t], t \in T = [t|-\infty < t < \infty]$. It is assumed that $E[x_t, y_t] = [0,0]$. The covariance function (assumed finite) is

$$(2.20) \quad R_\tau = E\, X'_{t+\tau} X_t = \begin{bmatrix} R_{xx}(\tau) & R_{xy}(\tau) \\ R_{yx}(\tau) & R_{yy}(\tau) \end{bmatrix} .$$

Clearly, $R_\tau = R'_{-\tau}$.

It is further assumed that

$$(2.21) \quad \lim_{\tau \to 0} R_{xx}(\tau) = R_{xx}(0) = \sigma_x^2 > 0 \text{ , and}$$

$$\lim_{\tau \to 0} R_{yy}(\tau) = R_{yy}(0) = \sigma_y^2 > 0 \text{ .}$$

The condition (2.21) implies that all the covariance functions $R_{xx}, R_{xy}, R_{yx}, R_{yy}$ are continuous for all values of τ. Furthermore

$$|R_{xx}(\tau)| \le \sigma_x^2 \quad , \quad |R_{xy}(\tau)| \le \sigma_x \sigma_y ,$$

(2.22)

$$|R_{yx}(\tau)| \le \sigma_x \sigma_y \quad , \quad |R_{yy}(\tau)| \le \sigma_y^2 .$$

The covariance functions $R_{xx}, R_{xy}, R_{yx}, R_{yy}$, can be expressed in the form

(2.23.1) $$R_{xx}(\tau) = \int_0^{\infty} \cos \tau\lambda \, dF_{xx}(\lambda) ,$$

(2.23.2) $$R_{xy}(\tau) = R_{yx}(-\tau) = \int_0^{\infty} \cos \tau\lambda \, dC(\lambda) + \int_0^{\infty} \sin \tau\lambda \, dQ(\lambda) ,$$

(2.23.3) $$R_{yy}(\tau) = \int_0^{\infty} \cos \tau\lambda \, dF_{yy}(\lambda) .$$

The functions $F_{xx}(\lambda), F_{yy}(\lambda), C(\lambda), Q(\lambda)$ are real functions of bounded variation in $(0 < \lambda < \infty)$. The functions $F_{xx}(\lambda)$ and $F_{yy}(\lambda)$ are the spectral functions of the processes x_t and y_t respectively. The function $C(\lambda)$ is called the cospectral function and the function $Q(\lambda)$ the quadrature spectral function. These spectral functions satisfy the following inequalities called Coherency Conditions:

For every (λ_1, λ_2), $0 < \lambda_1 < \lambda_2$,

(2.24) $$(\Delta C)^2 + (\Delta Q)^2 \le (\Delta F_{xx})(\Delta F_{yy})$$

where

$$\Delta C(\lambda) = C(\lambda_2) - C(\lambda_1) \quad \text{etc.}$$

As in the one-dimensional case the situation of physical interest is the one for which the spectral functions are absolutely continuous with continuous derivatives and unless otherwise specified consideration is restricted to this case. Thus, it is assumed that

$$F_{xx}(\lambda_0) = \int_0^{\lambda_0} f_{xx}(\lambda)d\lambda \ , \quad F_{yy}(\lambda_0) = \int_0^{\lambda_0} f_{yy}(\lambda)d\lambda \ ,$$

(2.25)

$$C(\lambda_0) = \int_0^{\lambda_0} c(\lambda)d\lambda \ , \quad Q(\lambda_0) = \int_0^{\lambda_0} q(\lambda)d\lambda \ ,$$

where the functions $f_{xx}(\lambda), f_{yy}(\lambda), c(\lambda)$, and $q(\lambda)$ are continuous. The functions $c(\lambda)$ and $q(\lambda)$ are termed the cospectral density function and quadrature spectral density function respectively. The cospectral density function $c(\lambda)$ (defined for $\lambda \geq 0$) is twice the real part of $f_{xy}(\lambda)$, and the quadrature spectral density function $q(\lambda)$ (defined for $\lambda \geq 0$) is twice the imaginary part of $f_{xy}(\lambda)$, where $f_{xy}(\lambda)$ is the cross spectral density function

(2.26) $$f_{xy}(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{xy}(\tau)e^{i\tau\lambda}d\tau \ .$$

Thus

(2.27) $$R_{xy}(\tau) = \int_{-\infty}^{\infty} f_{xy}(\lambda)e^{-i\tau\lambda}d\lambda \ .$$

Since $R_{xy}(\tau) = R_{yx}(-\tau)$, $f_{xy}(-\lambda) = f_{yx}(\lambda)$.

Since $R_{xy}(\tau)$ is real, $f_{xy}(\lambda) = \overline{f_{xy}(-\lambda)}$.

Thus

$$\text{(2.28)} \qquad \begin{aligned} \operatorname{Re} f_{xy}(\lambda) &= \operatorname{Re} f_{xy}(-\lambda) \\ \operatorname{Im} f_{xy}(\lambda) &= -\operatorname{Im} f_{xy}(-\lambda) \ . \end{aligned}$$

From (2.27) one obtains

$$\text{(2.29)} \qquad R_{xy}(\tau) = \int_{-\infty}^{\infty} \cos \tau\lambda \operatorname{Re} f_{xy}(\lambda)d\lambda + \int_{-\infty}^{\infty} \sin \tau\lambda \operatorname{Im} f_{xy}(\lambda)d\lambda \ ,$$

which by virtue of (2.28) reduces to (2.23.2) with $c(\lambda) = 2 \operatorname{Re} f_{xy}(\lambda)$ and $q(\lambda) = 2 \operatorname{Im} f_{xy}(\lambda)$.

A real two-dimensional process satisfying the continuity conditions (2.21) can be expressed in the canonical form (real spectral representation)

$$\text{(2.30)} \qquad \begin{aligned} x_t &= \int_0^{\infty} \cos t\lambda \, dU_x(\lambda) + \int_0^{\infty} \sin t\lambda \, dV_x(\lambda) \\ y_t &= \int_0^{\infty} \cos t\lambda \, dU_y(\lambda) + \int_0^{\infty} \sin t\lambda \, dV_y(\lambda) \end{aligned}$$

where $U_x(\lambda)$, $V_x(\lambda)$, $U_y(\lambda)$, $V_y(\lambda)$ are real processes satisfying the following relations:

$$\text{(2.31)} \qquad E\, dU_x(\lambda) = E\, dV_x(\lambda) = E\, dU_y(\lambda) = E\, dV_y(\lambda) = 0 \ .$$

$$\text{(2.32.1)} \qquad E\, dU_x(\lambda)dU_x(\lambda') = E\, dV_x(\lambda)dV_x(\lambda') = \begin{cases} 0 \text{ if } \lambda \neq \lambda' \\ dF_{xx}(\lambda) = f_{xx}(\lambda)d\lambda \text{ if } \lambda = \lambda' \ . \end{cases}$$

$$(2.32.2)\quad E\, dU_y(\lambda)dU_y(\lambda')$$

$$= E\, dV_y(\lambda)dV_y(\lambda') = \begin{cases} 0 & \text{if } \lambda \neq \lambda' \\ dF_{yy}(\lambda) = f_{yy}(\lambda)d\lambda & \text{if } \lambda = \lambda'. \end{cases}$$

$$(2.32.3)\quad E\, dU_x(\lambda)dV_x(\lambda') = E\, dU_y(\lambda)dV_y(\lambda') = 0\,.$$

$$(2.32.4)\quad E\, dU_x(\lambda)dU_y(\lambda')$$

$$= E\, dV_x(\lambda)dV_y(\lambda') = \begin{cases} 0 & \text{if } \lambda \neq \lambda' \\ dC(\lambda) = c(\lambda)d\lambda & \text{if } \lambda = \lambda'. \end{cases}$$

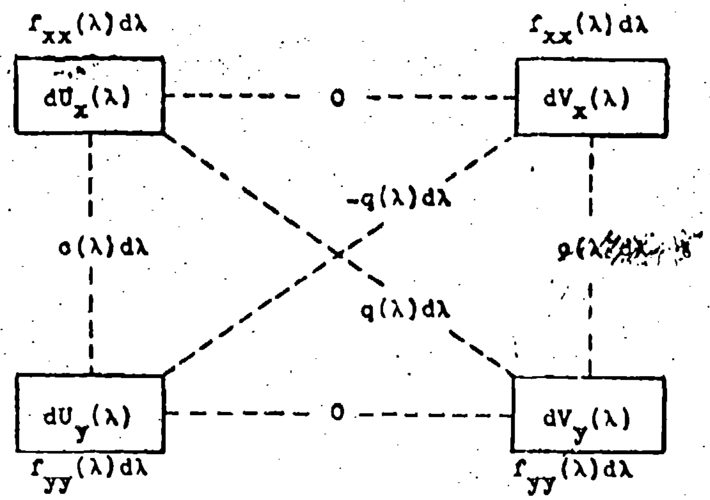
$$(2.32.5)\quad E\, dU_x(\lambda)dV_y(\lambda') = \begin{cases} 0 & \text{if } \lambda \neq \lambda' \\ dQ(\lambda) = q(\lambda)d\lambda & \text{if } \lambda = \lambda'. \end{cases}$$

$$(2.32.6)\quad E\, dV_x(\lambda)dU_y(\lambda') = \begin{cases} 0 & \text{if } \lambda \neq \lambda' \\ -dQ(\lambda) = -q(\lambda)d\lambda & \text{if } \lambda = \lambda'. \end{cases}$$

The integrals appearing in (2.30) are stochastic integrals. The spectral representation (2.30) exhibits the process $X_t = [x_t, y_t]$, $t \in T$ as a random superposition

$$(2.33)\quad \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \int_0^\infty \begin{bmatrix} dU_x(\lambda) & dV_x(\lambda) \\ dU_y(\lambda) & dV_y(\lambda) \end{bmatrix} \begin{bmatrix} \cos t\lambda \\ \sin t\lambda \end{bmatrix} d\lambda$$

of trigonometric functions. The increments $dU_x(\lambda)$ and $dV_x(\lambda)$ are the random amplitudes of $\cos t\lambda$ and $\sin t\lambda$ in the representation of x_t and similarly $dU_y(\lambda)$ and $dV_y(\lambda)$ are the random amplitudes of $\cos t\lambda$ and $\sin t\lambda$ in the representation of y_t. The equations of (2.32) express the covariance relations between these random amplitudes. One observes that any pair of random amplitudes at different angular frequencies are uncorrelated. At a fixed angular frequency λ, $dU_x(\lambda)$ and $dV_x(\lambda)$ are uncorrelated as are $dU_y(\lambda)$ and $dV_y(\lambda)$, and the spectral density $f_{xx}(\lambda)d\lambda$ gives the variance of $dU_x(\lambda)$ and $dV_x(\lambda)$ and the spectral density $f_{yy}(\lambda)d\lambda$ gives the variance of $dU_y(\lambda)$ and $dV_y(\lambda)$. The cospectral density and quadrature spectral density express the covariance relationships that exist between the random amplitudes $dU_x(\lambda)$, $dV_x(\lambda)$ of the x-component of the process and the random amplitudes $dU_y(\lambda)$, $dV_y(\lambda)$ of the y-component of the process. The pair of random amplitudes $dU_x(\lambda)$, $dU_y(\lambda)$ as well as the pair $dV_x(\lambda)$, $dV_y(\lambda)$ associate "in-phase" terms of the representation (2.30), i.e. each member of the first pair is an amplitude of $\cos t\lambda$ and each member of the second pair is an amplitude of $\sin t\lambda$. The pair of random amplitudes $dU_x(\lambda)$, $dV_y(\lambda)$ as well as the pair $dV_x(\lambda)$, $dU_y(\lambda)$ associate "in quadrature" (i.e. $\pi/2$ out of phase) terms of the representation (2.30), i.e. if one member of a pair is an amplitude of $\cos t\lambda$ the other is an amplitude of $\sin t\lambda$. With this terminology of "in-phase" and "in-quadrature" one can concisely express the meaning of the cospectral and quadrature spectral densities. The cospectral density $c(\lambda)d\lambda$ gives the covariance between two "in-phase" random amplitudes, and the quadrature spectral density $q(\lambda)d\lambda$ gives (to within a factor of $\pm$) the covariance between two "in-quadrature" random amplitudes. The diagram given below summarizes the covariance relations.



If the process $X_t, t \in T$ is Gaussian, the increments $dU_x(\lambda)$, $dV_x(\lambda)$, $dU_y(\lambda)$, $dV_y(\lambda)$, $(0 \leq \lambda < \infty)$ are Gaussian random variables and the mean and covariance relations (2.31) and (2.32) completely determine the probability structure of these random variables. Thus, it is seen that any pair of increments [as for example $dU_x(\lambda)$, $dV_y(\lambda')$] at different angular frequencies $\lambda, \lambda' (\lambda \neq \lambda')$ are independent. Furthermore, the variance covariance matrix of $[dU_x(\lambda), dV_x(\lambda), dU_y(\lambda), dV_y(\lambda)]$ is given by

$$(2.34) \qquad \begin{bmatrix} f_{xx}(\lambda) & 0 & c(\lambda) & q(\lambda) \\ 0 & f_{xx}(\lambda) & -q(\lambda) & c(\lambda) \\ c(\lambda) & -q(\lambda) & f_{yy}(\lambda) & 0 \\ q(\lambda) & c(\lambda) & 0 & f_{yy}(\lambda) \end{bmatrix} d\lambda .$$

so that $dU_x(\lambda)$ and $dV_x(\lambda)$ are independent as are $dU_y(\lambda)$ and $dV_y(\lambda)$.

2.4.3 The discrete parameter real one dimensional process.

The discrete parameter weakly stationary processes considered here are assumed to be of the form $X_{k\Delta t}, k = \ldots, -2, -1, 0, 1, 2, \ldots$, i.e. discrete parameter processes obtained by observing the sample functions of a weakly stationary continuous parameter process $X_t, t \in T$ at the discrete set of times $t = k\Delta t$, $k = \ldots -2, -1, 0, 1, 2, \ldots$. It is assumed that $E\, X_t = 0$, $t \in T$. Let $R_k^{(\Delta t)} = E\, X_{k\Delta t} X_0$.

The covariance sequence $R_k^{(\Delta t)}$, $k = \ldots -1,0,1,\ldots$ is even and positive definite and hence

(2.35) $$R_k^{(\Delta t)} = \int_0^{\pi} \cos k\lambda \, dF^{(\Delta t)}(\lambda)$$

where $F^{(\Delta t)}(\lambda)$ is a real bounded monotone non-decreasing function. As in the continuous parameter case, the function $F^{(\Delta t)}(\lambda)$ is called the spectral function of the process and is assumed (unless otherwise specified) to be absolutely continuous with a continuous derivative so that

(2.36) $$R_k^{(\Delta t)} = \int_0^{\pi} \cos k\lambda \, f^{(\Delta t)}(\lambda) d\lambda \quad , \quad f^{(\Delta t)}(\lambda) \geq 0 \, .$$

The process $X_{k\Delta t}$, $k = \ldots -2,-1,0,1,2\ldots$ can be expressed in the canonical form (spectral representation)

(2.37) $$X_{k\Delta t} = \int_0^{\pi} \cos k\lambda \, dU^{(\Delta t)}(\lambda) + \int_0^{\pi} \sin k\lambda \, dV^{(\Delta t)}(\lambda)$$

where $U^{(\Delta t)}(\lambda), V^{(\Delta t)}(\lambda)$ are real processes with orthogonal increments that are orthogonal to each other.

It is now desired to express $f^{(\Delta t)}(\lambda)$ the spectral density of the discrete parameter process $X_{k\Delta t}$, $k = \ldots,-1,0,1,\ldots$ in terms of $f(\lambda)$ the spectral density of the continuous parameter process $X_t, t \in T$ from which $X_{k\Delta t}$ is obtained. For the continuous process

(2.38) $$R(\tau) = \int_0^{\infty} \cos \tau\lambda \, f(\lambda) d\lambda \, .$$

Thus,

$$(2.39) \quad R_k^{(\Delta t)} = R(k\Delta t) = \int_0^{\infty} \cos(k\lambda\Delta t) f(\lambda) d\lambda$$

$$= \sum_{m=0}^{\infty} \int_{m\frac{\pi}{\Delta t}}^{(m+1)\frac{\pi}{\Delta t}} \cos(k\lambda\Delta t) f(\lambda) d\lambda \ .$$

If m is even,

$$(2.40) \quad \int_{m\frac{\pi}{\Delta t}}^{(m+1)\frac{\pi}{\Delta t}} \cos(k\lambda\Delta t) f(\lambda) d\lambda = \int_0^{\frac{\pi}{\Delta t}} \cos(k\lambda\Delta t) f(\lambda + m\frac{\pi}{\Delta t}) d\lambda \ .$$

If m is odd,

$$(2.41) \quad \int_{m\frac{\pi}{\Delta t}}^{(m+1)\frac{\pi}{\Delta t}} \cos(k\lambda\Delta t) f(\lambda) d\lambda = \int_0^{\frac{\pi}{\Delta t}} \cos(k\lambda\Delta t) f[-\lambda + (m+1)\frac{\pi}{\Delta t}] d\lambda \ .$$

Thus,

$$(2.42) \quad R_k^{(\Delta t)} = \int_0^{\frac{\pi}{\Delta t}} \cos(k\lambda\Delta t)[f(\lambda) + f(\frac{2\pi}{\Delta t} - \lambda) + f(\frac{2\pi}{\Delta t} + \lambda)$$

$$+ f(\frac{4\pi}{\Delta t} - \lambda) + f(\frac{4\pi}{\Delta t} + \lambda) + \ldots] d\lambda \ .$$

Upon making a change of variables,

$$(2.43) \quad R_k^{(\Delta t)} = \int_0^{\pi} \cos k\lambda' [f(\frac{\lambda'}{\Delta t}) + f(\frac{2\pi - \lambda'}{\Delta t}) + f(\frac{2\pi + \lambda'}{\Delta t})$$

$$+ f(\frac{4\pi - \lambda'}{\Delta t}) + f(\frac{4\pi + \lambda'}{\Delta t}) + \ldots] \frac{d\lambda'}{\Delta t} \ ,$$

where $\lambda' = \lambda\Delta t$.

Thus,

$$(2.44)\qquad f^{(\Delta t)}(\lambda') = \frac{1}{\Delta t}\left[f\left(\frac{\lambda'}{\Delta t}\right)+f\left(\frac{2\pi-\lambda'}{\Delta t}\right)+f\left(\frac{2\pi+\lambda'}{\Delta t}\right)+f\left(\frac{4\pi-\lambda''}{\Delta t}\right)+\ldots\right]$$

$$(0 \le \lambda'' \le \pi) .$$

The time interval Δt is called the sampling interval or <u>Nyquist interval</u> or <u>Nyquist time</u>, and the angular frequency $\pi/(\Delta t)$ is called the angular <u>turnover frequency</u> or angular <u>folding frequency</u> or angular <u>Nyquist frequency</u>. The angular frequencies

$$(2.45)\qquad \frac{\lambda'}{\Delta t},\ \frac{2\pi-\lambda'}{\Delta t},\ \frac{2\pi+\lambda'}{\Delta t},\ \frac{4\pi-\lambda'}{\Delta t},\ \ldots \qquad (0 \le \lambda'' < \pi)$$

$$\text{i.e.}\qquad \lambda,\ \frac{2\pi}{\Delta t}-\lambda,\ \frac{2\pi}{\Delta t}+\lambda,\ \frac{4\pi}{\Delta t}-\lambda,\ \ldots \qquad (0 \le \lambda \le \pi/\Delta t)$$

are said to be <u>aliases</u> of each other, and the angular frequency $\lambda'/(\Delta t),(0 \le \lambda' \le \pi)$ is called the <u>principal alias</u>. The reason for the term alias is that sinusoids of angular frequencies given by (2.45) are indistinguishable when observed at only the discrete set of times $k\Delta t$, $k = \ldots -1,0,1,\ldots$. One observes from (2.44) that the spectral density $f^{(\Delta t)}(\lambda')$ of the discrete parameter process at λ' is essentially the sum of the spectral densities of the continuous parameter process at those angular frequencies which are the aliases of $\lambda'/(\Delta t)$.

2.4.4 The discrete parameter real two-dimensional process.

The discrete parameter two-dimensional processes considered here are assumed to be of the form $X_{k\Delta t} = [x_{k\Delta t}, y_{k\Delta t}]$, $k = \ldots -1,0,1,\ldots$, i.e. discrete parameter processes obtained by observing the sample functions of a continuous parameter process $X_t = [x_t, y_t]$, $t \in T$, at the discrete set of times $t = k\Delta t$, $k = \ldots -1,0,1,\ldots$. It is assumed that $E\,X_t = [0,0]$, $t \in T$.

The discussion of the discrete parameter real two-dimensional process is analogous to that of the continuous parameter real two-dimensional process given in Section 2.4.2. The covariance function

$$(2.46) \quad R(k\Delta t) = E\,X'_{t+k\Delta t}X_t = \begin{bmatrix} E\,x_{t+k\Delta t}x_t & E\,x_{t+k\Delta t}y_t \\ E\,y_{t+k\Delta t}x_t & E\,y_{t+k\Delta t}y_t \end{bmatrix}$$

$$= \begin{bmatrix} R_{xx}(k\Delta t) & R_{xy}(k\Delta t) \\ R_{yx}(k\Delta t) & R_{yy}(k\Delta t) \end{bmatrix}$$

Furthermore, $R(k\Delta t) = R'(-k\Delta t)$.

The covariance function $R(k\Delta t)$ can be expressed in the form

$$(2.47) \quad R(k\Delta t) = \int_{-\pi}^{\pi} e^{ik\lambda}\, dF^{(\Delta t)}(\lambda) \; .$$

where

$$(2.48)\quad F^{(\Delta t)}(\lambda) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} R(k\Delta t)\int_{-\pi}^{\lambda} e^{-ik\lambda}\,d\lambda\,.$$

If $F^{(\Delta t)}(\lambda)$ is absolutely continuous and possesses a continuous derivative,

$$(2.49)\quad F^{(\Delta t)}(\lambda) = \int_{-\pi}^{\lambda}\begin{bmatrix} f_{xx}^{(\Delta t)}(\xi) & f_{xy}^{(\Delta t)}(\xi) \\ f_{yx}^{(\Delta t)}(\xi) & f_{yy}^{(\Delta t)}(\xi)\end{bmatrix} d\xi = \int_{-\pi}^{\lambda} f^{(\Delta t)}(\xi)d\xi\,.$$

Thus,

$$(2.50)\quad R(k\Delta t) = \int_{-\pi}^{\pi} e^{ik\lambda} f^{(\Delta t)}(\lambda)d\lambda\,.$$

The functions $f_{xx}^{(\Delta t)}(\lambda)$, $f_{yy}^{(\Delta t)}(\lambda)$ are the spectral densities of the $x_{k\Delta t}$, and $y_{k\Delta t}$, $k = \ldots -1,0,1,\ldots$ processes respectively and

$$(2.51)\quad f_{xy}^{(\Delta t)}(\lambda) = \overline{f_{yx}^{(\Delta t)}(\lambda)}$$

is the cross spectral density of the x_t process with the y_t process. The matrix-valued function $f^{(\Delta t)}(\lambda)$ is positive semi-definite, i.e.

$$(2.52.1)\quad f_{xx}^{(\Delta t)}(\lambda),\ f_{yy}^{(\Delta t)}(\lambda) \geq 0$$

$$(2.52.2)\quad f_{xx}^{(\Delta t)}(\lambda)\, f_{yy}^{(\Delta t)}(\lambda) - |f_{xy}^{(\Delta t)}(\lambda)|^2 \geq 0\,.$$

Since the process $X_t = [x_t, y_t]$, $t \in T$ is assumed to be real, one has

$$(2.53.1) \quad f_{xx}^{(\Delta t)}(\lambda) = f_{xx}^{(\Delta t)}(-\lambda) ,$$

$$(2.53.2) \quad f_{yy}^{(\Delta t)}(\lambda) = f_{yy}^{(\Delta t)}(-\lambda) ,$$

and

$$(2.53.3) \quad f_{xy}^{(\Delta t)}(\lambda) = \overline{f_{xy}^{(\Delta t)}(-\lambda)}.$$

Here, $f_{xy}^{(\Delta t)}(\lambda) = c^{(\Delta t)}(\lambda) + i\, q^{(\Delta t)}(\lambda)$, i.e. $c^{(\Delta t)}(\lambda)$, called the <u>cospectral density</u> of the discrete process $X_{k\Delta t}$, $k = \ldots -1,0,1,\ldots,$ is the real part of the <u>cross-spectral density</u> $f_{xy}^{(\Delta t)}(\lambda)$ and $q^{(\Delta t)}(\lambda)$, called the <u>quadrature spectral density</u>, is the imaginary part of $f_{xy}^{(\Delta t)}(\lambda)$. From (2.53.3),

$$(2.54.1) \quad c^{(\Delta t)}(-\lambda) = c^{(\Delta t)}(\lambda) ,$$

and

$$(2.54.2) \quad q^{(\Delta t)}(-\lambda) = -q^{(\Delta t)}(\lambda) ,$$

i.e. the cospectral density $c^{(\Delta t)}(\lambda)$ is an even function and the quadrature spectral density $q^{(\Delta t)}(\lambda)$ is an odd function. It is desired to express $f_{xx}^{(\Delta t)}(\lambda)$, $f_{yy}^{(\Delta t)}(\lambda)$, $c^{(\Delta t)}(\lambda)$, $q^{(\Delta t)}(\lambda)$ the spectral, cospectral, and quadrature spectral densities of the discrete parameter process in terms of $f_{xx}(\lambda)$, $f_{yy}(\lambda)$, $c(\lambda)$, $q(\lambda)$ the corresponding spectral densities of the continuous parameter process. Clearly $f_{xx}^{(\Delta t)}(\lambda)$ is related to $f_{xx}(\lambda)$ [and $f_{yy}^{(\Delta t)}(\lambda)$ is related to $f_{yy}(\lambda)$] by the relation (2.44). To obtain the relation between the cospectral densities (and the quadrature spectral densities) one proceeds as follows:

$$(2.55)\quad R_{xy}(k\Delta t) = \int_{-\infty}^{\infty} f_{xy}(\lambda)e^{-ik\lambda\Delta t}\, d\lambda$$

$$= \sum_{m=-\infty}^{\infty} \int_{(2m-1)\frac{\pi}{\Delta t}}^{(2m+1)\frac{\pi}{\Delta t}} f_{xy}(\lambda)e^{-ik\lambda\Delta t}\, d\lambda$$

$$= \sum_{m=-\infty}^{\infty} \int_{-\frac{\pi}{\Delta t}}^{\frac{\pi}{\Delta t}} f_{xy}(\lambda + \frac{2m\pi}{\Delta t})e^{-ik\lambda\Delta t}\, d\lambda .$$

Let $\lambda' = \lambda\Delta t$. Thus

$$(2.56)\quad R_{xy}(k\Delta t) = R_{xy}^{(\Delta t)}(k) = \sum_{m=-\infty}^{\infty} \int_{-\pi}^{\pi} f_{xy}(\frac{\lambda'+2m\pi}{\Delta t})e^{-ik\lambda'}\frac{1}{\Delta t}\, d\lambda'$$

$$= \int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} f_{xy}(\frac{\lambda'+2m\pi}{\Delta t})\frac{1}{\Delta t}\, e^{-ik\lambda'} d\lambda'.$$

That is, the cross-spectral density of the discrete parameter process is

$$\sum_{m=-\infty}^{\infty} f_{xy}(\frac{\lambda'+2m\pi}{\Delta t})\, \frac{1}{\Delta t} .$$

Thus,

$$(2.57)\quad c^{(\Delta t)}(\lambda') = \mathrm{Re} \sum_{m=-\infty}^{\infty} f_{xy}(\frac{\lambda'+2m\pi}{\Delta t})\frac{1}{\Delta t} = \sum_{m=-\infty}^{\infty} c(\frac{\lambda'+2m\pi}{\Delta t})\frac{1}{\Delta t}$$

and

$$(2.58)\qquad q^{(\Delta t)}(\lambda') = \operatorname{Im} \sum_{m=-\infty}^{\infty} f_{xy}\left(\frac{\lambda' + 2m\pi}{\Delta t}\right)\frac{1}{\Delta t}$$

$$= \sum_{m=-\infty}^{\infty} q\left(\frac{\lambda' + 2m\pi}{\Delta t}\right)\frac{1}{\Delta t} .$$

Here $-\pi < \lambda' < \pi$ and $c^{(\Delta t)}(\lambda')$ is an even function of λ' and $q^{(\Delta t)}(\lambda')$ is an odd function of λ', since $c(\lambda)$ is even and $q(\lambda)$ is odd. One thus observes that the cospectral density $c^{(\Delta t)}(\lambda')$ of the discrete process at λ' is essentially the sum of the cospectral densities of the continuous process at the angular frequencies $\frac{\lambda' + 2m\pi}{\Delta t}$, $m = \ldots,-1,0,1,\ldots$. A similar relation holds for the quadrature spectral density. One often considers the domain of the spectral, cospectral, and quadrature spectral densities to be $0 \leq \lambda < \infty$ for a continuous process and $0 \leq \lambda \leq \pi$ for a discrete process. With such a convention

$$(2.59)\qquad c^{(\Delta t)}(\lambda') = \frac{1}{\Delta t}\left[c\left(\frac{\lambda'}{\Delta t}\right)+c\left(\frac{2\pi-\lambda'}{\Delta t}\right)+c\left(\frac{2\pi+\lambda'}{\Delta t}\right)+c\left(\frac{4\pi-\lambda'}{\Delta t}\right)+\ldots\right]$$

and

$$(2.60)\qquad q^{(\Delta t)}(\lambda') = \frac{1}{\Delta t}\left[q\left(\frac{\lambda'}{\Delta t}\right)-q\left(\frac{2\pi-\lambda'}{\Delta t}\right)+q\left(\frac{2\pi+\lambda'}{\Delta t}\right)-q\left(\frac{4\pi-\lambda'}{\Delta t}\right)+\ldots\right]$$

$$(0 \leq \lambda' \leq \pi) .$$

Thus the cospectral density $c^{(\Delta t)}(\lambda')$ of the discrete process at λ' is essentially the sum of the cospectral densities of the continuous process over the angular frequencies which are aliases of $\lambda'/(\Delta t)$. A similar relation holds for the quadrature spectral density, except that for the quadrature spectral density certain terms in the sum, namely those corresponding to the aliases $\frac{2m\pi - \lambda'}{\Delta t}$, $m = 1,2,\ldots$ are taken negatively.

2.4.5 Weakly Stationary Gaussian Processes.

As was seen in Section 2.3 a Gaussian Process is specified by its mean function $\mu(t)$ and its covariance function $R(t,t')$. Thus if a weakly stationary process is Gaussian, the complete probability structure of the process is specified by the mean μ and covariance function $R(\tau)$. A weakly stationary Gaussian process is thus strongly stationary.

Chapter 3

Estimators for the Spectra, Cospectrum, and Quadrature Spectrum

3.0 Introduction

In this chapter estimators for the spectra, cospectrum, and quadrature spectrum of a two-dimensional stationary (zero mean) Gaussian vector process are studied. The estimators are such that it is possible to establish a one-to-one correspondence between the estimators and trigonometric polynomials (filters). Explicit expressions for certain "good" filters are derived by making use of results contained in [17]. Further, formulae for the means and covariances of the estimators are obtained. These formulae exhibit the means and covariances of the estimators as integrals of products of the trigonometric polynomials associated with the estimators and functions depending solely on the spectral, cospectral, and quadrature spectral densities of the process. The formulae can thus be thought of as spectral representations for the means and covariances of the estimators. Finally, a heuristic argument is presented to obtain an approximation to the joint sampling distribution of the estimators.

3.1 The Estimation Problem

A sample function of a continuous parameter two-dimensional stationary (zero mean) Gaussian vector process is observed at the discrete set of times $\Delta t, 2\Delta t, \ldots, N\Delta t$, that is

$$(3.1) \qquad X_{k\Delta t} = [x_{k\Delta t}, y_{k\Delta t}] \quad k = 1, 2, \ldots, N$$

is observed. It is desired to estimate the spectra $F_{xx}^{(\Delta t)}(\lambda)$, $F_{yy}^{(\Delta t)}(\lambda)$, the cospectrum $C^{(\Delta t)}(\lambda)$, and the quadrature spectrum $Q^{(\Delta t)}(\lambda)$, of the discrete process $X_{k\Delta t}$, $k = \ldots -2, -1, 0, 1, 2, \ldots$. It is intended that estimators for the spectra and cross spectra of the discrete parameter process serve to estimate the spectra and cross spectra of the continous parameter process. One notes from (2.44), (2.48), and (2.49) that if the spectral densities $f_{xx}(\lambda)$, $f_{yy}(\lambda)$, the cospectral density $c(\lambda)$, and the quadrature spectral density $q(\lambda)$ are negligibly smal for sufficiently large λ, say $\lambda > \lambda_0$, then the functions $\Delta t\, f_{xx}^{(\Delta t)}(\lambda\Delta t)$, $\Delta t\, f_{yy}^{(\Delta t)}(\lambda\Delta t)$, $\Delta t\, c^{(\Delta t)}(\lambda\Delta t)$, and $\Delta t\, q^{(\Delta t)}(\lambda\Delta t)$ are good approximations to the corresponding density functions $f_{xx}(\lambda)$, $f_{yy}(\lambda)$, $c(\lambda)$ and $q(\lambda)$ respectively, provided Δt is sufficiently small, namely $\Delta t < \frac{\pi}{\lambda_0}$. Thus, if one determines a means of estimating the spectra and cross spectra of a discrete parameter process from a sample of finite length, one can estimate the spectra and cross spectra of a continuous parameter process. This is accomplished by observing a sample function of the continuous parameter process at equally spaced intervals of time and then estimating the spectra and cross spectra of the resulting discrete parameter process. One has to assume that the spectra and cross spectra of the continuous parameter process effectively vanish for sufficiently large λ (an assumption valid for physically relevant processes) and that the sampling time Δt is sufficiently small so that "aliasing" becomes unimportant. With the above remarks in mind, attention will henceforth be restricted to the estimation of the spectra, cospectrum and quadrature spectrum of a discrete parameter process.

For the sake of simplicity the following abbreviated notation is introduced. The finite sample of length N from a discrete parameter real two-dimensional stationary (zero mean) Gaussian vector process given by (3.1) is simply denoted by

$$(3.2) \qquad [x_k, y_k], \quad k = 1, 2, \ldots, N.$$

Furthermore, the densities $f_{xx}^{(\Delta t)}(\lambda)$, $f_{yy}^{(\Delta t)}(\lambda)$, $c^{(\Delta t)}(\lambda)$, and $q^{(\Delta t)}(\lambda)$ are denoted by $f_x(\lambda)$, $f_y(\lambda)$, $c(\lambda)$, and $q(\lambda)$ respectively, and the corresponding spectral functions $F_{xx}^{(\Delta t)}(\lambda)$, $F_{yy}^{(\Delta t)}(\lambda)$, $C^{(\Delta t)}(\lambda)$, and $Q^{(\Delta t)}(\lambda)$ by $F_x(\lambda)$, $F_y(\lambda)$, $C(\lambda)$, and $Q(\lambda)$ respectively. It is desired to in some sense estimate the x and y spectral densities $f_x(\lambda)$ and $f_y(\lambda)$, the cospectral density $c(\lambda)$, and the quadrature spectral density $q(\lambda)$ from the sample (3.2). What is treated is not the estimation of the densities $f_x(\lambda)$, $f_y(\lambda)$, $c(\lambda)$, $q(\lambda)$ but rather the estimation problem for integrals of these densities over a set of frequency bands $\lambda_{i-1} \leq \lambda \leq \lambda_i$, $i = 1,2,\ldots, I$. That is, the frequency domain $0 \leq \lambda \leq \pi$ is partitioned into a finite set of frequency intervals $\lambda_{i-1} \leq \lambda \leq \lambda_i$, $i = 1,2,\ldots,I$ by the partition $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_I = \pi$. The integrated densities

$$(3.3.1) \qquad \alpha_i = \int_{\lambda_{i-1}}^{\lambda_i} f_x(\lambda)d\lambda = F_x(\lambda_i) - F_x(\lambda_{i-1}),$$

$$(3.3.2) \qquad \beta_i = \int_{\lambda_{i-1}}^{\lambda_i} f_y(\lambda)d\lambda = F_y(\lambda_i - F_y(\lambda_{i-1}),$$

$$(3.3.3) \qquad \gamma_i = \int_{\lambda_{i-1}}^{\lambda_i} c(\lambda)d\lambda = C(\lambda_i) - C(\lambda_{i-1}),$$

$$(3.3.4) \qquad \delta_i = \int_{\lambda_{i-1}}^{\lambda_i} q(\lambda)d\lambda = Q(\lambda_i) - Q(\lambda_{i-1}),\ i = 1,2,\ldots, I,$$

are thus the parameters which are to be estimated. In general it is desired that the frequency bands $\lambda_{i-1} \leq \lambda \leq \lambda_i$ be sufficiently narrow so that the integrals of (3.3), which apart from constant factors are the averages of the spectral, co-spectral, and quadrature spectral densities over the frequency bands, indicate the behavior of the respective densities in the frequency bands. As will be discussed subsequently the widths of the frequency bands are determined from statistical considerations, for generally speaking as the widths $\Delta_i = \lambda_i - \lambda_{i-1}$ of the frequency bands decrease, statistics serving as estimators for the integrals of (3.3) become less reliable, i.e. become relatively more biased or become relatively more variable. In the estimation problem to be now treated the partition $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_I = \pi$ is considered fixed and attention is directed toward obtaining estimators for the I integrals of (3.3) in the case when (3.2) is a sample from a Gaussian process. It is, however, shown in Section 3.2 that unbiased estimators for the integrals of (3.3) do not in general exist. A comment is therefore interjected here explaining that one should not too rigidly consider the integrals of (3.3) as the parameters to be estimated. One can obtain <u>unbiased</u> estimators for certain integrals, <u>not</u> the integrals (3.3) with rectangular kernels, but similar integrals with trigonometric polynomial kernels of bounded degree. Thus, the parameters that actually are estimated (unbiasedly) are certain integrals with trigonometric polynomial kernels of bounded degree that approximate the integrals (3.3). It is, however, convenient to speak of estimating the integrals (3.3), and this will be done throughout. The preceding remarks are somewhat anticipatory and will become clearer after one has read Sections 3.2 and 3.3.

3.2 The Estimators

The estimators for the α_i, β_i, γ_i, δ_i, $i = 1,2,\ldots, I$, are obtained as follows:

Let

$$A_0 = \frac{1}{N}(x_1^2 + x_2^2 + \ldots + x_N^2)$$

$$A_1 = \frac{1}{N-1}(x_1 x_2 + x_2 x_3 + \ldots + x_{N-1} x_N)$$

$$A_h = \frac{1}{N-h}(x_1 x_{1+h} + x_2 x_{2+h} + \ldots + x_{N-h} x_N) \tag{3.4}$$

$$A_{N-1} = x_1 x_N$$

and take as a class of estimators for the α_i, $i = 1,2,\ldots, I$ the quadratic forms

$$Q_{\alpha_i} = a^{(i)} A_0 + a_1^{(i)} A_1 + \ldots + a_{N-1}^{(i)} A_{N-1}, \quad i = 1,2,\ldots,I. \tag{3.5}$$

Similarly, let

$$B_h = \frac{1}{N-h}(y_1 y_{1+h} + y_2 y_{2+h} + \ldots + y_{N-h} y_N), \tag{3.6}$$
$$h = 0, 1, 2, \ldots, N-1$$

and take as a class of estimators for the B_i, $i = 1,2,\ldots, I$ the quadratic forms

$$Q_{\beta_i} = b_0^{(i)} B_0 + b_1^{(i)} B_1 + \ldots + b_{N-1}^{(i)} B_{N-1}, \quad i = 1,2,\ldots,I. \tag{3.7}$$

Let

$$C_0 = \frac{1}{N}[x_1y_1 + x_2y_2 + \dots + x_Ny_N]$$

$$C_1 = \frac{1}{2(N-1)}[(x_1y_2+x_2y_1)+(x_2y_3+x_3y_2)+\dots+(x_{N-1}y_N+x_Ny_{N-1})]$$

(3.8)

$$C_h = \frac{1}{2(N-h)}[(x_1y_{1+h}+x_{1+h}y_1)+\dots+(x_{N-h}y_N+x_Ny_{N-h})]$$

$$C_{N-1} = \frac{1}{2}[x_1y_N + x_Ny_1]$$

and take as a class of estimators for the γ_i, $i=1,2,\dots,I$ the bilinear forms

$$(3.9)\quad Q_{\gamma_i} = c_0^{(i)} C_0 + c_1^{(i)} C_1+\dots+c_{N-1}^{(i)} C_{N-1},\ i = 1,2,\ \dots,\ I.$$

Finally, let

$$(3.10)\ D_h = \frac{1}{2(N-h)}[(x_1y_{1+h}-x_{1+h}y_1)+\dots+(x_{N-h}y_N-x_Ny_{N-h})]$$

$$h = 1,2,\ \dots,\ N-1$$

and take as a class of estimators for the δ_i, $i=1,2,\dots,I$ the bilinear forms

$$(3.11)\quad Q_{\delta_i} = d_1^{(i)} D_1+d_2^{(i)} D_2+\dots+d_{N-1}^{(i)}D_{N-1},\ i = 1,2,\dots,I.$$

One observes that the matrices of the forms $Q_{\alpha_i}, Q_{\beta_i}, Q_{\gamma_i}, Q_{\delta_i}$, $i = 1,2,\dots,$ I introduced above are finite Laurent matrices.

3.3 Mean Values of the Estimators

There now remains the problem of how to choose the constants

$$
(3.12)\qquad
\begin{array}{l}
a_0^{(i)},\ a_1^{(i)},\ \ldots,\ a_{N-1}^{(i)} \\
b_0^{(i)},\ b_1^{(i)},\ \ldots,\ b_{N-1}^{(i)} \\
c_0^{(i)},\ c_1^{(i)},\ \ldots,\ c_{N-1}^{(i)} \\
\qquad d_1^{(i)},\ \ldots,\ d_{N-1}^{(i)} \qquad i = 1,2,\ldots,I
\end{array}
$$

so that "good" estimators for the $(\alpha_i,\beta_i,\gamma_i,\delta_i)$, $i=1,2,\ldots,I$ are obtained. It would be desirable to choose the constants (3.12) so that

$$(3.13)\qquad E\,Q_{\alpha_i} = \alpha_i;\ E\,Q_{\beta_i} = \beta_i;\ E\,Q_{\gamma_i} = \gamma_i;\ E\,Q_{\delta_i} = \delta_i .$$

Unfortunately, as will be seen by (3.25) and (3.28) there is no choice of the constants (3.12) that makes this possible.

In order to obtain certain simple and useful expressions for the expectations $E\,Q_{\alpha_i}, E\,Q_{\beta_i}, E\,Q_{\gamma_i}, E\,Q_{\delta_i}$, $i=1,2,\ldots,I$, expressions which will facilitate selecting the constants (3.12), it is first necessary to establish the following formulae:

$$(3.14.1)\qquad E\,x_k x_{k+h} = \int_{-\pi}^{\pi} \cos \lambda h\, f_x(\lambda)\, d\lambda,$$

$$(3.14.2)\qquad E\, y_k y_{k+h} = \int_{-\pi}^{\pi} \cos \lambda h\, f_y(\lambda)d\lambda,$$

$$(3.14.3)\qquad E\, \tfrac{1}{2}\,[x_k y_{k+h} + x_{k+h} y_k] = \int_{\pi}^{\pi} \cos \lambda h\, c(\lambda)d\lambda,$$

$$(3.14.4)\qquad E\, \tfrac{1}{2}\,[x_k y_{k+h} - x_{k+h} y_k] = \int_{-\pi}^{\pi} \sin \lambda h\, q(\lambda)d\lambda.$$

The formulas (3.14) follow essentially from the definition (2.46) of the covariance function and (2.50). From (2.46), (2.50), and (2.53)

$$(3.15)\qquad E\, x_k x_{k+h} = \int_{-\pi}^{\pi} e^{ih\lambda} f_x(\lambda)d\lambda = E\, x_k x_{k-h}$$

so that

$$(3.16)\qquad E\, x_k x_{k+h} = \int_{-\pi}^{\pi} \cos \lambda h\, f_x(\lambda)d\lambda.$$

Similarly,

$$(3.17)\qquad E\, y_k y_{k+h} = \int_{-\pi}^{\pi} \cos \lambda h\, f_y(\lambda)d\lambda.$$

From (2.46) and (2.50)

$$(3.18)\qquad E\, y_{k+h} x_k = \int_{-\pi}^{\pi} e^{ih\lambda} f_{yx}(\lambda)d\lambda.$$

Also by virtue of (2.51) and (2.53)

(3.19) $$E\, y_k x_{k+h} = \int_{-\pi}^{\pi} e^{ih\lambda} f_{xy}(\lambda)d\lambda = \int_{-\pi}^{\pi} e^{-ih\lambda} f_{xy}(-\lambda)d\lambda$$

$$= \int_{-\pi}^{\pi} e^{-ih\lambda}\overline{f_{xy}(\lambda)}d\lambda = \int_{-\pi}^{\pi} e^{-ih\lambda} f_{yx}(\lambda)d\lambda.$$

Thus,

(3.20) $$E\, \tfrac{1}{2}[y_{k+h}x_k + y_k x_{k+h}] = \int_{-\pi}^{\pi} \cos\lambda h\, f_{yx}(\lambda)d\lambda$$

$$= \int_{-\pi}^{\pi} (\cos\lambda h)[c(\lambda)-iq(\lambda)]d\lambda = \int_{-\pi}^{\pi} \cos\lambda h\, c(\lambda)d\lambda$$

since $q(\lambda)$ is an odd function.

Similarly,

(3.21) $$E\, \tfrac{1}{2}[y_{k+h}x_k - y_k x_{k+h}] = \int_{-\pi}^{\pi} i \sin\lambda h\, f_{yx}(\lambda)d\lambda$$

$$= \int_{-\pi}^{\pi} i \sin\lambda h\, [c(\lambda)-iq(\lambda)]d\lambda = \int_{-\pi}^{\pi} \sin\lambda h\, q(\lambda)d\lambda$$

since $c(\lambda)$ is an even function.

From (3.14) and the defining equations for the A_h, B_h, C_h and D_h, one observes that

(3.22.1) $$E\, A_h = \int_{-\pi}^{\pi} \cos\lambda h\, f_x(\lambda)d\lambda,$$

(3.22.2) $$E\, B_h = \int_{-\pi}^{\pi} \cos\lambda h\, f_y(\lambda)d\lambda,$$

$$(3.22.3)\quad E\,C_h = \int_{-\pi}^{\pi} \cos \lambda h \; c(\lambda)d\lambda,$$

$$(3.22.4)\quad E\,D_h = \int_{-\pi}^{\pi} \sin \lambda h \; q(\lambda)d\lambda.$$

Thus,

$$(3.23)\quad E\,Q_{\alpha_i} = E\sum_{h=0}^{N-1} a_h^{(i)} A_h = \sum_{h=0}^{N-1} a_h^{(i)} \int_{-\pi}^{\pi} \cos \lambda h \; f_x(\lambda)d\lambda$$

$$= \int_{-\pi}^{\pi} \Big(\sum_{h=0}^{N-1} a_h^{(i)} \cos \lambda h\Big) f_x(\lambda)d\lambda.$$

Thus,

$$(3.23.1)\quad E\,Q_{\alpha_i} = \int_{-\pi}^{\pi} P_a^{(i)}(\lambda) f_x(\lambda)d\lambda \quad \text{where}$$

$$P_a^{(i)}(\lambda) = \sum_{h=0}^{N-1} a_h^{(i)} \cos \lambda h.$$

Similarly,

$$(3.23.2)\quad E\,Q_{\beta_i} = \int_{-\pi}^{\pi} P_b^{(i)}(\lambda) f_y(\lambda)d\lambda \quad \text{where}$$

$$P_b^{(i)}(\lambda) = \sum_{h=0}^{N-1} b_h^{(i)} \cos \lambda h.$$

$$(3.23.3)\quad E\,Q_{\gamma_i} = \int_{-\pi}^{\pi} P_c^{(i)}(\lambda)\; c(\lambda)d\lambda \quad \text{where}$$

$$P_c^{(i)}(\lambda) = \sum_{h=0}^{N-1} c_h^{(i)} \cos \lambda h.$$

(3.23.4) $$E\,Q_{\delta_i} = \int_{-\pi}^{\pi} P_d^{(i)}(\lambda)\, q(\lambda)d\lambda \quad \text{where}$$

$$P_d^{(i)}(\lambda) = \sum_{h=1}^{N-1} d_h^{(i)} \sin \lambda h.$$

The equations (3.23) are to be considered as spectral representations for the expectation of the quadratic and bilinear forms introduced as estimators. For example, consider equation (3.23.4). One observes from equation (3.11) defining the Q_{δ_i} and equation (3.23.4) defining the $P_d^{(i)}(\lambda)$ that there is a one-to-one correspondence between the quadratic forms Q_{δ_i} and the trigonometric polynomials $P_d^{(i)}(\lambda)$, each being determined by the constants $d_0^{(i)}, d_1^{(i)}, \ldots, d_{N-1}^{(i)}$. Furthermore, the expectation of the quadratic form Q_{δ_i} is very simply expressed in terms of the associated trigonometric polynomial $P_d^{(i)}(\lambda)$, being an integral of the product of $P_d^{(i)}(\lambda)$ and the quadrature-spectral density $q(\lambda)$. Similar comments hold for the forms Q_{α_i}, Q_{β_i}, Q_{γ_i} and their associated trigonometric polynomials $P_a^{(i)}(\lambda)$, $P_b^{(i)}(\lambda)$, $P_c^{(i)}(\lambda)$ respectively. From equations (3.23) one gains some insight on how to choose the constants (3.12) or equivalently the trigonometric polynomials $P_a^{(i)}(\lambda)$, $P_b^{(i)}(\lambda)$, $P_c^{(i)}(\lambda)$, $P_d^{(i)}(\lambda)$, $i = 1, \ldots, I$. Take as a typical case the problem of how to choose $P_d^{(i)}(\lambda)$ so as to estimate δ_i. From (3.3.4), $\delta_i = \int_{\lambda_{i-1}}^{\lambda_i} q(\lambda)d\lambda$ and $0 < \lambda_{i-1} < \lambda_i \leq \pi$.

If it were possible to choose the constants $d_h^{(i)}$, $h=1,\ldots,N-1$ so that

$$(3.24)\quad P_d^{(i)}(\lambda) = \begin{cases} \tfrac{1}{2} \text{ for } \lambda_{i-1} \leq \lambda \leq \lambda_i \\ 0 \text{ for } 0 \leq \lambda < \lambda_{i-1} \text{ or } \lambda_i < \lambda \leq \pi, \end{cases}$$

then since $P_d^{(i)}(\lambda)$ is necessarily an odd function, $P_d^{(i)}(\lambda) = -P_d^{(i)}(-\lambda)$ for $\lambda < 0$, the associated bilinear form Q_{δ_i} would satisfy

$$(3.25)\quad E\, Q_{\delta_i} = \int_{-\pi}^{\pi} P_d^{(i)}(\lambda)\, q(\lambda)d\lambda = \tfrac{1}{2}\int_{\lambda_{i-1}}^{\lambda_i} q(\lambda)d\lambda - \tfrac{1}{2}\int_{-\lambda_i}^{-\lambda_{i-1}} q(\lambda)d\lambda$$

$$= \int_{\lambda_{i-1}}^{\lambda_i} q(\lambda)d\lambda = \delta_i,$$

i.e. Q_{δ_i} would then be an unbiased estimator of δ_i for every $q(\lambda)$. The function defined by (3.24) has the infinite Fourier series expansion

$$(3.25)\quad \sum_{h=1}^{\infty} \frac{2}{\pi h}(\cos \lambda_{i-1}h - \cos \lambda_i h)\sin \lambda h$$

and hence can not be exactly represented by any finite trigonometric polynomial of the form $P_d^{(i)}(\lambda) = \sum_{h=1}^{N-1} d_h^{(i)} \sin \lambda h$. Thus, all estimators Q_{α_i} of the α_{ii} are in general (i.e. without specific assumptions on $q(\lambda)$ biased. Similarly, the ideal $P_a^{(i)}(\lambda)$, $P_b^{(i)}(\lambda)$, and $P_c^{(i)}(\lambda)$ are equal to $P^{(i)}(\lambda)$ where

$$(3.26) \qquad P^{(i)}(\lambda) = \begin{cases} \frac{1}{2} \text{ for } \lambda_{i-1} \leq \lambda \leq \lambda_i \\ 0 \text{ for } 0 \leq \lambda < \lambda_{i-1} \text{ or } \lambda_i < \lambda \leq \pi \end{cases}$$

$$P^{(i)}(\lambda) = P^{(i)}(-\lambda) \text{ for } -\pi \leq \lambda \leq 0,$$

since with $P^{(i)}(\lambda)$ for $P_a^{(i)}(\lambda)$, $P_b^{(i)}(\lambda)$, $P_c^{(i)}(\lambda)$

$$E\, Q_{\alpha_i} = \int_{-\pi}^{\pi} P^{(i)}(\lambda)\, f_{xx}(\lambda)d\lambda = \alpha_i$$

$$(3.27) \qquad E\, Q_{\beta_i} = \int_{-\pi}^{\pi} P^{(i)}(\lambda)\, f_{yy}(\lambda)d\lambda = \beta_i$$

$$E\, Q_{\gamma_i} = \int_{-\pi}^{\pi} P^{(i)}(\lambda)\, c(\lambda)d\lambda = \gamma_i.$$

The function $P^{(i)}(\lambda)$ given by (3.26) has the infinite Fourier series expansion

$$(3.28) \qquad \frac{(\lambda_i - \lambda_{i-1})}{\pi} + \sum_{h=1}^{\infty} \frac{2}{\pi h}(\sin \lambda_i h - \sin \lambda_{i-1} h) \cos \lambda h$$

and hence can not be exactly represented by any finite trigonometric polynomial of the form $\sum_{h=0}^{N-1} p_h^{(i)} \cos \lambda h$. Thus, in general, the Q_{α_i}, Q_{β_i}, Q_{γ_i} are biased estimators for α_i, β_i, γ_i respectively.

From the foregoing discussion it is clear that if the Q_{α_i}, Q_{β_i}, Q_{γ_i}, Q_{δ_i}, $i=1, \ldots, I$ are to serve as reasonable estimators for α_i, β_i, γ_i, δ_i, $i=1, \ldots, I$ respectively, the trigonometric polynomials $P_a^{(i)}(\lambda)$, $P_b^{(i)}(\lambda)$, $P_c^{(i)}(\lambda)$, and $P_d^{(i)}(\lambda)$, $i=1, \ldots, I$ must reasonably approximate the ideal functions (3.26) and (3.24) respectively. Explicit expressions for certain "good" polynomial filters are derived in Section 3.5.

3.4 Variances and Covariances of the Estimators

In addition to the criterion of unbiasedness discussed in Section 3.3, the following should also be considered as ideal criteria in selecting the estimators Q_{α_i}, Q_{β_i}, Q_{γ_i}, Q_{δ_i}, $i=1,2, \ldots, I$:

(a) minimum variability - i.e., the variability of the estimators Q_{α_i}, Q_{β_i}, Q_{γ_i}, Q_{δ_i}, $i=1,2,\ldots,I$ should be a minimum, and

(b) no covariability between estimators pertaining to different frequency bands - i.e., the estimators Q_{α_i}, Q_{β_i}, Q_{γ_i}, Q_{δ_i}, and $Q_{\alpha_{i'}}$, $Q_{\beta_{i'}}$, $Q_{\gamma_{i'}}$, $Q_{\delta_{i'}}$ for $i \neq i'$ should be independent.

In order to investigate the variability and covariability of Q_{α_i}, Q_{β_i}, Q_{γ_i}, Q_{δ_i}, $i=1,2, \ldots, I$ it is necessary to establish formulae for the variances and covariances of these estimators. It would be desirable to have formulae for these variances and covariances similar to the spectral representation formulae (3.23) for the expectations, as such spectral representation formulae are particularly simple and concise, and indicate the functional dependence on the spectra, cospectrum and quadrature spectrum. Spectral representation formulae for the variances and covariances of the Q_{α_i}, Q_{β_i}, Q_{γ_i}, Q_{δ_i}, $i=1,2, \ldots, I$ are possible. However, if the Q_{α_i}, Q_{β_i}, Q_{γ_i}, Q_{δ_i}, $i=1,2,\ldots,I$ are replaced by certain modified forms $\hat{Q}_{\alpha_i}$, $\hat{Q}_{\beta_i}$, $\hat{Q}_{\gamma_i}$, $\hat{Q}_{\delta_i}$, $i=1, \ldots, I$, the spectral representation formulae (3.23) still hold, and the spectral representation formulae for the variances and covariances are particularly simple. The modified forms $\hat{Q}_{\alpha_i}$, $\hat{Q}_{\beta_i}$, $\hat{Q}_{\gamma_i}$, $\hat{Q}_{\delta_i}$, $i=1,2, \ldots, I$, are studied below and the spectral representation formulae for the variances and covariances of these forms derived.

It should be mentioned that only the assumption of weak stationarity was used to derive the expectation formulae (3.23), whereas in the derivation of the spectral representation of the variances and covariances of the $\hat{Q}_{\alpha_i}$, $\hat{Q}_{\beta_i}$, $\hat{Q}_{\gamma_i}$, $\hat{Q}_{\delta_i}$, $i=1,2,\ldots,I$ the assumption that (3.2) is a sample from a Gaussian process is used. The Gaussian assumption is used to express fourth moments in terms of second moments by means of the theorem of Isserlis which states that if z_1, z_2, z_3, z_4, have a joint Gaussian distribution with zero means, then

(3.29) $\quad \mathrm{Cov}(z_1z_2, z_3z_4) = \mathrm{Cov}(z_1z_3)\mathrm{Cov}(z_2z_4) + \mathrm{Cov}(z_1z_4)\mathrm{Cov}(z_2z_3)$

where

(3.30) $\quad \mathrm{Cov}(u,v) = E[(u-Eu)(v-Ev)] = E(uv)-(Eu)(Ev).$

The theorem of Isserlis is proved as follows: The characteristic function of the joint distribution of z_1, z_2, z_3, z_4, is by (2.7)

(3.31) $\quad E\, e^{i(t_1z_1+t_2z_2+t_3z_3+t_4z_4)} = \exp -\tfrac{1}{2} \sum_{m,n=1}^{4} \mu_{mn} t_m t_n$

where $E\, z_m z_n = \mu_{mn}$, $m,n = 1,\ldots,4$.

Upon equating the coefficients of $t_1t_2t_3t_4$ in the expansions of the right and left hand members of (3.31) one obtains

$$(3.32)\quad E(z_1z_2z_3z_4) = \mu_{12}\mu_{34} + \mu_{13}\mu_{24} + \mu_{14}\mu_{23}.$$

But,

$$(3.33)\quad \begin{aligned} \mathrm{Cov}(z_1z_2, z_3z_4) &= E(z_1z_2z_3z_4) - E(z_1z_2)\,E(z_3z_4) \\ &= E(z_1z_2z_3z_4)-\mu_{12}\mu_{34}=\mu_{13}\mu_{24}+\mu_{14}\mu_{23} \\ &=\mathrm{Cov}(z_1z_3)\mathrm{Cov}(z_2z_4)+\mathrm{Cov}(z_1z_4)\mathrm{Cov}(z_2z_3). \end{aligned}$$

In introducing the $\hat{Q}_{\alpha_i}$, $\hat{Q}_{\beta_i}$, $\hat{Q}_{\gamma_i}$, $\hat{Q}_{\delta_i}$, $i=1,2,\ldots,I$ and in deriving spectral representation formulas for the variances and covariances of these forms it is convenient to make the following change of notation. A translation of time is made so that time zero occurs at the center of the record, so that (assuming N to be odd) the finite sample (3.2) is then denoted by

$$(3.34)\quad x_j, y_j\ ,\ j = -\tfrac{1}{2}(N-1), \ldots, 0, 1, \ldots, \tfrac{1}{2}(N-1).$$

The forms $\hat{A}_h$, $\hat{B}_h$, $\hat{C}_h$, $\hat{D}_h$ are then defined as follows:

$$(3.35.1)\quad \hat{A}_h = \frac{1}{2M+1}\sum_{j=-M}^{M} x_{j-\frac{1}{2}h}\,x_{j+\frac{1}{2}h},$$

(3.35.2) $\hat{B}_h = \frac{1}{2M+1} \sum_{j=-M}^{M} y_{j-\frac{1}{2}h}\, y_{j+\frac{1}{2}h},$

(3.35.3) $\hat{C}_h = \frac{1}{2M+1} \sum_{j=-M}^{M} \tfrac{1}{2}(x_{j-\frac{1}{2}h} y_{j+\frac{1}{2}h} + x_{j+\frac{1}{2}h} y_{j-\frac{1}{2}h}),$

(3.35.4) $\hat{D}_h = \frac{1}{2M+1} \sum_{j=-M}^{M} \tfrac{1}{2}(x_{j-\frac{1}{2}h} y_{j+\frac{1}{2}h} - x_{j+\frac{1}{2}h} y_{j-\frac{1}{2}h}),$

for $h = 0,1,2, \ldots, N-2M-1$;

and

$\hat{A}_h = \hat{B}_h = \hat{C}_h = \hat{D}_h = 0$ for $h > N-2M-1$, where $M < N$.

The following remarks pertaining to (3.35) should be noted:

(a) When h is odd, $j \pm \frac{1}{2}h$ is half-integral so that $[x_{j\pm\frac{1}{2}h}, y_{j\pm\frac{1}{2}h}]$ refer to observations at times mid-way between the actual sampling times $k\Delta t$ of equation (3.1).

(b) All the $\hat{A}_h, \hat{B}_h, \hat{C}_h, \hat{D}_h$, $h=0,1,2,\ldots,N-2M-1$ are averages of the same number $(2M+1)$ of lagged-products, whereas more lagged-products are available when $h < N-2M-1$; also lagged products for $h > N-2M-1$ are neglected.

The $\hat{Q}_{\alpha_i}, \hat{Q}_{\beta_i}, \hat{Q}_{\gamma_i}, \hat{Q}_{\delta_i}$, $i=1,2,\ldots,I$ are defined in terms of the $\hat{A}_h, \hat{B}_h, \hat{C}_h, \hat{D}_h$, respectively by

(3.36.1) $$\hat{Q}_{\alpha_i} = \sum_{h=0}^{N-2M-1} a_h^{(i)} \hat{A}_h,$$

(3.36.2) $$\hat{Q}_{\beta_i} = \sum_{h=0}^{N-2M-1} b_h^{(i)} \hat{B}_h,$$

(3.36.3) $$\hat{Q}_{\gamma_i} = \sum_{h=0}^{N-2M-1} c_h^{(i)} \hat{C}_h,$$

(3.36.4) $$\hat{Q}_{\delta_i} = \sum_{h=0}^{N-2M-1} d_h^{(i)} \hat{D}_h.$$

It is desired to represent $\mathrm{Cov}(\hat{A}_m,\hat{A}_n)$, $\mathrm{Cov}(\hat{A}_m,\hat{B}_n)$,...., $\mathrm{Cov}(\hat{D}_m,\hat{D}_n)$ as follows:

(3.37)

(aa) $$\mathrm{Cov}(\hat{A}_m,\hat{A}_n) = \int_0^{\pi} \cos m\lambda \cos n\lambda \, V_{aa}(\lambda)d\lambda,$$

(ab) $$\mathrm{Cov}(\hat{A}_m,\hat{B}_n) = \int_0^{\pi} \cos m\lambda \cos n\lambda \, V_{ab}(\lambda)d\lambda,$$

(ac) $$\mathrm{Cov}(\hat{A}_m,\hat{C}_n) = \int_0^{\pi} \cos m\lambda \cos n\lambda \, V_{ac}(\lambda)d\lambda,$$

(ad) $$\mathrm{Cov}(\hat{A}_m,\hat{D}_n) = \int_0^{\pi} \cos m\lambda \sin n\lambda \, V_{ad}(\lambda)d\lambda,$$

(bb) $$\mathrm{Cov}(\hat{B}_m,\hat{B}_n) = \int_0^{\pi} \cos m\lambda \cos n\lambda \, V_{bb}(\lambda)d\lambda,$$

(3.37) cont'd

$$\text{(bc)}\quad \operatorname{Cov}(\hat{B}_m,\hat{C}_n) = \int_0^{\pi} \cos m\lambda \cos n\lambda\; V_{bc}(\lambda)d\lambda,$$

$$\text{(bd)}\quad \operatorname{Cov}(\hat{B}_m,\hat{D}_n) = \int_0^{\pi} \cos m\lambda \sin n\lambda\; V_{bd}(\lambda)d\lambda,$$

$$\text{(cc)}\quad \operatorname{Cov}(\hat{C}_m,\hat{C}_n) = \int_0^{\pi} \cos m\lambda \cos n\lambda\; V_{cc}(\lambda)d\lambda,$$

$$\text{(cd)}\quad \operatorname{Cov}(\hat{C}_m,\hat{D}_n) = \int_0^{\pi} \cos m\lambda \sin n\lambda\; V_{cd}(\lambda)d\lambda,$$

$$\text{(dd)}\quad \operatorname{Cov}(\hat{D}_m,\hat{D}_n) = \int_0^{\pi} \sin m\lambda \sin n\lambda\; V_{dd}(\lambda)d\lambda.$$

For, with the representation (3.37), the covariances $\operatorname{Cov}(\hat{Q}_{\alpha_i},\hat{Q}_{\alpha_j})$, $\operatorname{Cov}(\hat{Q}_{\alpha_i},\hat{Q}_{\beta_j})$, $\operatorname{Cov}(\hat{Q}_{\alpha_i},\hat{Q}_{\gamma_j})$, ..., $\operatorname{Cov}(\hat{Q}_{\gamma_i},\hat{Q}_{\delta_j})$, $\operatorname{Cov}(\hat{Q}_{\delta_i},\hat{Q}_{\delta_j})$ can be expressed as

$$\text{(aa)}\quad \operatorname{Cov}(\hat{Q}_{\alpha_i},\hat{Q}_{\alpha_j}) = \sum_{m,n=0}^{N-2M-1} a_m^{(i)} a_n^{(j)} \operatorname{Cov}(\hat{A}_m,\hat{A}_n)$$

$$= \sum_{m,n=0}^{N-2M-1} a_m^{(i)} a_n^{(j)} \int_0^{\pi} \cos m\lambda \cos n\lambda\; V_{aa}(\lambda)d\lambda$$

$$= \int_0^{\pi} \hat{P}_a^{(i)}(\lambda)\, \hat{P}_a^{(j)}(\lambda)\, V_{aa}(\lambda)d\lambda \quad\text{where}$$

$$\hat{P}_a^{(k)}(\lambda) = \sum_{m=0}^{N-2M-1} a_m^{(k)} \cos m\lambda.$$

Similarly,

(3.38)

(ab) $\mathrm{Cov}(\hat{Q}_{\alpha_i}, \hat{Q}_{\beta_j}) = \int_0^\pi \hat{P}_a^{(i)}(\lambda)\ \hat{P}_b^{(j)}(\lambda)\ V_{ab}(\lambda)d\lambda,$

(ac) $\mathrm{Cov}(\hat{Q}_{\alpha_i}, \hat{Q}_{\gamma_j}) = \int_0^\pi \hat{P}_a^{(i)}(\lambda)\ \hat{P}_c^{(j)}(\lambda)\ V_{ac}(\lambda)d\lambda,$

(ad) $\mathrm{Cov}(\hat{Q}_{\alpha_i}, \hat{Q}_{\delta_j}) = \int_0^\pi \hat{P}_a^{(i)}(\lambda)\ \hat{P}_d^{(j)}(\lambda)\ V_{ad}(\lambda)d\lambda,$

(bb) $\mathrm{Cov}(\hat{Q}_{\beta_i}, \hat{Q}_{\beta_j}) = \int_0^\pi \hat{P}_b^{(i)}(\lambda)\ \hat{P}_b^{(j)}(\lambda)\ V_{bb}(\lambda)d\lambda,$

(bc) $\mathrm{Cov}(\hat{Q}_{\beta_i}, \hat{Q}_{\gamma_j}) = \int_0^\pi \hat{P}_b^{(i)}(\lambda)\ \hat{P}_c^{(j)}(\lambda)\ V_{bc}(\lambda)d\lambda,$

(bd) $\mathrm{Cov}(\hat{Q}_{\beta_i}, \hat{Q}_{\delta_j}) = \int_0^\pi \hat{P}_b^{(i)}(\lambda)\ \hat{P}_d^{(j)}(\lambda)\ V_{bd}(\lambda)d\lambda,$

(cc) $\mathrm{Cov}(\hat{Q}_{\gamma_i}, \hat{Q}_{\gamma_j}) = \int_0^\pi \hat{P}_c^{(i)}(\lambda)\ \hat{P}_c^{(j)}(\lambda)\ V_{cc}(\lambda)d\lambda,$

(cd) $\mathrm{Cov}(\hat{Q}_{\gamma_i}, \hat{Q}_{\delta_j}) = \int_0^\pi \hat{P}_c^{(i)}(\lambda)\ \hat{P}_d^{(j)}(\lambda)\ V_{cd}(\lambda)d\lambda,$

(dd) $\mathrm{Cov}(\hat{Q}_{\delta_i}, \hat{Q}_{\delta_j}) = \int_0^\pi \hat{P}_d^{(i)}(\lambda)\ \hat{P}_d^{(j)}(\lambda)\ V_{dd}(\lambda)d\lambda,$

where

$$\hat{P}_b^{(k)}(\lambda) = \sum_{m=0}^{N-2M-1} b_m^{(k)} \cos m\lambda,$$

$$\hat{P}_c^{(k)}(\lambda) = \sum_{m=0}^{N-2M-1} c_m^{(k)} \cos m\lambda,$$

and $\hat{P}_d^{(k)}(\lambda) = \sum_{m=0}^{N-2M-1} d_m^{(k)} \sin m\lambda.$

Expressions for $Var(\hat{Q}_{\alpha_i})$, $Var(\hat{Q}_{\beta_i})$, $Var(\hat{Q}_{\gamma_i})$, $Var(\hat{Q}_{\delta_i})$ are then obtained from (3.38)(aa), (3.38)(bb), (3.38)(cc), (3.38)(dd) respectively by setting $j=i$. Thus,

(3.39.1) $Var(\hat{Q}_{\alpha_i}) = \int_0^\pi [\hat{P}_a^{(i)}(\lambda)]^2 V_{aa}(\lambda)d\lambda,$

(3.39.2) $Var(\hat{Q}_{\beta_i}) = \int_0^\pi [\hat{P}_b^{(i)}(\lambda)]^2 V_{bb}(\lambda)d\lambda.$

(3.39.3) $Var(\hat{Q}_{\gamma_i}) = \int_0^\pi [\hat{P}_c^{(i)}(\lambda)]^2 V_{cc}(\lambda)d\lambda,$

(3.39.4) $Var(\hat{Q}_{\delta_i}) = \int_0^\pi [\hat{P}_d^{(i)}(\lambda)]^2 V_{dd}(\lambda)d\lambda.$

The computations in the derivation of the covariance formulae (3.37) are somewhat lengthy and detailed and for convenience are set apart in Section 3.5. These computations show that $V_{aa}(\lambda)$, $V_{ab}(\lambda)$, $V_{ac}(\lambda)$, ... $V_{cd}(\lambda)$, $V_{dd}(\lambda)$ are given by:

(3.40)

(aa) $V_{aa}(\lambda) = \frac{8\pi}{2M+1} \int_{-(\pi-\lambda)}^{(\pi-\lambda)} K_{2M+1}(\lambda')f_x(\lambda+\lambda')f_x(\lambda-\lambda')d\lambda',$

(3.40) continued

$$\text{(ab)}\quad V_{ab}(\lambda)=\frac{8\pi}{2M+1}\int_{-(\pi-\lambda)}^{(\pi-\lambda)} K_{2M+1}(\lambda')[c(\lambda+\lambda')c(\lambda-\lambda')+q(\lambda+\lambda')q(\lambda-\lambda')]d\lambda',$$

$$\text{(ac)}\quad V_{ac}(\lambda)=\frac{8\pi}{2M+1}\int_{-(\pi-\lambda)}^{(\pi-\lambda)} K_{2M+1}(\lambda')f_x(\lambda+\lambda')c(\lambda-\lambda')d\lambda',$$

$$\text{(ad)}\quad V_{ad}(\lambda)=\frac{8\pi}{2M+1}\int_{-(\pi-\lambda)}^{(\pi-\lambda)} K_{2M+1}(\lambda')f_x(\lambda+\lambda')q(\lambda-\lambda')d\lambda',$$

$$\text{(bb)}\quad V_{bb}(\lambda)=\frac{8\pi}{2M+1}\int_{-(\pi-\lambda)}^{(\pi-\lambda)} K_{2M+1}(\lambda')f_y(\lambda+\lambda')f_y(\lambda-\lambda')d\lambda',$$

$$\text{(bc)}\quad V_{bc}(\lambda)=\frac{8\pi}{2M+1}\int_{-(\pi-\lambda)}^{(\pi-\lambda)} K_{2M+1}(\lambda')f_y(\lambda+\lambda')c(\lambda-\lambda')d\lambda',$$

$$\text{(bd)}\quad V_{bd}(\lambda)=\frac{8\pi}{2M+1}\int_{-(\pi-\lambda)}^{(\pi-\lambda)} K_{2M+1}(\lambda')f_y(\lambda+\lambda')q(\lambda-\lambda')d\lambda',$$

(3.40) continued

$$\text{(cc)}\quad V_{cc}(\lambda)=\frac{4\pi}{2M+1}\int_{-(\pi-\lambda)}^{(\pi-\lambda)} K_{2M+1}(\lambda')[f_x(\lambda+\lambda')f_y(\lambda-\lambda')$$

$$+c(\lambda+\lambda')c(\lambda-\lambda')-q(\lambda+\lambda')q(\lambda-\lambda')]d\lambda',$$

$$\text{(cd)}\quad V_{cd}(\lambda)=\frac{8\pi}{2M+1}\int_{-(\pi-\lambda)}^{(\pi-\lambda)} K_{2M+1}(\lambda')c(\lambda+\lambda')q(\lambda-\lambda')d\lambda',$$

$$\text{(dd)}\quad V_{dd}(\lambda)=\frac{4\pi}{2M+1}\int_{-(\pi-\lambda)}^{(\pi-\lambda)} K_{2M+1}(\lambda')[f_x(\lambda+\lambda')f_y(\lambda-\lambda')$$

$$+q(\lambda+\lambda')q(\lambda-\lambda')-c(\lambda+\lambda')c(\lambda-\lambda')]d\lambda',$$

where $K_N(\lambda')$ denotes the Fejer kernel

$$K_N(\lambda') = \frac{1}{\pi N}\left(\frac{\sin N\lambda'}{\sin\lambda'}\right)^2. \tag{3.41}$$

3.5 Derivation of the Covariance Formulae

In the derivation of the covariance formulae (3.37) certain summations of integrals repeatedly occur. These are listed and evaluated below.

$$(3.42.1)\quad S_1 = \Sigma\iint \cos[k-j+\tfrac{1}{2}(m-n)]\lambda_1 \cos[k-j-\tfrac{1}{2}(m-n)]\lambda_2$$

$$+ \cos[k-j-\tfrac{1}{2}(m+n)]\lambda_1 \cos[k-j+\tfrac{1}{2}(m+n)]\lambda_2\ g_1(\lambda_1)g_2(\lambda_2)d\lambda_1 d\lambda_2$$

$$= \int_0^{\pi} \cos m\lambda \cos n\lambda [8\pi(2M+1)\int_{-(\pi-\lambda)}^{(\pi-\lambda)} K_{2M+1}(\lambda')g_1(\lambda+\lambda')g_2(\lambda-\lambda')d\lambda']d\lambda.$$

$$(3.42.2)\quad S_2 = \Sigma\iint \cos[k-j+\tfrac{1}{2}(m-n)]\lambda_1 \cos[k-j-\tfrac{1}{2}(m-n)]\lambda_2$$

$$- \cos[k-j-\tfrac{1}{2}(m+n)]\lambda_1 \cos[k-j+\tfrac{1}{2}(m+n)]\lambda_2\ g_1(\lambda_1)g_2(\lambda_2)d\lambda_1 d\lambda_2$$

$$= \int_0^{\pi} \sin m\lambda \sin n\lambda [8\pi(2M+1)\int_{-(\pi-\lambda)}^{(\pi-\lambda)} K_{2M+1}(\lambda')g_1(\lambda+\lambda')g_2(\lambda-\lambda')d\lambda']d\lambda.$$

$$(3.42.3)\quad S_3 = \Sigma\iint \cos[k-j+\tfrac{1}{2}(m-n)]\lambda_1 \sin[k-j-\tfrac{1}{2}(m-n)]\lambda_2$$

$$+ \cos[k-j-\tfrac{1}{2}(m+n)]\lambda_1 \sin[k-j+\tfrac{1}{2}(m+n)]\lambda_2\ g(\lambda_1)h(\lambda_2)d\lambda_1 d\lambda_2$$

$$= \int_0^{\pi} \cos m\lambda \sin n\lambda [8\pi(2M+1)\int_{-(\pi-\lambda)}^{(\pi-\lambda)} K_{2M+1}(\lambda')g(\lambda+\lambda')h(\lambda-\lambda')d\lambda']d\lambda,$$

$$(3.42.4)\quad S_4 = \Sigma\iint \sin[k-j+\tfrac{1}{2}(m-n)]\lambda_1 \sin[k-j-\tfrac{1}{2}(m-n)]\lambda_2$$

$$+ \sin[k-j-\tfrac{1}{2}(m+n)]\lambda_1 \sin[k-j+\tfrac{1}{2}(m+n)]\lambda_2\ h_1(\lambda_1)h_2(\lambda_2)d\lambda_1 d\lambda_2$$

$$= \int_0^{\pi} \cos m\lambda \cos n\lambda [8\pi(2M+1)\int_{-(\pi-\lambda)}^{(\pi-\lambda)} K_{2M+1}(\lambda')h_1(\lambda+\lambda')h_2(\lambda-\lambda')d\lambda']d\lambda.$$

$$(3.42.5)\quad S_5 = \Sigma\!\iint \sin[k-j+\tfrac{1}{2}(m-n)]\lambda_1 \sin[k-j-\tfrac{1}{2}(m-n)]\lambda_2$$

$$-\sin[k-j-\tfrac{1}{2}(m+n)]\lambda_1 \sin[k-j+\tfrac{1}{2}(m+n)]\lambda_2 \; h_1(\lambda_1)h_2(\lambda_2)d\lambda_1 d\lambda_2$$

$$= \int_0^{\pi} \sin m\lambda \, \sin n\lambda [8\pi(2M+1)\int_{-(\pi-\lambda)}^{(\pi-\lambda)} K_{2M+1}(\lambda')h_1(\lambda+\lambda')h_2(\lambda-\lambda')d\lambda']d\lambda.$$

In (3.42) g denotes an even function, h denotes an odd function, and $\Sigma\!\iint$ denotes $\sum_{j,k=-M}^{M} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi}$.

The following trigonometric identities are needed in the proof of (3.42):

$$(3.43.1)\quad \cos A \cos B = \tfrac{1}{2}[\cos(A-B)+\cos(A+B)],$$

$$(3.43.2)\quad \sin A \sin B = \tfrac{1}{2}[\cos(A-B)-\cos(A+B)],$$

$$(3.43.3)\quad \sin A \cos B = \tfrac{1}{2}[\sin(A-B)+\sin(A+B)].$$

Case S_1

By using (3.43.1)

$$(3.44) \quad S_1 = E\iint \tfrac{1}{2}\Big[\cos[(k-j)(\lambda_1-\lambda_2)+\tfrac{1}{2}(m-n)(\lambda_1+\lambda_2)]$$
$$+ \cos[(k-j)(\lambda_1+\lambda_2)+\tfrac{1}{2}(m-n)(\lambda_1-\lambda_2)]$$
$$+ \cos[(k-j)(\lambda_1-\lambda_2)-\tfrac{1}{2}(m+n)(\lambda_1+\lambda_2)]$$
$$+ \cos[(k-j)(\lambda_1+\lambda_2)-\tfrac{1}{2}(m+n)(\lambda_1-\lambda_2)]\Big] g_1(\lambda_1) g_2(\lambda_2)\, d\lambda_1 d\lambda_2$$

Let

$$(3.45) \quad \begin{aligned} \lambda &= \tfrac{1}{2}(\lambda_1-\lambda_2) \\ \lambda' &= \tfrac{1}{2}(\lambda_1+\lambda_2) \end{aligned} \qquad \text{or} \qquad \begin{aligned} \lambda_1 &= \lambda + \lambda' \\ \lambda_2 &= -\lambda + \lambda' \end{aligned}$$

Thus,

$$(3.46) \quad d\lambda_1 d\lambda_2 = \begin{vmatrix} \frac{\partial\lambda_1}{\partial\lambda} & \frac{\partial\lambda_1}{\partial\lambda'} \\ \frac{\partial\lambda_2}{\partial\lambda} & \frac{\partial\lambda_2}{\partial\lambda'} \end{vmatrix} d\lambda d\lambda' = 2 d\lambda d\lambda'$$

and the region $-\pi \le \lambda_1, \lambda_2 \le \pi$ transforms into the region R of Fig. (1).

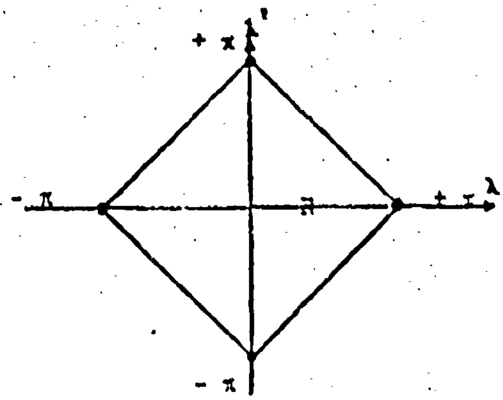


Fig. (1)

Thus,

$$(3.47)\quad S_1 = \int_R\int \Sigma\Big[\cos[2(k-j)\lambda+(m-n)\lambda'] + \cos[2(k-j)\lambda'+(m-n)\lambda]$$
$$+ \cos[2(k-j)\lambda-(m+n)\lambda']$$
$$+ \cos[2(k-j)\lambda'-(m+n)\lambda]\Big] g_1(\lambda+\lambda')g_2(\lambda-\lambda')d\lambda d\lambda'.$$

By expanding the cos[ ] terms in (3.47) and noting that terms involving $\sin 2(k-j)\lambda$ or $\sin 2(k-j)\lambda'$ vanish on summation,

$$(3.48)\quad S_1 = \int_R\int \Sigma\Big[\cos 2(k-j)\lambda\cos(m-n)\lambda' + \cos 2(k-j)\lambda'\cos(m-n)\lambda$$
$$+ \cos 2(k-j)\lambda\cos(m+n)\lambda'$$
$$+ \cos 2(k-j)\lambda'\cos(m+n)\lambda\Big] g_1(\lambda+\lambda')g_2(\lambda-\lambda')d\lambda d\lambda'.$$

But,

$$(3.49)\quad \sum_{j,k=-M}^{M} \cos 2(k-j)\lambda = \text{Real Part}\Big[\sum_{j,k=-M}^{M} e^{2i(k-j)\lambda}\Big]$$
$$= \text{Real Part}\Big[\Big(\sum_{k=-M}^{M} e^{2ik\lambda}\Big)\Big(\sum_{j=-M}^{M} e^{-2ij\lambda}\Big)\Big]$$
$$= \Big(\frac{\sin(2M+1)\lambda}{\sin\lambda}\Big)^2 = (2M+1)\pi K_{2M+1}(\lambda),$$

where $K_n(\lambda)$ denotes the Fejer kernel.

Thus, by performing the summation in (3.48) and using (3.43.1)

$$(3.50)\quad S_1 = 2\pi(2M+1)\int_R\int[K_{2M+1}(\lambda)\cos m\lambda' \cos n\lambda' + K_{2M+1}(\lambda')\cos m\lambda \cos n\lambda]g_1(\lambda+\lambda')g_2(\lambda-\lambda')d\lambda d\lambda',$$

which reduces to (3.42.1) when one considers the symmet. of the region R, and that g_1 and g_2 are even functions.

Case S_2

By the same argument as for case S_1 above

$$(3.51)\quad S_2 = \int_R\int\Sigma[\cos2(k-j)\lambda\cos(m-n)\lambda' + \cos2(k-j)\lambda'\cos(m-n)\lambda - \cos2(k-j)\lambda\cos(m+n)\lambda' - \cos2(k-j)\lambda'\cos(m+n)\lambda]g_1(\lambda+\lambda')g_2(\lambda-\lambda')d\lambda d\lambda'.$$

By performing the summation in (3.51) and using (3.43.2)

$$(3.52)\quad S_2 = 2\pi(2M+1)\int_R\int[K_{2M+1}(\lambda)\sin m\lambda' \sin n\lambda' + K_{2M+1}(\lambda')\sin m\lambda \sin n\lambda]g_1(\lambda+\lambda')g_2(\lambda-\lambda')d\lambda d\lambda',$$

which reduces to (3.42.2).

Case S_3

By using (3.43.3)

$$(3.53)\quad S_3 = \Sigma\iint \tfrac{1}{8}\Big[\sin[(k-j)(\lambda_2-\lambda_1)-\tfrac{1}{2}(m-n)(\lambda_2+\lambda_1)]$$

$$+\sin[(k-j)(\lambda_2+\lambda_1)-\tfrac{1}{2}(m-n)(\lambda_2-\lambda_1)]$$

$$+\sin[(k-j)(\lambda_2-\lambda_1)+\tfrac{1}{2}(m+n)(\lambda_2+\lambda_1)]$$

$$+\sin[(k-j)(\lambda_2+\lambda_1)+\tfrac{1}{2}(m+n)(\lambda_2-\lambda_1)]\Big]g(\lambda_1)h(\lambda_2)d\lambda_1 d\lambda_2.$$

By using the transformation (3.45)

$$(3.54)\quad S_3 = \int_R\int\Sigma\Big[-\sin[2(k-j)\lambda+(m-n)\lambda'] + \sin[2(k-j)\lambda'+(m-n)\lambda]\ .$$

$$-\sin[2(k-j)\lambda-(m+n)\lambda'$$

$$+\sin[2(k-j)\lambda'-(m+n)\lambda]\Big]g(\lambda+\lambda')h(-\lambda+\lambda')d\lambda d\lambda'.$$

By expanding the sin[ ] terms in (3.54) and noting that terms involving $\sin 2(k-j)\lambda$ or $\sin 2(k-j)\lambda'$ vanish on summation,

$$(3.55)\quad S_3 = \int_R\int\Sigma[\cos 2(k-j)\lambda\sin(m-n)\lambda' - \cos 2(k-j)\lambda'\ \sin(m-n)\lambda$$

$$-\cos 2(k-j)\lambda\sin(m+n)\lambda'$$

$$+\cos 2(k-j)\lambda'\ \sin(m+n)\lambda]g(\lambda+\lambda')h(\lambda-\lambda')d\lambda d\lambda'.$$

Thus, by performing the summation in (3.55) and using (3.43.3),

$$(3.56)\quad S_3 = 2\pi(2M+1)\int_R\int[-K_{2M+1}(\lambda)\cos m\lambda'\ \sin n\lambda'$$

$$+K_{2M+1}(\lambda')\cos m\lambda\sin n\lambda]g(\lambda+\lambda')h(\lambda-\lambda')d\lambda d\lambda'.$$

One obtains (3.42.3) from (3.56) by considering the region R and using the fact that g is even and h is odd.

Case S_4

By using (3.43.2)

$$(3.57)\quad S_4 = \Sigma\iint\tfrac{1}{2}\Big[\cos[(k-j)(\lambda_1-\lambda_2)+\tfrac{1}{2}(m-n)(\lambda_1+\lambda_2)]$$
$$-\cos[(k-j)(\lambda_1+\lambda_2)+\tfrac{1}{2}(m-n)(\lambda_1-\lambda_2)]$$
$$+\cos[(k-j)(\lambda_1-\lambda_2)-\tfrac{1}{2}(m+n)(\lambda_1+\lambda_2)]$$
$$-\cos[(k-j)(\lambda_1+\lambda_2)-\tfrac{1}{2}(m+n)(\lambda_1-\lambda_2)]\Big]h_1(\lambda_1)h_2(\lambda_2)d\lambda_1 d\lambda_2.$$

By using the transformation (3.45)

$$(3.58)\quad S_4 = \int_R\int\Sigma\Big[-\cos[2(k-j)\lambda+(m-n)\lambda']+\cos[2(k-j)\lambda'+(m-n)\lambda]$$
$$-\cos[2(k-j)\lambda-(m+n)\lambda']$$
$$+\cos[2(k-j)\lambda'-(m+n)\lambda]\Big]h_1(\lambda+\lambda')h_2(\lambda-\lambda')d\lambda d\lambda'.$$

By expanding the cos[ ] terms and noting that terms involving $\sin 2(k-j)\lambda$ or $\sin 2(k-j)\lambda'$ vanish on summation

$$(3.59)\quad S_4 = \int_R\int\Sigma[-\cos 2(k-j)\lambda\cos(m-n)\lambda'+\cos 2(k-j)\lambda'\cos(m-n)\lambda$$
$$-\cos 2(k-j)\lambda\cos(m+n)\lambda'$$
$$+\cos 2(k-j)\lambda'\cos(m+n)\lambda]h_1(\lambda+\lambda')h_2(\lambda-\lambda')d\lambda d\lambda'.$$

Case (ab)

$$(3.63)\quad \mathrm{Cov}(\hat{A}_m,\hat{B}_n)$$

$$= \mathrm{Cov}\left(\frac{1}{2M+1}\sum_{j=-M}^{M} x_{j-\frac{1}{2}m}x_{j+\frac{1}{2}m}, \frac{1}{2M+1}\sum_{k=-M}^{M} y_{k-\frac{1}{2}n}\, y_{k+\frac{1}{2}n}\right)$$

$$= \frac{1}{(2M+1)^2}\, \Sigma\, \mathrm{Cov}(x_{j-\frac{1}{2}m}\, x_{j+\frac{1}{2}m},\ y_{k-\frac{1}{2}n}\, y_{k+\frac{1}{2}n}).$$

Thus, by (3.29)

$$(3.64)\quad \mathrm{Cov}(\hat{A}_m,\hat{B}_n)$$

$$= \frac{1}{(2M+1)^2}\, \Sigma[\mathrm{Cov}(x_{j-\frac{1}{2}m}, y_{k-\frac{1}{2}n})\mathrm{Cov}(x_{j+\frac{1}{2}m}, y_{k+\frac{1}{2}n})$$

$$+ \mathrm{Cov}(x_{j-\frac{1}{2}m}, y_{k+\frac{1}{2}n})\mathrm{Cov}(x_{j+\frac{1}{2}m}, y_{k-\frac{1}{2}n})]\ .$$

From (3.14) by adding (3.14.3) and (3.14.4)

$$(3.65)\quad \mathrm{Cov}(x_k, y_{k+h}) = \int_{-\pi}^{\pi} [\cos\lambda h\; c(\lambda) + \sin\lambda h\; q(\lambda)]\, d\lambda.$$

Thus, from (3.64) if m and n are both even or both odd

$$(3.66)\quad \mathrm{Cov}\,(\hat{A}_m,\hat{B}_n) =$$

$$\frac{1}{(2M+1)^2}\,\Sigma\Bigg[\left(\int_{-\pi}^{\pi}[\cos[k-j+\tfrac{1}{2}(m-n)]\lambda_1\, c(\lambda_1)+\sin[k-j+\tfrac{1}{2}(m-n)]\lambda_1 q(\lambda_1)]d\lambda_1\right)$$

$$\cdot\left(\int_{-\pi}^{\pi}[\cos[k-j-\tfrac{1}{2}(m-n)]\lambda_2\, c(\lambda_2)+\sin[k-j-\tfrac{1}{2}(m-n)]\lambda_2 q(\lambda_2)]d\lambda_2\right)$$

$$+\left(\int_{-\pi}^{\pi}[\cos[k-j+\tfrac{1}{2}(m+n)]\lambda_1 c(\lambda_1)+\sin[k-j+\tfrac{1}{2}(m+n)]\lambda_1 q(\lambda_1)]d\lambda_1\right)$$

$$\cdot\left(\int_{-\pi}^{\pi}[\cos[k-j-\tfrac{1}{2}(m+n)]\lambda_2 c(\lambda_2)+\sin[k-j-\tfrac{1}{2}(m+n)]\lambda_2 q(\lambda_2)]d\lambda_2\right)\Bigg]\,.$$

By writing the right hand side of (3.66) as a double integral and collecting similar terms from the products, one obtains

$$(3.67)\quad \mathrm{Cov}(\hat{A}_m,\hat{B}_n) = \frac{1}{(2M+1)^2}\,\Sigma\iint[F_{jk}^{cc}(\lambda_1,\lambda_2)c(\lambda_1)c(\lambda_2)+F_{jk}^{cq}(\lambda_1,\lambda_2)c(\lambda_1)q(\lambda_2)$$

$$+F_{jk}^{qc}(\lambda_1,\lambda_2)q(\lambda_1)c(\lambda_2)+F_{jk}^{qq}(\lambda_1,\lambda_2)q(\lambda_1)q(\lambda_2)]\,d\lambda_1 d\lambda_2,$$

where

(3.68.1)

$$F_{jk}^{cc}(\lambda_1,\lambda_2) = \cos[k-j+\tfrac{1}{2}(m-n)]\lambda_1\cos[k-j-\tfrac{1}{2}(m-n)]\lambda_2+\cos[k-j+\tfrac{1}{2}(m+n)]\lambda_1\cos[k-j-\tfrac{1}{2}(m+n)]\lambda_2$$

(3.68.2)

$$F_{jk}^{cq}(\lambda_1,\lambda_2) = \cos[k-j+\tfrac{1}{2}(m-n)]\lambda_1\sin[k-j-\tfrac{1}{2}(m-n)]\lambda_2+\cos[k-j+\tfrac{1}{2}(m+n)]\lambda_1\sin[k-j-\tfrac{1}{2}(m+n)]\lambda_2$$

(3.68.3)

$$F_{jk}^{qc}(\lambda_1,\lambda_2) = \sin[k-j+\tfrac{1}{2}(m-n)]\lambda_1\cos[k-j-\tfrac{1}{2}(m-n)]\lambda_2+\sin[k-j+\tfrac{1}{2}(m+n)]\lambda_1\cos[k-j-\tfrac{1}{2}(m+n)]\lambda_2$$

(3.68.4)

$$F_{jk}^{qq}(\lambda_1,\lambda_2) = \sin[k-j+\tfrac{1}{2}(m-n)]\lambda_1\sin[k-j-\tfrac{1}{2}(m-n)]\lambda_2+\sin[k-j+\tfrac{1}{2}(m+n)]\lambda_1\sin[k-j-\tfrac{1}{2}(m+n)]\lambda_2.$$

One verifies by interchanging j and k, and λ_1 and λ_2 in $F_{jk}^{cq}(\lambda_1,\lambda_2)$ that

(3.69)

$$\Sigma\iint F_{jk}^{cq}(\lambda_1,\lambda_2)c(\lambda_1)q(\lambda_2)d\lambda_1 d\lambda_2 = -\Sigma\iint F_{jk}^{qc}(\lambda_1,\lambda_2)q(\lambda_1)c(\lambda_2)d\lambda_1 d\lambda_2.$$

Also, by (3.42.1)

(3.70)

$$\frac{1}{(2M+1)^2}\sum\iint F_{jk}^{cc}(\lambda_1,\lambda_2)c(\lambda_1)c(\lambda_2)d\lambda_1 d\lambda_2 = \int_0^{\pi}\cos m\lambda\,\cos n\lambda\left[\frac{8\pi}{2M+1}\int_{-(\pi-\lambda)}^{(\pi-\lambda)}K_{2M+1}(\lambda')c(\lambda+\lambda')c(\lambda-\lambda')d\lambda'\right]d\lambda$$

and by (3.42.4)

(3.71)

$$\frac{1}{(2M+1)^2}\sum\iint F_{jk}^{qq}(\lambda_1,\lambda_2)q(\lambda_1)q(\lambda_2)d\lambda_1 d\lambda_2 = \int_0^{\pi}\cos m\lambda\,\cos n\lambda\left[\frac{8\pi}{2M+1}\int_{-(\pi-\lambda)}^{(\pi-\lambda)}K_{2M+1}(\lambda')q(\lambda+\lambda')q(\lambda-\lambda')d\lambda'\right.$$

From (3.69), (3.70) and (3.71) one obtains (3.37)(ab) with $V_{ab}(\lambda)$ given by (3.40)(ab).

<u>Cases (aa) and (bb)</u>

For a stationary vector process whose x and y components are identical

(3.72) $\quad c(\lambda) = f_x(\lambda)$ and $q(\lambda) = 0$

as is easily seen from (2.32). Thus, Case (aa) is obtained from Case (ab) by setting $c(\lambda) = f_x(\lambda)$ and $q(\lambda)=0$. Similarly, Case (bb) is obtained from Case (ab) by setting

$f(\lambda) = f_y(\lambda)$ and $g(\lambda) = 0$.

<u>Cases (ac) and (ad)</u>

(3.73)

$$\mathrm{Cov}\left(\hat{A}_m, {\hat{C}_n \atop \hat{D}_n}\right) = \mathrm{Cov}\left(\frac{1}{2M+1}\sum x_{j-\frac{1}{2}m}x_{j+\frac{1}{2}m},\ \frac{1}{2M+1}\sum \tfrac{1}{2}(x_{k-\frac{1}{2}n}\, y_{k+\frac{1}{2}n} \pm x_{k+\frac{1}{2}n}\, y_{k-\frac{1}{2}n})\right)$$

$$= \frac{1}{2(2M+1)^2}\sum\left(\mathrm{Cov}(x_{j-\frac{1}{2}m}x_{j+\frac{1}{2}m}, x_{k-\frac{1}{2}n}y_{k+\frac{1}{2}n}) \pm \mathrm{Cov}(x_{j-\frac{1}{2}m}x_{j+\frac{1}{2}m}, x_{k+\frac{1}{2}n}y_{k-\frac{1}{2}n})\right)$$

where the notation is such that if two signs prefix a given term, the upper sign refers to $\mathrm{Cov}(\hat{A}_m,\hat{C}_n)$ and the lower to $\mathrm{Cov}(\hat{A}_n,\hat{D}_n)$.

By (3.29)

(3.74)

$$\mathrm{Cov}\left(\hat{A}_m, {\hat{C}_n \atop \hat{D}_n}\right) = \frac{1}{2(2M+1)^2}\sum\Big[\mathrm{Cov}(x_{j-\frac{1}{2}m}, x_{k-\frac{1}{2}n})\mathrm{Cov}(x_{j+\frac{1}{2}m}, y_{k+\frac{1}{2}n}) + \mathrm{Cov}(x_{j-\frac{1}{2}m}, y_{k+\frac{1}{2}n})\mathrm{Cov}(x_{j+\frac{1}{2}m}, x_{k-\frac{1}{2}n})$$

$$\pm \mathrm{Cov}(x_{j-\frac{1}{2}m}, x_{k+\frac{1}{2}n})\mathrm{Cov}(x_{j+\frac{1}{2}m}, \text{[illegible]}) \pm \mathrm{Cov}(x_{j-\frac{1}{2}m}, y_{k-\frac{1}{2}n})\mathrm{Cov}(x_{j+\frac{1}{2}m}, x_{k+\frac{1}{2}m})\Big]$$

(3.74) continued

$$= \frac{1}{2(2M+1)^2} \Sigma\iint \Big[\cos[k-j+\tfrac{1}{2}(m-n)]\lambda_1 f_x(\lambda_1)\left[\cos[k-j-\tfrac{1}{2}(m-n)]\lambda_2 c(\lambda_2)+\sin[k-j-\tfrac{1}{2}(m-n)]\lambda_2 q(\lambda_2)\right]$$

$$+\cos[k-j-\tfrac{1}{2}(m+n)]\lambda_1 f_x(\lambda_1)\left[\cos[k-j+\tfrac{1}{2}(m+n)]\lambda_2 c(\lambda_2)+\sin[k-j+\tfrac{1}{2}(m+n)]\lambda_2 q(\lambda_2)\right]$$

$$\pm\cos[k-j+\tfrac{1}{2}(m+n)]\lambda_1 f_x(\lambda_1)\left[\cos[k-j-\tfrac{1}{2}(m+n)]\lambda_2 c(\lambda_2)+\sin[k-j-\tfrac{1}{2}(m+n)]\lambda_2 q(\lambda_2)\right]$$

$$\pm\cos[k-j-\tfrac{1}{2}(m-n)]\lambda_1 f_x(\lambda_1)\left[\cos[k-j+\tfrac{1}{2}(m-n)]\lambda_2 c(\lambda_2)+\sin[k-j+\tfrac{1}{2}(m-n)]\lambda_2 q(\lambda_2)\right]\Big]d\lambda_1 d\lambda_2$$

$$= \frac{1}{2(2M+1)^2} \Sigma\iint [F_{jk}^{f_x c}(\lambda_1,\lambda_2) f_x(\lambda_1) c(\lambda_2) + F_{jk}^{f_x q}(\lambda_1,\lambda_2) f_x(\lambda_1) q(\lambda_2)]d\lambda_1 d\lambda_2,$$

where

(3.75.1)

$$F_{jk}^{f_x c}(\lambda_1,\lambda_2)=\cos[k-j+\tfrac{1}{2}(m-n)]\lambda_1\cos[k-j-\tfrac{1}{2}(m-n)]\lambda_2+\cos[k-j-\tfrac{1}{2}(m+n)]\lambda_1\cos[k-j+\tfrac{1}{2}(m+n)]\lambda_2$$

$$\pm\cos[k-j+\tfrac{1}{2}(m+n)]\lambda_1\cos[k-j-\tfrac{1}{2}(m+n)]\lambda_2\pm\cos[k-j-\tfrac{1}{2}(m-n)]\lambda_1\cos[k-j+\tfrac{1}{2}(m-n)]\lambda_2,$$

and

(3.75.2)

$$F_{jk}^{f_x q}(\lambda_1,\lambda_2) = \cos[k-j+\tfrac{1}{2}(m-n)]\lambda_1 \sin[k-j-\tfrac{1}{2}(m-n)]\lambda_2 + \cos[k-j-\tfrac{1}{2}(m+n)]\lambda_1 \sin[k-j+\tfrac{1}{2}(m+n)]\lambda_2$$

$$\pm \cos[k-j+\tfrac{1}{2}(m+n)]\lambda_1 \sin[k-j-\tfrac{1}{2}(m+n)]\lambda_2 \pm \cos[k-j-\tfrac{1}{2}(m-n)]\lambda_1 \sin[k-j+\tfrac{1}{2}(m-n)]\lambda_2.$$

For case (ac),

(3.76)

$$\Sigma F_{jk}^{f_x c}(\lambda_1,\lambda_2) = 2\Sigma\Big[\cos[k-j+\tfrac{1}{2}(m-n)]\lambda_1 \cos[k-j-\tfrac{1}{2}(m+n)]\lambda_2 + \cos[k-j-\tfrac{1}{2}(m+n)]\lambda_1 \cos[k-j+\tfrac{1}{2}(m+n)]\lambda_2\Big]$$

and

(3.77) $\Sigma F_{jk}^{f_x q}(\lambda_1,\lambda_2) = 0.$

From (3.74), (3.76),(3.77) and (3.42.1), one obtaines 3.37(ac) and 3.40(ac).

For case (ad),

(3.78) $\Sigma F_{jk}^{f_x c}(\lambda_1,\lambda_2) = 0$

and

(3.79)

$$\Sigma\, F^{f_x q}_{jk}(\lambda_1 \lambda_2) = 2\Sigma\Big[\cos[k-j+\tfrac{1}{2}(m-n)]\lambda_1 \sin[k-j-\tfrac{1}{2}(m-n)]\lambda_2 + \cos[k-j-\tfrac{1}{2}(m+n)]\lambda_1 \sin[k-j+\tfrac{1}{2}(m+n)]\lambda_2\Big]$$

One obtains (3.37)(ad) and (3.40)(ad) from (374), (3.78), (3.79), and (3.42.3).

Cases (bc) and (bd).

These are obtained by interchanging x and y in (ac) and (ad) respectively.

Cases (cc),(cd),and (dd)

The notation used is such that if three signs prefix a given term, the upper sign refers to case(cc), the middle sign to case (cd) and the lower sign to case (dd).

(3.80)

$$\mathrm{Cov}\begin{pmatrix}(\hat{C}_m,\hat{C}_n)\\(\hat{C}_m,\hat{D}_n)\\(\hat{D}_m,\hat{D}_n)\end{pmatrix} = \mathrm{Cov}\Big(\frac{1}{2(2M+1)}\Sigma[x_{j-\frac{1}{2}m}y_{j+\frac{1}{2}m} \overset{+}{\underset{-}{+}} x_{j+\frac{1}{2}m}y_{j-\frac{1}{2}m}], \frac{1}{2(2M+1)}\Sigma[x_{k-\frac{1}{2}n}y_{k+\frac{1}{2}n} \overset{+}{\underset{-}{-}} x_{k+\frac{1}{2}n}y_{k-\frac{1}{2}n}]\Big)$$

$$= \frac{1}{4(2M+1)^2}\Sigma[\mathrm{Cov}(x_{j-\frac{1}{2}m}y_{j+\frac{1}{2}m}, x_{k-\frac{1}{2}n}y_{k+\frac{1}{2}n}) \overset{+}{\underset{-}{-}} \mathrm{Cov}(x_{j-\frac{1}{2}m}y_{j+\frac{1}{2}m}, x_{k+\frac{1}{2}n}y_{k-\frac{1}{2}n})$$

$$\overset{+}{\underset{-}{+}} \mathrm{Cov}(x_{j+\frac{1}{2}m}y_{j-\frac{1}{2}m}, x_{k-\frac{1}{2}n}y_{k+\frac{1}{2}n}) \overset{+}{\underset{+}{-}} \mathrm{Cov}(x_{j+\frac{1}{2}m}y_{j-\frac{1}{2}m}, x_{k+\frac{1}{2}n}y_{k-\frac{1}{2}n})]$$

$$= \frac{1}{4(2M+1)^2} \Sigma[\mathrm{Cov}(x_{j-\frac{1}{2}m}, x_{k-\frac{1}{2}n})\mathrm{Cov}(y_{j+\frac{1}{2}m}, y_{k+\frac{1}{2}n}) + \mathrm{Cov}(x_{j-\frac{1}{2}m}, y_{k+\frac{1}{2}n})\mathrm{Cov}(y_{j+\frac{1}{2}m}, x_{k-\frac{1}{2}n})$$

$$\pm \mathrm{Cov}(x_{j-\frac{1}{2}m}, x_{k+\frac{1}{2}n})\mathrm{Cov}(y_{j+\frac{1}{2}m}, y_{k-\frac{1}{2}n}) \pm \mathrm{Cov}(x_{j-\frac{1}{2}m}, y_{k-\frac{1}{2}n})\mathrm{Cov}(y_{j+\frac{1}{2}m}, x_{k+\frac{1}{2}n})$$

$$\pm \mathrm{Cov}(x_{j+\frac{1}{2}m}, x_{k-\frac{1}{2}n})\mathrm{Cov}(y_{j-\frac{1}{2}m}, y_{k+\frac{1}{2}n}) \pm \mathrm{Cov}(x_{j+\frac{1}{2}m}, y_{k+\frac{1}{2}n})\mathrm{Cov}(y_{j-\frac{1}{2}m}, x_{k-\frac{1}{2}n})$$

$$\pm \mathrm{Cov}(x_{j+\frac{1}{2}m}, x_{k+\frac{1}{2}n})\mathrm{Cov}(y_{j-\frac{1}{2}m}, y_{k-\frac{1}{2}n}) \pm \mathrm{Cov}(x_{j+\frac{1}{2}m}, y_{k-\frac{1}{2}n})\mathrm{Cov}(y_{j-\frac{1}{2}m}, x_{k+\frac{1}{2}n})]$$

$$= \frac{1}{4(2M+1)^2} \Sigma\int\int\Big[\cos[k-j+\tfrac{1}{2}(m-n)]\lambda_1 f_x(\lambda_1)\cos[k-j-\tfrac{1}{2}(m-n)]\lambda_2 f_y(\lambda_2)$$

$$+\big[\cos[k-j+\tfrac{1}{2}(m+n)]\lambda_1 c(\lambda_1)+\sin[k-j+\tfrac{1}{2}(m+n)]\lambda_1 q(\lambda_1)\big]\big[\cos[j-k+\tfrac{1}{2}(m+n)]\lambda_2 c(\lambda_2)+\sin[j-k+\tfrac{1}{2}(m+n)]\lambda_2 q(\lambda_2)\big]$$

$$\pm \cos[k-j+\tfrac{1}{2}(m+n)]\lambda_1 f_x(\lambda_1)\cos[k-j-\tfrac{1}{2}(m+n)]\lambda_2 f_y(\lambda_2)$$

$$\pm\big[\cos[k-j+\tfrac{1}{2}(m-n)]\lambda_1 c(\lambda_1)+\sin[k-j+\tfrac{1}{2}(m-n)]\lambda_1 q(\lambda_1)\big]\big[\cos[j-k+\tfrac{1}{2}(m-n)]\lambda_2 c(\lambda_2)+\sin[j-k+\tfrac{1}{2}(m-n)]\lambda_2 q(\lambda_2)\big]$$

$$\pm \cos[k-j-\tfrac{1}{2}(m+n)]\lambda_1 f_x(\lambda_1)\cos[k-j+\tfrac{1}{2}(m+n)]\lambda_2 f_y(\lambda_2)$$

$$\pm\left[\cos[k-j-\tfrac{1}{2}(m-n)]\lambda_1 c(\lambda_1)+\sin[k-j-\tfrac{1}{2}(m-n)]\lambda_1 q(\lambda_1)\right]\left[\cos[j-k-\tfrac{1}{2}(m-n)]\lambda_2 c(\lambda_2)+\sin[j-k-\tfrac{1}{2}(m-n)]\lambda_2 q(\lambda_2)\right]$$

$$\pm\cos[k-j-\tfrac{1}{2}(m-n)]\lambda_1 f_x(\lambda_1)\cos[k-j+\tfrac{1}{2}(m-n)]\lambda_2 f_x(\lambda_2)$$

$$\pm\left[\cos[k-j-\tfrac{1}{2}(m+n)]\lambda_1 c(\lambda_1)+\sin[k-j-\tfrac{1}{2}(m+n)]\lambda_1 q(\lambda_1)\right]\left[\cos[j-k-\tfrac{1}{2}(m+n)]\lambda_2 c(\lambda_2)+\sin[j-k-\tfrac{1}{2}(m+n)]\lambda_2 q(\lambda_2)\right]\Big\}\, d\lambda_1 d\lambda_2$$

$$=\frac{1}{4(2M+1)^2}\Sigma\iint\Big[G_{jk}^{f_xf_y}(\lambda_1,\lambda_2)f_x(\lambda_1)f_y(\lambda_2)+G_{jk}^{cc}(\lambda_1,\lambda_2)c(\lambda_1)c(\lambda_2)+G_{jk}^{cq}(\lambda_1,\lambda_2)c(\lambda_1)q(\lambda_2)$$

$$+G_{jk}^{qc}(\lambda_1,\lambda_2)q(\lambda_1)c(\lambda_2)+G_{jk}^{qq}(\lambda_1,\lambda_2)q(\lambda_1)q(\lambda_2)\Big]\,d\lambda_1 d\lambda_2,$$

where

(3.81.1)

$$G_{jk}^{f_xf_y}(\lambda_1,\lambda_2)=\cos[k-j+\tfrac{1}{2}(m-n)]\lambda_1\cos[k-j-\tfrac{1}{2}(m-n)]\lambda_2\pm\cos[k-j+\tfrac{1}{2}(m+n)]\lambda_1\cos[k-j-\tfrac{1}{2}(m+n)]\lambda_2$$

$$\pm\cos[k-j-\tfrac{1}{2}(m+n)]\lambda_1\cos[k-j+\tfrac{1}{2}(m+n)]\lambda_2\pm\cos[k-j-\tfrac{1}{2}(m-n)]\lambda_1\cos[k-j+\tfrac{1}{2}(m-n)]\lambda_2\,,$$

(3.81.2)

$$G_{jk}^{cc}(\lambda_1,\lambda_2)=\cos[k-j+\tfrac{1}{2}(m+n)]\lambda_1\cos[k-j-\tfrac{1}{2}(m+n)]\lambda_2\pm\cos[k-j+\tfrac{1}{2}(m-n)]\lambda_1\cos[k-j-\tfrac{1}{2}(m-n)]\lambda_2$$

$$\pm\cos[k-j-\tfrac{1}{2}(m-n)]\lambda_1\cos[k-j+\tfrac{1}{2}(m-n)]\lambda_2\pm\cos[k-j-\tfrac{1}{2}(m+n)]\lambda_1\cos[k-j+\tfrac{1}{2}(m+n)]\lambda_2\,,$$

(3.81.3)

$$G^{cq}_{jk}(\lambda_1,\lambda_2) = \mp \cos[k-j+\tfrac{1}{2}(m+n)]\lambda_1 \sin[k-j-\tfrac{1}{2}(m+n)]\lambda_2 \mp \cos[k-j+\tfrac{1}{2}(m-n)]\lambda_1 \sin[k-j-\tfrac{1}{2}(m-n)]\lambda_2$$

$$\mp \cos[k-j-\tfrac{1}{2}(m-n)]\lambda_1 \sin[k-j+\tfrac{1}{2}(m-n)]\lambda_2 \mp \cos[k-j-\tfrac{1}{2}(m+n)]\lambda_1 \sin[k-j+\tfrac{1}{2}(m+n)]\lambda_2 ,$$

(3.81.4)

$$G^{qc}_{jk}(\lambda_1,\lambda_2) = \sin[k-j+\tfrac{1}{2}(m+n)]\lambda_1 \cos[k-j-\tfrac{1}{2}(m+n)]\lambda_2 \pm \sin[k-j+\tfrac{1}{2}(m-n)]\lambda_1 \cos[k-j-\tfrac{1}{2}(m-n)]\lambda_2$$

$$\pm \sin[k-j-\tfrac{1}{2}(m-n)]\lambda_1 \cos[k-j+\tfrac{1}{2}(m-n)]\lambda_2 \pm \sin[k-j-\tfrac{1}{2}(m+n)]\lambda_1 \cos[k-j+\tfrac{1}{2}(m-n)]\lambda_2 ,$$

(3.81.5)

$$G^{qq}_{jk}(\lambda_1,\lambda_2) = \mp \sin[k-j+\tfrac{1}{2}(m+n)]\lambda_1 s \quad \ldots -j-\tfrac{1}{2}(m+n)]\lambda_2 \mp \sin[k-j+\tfrac{1}{2}(m-n)]\lambda_1 \sin[k-j-\tfrac{1}{2}(m-n)]\lambda_2$$

$$\mp \sin[k-j-\tfrac{1}{2}(m-n)]\lambda_1 \sin[k-j+\tfrac{1}{2}(m+n)]\lambda_2 \mp \sin[k-j-\tfrac{1}{2}(m+n)]\lambda_1 \sin[k-j+\tfrac{1}{2}(m+n)]\lambda_2 .$$

For case (cc), by (3.42.1)

(3.82)

$$\frac{1}{4(2M+1)''} \Sigma\iint\left({}^{f_x f_y}_{--} \quad {}_1,\lambda_2\right) d\lambda_1 d\lambda_2 = \int_0^{\pi} \cos m\lambda \cos n\lambda \left[\frac{4\pi}{2M+1} \int_{-(\pi-\lambda)}^{(\pi-\lambda)} K_{2M+1}(\lambda')f_x(\lambda+\lambda')f_y(\lambda-\lambda')d\lambda'\right] d\lambda$$

and

(3.83)

$$\frac{1}{4(2M+1)^2}\sum\iint G_{jk}^{cc}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2=\int_0^{\pi}\cos m\lambda \cos n\lambda\Big[\frac{4\pi}{2M+1}\int_{-(\pi-\lambda)}^{(\pi-\lambda)}K_{2M+1}(\lambda')c(\lambda+\lambda')c(\lambda-\lambda')d\lambda'\Big]d\lambda$$

Furthermore,

(3.84) $$\sum G_{jk}^{cq}(\lambda_1,\lambda_2) = 0 = \sum G_{jk}^{qc}(\lambda_1,\lambda_2).$$

By (3.42.4)

(3.85)

$$\frac{1}{4(2M+1)^2}\sum\iint G_{jk}^{qq}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2=\int_0^{\pi}\cos m\lambda \cos n\lambda\Big[\frac{-4\pi}{2M+1}\int_{-(\pi-\lambda)}^{(\pi-\lambda)}K_{2M+1}(\lambda')q(\lambda+\lambda')q(\lambda-\lambda')d\lambda'\Big]d\lambda.$$

From (3.80), using (3.82),(3.83),(3.84) and (3.85) one obtains (3.37)(cc) and (3.40)(cc).

For case (cd),

(3.86) $$\sum G_{jk}^{f_x f_y}(\lambda_1,\lambda_2) = \sum G_{jk}^{cc}(\lambda_1,\lambda_2) = \sum G_{jk}^{qq}(\lambda_1,\lambda_2) = 0.$$

Also, since $G_{jk}^{qc}(\lambda_2,\lambda_1) = G_{jk}^{cq}(\lambda_1,\lambda_2)$,

(3.87)

$$\frac{1}{4(2M+1)^2}\Sigma\iint G_{jk}^{qc}(\lambda_1,\lambda_2)q(\lambda_1)c(\lambda_2)d\lambda_1 d\lambda_2 = \frac{1}{4(2M+1)^2}\Sigma\iint G_{jk}^{cq}(\lambda_1,\lambda_2)c(\lambda_1)q(\lambda_2)d\lambda_1 d\lambda_2$$

$$= \int_0^{\pi} \cos m\lambda \cos n\lambda \left[\frac{4\pi}{2M+1}\int_{-(\pi-\lambda)}^{(\pi-\lambda)} K_{2M+1}(\lambda')c(\lambda+\lambda')q(\lambda-\lambda')d\lambda'\right]d\lambda$$

by (3.42.3).

From (3.80), using (3.86) and (3.87) one obtains (3.37)(cd) and (3.40)(cd).

For case (dd), by (3.42.2)

(3.88)

$$\frac{1}{4(2M+1)^2}\Sigma\iint G_{jk}^{f_x f_y}(\lambda_1,\lambda_2)f_x(\lambda_1)f_y(\lambda_2)d\lambda_1 d\lambda_2 = \int_0^{\pi} \sin m\lambda \sin n\lambda\left[\frac{4\pi}{2M+1}\int_{-(\pi-\lambda)}^{(\pi-\lambda)} K_{2M+1}(\lambda')f_x(\lambda+\lambda')f_y(\lambda-\lambda')d\lambda'\right]d\lambda,$$

and

(3.89)

$$\frac{1}{4(2M+1)^2}\Sigma\iint G_{jk}^{cc}(\lambda_1,\lambda_2)c(\lambda_1)c(\lambda_2)d\lambda_1 d\lambda_2 = \int_0^{\pi} \sin m\lambda \sin n\lambda\left[\frac{-4\pi}{2M+1}\int_{-(\pi-\lambda)}^{(\pi-\lambda)} K_{2M+1}(\lambda')c(\lambda+\lambda')c(\lambda-\lambda')d\lambda'\right]d\lambda.$$

Furthermore,

(3.90) $$\Sigma\, G^{cq}_{jk}(\lambda_1,\lambda_2) = 0 = \Sigma\, G^{qc}_{jk}(\lambda_1,\lambda_2).$$

By (3.42.5)

(3.91)

$$\frac{1}{4(2M+1)^2}\,\Sigma\iint G^{qq}_{jk}(\lambda_1,\lambda_2)q(\lambda_1)q(\lambda_2)\,d\lambda_1 d\lambda_2 = \int_0^{\pi} \sin m\lambda \sin n\lambda \left[\frac{4\pi}{2M+1}\int_{-(\pi-\lambda)}^{(\pi-\lambda)} K_{2M+1}(\lambda')q(\lambda+\lambda')q(\lambda-\lambda')\,d\lambda'\right] d\lambda.$$

From (3.80),(3.89),(3.90) and (3.91) one obtains (3.37)(dd) and (3.40)(dd).

3.6 Polynomial Filters

As was seen in Section 3.3 and Section 3.4 there is a one-to-one correspondence between the estimators $(\hat{Q}_{\alpha_k},\hat{Q}_{\beta_k},\hat{Q}_{\gamma_k},\hat{Q}_{\delta_k})$ of $(\alpha_k,\beta_k,\gamma_k,\delta_k)$ and the trigonometric polynomials (filters) $(\hat{P}^{(k)}_a(\lambda),\ \hat{P}^{(k)}_b(\lambda),\ \hat{P}^{(k)}_c(\lambda),\ \hat{P}^{(k)}_d(\lambda))$ where

(3.92.1) $$\hat{P}^{(k)}_a(\lambda) = \sum_{h=0}^{m} a^{(k)}_h \cos \lambda h,$$

(3.92.2) $$\hat{P}^{(k)}_b(\lambda) = \sum_{h=0}^{m} b^{(k)}_h \cos \lambda h,$$

$$(3.92.3)\qquad \hat{P}_c^{(k)}(\lambda) = \sum_{h=0}^{m} c_h^{(k)} \cos \lambda h,$$

$$(3.92.4)\qquad \hat{P}_d^{(k)}(\lambda) = \sum_{h=0}^{m} d_h^{(k)} \sin \lambda h, \qquad (k = 1,2, \ldots, I)$$

and $m = N-2M-1$. The problem of how to choose the filters $(\hat{P}_a^{(k)}(\lambda), \ldots, \hat{P}_d^{(k)}(\lambda))$ so that "good" estimators for the $(\alpha_k, \beta_k, \gamma_k, \delta_k)$, $k=1,2,\ldots,I$ are obtained is now discussed. This discussion is restricted to the case where the frequency domain $0 \leq \lambda \leq \pi$ is partitioned into a set of frequency intervals $\lambda_{k-1} \leq \lambda \leq \lambda_k$, $k = 1,2,\ldots,I$ by the partition $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{I-1} < \lambda_I = \pi$ given by $0 < \frac{\pi}{2m} < 3\frac{\pi}{2m} < \ldots < (2m-3)\frac{\pi}{2m} < (2m-1)\frac{\pi}{2m} < \pi$ so that $I = m+1$ and the frequency intervals are (except for the two end intervals) of equal length. One can consider the $m+1$ frequency intervals to be centered at $0, \frac{\pi}{m}, 2\frac{\pi}{m}, \ldots, (m-1)\frac{\pi}{m}, \pi$ respectively. Consider the trigonometric sum

$$(3.93)\qquad \phi_0(\lambda) = \frac{1}{2m} \sum_{j=-m+1}^{m} e^{ij\lambda}.$$

One observes that

$$(3.94)\qquad \phi_0\left(k\frac{\pi}{m}\right) = \begin{cases} 1 \text{ for } k=0, \\ 0 \text{ for } k = -m+1, \ldots, -1, 1, \ldots, m. \end{cases}$$

Define

$$(3.95) \qquad \phi_k(\lambda) \equiv \phi_0(\lambda - k\tfrac{\pi}{m}).$$

(3.96)

$$\ell_0(\lambda) \equiv 2 \text{ Real Part } \phi_0(\lambda) = \frac{1}{m} + \frac{2}{m}\sum_{j=1}^{m-1} \cos j\lambda + \frac{1}{m}\cos m\lambda ,$$

$$\ell_k(\lambda) \equiv \text{Real Part}[\phi_k(\lambda) + \phi_{-k}(\lambda)] = \frac{1}{m} + \frac{2}{m}\sum_{j=1}^{m-1} \cos \frac{jk\pi}{m} \cos j\lambda + \frac{1}{m}\cos k\pi \cos m\lambda ,$$

$$(k = 1, 2, \ldots, m-1)$$

$$\ell_m(\lambda) \equiv 2 \text{ Real Part } \phi_0(\lambda - \pi) = \frac{1}{m} + \frac{2}{m}\sum_{j=1}^{m-1} (-)^j \cos j\lambda + \frac{1}{m}(-)^m \cos m\lambda ,$$

$$(3.97) \qquad \psi_0(\lambda) = K_1\phi_{-1}(\lambda) + K_0\,\phi_0(\lambda) + K_1\,\phi_1(\lambda).$$

Tukey[17] has shown that if K_0 and K_1 are chosen properly and $m \geq 12$, Real Part $\psi_0 = \frac{K_0}{2}\ell_0 + K_1\ell_1$ is a filter which approximates the ideal $\hat{P}_a^{(1)}(\lambda)$, $\hat{P}_b^{(1)}(\lambda)$, $\hat{P}_c^{(1)}(\lambda)$ filters given by (3.26).

"Good" choices of (K_0, K_1) are

(3.98.1) $(K_0, K_1) = (0.54, 0.23)$

(3.98.2) $(K_0, K_1) = (1/3, 1/3)$.

With (3.98.1) the filter (for m=12) is of the form sketched in Fig.(2) and with (3.98.2) the filter (for m=12) is of the form sketched in Fig.(3). It is seen that the first filter nearly vanishes outside of an interval centered at $\lambda=0$, but is not as constant within the interval as is the second filter. The second filter, however, does not as nearly vanish outside of an interval centered at $\lambda=0$ as does the first. Both filters fluctuate in sign outside of an interval of length $\frac{4\pi}{m}$ ($\frac{\pi}{3}$ for m=12) centered at $\lambda=0$.

If Real Part ψ_0 is a "good" approximation to the ideal $\hat{P}_a^{(1)}(\lambda),\ldots,\hat{P}_c^{(1)}(\lambda)$ filters, then for $k = 1,2,\ldots,m-1$,

$\frac{1}{2}$ Real Part $[\psi_0(\lambda-k\frac{\pi}{m}) + \psi_0(\lambda+k\frac{\pi}{m})]$ is a "good" approximation to the ideal $\hat{P}_a^{(k+1)}(\lambda)$, ..., $\hat{P}_c^{(k+1)}(\lambda)$ filters, and

$\frac{1}{2}$ Real Part $[\psi_0(\lambda-k\frac{\pi}{m}) - \psi_0(\lambda+k\frac{\pi}{m})]$ is a "good" approximation to the ideal $\hat{P}_d^{(k+1)}(\lambda)$ filters. A computation shows that

$$(3.99)\quad \tfrac{1}{2}\text{ Real Part }[\psi_0(\lambda-k\tfrac{\pi}{m}) + \psi_0(\lambda+k\tfrac{\pi}{m})] =$$

$$\tfrac{1}{2}\text{ Real Part}[K_1\,\phi_{-1}(\lambda-k\tfrac{\pi}{m})+K_0\,\phi_0(\lambda-k\tfrac{\pi}{m})+K_1\,\phi_1(\lambda-k\tfrac{\pi}{m})$$

$$+\,K_1\phi_{-1}(\lambda+k\tfrac{\pi}{m})+K_0\,\phi_0(\lambda+k\tfrac{\pi}{m})+K_1\,\phi_1(\lambda+k\tfrac{\pi}{m})] =$$

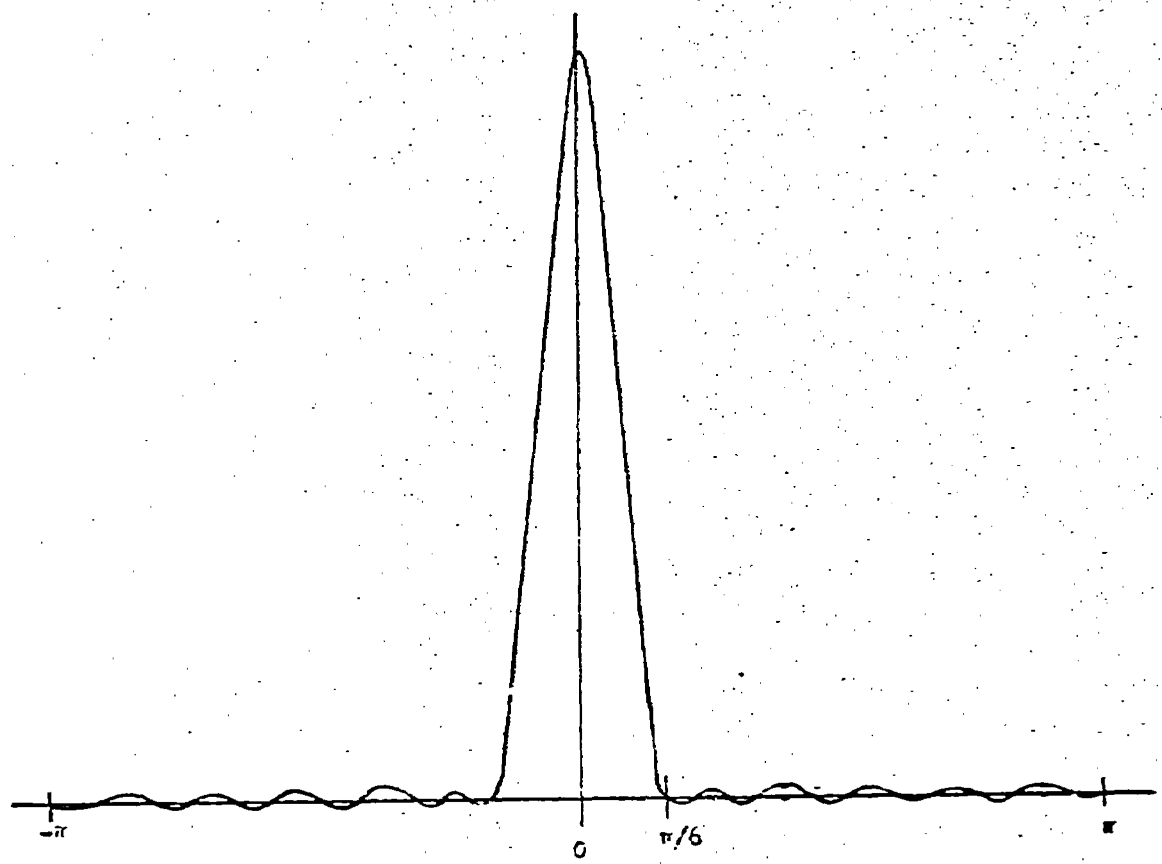


Fig.(2). Polynomial Filter $0.27\,\ell_0 + 0.23\,\ell_1$. (From Tukey[17]).

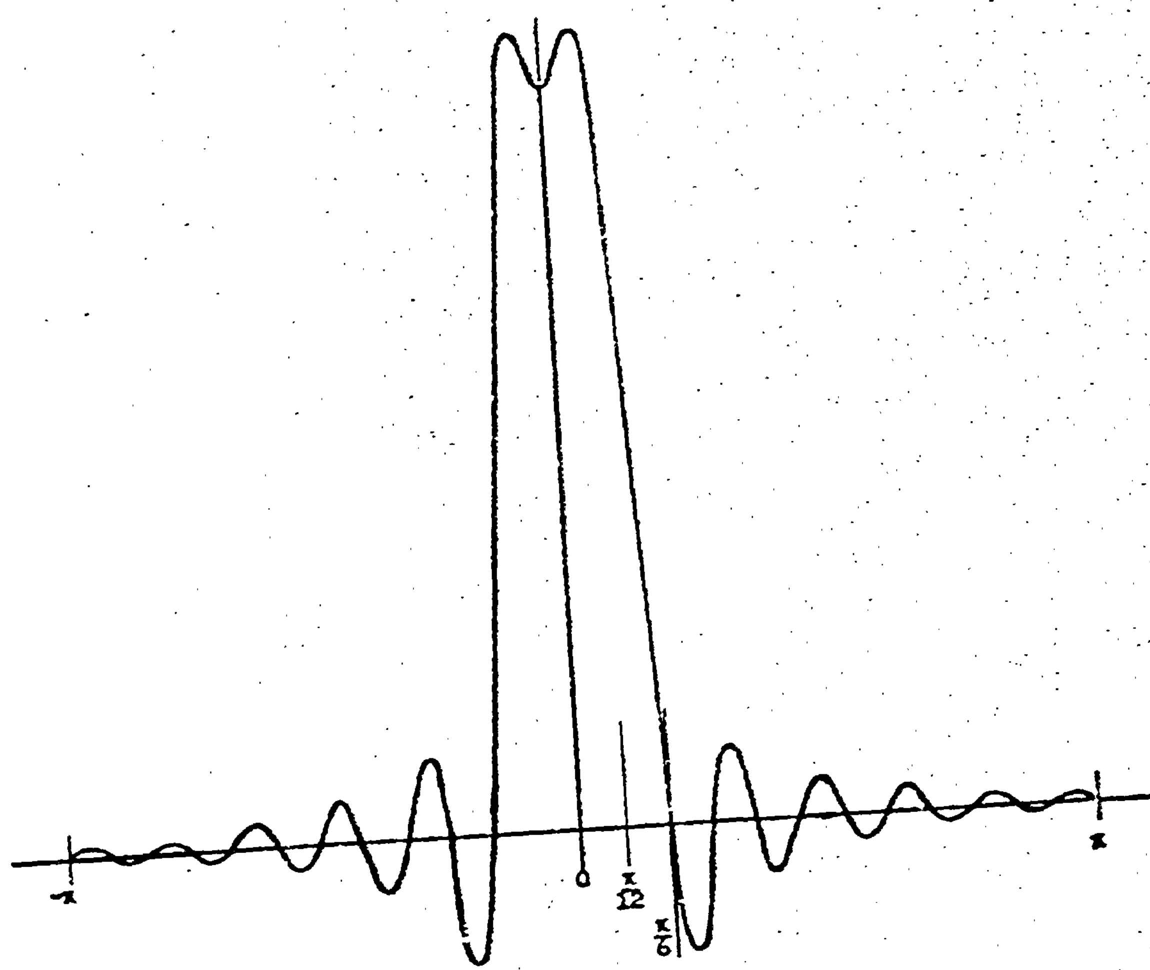


Fig.(3). Polynomial Filter $\frac{1}{6}\ell_0 + \frac{1}{3}\ell_1$. (From Tukey [17]).

$$\tfrac{1}{2}\text{ Real Part}[K_1\phi_{k-1}(\lambda)+K_0\phi_k(\lambda)+K_1\phi_{k+1}(\lambda)+K_1\phi_{-k-1}(\lambda)$$

$$+K_0\phi_{-k}(\lambda)+K_1\phi_{-k+1}(\lambda)]$$

$$=\tfrac{1}{2}[K_1\ell_{k-1}(\lambda)+K_0\ell_k(\lambda)+K_1\ell_{k+1}(\lambda)].$$

Similarly,

$$(3.100)\quad \tfrac{1}{2}[\text{Real Part } \psi_0(\lambda-k\tfrac{\pi}{m})-\psi_0(\lambda+k\tfrac{\pi}{m})]=$$

$$\tfrac{1}{2}\text{ Real Part}[K_1\phi_{k-1}(\lambda)+K_0\phi_k(\lambda)+K_1\phi_{k+1}(\lambda)$$

$$-K_1\phi_{-k-1}(\lambda)-K_0\phi_{-k}(\lambda)-K_1\phi_{-k+1}(\lambda)]$$

$$=\tfrac{1}{2}[K_1\ell'_{k-1}(\lambda)+K_0\ell'_k(\lambda)+K_1\ell'_{k+1}(\lambda)]$$

where

$$(3.101)\quad \ell'_k(\lambda)\equiv\text{Real Part}[\phi_k(\lambda)-\phi_{-k}(\lambda)],\quad k=1,2,\ldots,m-1$$

and explicitly

$$(3.102)\quad \ell'_k(\lambda)=\text{Real Part}[\frac{1}{2m}\sum_{j=-m+1}^{m}(e^{ij(\lambda-k\frac{\pi}{m})}-e^{ij(\lambda+k\frac{\pi}{m})})]$$

$$=\text{Real Part}[\frac{-i}{m}\sum_{j=-m+1}^{m}e^{ij\lambda}\sin\frac{jk\pi}{m}]$$

$$= \text{Imaginary Part} \left[\frac{1}{m} \sum_{j=-m+1}^{m} e^{ij\lambda} \sin \frac{jk\pi}{m} \right]$$

$$= \frac{1}{m} \sum_{j=-m+1}^{m} \sin \frac{jk\pi}{m} \sin j\lambda = \frac{2}{m} \sum_{j=1}^{m-1} \sin \frac{jk\pi}{m} \sin j\lambda .$$

The filters centered at $\lambda = 0$ and $\lambda = \pi$ (i.e. cases $k=1, m+1$) require special treatment. Thus,

(3.103) Real Part $\psi_0(\lambda-\pi)$ = Real Part$[K_1\phi_{-1}(\lambda-\pi)+K_0\,\phi_0(\lambda-\pi)+K_1\,\phi_1(\lambda-\pi)]$

$= \frac{1}{2} K_0 \ell_m' + K_1 \ell_{m-1}$ is a "good" approximation to the ideal $\hat{P}_a^{(m+1)}(\lambda), \ldots, \hat{P}_c^{(m+1)}(\lambda)$. Since the quadrature spectral density $q(\lambda)$ is an odd function, $q(0) = q(\pi) = 0$. Thus, if m is sufficiently large, one may take $\hat{P}_d^{(1)}(\lambda) = \hat{P}_d^{(m+1)}(\lambda) = 0$. For smaller m the determination of a "good" approximation to the ideal $\hat{P}_d^{(1)}(\lambda)$ and $\hat{P}_d^{(m+1)}(\lambda)$ would require a study similar to the one carried out by Tukey [17] to determine a "good" approximation to the ideal $\hat{P}_a^{(1)}(\lambda)$ and is here omitted.

3.7 Sampling Distribution of the Estimators

In Sections 3.3, 3.4, and 3.5 the mean values and variances and covariances of the estimators $\hat{Q}_{\alpha_i}, \hat{Q}_{\beta_i}, \hat{Q}_{\gamma_i}, \hat{Q}_{\delta_i}$, $i=1,2,\ldots,I$ were studied. A heuristic discussion of the distribution of the estimators $\hat{Q}_{\alpha_i}, \hat{Q}_{\beta_i}, \hat{Q}_{\gamma_i}, \hat{Q}_{\delta_i}$, $i = 1,2,\ldots,I$ is now presented.

The analogue of the spectral representation (2.33) for a discrete parameter two-dimensional stationary vector process is

$$(3.104)\quad \begin{bmatrix} x_k \\ y_k \end{bmatrix} = \int_0^\pi \cos k\lambda \begin{bmatrix} dU_x(\lambda) \\ dU_y(\lambda) \end{bmatrix} + \int_0^\pi \sin k\lambda \begin{bmatrix} dV_x(\lambda) \\ dV_y(\lambda) \end{bmatrix}, \quad k=\ldots,-1,0,1,2,\ldots$$

Let

$$(3.105)\quad Z(\lambda) = \begin{bmatrix} Z_x(\lambda) \\ Z_y(\lambda) \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} U_x(\lambda) \\ U_y(\lambda) \end{bmatrix} - \tfrac{1}{2} i \begin{bmatrix} V_x(\lambda) \\ V_y(\lambda) \end{bmatrix}, \quad Z(-\lambda) = \overline{Z(\lambda)}$$

$$(0 \le \lambda \le \pi).$$

The representation (3.104) can then be rewritten in the complex form

$$(3.106)\quad \begin{bmatrix} x_k \\ y_k \end{bmatrix} = \int_{-\pi}^{\pi} e^{ik\lambda}\, dZ(\lambda) = \int_{-\pi}^{\pi} e^{ik\lambda}\, d \begin{bmatrix} Z_x(\lambda) \\ Z_y(\lambda) \end{bmatrix} \quad k=\ldots,-1,0,1,\ldots$$

Thus,

$$(3.107)\quad x_j x_k = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{ij\lambda_1 + ik\lambda_2}\, dZ_x(\lambda_1)\, dZ_x(\lambda_2).$$

Make the approximation that (3.104) also holds for half-integral values, so that

(3.108)

$$\hat{A}_h = \frac{1}{2M+1} \sum_{m=-M}^{M} x_{m-\frac{1}{2}h} x_{m+\frac{1}{2}h} = \frac{1}{2M+1} \int_{-\pi}^{\pi}\int \sum_{m=-M}^{M} e^{i(m-\frac{1}{2}h)\lambda_1 + i(m+\frac{1}{2}h)\lambda_2}\, dZ_x(\lambda_1)\, dZ_x(\lambda_2).$$

$$h = 0,1,2,\ \ldots,\ N-2M+1.$$

On making the substitution (3.45) in the right hand side of (3.108),

$$(3.109) \quad \hat{A}_h \quad \frac{1}{2M+1} \iint_R \sum_{m=-M}^{M} e^{2im\lambda' - ih\lambda} dZ_x(\lambda+\lambda') dZ_x(-\lambda+\lambda').$$

Thus,

(3.110)

$$\hat{Q}_{a_1} = \sum_{h=0}^{N-2M+1} a_h^{(1)} \hat{A}_h = \frac{1}{2M+1} \iint_R \sum_{m=-M}^{M} \sum_{h=0}^{N-2M-1} a_h^{(1)} e^{2im\lambda' - ih\lambda} dZ_x(\lambda+\lambda') dZ_x(-\lambda+\lambda').$$

But,

$$(3.111) \quad \sum_{m=-M}^{M} e^{2im\lambda'} = \frac{\sin(2M+1)\lambda'}{\sin\lambda'},$$

and

(3.112)

$$\sum_{h=0}^{N-2M-1} a_h^{(1)} e^{-ih\lambda} = \sum_{h=0}^{N-2M-1} a_h^{(1)} \cos h\lambda - i \sum_{h=0}^{N-2M-1} a_h^{(1)} \sin h\lambda = \hat{P}_a^{(1)}(\lambda) - i\, \hat{P}_{a,conj.}^{(1)}(\lambda).$$

Thus,

(3.113)

$$\hat{Q}_{a_1} = \frac{1}{2M+1} \iint_R \left[\frac{\sin(2M+1)\lambda'}{\sin\lambda'}\right] \left[\hat{P}_a^{(1)}(\lambda) - i\, \hat{P}_{a,conj.}^{(1)}(\lambda)\right] dZ_x(\lambda+\lambda') dZ_x(-\lambda+\lambda').$$

Using $dZ(-\lambda) = \overline{dZ(\lambda)}$, $\hat{P}^{(1)}_{a,conj.}(-\lambda) = \hat{P}^{(1)}_{a,conj.}(\lambda)$, and the symmetries of the region R,

(3.114) $$\iint_R \frac{\sin(2M+1)\lambda'}{\sin\lambda'}\, \hat{P}^{(1)}_{a,conj.}(\lambda)\, dZ_X(\lambda+\lambda')\, dZ_X(-\lambda+\lambda') =$$

$$\int_0^{\pi}\int_0^{\pi-\lambda} \frac{\sin(2M+1)\lambda'}{\sin\lambda'}\, \hat{P}^{(1)}_{a,conj.}(\lambda) \left[\begin{array}{l} dZ_X(\lambda+\lambda')\overline{dZ_X(\lambda-\lambda')} - dZ_X(-\lambda+\lambda')\overline{dZ_X(-\lambda-\lambda')} \\ + dZ_X(\lambda-\lambda')\overline{dZ_X(\lambda+\lambda')} - dZ_X(-\lambda-\lambda')\overline{dZ_X(-\lambda+\lambda')} \end{array} \right] = 0 ,$$

and thus,

(3.115)

$$\hat{Q}_{a_1} = \frac{2}{2M+1} \int_0^{\pi}\int_{-(\pi-\lambda)}^{(\pi-\lambda)} \frac{\sin(2M+1)\lambda'}{\sin\lambda'}\, \hat{P}^{(1)}_{a}(\lambda)\, dZ_X(\lambda+\lambda')\, \overline{dZ_X(\lambda-\lambda')} .$$

The function $\frac{\sin(2M+1)\lambda'}{\sin\lambda'}$ is apart from a factor of $\frac{1}{\pi}$ the Dirichlet kernel and approximates a δ-function centered at $\lambda'=0$, for M large.

Thus,

(3.1?)

$$\hat{Q}_{a_1} \sim \frac{2\pi}{2M+1} \int_0^{\pi} \hat{P}^{(1)}_{a}(\lambda)\, dZ_X(\lambda)\, \overline{dZ_X(\lambda)} = \frac{\pi}{2(2M+1)} \int_0^{\pi} \hat{P}^{(1)}_{a}(\lambda)\left[(dU_X(\lambda))^2 + (dV_X(\lambda))^2\right] .$$

Similarly,

(3.117)

$$\hat{Q}_{\beta_1} \approx \frac{2\pi}{2M+1} \int_0^{\pi} \hat{\beta}_b^{(1)}(\lambda)\, dZ_y(\lambda)\overline{dZ_y(\lambda)} = \frac{\pi}{2(2M+1)} \int_0^{\pi} \hat{\beta}_b^{(1)}(\lambda)\, [(dU_y(\lambda))^2 + (dV_y(\lambda))^2].$$

From (3.106)

(3.118) $$x_j x_k = \int_{-\pi}^{\pi}\!\!\int e^{ij\lambda_1 + ik\lambda_2}\, dZ_x(\lambda_1)\, dZ_y(\lambda_2).$$

Thus,

(3.119) $$\hat{C}_h = \frac{1}{2M+1} \sum_{m=-M}^{M} \tfrac{1}{2}(x_{m-\frac{1}{2}h} y_{m+\frac{1}{2}h} + x_{m+\frac{1}{2}h} y_{m-\frac{1}{2}h})$$

$$= \frac{1}{2M+1} \int_{-\pi}^{\pi}\!\!\int \sum_{m=-M}^{M} \tfrac{1}{2}[e^{i(m-\frac{1}{2}h)\lambda_1 + i(m+\frac{1}{2}h)\lambda_2} + e^{i(m+\frac{1}{2}h)\lambda_1 + i(m-\frac{1}{2}h)\lambda_2}]\, dZ_x(\lambda_1)\, dZ_y(\lambda_2)$$

$$= \frac{1}{2M+1} \iint_R \sum_{m=-M}^{M} \tfrac{1}{2}[e^{2im\lambda' - ih\lambda} + e^{2im\lambda' + ih\lambda}]\, dZ_x(\lambda + \lambda')\, dZ_y(-\lambda + \lambda')$$

$$= \frac{1}{2M+1} \iint_R \sum_{m=-M}^{M} e^{2im\lambda'} \cos h\lambda\, dZ_x(\lambda + \lambda')\, \overline{dZ_y(\lambda - \lambda')}.$$

Thus,

(3.120) $$\hat{Q}_{Y_1} = \sum_{h=0}^{N-2M-1} c_h^{(1)} \hat{C}_h = \frac{1}{2M+1} \iint_R \frac{\sin(2M+1)\lambda'}{\sin \lambda'}\, \hat{\beta}_c^{(1)}(\lambda)\, dZ_x(\lambda+\lambda')\, \overline{dZ_y(\lambda-\lambda')}.$$

$$(3.121)\quad \hat{Q}_{\gamma_i} \approx \frac{\pi}{2M+1} \int_{-\pi}^{\pi} \hat{P}_c^{(i)}(\lambda)\, dZ_x(\lambda)\, \overline{dZ_y(\lambda)}$$

$$= \frac{\pi}{2M+1} \int_{0}^{\pi} \hat{P}_c^{(i)}(\lambda)\,[dZ_x(\lambda)\,\overline{dZ_y(\lambda)} + \overline{dZ_x(\lambda)}\,dZ_y(\lambda)]$$

$$= \frac{\pi}{2(2M+1)} \int_{0}^{\pi} \hat{P}_c^{(i)}(\lambda)\,[(dU_x(\lambda))dU_y(\lambda)+(dV_x(\lambda))(dV_y(\lambda))] .$$

A similar argument for $\hat{Q}_{\delta_i}$ yields,

(3.122)

$$\hat{Q}_{\delta_i} \approx \frac{\pi}{2(2M+1)} \int_{0}^{\pi} \hat{P}_d^{(i)}(\lambda)\,[(dU_x(\lambda))(dV_y(\lambda))-(dV_x(\lambda))(dU_y(\lambda))] .$$

The expressions (3.116),(3.117),(3.121),(3.122) should be regarded as asymptotic forms of $\hat{Q}_{\alpha_i}, \hat{Q}_{\beta_i}, \hat{Q}_{\gamma_i}, \hat{Q}_{\delta_i}$ respectively. From these expressions one observes the following:

(a) If the filters $\hat{P}_a^{(i)}(\lambda)$, $\hat{P}_b^{(i)}(\lambda)$, $\hat{P}_c^{(i)}(\lambda)$, $\hat{P}_d^{(i)}(\lambda)$, $i = 1,2, \ldots, I$ had an ideally sharp cut-off (i.e. vanished outside the intervals $\lambda_{i-1} \leq \lambda \leq \lambda_i$, $-\lambda_i \leq \lambda \leq -\lambda_{i-1}$), then for $i \neq j$, $\hat{Q}_{\alpha_i}, \hat{Q}_{\beta_i}, \hat{Q}_{\gamma_i}, \hat{Q}_{\delta_i}$ and $\hat{Q}_{\alpha_j}, \hat{Q}_{\beta_j}, \hat{Q}_{\gamma_j}, \hat{Q}_{\delta_j}$ would (in the Gaussian case) be independent, since each estimator indexed by i and each indexed by j are functions of spectral random variables in disjoint frequency intervals, and such random variables (in the Gaussian case) are independent. Thus, if the filters $\hat{P}_a^{(i)}(\lambda), \ldots, \hat{P}_d^{(i)}(\lambda)$ have a reasonably sharp cut-off (this is attainable if (2M+1) is large) the estimators $\hat{Q}_{\alpha_i}, \ldots, Q_{\delta_i}$ and $\hat{Q}_{\alpha_j}, \ldots, \hat{Q}_{\delta_j}$, $i \neq j$ are approximately independent for large (2M+1).

(b) If the filters $\hat{P}_a^{(i)}(\lambda), \ldots, \hat{P}_d^{(i)}(\lambda)$ were ideal and the frequency interval $\lambda_{i-1} \leq \lambda \leq \lambda_i$ sufficiently narrow so that $f_x(\lambda), f_y(\lambda), c(\lambda), q(\lambda)$ could be considered constant in the interval, then

(3.123.1) $$\hat{Q}_{\alpha_i} \approx K \int_{\lambda_{i-1}}^{\lambda_i} [(dU_x(\lambda))^2 + (dV_x(\lambda))^2]$$

(3.123.2) $$\hat{Q}_{\beta_i} \approx K \int_{\lambda_{i-1}}^{\lambda_i} [(dU_y(\lambda))^2 + (dV_y(\lambda))^2]$$

(3.123.3) $$\hat{Q}_{\gamma_i} \approx K \int_{\lambda_{i-1}}^{\lambda_i} [(dU_x(\lambda))(dU_y(\lambda)) + (dV_x(\lambda))(dV_y(\lambda))]$$

(3.123.4) $$\hat{Q}_{\delta_i} \approx K \int_{\lambda_{i-1}}^{\lambda_i} [(dU_x(\lambda))(dV_y(\lambda)) - (dV_x(\lambda))(dU_y(\lambda))],$$

thus $\hat{Q}_{\alpha_i}, \ldots, \hat{Q}_{\delta_i}$ would (in the Gaussian case) be respectively sums of infinitely many independent indentically distributed random variables, and hence by the central limit theorem $(\hat{Q}_{\alpha_i}, \hat{Q}_{\beta_i}, \hat{Q}_{\gamma_i}, \hat{Q}_{\delta_i})$ would be distributed four-variate Gaussian. Thus, if the filters $\hat{P}_a^{(i)}(\lambda), \ldots, \hat{P}_d^{(i)}(\lambda)$, are reasonably flat, have a reasonably sharp cut-off, and are not too wide,

$(\hat{Q}_{\alpha_i}, \hat{Q}_{\beta_i}, \hat{Q}_{\gamma_i}, \hat{Q}_{\delta_i})$ is distributed approximately four-variate Gaussian for large 2M+1.

In the case when 2M+1 is not too large, yet the filters $\hat{P}_a^{(i)}(\lambda), \ldots, \hat{P}_d^{(i)}(\lambda)$ can be considered sensibly narrow (so that $f_x(\lambda)$, $f_y(\lambda)$, $c(\lambda)$, $q(\lambda)$ do not vary greatly in the frequency interval $\lambda_{i-1} \leq \lambda \leq \lambda_i$), one obtains a better approximation to the joint distribution of the estimators $\hat{Q}_{\alpha_i}, \ldots, \hat{Q}_{\delta_i}$ as follows.

Guided by the asymptotic forms (3.116),(3.117),(3.121), (3.122), one introduces analogous finite sums

$$
\begin{aligned}
a &= \frac{1}{2n} \sum_{i=1}^{n} (u_{xi}^2 + v_{xi}^2) \\
b &= \frac{1}{2n} \sum_{i=1}^{n} (u_{yi}^2 + v_{yi}^2) \qquad (3.124) \\
c &= \frac{1}{2n} \sum_{i=1}^{n} (u_{xi}u_{yi} + v_{xi}v_{yi}) \\
d &= \frac{1}{2n} \sum_{i=1}^{n} (u_{xi}v_{yi} - v_{xi}u_{yi}),
\end{aligned}
$$

where

$(u_{xi}, v_{xi}, u_{yi}, v_{yi})$ is distributed four-variate Gaussian with mean (o,o,o,o) and variance-covariance matrix $\mathbf{v}$

$$(3.125) \qquad V = \begin{bmatrix} 1 & 0 & \alpha & \beta \\ 0 & 1 & -\beta & \alpha \\ \alpha & -\beta & 1 & 0 \\ \beta & \alpha & 0 & 1 \end{bmatrix}, \quad (\alpha^2+\beta^2 \leq 1),$$

for fixed i, and $(u_{xi}, v_{xi}, u_{yi}, v_{yi})$, $i = 1,2, \ldots, n$ are independent. One then approximates the joint distribution of

$$(3.126) \qquad \left[\frac{\hat{Q}_{\alpha_i}}{E\,\hat{Q}_{\alpha_i}}, \frac{\hat{Q}_{\beta_i}}{E\,\hat{Q}_{\beta_i}}, \frac{\hat{Q}_{\gamma_i}}{\sqrt{E\,\hat{Q}_{\alpha_i}\;E\,\hat{Q}_{\beta_i}}}, \frac{\hat{Q}_{\delta_i}}{\sqrt{E\,\hat{Q}_{\alpha_i}\;E\,\hat{Q}_{\beta_i}}} \right]$$

by the joint distribution of (a,b,c,d). In reference to (3.124), one naturally associates $(u_{xi}, v_{xi}, u_{yi}, v_{yi})$ with

$$(3.127) \qquad \left[\frac{dU_x(\lambda_i)}{\sqrt{f_x(\lambda_i)d\lambda}}, \frac{dV_x(\lambda_i)}{\sqrt{f_x(\lambda_i)d\lambda}}, \frac{dU_y(\lambda_i)}{\sqrt{f_y(\lambda_i)d\lambda}}, \frac{dV_y(\lambda_i)}{\sqrt{f_y(\lambda_i)d\lambda}} \right]$$

so that the parameters α, β in (3.125) are identified as

$$(3.128) \qquad \alpha = \frac{c(\lambda)}{\sqrt{f_x(\lambda)f_y(\lambda)}}, \quad \beta = \frac{q(\lambda)}{\sqrt{f_x(\lambda)f_y(\lambda)}}.$$

The condition $\alpha^2+\beta^2 \leq 1$ which insures that v is positive semi-definite is the analogue of the coherency condition (2.52.2) that $c^2(\lambda) + q^2(\lambda) - f_x(\lambda) f_y(\lambda) \leq 0$.

The joint distribution of (a,b,c,d) and related distributions are studied in Chapter 4. As indicated in the preceding discussion (and later discussed in Chapter 5) these distributions approximate the joint distribution of the relative or dimensionless estimators (3.126) and the distributions of certain functions of these estimators.

Chapter 4

The Complex Wishart Distribution and Related Distributions

4.0 Introduction

This chapter concerns itself with the derivation of the distributions of certain functions of complex Gaussian random variables. The fundamental distribution derived is a complex analogue of a Wishart distribution. This distribution, named the Complex Wishart distribution serves as the starting point in the derivation of the other distributions.

4.1 Introductory Specification of the Complex Wishart Distribution and Related Distributions.

Let X and Y denote the complex Gaussian random variables

$$X = U_x + i\,V_x \tag{4.1}$$

$$Y = U_y + i\,V_y \quad ,$$

where the real random variables U_x, V_x, U_y, V_y are distributed four-variate Gaussian with mean (0,0,0,0) and variance-covariance matrix V

$$V = \begin{bmatrix} \sigma_x^2 & 0 & \alpha\sigma_x\sigma_y & \beta\sigma_x\sigma_y \\ 0 & \sigma_x^2 & -\beta\sigma_x\sigma_y & \alpha\sigma_x\sigma_y \\ \alpha\sigma_x\sigma_y & -\beta\sigma_x\sigma_y & \sigma_y^2 & 0 \\ \beta\sigma_x\sigma_y & \alpha\sigma_x\sigma_y & 0 & \sigma_y^2 \end{bmatrix} \tag{4.2}$$

with $\alpha^2+\beta^2 = \gamma^2 \le 1$. (The condition $\gamma^2 \le 1$ insures that V is positive semi-definite).

Now, define and evaluate

(4.3.1) $$A_0 \equiv \underline{\text{Variance}(X)} \equiv \tfrac{1}{2} EX^*X = \sigma_x^2 ,$$

(4.3.2) $$B_0 \equiv \underline{\text{Variance }(Y)} \equiv \tfrac{1}{2} EY^*Y \equiv \sigma_y^2 ,$$

(4.3.3) $$C_0+i\,D_0 \equiv \underline{\text{Cross-variance of } X \text{ with } Y} \equiv \tfrac{1}{2} EX^*Y = (\alpha+i\beta)\sigma_x\sigma_y ,$$

and

(4.4.1) $$C_0 \equiv \underline{\text{Co-Variance of } X \text{ with } Y} \equiv \text{Real Part } (\tfrac{1}{2} EX^*Y) = \alpha\sigma_x\sigma_y ,$$

(4.4.2) $$D_0 \equiv \underline{\text{Quad-Variance of } X \text{ with } Y} \equiv \text{Imaginary Part } (\tfrac{1}{2} EX^*Y) = \beta\sigma_x\sigma_y ,$$

(4.4.3) $$R_0 \equiv \underline{\text{Amplitude of Cross-Variance of } X \text{ with } Y} \equiv \text{Amp. } (\tfrac{1}{2} EX^*Y) = \gamma\,\sigma_x\sigma_y ,$$

(4.4.4) $$\phi_0 \equiv \underline{\text{Argument of Cross-Variance of } X \text{ with } Y} \equiv \text{Arg } (\tfrac{1}{2} EX^*Y) = \text{Arg } (\alpha+i\beta) ,$$

(4.4.5) $$F_0 \equiv \underline{\text{Variance Ratio of } Y \text{ and } X} \equiv \frac{\text{Var}(Y)}{\text{Var}(X)} = \frac{B_0}{A_0} = \frac{\sigma_y^2}{\sigma_x^2} ,$$

(4.4.6) $$Z_0 \equiv \underline{\text{Coherency of } X \text{ and } Y} \equiv \frac{\left|\tfrac{1}{2} EX^*Y\right|^2}{\text{Var}(X)\ \text{Var}(Y)} = \gamma^2,$$

(4.4.7) $\quad L_0 \equiv$ Complex Regression Coefficient of Y on X

$$= \frac{EX^* Y}{Var(X)} = \frac{(\alpha+i\beta)\sigma_y}{\sigma_x} .$$

Let (X_j, Y_j), with

(4.5) $$X_j = U_{xj} + i V_{xj}$$

$$Y_j = U_{yj} + i V_{yj}, \quad j = 1, 2, \ldots, n ,$$

denote independent random variables, where U_{xj}, V_{xj}, U_{yj}, V_{yj} for fixed j are distributed as are U_x, V_x, U_y, V_y .

Define and evaluate

(4.6.1) $$A \equiv \underline{\text{Sample Variance (X)}} = \frac{1}{2n} \sum_{j=1}^{n} X_j^* X_j$$

$$= \frac{1}{2n} \sum_{j=1}^{n} (U_{xj}^2 + V_{xj}^2) ,$$

(4.6.2) $$B \equiv \underline{\text{Sample Variance (Y)}} \equiv \frac{1}{2n} \sum_{j=1}^{n} Y_j^* Y_j$$

$$= \frac{1}{2n} \sum_{j=1}^{n} (U_{yj}^2 + V_{yj}^2) ,$$

(4.6.3) $$C + i D \equiv \underline{\text{Sample Cross-Variance of X with Y}}$$

$$\equiv \frac{1}{2n} \sum_{j=1}^{n} X_j^* Y_j = \frac{1}{2n} \sum_{j=1}^{n} (U_{xj}U_{yj} + V_{xj}V_{yj})$$

$$+ \frac{i}{2n} \sum_{j=1}^{n} (-V_{xj}U_{yj} + U_{xj}V_{yj}) .$$

For independent real bivariate Gaussian random variables (X_j, Y_j), $j=1,\ldots,n$ such that $E(X_j, Y_j) = (0,0)$, the joint distribution of

$$(4.7) \qquad \sum_{j=1}^{n} X_j^2 \,, \quad \sum_{j=1}^{n} Y_j^2 \,, \quad \sum_{j=1}^{n} X_j Y_j$$

is a Wishart distribution. Thus, (4.6) suggests that the joint distribution of $A, B, C + i\,D$ is a complex analogue of a Wishart distribution. The joint distribution of (A,B,C,D) is therefore called a Complex Wishart distribution.

Now, in direct analogy with (4.4) define and evaluate

$$(4.8.1) \qquad C \equiv \underline{\text{Sample Co-Variance of } X \text{ with } Y} \equiv \text{Real Part } \left(\frac{1}{2n}\sum_{j=1}^{n} X_j^* \, Y_j\right)$$

$$= \frac{1}{2n} \sum_{j=1}^{n} (U_{xj}U_{yj} + V_{xj}V_{yj}) \,,$$

$$(4.8.2) \qquad D \equiv \underline{\text{Sample Quad-Variance of } X \text{ with } Y} \equiv \text{Imag.Part } \left(\frac{1}{2n}\sum_{j=1}^{n} X_j^* Y_j\right)$$

$$= \frac{1}{2n} \sum_{j=1}^{n} (-V_{xj}U_{yj} + U_{xj}V_{yj}) \,,$$

$$(4.8.3) \qquad R \equiv \underline{\text{Sample Amplitude of Cross-Variance of } X \text{ with } Y}$$

$$\equiv \text{Amp } \left(\frac{1}{2n}\sum_{j=1}^{n} X_j^* Y_j\right) = \sqrt{C^2+D^2} \,,$$

$$(4.8.4) \qquad \equiv \underline{\text{Sample Argument of Cross-Variance of } X \text{ with } Y}$$

$$\equiv \text{Arg } \left(\frac{1}{2n}\sum_{j=1}^{n} X_j^* Y_j\right) = \text{Arg } (C + i\,D) \,,$$

(4.8.5) $F \equiv$ Sample Variance Ratio of Y and X $= \dfrac{\sum_{j=1}^{n} Y_j^* Y_j}{\sum_{j=1}^{n} X_j^* X_j} = \dfrac{B}{A}$.

(4.8.6) $Z \equiv$ Sample Coherency of X and Y $\equiv \dfrac{\left|\sum_{j=1}^{n} X_j^* Y_j\right|^2}{\left(\sum_{j=1}^{n} X_j^* X_j\right)\left(\sum_{j=1}^{n} Y_j^* Y_j\right)}$

$$= \frac{C^2+D^2}{AB} .$$

(4.8.7) $L \equiv$ Sample Complex Regression Coefficient of Y on X

$$\equiv \frac{\sum_{j=1}^{n} X_j^* Y_j}{\sum_{j=1}^{n} X_j^* X_j} = \frac{C + i D}{A} .$$

The sample complex regression coefficient of Y on X can be expressed in various coordinate systems. The following coordinate systems [see Fig.(4)] are of importance:

(4.9.1) Rectangular (L_{Re}, L_{Im}) where $L = L_{Re} + i\, L_{Im}$,

(4.9.2) Polar (G, ϕ) where $L = G\, e^{i\phi}$,

(4.9.3) Radial-Transverse (L_{Ra}, L_{Tr}) where $L = (L_{Ra} + i\, L_{Tr})e^{i\phi_0}$,

(4.9.4) Centered Polar (L_c, θ) where $L = (1 + \frac{L_c}{|L_0|} e^{i\theta})\, L_0$.

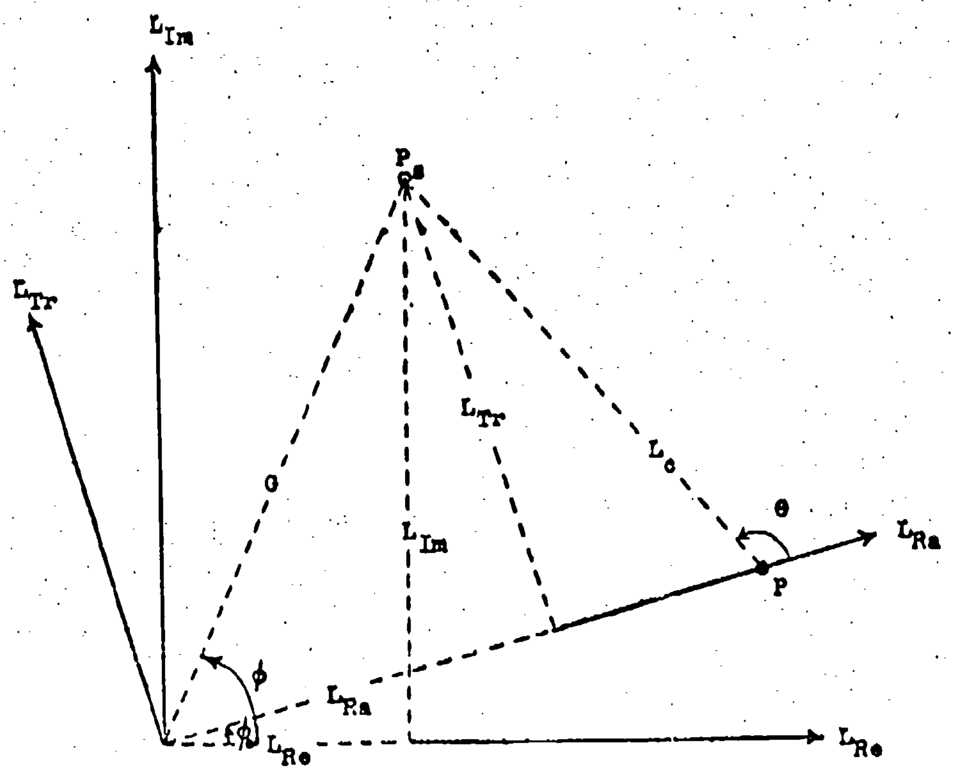


Fig. (4). Coordinate systems for the sample complex regression coefficient of Y on X.

The point P designates the complex regression coefficient of Y on X.

The point P_s designates the sample complex regression coefficient of Y on X.

With reference to (4.9) define and evaluate

(4.10.1) $L_{Re_o} \equiv$ Real Part of Complex Regression Coefficient of Y on X

$$L_{Re_o} = \frac{\alpha\sigma_Y}{\sigma_X} ,$$

(4.10.2) $L_{Im_o} \equiv$ Imaginary Part of Complex Regression Coefficient of Y on X

$$L_{Im_o} = \frac{\beta\sigma_Y}{\sigma_X} ,$$

(4.10.3) $G_o \equiv$ Gain of Complex Regression Coefficient of Y on X

$$G_o = \frac{\gamma\sigma_Y}{\sigma_X} ,$$

(4.10.4) $\phi_o \equiv$ Phase of Complex Regression Coefficient of Y on X

$$\phi_o = \text{Arg } (\alpha + i\beta) ,$$

(4.10.5) $L_{Ra_o} \equiv$ Radial Component of Complex Regression Coefficient of Y on X

$$L_{Ra_o} = \frac{\gamma\sigma_Y}{\sigma_X} ,$$

(4.10.6) $L_{Tr_o} \equiv$ Transverse Component of Complex Regression Coefficient of Y on X

$$L_{Tr_o} = 0 ,$$

(4.10.7) $L_{c_o} \equiv$ Centered Amplitude of Complex Regression Coefficient of Y on X

$$L_{c_o} = 0 ;$$

and

(4.11.1) $L_{Re} \equiv$ Sample Real Part of Complex Regression Coefficient of Y on X

$$L_{Re} = \frac{C}{A},$$

(4.11.2) $L_{Im} \equiv$ Sample Imaginary Part of Complex Regression Coefficient of Y on X

$$L_{Im} = \frac{D}{A},$$

(4.11.3) $G \equiv$ Sample Gain of Complex Regression Coefficient of Y on X

$$G = \frac{\sqrt{C^2+D^2}}{A},$$

(4.11.4) $\phi \equiv$ Sample Phase of Complex Regression Coefficient of Y on X

$$\phi = \operatorname{Arg}\left(\frac{C+iD}{A}\right),$$

(4.11.5) $L_{Ra} \equiv$ Sample Radial Component of Complex Regression Coefficient of Y on X

$$L_{Ra} = \frac{C}{A}\cos\phi_o + \frac{D}{A}\sin\phi_o,$$

(4.11.6) $L_{Tr} \equiv$ Sample Transverse Component of Complex Regression Coefficient of Y on X

$$L_{Tr} = \frac{D}{A}\cos\phi_o - \frac{C}{A}\sin\phi_o,$$

(4.11.7) $L_c \equiv$ Sample Centered Amplitude of Complex Regression Coefficient of Y on X

$$L_c = \sqrt{\left(\frac{C}{A} - \alpha\frac{\sigma_Y}{\sigma_X}\right)^2 + \left(\frac{D}{A} - \beta\frac{\sigma_Y}{\sigma_X}\right)^2},$$

(4.11.8) θ = Sample Centered Polar Angle of Complex Regression Coefficient of Y on X

$$\theta = \operatorname{Arg}(L-L_o) - \phi_o = \operatorname{Arg}\left[\left(\frac{C}{A} - \alpha\frac{\sigma_Y}{\sigma_X}\right) + i\left(\frac{D}{A} - \beta\frac{\sigma_Y}{\sigma_X}\right)\right] - \phi_o .$$

In the remaining sections of this chapter the distribution functions of the random variables defined by (4.6), (4.8), and (4.11) as well as many joint distributions of these random variables are derived. An index of these distributions is given in Table I.

4.2 Dimensionless Random Variables.

In deriving the distribution functions listed in Table I it is convenient to work with dimensionless or relative random variables. Thus, define and evaluate

$$(4.12)\quad \begin{aligned} x_j &= X_j/\sigma_x = \frac{U_{xj}}{\sigma_x} + i\,\frac{V_{xj}}{\sigma_x} = u_{xj} + i\,v_{xj} \\ y_j &= Y_j/\sigma_y = \frac{U_{yj}}{\sigma_y} + i\,\frac{V_{yj}}{\sigma_y} = u_{yj} + i\,v_{yj} . \end{aligned}$$

The real random variables $u_{xj}, v_{xj}, u_{yj}, v_{yj}$ (for fixed j) are distributed four-variate Gaussian with mean (0,0,0,0) and variance-covariance matrix v where

$$(4.13)\quad v = \begin{bmatrix} 1 & 0 & \alpha & \beta \\ 0 & 1 & -\beta & \alpha \\ \alpha & -\beta & 1 & 0 \\ \beta & \alpha & 0 & 1 \end{bmatrix} .$$

TABLE I

Index of Distributions

Joint Distribution of	Probability Density
(A,B,C,D)	(4.43) with (4.44).
(B,C,D)	(4.75) with (4.44) and (4.14).
(A,C,D)	(4.74) with (4.44) and (4.14).
(A,B,D)	(4.78) with (4.44) and (4.14).
(A,B,C)	(4.77) with (4.44) and (4.14).
(A,B)	(4.110) with (4.44).
(A,C)	(4.111) with (4.44).
(A,D)	(4.112) with (4.44).
(B,C)	(4.113) with (4.44).
(B,D)	(4.114) with (4.44).
(C,D)	(4.115) with (4.44).
(A)	(4.64) with (4.14).
(B)	(4.67 with (4.14).
(C)	(4.70) with (4.44) and (4.16).
(D)	(4.72) with (4.44) and (4.16).
(R, ϕ)	(4.118) with (4.117) and (4.16).
(R)	(4.119) with (4.117) and (4.16).
(A, B, Z, ϕ)	(4.52) with (4.44) and (4.16).
(A,B,Z)	(4.59) with (4.44) and (4.16).
(A, B, ϕ)	(4.62) with (4.44) and (4.16).
(Z, ϕ)	(4.55) with (4.16).
(Z)	(4.60) with (4.16).

TABLE I (continued)

Joint Distribution of	Probability Density
(ϕ)	(4.58) with (4.16).
(A, F, Z)	(4.105) with (4.16) and (4.44).
(F, Z)	(4.107) with (4.16).
(F)	(4.108) with (4.16).
(A, G, ϕ)	(4.80) with (4.44) and (4.17).
(G, ϕ)	(4.81) with (4.17).
(G)	(4.97) with (4.17).
(ψ)	(4.104) with (4.16).
(L_{Ra}, L_{Tr})	(4.83) with (4.17).
(L_{Ra})	(4.84) with (4.17).
(L_{Tr})	(4.85) with (4.17).
(L_o, θ)	([illegible]7) with (4.17).
(L_o).	(4.88) with (4.17).
(L_{Re}, L_{Im})	(4.91) with (4.17).
(L_{Re})	(4.92) with (4.17).
(L_{Im})	(4.93) with (4.17).

Now, define and evaluate

$$(4.14.1) \quad a \equiv \frac{1}{2n} \sum_{j=1}^{n} (u_{xj}^2 + v_{xj}^2) = \frac{A}{\sigma_x^2} = \frac{A}{A_0} ,$$

$$(4.14.2) \quad b \equiv \frac{1}{2n} \sum_{j=1}^{n} (u_{yj}^2 + v_{yj}^2) = \frac{B}{\sigma_y^2} = \frac{B}{B_0} ,$$

(4.14.3)

$$c+id \equiv \frac{1}{2n} \sum_{j=1}^{n} (u_{xj}u_{yj}+v_{xj}v_{yj}) + \frac{i}{2n} \sum_{j=1}^{n} (-v_{xj}u_{yj}+u_{xj}v_{yj})$$

$$= \frac{C + iD}{\sigma_x\sigma_y} = \alpha(\frac{C}{C_0}) + i\beta(\frac{D}{D_0}) .$$

The joint distribution of (a,b,c,d) is called a <u>Unit Complex Wishart</u> distribution.

From (4.14) and (4.13)

$$(4.15.1) \quad E\, a = \frac{EA}{\sigma_x^2} = 1 ,$$

$$(4.15.2) \quad E\, b = \frac{EB}{\sigma_y^2} = 1 ,$$

$$(4.15.3) \quad E(c + id) = \frac{E(C + iD)}{\sigma_x\sigma_y} = \alpha + i\beta .$$

Finally, define and evaluate

$$(4.16.1) \quad c = \frac{1}{2n} \sum_{j=1}^{n} (u_{xj}u_{yj} + v_{xj}v_{yj}) = \frac{C}{\sigma_x\sigma_y} = \alpha(\frac{C}{C_0}) ,$$

$$(4.16.2) \quad d = \frac{1}{2n} \sum_{j=1}^{n} (-v_{xj}u_{yj} + u_{xj}v_{yj}) = \frac{D}{\sigma_x\sigma_y} = \beta(\frac{D}{D_0}) ,$$

$$(4.16.3)\quad r \equiv \sqrt{c^2 + d^2} = \frac{R}{\sigma_x\sigma_y} = \gamma\left(\frac{R}{R_0}\right) ,$$

$$(4.16.4)\quad \phi = \mathrm{Arg}\,(c + id) = \mathrm{Arg}\,(C + iD) = \mathrm{Arg}\left(\frac{C + iD}{A}\right) ,$$

$$(4.16.5)\quad f = \frac{b}{a} = \left(\frac{B}{A}\right)/\left(\frac{\sigma_y^2}{\sigma_x^2}\right) = F/\left(\frac{\sigma_y^2}{\sigma_x^2}\right) = \frac{F}{F_0} ,$$

$$(4.16.6)\quad z \equiv \frac{c^2 + d^2}{ab} = \frac{C^2 + D^2}{AB} = Z = \gamma^2\left(\frac{Z}{Z_0}\right) ,$$

$$(4.16.7)\quad \ell \equiv \frac{c + id}{a} = \left(\frac{C + iD}{A}\right)/\left(\frac{\sigma_y}{\sigma_x}\right) = L/\left(\frac{\sigma_y}{\sigma_x}\right) ;$$

and

$$(4.17.1)\quad \ell_{Re} \equiv \frac{c}{a} = \left(\frac{C}{A}\right)/\left(\frac{\sigma_y}{\sigma_x}\right) = L_{Re}/\left(\frac{\sigma_y}{\sigma_x}\right) = \alpha\left(\frac{L_{Re}}{L_{Re_0}}\right) ,$$

$$(4.17.2)\quad \ell_{Im} \equiv \frac{d}{a} = \left(\frac{D}{A}\right)/\left(\frac{\sigma_y}{\sigma_x}\right) = L_{Im}/\left(\frac{\sigma_y}{\sigma_x}\right) = \beta\left(\frac{L_{Im}}{L_{Im_0}}\right) ,$$

$$(4.17.3)\quad g \equiv \frac{\sqrt{c^2 + d^2}}{a} = \left(\frac{\sqrt{C^2 + D^2}}{A}\right)/\left(\frac{\sigma_y}{\sigma_x}\right) = G/\left(\frac{\sigma_y}{\sigma_x}\right) = \gamma\left(\frac{G}{G_0}\right) ,$$

$$(4.17.4)\quad \phi = \mathrm{Arg}\left(\frac{c + id}{a}\right) = \mathrm{Arg}\left(\frac{C + iD}{A}\right) = \mathrm{Arg}(C + iD) ,$$

$$(4.17.5)\quad \ell_{Ra} \equiv \frac{c}{a}\cos\phi_0 + \frac{d}{a}\sin\phi_0 = \left(\frac{C}{A}\cos\phi_0 + \frac{D}{A}\sin\phi_0\right)/\left(\frac{\sigma_y}{\sigma_x}\right)$$
$$= L_{Ra}/\left(\frac{\sigma_y}{\sigma_x}\right) = \left(\frac{L_{Ra}}{L_{Ra_0}}\right) ,$$

$$(4.17.6)\quad \ell_{Tr} \equiv \frac{d}{a}\cos\phi_0 - \frac{c}{a}\sin\phi_0 = \left(\frac{D}{A}\cos\phi_0 - \frac{C}{A}\sin\phi_0\right)/\left(\frac{\sigma_y}{\sigma_x}\right)$$
$$= L_{Tr}/\left(\frac{\sigma_y}{\sigma_x}\right) ,$$

(4.17.7) $\ell_0 = \sqrt{(\frac{c}{a} - \alpha)^2 + (\frac{d}{a} - \beta)^2} = L_0/(\frac{\sigma_Y}{\sigma_X})$,

(4.17.8) $\theta = \text{Arg}\,[(\frac{c}{a} - \alpha) + i(\frac{d}{a} - \beta)] - \phi_0$.

4.3 Remarks on Characteristic Functions

The following properties of characteristic functions are used in the derivation of the characteristic function of the Complex Wishart distribution:

(a) If the joint distribution of the random variables X_1, X_2 has the characteristic function $\phi(\theta_1,\theta_2)$, i.e.

(4.18) $E\, e^{i\theta_1 X_1 + i\theta_2 X_2} = \iint e^{i\theta_1 x_1 + i\theta_2 x_2} dF(x_1,x_2) = \phi(\theta_1,\theta_2)$

then, since

(4.19) $\phi(\theta_1, 0) = \int_{x_1} e^{i\theta_1 x_1} [\int_{x_2} dF(x_1,x_2)]$,

the characteristic function of the distribution of X_1 is $\phi(\theta_1, 0)$.

(b) If $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$, and $\psi(\theta_1,\theta_2)$ denotes the characteristic function of the joint distribution of Y_1, Y_2, then

(4.20) $\psi(\theta_1,\theta_2) = E\, e^{i\theta_1(X_1+X_2) + i\theta_2(X_1-X_2)} = E\, e^{i(\theta_1+\theta_2)X_1 + i(\theta_1-\theta_2)X_2}$
$= \phi(\theta_1+\theta_2, \theta_1-\theta_2)$.

(c) If $\phi(\theta)$ denotes the characteristic function of the random variable X, then

$$(4.21)\quad E\, e^{i(kX)\theta} = E\, e^{iX(k\theta)} = \phi(k\theta)$$

where k denotes a constant, is the characteristic function of the distribution of the random variables kX.

4.4 Determination of the Characteristic Function of the Unit Complex Wishart Distribution

Consider $b_{11}, b_{12}, b_{13}, b_{14}, b_{22}, b_{23}, b_{24}, b_{33}, b_{34}, b_{44}$ defined by

(4.22)

$$b_{11} = \sum_{j=1}^{n} u_{xj}^2 ,\quad b_{12} = \sum_{j=1}^{n} u_{xj} v_{xj} ,\quad b_{13} = \sum_{j=1}^{n} u_{xj} u_{yj} ,\quad b_{14} = \sum_{j=1}^{n} u_{xj} v_{yj} ,$$

$$b_{22} = \sum_{j=1}^{n} v_{xj}^2 ,\quad b_{23} = \sum_{j=1}^{n} v_{xj} u_{yj} ,\quad b_{24} = \sum_{j=1}^{n} v_{xj} v_{yj} ,$$

$$b_{33} = \sum_{j=1}^{n} u_{yj}^2 ,\quad b_{34} = \sum_{j=1}^{n} u_{yj} v_{yj} ,$$

$$b_{44} = \sum_{j=1}^{n} v_{yj}^2 .$$

where $u_{xj}, \cdots, v_{yj}$ are as specified in (4.12). Thus, $b_{11}, b_{12}, \ldots, b_{44}$ are distributed[1] with a Wishart distribution and the characteristic function of this Wishart distribution is

$$(4.23)\quad \Phi(\theta_{ij}) = |v|^{-\frac{n}{2}} |v^{ij} - i\, e_{ij}\theta_{ij}|^{-\frac{n}{2}} ,$$

[1] See Chapter XI of [19].

where

$$(4.24) \qquad \|v^{ij}\| = v^{-1},\ \Theta_{ij} = \Theta_{ji},\quad e_{ij} = \begin{matrix} 2 \text{ if } i=j \\ 1 \text{ if } i \neq j \end{matrix}$$

and v is given by (4.13).

A computation shows that

$$(4.25) \qquad |v| = \delta^2 = (1-\gamma^2)^2, \text{ where } \gamma^2 = \alpha^2+\beta^2 \text{ and } \delta = 1-\gamma^2,$$

and that (when $0 < \delta$)

$$(4.26) \qquad v^{-1} = \frac{1}{\delta}\begin{bmatrix} 1 & 0 & -\alpha & -\beta \\ 0 & 1 & \beta & -\alpha \\ -\alpha & \beta & 1 & 0 \\ -\beta & -\alpha & 0 & 1 \end{bmatrix}.$$

From the characteristic function $\Phi(\Theta_{ij})$ of the Wishart distribution the characteristic function of the joint distribution of (a,b,c,d) is obtained as follows: By applying property (b) of Section 4.3, the characteristic function of the transformed variables

(4.27)

$$B_{11} = b_{11} + b_{22} \qquad B_{22} = b_{11} - b_{22}$$

$$B_{33} = b_{33} + b_{44} \qquad B_{44} = b_{33} - b_{44}$$

$$B_{13} = b_{13} + b_{24} \qquad B_{24} = b_{13} - b_{24}$$

$$B_{23} = b_{14} - b_{23} \qquad B_{14} = b_{14} + b_{23}$$

$$B_{12} = b_{12}$$

$$B_{34} = b_{34}$$

is

(4.28)

$$|v|^{-\frac{n}{2}} \begin{vmatrix} [v^{11}-2i(\theta_{11}+\theta_{22})] & [v^{12}-i\theta_{12}] & [v^{13}-i(\theta_{13}+\theta_{24})] & [v^{14}-i(\theta_{14}+\theta_{23})] \\ [v^{12}-i\theta_{12}] & [v^{22}-2i(\theta_{11}-\theta_{22})] & [v^{23}-i(\theta_{14}-\theta_{23})] & [v^{24}-i(\theta_{13}-\theta_{24})] \\ [v^{13}-i(\theta_{13}+\theta_{24})] & [v^{23}-i(\theta_{14}-\theta_{23})] & [v^{33}-2i(\theta_{33}+\theta_{44})] & [v^{34}-i\theta_{34}] \\ [v^{14}-i(\theta_{14}+\theta_{23})] & [v^{24}-i(\theta_{13}-\theta_{24})] & [v^{34}-i\theta_{34}] & [v^{44}-2i(\theta_{33}-\theta_{44})] \end{vmatrix}^{-\frac{n}{2}}$$

where now θ_{ij} is associated with B_{ij}, $i \le j = 1,2,3,4$.

But

(4.29) $a = \frac{1}{2n} B_{11}$, $b = \frac{1}{2n} B_{33}$, $c = \frac{1}{2n} B_{13}$, and $d = \frac{1}{2n} B_{23}$.

Thus, by applying properties (a) and (c) of Section 4.3, the characteristic function of the joint distribution of (a,b,c,d) is

(4.30) $$\phi(\theta_a,\theta_b,\theta_c,\theta_d) = |v|^{-\frac{n}{2}} \begin{vmatrix} (v^{11}-i\frac{\theta_a}{n}) & v^{12} & (v^{13}-i\frac{\theta_c}{n}) & (v^{14}-i\frac{\theta_d}{2n}) \\ v^{12} & (v^{22}-i\frac{\theta_a}{n}) & (v^{23}+i\frac{\theta_d}{2n}) & (v^{24}-i\frac{\theta_c}{2n}) \\ (v^{13}-i\frac{\theta_c}{2n}) & (v^{23}+i\frac{\theta_d}{2n}) & (v^{33}-i\frac{\theta_b}{n}) & v^{34} \\ (v^{14}-i\frac{\theta_d}{2n}) & (v^{24}-i\frac{\theta_c}{2n}) & v^{34} & (v^{44}-i\frac{\theta_b}{2n}) \end{vmatrix}^{-\frac{n}{2}}$$

Let (when $\delta > 0$)

$$
(4.31)\qquad
\begin{aligned}
\theta_a\delta &= n\,\theta_a' \\
\theta_b\delta &= n\,\theta_b' \\
\theta_c\delta &= 2n\,\theta_c' \\
\theta_d\delta &= 2n\,\theta_d' \,.
\end{aligned}
$$

Thus, from (4.26) and (4.30)

$$
(4.32)\qquad
\begin{aligned}
\phi(\theta_a,\theta_b,\theta_c,\theta_d) &= \delta^{n} \begin{vmatrix} (1-i\theta_a') & 0 & (-\alpha-i\theta_c') & (-\beta-i\theta_d') \\ 0 & (1-i\theta_a') & (\beta+i\theta_d') & (-\alpha-i\theta_c') \\ (-\alpha-i\theta_c') & (\beta+i\theta_d') & (1-i\theta_b') & 0 \\ (-\beta-i\theta_d') & (-\alpha-i\theta_c') & 0 & (1-i\theta_b') \end{vmatrix}^{-\frac{n}{2}} \\
&= \delta^{n}[(1-i\theta_a')(1-i\theta_b')-(\alpha+i\theta_c')^{2}-(\beta+i\theta_d')^{2}]^{-n} \\
&= [1-\tfrac{i}{n}(\theta_a+\theta_b+\alpha\theta_c+\beta\theta_d)-\frac{\delta}{4n^{2}}(4\theta_a\theta_b-\theta_c^{2}-\theta_d^{2})]^{-n} \,.
\end{aligned}
$$

From (4.32) it is easily verified that the mean and variance-covariance matrix of the joint distribution of (a,b,c,d) are respectively

$$
(4.33)\qquad (\mu_a,\mu_b,\mu_c,\mu_d) = (1,1,\alpha,\beta)
$$

and

$$
(4.34)\qquad
\begin{bmatrix} \sigma_{aa} & \sigma_{ab} & \sigma_{ac} & \sigma_{ad} \\ & \sigma_{bb} & \sigma_{bc} & \sigma_{bd} \\ & & \sigma_{cc} & \sigma_{cd} \\ & & & \sigma_{dd} \end{bmatrix}
= \frac{1}{n}
\begin{bmatrix} 1 & \gamma^{2} & \alpha & \beta \\ & 1 & \alpha & \beta \\ & & \frac{1}{2}(1+\alpha^{2}-\beta^{2}) & \alpha\beta \\ & & & \frac{1}{2}(1-\alpha^{2}+\beta^{2}) \end{bmatrix}
$$

Let (when $\delta > 0$)

$$
(4.35) \qquad
\begin{aligned}
a'\delta &= n\,a \\
b'\delta &= n\,b \\
c'\delta &= 2n\,c \\
d'\delta &= 2n\,d
\end{aligned}
$$

so that the characteristic function of the joint distribution of a', b', c', d' is

$$
(4.36) \qquad \Psi(\theta_a', \theta_b', \theta_c', \theta_d') = \delta^n [(1-i\theta_a')(1-i\theta_b') - (\alpha+i\theta_c')^2 - (\beta+i\theta_d')^2]^{-n} .
$$

Since a', b', c', d' are proportional to a, b, c, d respectively, the characteristic function of the Unit Complex Wishart distribution can be considered as given by (4.36).

4.5 Determination of the Probability Density Function of the Unit Complex Wishart Distribution

The probability density function $p(a', b', c', d')$ of the joint distribution $f(a', b', c', d')$ is now derived.[1] The Fourier transform formulae used in this derivation (and in the derivations that follow) are listed in Table II at the end of this chapter.

[1] The notation employed in this section and in subsequent sections does not distinguish between random variables and the variables in the probability density function for the random variables. Such a notation very simply identifies the random variables with their probability density functions, and since many (see Table I) probability density functions are derived, such ease of identification is important. Had it been possible, random variables would have been printed in boldface type; however even though this has not been done it is usually clear from the context which variables are random variables.

The probability density function $p(a',b',c',d')$ of the joint distribution of (a',b',c',d') is the inverse Fourier transform of $\psi(\theta_a',\theta_b',\theta_c',\theta_d')$, i.e.

$$(4.37)\quad p(a',b',c',d') = \frac{1}{(2\pi)^4}\iiiint_{-\infty}^{\infty} e^{-i(a'\theta_a'+b'\theta_b'+c'\theta_c'+d'\theta_d')} \cdot \psi(\theta_a',\theta_b',\theta_c',\theta_d')\,d\theta_a'\,d\theta_b'\,d\theta_c'\,d\theta_d' .$$

Exponential Fourier transforms of the type listed in Table II, in one or several variables, are simply denoted by T, so that for example equation (4.37) is written as

$$(4.38)\quad p(a',b',c',d') = T\ \psi(\theta_a',\theta_b',\theta_c',\theta_d') .$$

From (4.38), (4.36), and Table II (2)

$$(4.39)\quad p(a',b',c',d') = \delta^n e^{\alpha c'+\beta d'} T[(1-i\theta_a')(1-i\theta_b')+\theta_c'^2+\theta_d'^2]^{-n}$$

$$= \delta^n e^{\alpha c'+\beta d'} T\left[(1-i\theta_b')^{-n}[1-i\theta_a'+\frac{\theta_c'^2+\theta_d'^2}{1-i\theta_b'}]^{-n}\right].$$

By using Table II (4)

$$(4.40)\quad p(a',b',c',d') = \frac{\delta^n e^{\alpha c'+\beta d'}}{\Gamma(n)}(a')^{n-1} e^{-a'} \cdot T\left[(1-i\theta_b')^{-n} e^{-a'\left(\frac{\theta_c'^2+\theta_d'^2}{1-i\theta_b'}\right)}\right]$$

By using Table II (5)

$$(4.41)\quad p(a',b',c',d') = \frac{\delta^n}{4\pi\,\Gamma(n)}(a')^{n-2} e^{\alpha c'+\beta d'-a'} e^{-\left(\frac{c'^2+d'^2}{4a'}\right)} T(1-i\theta_b')^{-n+1} e^{i\left(\frac{c'^2+d'^2}{4a'}\right)\theta_b'} .$$

By using Table II (3) and Table II (4)

$$(4.42)\quad T(1-i\theta_b')^{-n+1} e^{i\left(\frac{c'^2+d'^2}{4a'}\right)\theta_b'} = \frac{1}{\Gamma(n-1)}\left(b' - \frac{c'^2+d'^2}{4a'}\right)^{n-2} e^{-\left(b' - \frac{c'^2+d'^2}{4a'}\right)}$$

$$\text{if } b' - \frac{c'^2+d'^2}{4a'} > 0 ,$$

$$= 0 \qquad \text{otherwise} .$$

On substituting (4.42) in (4.41) and simplifying

$$(4.43)\quad p(a',b',c',d') = \frac{\delta^n}{4^{n-1}\pi\Gamma(n)\Gamma(n-1)}(4a'b'-c'^2-d'^2)^{n-2} e^{\alpha c'+\beta d'-a'-b'}$$

$$\text{if } 4a'b'-c'^2-d'^2 > 0 ,$$

$$= 0 \qquad \text{otherwise},$$

Since a,b,c,d are proportional to a',b',c',d' respectively, one may consider (4.43) as the probability density function of the Unit Complex Wishart distribution. From (4.35) and (4.14)

$$a' = \frac{n}{\delta}\left(\frac{A}{A_0}\right),$$
$$b' = \frac{n}{\delta}\left(\frac{B}{B_0}\right),$$
$$(4.44) \qquad c' = \frac{2\alpha n}{\delta}\left(\frac{C}{C_0}\right),$$
$$d' = \frac{2\beta n}{\delta}\left(\frac{D}{D_0}\right).$$

4.6 The Unit Complex Wishart Distribution in the Perfectly Coherent Case

The Unit Complex Wishart distribution in the perfectly coherent case, i.e. in the case when $\gamma^* = 1$ requires special discussion. In the perfectly coherent case

$$(4.45) \qquad \alpha = \cos \phi_0 , \quad \beta = \sin \phi_0$$

so that the variance-covariance matrix v of the four-variate Gaussian random variable $(u_{xj}, v_{xj}, u_{yj}, v_{yj})$ becomes

$$(4.46) \qquad v = \begin{bmatrix} 1 & 0 & \cos \phi_0 & \sin \phi_0 \\ 0 & 1 & -\sin \phi_0 & \cos \phi_0 \\ \cos \phi_0 & -\sin \phi_0 & 1 & 0 \\ \sin \phi_0 & \cos \phi_0 & 0 & 1 \end{bmatrix}$$

The matrix v is of rank two, and it is easily verified that the correlation between u_{yj} and $u_{xj}\cos \phi_0 - v_{xj}\sin \phi_0$ is unity, and that the correlation between v_{yj} and $u_{xj}\sin \phi_0 + v_{xj}\cos \phi_0$ is unity. Thus,

$$(4.47)\qquad \begin{bmatrix} u_{yj} \\ v_{yj} \end{bmatrix} = \begin{bmatrix} \cos\phi_0 & -\sin\phi_0 \\ \sin\phi_0 & \cos\phi_0 \end{bmatrix} \begin{bmatrix} u_{xj} \\ v_{xj} \end{bmatrix} \quad \text{(with probability one).}$$

Thus, in the notation of (4.12)

$$x_j = u_{xj} + i\, v_{xj}$$
$$y_j = u_{yj} + i\, v_{yj} = e^{i\phi_0} x_j \, .$$

Thus, from (4.14) and (4.48)

$$a = \frac{1}{2n} \sum_{j=1}^{n} x_j^* x_j$$
$$(4.49)\qquad b = \frac{1}{2n} \sum_{j=1}^{n} y_j^* y_j = a$$
$$c + id = \frac{1}{2n} \sum_{j=1}^{n} x_j^* y_j = e^{i\phi_0} a \, ,$$

so that the Unit Complex Wishart distribution is degenerate in the perfectly coherent case. It is easily seen that $2n\ a$ is distributed χ^2_{2n} . Furthermore,

$$b = a\ , \ c = a\cos\phi_0 , \quad d = a\sin\phi_0 \ ;$$
$$\frac{c^2+d^2}{ab} = 1 = \frac{\sqrt{c^2+d^2}}{a} \ ;$$
$$\phi = \phi_0 \qquad \text{(with probability one).}$$

4.7 Probability Density Functions of Distributions Related to the Complex Wishart Distribution

4.7.1 $p(a', b', z, \phi)$

From (4.16.4),(4.16.6) and (4.35)

$$(4.51)\quad \begin{aligned} c' &= 2\sqrt{a'b'}\, z \cos\phi \\ d' &= 2\sqrt{a'b'}\, z \sin\phi . \end{aligned}$$

Thus, from (4.43)

$$(4.52)\quad p(a',b',z,\phi)$$
$$= \frac{\delta^n}{\pi\Gamma(n)\Gamma(n-1)} (a'b')^{n-1} z(1-z^2)^{n-2}\, e^{-a'-b'+2z\sqrt{a'b'}\gamma\cos(\phi-\phi_0)}$$

where $a',b' > 0,\ 0 < z < 1,\ 0 < \phi < 2\pi$.

4.7.2 $\underline{p(z,\phi)}$

The density $p(z,\phi)$ is obtained by integrating out a',b' in (4.52). Let

$$(4.53)\qquad s = 2\gamma z \cos(\phi-\phi_0) .$$

Then,

$$(4.54)\ \int_0^\infty\int_0^\infty (a'b')^{n-1} e^{s\sqrt{a'b'}-a'-b'}\, da'\,db' =$$
$$= \sum_{k=0}^\infty \int_0^\infty\int_0^\infty (a'b')^{n+\frac{k}{2}-1} e^{-a'-b'} \frac{s^k}{k!}\, da'\,db'$$
$$= \sum_{k=0}^\infty \frac{\Gamma^2(n+\frac{k}{2})}{\Gamma(k+1)} s^k$$

Thus,

$$(4.55)\quad p(z,\phi) = \frac{\delta^n}{\pi\Gamma(n)\Gamma(n-1)} z(1-z^2)^{n-2} \sum_{k=0}^\infty \frac{2^k\gamma^k\Gamma^2(n+\frac{k}{2})}{\Gamma(k+1)} z^k \cos^k(\phi-\phi_0) .$$

By using the ratio test and the asymptotic formula for the Gamma function $\Gamma(x) \approx \sqrt{2\pi}\, e^{-x} x^{x-\frac{1}{2}}$ one verifies that the series in (4.55) converges for $|\gamma z \cos(\phi - \phi_0)| < 1$.

4.7.3 $\underline{p(\phi)}$

From (4.55) and the definition of the incomplete beta function (see Table III (1) at end of chapter)

$$(4.56) \quad \int_0^{z_1} p(z,\phi)\,dz = \frac{\delta^n}{\pi\Gamma(n)\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{2^{k-1}\gamma^k \Gamma^2(n+\frac{k}{2})}{\Gamma(k+1)} B_{z_1^2}(\tfrac{k}{2}+1, n-1)\cos^k(\phi - \phi_0) .$$

By definition of the beta function [see Table III (2)]

$$(4.57) \quad \lim_{z \to 1} B_{z_1^2}(\tfrac{k}{2}+1, n-1) = B(\tfrac{k}{2}+1, n-1) = \frac{\Gamma(\frac{k}{2}+1)\,\Gamma(n-1)}{\Gamma(\frac{k}{2}+n)} .$$

Thus,

$$(4.58) \quad p(\phi) = \frac{\delta^n}{\pi\Gamma(n)} \sum_{k=0}^{\infty} \frac{2^{k-1}\gamma^k\, \Gamma(n+\frac{k}{2})\,\Gamma(1+\frac{k}{2})}{\Gamma(k+1)} \cos^k(\phi - \phi_0) .$$

The series in (4.58) converges for $|\gamma \cos(\phi - \phi_0)| < 1$.

4.7.4 $\underline{p(a',b',z)}$

By integrating (4.52) on ϕ [see Table III (3)]

$$(4.59) \quad p(a',b',z) = \frac{2\delta^n}{\Gamma(n)\Gamma(n-1)} (a'b')^{n-1} z(1-z^2)^{n-2} e^{-a'-b'} I_0(2\gamma z\sqrt{a'b'})$$

$$= \frac{2\delta^n}{\Gamma(n)\Gamma(n-1)} (a'b')^{n-1} z(1-z^2)^{n-2} e^{-a'-b'} \cdot \sum_{k=0}^{\infty} \frac{\gamma^{2k}}{\Gamma^2(k+1)} (a'b')^k z^{2k} .$$

4.7.5 p(z)

From (4.59)

$$(4.60)\quad p(z) = \int_0^\infty\int_0^\infty p(a',b',z)\,da'\,db'$$

$$= \frac{2\delta^n}{\Gamma(n)\Gamma(n-1)}\, z(1-z^2)^{n-2} \sum_{k=0}^{\infty} \frac{\gamma^{2k}\,\Gamma^2(n+k)}{\Gamma^2(k+1)}\, z^{2k} \, .$$

4.7.6 $p(a',b',\phi)$

From (4.52)

$$(4.61)\quad p(a',b',z,\phi) = \frac{\delta^n}{\pi\Gamma(n)\Gamma(n-1)}(a'b')^{n-1} e^{-a'-b'} \sum_{k=0}^{\infty} z^{k+1}(1-z^2)^{n-2} \frac{t^k}{k!}$$

where here $t = 2\sqrt{a'b'}\,\gamma \cos(\phi - \phi_0)$.

Thus, by integrating (4.61) on z [see Table III (3)]

$$(4.62)\quad p(a',b',\phi) = \frac{\delta^n}{\pi\Gamma(n)}(a'b')^{n-1}e^{-a'-b'} \sum_{k=0}^{\infty} \frac{2^{k-1}\gamma^k\Gamma(\frac{k}{2}+1)}{k!\,\Gamma(n+\frac{k}{2})}(a'b')^{\frac{k}{2}}\cos^k(\phi-\phi_0)$$

4.7.7 p(a)

From (4.32)

$$(4.63)\quad \phi(\theta_a) = \phi(\theta_a,0,0,0) = (1 - \tfrac{1}{n}\,\theta_a)^{-n}$$

Thus, by Table II (4)

$$(4.64)\quad \begin{aligned} p(a) &= T\,(1-\tfrac{1}{n}\theta_a)^{-n} = \frac{n^n a^{n-1} e^{-na}}{\Gamma(n)} && \text{for } a \ge 0 \\ &= 0 && \text{for } a < 0 \, . \end{aligned}$$

Thus, 2na is distributed χ^2_{2n}, i.e. with probability density function

$$p(u) = \frac{1}{2^n \Gamma(n)} u^{n-1} e^{-\frac{1}{2}u} \quad \text{for } u \geq 0$$

(4.65)

$$= 0 \quad \text{for } u < 0$$

4.7.8 p(b)

From (4.32)

(4.66) $$\phi(\theta_b) = (1 - \frac{1}{n}\theta_b)^{-n}$$

so that

(4.67) $$p(b) = \frac{n^n b^{n-1} e^{-na}}{\Gamma(n)} \quad \text{for } a \geq 0$$

$$= 0 \quad \text{for } a < 0 .$$

i.e. 2nb is distributed χ^2_{2n} .

4.7.9 p(c')

From (4.32)

(4.68) $$\phi(\theta_c') = \phi(0,0,\theta_c',0) = \delta^n[1 - \beta^2 - (a+i\theta_c')^2]^{-n} .$$

Thus, by Table II (2)

(4.69) $$p(c') = T\,\phi(\theta_c') = \delta^n e^{ac'} T[1 - \beta^2 + \theta_c'^2]^{-n}$$

Thus, by Table II (6)

(4.70) $$p(c') = \frac{\delta^n e^{ac'} |c'|^{n-\frac{1}{2}}}{2^{n-\frac{1}{2}} \sqrt{\pi}\,\Gamma(n)(1-\beta^2)^{\frac{n}{2}-\frac{1}{4}}} K_{n-\frac{1}{2}}(\sqrt{1-\beta^2}\,|c'|) ,$$

where $K_\nu(\cdot)$ denotes a modified Bessel function of the third kind. The Bessel functions and modified Bessel functions reduce to combinations of elementary functions when the order ν is half an odd integer. [See Table III (4)] .

4.7.10 $p(d')$

From (4.32)

(4.71) $$\phi(\theta_d') = \delta^n[1-\alpha^2-(\beta+i\theta_d')^2]^{-n} ,$$

so that

(4.72) $$p(d') = \frac{\delta^n e^{\beta d'}|d'|^{n-\frac{1}{2}}}{2^{n-\frac{1}{2}}\sqrt{\pi}\,\Gamma(n)(1-\alpha^2)^{\frac{n}{2}-\frac{1}{4}}} K_{n-\frac{1}{2}}(\sqrt{1-\alpha^2}\,|d'|) .$$

4.7.11 $p(a',c',d')$

From (4.43), by integrating on b'

(4.73) $$p(a',c',d') = \frac{\delta^n(4a')^{n-2}e^{\alpha c'+\beta d'-a'}}{4^{n-1}\pi\Gamma(n)\Gamma(n-1)} \int_{\frac{c'^2+d'^2}{4a'}}^{\infty} (u-\frac{c'^2+d'^2}{4a'})^{n-2} e^{-u}du$$

Thus, by Table III (5)

(4.74) $$p(a',c',d') = \frac{\delta^n(a')^{n-2}}{4\pi\,\Gamma(n)} \exp[\alpha c'+\beta d'-a'-\frac{c'^2+d'^2}{4a'}] .$$

4.7.12 $p(b',c',d')$

From (4.43), by integrating on a'

(4.75) $$p(b',c',d') = \frac{\delta^n(b')^{n-2}}{4\pi\,\Gamma(n)} \exp[\alpha c'+\beta d'-b'-\frac{c'^2+d'^2}{4b'}] .$$

4.7.13 $\underline{p(a',b',c')}$

From (4.43), by integrating on d'

$$(4.76)\quad p(a',b',c') = \frac{\delta^n e^{\alpha c'-a'-b'}}{4^{n-1}\,\pi\Gamma(n)\Gamma(n-1)} \int_{-\sqrt{4a'b'-c'^2}}^{\sqrt{4a'b'-c'^2}} [(4a'b'-c'^2)-u^2]^{n-2}\, e^{\beta u}\,du$$

$$= \frac{\delta^n e^{\alpha c'-a'-b'}}{4^{n-1}\,\pi\Gamma(n)\Gamma(n-1)} (4a'b'-c'^2)^{n-\frac{3}{2}} \int_{-1}^{1} (1-v^2)^{n-2}\, e^{\beta\sqrt{4a'b'-c'^2}\,v}\,dv\,.$$

Thus, by Table III (6)

$$(4.77)\quad p(a',b',c') = \frac{\delta^n e^{\alpha c'-a'-b'}\left(\sqrt{4a'b'-c'^2}\right)^{n-\frac{3}{2}}}{2^{n-\frac{3}{2}}\sqrt{\pi}\,\Gamma(n)\,\beta^{n-\frac{3}{2}}}\, I_{n-\frac{3}{2}}\left(\beta\sqrt{4a'b'-c'^2}\right) \quad \text{if } 0 < 4a'b'-c'^2\,,$$

$$= 0 \qquad \text{otherwise.}$$

Since the order $n-\frac{3}{2}$ of the Bessel function occurring in (4.77) is half an odd integer, $p(a',b',c')$ can be expressed in terms of elementary functions. [See Table III (6)].

4.7.14 $\underline{p(a',b',d')}$

By an argument similar to that of Section 4.7.13

$$(4.78)\quad p(a',b',d') = \frac{\delta^n e^{\beta d'-a'-b'}\left(\sqrt{4a'b'-d'^2}\right)^{n-\frac{3}{2}}}{2^{n-\frac{3}{2}}\sqrt{\pi}\,\Gamma(n)\,\alpha^{n-\frac{3}{2}}}\, I_{n-\frac{3}{2}}\left(\alpha\sqrt{4a'b'-d'^2}\right) \quad \text{if } 0 < 4a'b'-d'^2,$$

$$= 0 \qquad \text{otherwise.}$$

4.7.15 $p(a',g,\phi)$

From (4.17.3) and (4.35)

$$(4.79) \qquad c' = 2a'g\cos\phi \quad , \quad d' = 2a'g\sin\phi \ .$$

Thus, from (4.74)

$$(4.80) \quad p(a',g,\phi) = \frac{\delta^n (a')^n g}{\pi\Gamma(n)} \exp\left[-a'[1-2\gamma g\cos(\phi-\phi_0)+g^2]\right] .$$

The condition $\gamma^2 < 1$ insures that $[1-2\gamma g\cos(\phi-\phi_0)+g^2] > 0$ for $0 \leq \phi < 2\pi$, $0 \leq g < \infty$.

4.7.16 $p(g,\phi)$

From (4.80), by integrating on a' [see Table III (7)]

$$(4.81) \quad p(g,\phi) = \frac{n\,\delta^n\, g}{\pi[1-2\gamma g\cos(\phi-\phi_0)+g^2]^{n+1}} \ .$$

4.7.17 $p(\ell_{Ra}, \ell_{Tr})$

From (4.17.3), (4.17.4), (4.17.5), and (4.17.6)

$$(4.82) \quad \ell_{Ra} = g\cos(\phi-\phi_0) \ , \quad \ell_{Tr} = g\sin(\phi-\phi_0) \ .$$

Thus, from (4.81)

$$(4.83) \quad p(\ell_{Ra},\ell_{Tr}) = \frac{n\,\delta^n}{\pi[\delta+(\ell_{Ra}-\gamma)^2+\ell_{Tr}^2]^{n+1}} \ .$$

4.7.18 $p(\ell_{Ra})$

By integrating on ℓ_{Tr} in (4.83) [see Table III (8)]

$$(4.84)\quad p(\ell_{Ra}) = \frac{\delta^n \Gamma(n+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(n)[\delta+(\ell_{Ra}-\gamma)^2]^{n+\frac{1}{2}}} \; .$$

4.7.19 $p(\ell_{Tr})$

By integrating on ℓ_{Ra} in (4.83) [see Table III (8)]

$$(4.85)\quad p(\ell_{Tr}) = \frac{\delta^n \Gamma(n+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(n)[\delta+\ell_{Tr}^2]^{n+\frac{1}{2}}} \; .$$

4.7.20 $p(\ell_c,\theta)$

From (4.17.5),(4.17.6),(4.17.7) and (4.10.5)

$$(4.86)\quad \begin{aligned} \ell_{Ra} - \gamma &= \ell_c \cos\theta \\ \ell_{Tr} &= \ell_c \sin\theta \end{aligned}$$

Thus, from (4.83)

$$(4.87)\quad p(\ell_c,\theta) = \frac{n\delta^n \ell_c}{\pi[\delta + \ell_c^2]^{n+1}} \; .$$

4.7.21 $p(\ell_c)$

By integrating on θ in (4.87)

$$(4.88)\quad p(\ell_c) = \frac{2n\,\delta^n \ell_c}{[\delta + \ell_c^2]^{n+1}} \; .$$

Furthermore,

$$(4.89)\quad P(\ell'_c) = \int_0^{\ell'_c} p(\ell_c)\,d\ell_c = 1-[1+\delta^{-1}\ell_c'^2]^{-n} \; .$$

4.7.22 $p(\ell_{Re},\ell_{Im})$

From (4.17.1), (4.17.2), (4.17.3), (4.16.4)

(4.90) $\ell_{Re} = g \cos\phi \quad , \quad \ell_{Im} = g \sin\phi \quad .$

Thus, from (4.81)

(4.91) $p(\ell_{Re}, \ell_{Im}) = \dfrac{n\,\delta^n}{\pi[\delta+(\ell_{Re}-\alpha)^2+(\ell_{Im}-\beta)^2]^{n+1}} \quad .$

4.7.23 $\underline{p(\ell_{Re})}$

By integrating on ℓ_{Im} in (4.91) [see Table III (8)]

(4.92) $p(\ell_{Re}) = \dfrac{\delta^n \Gamma(n+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(n)[\delta+(\ell_{Re}-\alpha)^2]^{n+\frac{1}{2}}} \quad .$

4.7.24 $\underline{p(\ell_{Im})}$

By integrating on ℓ_{Re} in (4.91) [see Table III (8)]

(4.93) $p(\ell_{Im}) = \dfrac{\delta^n \Gamma(n+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(n)[\delta+(\ell_{Im}-\beta)^2]^{n+\frac{1}{2}}} \quad .$

4.7.25 $\underline{p(g)}$

In (4.81) let

(4.94) $u = \tan\left(\dfrac{\phi-\phi_0}{2}\right)$

so that

(4.95) $\cos(\phi-\phi_0) = \dfrac{1-u^2}{1+u^2} \; , \quad d(\phi-\phi_0) = \dfrac{2du}{1+u^2} \quad ,$

and therefore

$$(4.96)\quad p(g) = 2\int_{\phi_0}^{\phi_0+\pi} p(g,\phi)\,d\phi = \int_0^\infty \frac{4n\ldots g(1+u^2)^n\,du}{\pi[(1-2\gamma g+g^2)+(1+2\gamma g+g^2)u^2]^{n+1}} .$$

Thus, [see Table III (9)]

$$(4.97)\quad p(g) = \frac{2n\delta^n g}{\pi(1-2\gamma g+g^2)^{n+\frac{1}{2}}(1+2\gamma g+g^2)^{\frac{1}{2}}}$$

$$\cdot \sum_{k=0}^{\infty} \frac{\Gamma(k+\frac{1}{2})\Gamma(n-k+\frac{1}{2})}{\Gamma(k+1)\Gamma(n-k+1)} \left(\frac{1-2\gamma g+g^2}{1+2\gamma g+g^2}\right)^k .$$

4.7.26 $\underline{p(\phi)}$

Let

$$(4.98)\quad s = -\gamma\cos(\phi-\phi_0) .$$

Then, (4.81) can be expressed in the form

$$(4.99)\quad p(g,\phi) = \frac{n\delta^n g}{\pi[1-s^2+(g+s)^2]^{n+1}}$$

$$= \frac{n\delta^n(g+s)}{\pi[1-s^2+(g+s)^2]^{n+1}} - \frac{n\delta^n s}{\pi[1-s^2+(g+s)^2]^{n+1}} .$$

Thus,

$$(4.100)\quad p(\phi) = \int_0^\infty p(g,\phi)\,dg$$

$$= \frac{\delta^n}{2\pi} - \int_0^\infty \frac{n\delta^n s\,dg}{\pi[1-s^2+(g+s)^2]^{n+1}} .$$

Now let

$$(4.101)\quad g+s = (1-s^2)^{\frac{1}{2}}\tan v .$$

Thus,

$$(4.102)\quad \int_0^{\infty} \frac{n\delta^n\, s\, dg}{\pi(1-s^2+(g+s)^2)^{n+1}} = \int_{\text{arc sin } s}^{\frac{1}{2}\pi} \frac{n\delta^n s \cos^{2n} v\, dv}{\pi(1-s^2)^{n+\frac{1}{2}}}$$

where $-\frac{1}{2}\pi \leq \text{arc sin } s \leq \frac{1}{2}\pi$.

From Table III (10)

$$(4.103)\quad \int_{\text{arc sin } s}^{\frac{1}{2}\pi} \cos^{2n} v\, dv = \frac{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}{2\Gamma(n+1)} \pm \frac{1}{2} B_{s^2}(\tfrac{1}{2}, n+\tfrac{1}{2})$$

where $B_{s^2}(,)$ denotes the Incomplete Beta function [see Table III (1)], and the plus sign applies if $s \leq 0$ and the minus sign applies if $s \geq 0$.

Thus,

$$(4.104)\quad p(\phi) = \frac{\delta^n}{2\pi} - \frac{n\delta^n s}{2\pi(1-s^2)^{n+\frac{1}{2}}}\left[\frac{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \pm B_{s^2}(\tfrac{1}{2}, n+\tfrac{1}{2})\right]$$

where $s = -\gamma \cos(\phi - \phi_0)$, $B_{s^2}(,)$ denotes the Incomplete Beta function [see Table III (1)], the plus sign applies if $|\phi - \phi_0| \leq \frac{1}{2}\pi$, and the minus sign applies if $\frac{1}{2}\pi \leq |\phi - \phi_0| \leq \pi$.

4.7.27 $\underline{p(a', f, z)}$

From (4.16.5), (4.35), and (4.59)

$$(4.105)\quad p(a', f, z) = \frac{2\delta^n}{\Gamma(n)\Gamma(n-1)} (a')^{2n-1} f^{n-1} z(1-z^2)^{n-2} e^{-a'(1+f)} I_0(2\gamma z a' \sqrt{f}) .$$

4.7.28 $\underline{p(f, z)}$

From (4.105) and Table III (3)

(4.106) $$p(a',f,z) = \sum_{k=0}^{\infty} \frac{2\gamma^{2k}\delta^{n}}{\Gamma(n)\Gamma(n-1)(k!)^2}(a')^{2n+2k-1} \cdot f^{n+k-1}z^{2k+1}(1-z^2)^{n-2}e^{-a'(1+f)}.$$

Thus by integrating on a' in (4.106)

(4.107) $$p(f,z) = \sum_{k=0}^{\infty} \frac{2\gamma^{2k}\delta^{n}\Gamma(2n+2k)}{\Gamma(n)\Gamma(n-1)(k!)^2} \frac{f^{n+k-1}z^{2k+1}(1-z^2)^{n-2}}{(1-f)^{2n+2k}} .$$

4.7.29 p(f)

From (4.107) by integrating on z[see Table III(2)]

(4.108) $$p(f) = \sum_{k=0}^{\infty} \frac{\Gamma(2n+2k)\gamma^{2k}\delta^{n}}{\Gamma(n)\Gamma(k+1)\Gamma(k+n)} \cdot \frac{f^{n+k-1}}{(1+f)^{2n+2k}} .$$

By using the ratio test one establishes that the series in (4.108) converges for $\frac{f}{(1+f)^2} < \frac{1}{4\gamma^2}$, i.e. for $0 \le f < \infty$ if $\gamma^2 < 1$.

4.7.30 p(a',b')

From (4.59) by integrating on z[see Table III (2)]

(4.109) $$p(a',b') = \frac{\delta^{n}e^{-a'-b'}}{\Gamma(n)} \sum_{k=0}^{\infty} \frac{\gamma^{2k}(a'b')^{k+n-1}}{\Gamma(k+1)\Gamma(k+n)} .$$

From Table III (11)

(4.110) $$p(a',b') = \frac{\delta^{n}(a'b')^{\frac{1}{2}(n-1)}e^{-a'-b'}}{\gamma^{n-1}\Gamma(n)} I_{n-1}(2\gamma\sqrt{a'b'}) .$$

4.7.31 p(a',c')

From (4.74) by integrating on d'[see Table III (11)]

$$(4.111)\quad p(a',c') = \frac{\beta^n (a')^{n-\frac{3}{2}}}{2\sqrt{\pi}\,\Gamma(n)} \exp[(\beta^2-1)a' + \alpha c' - \frac{c'^2}{4a'}] \;.$$

4.7.32 <u>$p(a',d')$</u>

From (4.74) by integrating on c' [see Table III (13)]

$$(4.112)\quad p(a',d') = \frac{\alpha^n (a')^{n-\frac{3}{2}}}{2\sqrt{\pi}\,\Gamma(n)} \exp[(\alpha^2-1)a' + \beta d' - \frac{d'^2}{4a'}] \;.$$

4.7.33 <u>$p(b',c')$</u>

From (4.75) by integrating on d' [see Table III (13)]

$$(4.113)\quad p(b',c') = \frac{\beta^n (b')^{n-\frac{3}{2}}}{2\sqrt{\pi}\,\Gamma(n)} \exp[(\beta^2-1)b' + \alpha c' - \frac{c'^2}{4b'}] \;.$$

4.7.34 <u>$p(b',d')$</u>

From (4.75) by integrating on c' [see Table III (13)]

$$(4.114)\quad p(b',d') = \frac{\alpha^n (b')^{n-\frac{3}{2}}}{2\sqrt{\pi}\,\Gamma(n)} \exp[(\alpha^2-1)b' + \beta d' - \frac{d'^2}{4b'}] \;.$$

4.7.35 <u>$p(c',d')$</u>

From (4.74), by integrating on a' [see Table III (12), (14)]

$$(4.115)\quad p(c',d') = \frac{\alpha^n e^{\alpha c'+\beta d'}}{2^n \pi\,\Gamma(n)} (c'^2+d'^2)^{\frac{1}{2}(n-1)} K_{n-1}(\sqrt{c'^2+d'^2}) \;.$$

4.7.36 <u>$p(r',\phi)$</u>

Let

$$(4.116)\quad r' = \sqrt{c'^2+d'^2} \;,$$

so that from (4.16.3), (4.16.4), and (4.35)

(4.117) $c' = r'\cos\phi$, $d' = r'\sin\phi$; and $r'\delta = 2nr$.

Thus, from (4.115)

$$(4.118)\quad p(r',\phi) = \frac{\delta^{n} e^{\gamma r'\cos(\phi-\phi_0)}}{2^{n}\pi\,\Gamma(n)}\,(r')^{n} K_{n-1}(r').$$

4.7.37 <u>$p(r')$</u>

From (4.118), by integrating on ϕ [see Table III (3)]

$$(4.119)\quad p(r') = \frac{\delta^{n}}{2^{n-1}\Gamma(n)}\, I_0(\gamma r')(r')^{n} K_{n-1}(r').$$

4.8 <u>Simultaneous Confidence Statements for the Sample Complex Regression Coefficient of Y on X.</u>

From (4.17) and Fig. (4) one is justified in drawing Fig. (5). (In Fig. (5), $\Delta\ell_0 < \gamma$).

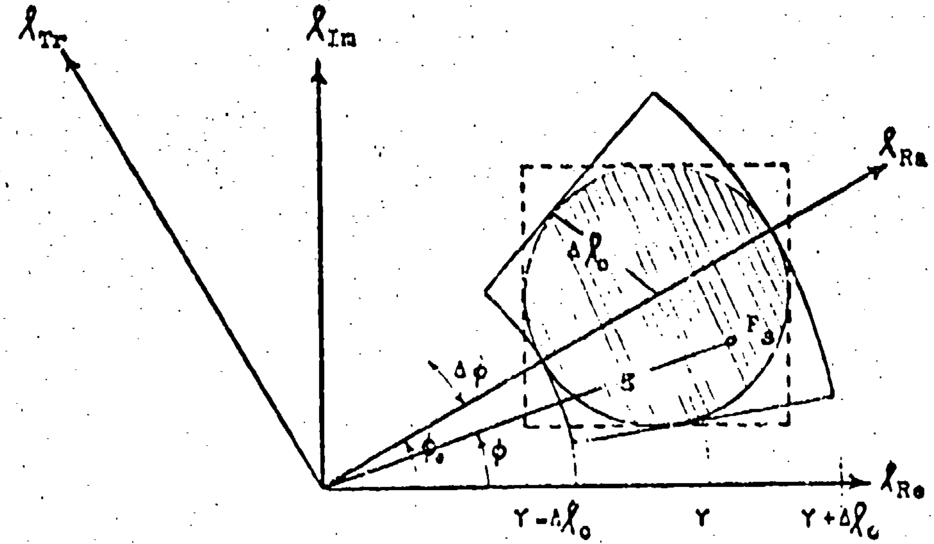


Fig. (5).

From Fig. (5) and (4.69)

$$(4.120)\quad \text{Prob.}\,(|\phi - \phi_0| < \Delta\phi,\ |\kappa - \gamma| < \Delta\ell_c)$$

$$> \text{Prob.}\,(\ell_c < \Delta\ell_c) = 1 - [1 + \delta^{-1}(\Delta\ell_c)^2]^{-n}\ ,$$

where $\Delta\ell_c = \gamma \sin \Delta\phi$.

A set S of similar inequalities can be obtained by enclosing the circular region of Fig. (5) in other regions. Thus (see square region of Fig. (5)),

$$(4.121)\quad \text{Prob.}\,(|\ell_{Re} - \alpha| < \Delta\ell_c,\ |\ell_{Im} - \beta| < \Delta\ell_c) > 1 - [1 + \delta^{-1}(\Delta\ell_c)^2]^{-n}\ .$$

Furthermore,

$$(4.122)\quad \text{Prob.}\,(\text{Set of Inequalities } S) \geq 1 - [1 + \delta^{-1}(\Delta\ell_c)^2]^{-n}\ .$$

TABLE II

Exponential Fourier Transforms

	$f(x)$	$g(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-ixy}\, dx$
(1)	$f(x) = f(-x)$	$\frac{1}{\pi} \int_0^{\infty} f(x) \cos x y\, dx$
(2)	$f(a^{-1}x + b) \quad a > 0$	$a\, e^{iaby}\, g(ay)$
(3)	$f(ax)\, e^{ibx} \quad a > 0$	$\frac{1}{a}\, g\left(\frac{y-b}{a}\right)$
(4)	$(a-ix)^{-\nu} \quad \mathrm{Re}\ a > 0$, $\mathrm{Re}\ \nu > 0$	$y^{\nu-1}\, e^{-ay}/\Gamma(\nu) \quad y > 0$; $0 \quad y < 0$
(5)	$e^{-\frac{x^2}{a}} \quad \mathrm{Re}\ a > 0$	$\frac{1}{2} \sqrt{\frac{a}{\pi}}\, e^{-\frac{1}{4} a y^2}$
(6)	$(a^2+x^2)^{-n} \quad \mathrm{Re}\ a > 0$, $\mathrm{Re}\ n > 0$	$\frac{1}{\sqrt{\pi}\, \Gamma(n)} \left(\frac{\lvert y \rvert}{2a}\right)^{n-\frac{1}{2}} K_{n-\frac{1}{2}}(a \lvert y \rvert)$

TABLE III

Functions and Integrals

(1)	Incomplete beta function[1]	$B_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}dt$
(2)	Beta function	$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt$ $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$
(2')	Gamma function	$\Gamma(p) = \int_0^\infty e^{-t}t^{p-1}dt$
(3)	Bessel function with purely imaginary argument $I_0(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{z\cos\phi}\, d\phi = J_0(iz) = \sum_{r=0}^{\infty} \frac{(\frac{1}{2}z)^{2r}}{(r!)^2}$.	
(4)	Modified Bessel function of the third kind of order half an odd integer $K_{n-\frac{1}{2}}(z) = (\frac{\pi}{2z})^{\frac{1}{2}} e^{-z} \sum_{k=0}^{n-1} (2z)^{-k} \frac{\Gamma(n+k)}{k!\Gamma(n-k)}$, $n=1,2,\ldots$.	
(5)	$\int_a^\infty (u-a)^n e^{-u} du = \Gamma(n+1)\, e^{-a}$	

[1] This definition of the incomplete beta function is the one given in [4]. In mathematical statistics it is customary to refer to the function $I_x(p,q) \equiv B_x(p,q)/B_1(p,q)$ as the incomplete beta function.

Table III (continued)

(6)	**Bessel function with purely imaginary argument of order half an odd integer** $$\int_{-1}^{1} e^{zt}(1-t^2)^n dt = 2^{n+\frac{1}{2}} \sqrt{\pi}\,\Gamma(n+1) z^{-n-\frac{1}{2}} I_{n+\frac{1}{2}}(z)$$ where $$I_{n+\frac{1}{2}}(z) = \frac{(2z)^{n+\frac{1}{2}}}{\sqrt{\pi}} \frac{d^n}{d(z^2)^n}\left(\frac{\sinh z}{z}\right), \quad n=0,1,2,3,\ldots .$$ so that $$I_{n+\frac{1}{2}}(z) = P_n \sinh z + Q_n \cosh z$$ where P_n, Q_n are polynomials in $z^{\frac{1}{2}}$.
(7)	$\int_0^{\infty} u^n e^{-bu}\, du = \frac{\Gamma(n+1)}{b^{n+1}}$ (Re $b > 0$) .
(8)	$\int_{-\infty}^{\infty} \frac{du}{(a^2+b^2u^2)^{n+1}} = \frac{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}{a^{2n+1} b\, \Gamma(n+1)}$.
(9)	$\int_0^{\infty} \frac{(1+u^2)^n du}{(a^2+u^2)^{n+1}} = \frac{1}{2a^{2n+1}} \sum_{k=0}^{n} \frac{\Gamma(k+\frac{1}{2})\Gamma(n-k+\frac{1}{2})}{\Gamma(k+1)\Gamma(n-k+1)} a^{2k}$.
(10)	$\int_0^{\theta_0} \cos^n \theta d\theta = \frac{1}{2}\int_0^{\sin^2\theta_0} u^{-\frac{1}{2}}(1-u)^{\frac{n-1}{2}} du = \frac{1}{2}B_{\sin^2\theta_0}\left(\frac{1}{2},\frac{n+1}{2}\right)$ $(0 \le \theta_0 \le \frac{\pi}{2})$.

TABLE III (continued)

(11)	Modified Bessel function of the first kind $$I_{n-1}(z) = \sum_{k=0}^{\infty} \frac{(\frac{1}{2}z)^{2k+n-1}}{k!\,\Gamma(k+n)}$$
(12)	Modified Bessel function of the third kind $$K_\nu(z) = \frac{\pi}{2}\,\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}$$ and $K_n(z) = \lim_{\nu \to n} K_\nu(z)$ for integer n . $$K_n(z) = \frac{1}{2}\sum_{m=0}^{n-1} \frac{(-)^m (n-m-1)!}{m!(\frac{1}{2}z)^{n-2m}}$$ $$+(-)^{n+1}\sum_{m=0}^{\infty}\frac{(\frac{1}{2}z)^{n+2m}}{m!(n+m)!}\left[\log(\tfrac{1}{2}z) - \tfrac{1}{2}\psi(m+1) - \tfrac{1}{2}\psi(n+m+1)\right]$$ where $\psi(1) = -\gamma$ (γ denotes Euler's constant 0.5772157 ...) $\psi(m+1) = \frac{1}{1} + \frac{1}{2} + \ldots\ldots + \frac{1}{m} - \gamma$.
(13)	$$\int_{-\infty}^{\infty} e^{-(ax^2+bx)}\,dx = \sqrt{\frac{\pi}{a}}\; e^{\frac{b^2}{4a}} \qquad (a > 0) .$$
(14)	$$\int_0^{\infty} e^{-px}\, x^{n-2}\, e^{-\frac{a}{4x}}\, dx = 2\left(\frac{a}{4p}\right)^{\frac{n-1}{2}} K_{n-1}(\sqrt{a\,p}\,)$$ (Re a > 0) (Re p > 0) .

Chapter 5

Applications

5.0 Introduction

In this chapter it is shown how the results of the previous chapters may be used to discuss the sampling variability of estimators for the spectral, cospectral, and quadrature spectral densities of a two-dimensional stationary (zero mean) Gaussian vector process. The response of a linear time invariant system to random inputs is studied, and it is shown how certain functions of the spectral, cospectral, and quadrature spectral densities of measured input-output records are related to the frequency response function of the system, extraneous inputs to the system, and errors in measuring input and output. It is then indicated how the sampling variability of estimators for these functions can be discussed by means of the results of Chapter 4.

5.1 The Joint Distribution of Estimators for the Spectral, Cospectral, and Quadrature Spectral Densities.

At the end of Section 3.7 it was indicated that the joint distribution of (a,b,c,d) with appropriate parameters α,β,n approximates the joint distribution of the relative or dimensionless estimators for the spectra, cospectrum, and quadrature spectrum

$$(5.1)\quad \left[\frac{\hat{Q}_{\alpha_i}}{E\hat{Q}_{\alpha_i}},\ \frac{\hat{Q}_{\beta_i}}{E\hat{Q}_{\beta_i}},\ \frac{\hat{Q}_{\gamma_i}}{\sqrt{(E\hat{Q}_{\alpha_i})(E\hat{Q}_{\beta_i})}},\ \frac{\hat{Q}_{\delta_i}}{\sqrt{(E\hat{Q}_{\alpha_i})(E\hat{Q}_{\beta_i})}}\right]$$

In Chapter 4 the probability density function of the joint distribution of (a,b,c,d) and the probability density functions of many related distributions were derived. In order to apply these results to the joint distribution of estimators for the spectra, cospectrum, and quadrature spectrum it is necessary to determine the parameters α,β,n. A discussion on how the parameters α,β,n can be determined is presented below.

On the assumption that (5.1) is distributed as is (a,b,c,d) i.e. with a Unit Complex Wishart distribution it follows from (4.33) that

$$(5.2)\qquad \frac{E\,\hat{Q}_{\gamma_i}}{\sqrt{(E\hat{Q}_{\alpha_i})(E\hat{Q}_{\beta_i})}} = \alpha, \qquad \frac{E\,\hat{Q}_{\delta_i}}{\sqrt{(E\hat{Q}_{\alpha_i})(E\hat{Q}_{\beta_i})}} = \beta\ ;$$

so that from (4.34)

(5.3)

$$(aa)\qquad n_{aa}^{(i)} \equiv \frac{(E\hat{Q}_{\alpha_i})^2}{\mathrm{Var}(\hat{Q}_{\alpha_i})} = n,$$

$$(ab)\qquad n_{ab}^{(i)} \equiv \frac{(E\hat{Q}_{\gamma_i})^2 + (E\hat{Q}_{\delta_i})^2}{\mathrm{Cov}(\hat{Q}_{\alpha_i},\hat{Q}_{\beta_i})} = n,$$

(5.3) continued

$$\text{(ac)} \qquad n_{ac}^{(i)} = \frac{(E\,\hat{Q}_{\alpha_i})(E\,\hat{Q}_{\gamma_i})}{\mathrm{Cov}(\hat{Q}_{\alpha_i},\hat{Q}_{\gamma_i})} = n,$$

$$\text{(ad)} \qquad n_{ad}^{(i)} = \frac{(E\,\hat{Q}_{\alpha_i})(E\,\hat{Q}_{\delta_i})}{\mathrm{Cov}(\hat{Q}_{\alpha_i},\hat{Q}_{\delta_i})} = n,$$

$$\text{(bb)} \qquad n_{bb}^{(i)} = \frac{(E\,\hat{Q}_{\beta_i})^2}{\mathrm{Var}(\hat{Q}_{\beta_i})} = n,$$

$$\text{(bc)} \qquad n_{bc}^{(i)} = \frac{(E\,\hat{Q}_{\beta_i})(E\,\hat{Q}_{\gamma_i})}{\mathrm{Cov}(\hat{Q}_{\beta_i},\hat{Q}_{\gamma_i})} = n,$$

$$\text{(bd)} \qquad n_{bd}^{(i)} = \frac{(E\,\hat{Q}_{\beta_i})(E\,\hat{Q}_{\delta_i})}{\mathrm{Cov}(\hat{Q}_{\beta_i},\hat{Q}_{\delta_i})} = n,$$

$$\text{(cc)} \qquad n_{cc}^{(i)} = \frac{(E\,\hat{Q}_{\alpha_i})(E\,\hat{Q}_{\beta_i})+(E\,\hat{Q}_{\gamma_i})^2-(E\,\hat{Q}_{\delta_i})^2}{2\,\mathrm{Var}(\hat{Q}_{\gamma_i})} = n,$$

$$\text{(cd)} \qquad n_{cd}^{(i)} = \frac{(E\,\hat{Q}_{\gamma_i})(E\,\hat{Q}_{\delta_i})}{\mathrm{Cov}(\hat{Q}_{\gamma_i},\hat{Q}_{\delta_i})} = n,$$

(dd) $$n_{dd}^{(1)} \approx \frac{(E\,\hat{Q}_{\alpha_1})(E\,\hat{Q}_{\beta_1})-(E\,\hat{Q}_{\gamma_1})^2+(E\,\hat{Q}_{\delta_1})^2}{2\,\mathrm{Var}(\hat{Q}_{\delta_1})} = n.$$

From (3.40) on the assumption that 2M+1 is sufficiently large so that the Fejér kernel $K_{2M+1}(\lambda')$ acts effectively as a delta function centered at $\lambda'=0$

(5.4)

(aa) $$V_{aa}(\lambda) \approx \frac{8\pi}{2M+1}\; f_x^2(\lambda),$$

(ab) $$V_{ab}(\lambda) \approx \frac{8\pi}{2M+1}\; [c^2(\lambda)+q^2(\lambda)],$$

(ac) $$V_{ac}(\lambda) \approx \frac{8\pi}{2M+1}\; f_x(\lambda)c(\lambda),$$

(ad) $$V_{ad}(\lambda) \approx \frac{8\pi}{2M+1}\; f_x(\lambda)q(\lambda),$$

(bb) $$V_{bb}(\lambda) \approx \frac{8\pi}{2M+1}\; f_y^2(\lambda),$$

(bc) $$V_{bc}(\lambda) \approx \frac{8\pi}{2M+1}\; f_y(\lambda)c(\lambda),$$

(bd) $$V_{bd}(\lambda) \approx \frac{8\pi}{2M+1}\; f_y(\lambda)q(\lambda),$$

(cc) $$V_{cc}(\lambda) \approx \frac{4\pi}{2M+1}\; [f_x(\lambda)f_y(\lambda)+c^2(\lambda)-q^2(\lambda)],$$

(5.4) continued

(cd) $V_{cd}(\lambda) \approx \frac{8\pi}{2M+1} c(\lambda)q(\lambda)$,

(dd) $V_{dd}(\lambda) \approx \frac{4\pi}{2M+1} [f_x(\lambda)f_y(\lambda)+q^2(\lambda)-c^2(\lambda)]$.

Thus, from (3.23) and (3.38) on the assumption that $f_x(\lambda)$, $f_y(\lambda)$, $c(\lambda)$, $q(\lambda)$ are approximately constant over the widths of the filters $\hat{P}_a^{(i)}(\lambda)$, $\hat{P}_b^{(i)}(\lambda)$, $\hat{P}_c^{(i)}(\lambda)$, $\hat{P}_d^{(i)}(\lambda)$

(5.5.1) $E\,\hat{Q}_{\alpha_i} \approx 2\, f_x(\lambda^{(i)}) \int_0^\pi \hat{P}_a^{(i)}(\lambda)d\lambda$,

(5.5.2) $E\,\hat{Q}_{\beta_i} \approx 2\, f_y(\lambda^{(i)}) \int_0^\pi \hat{P}_b^{(i)}(\lambda)d\lambda$,

(5.5.3) $E\,\hat{Q}_{\gamma_i} \approx 2\, c(\lambda^{(i)}) \int_0^\pi \hat{P}_c^{(i)}(\lambda)d\lambda$,

(5.5.4) $E\,\hat{Q}_{\delta_i} \approx 2\, q(\lambda^{(i)}) \int_0^\pi \hat{P}_d^{(i)}(\lambda)d\lambda$,

and

(5.6)

(aa) $\mathrm{Var}(\hat{Q}_{\alpha_i}) \approx \frac{8\pi}{2M+1} f_x^2(\lambda^{(i)}) \int_0^\pi (\hat{P}_a^{(i)}(\lambda))^2\, d\lambda$,

(5.6) continued

(ab) $\mathrm{Cov}(\hat{Q}_{\alpha_i},\hat{Q}_{\beta_i}) \approx \frac{8\pi}{2M+1}\left[c^2(\lambda^{(i)})+q^2(\lambda^{(i)})\right]\int_0^{\pi}\hat{P}_a^{(i)}(\lambda)\hat{P}_b^{(i)}(\lambda)\,d\lambda,$

(ac) $\mathrm{Cov}(\hat{Q}_{\alpha_i},\hat{Q}_{\gamma_i}) \approx \frac{8\pi}{2M+1} f_x(\lambda^{(i)})c(\lambda^{(i)}) \int_0^{\pi}\hat{P}_a^{(i)}(\lambda)\hat{P}_c^{(i)}(\lambda)\,d\lambda,$

(ad) $\mathrm{Cov}(\hat{Q}_{\alpha_i},\hat{Q}_{\delta_i}) \approx \frac{8\pi}{2M+1} f_x(\lambda^{(i)})q(\lambda^{(i)}) \int_0^{\pi}\hat{P}_a^{(i)}(\lambda)\hat{P}_d^{(i)}(\lambda)\,d\lambda,$

(bb) $\mathrm{Var}(\hat{Q}_{\beta_i}) \approx \frac{8\pi}{2M+1} f_y^2(\lambda^{(i)}) \int_0^{\pi}\left(\hat{P}_b^{(i)}(\lambda)\right)^2 d\lambda,$

(bc) $\mathrm{Cov}(\hat{Q}_{\beta_i},\hat{Q}_{\gamma_i}) \approx \frac{8\pi}{2M+1} f_y(\lambda^{(i)})c(\lambda^{(i)}) \int_0^{\pi}\hat{P}_b^{(i)}(\lambda)\hat{P}_c^{(i)}(\lambda)\,d\lambda,$

(bd) $\mathrm{Cov}(\hat{Q}_{\beta_i},\hat{Q}_{\delta_i}) \approx \frac{8\pi}{2M+1} f_y(\lambda^{(i)})q(\lambda^{(i)}) \int_0^{\pi}\hat{P}_b^{(i)}(\lambda)\hat{P}_d^{(i)}(\lambda)\,d\lambda,$

(cc) $\mathrm{Var}(\hat{Q}_{\gamma_i}) \approx \frac{8\pi}{2M+1}\Big[f_x(\lambda^{(i)})f_y(\lambda^{(i)})+c^2(\lambda^{(i)})$

$\qquad - q^2(\lambda^{(i)})\Big]\int_0^{\pi}\left(\hat{P}_c^{(i)}(\lambda)\right)^2 d\lambda,$

(cd) $\mathrm{Cov}(\hat{Q}_{\gamma_i},\hat{Q}_{\delta_i}) \approx \frac{8\pi}{2M+1} c(\lambda^{(i)})q(\lambda^{(i)}) \int_0^{\pi}\hat{P}_c^{(i)}(\lambda)\hat{P}_d^{(i)}(\lambda)\,d\lambda,$

(5.6) continued

(dd) $\mathrm{Var}(\hat{Q}_{\delta_i}) \approx \frac{4\pi}{2M+1}\Big[f_x(\lambda^{(i)}) f_y(\lambda^{(i)}) + q^2(\lambda^{(i)})$

$$- c^2(\lambda^{(i)})\Big] \int_0^{\pi} (\hat{\beta}_d^{(i)}(\lambda))^2 d\lambda,$$

where $\lambda^{(i)}$ denotes the frequency at which the filters $\hat{\beta}_a^{(i)}(\lambda)$, $\hat{\beta}_b^{(i)}(\lambda)$, $\hat{\beta}_c^{(i)}(\lambda)$, $\hat{\beta}_d^{(i)}(\lambda)$ are centered.

Equations (5.2), (5.3), (5.4), (5.5) and (5.6) indicate how the parameters α, β, n can be determined. From (5.2) and (5.5)

$$(5.7) \quad \alpha = \frac{c(\lambda^{(i)})}{\sqrt{f_x(\lambda^{(i)}) f_y(\lambda^{(i)})}}, \quad \beta = \frac{q(\lambda^{(i)})}{\sqrt{f_x(\lambda^{(i)}) f_y(\lambda^{(i)})}} .$$

If sufficient a priori information about $f_x(\lambda)$, $f_y(\lambda)$, $c(\lambda)$, $q(\lambda)$ is available it may be possible to determine α, β from (5.7). Such information will rarely be available, however, it is thus suggested that α, β be taken respectively as

$$(5.8) \quad \alpha \approx \frac{\hat{Q}_{\gamma_i}}{\sqrt{\hat{Q}_{\alpha_i} \hat{Q}_{\beta_i}}}, \quad \beta \approx \frac{\hat{Q}_{\delta_i}}{\sqrt{\hat{Q}_{\alpha_i} \hat{Q}_{\beta_i}}} ;$$

i.e. that α, β be taken as equal to their sample values.

If the ideal filters given by (3.24) and (3.26) were attainable, then from each of the ten cases of (5.3) using (5.5) and (5.6) it would follow that

$$(5.9) \qquad 2n \approx \frac{(\lambda_i - \lambda_{i-1})}{\pi} (2M+1).$$

For the polynomial filters of Section 3.6 (where K_0 and K_1 are defined) it can be shown that for m sufficiently large so that $\frac{1}{m^2}$ is small compared to $\frac{1}{m}$,

$$(5.10.1) \qquad \int_0^{\pi} \hat{p}_u^{(i)}(\lambda) d\lambda \approx \frac{\pi}{2m} (K_0 + 2K_1) = \frac{\pi}{2m},$$

$$(5.10.2) \qquad \int_0^{\pi} \hat{p}_u^{(i)}(\lambda)\, \hat{p}_v^{(i)}(\lambda) d\lambda \approx \frac{\pi}{4m} (K_0^* + 2K_1^*)$$

$$u,v = a,b,c,d; \; i = 2,3,\ldots,m-2.$$

Thus, from each of the ten cases of (5.3) using (5.5) and (5.6)

$$(5.11) \qquad 2n^{(i)} \approx \frac{1}{K_0^* + 2K_1^*} \cdot \frac{2M+1}{m}$$

for $i = 2,3,\ldots,m-2$.

The assumptions on which (5.11) was obtained are

(1) the densities $f_x(\lambda)$, $f_y(\lambda)$, $c(\lambda)$, $q(\lambda)$ are approximately constant over the widths of the filters,

(2) $2M+1$ is sufficiently large that the Fejér kernel $K_{2M+1}(\lambda')$ acts effectively as a delta function centered at $\lambda'=0$,

(3) m sufficiently large ($\frac{1}{m^2}$ small compared to $\frac{1}{m}$).

That condition (1) above is satisfied can often be established from a priori knowledge of $f_x(\lambda)$, $f_y(\lambda)$, $c(\lambda)$, $q(\lambda)$. Thus, when the conditions (1), (2), (3) are satisfied and the filters of Section 3.6 are used, it is suggested that n be determined from (5.11). When condition (1) is <u>not</u> satisfied, the determination of an appropriate n to use in the Complex Wishart distribution becomes difficult. If condition (1) is not too flagrantly violated it is suggested that $n_{aa}^{(i)}$, $n_{ab}^{(i)}$, ..., $n_{dd}^{(i)}$, be computed from (5.3) using (3.23), (3.38), (3.40) and that $n^{(i)}$ be taken as the minimum of the $n_{aa}^{(i)}$, $n_{ab}^{(i)}$, ..., $n_{dd}^{(i)}$ so determined. When the densities $f_x(\lambda)$, $f_y(\lambda)$, $c(\lambda)$, $q(\lambda)$ vary greatly over the widths of the filters, the use of the Complex Wishart distribution is no longer justified by the argument of Section 3.7. In such cases it may be possible (from a priori knowledge of $f_x(\lambda)$, $f_y(\lambda)$, $c(\lambda)$, $q(\lambda)$, to transform $[x_k, y_k]$ into $[\hat{x}_k, \hat{y}_k]$ by a linear transformation of the form where $K(\tau)$ denotes the impulse response function of the system and L denotes the linear operator expressed on the right hand side of (5.13).

The exponentials $e^{i\lambda t}$ are eigenfunctions of the operator L and

$$(5.14) \qquad L\, e^{i\lambda t} = \ell(\lambda) e^{i\lambda t},$$

where

$$(5.15) \qquad \ell(\lambda) = u(\lambda) + iv(\lambda) = \int_0^\infty K(\tau) e^{-i\lambda\tau} d\tau .$$

The function $\ell(\lambda)$ is called the frequency response function or the transfer function of the system L and characterizes the system. From (5.14) one observes that $\ell(\lambda)$ essentially gives the output of the system L to a sinusoidal input of frequency λ. A simple computation shows that

$$(5.16) \qquad L \sin \lambda t = u(\lambda) \sin \lambda t + v(\lambda) \cos \lambda t = \sqrt{u^2+v^2}\, \sin[\, t+\phi(\lambda)]$$

where $\ell(\lambda) = \sqrt{u^2(\lambda) + v^2(\lambda)}\; e^{i\phi(\lambda)}$.

Also, by taking Fourier transforms in (5.13)

$$(5.17) \qquad Y(\lambda) = \ell(\lambda)\, X(\lambda).$$

$$X(\lambda) = \int_{-\infty}^{\infty} x(\tau)e^{-i\lambda\tau}d\tau \text{ and } Y(\lambda) = \int_{-\infty}^{\infty} y(\tau)e^{-i\lambda\tau}d\tau.$$

The determination of the frequency response function $\mathcal{K}(\lambda)$ of a system is of considerable practical interest. Much attention has been devoted to the problem of experimentally determining the frequency response function of a system and several methods have been employed. The methods customarily employed differ essentially in the choice of the input excitation used.

The classical method is suggested by (5.16). If the system L is subjected to a sinusoidal input of frequency λ, the resultant output is also sinusoidal of frequency λ but possibly of different amplitude and phase. The amplitude $\sqrt{u^2(\lambda)+v^2(\lambda)}$ and phase $\phi(\lambda)$ determine $\mathcal{K}(\lambda)$. Thus, the value of the frequency response function $\mathcal{K}(\lambda)$ at frequency λ is determined by measuring the amplitude and phase of the output which results from a sinusoidal input of frequency λ. The function $\mathcal{K}(\lambda)$ is explored by letting λ vary through a set of frequencies $\lambda_0, \lambda_1, \ldots, \lambda_n$.

A second method is suggested by (5.13) and (5.15). If the system L is subjected to a unit-impulse input (i.e. ideally $x(t) = \delta(t)$ where $\delta(t)$ denotes the Dirac delta function) then as is seen from (5.13) the output is $y(t) = K(t)$. Thus, if the output resulting from a unit-impulse input is measured, the frequency response function $\mathcal{K}(\lambda)$ is calculated by using (5.15).

A third method (which includes the first two as special cases) is suggested by (5.17). If $y(t)$ denotes the output resulting from an arbitrary input $x(t)$ (for which a Fourier transform exists) then the frequency response function $\ell(\lambda)$ is calculated by using (5.17) provided that $X(\lambda) \neq 0$.

In the preceding discussion of the three methods no mention is made of difficulties that may be encountered in attempting to employ the methods. Such difficulties are:

(5.18)

(1) difficulties in exciting the system by the desired input or inputs

(2) existence of extraneous noise inputs to the system which are not measured.

(3) errors in measuring the input and output

(4) computational difficulties.

In the applications of the methods discussed above consideration is usually directed to difficulties (5.18.1) and (5.18.4) and the particular method employed in a given situation reflects the results of such consideration. When the errors and extraneous noise are small, difficulties (5.18.2) and (5.18.3) are ignored, but when the errors and extraneous noise are no longer sensibly negligible the methods become inapplicable as no means of eliminating or assessing the effect of the errors and extraneous noise on the computed frequency response function is provided. A method of determining the

the frequency response function which in certain cases circumvents these difficulties to some degree is suggested by Y.W.Lee in [9]. The method essentially involves subjecting the system L to a random noise input, measuring the input and output, and from the measured input-output record "determining" the input spectral density and the cross-spectral density between input and output. Under suitable assumptions (input measured without error, input incoherent with extraneous noise), the ratio of the cross-spectral density to the input spectral density gives the frequency response function of the system. Lee [9] essentially assumes that the covariance functions and spectral densities required to determine the frequency response function are known exactly (or equivalently that an infinite sample record is available). A quantitative basis for applying the method with only a finite sample record of input and output and when errors in the measured input exist is not provided. Such a quantitative basis can be provided by recognizing the problem as one of the joint estimation of the spectral, cospectral and quadrature-spectral densities of a certain two dimensional stationary Gaussian vector process and utilizing the results of Chapters 3 and 4 which deal with the statistical estimation of functions of these densities. A discussion of the statistical estimation of the frequency response function of a linear time-invariant system is now presented.

Consider the following model:

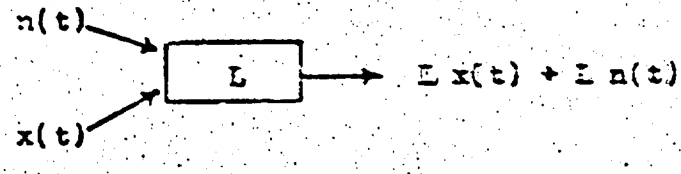


Fig. (6)

With reference to Fig. (6),

$$
\begin{aligned}
x(t) &= \text{noise input} \\
n(t) &= \text{extraneous noise input} \\
y(t) &= L\,x(t) + L\,n(t) = \text{noise output} \\
e_i(t) &= \text{error in measuring input } x(t) \qquad (5.19) \\
e_o(t) &= \text{error in measuring output } y(t) \\
x^*(t) &= x(t) + e_i(t) = \text{measured input noise} \\
y^*(t) &= y(t) + e_o(t) = \text{measured output noise.}
\end{aligned}
$$

The functions $x(t)$, $n(t)$, $e_i(t)$, $e_o(t)$ are assumed to be sample functions of stationary (zero mean) Gaussian processes incoherent with each other. Thus, $[x^*(t), y^*(t)]$, $-\infty < t < \infty$ is a sample func~~tion of~~ a stationary (zero mean) Gaussian process and the density functions characterizing this process are given by

$$
\begin{aligned}
f_{xx} &= s_x + s_i \\
f_{yy} &= (u^2+v^2)\, s_x + (u^2+v^2)\, s_n + s_o \qquad (5.20) \\
c &= u s_x \\
q &= -v s_x
\end{aligned}
$$

where $s_x(\lambda)$, $s_n(\lambda)$, $s_1(\lambda)$, $s_0(\lambda)$ are the spectral densities of the $x(t)$, $n(t)$, $s_1(t)$, $s_0(t)$ processes respectively.

Let

$$(5.21.1) \qquad \rho\, e^{i\phi} = \frac{c - i\, q}{f_{xx}}$$

$$(5.21.2) \qquad z = \frac{c^2 + q^2}{f_{xx} f_{yy}}$$

$$(5.21.3) \qquad r = \frac{z}{1-z} \,.$$

Thus,

$$\left.\begin{aligned}
&(5.22.1) \qquad \frac{c}{f_{xx}} = \frac{u}{1 + \frac{s_1}{s_x}} \approx u \\
&(5.22.2) \qquad \frac{-q}{f_{xx}} = \frac{v}{1 + \frac{s_1}{s_x}} \approx v \\
&(5.22.3) \qquad \rho = \frac{\sqrt{u^2 + v^2}}{1 + \frac{s_1}{s_x}} \approx \sqrt{u^2 + v^2}
\end{aligned}\right] \quad \left(\text{if } \frac{s_1}{s_x} \text{ is small}\right),$$

$$(5.22.4) \qquad \phi = \mathrm{Arg}(u + iv) \,, \ (\text{if } s_x \neq 0),$$

$$(5.22.5)\quad z = \frac{1}{1 + \frac{s_n}{s_x} + \frac{s_i}{s_x} + \frac{s_n}{s_x}\cdot\frac{s_i}{s_x} + \frac{s_o}{(u^2+v^2)s_x} + \frac{s_o}{(u^2+v^2)s_x}\cdot\frac{s_i}{s_x}}$$

$$\text{(if } (u^2+v^2)s_x \neq 0\text{)},$$

$$(5.22.6)\quad r = \frac{(u^2+v^2)s_x}{(1+\frac{s_i}{s_x})(u^2+v^2)s_n + (1+\frac{s_i}{s_x})s_o + (u^2+v^2)s_i}$$

$$\approx \frac{\text{output signal power spectral density}}{\text{output noise power spectral density}}$$

$$\text{(if } \frac{s_i}{s_x} \text{ is small)},$$

$$(5.22.7)\quad \frac{f_{yy}}{f_{xx}} = \frac{u^2+v^2}{1+\frac{s_i}{s_x}} + \frac{u^2+v^2}{\frac{s_x}{s_n}+\frac{s_i}{s_n}} + \frac{1}{\frac{s_x}{s_o}+\frac{s_i}{s_o}} .$$

It is clear from (5.21) and (5.22) that the functions $f_{xx}, f_{yy}, c, q, \frac{c}{f_{xx}}, \frac{q}{f_{xx}}, \rho, \phi$, z, and r are of particular interest in studying the noises and the system L. These functions can be estimated from a finite part of the measured [input, output] record [$x^*(t)$, $y^*(t)$] and the sampling variability of the estimators discussed by means of the distributions given in Chapter 4. More precisely, what can be estimated are not the functions themselves, but corresponding functions of averages of the densities f_{xx}, f_{yy}, c, q where the averages are such as to concentrate most of their weight in narrow frequency bands. Given below in Table IV is a list of the functions to be estimated (in the sense just discussed) and the estimators for these functions. The Complex-Wishart distribution and related distributions derived in Chapter 4 give the distributions of the estimators. The distributions which are relevant to discussing the variability of a given estimator are indicated in the last column of a Table IV.

The results of Section 4.8 can be used to plan experiments to "measure" the frequency response function. To facilitate such use the functions

(5.23)
$$\sin\Delta\phi = \left[\frac{1-\gamma^2}{\gamma^2}\left[(1-P)^{\frac{-m}{N'}} - 1\right]\right]^{\frac{1}{2}}$$
$$\Delta\phi = \text{arc}\sin[\quad]^{\frac{1}{2}}$$

were computed for various values of the arguments γ^2, P, m, N' and listed in Table V. From Sections 3.6, 3.7, 4.8 and 5.1,

(5.24)
$$\text{Prob.}\left[\begin{matrix} 1 - \sin\Delta\phi < \dfrac{\text{Sample gain}}{\text{True gain}} < 1 + \sin\Delta\phi \\ -\Delta\phi < (\text{Sample phase}) - (\text{True phase}) < \Delta\phi \end{matrix}\right] > P \quad .$$

In (5.24), $\Delta\phi$ and $\sin\Delta\phi$ are given by (5.23), where

N = length of record,

m = highest order lagged products used in computing the $\hat{Q}_{\alpha_1}, \hat{Q}_{\beta_1}, \hat{Q}_{\gamma_1}, \hat{Q}_{\delta_1}$ (See (3.36);

$m = N-2M-1$),

(5.25) $N' = N-m$ = effective length of record,

γ^2 = a priori estimate of coherency,

P = confidence level.

Furthermore, in (5.24)

$$\text{Sample gain} = \frac{\sqrt{\hat{Q}^2_{\gamma_1} + \hat{Q}^2_{\delta_1}}}{\hat{Q}_{\alpha_1}}$$

(5.26)

$$\text{Sample phase} = \mathrm{Arg}(\hat{Q}_{\gamma_1} + i\hat{Q}_{\delta_1}),$$

where the polynomial filters used in computing the $\hat{Q}_{\alpha_1}, \hat{Q}_{\gamma_1}$ are given by (3.99) and the polynomial filters used in computing the $\hat{Q}_{\delta_1}$ are given by (3.100), and in each case (K_0, K_1) is given by (3.98.1). With $P = 0.75$, $\gamma^2 = 0.50$, $m=30$, $N'=1000$ one has (see Table V) $\sin\Delta\phi \approx 0.21$, $\Delta\phi \approx 12^\circ$. What is achieved with a record twice as long? With $P = 0.75$, $\gamma^2=0.50$, $m=30$, $N'=2000$; $\sin\Delta\phi \approx 0.15$, $\Delta\phi \approx 8^\circ$.

Table IV

Estimators of Interest in Studying A Linear Time-Invariant System[1]

Function Estimated	Estimator	Complex Wishart Variable Corresponding to Estimator	Relevant Distributions
f_{xx}	$\hat{Q}_{\alpha_1}$	A	(4.64),(4.43)
f_{yy}	$\hat{Q}_{\beta_1}$	B	(4.67),(4.43)
c	$\hat{Q}_{\gamma_1}$	C	(4.70),(4.115)
q	$\hat{Q}_{\delta_1}$	D	(4.72),(4.115)
c/f_{xx}	$\hat{Q}_{\gamma_1}/\hat{Q}_{\alpha_1}$	C/A	(4.92),(4.91)
q/f_{xx}	$\hat{Q}_{\delta_1}/\hat{Q}_{\alpha_1}$	D/A	(4.93),(4.91)
$\rho = \frac{\sqrt{c^2+q^2}}{f_{xx}}$	$\frac{\sqrt{\hat{Q}_{\gamma_1} + \hat{Q}_{\delta_1}}}{\hat{Q}_{\alpha_1}}$	$\frac{\sqrt{C^2 + D^2}}{A}$	(4.97),(4.81)
$\phi_0 = \mathrm{Arg}(c-iq)$	$\mathrm{Arg}(\hat{Q}_{\gamma_1} - i\hat{Q}_{\delta_1})$	Arg(C−iD)	(4.104),(4.58),(4.81),(4.55)
$s = \frac{c^2+q^2}{f_{xx}f_{yy}}$	$\frac{\hat{Q}_{\gamma_1}^2 + \hat{Q}_{\delta_1}^2}{\hat{Q}_{\alpha_1}\hat{Q}_{\beta_1}}$	$\frac{C^2 + D^2}{AB}$	(4.60),(4.55)
$r = \frac{s}{1-s}$	$\frac{\hat{Q}_{\gamma_1}^2 + \hat{Q}_{\delta_1}^2}{\hat{Q}_{\alpha_1}\hat{Q}_{\beta_1} - \hat{Q}_{\gamma_1}^2 - \hat{Q}_{\delta_1}^2}$	$\frac{C^2 + D^2}{AB - C^2 - D^2}$	(4.60)

[1] For additional estimators refer to the material of Chapter 4 dealing with the complex regression coefficient. In particular see (4.9), (4.10), and (4.11).

Table V

Approximate Confidence Bands for Estimators of Frequency Response Function

n	10				20		30		40		50		60	
	sin Δφ	Δφ	sin Δφ	Δφ	sin Δφ	Δφ	sin Δφ	Δφ	sin Δφ	Δφ	sin Δφ	Δφ	sin Δφ	Δφ
100	.454	.483	.573	.611	.668	[illegible]	.833	.964	.979					
200	.325	.331	.400	.412	.464	[illegible]	.573	.611	.668	.731	.753	.853	.833	.984
300	.255	.268	.325	.331	.377	[illegible]	.454	.483	.539	.569	.606	.651	.668	.731
400	.229	.231	.251	.265	.325	[illegible]	.400	.412	.464	.483	.521	.546	.573	.611
500	.205	.206	.251	.254	.290	[illegible]	.357	.365	.414	.426	.464	.483	.510	.535
1000	.145	.145	.177	.178	.225	[illegible]	.251	.254	.290	.295	.325	.331	.357	.365
1500	.118	.118	.145	.145	.157	[illegible]	.205	.206	.237	.239	.265	.268	.290	.295
2000	.102	.102	.125	.125	.145	[illegible]	.177	.178	.205	.206	.229	.231	.251	.254
100	.668	.731	.833	.984	.979									
200	.464	.483	.573	.611	.668	.731	.833	.984	.979					
300	.377	.386	.464	.483	.539	.569	.668	.731	.780	.895	.849		.979	
400	.325	.331	.400	.412	.464	.483	.573	.611	.668	.731	.753	.853	.833	.984
500	.290	.295	.357	.365	.414	.426	.510	.535	.573	.635	.668	.731	.737	.828
1000	.255	.206	.251	.254	.270	.295	.357	.365	.414	.426	.464	.483	.510	.535
1500	.167	.168	.205	.206	.237	.237	.270	.275	.316	.343	.377	.386	.414	.426
2000	.145	.145	.177	.178	.205	.206	.251	.254	.270	.275	.325	.331	.357	.365
100	.724	.807	.905											
200	.501	.525	.620	.649	.724	.809	.905							
300	.467	.417	.501	.525	.593	.623	.774	.807	.847		.961			
400	.351	.359	.432	.447	.501	.525	.620	.667	.724	.809	.818	.957	.905	
500	.313	.317	.385	.395	.447	.463	.552	.584	.647	.671	.724	.809	.809	.974
1000	.271	.272	.271	.274	.313	.319	.385	.395	.447	.463	.501	.525	.552	.584
1500	.185	.181	.221	.222	.235	.234	.313	.319	.353	.371	.407	.419	.447	.463
2000	.154	.154	.191	.192	.221	.227	.271	.274	.313	.319	.351	.359	.385	.395

n	10		15		20		30		40		50		60	
	sin Δφ	Δφ	sin Δφ	Δφ	sin Δφ	Δφ	sin Δφ	Δφ	sin Δφ	Δφ	sin Δφ	Δφ	sin Δφ	Δφ
100	.792	.914	.994											
200	.546	.578	.677	.744	.792	.914	.994							
300	.443	.458	.546	.578	.636	.689	.792	.914	.929					
400	.332	.392	.470	.490	.546	.579	.677	.744	.792	.914	.896		.994	
500	.341	.348	.419	.433	.486	.508	[illegible]	.645	.701	.777	.792	.914	.876	
1000	.240	.242	.294	.299	.341	.348	.427	.433	.486	.508	.546	.578	.602	.645
1500	.195	.197	.240	.242	.277	.281	.341	.348	.395	.406	.443	.458	.486	.506
2000	.169	.170	.207	.209	.240	.242	.294	.279	.341	.348	.382	.392	.419	.433
100	.861													
200	.605	.652	.752	.851	.861									
300	.407	.511	.605	.650	.706	[illegible]	.861							
400	.422	.435	.520	.597	.605	.650	.752	.851	.861					
500	.376	.345	.443	.462	.538	.568	.627	.730	.779	.893	.861		.977	
1000	.264	.247	.325	[illegible]	.376	.385	.463	.487	.538	.568	.675	.650	.667	[illegible]
1500	.215	[illegible]	.254	[illegible]	.306	.311	.376	.385	.436	.451	.497	.511	.538	.566
2000	.166	[illegible]	.229	[illegible]	[illegible]	.267	.325	.331	.376	.385	.472	.435	.443	.462
100	.646	.773	.809											
200	.541	.596	.696	[illegible]	.814	.952								
300	.453	.564	.577	.540	.696	.770	.857							
400	.410	.445	.531	[illegible]	.618	.664	.769	.877	.961					
500	.367	.367	.371	.310	.430	.445	.551	.566	.616	.644	.676	[illegible]	.746	.877
1000	.244	.249	.302	.377	.371	.357	.436	.415	.506	.573	.541	[illegible]	.416	[illegible]
1500	.213	.214	.241	.244	.302	.307	.371	.380	.430	.449	.463	.562	.531	[illegible]
2000														

Table V (continued)

$\gamma' = 0.50$

	10 sinΔφ	10 Δφ	15 sinΔφ	15 Δφ	20 sinΔφ	20 Δφ	30 sinΔφ	30 Δφ	40 sinΔφ	40 Δφ	50 sinΔφ	50 Δφ	60 sinΔφ	60 Δφ
100	.268	.271	.331	.337	.386	.396	.461	.502	.565	.601	.644	.699	.718	.801
200	.168	.169	.231	.233	.268	.271	.331	.337	.386	.396	.435	.450	.481	.502
300	.153	.154	.188	.189	.218	.219	.268	.271	.311	.316	.350	.357	.386	.396
400	.132	.133	.162	.163	.185	.189	[illegible]	.233	.268	.271	.301	.306	.331	.337
500	.118	.118	.145	.146	.162	.168	[illegible]	.238	.237	.241	.268	.271	.295	.299
1000	.083	.083	.102	.102	.118	.114	.145	.146	.168	.168	.188	.189	.206	.208
1500	.068	.068	.083	.083	.096	.097	.118	.118	.137	.137	.153	.154	.168	.168
2000	.059	.059	.072	.072	.083	.083	.102	.102	.118	.118	.132	.133	.145	.146
100	.386	.396	.481	.502	.565	.601	.718	.801	.841					
200	.268	.271	.331	.337	.386	.396	.481	.502	.565	.601	.644	.699	.718	.801
300	.218	.219	.268	.271	.311	.316	.386	.396	.451	.467	.510	.535	.565	.601
400	.186	.187	.231	.233	.268	.271	.331	.337	.384	.396	.435	.450	.481	.502
500	.172	.172	.214	.208	.237	.241	.295	.299	.341	.355	.386	.396	.425	.437
1000	.118	.118	.145	.146	.168	.168	.206	.208	.237	.241	.268	.271	.295	.299
1500	.096	.097	.118	.118	.137	.137	.168	.168	.194	.195	.218	.219	.239	.241
2000	.083	.083	.102	.102	.118	.118	.145	.146	.168	.168	.188	.189	.206	.208
100	.418	.411	.521	.550	.616	.664	.795	.907	.951					
200	.279	.274	.356	.366	.418	.411	.571	[illegible]	.616	.664	.706	.781	.798	.907
300	.215	.217	.270	.274	.317	.341	.418	.411	.467	.471	.555	.560	.616	.664
400	.204	.204	.249	.257	.296	.294	.356	.366	.418	.431	.477	.477	.521	.550
500	.161	.167	.222	.224	.256	.261	.318	.324	.371	.386	.418	.431	.447	.466
1000	.137	.128	.156	.157	.181	.187	.222	.224	.258	.261	.290	.294	.318	.324
1500	.104	.104	.127	.128	.147	.148	.181	.182	.209	.211	.235	.237	.258	.261
2000	.090	.090	.110	.110	.127	.128	.154	.157	.181	.182	.203	.204	.222	.224

	10 sinΔφ	10 Δφ	15 sinΔφ	15 Δφ	20 sinΔφ	20 Δφ	30 sinΔφ	30 Δφ	40 sinΔφ	40 Δφ	50 sinΔφ	50 Δφ	60 sinΔφ	60 Δφ
100	.157	.475	.574	.611	.679	.747	.876							
200	.316	.321	.391	.402	.457	.475	.574	.611	.679	.747	.779	.893	.876	
300	.256	.251	.316	.321	.367	.376	.457	.475	.537	.566	.610	.656	.679	.747
400	.220	.221	.272	.275	.316	.321	.391	.402	.457	.475	.517	.544	.574	.611
500	.197	.193	.242	.244	.281	.285	.347	.355	.405	.417	.457	.475	.506	.530
1000	.138	.137	.170	.171	.197	.198	.242	.244	.281	.285	.316	.321	.347	.355
1500	.113	.111	.138	.137	.160	.161	.197	.198	.228	.230	.256	.258	.231	.235
2000	.098	.097	.120	.120	.138	.139	.170	.171	.197	.198	.220	.222	.242	.244
100	.509	.534	.642	.697	.765	.871	.998							
200	.349	.357	.434	.447	.509	.534	.642	.697	.765	.871	.882		.998	
300	.282	.285	.349	.357	.407	.419	.509	.534	.600	.643	.684	.753	.765	.871
400	.243	.244	.300	.306	.347	.357	.434	.447	.509	.534	.578	.616	.642	.697
500	.217	.219	.267	.271	.311	.316	.385	.395	.450	.466	.509	.534	.564	.590
1000	.153	.153	.188	.189	.217	.219	.267	.271	.311	.316	.349	.357	.384	.395
1500	.124	.125	.153	.153	.177	.178	.217	.219	.252	.254	.282	.286	.311	.316
2000	.108	.108	.132	.132	.153	.153	.188	.189	.217	.219	.243	.244	.267	.271
100	.591	.612	.751	.851	.9[illegible]									
200	.401	.414	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	.796					
300	.324	.317	.401	[illegible]	[illegible]	[illegible]	.591	.612	.701	.712	.806	.935	.914	
400	.277	.281	.345	[illegible]	.405	.414	.509	.474	.491	.517	.476	[illegible]	[illegible]	.651
500	.249	.251	.307	.312	.357	.344	.444	.466	.526	.547	.591	[illegible]	[illegible]	[illegible]
1000	.174	.175	.214	.214	.249	.251	.307	.312	.357	.365	.407	.414	.444	[illegible]
1500	.142	.143	.174	.176	.202	.203	.249	.251	.288	.293	.324	.334	.357	.365
2000	.123	.124	.151	.151	.174	.175	.214	.214	.249	.251	.279	.283	.307	.313

Table V (continued)

$\gamma^2 = 0.75$

n \ m	P	10 sinΔφ	10 Δφ	15 sinΔφ	15 Δφ	20 sinΔφ	20 Δφ	30 sinΔφ	30 Δφ	40 sinΔφ	40 Δφ	50 sinΔφ	50 Δφ	60 sinΔφ	60 Δφ
100	P=0.5[illegible]	.155	.155	.191	.192	.223	.225	.278	.281	.326	.332	.372	.381	.415	.428
200		.108	.109	.133	.134	.155	.155	.191	.192	.223	.225	.251	.254	.278	.281
300		.088	.088	.108	.109	.125	.126	.155	.155	.180	.181	.202	.203	.223	.225
400		.076	.076	.074	.094	.103	.109	.133	.134	.155	.155	.174	.175	.191	.192
500		.068	.068	.084	.084	.097	.097	.119	.119	.138	.138	.155	.155	.170	.171
1000		.048	.048	.059	.059	.068	.068	.084	.084	.097	.097	.108	.109	.119	.119
1500		.039	.039	.048	.048	.056	.056	.068	.068	.079	.079	.088	.088	.097	.097
2000		.034	.034	.042	.042	.048	.042	.059	.059	.068	.068	.076	.076	.084	.084
100	P=0.75	.223	.225	.278	.261	.326	.332	.415	.428	.497	.529	.577	.616	.650	.718
200		.155	.155	.191	.192	.223	.225	.278	.281	.326	.332	.372	.381	.415	.428
300		.126	.125	.155	.155	.180	.181	.223	.225	.260	.263	.294	.299	.326	.332
400		.108	.109	.133	.134	.155	.155	.191	.192	.223	.225	.251	.254	.278	.281
500		.097	.097	.119	.119	.138	.138	.173	.171	.198	.199	.223	.225	.246	.248
1000		.068	.068	.084	.084	.097	.097	.119	.119	.138	.138	.155	.155	.170	.171
1500		.056	.056	.068	.068	.079	.079	.097	.097	.112	.112	.126	.126	.138	.138
2000		.048	.048	.059	.059	.068	.068	.084	.084	.097	.097	.108	.109	.119	.119
100	P=0.[illegible]0	.241	.244	.302	.307	.356	.364	.455	.472	.549	.581	.642	.697	.736	.828
200		.167	.168	.207	.208	.241	.244	.272	.307	.356	.354	.406	.409	.455	.472
300		.136	.136	.167	.168	.194	.196	.241	.244	.233	.256	.320	.326	.356	.364
400		.117	.117	.144	.145	.167	.168	.207	.208	.241	.244	.273	.276	.302	.307
500		.104	.105	.128	.129	.149	.149	.184	.165	.214	.216	.241	.244	.267	.270
1000		.074	.074	.090	.090	.104	.105	.128	.129	.149	.149	.167	.168	.184	.185
1500		.060	.060	.074	.074	.085	.085	.104	.105	.121	.121	.136	.136	.149	.149
2000		.052	.052	.064	.064	.074	.074	.090	.090	.104	.105	.117	.117	.128	.129

n \ m	P	10 sinΔφ	10 Δφ	15 sinΔφ	15 Δφ	20 sinΔφ	20 Δφ	30 sinΔφ	30 Δφ	40 sinΔφ	40 Δφ	50 sinΔφ	50 Δφ	60 sinΔφ	60 Δφ
100	P=0.[illegible]5	.264	.267	.331	.338	.392	.403	.506	.530	.615	.643	.726	.813	.841	.999
200		.182	.183	.225	.228	.764	.767	.331	.338	[illegible]	.430	.450	.467	.506	.530
300		.148	.148	.182	.183	.213	.214	.264	.267	[illegible]	.315	.352	.360	.392	.403
400		.127	.128	.157	.157	.182	.183	.226	.228	.264	.267	.299	.303	.331	.338
500		.114	.114	.140	.140	.162	.163	.201	.202	.234	.236	.264	.267	.292	.276
1000		.080	.080	.078	.078	.114	.114	.140	.140	.162	.163	.182	.183	.201	.202
1500		.065	.065	.080	.080	.092	.093	.114	.114	.132	.132	.148	.148	.163	.163
2000		.056	.056	.069	.069	.080	.080	.098	.098	.114	.114	.127	.128	.140	.141
100	P=0.90	.294	.291	.435	.450	.442	.457	.576	.614	.710	.789	.849		.997	
200		.202	.201	.290	.294	.294	.298	.371	.380	.442	.457	.509	.534	.576	.614
300		.163	.161	.232	.234	.235	.237	.294	.298	.346	.353	.395	.406	.442	.457
400		.141	.141	.197	.200	.202	.203	.251	.253	.294	.298	.333	.340	.371	.380
500		.125	.125	.177	.178	.179	.180	.222	.224	.260	.263	.294	.298	.326	.332
1000		.088	.081	.124	.124	.125	.126	.154	.155	.179	.180	.202	.203	.222	.224
1500		.072	.072	.101	.101	.102	.102	.125	.126	.145	.146	.163	.164	.179	.180
2000		.062	.061	.087	.087	.088	.088	.108	.108	.125	.126	.141	.141	.154	.155
100	P=0.95	[illegible]	[illegible]	[illegible]	.380	.523	.550	.677	.771	.278					
200		.232	.231	[illegible]	.253	.341	.348	.435	.450	.523	.550	.610	.656	.697	.771
300		.187	.181	.207	.203	.272	.275	.341	.348	.405	.417	.465	.483	.523	.550
400		.161	.162	.172	.174	.232	.234	.290	.294	.341	.348	.389	.400	.435	.450
500		.143	.144	[illegible]	.155	.206	.208	.256	.259	.301	.305	.341	.348	.380	.390
1000		.101	.101	[illegible]	.108	.144	.144	.177	.178	.206	.208	.232	.234	.256	.259
1500		.082	.082	.083	.088	.117	.117	.144	.144	.167	.167	.187	.188	.206	.208
2000		.071	.071	.076	[illegible]	.101	.101	.124	.124	.144	.144	.161	.162	.177	.178

Table V (continued)

$\gamma^2 = 0.90$

N	10 sin Δφ	10 Δφ	15 sin Δφ	15 Δφ	20 sin Δφ	20 Δφ	30 sin Δφ	30 Δφ	40 sin Δφ	40 Δφ	50 sin Δφ	50 Δφ	60 sin Δφ	60 Δφ
100	.098	.098	.121	.121	.141	.141	.176	.176	.204	.208	.235	.237	.262	.265
200	.069	.069	.084	.085	.098	.098	.121	.121	.141	.141	.159	.159	.176	.176
300	.056	.056	.069	.069	.079	.079	.098	.098	.114	.114	.128	.128	.141	.141
400	.048	.048	.059	.059	.069	.069	.084	.085	.098	.098	.110	.110	.121	.121
500	.043	.043	.053	.053	.061	.061	.075	.075	.087	.087	.096	.098	.108	.108
1000	.031	.031	.037	.037	.043	.043	.053	.053	.061	.061	.069	.069	.075	.075
1500	.025	.025	.031	.031	.035	.035	.043	.043	.050	.050	.056	.056	.061	.061
2000	.022	.022	.026	.026	.031	.031	.037	.037	.043	.043	.048	.048	.053	.053
100	.141	.141	.176	.176	.206	.208	.262	.265	.314	.320	.365	.374	.416	.427
200	.098	.098	.121	.121	.141	.141	.176	.176	.206	.206	.235	.237	.262	.265
300	.079	.083	.098	.098	.114	.114	.141	.141	.165	.165	.186	.187	.206	.208
400	.069	.069	.084	.085	.098	.098	.121	.121	.141	.141	.159	.159	.176	.176
500	.061	.061	.075	.075	.087	.087	.108	.108	.125	.125	.141	.141	.155	.156
1000	.043	.043	.053	.053	.061	.061	.075	.075	.087	.087	.098	.098	.108	.108
1500	.035	.035	.043	.043	.050	.050	.061	.061	.071	.071	.079	.080	.087	.087
2000	.031	.031	.037	.037	.043	.043	.053	.053	.061	.061	.069	.069	.075	.075
100	.153	.153	.191	.192	.225	.227	.288	.292	.347	.355	.406	.418	.466	.484
200	.106	.106	.131	.131	.153	.153	.191	.192	.225	.227	.257	.260	.288	.292
300	.086	.086	.106	.106	.123	.123	.153	.153	.179	.180	.203	.204	.225	.227
400	.074	.074	.091	.091	.106	.106	.131	.131	.153	.153	.172	.173	.191	.192
500	.066	.066	.081	.081	.094	.094	.116	.117	.135	.136	.153	.153	.169	.169
1000	.047	.047	.057	.057	.066	.066	.081	.081	.094	.094	.106	.106	.116	.117
1500	.038	.038	.047	.047	.054	.054	.066	.066	.077	.077	.086	.086	.094	.094
2000	.033	.033	.040	.040	.047	.047	.057	.057	.066	.066	.074	.074	.081	.081

N	10 sin Δφ	10 Δφ	15 sin Δφ	15 Δφ	20 sin Δφ	20 Δφ	30 sin Δφ	30 Δφ	40 sin Δφ	40 Δφ	50 sin Δφ	50 Δφ	60 sin Δφ	60 Δφ
100	.147	.148	.210	.211	.248	.251	.320	.326	.397	.410	.459	.477	.532	.561
200	.115	.115	.143	.143	.167	.168	.210	.211	.248	.251	.285	.288	.320	.324
300	.093	.093	.115	.115	.134	.135	.167	.168	.196	.197	.223	.225	.248	.251
400	.081	.081	.099	.099	.115	.115	.143	.143	.167	.168	.189	.190	.210	.211
500	.072	.072	.068	.069	.103	.103	.127	.127	.148	.148	.167	.168	.165	.166
1000	.051	.051	.062	.062	.072	.072	.088	.089	.103	.103	.115	.115	.127	.127
1500	.041	.041	.051	.051	.058	.059	.072	.072	.083	.083	.093	.093	.103	.103
2000	.036	.036	.044	.044	.051	.051	.062	.062	.072	.072	.081	.081	.088	.089
100	.186	.187	.235	.237	.279	.283	.364	.373	.447	.466	.537	.567	.636	.663
200	.128	.129	.159	.159	.186	.187	.235	.237	.279	.283	.322	.328	.364	.373
300	.103	.103	.128	.128	.149	.149	.186	.187	.219	.221	.250	.252	.277	.283
400	.089	.089	.110	.110	.128	.128	.159	.159	.186	.187	.211	.213	.235	.237
500	.079	.079	.098	.098	.113	.114	.141	.141	.164	.165	.186	.187	.206	[illegible]
1000	.056	.056	.069	.069	.079	.079	.098	.098	.113	.114	.128	.128	.141	.141
1500	.045	.045	.056	.056	.065	.065	.079	.079	.092	.092	.103	.103	.113	.114
2000	.039	.039	.048	.048	.056	.056	.069	.069	.079	.079	.087	.089	.098	.098
100	.216	.218	.275	.279	.331	.337	.441	.456	.556	.587	.680	.748	.819	[illegible]
200	.147	.147	.183	.184	.216	.218	.275	.279	.331	.337	.386	.396	.441	.456
300	.118	.119	.147	.147	.172	.173	.216	.218	.256	.259	.294	.298	.331	.337
400	.102	.102	.126	.126	.147	.147	.183	.184	.216	.218	.246	.249	.275	.279
500	.091	.091	.112	.112	.130	.131	.162	.163	.190	.191	.216	.218	.240	.243
1000	.064	.064	.078	.078	.091	.091	.112	.112	.130	.131	.147	.147	.162	.163
1500	.052	.052	.064	.064	.074	.074	.091	.091	.106	.106	.118	.119	.130	.131
2000	.045	.045	.055	.055	.064	.064	.078	.078	.091	.091	.102	.102	.112	.112

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