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THE OSCILLATORY MOTION OF CABLED - TOWED BODIES

BY

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THE OSCILLATORY MOTION OF CABLE-TOWED BODIES

by

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NOTATION

u	horizontal velocity component
v	vertical velocity component
U	normal velocity component
V	tangential velocity component
C	resultant velocity
x, y	rectangular coordinates of point on cable
s	distance along cable
t	time
F	normal loading function
G	tangential loading function
T	cable tension
L	lift force and weight of body
D	drag of body
W	weight/unit length of cable
R	drag/unit length of cable when cable is normal to the direction of motion
c	chord of cable
t	thickness of cable
t/c	thickness/chord ratio
m_x	virtual mass of body for motion in x direction
m_y	virtual mass of body for motion in y direction
θ	angle measured from resultant velocity to cable
ϕ	angle measured from direction of motion to cable
ϕ_c	critical angle
ω	perturbation frequency
$\tau, \sigma,$ [illegible] $, \eta$	cable functions [see equation (3.1.12)]

μ mass/unit length of cable

ρ mass density of fluid

Other symbols are defined where used.

LIST OF ILLUSTRATIONS

ABSTRACT

A theoretical investigation of the oscillatory motion of a flexible cable-body system is presented. Cable-towed devices of this type have both hydrodynamic and aerodynamic applications. For example, cable-towed devices are used for underwater minesweeping gear and air-to-air refueling of jet aircraft.

In this analysis the general equations of motion of a cable-body system are derived in terms of generalized loading functions. A solution of the resulting four, first order, quasi-linear, partial differential equations was obtained by the method of characteristics which is valid for any arbitrary motion. The cable was assumed to be finite in length, and boundary conditions are considered for both ends of the cable. Boundary conditions at the upper end of the cable are derived in terms of the disturbances produced by the towing vehicle, whereas the boundary conditions at the lower end of the cable are in terms of the forces acting on the towed body. Also, conclusions about how the disturbances propagate up and down the cable are made.

When the motion of the system is steady, not a function of time, the equations of motion are greatly simplified. The solution for this special case is presented in the form of cable functions which can be used to determine the cable-body equilibrium configuration.

In order to facilitate obtaining a solution when the perturbation forces and motions are small, the general equations are linearized. Asymptotic expansions for both small and large values of the perturbation frequency are derived. Also, an example is presented to illustrate the method.

1.0 INTRODUCTION

The problems associated with the design of towed devices require a means of determining the position of the towed body, the tension in the cable and the effect of the motion of the towing craft on the behavior of the cable-body system. Problems of this type arise, for example, when towed devices are used for minesweeping, sonar housings, and anti-submarine warfare. Also, related problems arise when laying submarine cables, anchoring balloons, buoys and off-shore installations.

A great many investigators have studied various aspects of the general problem, References 1 to 18. In one of the earlier investigations, performed by McLeod (1)*, the steady-state shape of a heavy, lexible, circular cable was derived neglecting the tangential force acting along the cable. This analysis did not lead to a simple result and no attempt was ade to obtain numerical results. The meaning associated with the equilibrium or steady-state configuration throughout this investigation is that the resultant velocity is constant.

Later, Glauert (2) investigated the same problem and presented a series of charts for solving problems which involved towing a heavy body in the vertical plane. However, these charts have been found inadequate for treating cases where the body is towed by a long cable at moderate or high speeds since the tangential component of the force along the cable was neglected. Both of these analyses assumed the "sine-squared" law for the normal force acting on the circular cable as

* References are listed on page 80

obtained experimentally by Relf [12] in 1917.

Shortly before and during World War II several special cable-body configurations were investigated where assumptions were made about the tangential force component and the weight of the cable. The most comprehensive work was a report published by Pode [11] in 1951. In this report the equilibrium configuration of a circular cable immersed in a steady, uniform stream was treated. The sine-squared law was assumed for the normal force whereas a constant tangential force was used to determine the steady-state shape of a heavy cable.

The stability of a body towed by a weightless cable was analyzed by Glauert [13]. However, the tension was assumed constant and the damping of the cable was neglected. Phillips [14] showed that these assumptions lead to erroneous conclusions with respect to the boundaries of stability. In both of these investigations oscillations of the cable were neglected. These theories failed to predict the violent motion of these bodies which had been observed when they were towed faster than a certain speed from an airplane. The resulting motion involved oscillations of the cable. In operation, such towed bodies have been observed to generally remain stable up to a certain speed, but above this speed violent short-period oscillations have been observed involving pitching and vertical motion of the body as well as oscillations of the cable. These oscillations have frequently resulted in separating the body from the cable even though the cable attachment was designed to withstand a load of 25 times the weight of the body [15].

Phillips [15], in a later investigation, considered the possibility that oscillations of the cable generated near the towing craft may be amplified, as the oscillations are propagated down the cable, due to the action of the forces acting on the cable. In this investigation the tangential force along the cable was neglected, the cable tension was assumed to be constant, and the equilibrium configuration of the cable was assumed to be a straight line. Also, the effect of the body on the cable was neglected, that is, the cable was assumed to be infinitely long. In a report by Miles (16) similar assumptions were made, however, a finite length of cable was considered.

In the present investigation the general equations of motion for the cable-body system are derived in terms of arbitrary normal and tangential loading functions. The possibility of self-excited lateral vibrations of the cable, sometimes called "transmission line gallop"(19), is ruled out due to the method of attaching the fairing to the load carrying member. Usually the fairing is free to rotate about a circular cable which is placed well forward of the center of pressure of the fairing. Hence, the fairing will assume a position such that the side force is zero. This phenomenon of transmission line gallop, which is due to aerodynamic instability, is usually one of very low frequency and large amplitude. However, a high frequency, small amplitude lateral vibration is also possible. This type of vibration, which is due to the formation of the "Karman Vortex street", is produced by the alternate shedding of vortices from the member. This shedding of vortices produces a side force which alternates from one side of the member to the other side. Also, this is the same phenomenon that causes "singing" propellers and telephone wires. Since the amplitude of this vibration is usually small in comparison to the disturbance produced by the towing vehicle it will be neglected. Also, this vibration is always damped by the viscous forces acting on the cable. Therefore, only vibrations in a plane parallel to the direction of motion will be considered in this analysis.

Simple relations, which are based partially on theoretical results and partially on experimental results, are proposed for the loading functions which are applicable to either circular or faired cables. The resulting equations of motion consist of four, first order, quasi-linear, partial differential equations. This system of equations is simplified by neglecting the time dependence of the functions to study the important steady-state case. The results of this steady-state investigation are presented in the form of cable functions which must be evaluated by numerical integration.

Next, the general, quasi-linear system of equations which describe the motion of the cable-body system are solved for any arbitrary disturbing force. This solution is achieved by the method of characteristics.

In an effort to emphasize the important factors in determining the motion which results from a small disturbance, the system of equations is linearized. However, the resulting equations are still very difficult to solve since the variables are functions of both cable length and time. Hence, an asymptotic expansion is made for both large and small values of the disturbance frequency. Even though the resulting equations, which are now ordinary differential equations, have non-constant coefficients, certain assumptions may be made and then the equations can be solved. From these asymptotic expansions the effect of the two important parameters can be noted.

2.0 MATHEMATICAL FORMULATION OF PROBLEM

In the mathematical formulation of the cable problem the forces acting on an element of the cable will be considered. The hydrodynamic forces acting on the element of the cable are assumed to depend only on the velocity at the element, and are not affected by such matters as curvature of the cable or the flow at nearby elements. In other words, the form of the hydrodynamic force is the same for a small element of the cable as for an infinitely long cable of the same size and shape and inclined at the same angle to the free-stream. In addition to the hydrodynamic forces, there is a gravitational force and towline tension acting on the cable element. From practical considerations, it will be assumed that the cable offers no resistance to bending and cannot support a compressive force.

Also, the hydrodynamic forces, the weight of the cable, and the forces applied to the cable ends are assumed to lie in a plane. Therefore, the entire cable will lie in the plane of the direction of motion and the direction of gravity. This requirement is satisfied by smooth symmetrical profiles, either circular or faired, and is approximately satisfied by a stranded cable.

The free surface effect, aeration cavity formed behind the cable at and near the water surface is neglected. Normally this cavity extends over a relatively small percentage of the total cable length for most applications. However, this free surface effect may become important for some special cases. For example, this effect is important when towing a body on a short length of cable as is done quite often in towing basins.

2.1 Derivation of Equations

The horizontal and vertical forces acting on the cable element, subject to the assumptions made in section 2.0, are obtained by applying Newton's Law to the cable element shown on Figure 1. From consideration of this figure it is seen that

$$u = \frac{\partial x}{\partial t} \qquad \text{and} \qquad v = \frac{\partial y}{\partial t}$$

Then the horizontal and vertical forces can be written as

$$\mu\, ds \frac{\partial u}{\partial t} = d(T\cos\phi) + G\,ds\cos\phi - F\,ds\sin\phi$$

and

$$\mu\, ds \frac{\partial v}{\partial t} = d(T\sin\phi) + F\,ds\cos\phi + G\,ds\sin\phi - W\,ds$$

Dividing these two equations by ds yields

$$\mu \frac{\partial u}{\partial t} = \frac{\partial}{\partial s}(T\cos\phi) + G\cos\phi - F\sin\phi \qquad \ldots\ldots\ldots \text{(2.1.1)}$$

$$\mu \frac{\partial v}{\partial t} = \frac{\partial}{\partial s}(T\sin\phi) + F\cos\phi + G\sin\phi - W \qquad \ldots\ldots\ldots \text{(2.1.2)}$$

From the geometry of the configuration

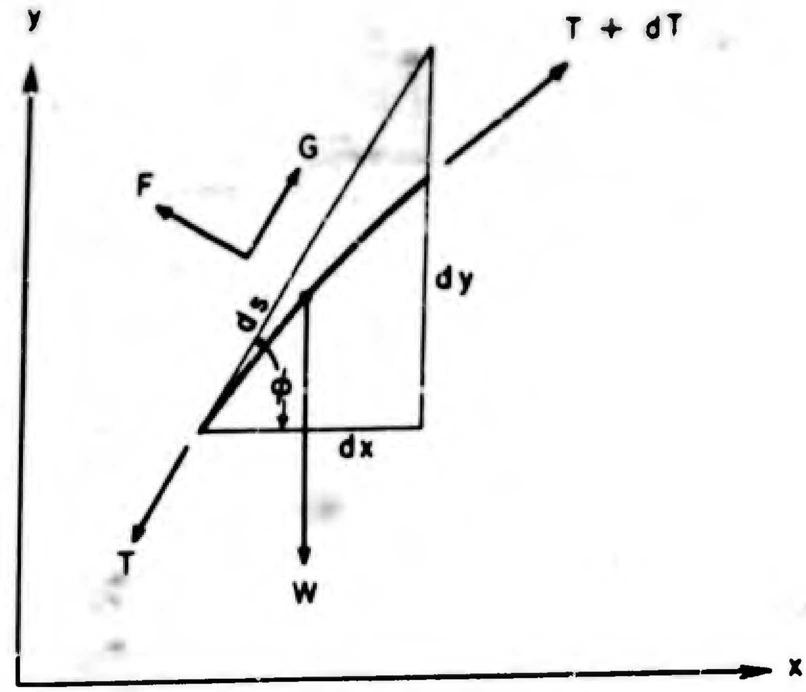


FIG. 1a FORCES ACTING ON ELEMENT OF CABLE

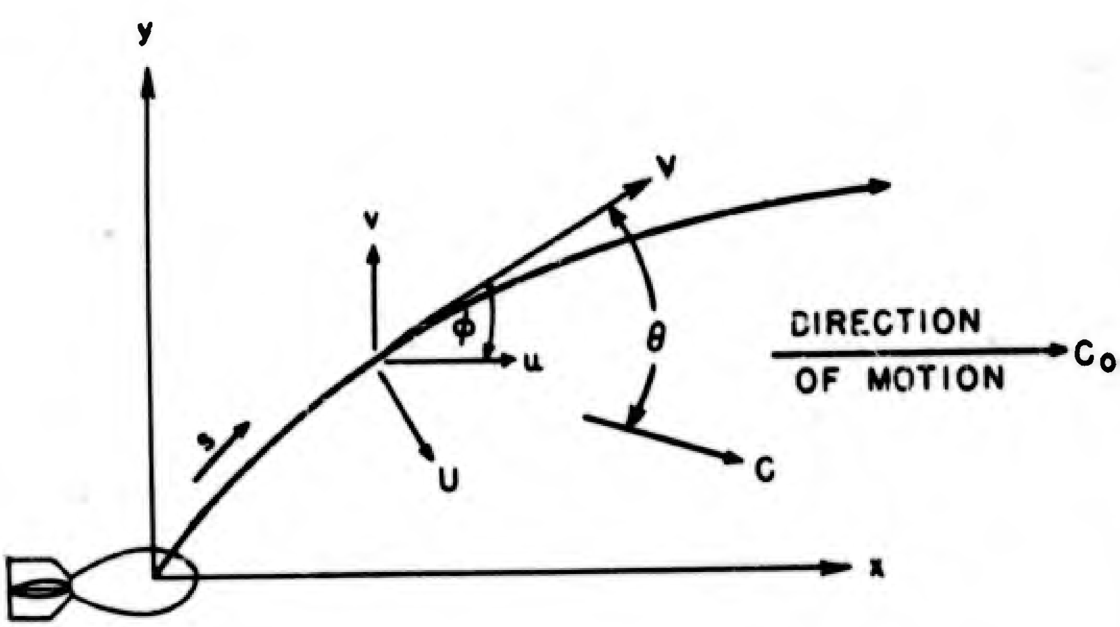


FIG. 1b VELOCITIES ACTING ON ELEMENT OF CABLE

FIG. 1 FORCES AND VELOCITIES ACTING ON CABLE

$$\left.\begin{array}{l} \dfrac{\partial x}{\partial s} = \cos\phi \qquad \text{and} \qquad \dfrac{\partial y}{\partial s} = \sin\phi \\ \\ \text{then} \\ \\ \dfrac{\partial u}{\partial s} = -\sin\phi\,\dfrac{\partial \phi}{\partial t} \qquad \text{and} \qquad \dfrac{\partial v}{\partial s} = \cos\phi\,\dfrac{\partial \phi}{\partial t} \end{array}\right\} \quad \ldots (2.1.3)$$

Multiply equation (2.1.1) by $\sin\phi$ and (2.1.2) by $\cos\phi$ and subtract (2.1.2) from (2.1.1) yields

$$\mu\left[\frac{\partial u}{\partial t}\sin\phi - \frac{\partial v}{\partial t}\cos\phi\right] = -T\frac{\partial \phi}{\partial s} - F + W\cos\phi \quad \ldots\ldots\ldots (2.1.4)$$

and by a similar process

$$\mu\left[\frac{\partial u}{\partial t}\cos\phi + \frac{\partial v}{\partial t}\sin\phi\right] = \frac{\partial T}{\partial s} + G - W\sin\phi \quad \ldots\ldots\ldots (2.1.5)$$

Equations (2.1.4) and (2.1.5) are summations of the forces normal and tangential to the cable element respectively.

The normal and tangential velocities are given by

$$\left.\begin{array}{l} U = u\sin\phi - v\cos\phi \\ V = u\cos\phi + v\sin\phi \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots (2.1.6)$$

then

$$\frac{\partial U}{\partial t} = \frac{\partial u}{\partial t}\sin\phi - \frac{\partial v}{\partial t}\cos\phi + V\frac{\partial \phi}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots (2.1.7)$$

and

$$\frac{\partial V}{\partial t} = \frac{\partial u}{\partial t}\cos\phi + \frac{\partial v}{\partial t}\sin\phi - U\frac{\partial \phi}{\partial t} \quad \ldots\ldots\ldots\ldots (2.1.8)$$

Rewriting the last two equations

$$\frac{\partial u}{\partial t}\sin\phi - \frac{\partial v}{\partial t}\cos\phi = \frac{\partial U}{\partial t} - V\frac{\partial \phi}{\partial t}$$

and

$$\frac{\partial u}{\partial t}\cos\phi + \frac{\partial v}{\partial t}\sin\phi = \frac{\partial V}{\partial t} + U\frac{\partial\phi}{\partial t}$$

and substituting these relationships into equations (2.1.4) and (2.1.5) yield

$$\mu\left[\frac{\partial U}{\partial t} - V\frac{\partial\phi}{\partial t}\right] = -T\frac{\partial\phi}{\partial s} - F + W\cos\phi \qquad \text{(2.1.9)}$$

$$\mu\left[\frac{\partial V}{\partial t} + U\frac{\partial\phi}{\partial t}\right] = \frac{\partial T}{\partial s} + G - W\sin\phi \qquad \text{(2.1.10)}$$

Two other equations are obtained from equation (2.1.6) by differentiation

$$\frac{\partial U}{\partial s} = \frac{\partial u}{\partial s}\sin\phi - \frac{\partial v}{\partial s}\cos\phi + V\frac{\partial\phi}{\partial s} \qquad \text{(2.1.11)}$$

and

$$\frac{\partial V}{\partial s} = \frac{\partial u}{\partial s}\cos\phi + \frac{\partial v}{\partial s}\sin\phi - U\frac{\partial\phi}{\partial s} \qquad \text{(2.1.12)}$$

These last two equations can be simplified by using equation (2.1.3). Then

$$\frac{\partial u}{\partial s}\sin\theta - \frac{\partial v}{\partial s}\cos\phi = -\frac{\partial\phi}{\partial t}$$

and

$$\frac{\partial u}{\partial s}\cos\phi + \frac{\partial v}{\partial s}\sin\phi = 0$$

Substitution of these two equations into equation (2.1.11) and (2.1.12) give

$$\frac{\partial U}{\partial s} - V\frac{\partial\phi}{\partial s} = -\frac{\partial\phi}{\partial t} \qquad \text{(2.1.13)}$$

and

$$\frac{\partial V}{\partial s} + U\frac{\partial \phi}{\partial s} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.1.14)$$

Hence, the four equations of motion for the cable are

$$\left.\begin{aligned}
\mu\left[\frac{\partial U}{\partial t} - V\frac{\partial \phi}{\partial t}\right] &= -T\frac{\partial \phi}{\partial s} - F + W\cos\phi \\
\mu\left[\frac{\partial V}{\partial t} + U\frac{\partial \phi}{\partial t}\right] &= \frac{\partial T}{\partial s} + G - W\sin\phi \\
\frac{\partial \phi}{\partial t} &= V\frac{\partial \phi}{\partial s} - \frac{\partial U}{\partial s} \\
0 &= \frac{\partial V}{\partial s} + U\frac{\partial \phi}{\partial s}
\end{aligned}\right\} \quad \ldots\ldots\ldots (2.1.15)$$

For ease of writing this system of quasi-linear, partial differential equations, differentiation with respect to s and t will be denoted by subscripts. In this notation equation (2.1.15) becomes

$$\left.\begin{aligned}
\mu\left[U_t - V\phi_t\right] &= -T\phi_s - F + W\cos\phi \\
\mu\left[V_t + U\phi_t\right] &= T_s + G - W\sin\phi \\
\phi_t &= V\phi_s - U_s \\
0 &= V_s + U\phi_s
\end{aligned}\right\} \quad \ldots\ldots\ldots (2.1.16)$$

The above system of equations describe the velocity, tension, and shape of the cable at any time. Before discussing solutions to this system of equations, the loading functions and boundary conditions will be investigated.

2.2 Loading Functions

In order to determine the resistance of the cable moving through a fluid, it is necessary to study the mechanics of real fluids. The early development of the science of fluid mechanics followed two very different lines. One was the theoretical approach based on Euler's equation of motion for a perfect fluid, whereas the other was an empirical approach which developed into the science of hydraulics. However, L. Prandtl[20] in 1904 introduced the concept of a thin friction layer, called the boundary layer, which brought these two branches of fluid mechanics into closer agreement. Application of the boundary layer theory makes it possible to separate the flow about a solid body into two regions. One region consists of a thin layer in the neighborhood of the body where friction plays an essential part, and the other region is composed of the fluid outside the boundary layer. In this second region the effect of friction can be neglected, hence potential theory can be applied.

The resistance of a submerged body moving through a fluid can be separated into two components which will be called the skin-friction component and the pressure component. The skin-friction component is a tangential force produced by the fluid moving past the surface of the body, whereas the pressure component is due to the modification of the potential theory pressure distribution by the boundary layer. The flow in the boundary layer has the property that under certain conditions a reversal of flow direction occurs in the neighborhood of the body which causes the boundary layer to separate from the body. This is usually accompanied by the formation of eddies in the wake of the body which changes the pressure distribution about the body from that which is computed from potential theory. This variation in the pressure distribution produces the pressure component of the resistance. These two types of resistance may be vizualized by comparison of the resistance of a flat-plate and a circular cylinder. The resistance produced by the flat-plate is mainly friction drag, whereas the resistance produced by the circular cylinder is mainly pressure drag.

The problem of determining suitable loading functions for bodies of arbitrary shape has been studied by a great many investigators in both hydrodynamics and aerodynamics. Semi-empirical relationships, which depend on the Reynolds Number, have been developed for the skin-friction resistance of a flat-plate.[21,22] However, the problem is more difficult for bodies of arbitrary shape.[23] Usually some assumption about the pressure component of the resistance is made. For example, in naval architecture [24] the pressure component is lumped into the residual resistance which is assumed to be independent of Reynolds Number and the frictional component is computed using an empirical flat-plate law. Several theoretical investigations of the profile resistance (pressure and friction components) of two-dimensional wings have been made, however, the results of these investigations are difficult to apply.[21,23,25] Also, studies of three-dimensional boundary layer problems have been made but as yet the results are not in a form suitable for application to the cable problem.[21,26,27] For example, it is necessary to know the location of the transition point, shearing stress in the turbulent layer and other quantities depending on which method is employed.

The form of the normal hydrodynamic loading force acting on a cable towed with uniform velocity through a fluid has, in general, been taken as a function of the sine-squared of the angle of inclination, Φ, which the cable makes with the horizontal. The resultant steady-state velocity was taken to be horizontal; however, it is not necessary to make this choice. More general loading functions than those used by previous investigators will be developed. The difference between the proposed loading functions and the loading functions used by other investigators will be clearly indicated and the results will be compared with experimental data.

As shown on Figure 1, the cable element is inclined at an arbitrary angle, θ, to the resultant velocity, C. The normal loading function, F, and the tangential loading function, G, will be derived in terms of the normal and tangential velocity components of the resultant velocity. These loading functions will be assumed to be proportional to the square of the respective velocities, neglecting any spanwise pressure gradients, as is usually done in hydrodynamics and aerodynamics. The spanwise distribution of velocity in the boundary layer of a yawed cylinder, whether laminar or turbulent, was shown to be insensitive to the chordwise pressure distribution in Reference 25. Hence, using this independence principle, the forces normal and tangential to the cable will be written as

$$\left.\begin{aligned} F &= C_N \frac{\rho}{2} c\, U^2 \\ G &= -C_T \frac{\rho}{2} c\, V^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.2.1)$$

where C_N and C_T are commonly called drag coefficients. The negative sign was chosen for the second equation since the tangential force acts in the negative s direction. The drag coefficient, for most body shapes, is a function of the Reynolds Number as well as surface roughness. As was suggested earlier, the drag coefficients will be considered as though they were composed of two parts. One part of the drag coefficient will be associated with the pressure drag and the other part with the friction drag. Then the drag coefficient will be written as

$$\left.\begin{aligned} C_N &= C_R\left[a + \frac{c}{U} b\right] \\ C_T &= C_R\left[d + \frac{c}{V} e\right] \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (2.2.2)$$

where C_R is the drag coefficient of the cable when the cable is perpendicular to the free-stream. Therefore, the quantities in the brackets reflect the Reynolds Number effect on the drag coefficients. This form for the drag coefficients was chosen so that

1 - a reference Reynolds Number range can be selected by choosing C_R when the cable is perpendicular to the free-stream.

2 - the components of the drag coefficients due to the pressure forces, a and d, are independent of the variations in Reynolds Number along the cable.

3 - the components of the drag coefficients due to the friction force, $\frac{c}{U} b$ and $\frac{c}{V} e$, are functions of the component of the velocity parallel to the surface.

4 - the coefficients, a, b, d, e, can be determined from experimental data for the particular cable section shape, t/c, and the roughness of the surface can be considered.

When equation (2.2.2) is substituted into equation (2.2.1), the following equations for the loading functions are obtained.

$$\left.\begin{aligned} F &= C_R \frac{\rho}{2} r\left[aU^2 + bCU\right] = R\left[a\left(\frac{U}{C}\right)^2 + b\left(\frac{U}{C}\right)\right] \\ G &= -C_R \frac{\rho}{2} r\left[dV^2 + eCV\right] = -R\left[d\left(\frac{V}{C}\right)^2 + e\left(\frac{V}{C}\right)\right] \end{aligned}\right\} \quad \ldots\ldots\ldots (2.2.3)$$

where

$$R = C_R \frac{\rho}{2} r C^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.2.4)$$

is the drag/unit length when the cable is perpendicular to the free-stream. A similar loading function was used by Kemp in Reference 28 for analyzing resistance data for long cylinders and flat-plates. The above loading functions may be written in terms of the effective angle of inclination, ϑ , since

$$U = C \sin \vartheta \quad \text{and} \quad V = C \cos \vartheta \quad \ldots\ldots\ldots (2.2.5)$$

as can be seen from Figure 1. Substituting these values of U and V in equation (2.2.3) yields

$$\left.\begin{aligned} F &= R\left[a \sin^2\vartheta + b \sin\vartheta\right] \\ G &= -R\left[d \cos^2\vartheta + e \cos\vartheta\right] \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (2.2.6)$$

Now it is possible to compare this form for the loading functions with the types used by previous investigators. For the steady-state case, when the resultant velocity is steady, uniform and directed along the x-axis

$$\vartheta = \varphi \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.2.7)$$

Then the loading functions become

$$\left.\begin{aligned} F &= R(a \sin^2\phi + b \sin\phi) \\ G &= -R(d \cos^2\phi + e \cos\phi) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (2.2.8)$$

The form of the normal loading function used by most previous investigators was

$$F = R \sin^2\phi$$

For cables of circular cross-section, this relationship is satisfactory since most of the drag is due to the pressure term. However, this is not the case for a faired cable. The primary purpose in fairing the cable is to reduce the pressure drag component, but, unfortunately, this increases the friction drag component. Since the sine-squared law neglects the frictional component, the use of this loading function to compute the equilibrium configuration of a faired cable, as was suggested in Reference 11, is of doubtful validity.

The tangential loading function, G, has been assumed to be independent of the angle of inclination of the cable by most investigators. McLeod (1) and Glauert (2) and others (15,16) neglected this component entirely, whereas Landweber (7) and Pode (11) assumed G/R to be constant.

The coefficients, a, b, d, e, in equation (2.2.8) were evaluated from experimental data in References 12 and 29 for values of t/c of 1.0 and 0.25. The numerical values of these coefficients for $t/c = 1.0$ were found to be $a = 1.0$, $b = 0$, $d = -0.035$, $e = 0.083$ and for $t/c = 0.25$, $a = 0.25$, $b = 0.75$, $d = -0.050$, $e = 0.312$. A comparison of equation (2.2.8), utilizing these numerical values for the coefficients, with experimental data is shown on Figures 2 to 5. From a study of these figures it is seen that the equations fit the experimental data reasonably

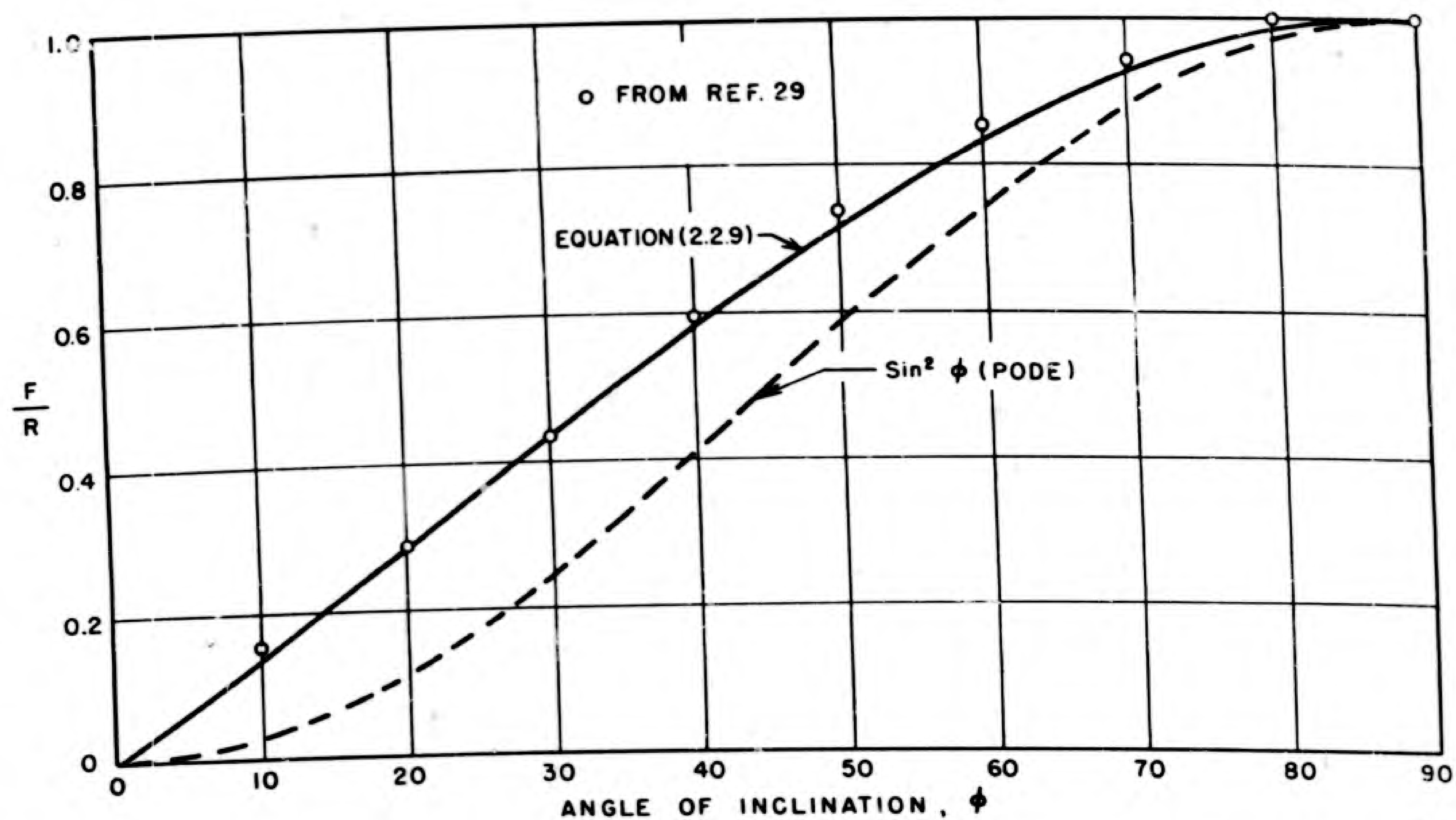


FIG. 2 NORMAL LOADING FUNCTIONS FOR A FAIRED STRUT WITH $t/c = 0.25$

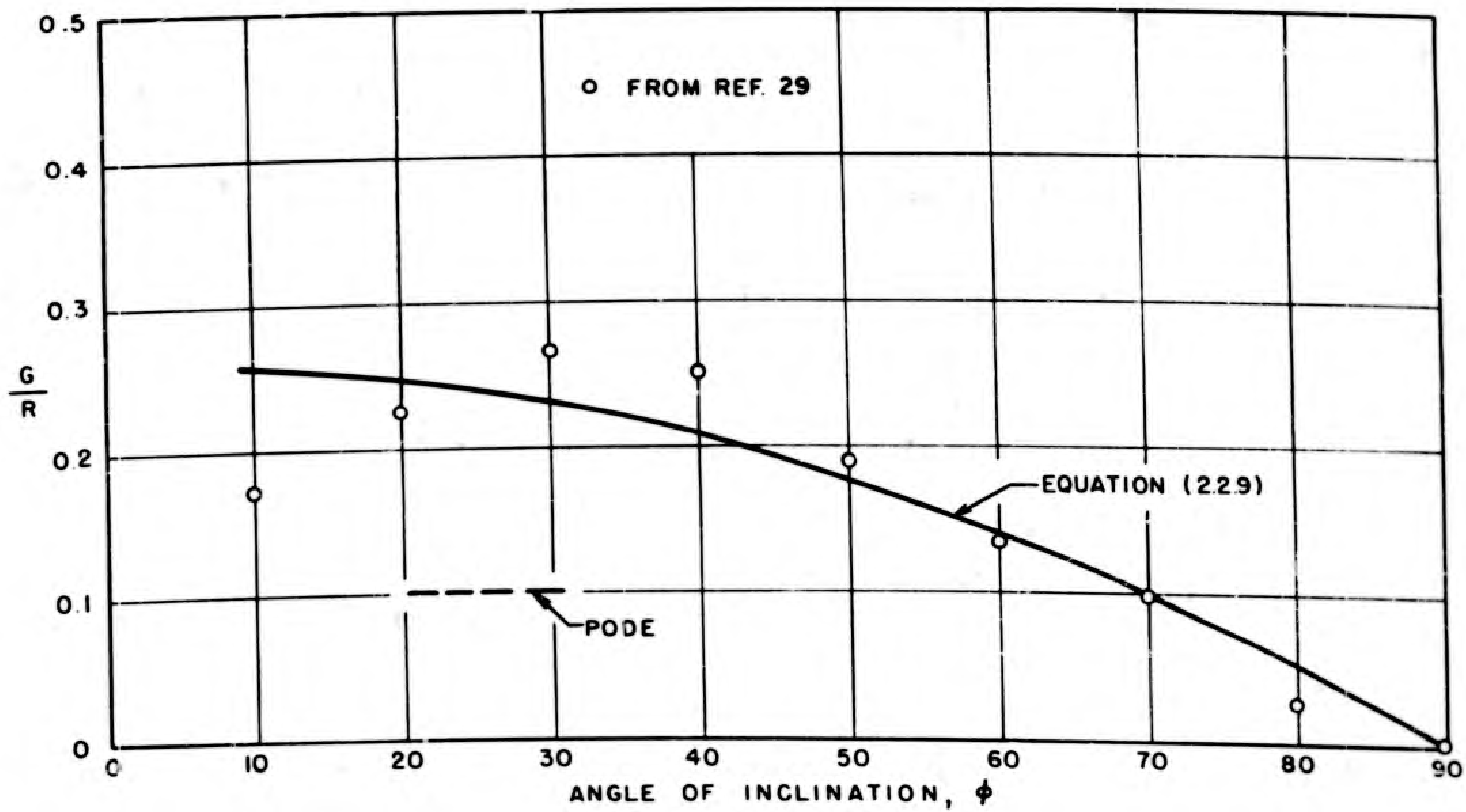


FIG. 3 TANGENTIAL LOADING FUNCTIONS FOR A FAIRED STRUT WITH $t/c = 0.25$

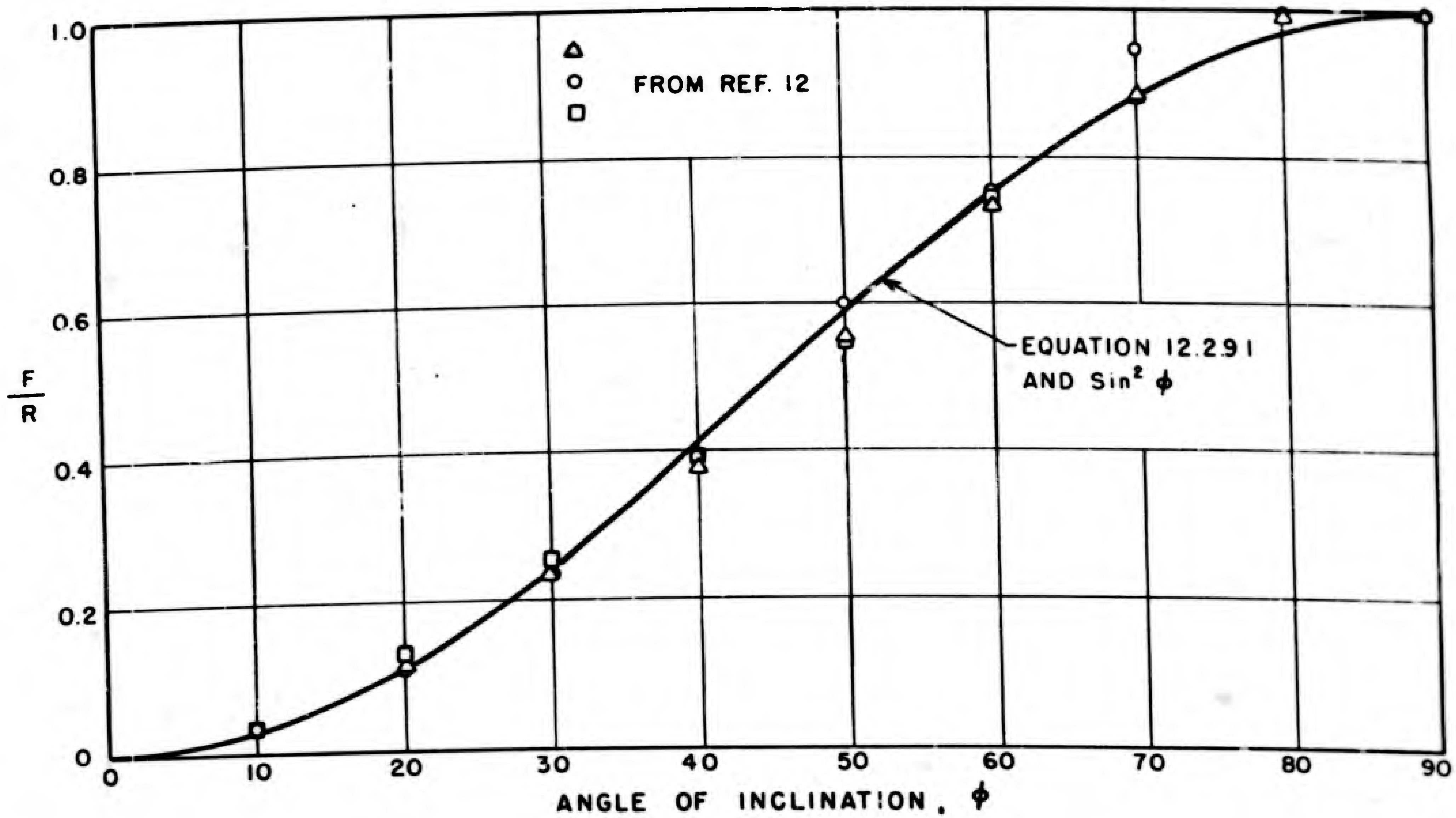


FIG. 4 NORMAL LOADING FUNCTIONS FOR A CIRCULAR CYLINDER

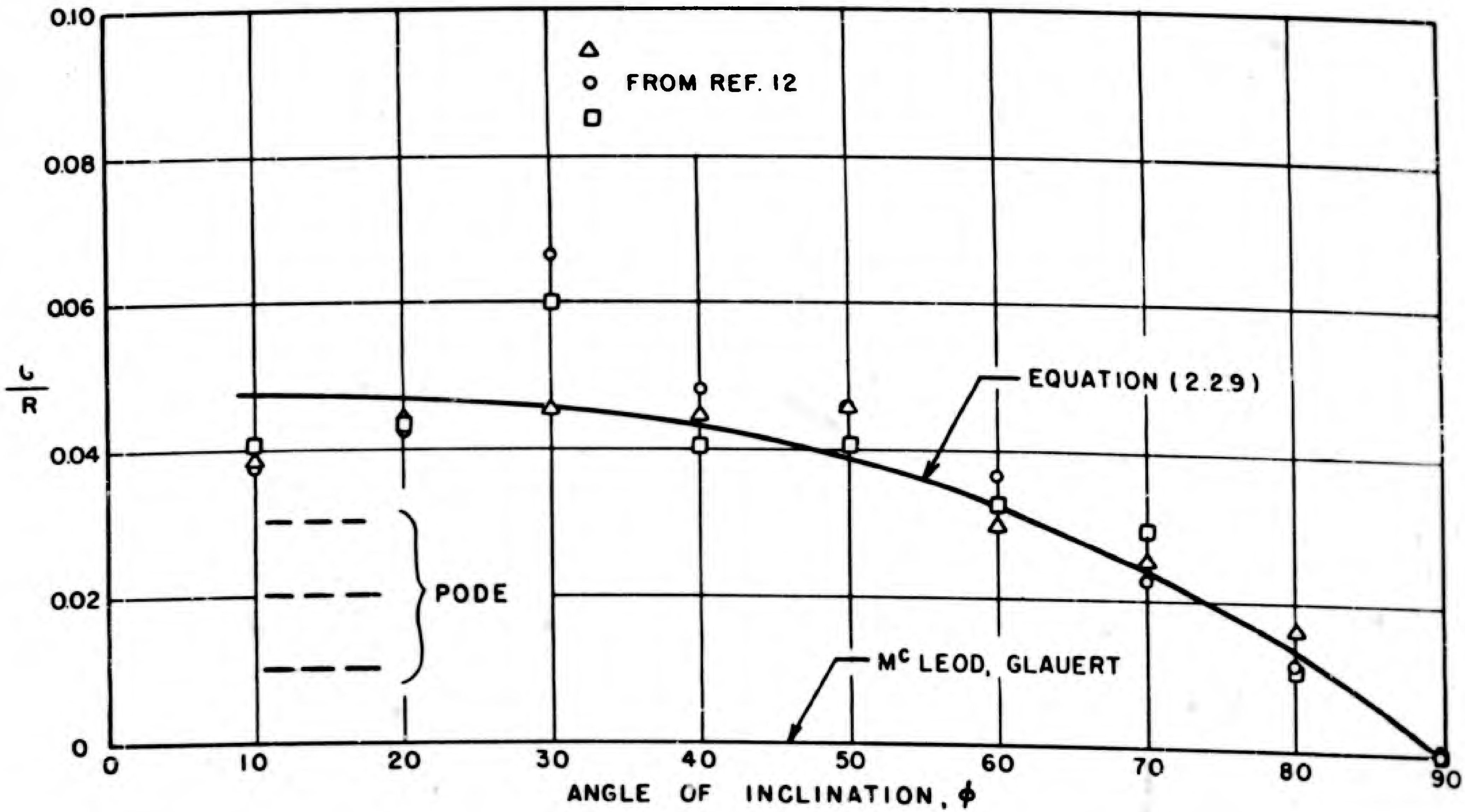


FIG. 5 TANGENTIAL LOADING FUNCTIONS FOR A CIRCULAR CYLINDER

well. Also plotted on these figures are some of the relationships assumed by other investigators for the loading functions.

If a linear ratio is assumed for the various t/c values, the coefficients of equation (2.2.8) may be expressed as

$$a = t/c \quad , \qquad b = 1 - t/c$$

$$d = -(0.055 - 0.020\, t/c) \quad , \quad e = (0.386 - 0.303\, t/c)$$

Since most of the normal drag of a circular cylinder is pressure drag, the simple relationship for the coefficients, a and b, was to be expected. Inserting these relationships for the coefficients into the equations for the loading functions yields

$$\left.\begin{aligned} \frac{F}{R} &= \left(1 - \frac{t}{c}\right)\sin\phi + \frac{t}{c}\sin^2\phi \\ \frac{G}{R} &= -\left[\left(0.386 - 0.303\frac{t}{c}\right)\cos\phi - \left(0.055 - 0.020\frac{t}{c}\right)\cos^2\phi\right] \end{aligned}\right\} \quad . . . . (2.2.9)$$

Since these loading functions agree reasonably well with the experimental data, it is anticipated that computation of the cable configuration can be more accurately performed using these functions than was possible with the previous functions.

2.3 General Boundary Conditions

For the boundary conditions at the towed body, $s = 0$, it will be assumed that the line of action of all forces acting on the body pass through the towpoint. An attempt is made to satisfy this condition when designing a towed body since, for most applications, it is desirable to have the body remain at a prescribed angle to the flow. However, if this condition is not satisfied by a specific body, the resulting effect on the towpoint can be prescribed and substituted into the boundary conditions. For example, suppose the line of action of the drag force did not pass through the towpoint. Then the angle of the body will change with variations in velocity. Usually a change in angle will result in a change in the downward force also. Regardless of the specific body configuration, the resulting effect on the towpoint can be prescribed and these conditions substituted into the boundary conditions.

The downward force developed by the body, whether by weight, dynamic lift or a combination of both, will be denoted by L and the horizontal force by D. Then summing forces at the towpoint, first horizontally and then vertically yields

$$\left.\begin{aligned} T\cos\phi - D &= m_x u_t \\ T\sin\phi - L &= m_y v_t \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.3.1)$$

where m_x and m_y are the virtual masses of the body in the x and y directions respectively. The virtual mass of the body must be used instead of the actual mass of the body since the presence of the fluid effectively increases the mass of a moving body. This virtual mass of the body is obtained by increasing the actual mass by the added mass (hydrodynamic mass). In the notation used by Lamb [30], the virtual mass can be written as

$$\left.\begin{aligned} m_x &= M(1 + k_1) \\ m_y &= M(1 + k_2) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.3.2)$$

where M is the actual mass and k_1, and k_2 are the inertia coefficients. Values of the inertia coefficients are tabulated in Reference 30 for various bodies of revolution.

When the equations numbered (2.3.1) are multiplied by $\sin\phi$ and $\cos\phi$ respectively, and the second equation is subtracted from the first, the following equation is obtained

$$T \sin\phi \cos\phi \left(\frac{m_y - m_x}{m_x m_y}\right) - \frac{D}{m_x}\sin\phi + \frac{L}{m_y}\cos\phi = u_t \sin\phi - v_t \cos\phi$$

and in a similar manner

$$T\left[\frac{m_y \cos^2\phi + m_x \sin^2\phi}{m_x m_y}\right] - \frac{D}{m_x}\cos\phi - \frac{L}{m_y}\sin\phi = u_t \cos\phi + v_t \sin\phi$$

The right hand side of these two equations can be written in terms of the velocities normal and parallel to the cable by using equations (2.1.7) and (2.1.8). Then

$$\left.\begin{aligned} &T \sin\phi \cos\phi \left(\frac{m_y - m_x}{m_x m_y}\right) - \frac{D}{m_x}\sin\phi + \frac{L}{m_y}\cos\phi = U_t - V\phi_t \\ &\text{and} \\ &T\left(\frac{m_y \cos^2\phi - m_x \sin^2\phi}{m_x m_y}\right) - \frac{D}{m_x}\cos\phi - \frac{L}{m_y}\sin\phi = V_t + U\phi_t \end{aligned}\right\} \quad \ldots\ldots (2.3.3)$$

are the boundary conditions that must be satisfied at the body, $s = 0$.

The disturbing force generated at the towing vehicle, $s = \ell$, will be assumed to be proportional to the real part of $\epsilon e^{i\omega t}$ where ϵ is a small parameter. For a cable-body system towed by a ship, a suitable choice of ϵ would be the wave height/wave length ratio since the motion of the ship is a function of this ratio. This assumption of a periodic disturbing force is not necessary for the method of solution developed later. Two arbitrary functions of time, $f(t)$ and $g(t)$, could have been prescribed. However, for convenience the disturbing force will be assumed to be periodic in time. Hence, the disturbing function, at $s = \ell$, will be written as

$$\left.\begin{aligned} f_x &= a\epsilon e^{i\omega t} \\ f_y &= b\epsilon e^{i\omega t} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.3.4)$$

where ω is the frequency of the disturbance and a and b are parameters which may depend on the frequency ω. Combining these disturbance velocities with the towing velocity, C_o , shown on Figure 1, yields

$$\left.\begin{aligned} u &= C_o + a\epsilon e^{i\omega t} \\ v &= b\epsilon e^{i\omega t} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.3.5)$$

for the horizontal and vertical components of the velocity. Resolving these equations into components normal and parallel to the cable gives

$$\left.\begin{aligned} U &= C_o \sin\phi + \epsilon e^{i\omega t}[a \sin\phi - b\cos\phi] \\ V &= C_o \cos\phi + \epsilon e^{i\omega t}[a\cos\phi + b\sin\phi] \end{aligned}\right\} \quad \ldots\ldots\ldots (2.3.6)$$

These are the boundary conditions that must be satisfied at the towing vehicle, $s = \ell$.

An important case to be considered later is when the motion is not a function of time, i.e., disturbances approach zero ($\epsilon \to 0$). This is the so-called steady-state or equilibrium configuration. For this case the boundary conditions reduce to

$$\left.\begin{aligned} D &= T\cos\phi \\ L &= T\sin\phi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.3.7)$$

at $s = 0$ and at the towing vehicle,

$$\left.\begin{aligned} U &= C_o \sin\phi \\ V &= C_o \cos\phi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.3.3)$$

where $s = \ell$.

Other simplifications to the general boundary conditions, equations (2.3.3) and (2.3.6), will be discussed when the need arises.

3.0 EQUILIBRIUM CONFIGURATION

In some applications of cable-towed bodies, the time dependence of U, V, φ, and T can be neglected. For example, this time dependence can be neglected for applications where the body is towed through a uniform stream from a steady platform. For such cases, the general system of equations (2.1.16) can be solved numerically. Since most investigators in this field are familiar with the work of Glauert, Landweber and Pode, the same notation developed in their reports will be used here. Except for the form of the loading functions, the development for the steady-state case will be similar to Reference 11 and is included so that this paper will be self-contained.

3.1 Steadv-State Equations

The general system of equations (2.1.15) can be simplified when the variables are not a function of time. This simplified system of equations can be written as

$$\left.\begin{aligned} T\frac{d\varphi}{ds} + F - W\cos\varphi &= 0 \\ \frac{dT}{ds} + G - W\sin\varphi &= 0 \\ \frac{dU}{ds} - V\frac{d\varphi}{ds} &= 0 \\ \frac{dV}{ds} + U\frac{d\varphi}{ds} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (3.1.1)$$

The last two equations of (3.1.1) can be combined and integrated to

$$U^2 + V^2 = C_0^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3.1.2)$$

If the first two equations are multiplied by the element of cable length, ds , they can be written as

$$T\,d\varphi = -(F - W\cos\varphi)\,ds = -Q(\varphi)\,ds \qquad \text{. . . . . (3.1.3)}$$

$$dT = -(G - W\sin\varphi)\,ds = -P(\varphi)\,ds \qquad \text{. . . . . (3.1.4)}$$

then

$$\frac{dT}{T} = \left[\frac{G - W\sin\varphi}{F - W\cos\varphi}\right] d\varphi$$

or

$$\frac{dT}{T} = \frac{P(\varphi)}{Q(\varphi)}\,d\varphi \qquad \text{. . . . . (3.1.5)}$$

When this equation is integrated from some point P_0 on the cable where the tension, T_0 , and the angle, φ_0 , are known to an arbitrary point P , it becomes

$$\frac{T}{T_0} = e^{\int_{\varphi_0}^{\varphi}\frac{P}{Q}d\varphi} \qquad \text{. . . . . (3.1.6)}$$

Substituting this value of T into equation (3.1.3) the element of cable length becomes

$$ds = \frac{T_0}{-Q}\,e^{\int_{\varphi_0}^{\varphi}\frac{P}{Q}d\varphi}\,d\varphi \qquad \text{. . . . . (3.1.7)}$$

Then the distance along the cable from the point P_0 to P is given by

$$s = \int_{\varphi_0}^{\varphi}\frac{T_0}{-Q}\,e^{\int_{\varphi_0}^{\varphi}\frac{P}{Q}d\varphi}\,d\varphi \qquad \text{. . . . . (3.1.8)}$$

In addition to the tension and length of the cable, the location of point P with respect to P_0 can be found in terms of coordinates parallel, x , and perpendicular, y , to the direction of motion. From equation (2.1.3), which is the parametric equation of a curve in terms of arc length

$$dx = ds \cos\varphi \quad , \quad dy = ds \sin\varphi$$

hence

$$x = \int_{\varphi_0}^{\varphi} \frac{T_0}{-Q} \, e^{\int_{\varphi_0}^{\varphi} \frac{P}{Q} d\varphi} \cos\varphi \, d\varphi \quad \ldots\ldots\ldots\ldots (3.1.9)$$

and

$$y = \int_{\varphi_0}^{\varphi} \frac{T_0}{-Q} \, e^{\int_{\varphi_0}^{\varphi} \frac{P}{Q} d\varphi} \sin\varphi \, d\varphi \quad \ldots\ldots\ldots\ldots (3.1.10)$$

To facilitate the solution of problems, it is desirable to transpose equations (3.1.6) through (3.1.10) to nondimensional form. The equation for the tension is already in nondimensional form in terms of the known tension, T_0 , at P_0 . It has been found convenient to make s , x , and y nondimensional by using a characteristic length, $\frac{T_0}{R}$. Dividing equations (3.1.8), (3.1.9), and (3.1.10) by $\frac{T_0}{R}$ and letting

$$\left.\begin{aligned} \sigma &= \frac{Rs}{T_0} \\ \xi &= \frac{Rx}{T_0} \\ \eta &= \frac{Ry}{T_0} \\ \tau &= \frac{T}{T_0} \\ &\text{and} \\ p &= \frac{P(\varphi)}{R} \\ q &= \frac{Q(\varphi)}{R} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3.1.11)$$

the following so-called cable functions are obtained.

$$\left.\begin{aligned}
\tau &= \frac{T}{T_0} = e^{\int_{\varphi_0}^{\varphi} \frac{f}{g} d\varphi} \\
\sigma &= \frac{Rs}{T_0} = \int_{\varphi_0}^{\varphi} \frac{\tau}{-g} d\varphi \\
\xi &= \frac{RX}{T_0} = \int_{\varphi_0}^{\varphi} \frac{\tau \cos\varphi}{-g} d\varphi \\
\eta &= \frac{RY}{T_0} = \int_{\varphi_0}^{\varphi} \frac{\tau \sin\varphi}{-g} d\varphi
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (3.1.12)$$

After inserting the loading functions in terms of φ as was discussed in section 2.2, the cable functions can be evaluated. Since these functions are in nondimensional form, the values obtained from numerical integration can be tabulated to facilitate solution of problems as was done in Reference 11 for circular cables.

3.2 Critical Angle

At this point in the development it will be convenient to consider a cable towed with a constant velocity through a steady, uniform stream without a body on the lower end of the cable. If end effects are neglected so that identical forces act on each element of the cable, the cable configuration will be a straight line. Physically this means that the configuration of the cable towed without a body will be a straight line inclined to the free stream at an angle $\varphi = \varphi_c$. Hence the critical angle, φ_c, is defined as the angle at which the normal hydrodynamic force is exactly balanced by the normal component of the weight of the cable.

From consideration of equation (3.1.3), which is rewritten here

$$T\frac{d\varphi}{ds} = -(F - W\cos\varphi) = -Q \quad \ldots\ldots\ldots\ldots \quad (3.2.1)$$

where $F = R(a\sin^2\varphi + b\sin\varphi)$ from equation (2.2.8)

the critical angle for the two limiting cases can be easily determined. Since $T = 0$ at the lower end of the cable, $F - W\cos\varphi$ must be zero at this point. Then if $F = 0$, the critical angle is 90° since $\cos\varphi \equiv 0$. However, when $W = 0$, the critical angle is 0° since F is a function of $\sin\varphi$. These are the two limiting cases; however, F and W are not zero generally.

When F and W are not zero, it is seen from equation (3.2.1) that

$$F = W\cos\varphi_c \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.2.2)$$

at the lower end of the cable since $T = 0$ at this point. Then, if the forces acting on each element of the cable are identical, the cable will be a straight line and inclined at an angle φ_c to the free stream as defined by equation (3.2.2). A similar assumption was made by McLeod(1), Glauert(2), and Pode(11) in their investigations of the steady-state cable problem. Pode presented experimental verification of this assumption in Reference 10.

The mathematical significance of the critical angle can be seen from consideration of a cable-body configuration such as the one shown on Figure 1. If a downward force is applied at $s = 0$, the cable configuration will lie below the straight line configuration of the cable towed without a body on the lower end. Likewise, if an upward force is exerted on the lower end of the cable, the configuration will lie above the straight line configuration of the cable without a body on the lower end. Only as the length of the cable is indefinitely increased will the effect of the body die out and then the cable angle will approach the critical angle far from the body. Hence, for the evaluation of the cable functions as a function of φ, the critical angle is a suitable upper limit. Since these cable functions are defined with Q in the denominator, a special technique must be used for their evaluation in the neighborhood of φ_c. However, no difficulty was encountered in evaluating these functions in Reference 11 for circular cables.

Several important conclusions may be deduced from this discussion of the critical angle and consideration of equation (3.2.1).

1 - If the angle of the cable is equal to the critical angle anywhere on the cable, the angle of the cable must be equal to the critical angle everywhere.

2 - Conversely, if the angle of the cable is different from the critical angle anywhere, it is not equal to the critical angle anywhere for a finite length of cable.

3 - The angle of the cable at the upper end will approach the critical angle as the length of the cable is indefinitely increased.

If the relationship for the normal loading function is substituted into equation (3.2.1), an equation for the critical angle in terms of R and W can be obtained. When the sine-squared law, which was assumed by other investigators, is used

$$\frac{W}{R} = \tan\varphi_c \sin\varphi_c \qquad \text{(3.2.3)}$$

However, using equation (2.2.9) which is valid for either circular or faired cables

$$\frac{W}{R} = \tan\varphi_c \left[\frac{t}{c}\sin\varphi_c + \left(1-\frac{t}{c}\right)\right] \qquad \text{(3.2.4)}$$

These last two equations are plotted on Figure 6 to illustrate the effect of the normal loading function on the upper limit of integration. It is

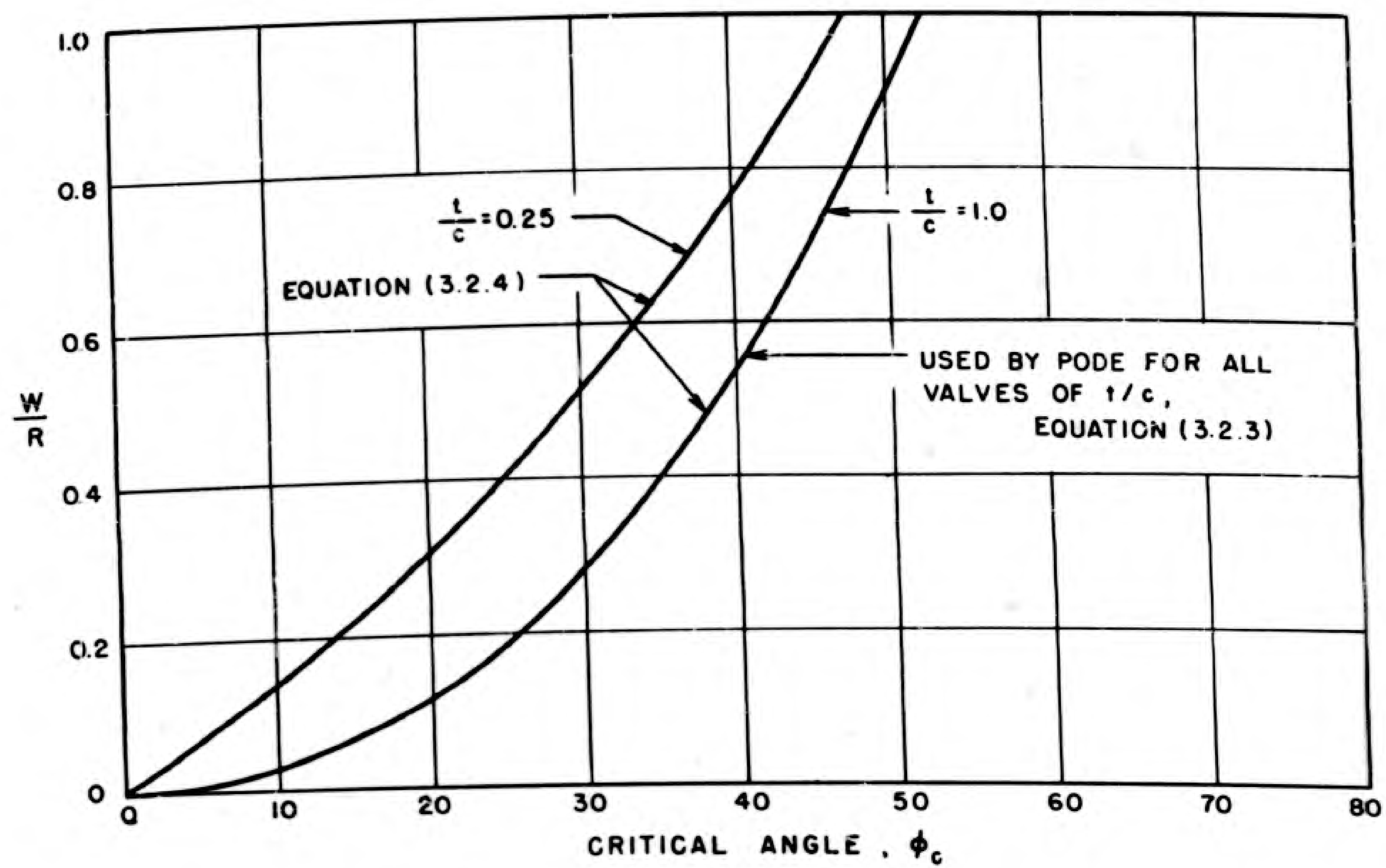


FIG. 6 VARIATION OF THE CRITICAL ANGLE WITH W/R FOR TWO FORMS OF THE NORMAL LOADING FUNCTION

seen that for a circular cable, $\frac{t}{c} = 1.0$; the equations are identical. However, if a faired cable is being considered, for example, $\frac{t}{c} = 0.25$, a difference in the two equations for the critical angle exists.

Since the loading functions that were used in Reference 11 do not fit the experimental data for faired cables, as was shown in section 2.2, use of the tabulated values of the cable functions given in Reference 11 is not recommended for computing the equilibrium configuration of a faired cable-body system. It is anticipated that computation based on the loading functions given by equation (2.2.9) will yield a more accurate determination of the cable configuration.

3.3 General Steady-State Configuration

In order to determine a suitable lower limit of integration, it will be convenient to consider a configuration that includes all possible cable-body configurations. Such a general configuration is obtained if a heavy string, where both ends of the string are fixed at different heights, is towed in a uniform stream. Figure 7 illustrates this problem graphically.

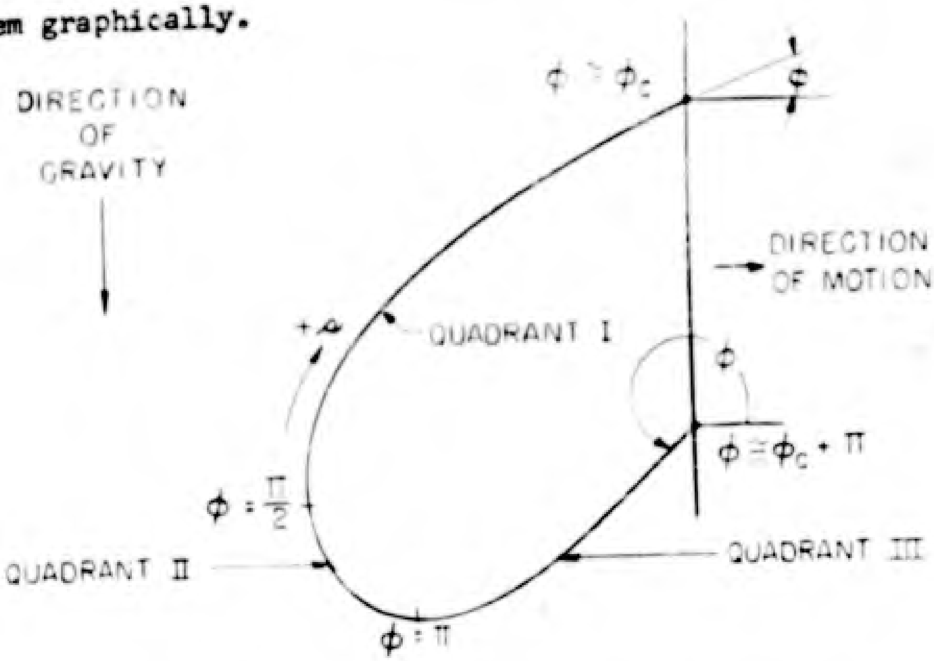


Figure 7 - General Cable Configuration

As was discussed in Section 3.2, the angle at both ends of the cable will approach the critical angle. Hence at some point on the cable, the angle must take on the value of $\frac{\pi}{2}$ and at another point the value π. It will be convenient to break the problem into three regions which are denoted as Quadrants I, II, and III. Most towing problems where faired cables are used fall into Quadrant I. A buoy moored to the ocean bottom would assume a configuration similar to Quadrant II. Some problems may involve two or even all three quadrants. For example, a surface ship towing another surface ship with a heavy cable would involve both Quadrants II and III. If the general configuration is separated into three quadrants, the quadrant in which the problem under consideration falls can be easily determined by comp ing the value of the critical angle. Recalling that φ is measured counterclockwise from the direction of motion to the positive direction along the cable, it is seen that

$$\left.\begin{array}{ll} \varphi_c < \varphi \leq \frac{\pi}{2} & \text{in Quadrant I} \\ \frac{\pi}{2} \leq \varphi \leq \pi & \text{in Quadrant II} \\ \pi \leq \varphi \leq \varphi_c + \pi & \text{in Quadrant III} \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (3.3.1)$$

Hence a convenient lower limit (reference point) for Quadrant I would be $\frac{\pi}{2}$. For Quadrant II either $\frac{\pi}{2}$ or π could be used, and for Quadrant III, π is a satisfactory reference point. Also for problems where the cable configuration lies in all three quadrants, it is convenient to have all the integrations based on the same reference point, π, for example. All of the reference points discussed above were used in Reference 11 to compute the cable functions for a circular cable.

As was mentioned above, most problems involving the use of faired cables fall into Quadrant I from practical considerations of the cable. Therefore, only that case need be computed and tabulated for faired cables. Hence, a series of tables of the cable functions vs. ϕ for a range of ϕ_c, all with the same lower limit of $\frac{\pi}{2}$, is needed.

3.4 Reference Point

If the cable functions are tabulated, specific information about the problem is needed to start the solution. The boundary conditions at $s = 0$, equation (2.3.7), are

$$\left.\begin{aligned} D &= T\cos\phi \\ L &= T\sin\phi \end{aligned}\right\} \qquad \text{(3.4.1)}$$

These two equations can be written as a single equation which is

$$\tan\phi = L/D \qquad \text{(3.4.2)}$$

where the lift and drag of the towed body are usually known. Hence, this value of ϕ, which was called ϕ_0 when the cable functions were integrated, is sufficient to start the solution. For example, the weight and drag per unit length of the cable are known, therefore the critical angle could be computed. Then, knowing the critical angle, the proper table could be selected and the values of τ_0, σ_0, ξ_0, and η_0 would be known for the value of ϕ_0 defined by equation (3.4.2). The zero subscript indicates the point P_0 shown on Figure 8.

In order to determine the corresponding values at point P_1, it must be remembered that the reference point is at $P_R = \frac{\pi}{2}$ and not at P_0. Hence, the values of s, x, y, T are measured from P_R and not from P_0. However, the reference point can be easily shifted to any arbitrary point. The nondimensional tension at point P_0 can be written as

$$\tau_0 = \frac{T_0}{T_R}$$

and at point P_1 as

$$\tau_1 = \frac{T_1}{T_R}$$

Therefore

$$\frac{\tau_1}{\tau_0} = \frac{T_1}{T_0}$$

which represents the tension at P_1 in terms of P_0 only. Also, the horizontal displacement of point P_0 with respect to P_1 can be determined from

$$\xi_1 = \frac{RX_{1,P_R}}{T_{P_R}}$$

and

$$\xi_0 = \frac{RX_{0,P_R}}{T_{P_R}}$$

then

$$\frac{\xi_1 - \xi_0}{\tau_0} = \frac{R}{T_{P_R}}\left[\frac{X_{1,P_R} - X_{0,P_R}}{T_0/T_{P_R}}\right] = \frac{RX_{1,0}}{T_0}$$

where the subscripts indicate the points shown on Figure 8.

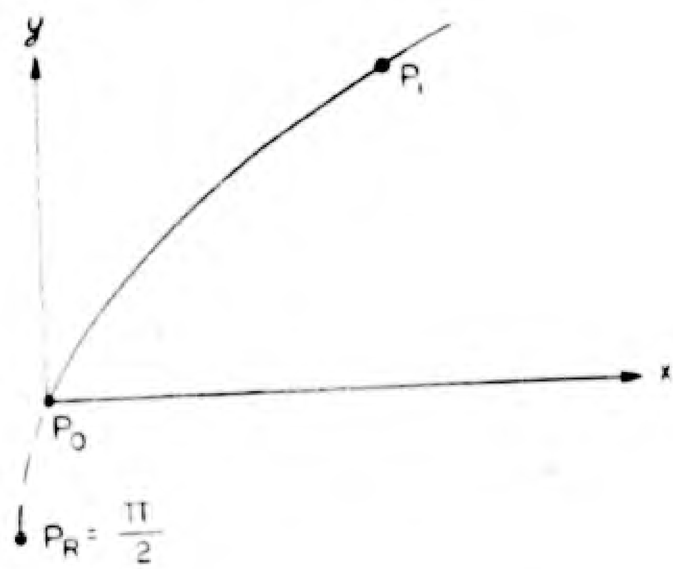


Figure 8 - Shifting Reference Point

From the last equation it is seen that ξ_1 , is expressed in terms of the cable functions at P_0 only. Likewise, σ and η could be shifted, then the value at any point can be found from

$$\left.\begin{aligned} \frac{\tau}{\tau_0} &= \frac{T}{T_0} \\ \frac{\sigma-\sigma_0}{\tau_0} &= \frac{Rs}{T_0} \\ \frac{\xi-\xi_0}{\tau_0} &= \frac{Rx}{T_0} \\ \frac{\eta-\eta_0}{\tau_0} &= \frac{Ry}{T_0} \end{aligned}\right\} \qquad (3.4.3)$$

where the distances s , x , and y are measured from the arbitrary point P_0 where the tension is known.

Using the relationships defined by equation (3.4.3) and knowing, for example, the angle of the cable at the towed body, the required towing depth, and the type of cable to be used, the length of cable, the horizontal displacement aft of the towpoint, and the tension at the towpoint could be computed. Problems where other facts are known can also be solved as can be seen from consideration of the equations. Hence, the one mentioned above was merely an illustration.

4.0 SOLUTION OF QUASI-LINEAR SYSTEM OF EQUATIONS

The system of quasi-linear, partial differential equations of the first order, derived in Section 2.1, describes the oscillatory motion of the configuration as a function of position and time. A method of solution for this system of equations will be presented where assumptions such as small perturbations are not necessary. This method of solution, generally called the method of characteristics, can be used provided the system of equations are of the hyperbolic type. The general mathematical theory of hyperbolic equations, which can be found in References 31 to 34, will not be developed here. Instead, the theory will be applied to the system of equations and a complete discussion of the method will be presented.

Rewriting the system of equations (2.1.16)

$$\left.\begin{aligned} & U_t - V\phi_t + \frac{T}{\mu}\phi_s + \frac{1}{\mu}(F - W\cos\phi) = 0 \\ & V_t + U\phi_t - \frac{1}{\mu}T_s - \frac{1}{\mu}(G - W\sin\phi) = 0 \\ & U_s + \phi_t - V\phi_s = 0 \\ & V_s + U\phi_s = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (4.0.1)$$

it is seen that there are four dependent variables, U, V, ϕ, T, and two independent variables, s and t. Since there are four dependent variables, four differential equations with the appropriate initial and boundary conditions are sufficient to determine a complete solution. The essential step in the method of characteristics is the determination of the characteristic equations which are ordinary differential equations. Then the resulting ordinary differential equations can be solved by finite differences or other methods applicable to ordinary differential equations. The characteristic equations will be discussed in more detail in the following sections.

4.1 Derivation of the Characteristic Equations

The general form of the original system of equations, (4.0.1), may be written as

$$L_m = A_m U_t + B_m U_s + C_m V_t + D_m V_s + E_m \phi_t + F_m \phi_s + G_m T_t + H_m T_s + K_m$$

where $m = 1, 2, 3, 4$. . . . .(4.1.1)

and the subscripts s and t denote differentiation with respect to s and t. Rewriting equation (4.0.1) in this notation

$$\left.\begin{aligned} L_1 &= A_1 U_t + F_1 \varphi_t + F_1 \varphi_s + K_1 = 0 \\ L_2 &= C_2 V_t + E_2 \varphi_t + H_2 T_s + K_2 = 0 \\ L_3 &= B_3 U_s + E_3 \varphi_t + F_3 \varphi_s = 0 \\ L_4 &= D_4 V_s + F_4 \varphi_s = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (4.1.2)$$

where the coefficients, A_1 ,..., are known functions of U , V , φ , T , s and t . Such a system of equations is called quasi-linear since the derivatives occur linearly while the coefficients may be functions of both the dependent and the independent variables.

A linear combination of the L's will be formed

$$L = \lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 + \lambda_4 L_4 \quad \ldots\ldots\ldots\ldots (4.1.3)$$

such that all derivatives of U , V , φ , T combine to form derivatives in the same direction in the $s-t$ plane. This variable for which the derivatives combine in this way will be denoted by σ which, in general, depends on s and t as well as U , V , φ , T and is called a characteristic. Suppose this direction is given by the ratio $\frac{ds}{d\sigma}\Big/\frac{dt}{d\sigma}$ which will be written as s_σ / t_σ . With this definition of the characteristics, $t_\sigma \neq 0$. The special case, $t_\sigma = 0$, will be discussed later. Recall that the total derivative of a function can be written as

$$\frac{df}{d\sigma} = \frac{\partial f}{\partial s}\frac{ds}{d\sigma} + \frac{\partial f}{\partial t}\frac{dt}{d\sigma} = f_s s_\sigma + f_t t_\sigma$$

and let

$$\begin{aligned} L = {} & R_1[U_s s_\sigma + U_t t_\sigma] + R_2[V_s s_\sigma + V_t t_\sigma] \\ & + R_3[\varphi_s s_\sigma + \varphi_t t_\sigma] + R_4[T_s s_\sigma + T_t t_\sigma] + R_5 \end{aligned}$$

where the R_1,... are unknown functions. Rewriting equation (4.1.3) and substituting for L's from equation (4.1.2) yields

$$\begin{aligned} L = {} & \lambda_1[A_1 U_t + E_1 \varphi_t + F_1 \varphi_s + K_1] + \lambda_2[C_2 V_t + E_2 \varphi_t + H_2 T_s + K_2] \\ & + \lambda_3[B_3 U_s + E_3 \varphi_t + F_3 \varphi_s] + \lambda_4[D_4 V_s + F_4 \varphi_s] \end{aligned}$$

Then if all derivatives of U, V, φ, T are to combine to form derivatives in only one direction in the $s - t$ plane the last two expressions for L must be equal. Hence, equating the coefficients of the respective derivatives yields

$$\left.\begin{aligned}
&R_1 s_\sigma = \lambda_3 B_3 \quad , \quad R_1 t_\sigma = \lambda_1 A_1 \\
&R_2 s_\sigma = \lambda_4 D_4 \quad , \quad R_2 t_\sigma = \lambda_2 C_2 \\
&R_3 s_\sigma = \lambda_1 F_1 + \lambda_3 F_3 + \lambda_4 F_4 \\
&R_3 t_\sigma = \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 \\
&R_4 s_\sigma = \lambda_2 H_2 \quad , \quad R_4 t_\sigma = 0 \\
&R_5 = 0
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (4.1.4)$$

The unknown functions, $R_1, \ldots$, can be eliminated from the above system of equations. As can be seen from the $R_4 t_\sigma = 0$ equation of the above system, two cases must be considered. Either $R_4 = 0$ or $t_\sigma = 0$. The first case to be considered is when $R_4 = 0$, that is, $t_\sigma \neq 0$. Then from equation (4.1.4) it is seen that $\lambda_2 H_2 t_\sigma = 0$ and $H_2 = -\frac{1}{\mu} \neq 0$, therefore $\lambda_2 = 0$. Then consider

$$\lambda_4 D_4 t_\sigma = \lambda_2 C_2 s_\sigma$$

hence $\lambda_4 = 0$ since $D_4 = 1.0$. The remaining two conditions are

$$\left.\begin{aligned}
&\lambda_1 A_1 s_\sigma = \lambda_3 B_3 t_\sigma \\
&\lambda_1 (E_1 s_\sigma - F_1 t_\sigma) = -\lambda_3 (E_3 s_\sigma - F_3 t_\sigma)
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (4.1.5)$$

Since λ_1, $\lambda_3 \neq 0$, the determinant of the coefficients must vanish. Hence

$$\begin{vmatrix} A_1 s_\sigma & -B_3 t_\sigma \\ E_1 s_\sigma - F_1 t_\sigma & E_3 s_\sigma - F_3 t_\sigma \end{vmatrix} = 0$$

and expanding

$$A_1 E_3 (s_\sigma)^2 - (A_1 F_3 - B_3 E_1) s_\sigma t_\sigma - B_3 F_1 (t_\sigma)^2 = 0 \quad \ldots\ldots\ldots\ldots (4.1.6)$$

For the characteristic direction to be real and distinct, the equations must be of the hyperbolic type. For an equation to be of the hyperbolic type the coefficients must satisfy the condition that

$$ac - b^2 < 0$$

For this case $a = A_1E_3$, $b = -(A_1F_3 - B_3E_1)$, $c = -B_3F_1$. These coefficients, A_1, etc., are known and can be determined by comparison of equations (4.0.1) and (4.1.2). Then the above condition becomes $-\frac{T}{\mu}$ which must be less than zero. This condition is always satisfied since T and μ are always positive. Hence the equations are of the hyperbolic type, therefore; the characteristic directions will be real and distinct.

However, if t = constant then $t_\sigma = 0$ and from equation (4.1.4) it is seen that $\lambda_1 = \lambda_2 = \lambda_3 = 0$ but $\lambda_4 \neq 0$. Hence, constant time lines in the $s - t$ plane denote characteristics. Then the L_4 equation of (4.1.2) is a characteristic.

If the slope of the characteristics, when $t_\sigma \neq 0$, is denoted by

$$\zeta = \frac{s_\sigma}{t_\sigma} \quad \text{.......................(4.1.7)}$$

equation (4.1.6), with the coefficients, A_1, etc., replaced by their corresponding value, can be written as

$$\zeta^2 - (-V + V)\zeta - \frac{T}{\mu} = 0$$

hence

$$\zeta = \pm\sqrt{\frac{T}{\mu}}$$

If the two roots of this equation are denoted by ζ_+ and ζ_-, two different characteristic directions are given by

$$\left.\begin{aligned} \frac{ds}{dt} &= \zeta_+ = \sqrt{T/\mu} \\ \frac{ds}{dt} &= \zeta_- = -\sqrt{T/\mu} \end{aligned}\right\} \quad \text{...............(4.1.8)}$$

From consideration of the other two conditions, $\lambda_2 = \lambda_4 = 0$, equation (4.1.3) may be written as $L_1 = \lambda_1 L_1 + \lambda_3 L_3 = 0$.

When the relationships for L_1 and L_2 are substituted into the above equation and the resulting equation is multiplied first by t_σ and then by s_σ, the following equations are obtained.

$$\left.\begin{aligned} Lt_\sigma &= \lambda_1(A_1 u_\sigma + E_1\phi_\sigma + K_1 t_\sigma) + \lambda_3 E_3\phi_\sigma = 0 \\ Ls_\sigma &= \lambda_1(F_1\phi_\sigma + K_1 s_\sigma) + \lambda_3(B_3 u_\sigma + F_3\phi_\sigma) = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (4.1.9)$$

If the first equation of (4.1.9) is considered along with the first equation of (4.1.5), the following determinate of the coefficients must vanish.

$$\begin{vmatrix} A_1 s_\sigma & -B_3 t_\sigma \\ A_1 u_\sigma + E_1\phi_\sigma + K_1 t_\sigma & E_3\phi_\sigma \end{vmatrix} = 0$$

Expanding this determinant and substituting for the coefficients gives

$$u_\sigma - \phi_\sigma(V - \zeta) + \frac{1}{\mu}(F - W\cos\phi)\, t_\sigma = 0 \quad \ldots\ldots\ldots\ldots (4.1.10)$$

Since ζ can assume two values, ζ_+ and ζ_-, equation (4.1.10) can be written as two equations. If the other equations of (4.1.9) and (4.1.5) are used to determine characteristic equations, the resulting equations will be proportional to the ones already obtained.

If a curvilinear coordinate net, C_+ and C_-, is introduced such that ζ_+ is associated with C_+ along which $\beta(s,t)$ is constant and ζ_- is associated with C_- along which $\alpha(s,t)$ is constant, equations (4.1.8) and (4.1.10) can be written as

$$\left.\begin{aligned} C_+ &: \quad s_\alpha - \zeta_+ t_\alpha = 0 \\ C_- &: \quad s_\beta - \zeta_- t_\beta = 0 \\ C_+ &: \quad u_\alpha - \phi_\alpha(V - \zeta_+) + \frac{1}{\mu}(F - W\cos\phi)\, t_\alpha = 0 \\ C_- &: \quad u_\beta - \phi_\beta(V - \zeta_-) + \frac{1}{\mu}(F - W\cos\phi)\, t_\beta = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (4.1.11)$$

These four characteristic equations in conjunction with the two equations of the original system, L_2 and L_4, and the appropriate initial and boundary conditions are sufficient to determine U, V, φ, T, as a function of s and t, as will be shown. The other two equations, L_2 and L_4, rewritten from (4.0.1) are

$$\left.\begin{aligned} &V_t + U\varphi_t - \frac{1}{\mu}T_s - \frac{1}{\mu}(G - W\sin\varphi) = 0 \\ &V_s + U\varphi_s = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (4.1.12)$$

A method of solution of these six equations will be presented in the next section. The boundary conditions, at $s = 0$ and $s = \ell$, which these equations must satisfy, were developed in section 2.3.

4.2 Solution of Characteristic Equations

The four characteristic equations derived in the last section along with the two equations of (4.1.12) will be solved by a finite difference method. For convenience these six equations are rewritten here.

$$\left.\begin{aligned} C_+ &: \; s_\alpha - \zeta_+ t_\alpha = 0 \\ C_- &: \; s_\beta - \zeta_- t_\beta = 0 \\ C_+ &: \; U_\alpha - \varphi_\alpha(V - \zeta_+) + \frac{1}{\mu}(F - W\cos\varphi)\,t_\alpha = 0 \\ C_- &: \; U_\beta - \varphi_\beta(V - \zeta_-) + \frac{1}{\mu}(F - W\cos\varphi)\,t_\beta = 0 \\ &\quad V_t + U\varphi_t - \frac{1}{\mu}T_s - \frac{1}{\mu}(G - W\sin\varphi) = 0 \\ &\quad V_s + U\varphi_s = 0 \\ \text{where} &\qquad \zeta_\pm = \pm\sqrt{T/\mu} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (4.2.1)$$

Using this system of equations and the following boundary and initial conditions, the cable problem can be completely solved. The boundary conditions, equations (2.3.3) and (2.3.6) from section 2.3 are at $s = 0$

$$T\sin\phi\cos\phi\left[\frac{m_y - m_x}{m_x m_y}\right] - \frac{D}{m_x}\sin\phi + \frac{L}{m_y}\cos\phi = U_t - V\phi_t$$

$$T\left[\frac{m_y\cos^2\phi + m_x\sin^2\phi}{m_x m_y}\right] - \frac{D}{m_x}\cos\phi - \frac{L}{m_y}\sin\phi = V_t + U\phi_t$$

. . . . (4.2.2)

and at $s = l$

$$U = C_0\sin\phi + e^{i\omega t}\epsilon\left[a\sin\phi - b\cos\phi\right]$$

$$V = C_0\cos\phi + e^{i\omega t}\epsilon\left[a\cos\phi + b\sin\phi\right]$$

For the cable problem initial conditions are prescribed for U, V, ϕ, T for all values of s when $t = 0$. These initial conditions, at $t = 0$, will be written as

$$\begin{aligned} U(s,0) &= \bar{U}^*(s) \\ V(s,0) &= \bar{V}^*(s) \\ \phi(s,0) &= \bar{\phi}^*(s) \\ T(s,0) &= \bar{T}^*(s) \end{aligned}$$

where $\bar{U}^*$, $\bar{V}^*$, $\bar{\phi}^*$, $\bar{T}^*$ are known functions of s. Then an approximation for U, V, ϕ, T can be made for small values of t by considering a series of points on the s-axis. These points are selected such that either the characteristics drawn from these points intersect on a common time line, or drawn such that the values may be determined on a common time line. Also, this time interval must be selected so that the point to be investigated is located at the intersection or within the characteristic triangle (domain of dependence on Figure 16) formed by the two characteristics. The latter method will be used in this solution.

The cable will be separated into three regions and each region will be considered separately. These three regions are shown on Figure 9.

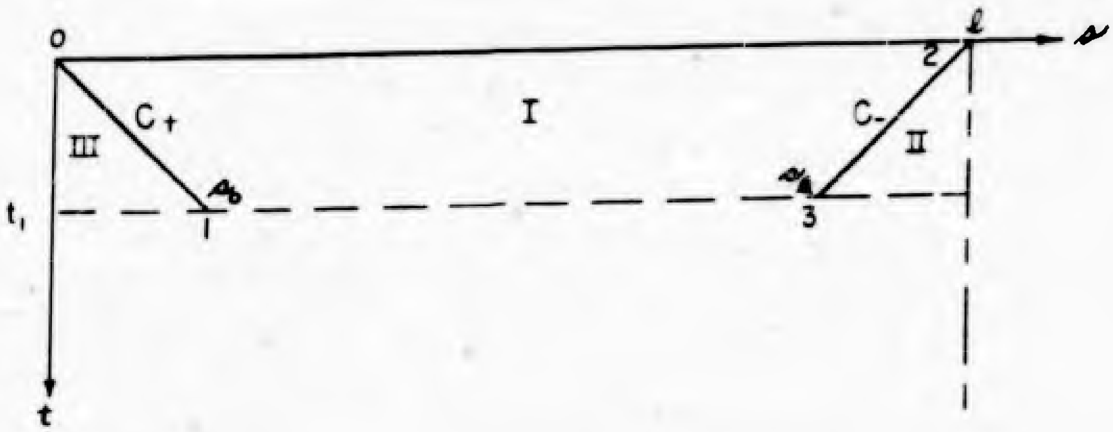


Figure 9 - Three Regions of $s - t$ Plane

Region I, which is interior to the two boundaries will be considered first. The initial conditions prescribe the values of U, V, φ, T along the s -axis and the slopes of the characteristic lines drawn from points on the s -axis are determined from equation (4.2.1). These equations for the slope on the $s - t$ plane are

$$\left.\begin{aligned} C_+ &: \quad \Delta s = \zeta_+ \Delta t \\ C_- &: \quad \Delta s = \zeta_- \Delta t \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.2.3)$$

where $\zeta_\pm = \pm\sqrt{T/\mu}$ and Δ denotes the difference between two points, for example $\Delta s = s_1 - s_0$ or $s_2 - s_3$. The second pair of relations given by equation (4.2.1) can be written in finite difference notation as

$$\left.\begin{aligned} C_+ &: \quad \Delta U - \Delta\varphi(V - \zeta_+) + \frac{1}{\mu}(F - W\cos\varphi)\Delta t = 0 \\ C_- &: \quad \Delta U - \Delta\varphi(V - \zeta_-) + \frac{1}{\mu}(F - W\cos\varphi)\Delta t = 0 \end{aligned}\right\} \quad \ldots\ldots \quad (4.2.4)$$

In general, if the characteristics are drawn from arbitrary points along the s -axis, these characteristics will not intersect on a common time line. Hence, consider a series of C_+ characteristics drawn from arbitrary points along the s -axis. For the present only points interior to the boundary, region I, will be considered as shown on Figure 10. The C_+ equation of (4.2.4), written for points 0 and 5,

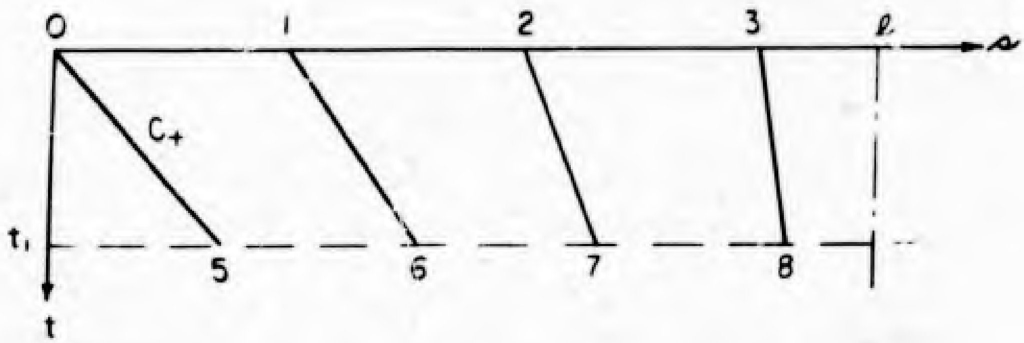


Figure 10 - C_+ Characteristic Lines

is

$$U_5 - U_0 - (\varphi_5 - \varphi_0)(V - \xi_+)_0 + \frac{1}{\mu}(F - W\cos\varphi)_0(t_1 - t_0)$$

or

$$U_5 - \varphi_5(V - \xi_+)_0 = U_0 - \varphi_0(V - \xi_+)_0 - \frac{1}{\mu}(F - W\cos\varphi)_0(t_1 - t_0)$$

All of the quantities are known except U_5 and φ_5 hence this last equation can be written as

$$U_5 - A_+(s)\,\varphi_5 = B_+(s) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.2.5)$$

where A_+ and B_+ are the corresponding quantities in the above equation. Similar equations can be written for points 1 , 2 ,... .

If the C_- characteristics are drawn from points along the s -axis, interior to the boundaries as shown on Figure 11, an equation similar to (4.2.5) can be written. For example, for points 10 and 14 the equation becomes

$$U_{14} - \varphi_{14}(V - \xi_-)_{10} = U_{10} - \varphi_{10}(V - \xi_-)_{10} - \frac{1}{\mu}(F - W\cos\varphi)_{10}(t_1 - t_0)$$

or

$$U_{14} - A_-\varphi_{14} = B_- \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.2.6)$$

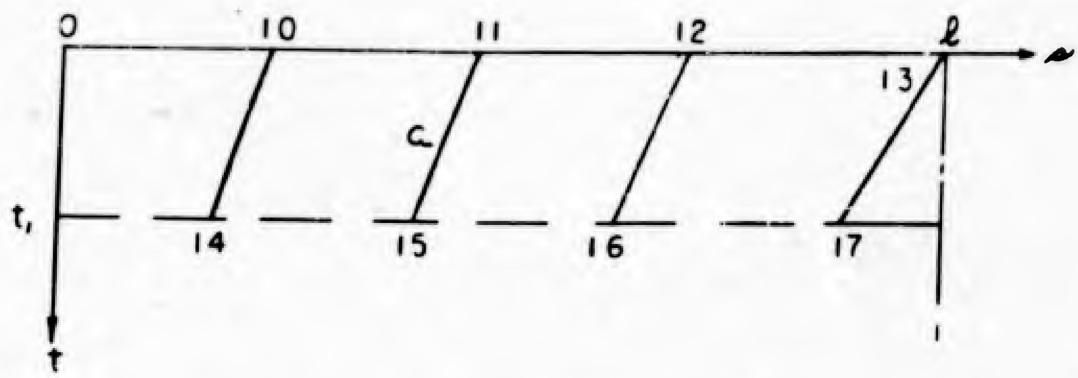


Figure 11 - C_- Characteristic Lines

When A_+ , A_- , B_+ , B_- are evaluated for several points along the s -axis, curves may be drawn, as shown on Figure 12, from which U and ϕ can be evaluated. For example, at $s = s_1$, the coefficients of equations (4.2.5) and (4.2.6) are known and at this value of s_1 , along the line t_1 , the two values of U and ϕ are the same. Therefore,

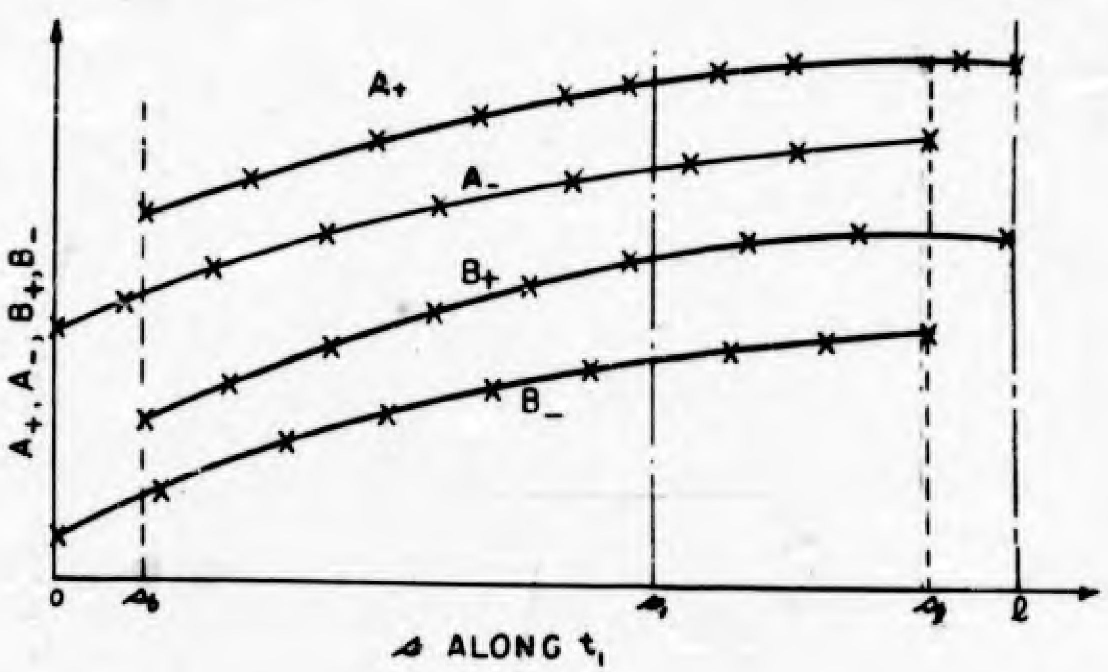


Figure 12 - Curves for Determination of U and ϕ

φ and U are given by

$$\left.\begin{aligned} \varphi &= \frac{B_+ - B_-}{A_- - A_+} \\ U &= \frac{A_- B_+ - A_+ B_-}{A_- - A_+} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.2.7)$$

for the point s_1, measured along t_1. Hence U and φ can be determined for all points along t_1, between s_0 and s_ℓ as shown on Figure 12. Points exterior to this region can be determined from the boundary conditions and then the curves could be extended over the entire cable length, ℓ.

In order to determine U and φ at the boundaries, it is necessary to consider the boundary conditions which were given by equation (4.2.2). Region II, the boundary at $s = \ell$ as shown on Figure 9, will now be considered. If the C_+ characteristic is drawn from point 15, as shown on Figure 13, equation (4.2.5) can be applied between these two points.

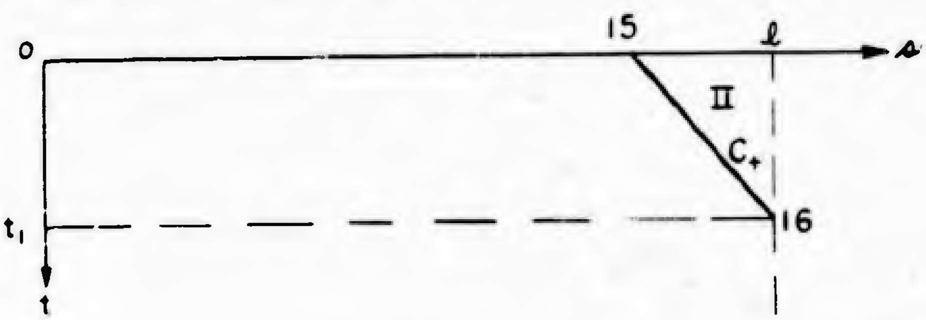


Figure 13 - Region II in s - t Plane

Hence

$$U_{16} - A_+ \varphi_{16} = B_+ \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.2.8)$$

The U boundary condition, from equation (4.2.2) can be used for the second equation so that

$$U_{16} = C_0 \sin \varphi_{16} + \epsilon e^{i\omega t_1} (a \sin \varphi_{16} - b \cos \varphi_{16}) \quad \ldots\ldots \quad (4.2.9)$$

and then U and ϕ can be determined at the boundary $s = \ell$. This boundary condition can be written in a different form which is easier to apply by differentiating equation (4.2.2) with respect to time and retaining the real part. Then, for these two points, ℓ and 16, the boundary condition can be written as

$$U_{16} - U_{\ell} = (\phi_{16} - \phi_{\ell})\Big[C_0 \cos\phi_{\ell} + \epsilon(a\cos\phi_{\ell} + b\sin\phi_{\ell})\cos\omega t$$
$$-\epsilon\omega(a\sin\phi_{\ell} - b\cos\phi_{\ell})\sin\omega t\Big] \quad \ldots\ldots\ldots \quad (4.2.9a)$$

Either of these two equations can be used; however, the latter will probably be easier to use.

Before considering the boundary at $s = 0$, it is necessary to apply the last equation of (4.2.1). Using this equation, which is rewritten here

$$V_s + U\phi_s = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.2.10)$$

the value of V may be determined along the line t_1, in regions I and II where ϕ is known. For the line t_1, shown on Figure 14, U and ϕ are known for all points exterior to region III.

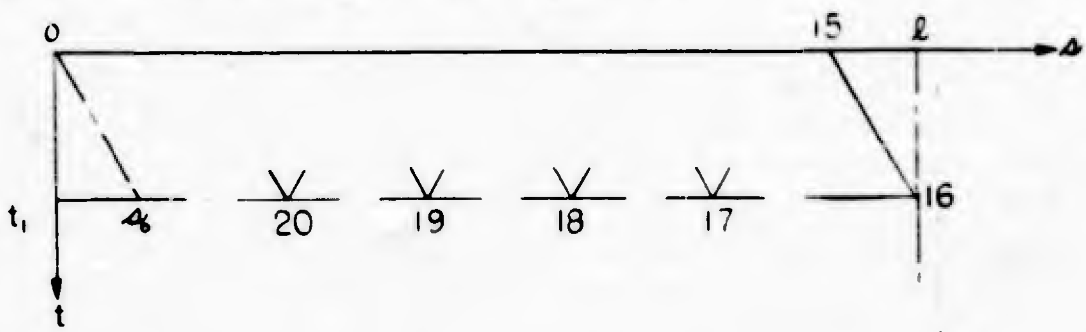


Figure 14 - Application of V Boundary Condition

Applying the V boundary condition, equation (4.2.2), to point 16 yields

$$V_{16} = C_0 \cos\phi_{16} + \epsilon e^{i\omega t_1}(a\cos\phi_{16} + b\sin\phi_{16}) \quad \ldots\ldots \quad (4.2.11)$$

Hence the tangential velocity at point 16 can be computed. Then from equation (4.2.10)

$$V_{16} - V_{17} + U_{16}(\varphi_{16} - \varphi_{17}) = 0$$

or

$$V_{17} = V_{16} + U_{16}(\varphi_{16} - \varphi_{17}) \quad \ldots\ldots\ldots\ldots\ldots \quad (4.2.12)$$

Since U and φ are not known in region III, V can not be computed in this region until the boundary conditions at $s = 0$ are considered.

The C_- characteristic and the boundary conditions at $s = 0$ will be needed to determine the value of U, V, φ, T at point 21 as shown on Figure 15.

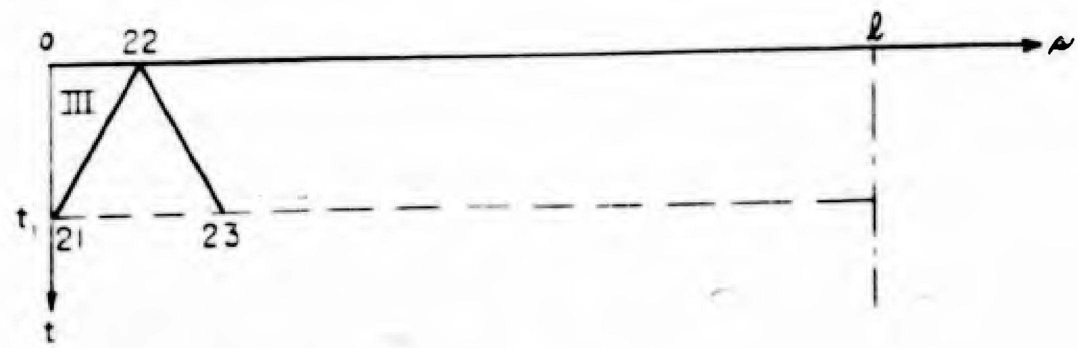


Figure 15 - Region III in s - t Plane

Writing the C_- characteristic, equation (4.2.6) for points 21 and 22 gives

$$U_{21} - A_- \varphi_{21} = B_- \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.2.13)$$

and equation (4.2.12) yields

$$V_{21} = V_{23} + U_{23}(\varphi_{23} - \varphi_{21}) \quad \ldots\ldots\ldots\ldots\ldots \quad (4.2.14)$$

For a third equation, since U_{21}, V_{21}, φ_{21}, are unknown, the tension, T, is eliminated from the boundary condition (4.2.2) and becomes

$$(V\varphi_t - U_t)(m_y \cos^2\varphi + m_x \sin^2\varphi) + (U\varphi_t + V_t)\sin\varphi\cos\varphi\,(m_y - m_x)$$

$$+ L, \quad \varphi - D\sin\varphi = 0$$

Writing this equation in finite difference notation for point 21

$$\left[V_0(\phi_{21}-\phi_0)-(U_{21}-U_0)\right]\left[m_y\cos^2\phi_0+m_x\sin^2\phi_0\right]$$

$$+\left[U_0(\phi_{21}-\phi_0)+(V_{21}-V_0)\right]\left[(m_y-m_x)\sin\phi_0\cos\phi_0\right]$$

$$+\left[L\cos\phi_0-D\sin\phi_0\right]\left[t_1-t_0\right] \quad \ldots\ldots\ldots\ldots\ldots \quad (4.2.15)$$

Therefore, using equations (4.2.13), (4.2.14) and (4.2.15), the values of U_{21}, V_{21}, and ϕ_{21} can be computed. The value of T_{21} can be computed from either of the boundary conditions, equation (4.2.2) at $s=0$. Then to compute the value of T at other points along the line t_1, the fifth relationship of equation (4.2.1) is used. In finite difference notation this equation for point 23 becomes

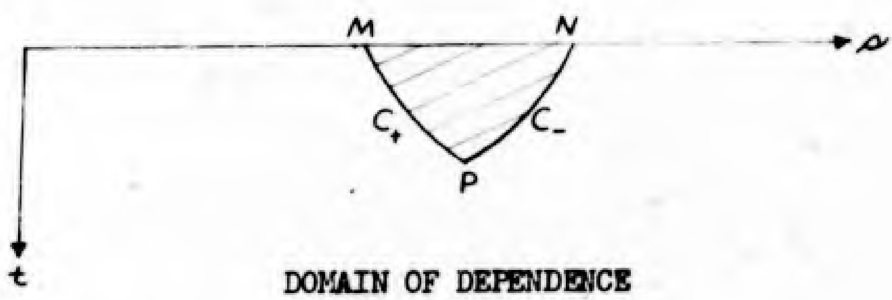
$$T_{23}-T_{21}=\mu\left\{\frac{\Delta s}{\Delta t}\left[(V_{21}-V_0)+U_0(\phi_{21}-\phi_0)\right]-\frac{1}{\mu}(G-W\sin\phi)_{21}\,\Delta s\right\}$$

where

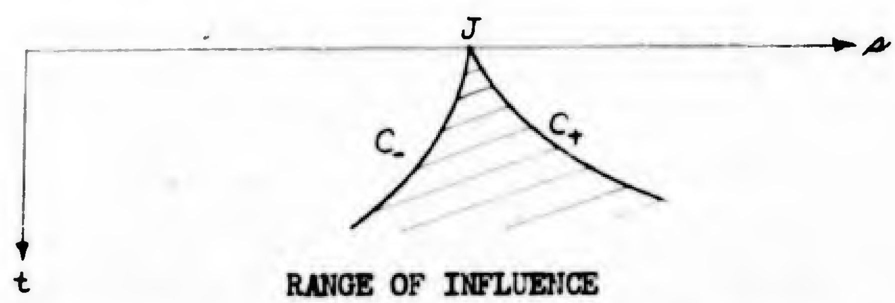
$$\Delta s = s_{23}-s_{21} \quad , \quad \Delta t = t_1-t_0 \quad \ldots\ldots\ldots\ldots \quad (4.2.16)$$

The values of U, V, ϕ, T have been determined for all points along the line t_1. Since t_1 is an arbitraty value of time, so long as it is chosen small enough that the point is within the domain of dependence, this same procedure can be repeated for a new time, t_2. This is possible since the value of the functions is known along the line t_1, hence, for each different constant time line the initial value problem can be solved.

It is interesting to note how the disturbances are propagated along the cable. From consideration of Figure 16 it is clear that the values of U and ϕ at the point $P(s,t)$, within the range of the solution, are determined solely from the initial values prescribed on the segment of the s-axis labeled MN which is subtended by the two characteristics issuing from P . The shaded region is often called the domain of dependence. Also, the range of influence of the point J is shown on Figure 16.



DOMAIN OF DEPENDENCE



RANGE OF INFLUENCE

Figure 16 - Propagation of Disturbances in s - t Plane

The range of influence in the s , t plane is the region in which the functions, U and ϕ , are influenced by the initial values assigned to point J . From consideration of Figure 16 and equation (4.2.1) several conclusions about the propagation of disturbances up and down the cable can be made. Consider first the C_+ characteristic of equation (4.2.1) after it has been multiplied by $d\alpha/ds$. Then

$$\frac{dU}{ds} = \frac{d\phi}{ds}(V - \zeta_+) - \frac{1}{\mu}(F - W\cos\phi)\frac{dt}{ds} \quad \ldots\ldots\ldots\ldots\ldots \quad (4.2.17)$$

along the C_+ characteristic where $\zeta_+ = \frac{ds}{dt} = \sqrt{\frac{T}{\mu}}$. As shown on Figure 16, the C_+ characteristic denotes a wave traveling up the cable.

Since $d\varphi/ds$, U and V are negative for this case, as can be seen from Figure 1, equation (4.2.17) can be written as

$$\frac{dU}{ds} = \left|\frac{d\varphi}{ds}\right|\left(|V| + \sqrt{\frac{T}{\mu}}\right) - \frac{1}{\mu}(F - W\cos\varphi)\frac{1}{\sqrt{T/\mu}} \quad \ldots\ldots\ldots\ldots (4.2.18)$$

Therefore, for the waves traveling up the cable to be damped, $\frac{dU}{ds} > 0$ which can be written as

$$\left|\frac{d\varphi}{ds}\right|\left(|V| + \sqrt{\frac{T}{\mu}}\right) > \frac{1}{\mu}(F - W\cos\varphi)\frac{1}{\sqrt{T/\mu}} \quad \ldots\ldots\ldots\ldots (4.2.19)$$

For waves traveling down the cable, consider the C_- characteristic of equation (4.2.4) in the following form

$$\frac{dU}{ds} = \frac{d\varphi}{ds}(V - \zeta_-) - \frac{1}{\mu}(F - W\cos\varphi)\frac{dt}{ds} \quad \ldots\ldots\ldots\ldots (4.2.20)$$

where $\zeta_- = ds/dt = -\sqrt{T/\mu}$. Just as for the previous case $\frac{d\varphi}{ds}$, U and V are negative, hence equation (4.2.20) can be written as

$$\frac{dU}{ds} = \left|\frac{d\varphi}{ds}\right|\left(|V| - \sqrt{T/\mu}\right) + \frac{1}{\mu}(F - W\cos\varphi)\frac{1}{\sqrt{T/\mu}} \quad \ldots\ldots\ldots\ldots (4.2.21)$$

Then, for the waves traveling down the cable to be damped $\frac{dU}{ds} < 0$ which can be written as

$$\left|V\frac{d\varphi}{ds}\right| + \frac{1}{\mu\sqrt{T/\mu}}(F - W\cos\varphi) < \left|\frac{d\varphi}{ds}\right|\sqrt{\frac{T}{\mu}} \quad \ldots\ldots\ldots\ldots (4.2.22)$$

These conclusions are similar to the ones made by Phillips[15] from a linearized analysis of an infinitely long cable. However, in his analysis it was found that the waves traveling up the cable were always damped whereas the waves traveling down the cable were damped when the free-stream velocity was less than $\sqrt{T/\mu}$.

As was noted in the solution, the boundary conditions enter when a characteristic line intersects the boundary at $s = 0$ and $s = \ell$.

The essential steps of the solution will be sumarized to emphasize the method.

1. Compute the slope of the characteristic line from $\zeta_\pm = \pm\sqrt{\frac{T}{\mu}}$.

2. Determine values of U and φ along a constant time line from equations (4.2.5) and (4.2.6).

3. Use the U boundary condition, equation (4.2.2) and equation (4.2.5) to determine U and φ at $s = \ell$.

4. Use the other boundary condition, equation (4.2.2) to compute V at $s = \ell$.

5. Compute V for all points along a constant time line exterior to region III by using equation (4.2.10).

6. The values of U, V, φ, at $s = 0$ are computed from equations (4.2.13), (4.2.14) and (4.2.15).

7. The value of T at $s = 0$ is obtained from the boundary condition, equation (4.2.2).

8. Use equation (4.2.16) to compute T at other points along the constant time line.

When the values of U, V, φ, T are obtained for the line t = constant, the process can be repeated to evaluate the functions at a new time.

5.0 LINEARIZATION OF EQUATIONS

The four, quasi-linear, partial differential equations of the first order, which were derived in section 2.1, will be linearized to facilitate obtaining an approximate solution. For this type of analysis the disturbing forces and the resulting motions must be small. Hence, it will be convenient to expand the functions U, V, φ, T about the steady-state value. Then these functions can be written as the sum of an infinite series consisting of a steady-state part which depends on the position along the cable and a part which depends on both position and time. Therefore

$$
\left.\begin{aligned}
U &= U_0(s) + \epsilon U_1(s,t) + \epsilon^2 U_2(s,t) + \cdots = U_0 + \sum_{n=1}^{\infty} \epsilon^n U_m \\
V &= V_0(s) + \epsilon V_1(s,t) + \epsilon^2 V_2(s,t) + \cdots = V_0 + \sum_{n=1}^{\infty} \epsilon^n U_m \\
\phi &= \phi_0(s) + \epsilon \phi_1(s,t) + \epsilon^2 \phi_2(s,t) + \cdots = \phi_0 + \sum_{n=1}^{\infty} \epsilon^n \phi_m \\
T &= T_0(s) + \epsilon T_1(s,t) + \epsilon^2 T_2(s,t) + \cdots = T_0 + \sum_{n=1}^{\infty} \epsilon^n T_m
\end{aligned}\right\} \quad \ldots\ldots (5.0.1)
$$

where the zero subscript denotes the steady-state value and ϵ is a small parameter which, for the ship-towed problem, may be taken as the wave height/wave length ratio. Since ϵ is assumed to be small, terms involving ϵ^2 and higher powers will be neglected in comparison with terms involving ϵ^0 and ϵ^1. In this notation $\sin\phi$ and $\cos\phi$ become

$$
\left.\begin{aligned}
\sin\phi &= \sin\phi_0 + \epsilon\phi_1 \cos\phi_0 \\
\cos\phi &= \cos\phi_0 - \epsilon\phi_1 \sin\phi_0
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (5.0.2)
$$

When these relationships are substituted into the original system of equations (2.1.16) the following equations are obtained

$$
\left.\begin{aligned}
\epsilon\left(V_0 \frac{\partial \phi_1}{\partial t} - \frac{\partial U_1}{\partial t}\right) &= \frac{1}{\mu}\left[T_0 \frac{d\phi_0}{ds} + F - W\cos\phi_0\right] + \frac{\epsilon}{\mu}\left[T_0 \frac{\partial \phi_1}{\partial s} + T_1 \frac{d\phi_0}{ds}\right. \\
&\quad \left. + (F_U U_1 + F_V V_1) + W\phi_1 \sin\phi_0\right] + O(\epsilon^2) \\
\epsilon\left(\frac{\partial V_1}{\partial t} + U_0 \frac{\partial \phi_1}{\partial t}\right) &= \frac{1}{\mu}\left[\frac{dT_0}{ds} + G - W\sin\phi_0\right] + \frac{\epsilon}{\mu}\left[\frac{\partial T_1}{\partial s} + (G_U U_1 + G_V V_1)\right. \\
&\quad \left. - W\phi_1 \cos\phi_0\right] + O(\epsilon^2) \\
\epsilon \frac{\partial \phi_1}{\partial t} &= \left[V_0 \frac{d\phi_0}{ds} - \frac{dU_0}{ds}\right] + \epsilon\left[V_1 \frac{d\phi_0}{ds} + V_0 \frac{\partial \phi_1}{\partial s} - \frac{\partial U_1}{\partial s}\right] + O(\epsilon^2) \\
0 &= \left[\frac{dV_0}{ds} + U_0 \frac{d\phi_0}{ds}\right] + \epsilon\left[\frac{\partial V_1}{\partial s} + U_1 \frac{d\phi_0}{ds} + U_0 \frac{\partial \phi_1}{\partial s}\right] + O(\epsilon^2)
\end{aligned}\right\} \quad \ldots (5.0.3)
$$

The loading functions, F and G, were assumed to be functions of both U and V and were expanded in a Taylor's Series about the steady-state values, U_0 and V_0. When the proposed loading functions, equation (2.2.3), are used, F_V and G_U are zero. However, these two terms will be retained to make the analysis general.

Taking the limit of equation (5.0.3) as ϵ approaches zero yields

$$\left.\begin{aligned} &T_0 \frac{d\phi_0}{ds} + F - W\cos\phi_0 = 0 \\ &\frac{dT_0}{ds} + G - W\sin\phi_0 = 0 \\ &\frac{dU_0}{ds} - V_0\frac{d\phi_0}{ds} = 0 \\ &\frac{dV_0}{ds} + U_0\frac{d\phi_0}{ds} = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5.0.4)$$

Comparison of this system of equations with the steady-state equations (3.1.1), which were solved in section 3.1, verifies that as the disturbance parameter, ϵ, approaches zero the motion approaches the steady-state condition.

Collecting the terms that involve the first power of ϵ gives

$$\left.\begin{aligned} &V_0\frac{\partial\phi_1}{\partial t} - \frac{\partial U_1}{\partial t} = \frac{1}{\mu}\Big[T_0\frac{\partial\phi_1}{\partial s} + T_1\frac{d\phi_0}{ds} + (F_U U_1 + F_V V_1) \\ &\qquad\qquad + W\phi_1\sin\phi_0\Big] \\ &\frac{\partial V_1}{\partial t} + U_0\frac{\partial\phi_1}{\partial t} = \frac{1}{\mu}\Big[\frac{\partial T_1}{\partial s} + (G_U U_1 + G_V V_1) \\ &\qquad\qquad - W\phi_1\cos\phi_0\Big] \\ &\frac{\partial\phi_1}{\partial t} = V_1\frac{d\phi_0}{ds} + V_0\frac{\partial\phi_1}{\partial s} - \frac{\partial U_1}{\partial s} \\ &0 = \frac{\partial V_1}{\partial s} + U_1\frac{d\phi_0}{ds} + U_0\frac{\partial\phi_1}{\partial s} \end{aligned}\right\} \quad \ldots\ldots (5.0.5)$$

Equation (5.0.5) is a system of four, first-order, linear partial differential equations with non-constant coefficients. These equations can be reduced to one, fourth-order equation with non-constant coefficients; however, this is of no advantage since the general solution of such an equation is not known.

When the disturbance velocities applied at $s = \ell$ are periodic, with a period of ω, the steady-state oscillatory motion will have the same period ω. Then assuming that these disturbances can be represented as

$$\left.\begin{aligned} U_1 &= \bar{U}(s)\, e^{i\omega t} \\ V_1 &= \bar{V}(s)\, e^{i\omega t} \\ \phi_1 &= \bar{\phi}(s)\, e^{i\omega t} \\ T_1 &= \bar{T}(s)\, e^{i\omega t} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5.0.6)$$

equation (5.0.5) can be written, after rearrangement of terms as

$$\left.\begin{aligned} \frac{d\bar{\phi}}{ds} &= \frac{1}{T_0}\left[-\bar{U}(i\omega\mu + \xi) - \xi\bar{V} + \bar{\phi}(i\omega\mu V_0 - W\sin\phi_0) - \bar{T}\frac{d\phi_0}{ds}\right] \\ \frac{d\bar{T}}{ds} &= -G_u\bar{U} + (i\omega\mu - G_v)\bar{V} + \bar{\phi}(i\omega\mu U_0 + W\cos\phi_0) \\ \frac{d\bar{U}}{ds} &= \bar{V}\frac{d\phi_0}{ds} - i\omega\bar{\phi} + V_0\frac{d\bar{\phi}}{ds} \\ \frac{d\bar{V}}{ds} &= -\bar{U}\frac{d\phi_0}{ds} - U_0\frac{d\bar{\phi}}{ds} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (5.0.7)$$

The last two equations of the above system can be written in canonical form by substitution of the value $d\bar{\phi}/ds$ from the first equation. When this substitution is made, equation (5.0.7) becomes

$$\left.\begin{aligned}
\frac{d\bar{\Phi}}{ds} &= -\frac{1}{T_0}\left[\bar{U}(i\omega\mu + E_u) + E_v\bar{V} - \bar{\Phi}(i\omega\mu V_0 - W\sin\phi_0) + \bar{T}\frac{d\phi_0}{ds}\right] \\
\frac{d\bar{T}}{ds} &= -G_u\bar{U} + \bar{V}(i\omega\mu - G_v) + \bar{\Phi}(i\omega\mu U_0 - W\cos\phi_0) \\
\frac{d\bar{U}}{ds} &= \frac{V_0}{T_0}\Big\{-\bar{U}(i\omega\mu + E_u) + \bar{V}\Big(\frac{T_0}{V_0}\frac{d\phi_0}{ds} - E_v\Big) \\
&\qquad + \bar{\Phi}\Big[i\omega\Big(\mu V_0 - \frac{T_0}{V_0}\Big) - W\sin\phi_0\Big] - \bar{T}\frac{d\phi_0}{ds}\Big\} \\
\frac{d\bar{V}}{ds} &= \frac{U_0}{T_0}\Big\{\bar{U}\Big(i\omega\mu + E - \frac{T_0}{U_0}\frac{d\phi_0}{ds}\Big) + E\bar{V} - \bar{\Phi}(i\omega\mu V_0 - W\sin\phi_0) + \bar{T}\frac{d\phi_0}{ds}\Big\}
\end{aligned}\right\} \quad \ldots\ldots (5.0.7)$$

These equations are now ordinary linear differential equations, however, the coefficients are not constant. The linearized boundary conditions at $s = 0$ which equation (5.0.3) must satisfy are obtained from equations (2.3.3) and (5.0.1). These boundary conditions, neglecting ϵ^2 and higher order terms, are

$$\left.\begin{aligned}
&\left[T_0\sin\phi_0\cos\phi_0(m_y - m_x) - Dm_y\sin\phi_0 + Lm_x\cos\phi_0\right] \\
&+ \epsilon\Big\{\left[T_0\phi_1(\cos^2\phi_0 - \sin^2\phi_0) + T_1\sin\phi_0\cos\phi_0\right](m_y - m_x) \\
&- \phi_1\left[Dm_y\cos\phi_0 + Lm_x\sin\phi_0\right] - m_y(D_u U_1 + D_v V_1)\sin\phi_0 \\
&+ m_x(L_u U_1 + L_v V_1)\cos\phi_0\Big\} = \epsilon m_x m_y\Big(\frac{\partial U_1}{\partial t} - V_0\frac{\partial \phi_1}{\partial t}\Big) \\
&\text{and} \\
&\left[T_0(m_y\cos^2\phi_0 + m_x\sin^2\phi_0) - Dm_y\cos\phi_0 - Lm_x\sin\phi_0\right] \\
&+ \epsilon\Big\{T_1(m_y\cos^2\phi_0 + m_x\sin^2\phi_0) - 2T_0\phi_1\sin\phi_0\cos\phi_0(m_y - m_x) \\
&+ \phi_1(Dm_y\sin\phi_0 - Lm_x\cos\phi_0) - m_y(D_u U_1 + D_v V_1)\cos\phi_0 \\
&- m_x(L_u U_1 + L_v V_1)\sin\phi_0\Big\} = \epsilon m_x m_y\Big(\frac{\partial V_1}{\partial t} + U_0\frac{\partial \phi_1}{\partial t}\Big)
\end{aligned}\right\} \quad \ldots\ldots (5.0.8)$$

where L and D were expanded in a Taylor's Series about the steady-state values of U_0 and V_0. Also, as was done in Lamb[30], m_x and m_y are assumed to be constant even though there might be a slight dependence of these functions on frequency.

The linearized boundary conditions, obtained from equations (2.3.6) and (5.0.1) for $s = \ell$ are

$$\left.\begin{aligned} U_0 - C_0 \sin\phi_0 &= \epsilon(C_0\phi_1\cos\phi_0 - U_1) + \epsilon e^{i\omega t}(a\sin\phi_0 - b\cos\phi_0) \\ V_0 - C_0\cos\phi_0 &= -\epsilon(C_0\phi_1\sin\phi_0 + V_1) + \epsilon e^{i\omega t}(a\cos\phi_0 + b\sin\phi_0) \end{aligned}\right\} \quad . . . (5.0.9)$$

Just as for the linearized differential equations, taking the limit of equations (5.0.8) and (5.0.9) as ϵ approaches zero yields the steady-state case. Equation (5.0.8) reduces to

$$D = T_0 \cos\phi_0$$
$$L = T_0 \sin\phi_0$$

since $m_x = m_y = M$ for the steady-state condition. From equation (5.0.9)

$$U_0 = C_0 \sin\phi_0$$
$$V_0 = C_0 \cos\phi_0$$

when $\epsilon = 0$. These are the boundary conditions for the steady-state case derived in section 2.3.

Considering the terms involving the first power of ϵ of equation (5.0.8) and substituting for U_1, V_1, ϕ_1, T_1 from equation (5.0.6), the following linearized boundary conditions for a periodic disturbance are obtained for $s = 0$.

$$\left.\begin{aligned}
&\bar{U}(L_v m_x \cos\phi_0 - Q_v m_y \sin\phi_0 - i\omega m_x m_y) + \bar{V}(L_v m_x \cos\phi_0 \cdot Q_v m_y \sin\phi_0) \\
&+\bar{\Phi}[i\omega m_x m_y V_0 - D m_y \cos\phi_0 - L m_x \sin\phi_0 + T_0(m_y - m_x)(\cos^2\phi_0 - \sin^2\phi_0)) \\
&+\bar{T}\sin\phi_0\cos\phi_0(m_y - m_x) = 0 \\
&\text{and} \\
&\bar{U}(L_v m_x \sin\phi_0 + Q_v m_y \cos\phi_0) + \bar{V}(i\omega m_x m_y + L_v m_x \sin\phi_0 + Q_v m_y \cos\phi_0) \\
&-\bar{\Phi}[D m_y \sin\phi_0 - L m_x \cos\phi_0 - 2T_0 \sin\phi_0\cos\phi_0(m_y - m_x) - i\omega m_x m_y U_0] \\
&+\bar{T}(m_y\cos^2\phi_0 + m_x\sin^2\phi_0) = 0
\end{aligned}\right\} . \quad (5.0.10)$$

The linearized boundary conditions for a periodic disturbance, obtained from equations (5.0.9) and (5.0.6) for $s = \ell$ are

$$\left.\begin{aligned}
&C_0\bar{\Phi}\cos\phi_0 - \bar{U} + a\sin\phi_0 - b\cos\phi_0 = 0 \\
&C_0\bar{\Phi}\sin\phi_0 + \bar{V} - a\cos\phi_0 - b\sin\phi_0 = 0
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (5.0.11)$$

These equations, (5.0.10) and (5.0.11), are the boundary conditions for the system of linearized equations of motion of the cable-body configuration, equation (5.0.7), when the system is subjected to a periodic disturbance. In general, the solution of such a system of equations is not known. However, approximate solutions can be obtained when assumptions are made about the coefficients [15,16]. Also, asymptotic expansions for both small and large values of the perturbation frequency, ω, can be made. This latter method will be developed in the next section. An illustrative example, utilizing the linearized equations developed in this section, is presented in section 7.0.

6.0 ASYMPTOTIC EXPANSIONS FOR SMALL AND LARGE VALUES OF THE PERTURBATION FREQUENCY

A general solution for the linearized equations developed in the last section is not known. However, an approximate solution can be obtained by making an asymptotic expansion (35) in terms of the frequency parameter, ω . Both small and large values of ω will be investigated even though most ship-towed systems will be covered by a small value of ω . For example, the period for rolling motion of an average ship is about 8 seconds, hence, the frequency is less than one cycle/second. However, the frequency may be greater than one cycle/second for some ship-towed problems involving slamming as well as devices towed by aircraft which may encounter high frequencies. Therefore, both cases will be developed.

6.1 Asymptotic Expansion for Small Values of ω

In the development of the linearized equations in section 5, the perturbation frequency was denoted by ω . The canonical form of the linearized equations (5.0.7) will be used for this asymptotic expansion. The functions $\bar{U}$, $\bar{V}$, $\bar{\Phi}$, $\bar{T}$ as defined in section 5.0 will be written as a power series in ω . Then

$$\left.\begin{aligned}
\bar{U} &= U_0^* + U_1^*\omega + U_2^*\omega^2 + \cdots = \sum_{m=0}^{\infty} U_m^*\omega^m \\
\bar{V} &= V_0^* + V_1^*\omega + V_2^*\omega^2 + \cdots = \sum_{m=0}^{\infty} V_m^*\omega^m \\
\bar{\Phi} &= \Phi_0^* + \Phi_1^*\omega + \Phi_2^*\omega^2 + \cdots = \sum_{m=0}^{\infty} \Phi_m^*\omega^m \\
\bar{T} &= T_0^* + T_1^*\omega + T_2^*\omega^2 + \cdots = \sum_{m=0}^{\infty} T_m^*\omega^m
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (6.1.1)$$

where $\omega \ll 1$. When these functions are substituted into

$$\left.\begin{aligned}
\frac{d\bar{\Phi}}{ds} &= -\frac{i\omega\mu}{T_0}(\bar{U}-V_0\bar{\Phi}) - \frac{1}{T_0}\left[F_U\bar{U}+F_V\bar{V}+\bar{\Phi}W\sin\phi_0+\bar{T}\frac{d\phi_0}{ds}\right] \\
\frac{d\bar{T}}{ds} &= i\omega\mu(\bar{V}+U_0\bar{\Phi}) - G_U\bar{U} - G_V\bar{V} + \bar{\Phi}W\cos\phi_0 \\
\frac{d\bar{U}}{ds} &= -\frac{i\omega\mu V_0}{T_0}\left[\bar{U}-(V_0-\frac{T_0}{\mu V_0})\bar{\Phi}\right] - \frac{V_0}{T_0}\left[F_U\bar{U}+\bar{V}(F_V-\frac{T_0}{V_0}\frac{d\phi_0}{ds})+\bar{\Phi}W\sin\phi_0+\bar{T}\frac{d\phi_0}{ds}\right] \\
\frac{d\bar{V}}{ds} &= \frac{i\omega\mu U_0}{T_0}\left[\bar{U}-V_0\bar{\Phi}\right] + \frac{U_0}{T_0}\left[(F_U-\frac{T_0}{U_0}\frac{d\phi_0}{ds})\bar{U}+F_V\bar{V}+\bar{\Phi}W\sin\phi_0+\bar{T}\frac{d\phi_0}{ds}\right]
\end{aligned}\right\} \quad . . . (6.1.2)$$

which is equation (5.0.7) written in a different form, the following system of equations is obtained

$$\left.\begin{aligned}
\sum_{n=0}^{\infty}\omega^n\frac{d\phi_n^x}{ds} &= -\frac{i\mu}{T_0}\sum_{n=1}^{\infty}\omega^n(U_{n-1}^x-V_0\phi_{n-1}^x) - \frac{1}{T_0}\sum_{n=0}^{\infty}\omega^n\Big[F_U U_n^x \\
&\qquad + F_V V_n^x + \phi_n^x W\sin\phi_0 + T_n^x\frac{d\phi_0}{ds}\Big] \\
\sum_{n=0}^{\infty}\omega^n\frac{dT_n^x}{ds} &= i\mu\sum_{n=1}^{\infty}\omega^n(V_{n-1}^x + U_0\phi_{n-1}^x) - \sum_{n=0}^{\infty}\omega^n\Big[G_U U_n^x \\
&\qquad + G_V V_n^x - \phi_n^x W\cos\phi_0\Big] \\
\sum_{n=0}^{\infty}\omega^n\frac{dU_n^x}{ds} &= -i\mu\frac{V_0}{T_0}\sum_{n=1}^{\infty}\omega^n\Big[U_{n-1}^x-(V_0-\frac{T_0}{\mu V_0})\phi_{n-1}^x\Big] - \frac{V_0}{T_0}\sum_{n=0}^{\infty}\omega^n\Big[F_U U_n^x \\
&\qquad + V_n^x(F_V-\frac{T_0}{V_0}\frac{d\phi_0}{ds}) + \phi_n^x W\sin\phi_0 + T_n^x\frac{d\phi_0}{ds}\Big] \\
\sum_{n=0}^{\infty}\omega^n\frac{dV_n^x}{ds} &= \frac{i\mu U_0}{T_0}\sum_{n=1}^{\infty}\omega^n(U_{n-1}^x-V_0\phi_{n-1}^x) + \frac{U_0}{T_0}\sum_{n=0}^{\infty}\omega^n\Big[(F_U-\frac{T_0}{U_0}\frac{d\phi_0}{ds})U_n^x \\
&\qquad + F_V V_n^x + \phi_n^x W\sin\phi_0 + T_n^x\frac{d\phi_0}{ds}\Big]
\end{aligned}\right\} \quad . \ (6.1.3)$$

The coefficients of like powers of ω must vanish, hence for $n = 0$ these equations are

$$\left.\begin{aligned}
&\frac{d\phi_0^*}{ds} + \phi_0^* \frac{W \sin\phi_0}{T_0} = -\frac{1}{T_0}\left[F_u U_0^* + F_v V_0^* + T_0^* \frac{d\phi_0}{ds}\right] \\
&\frac{dT_0^*}{ds} = -\left[G_u U_0^* + G_v V_0^* - \phi_0^* W \cos\phi_0\right] \\
&\frac{dU_0^*}{ds} + \frac{F_v V_0}{T_0} U_0^* = -\frac{V_0}{T_0}\left[V_0^*\left(F_v - \frac{T_0}{V_0}\frac{d\phi_0}{ds}\right) + \phi_0^* W \sin\phi_0 + T_0^* \frac{d\phi_0}{ds}\right] \\
&\frac{dV_0^*}{ds} - \frac{F_v U_0}{T_0} V_0^* = \frac{U_0}{T_0}\left[\left(F_v - \frac{T_0}{U_0}\frac{d\phi_0}{ds}\right)U_0^* + \phi_0^* W \sin\phi_0 + T_0^* \frac{d\phi_0}{ds}\right]
\end{aligned}\right\} \quad \ldots . \; (6.1.4)$$

For $n = 1, 2, 3, \ldots$ the equations become

$$\left.\begin{aligned}
&\frac{d\phi_n^*}{ds} + \phi_n^* \frac{W \sin\phi_0}{T_0} = -\frac{i\mu}{T_0}\left(U_{n-1}^* - V_0 \phi_{n-1}^*\right) - \frac{1}{T_0}\left[F_u U_n^* + F_v V_n^* + T_n^* \frac{d\phi_0}{ds}\right] \\
&\frac{dT_n^*}{ds} = i\mu\left(V_{n-1}^* + U_0 \phi_{n-1}^*\right) - \left[G_u U_n^* + G_v V_n^* - \phi_n^* W \cos\phi_0\right] \\
&\frac{dU_n^*}{ds} + \frac{V_0}{T_0} F_v U_n^* = -\frac{i\mu V_0}{T_0}\left[U_{n-1}^* - \left(V_0 - \frac{T_0}{\mu V_0}\right)\phi_{n-1}^*\right] - \frac{V_0}{T_0}\left[V_n^*\left(F_v - \frac{T_0}{V_0}\frac{d\phi_0}{ds}\right)\right. \\
&\qquad\qquad \left. + \phi_n^* W \sin\phi_0 + T_n^* \frac{d\phi_0}{ds}\right] \\
&\frac{dV_n^*}{ds} - \frac{F_v U_0}{T_0} V_n^* = \frac{i\mu U_0}{T_0}\left(U_{n-1}^* - V_0 \phi_{n-1}^*\right) + \frac{U_0}{T_0}\left[\left(F_v - \frac{T_0}{U_0}\frac{d\phi_0}{ds}\right)U_n^*\right. \\
&\qquad\qquad \left. + \phi_n^* W \sin\phi_0 + T_n^* \frac{d\phi_0}{ds}\right]
\end{aligned}\right\} \quad . \, . (6.1.5)$$

Equations (6.1.4) and (6.1.5) are formulae for the members of the series of equations (6.1.1). For this case, when $\omega << 1$, it is necessary to solve a differential equation for each member of the series.

The boundary conditions are obtained from equation (6.1.1), (5.0.10) and (5.0.11). These boundary conditions for equation (6.1.4) at $s = 0$ are

$$\left.\begin{aligned}
&U_0^*(L_v m_x \cos\phi_0 - D_v m_y \sin\phi_0) + V_0^*(L_v m_x \cos\phi_0 - D_v m_y \sin\phi_0) \\
&+\phi_0^*[T_0(\cos^2\phi_0 - \sin^2\phi_0)(m_y - m_x) - (D m_y \cos\phi_0 + L m_x \sin\phi_0)] \\
&+ T_0^* \sin\phi_0 \cos\phi_0 (m_y - m_x) = 0 \\
&\text{and} \\
&U_0^*(D_v m_y \cos\phi_0 + L_v m_x \sin\phi_0) + V_0^*(D_v m_y \cos\phi_0 + L_v m_x \sin\phi_0) \\
&-\phi_0^*[D m_y \sin\phi_0 - L m_x \cos\phi_0 - 2T_0 \sin\phi_0 \cos\phi_0 (m_y - m_x)] \\
&+ T_0^*(m_y \cos^2\phi_0 + m_x \sin^2\phi_0) = 0
\end{aligned}\right\} \quad \ldots (6.1.6a)$$

At the boundary, $s = \ell$, when the parameters a and b are assumed to be functions of ω

$$\left.\begin{aligned}
&a = a_0 + a_1\omega + a_2\omega^2 + \cdots = \sum_{n=0}^{\infty} \omega^n a_n \\
&b = b_0 + b_1\omega + b_2\omega^2 + \cdots = \sum_{n=0}^{\infty} \omega^n b_n \\
&\text{equation (5.0.11) becomes} \\
&U_0^* - \phi_0^* C_0 \cos\phi_0 - (a_0 \sin\phi_0 - b_0 \cos\phi_0) = 0 \\
&V_0^* + \phi_0^* C_0 \sin\phi_0 - (a_0 \cos\phi_0 + b_0 \sin\phi_0) = 0
\end{aligned}\right\} \quad \ldots\ldots (6.1.6b)$$

These are the boundary conditions for the zero order equations which are given by equation (6.1.4). The boundary conditions for the higher order equations (6.1.5) can be written for $s = 0$ as

$$
\left.
\begin{aligned}
&U_m^*(L_u m_x \cos\phi_0 - D_u m_y \sin\phi_0) + V_m^*(L_v m_x \cos\phi_0 - D_v m_y \sin\phi_0) \\
&+\phi_m^*[T_0(\cos^2\phi_0 - \sin^2\phi_0)(m_y - m_x) - (D m_y \cos\phi_0 + L m_x \sin\phi_0)] \\
&-T_m^* \sin\phi_0 \cos\phi_0 (m_y - m_x) = i m_x m_y (U_{m-1}^* - V_0 \phi_{m-1}^*) \\
&\text{and} \\
&U_m^*(D_u m_y \cos\phi_0 + L_u m_x \sin\phi_0) + V_m^*(D_v m_y \cos\phi_0 + L_v m_x \sin\phi_0) \\
&-\phi_m^*[D m_y \sin\phi_0 - L m_x \cos\phi_0 - 2T_0 \sin\phi_0 \cos\phi_0 (m_y - m_x)] \\
&-T_m^*(m_y \cos^2\phi_0 + m_x \sin^2\phi_0) = -i m_x m_y (V_{m-1}^* + U_0 \phi_{m-1}^*) \\
&\text{for } n = 1, 2, 3, \ldots \\
&\text{and at } s = \ell \\
&U_m^* - \phi_m^* C_0 \cos\phi_0 - (a_m \sin\phi_0 - b_m \cos\phi_0) = 0 \\
&\text{and} \\
&V_m^* + \phi_m^* C_0 \sin\phi_0 - (a_m \cos\phi_0 + b_m \sin\phi_0) = 0 \\
&\text{for } n = 1, 2, 3, \ldots
\end{aligned}
\right\} \quad \ldots\ldots (6.1.7)
$$

Equations (6.1.6) and (6.1.7) are the boundary conditions for the four, linear, ordinary differential equations denoted as equations (6.1.4) and (6.1.5). The coefficients of these equations of motion are known from the steady-state solution. Since there are four, first-order equations the general solution will contain four constants which can be determined from the four boundary conditions, two at $s = 0$ and two at $s = \ell$. When U_m^*, V_m^*, ϕ_m^*, T_m^* have been determined, the functions U , V , ϕ , T can be found from

$$\left.\begin{aligned}
U &= U_0 + \epsilon e^{i\omega t}[U_0^{x} + U_1^{x}\omega + U_2^{x}\omega^2 + \cdots] \\
V &= V_0 + \epsilon e^{i\omega t}[V_0^{x} + V_1^{x}\omega + V_2^{x}\omega^2 + \cdots] \\
\varphi &= \varphi_0 + \epsilon e^{i\omega t}[\varphi_0^{x} + \varphi_1^{x}\omega + \varphi_2^{x}\omega^2 + \cdots] \\
T &= T_0 + \epsilon e^{i\omega t}[T_0^{x} + T_1^{x}\omega + T_2^{x}\omega^2 + \cdots]
\end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (6.1.8)$$

where only the real part of the above equations is retained.

6.2 Asymptotic Expansion for Large Values of ω

In a manner similar to the method used in section 6.1, the functions $\bar{U}$, $\bar{V}$, $\bar{\varphi}$, $\bar{T}$ will be written as an infinite series in powers of $\frac{1}{\omega}$.

$$\left.\begin{aligned}
\bar{U} &= U_0^{x} + \frac{U_1^{x}}{\omega} + \frac{U_2^{x}}{\omega^2} + \cdots = \sum_{n=0}^{\infty} \frac{U_n^{x}}{\omega^n} \\
\bar{V} &= V_0^{x} + \frac{V_1^{x}}{\omega} + \frac{V_2^{x}}{\omega^2} + \cdots = \sum_{n=0}^{\infty} \frac{V_n^{x}}{\omega^n} \\
\bar{\varphi} &= \varphi_0^{x} + \frac{\varphi_1^{x}}{\omega} + \frac{\varphi_2^{x}}{\omega^2} + \cdots = \sum_{n=0}^{\infty} \frac{\varphi_n^{x}}{\omega^n} \\
\bar{T} &= T_0^{x} + \frac{T_1^{x}}{\omega} + \frac{T_2^{x}}{\omega^2} + \cdots = \sum_{n=0}^{\infty} \frac{T_n^{x}}{\omega^n}
\end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (6.2.1)$$

where $\omega \gg 1$. When equation (6.1.2) is divided by ω and equation (6.2.1) is substituted into the resulting equations, the following system of equations is obtained

$$\left.\begin{aligned}
&\sum_{m=1}^{\infty}\frac{1}{W^m}\frac{d\Phi^x_{m-1}}{d\sigma} = -\frac{i\mu}{T_0}(U_0^x - V_0\Phi_0^x) - \frac{1}{T_0}\sum_{m=1}^{\infty}\frac{1}{W^m}\Big[i\mu(U_m^x - V_0\Phi_m^x) + E_v U_{m-1}^x \\
&\qquad\qquad + E_v V_{m-1}^x + \Phi_{m-1}^x W\sin\Phi_0 + \frac{d\Phi_0}{d\sigma}T_{m-1}^x\Big] \\
&\sum_{m=1}^{\infty}\frac{1}{W^m}\frac{dT^x_{m-1}}{d\sigma} = i\mu(V_0^x + U_0\Phi_0^x) + \sum_{m=1}^{\infty}\frac{1}{W^m}\Big[i\mu(V_m^x + U_0\Phi_m^x) \\
&\qquad\qquad - G_v U_{m-1}^x - G_v V_{m-1}^x + \Phi_{m-1}^x W\cos\Phi_0\Big] \\
&\sum_{m=1}^{\infty}\frac{1}{W^m}\frac{dU^x_{m-1}}{d\sigma} = -\frac{i\mu V_0}{T_0}\Big[U_0^x - \Big(V_0 - \frac{T_0}{\mu V_0}\Big)\Phi_0^x\Big] - \frac{V_0}{T_0}\sum_{m=1}^{\infty}\frac{1}{W^m}\Big\{i\mu\Big[U_m^x - \Big(V_0 - \frac{T_0}{\mu V_0}\Big)\Phi_m^x\Big] \\
&\qquad\qquad + E_v U_{m-1}^x + \Big(E_v - \frac{d\Phi_0}{d\sigma}\frac{T_0}{V_0}\Big)V_{m-1}^x + \Phi_{m-1}^x W\sin\Phi_0 + \frac{d\Phi_0}{d\sigma}T_{m-1}^x\Big\} \\
&\sum_{m=1}^{\infty}\frac{1}{W^m}\frac{dV^x_{m-1}}{d\sigma} = \frac{i\mu U_0}{T_0}\Big[U_0^x - V_0\Phi_0^x\Big] + \frac{U_0}{T_0}\sum_{m=1}^{\infty}\frac{1}{W^m}\Big\{i\mu(U_m^x - V_0\Phi_m^x) \\
&\qquad\qquad + \Big(E_v - \frac{d\Phi_0}{d\sigma}\frac{T_0}{U_0}\Big)U_{m-1}^x + E_v V_{m-1}^x + \Phi_{m-1}^x W\sin\Phi_0 + \frac{d\Phi_0}{d\sigma}T_{m-1}^x\Big\}
\end{aligned}\right\}\qquad (6.2.2)$$

The coefficients of like powers of W must vanish, hence for the zero order terms

$$\left.\begin{aligned}
&-\frac{i\mu}{T_0}\left[U_0^x - V_0\Phi_0^x\right] = 0 \\
&i\mu\left[V_0^x + U_0\Phi_0^x\right] = 0 \\
&-\frac{i\mu V_0}{T_0}\left[U_0^x - \left(V_0 - \frac{T_0}{\mu V_0}\right)\Phi_0^x\right] = 0 \\
&\frac{i\mu U_0}{T_0}\left[U_0^x - V_0\Phi_0^x\right] = 0
\end{aligned}\right\}\quad \cdots\cdots\cdots\cdots\quad (6.2.3)$$

By subtracting the third equation from the first equation it is seen that

$$\phi_0^* = U_0^* = V_0^* = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6.2.4)$$

Since these functions are zero the derivative of the functions must also be zero. Hence

$$\frac{d\phi_0^*}{ds} = \frac{dU_0^*}{ds} = \frac{dV_0^*}{ds} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6.2.5)$$

By substituting equations (6.2.4) and (6.2.5) into equation (6.2.2) and combining the separate equations of (6.2.2), the system of equations for $\omega \gg 1$ can be simplified to

$$\left.\begin{aligned}
i\phi_m^* &= \frac{d\phi_0}{ds} V_{m-1}^* + V_0 \frac{d\phi_{m-1}^*}{ds} - \frac{dU_{m-1}^*}{ds} \\
i\mu V_m^* &= \frac{dT_{m-1}^*}{ds} + G_U U_{m-1}^* + G_V V_{m-1}^* - \phi_{m-1}^* W\cos\phi_0 \\
&\quad - U_0\mu \frac{d\phi_0}{ds} V_{m-1}^* - U_0 V_0 \mu \frac{d\phi_{m-1}^*}{ds} + U_0\mu \frac{dU_{m-1}^*}{ds} \\
i\mu U_m^* &= -U_{m-1}^* F_U + \left(\mu V_0 \frac{d\phi_0}{ds} - F_V\right) V_{m-1}^* - \phi_{m-1}^* W\sin\phi_0 \\
&\quad - \frac{d\phi_0}{ds} T_{m-1}^* - \mu V_0 \frac{dU_{m-1}^*}{ds} + \mu \frac{d\phi_{m-1}^*}{ds}\left(V_0^2 - \frac{T_0}{\mu}\right) \\
U_0 \frac{d\phi_{m-1}^*}{ds} &+ \frac{dV_{m-1}^*}{ds} + \frac{d\phi_0}{ds} U_{m-1}^* = 0
\end{aligned}\right\} \quad \ldots\ldots \quad (6.2.6)$$

for $m = 1, 2, 3, \ldots$

Using the conditions that $U_0^* = V_0^* = \phi_0^* = 0$, equation (6.2.4), and the linearized boundary conditions given by equations (5.0.10) and (5.0.11), the boundary conditions for the case when $\omega \gg 1$ can be written at $s = 0$ as

$$\left.\begin{aligned} U_0^{x} - V_0 \phi_0^{x} &= 0 \\ V_0^{x} + U_0 \phi_0^{x} &= 0 \end{aligned}\right\} \qquad (6.2.7)$$

However, it has been shown that $U_0^{x} = V_0^{x} = \phi_0^{x} = 0$ everywhere along the cable for the case when $\omega \gg 1$.

The boundary conditions for the zero order terms at $s = \ell$ are

$$\left.\begin{aligned} U_0^{x} - \phi_0^{x} C_0 \cos\phi_0 - (a_0 \sin\phi_0 - b_0 \cos\phi_0) &= 0 \\ V_0^{x} + \phi_0^{x} C_0 \sin\phi_0 - (a_0 \cos\phi_0 + b_0 \sin\phi_0) &= 0 \end{aligned}\right\} \qquad (6.2.8)$$

The boundary conditions for the higher order terms can be written, at $s = 0$, as

$$\left.\begin{aligned} & U_{m-1}^{x}(L_U m_x \cos\phi_0 - D_U m_y \sin\phi_0) + V_{m-1}^{x}(L_V m_x \cos\phi_0 - D_V m_y \sin\phi_0) \\ & + \phi_{m-1}^{x}\left[T_0(\cos^2\phi_0 - \sin^2\phi_0)(m_y - m_x) - D m_y \cos\phi_0 - L m_x \sin\phi_0\right] \\ & + T_{m-1}^{x} \sin\phi_0 \cos\phi_0 (m_y - m_x) = i m_x m_y \left[U_m^{x} - V_0 \phi_m^{x}\right] \\ & \text{and} \\ & U_{m-1}^{x}(D_U m_y \cos\phi_0 + L_U m_x \sin\phi_0) + V_{m-1}^{x}(D_V m_y \cos\phi_0 + L_V m_x \sin\phi_0) \\ & - \phi_{m-1}^{x}\left[D m_y \sin\phi_0 - L m_x \cos\phi_0 - 2 T_0 \sin\phi_0 \cos\phi_0 (m_y - m_x)\right] \\ & + T_{m-1}^{x}(m_y \cos^2\phi_0 + m_x \sin^2\phi_0) = -i m_x m_y \left[V_m^{x} + U_0 \phi_m^{x}\right] \end{aligned}\right\} \qquad (6.2.9)$$

for $m = 1, 2, 3, \ldots$

At $s = \ell$ the boundary conditions are

$$\left.\begin{aligned} U_m^{x} - \phi_m^{x} C_0 \cos\phi_0 - (a_m \sin\phi_0 - b_m \cos\phi_0) &= 0 \\ V_m^{x} + \phi_m^{x} C_0 \sin\phi_0 - (a_m \cos\phi_0 + b_m \sin\phi_0) &= 0 \end{aligned}\right\} \qquad (6.2.10)$$

for $m = 1, 2, 3, \ldots$

For this case, $\omega >> 1$, it is seen from equation (6.2.8) that a_0 and b_0 are zero since $U_0^x = V_0^x = \Phi_0^x = 0$. However, a_1, b_1, etc. are not necessarily zero as can be seen from equation (6.2.10).

Equation (6.2.6) with the appropriate boundary conditions, equations (6.2.7) through (6.2.10), is sufficient to determine the members of the series. Several general results can be noted from this asymptotic expansion for large values of ω. It was found that

$$U_0^x = V_0^x = \Phi_0^x = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6.2.4)$$

and

$$\frac{dU_0^x}{ds} = \frac{dV_0^x}{ds} = \frac{d\Phi_0^x}{ds} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6.2.5)$$

Using the first equation of (6.2.6) it is found that

$$\Phi_1^x = 0$$

and from the second and third equations of (6.2.6)

$$i\mu U_1^x = -\frac{d\Phi_0}{ds} T_0^x$$

and

$$i\mu V_1^x = \frac{dT_0^x}{ds}$$

Substitution of these last three equations into the last equation of (6.2.6) gives

$$\frac{d^2 T_0^x}{ds^2} - \left(\frac{d\Phi_0}{ds}\right)^2 T_0^x = 0$$

When this differential equation for T_0^x is solved, the values of U_1^x and V_1^x can be easily computed. Similar equations for the higher order terms can be obtained from equation (6.2.6). Hence, for $\omega >> 1$, the functions U, V, Φ, T can be written as

$$\left.\begin{aligned} U &= U_0 + \epsilon e^{i\omega t}\left[\frac{U_1^*}{\omega} + \frac{U_2^*}{\omega^2} + \cdots\right] \\ V &= V_0 + \epsilon e^{i\omega t}\left[\frac{V_1^*}{\omega} + \frac{V_2^*}{\omega^2} + \cdots\right] \\ \phi &= \phi_0 + \epsilon e^{i\omega t}\left[\frac{\phi_2^*}{\omega^2} + \frac{\phi_3^*}{\omega^3} + \cdots\right] \\ T &= T_0 + \epsilon e^{i\omega t}\left[T_0^* + \frac{T_1^*}{\omega} + \cdots\right] \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (6.2.11)$$

where only the real part of the above equations is retained.

7.0 ILLUSTRATIVE EXAMPLE

The linearized equations of section 5.0 will be used to investigate the oscillatory motion of a simplified cable-body system. The equilibrium configuration will be assumed to be a straight line and the tangential loading function as well as the tension will be assumed to be constant. With these assumptions, the steady-state equations (5.0.4) become

$$\left.\begin{aligned} &F - W\cos\phi_0 = 0 \\ &G - W\sin\phi_0 = 0 \\ &\frac{dU_0}{ds} = \frac{dV_0}{ds} = 0 \\ &\frac{d\phi_0}{ds} = \frac{dT_0}{ds} = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (7.0.1)$$

since

Then equation (5.0.7) can be written, after rearrangement of terms, as

$$\frac{d\bar{\Phi}}{ds} = -\frac{1}{T_0}\left[\bar{U}(i\omega\mu + F_v) - \bar{\Phi}(i\omega\mu V_0 - W\sin\phi_0)\right] \quad \ldots\ldots\ldots (7.0.2)$$

$$0 = i\omega\mu\bar{V} + \bar{\Phi}(i\omega\mu U_0 + W\cos\phi_0) \quad \ldots\ldots\ldots (7.0.3)$$

and

$$\frac{d\bar{U}}{ds} = \frac{V_0}{T_0}\left[-\bar{U}(i\omega\mu + F_v) + \bar{\Phi}\left\{i\omega\left(\mu V_0 - \frac{T_0}{V_0}\right) - W\sin\phi_0\right\}\right] \quad \ldots\ldots \quad (7.0.4)$$

$$\frac{d\bar{V}}{ds} = \frac{U_0}{T_0}\left[\bar{U}(i\omega\mu + F_v) - \bar{\Phi}(i\omega\mu V_0 - W\sin\phi_0)\right] \quad \ldots\ldots\ldots \quad (7.0.5)$$

As can be seen from the second equation of (7.0.1) these assumptions cannot be satisfied by a cable-body system since the tangential force, G , acts in the same direction as $W\sin\phi_0$. However, the assumption of a constant tension greatly simplifies the problem, and since the purpose of this example is to illustrate the method, this assumption will be retained.

When $\bar{\Phi}$, from equation (7.0.4), is substituted into equation (7.0.2) and the resulting equation for $d\bar{\Phi}/ds$ is substituted into equation (7.0.4) after differentiation with respect to s , the following equation is obtained.

$$\frac{d^2\bar{U}}{ds^2} + \frac{d\bar{U}}{ds}\left(\frac{F_v V_0 + W\sin\phi_0}{T_0}\right) + \bar{U}\left(\frac{\omega^2\mu - i\omega F_v}{T_0}\right) = 0 \quad \ldots\ldots\ldots \quad (7.0.6)$$

The general solution for this equation can be written as

$$\bar{U} = A e^{m_1 s} + B e^{m_2 s} \quad \ldots\ldots\ldots\ldots\ldots \quad (7.0.7)$$

where

$$\left.\begin{aligned} m_1 &= -\left[\frac{F_v V_0 + W\sin\phi_0}{2T_0}\right] + \left[\left(\frac{F_v V_0 + W\sin\phi_0}{2T_0}\right)^2 - \left(\frac{\omega^2\mu - i\omega F_v}{T_0}\right)\right]^{\frac{1}{2}} \\ m_2 &= -\left[\frac{F_v V_0 + W\sin\phi_0}{2T_0}\right] - \left[\left(\frac{F_v V_0 + W\sin\phi_0}{2T_0}\right)^2 - \left(\frac{\omega^2\mu - i\omega F_v}{T_0}\right)\right]^{\frac{1}{2}} \end{aligned}\right\} \quad \ldots (7.0.8)$$

and

$F_v = C_R \frac{\rho}{2} c\left[2\left(\frac{t}{c}\right)U_0 + \left(1 - \frac{t}{c}\right)C\right]$ from equation (2.2.3).

The coefficients, A and B, can be determined from the boundary conditions given by equations (5.0.10) and (5.0.11). In order to apply these boundary conditions to this example, they must be written in terms of $\bar{U}$ only. This can be easily done. For example, if $\bar{\Phi}$ from equation (7.0.4) is substituted into the first equation of (5.0.11), the boundary condition at $s = \ell$ is in terms of $\bar{U}$ only. The boundary condition at $s = 0$ can be transformed in a similar manner. Since the determination of the coefficients, A and B, is simply an algebraic computation, it will be omitted here. Instead, the oscillatory motion determined from m_1 and m_2 will be discussed.

Suppose, for example, that F_U and W were both zero. Then

$$m_1 = \frac{i\omega}{\sqrt{T/\mu}} \qquad \text{and} \qquad m_2 = -\frac{i\omega}{\sqrt{T/\mu}}$$

hence

$$\bar{U} = A e^{\frac{i\omega}{\sqrt{T/\mu}} s} + B e^{-\frac{i\omega}{\sqrt{T/\mu}} s}$$

which describes pure oscillatory motion of a weightless cable in a vacuum.

A more interesting case is obtained when only F_U is assumed to be zero. Then

$$m_1 = -\frac{W \sin\phi_0}{2T_0} + \left[\left(\frac{W \sin\phi_0}{2T_0}\right)^2 - \frac{\mu\omega^2}{T_0}\right]^{\frac{1}{2}}$$

$$m_2 = -\frac{W \sin\phi_0}{2T_0} - \left[\left(\frac{W \sin\phi_0}{2T_0}\right)^2 - \frac{\mu\omega^2}{T_0}\right]^{\frac{1}{2}}$$

Since m_1 and m_2 can assume two different values, it is necessary to investigate both cases.

When

$$\left(\frac{W \sin \phi_0}{2T_0}\right)^2 > \frac{\mu \omega^2}{T_0}$$

the square root term in both m_1 and m_2 is positive, hence m_1 and m_2 are always negative and real. Therefore, the motion is critically damped.

When

$$\left(\frac{W \sin \phi_0}{2T_0}\right)^2 < \frac{\mu \omega^2}{T_0}$$

the square root term of m_1 and m_2 becomes imaginary. Then the solution is proportional to

$$\bar{v} \propto e^{-\frac{W \sin \phi_0}{2T_0} s} \cos \sqrt{\frac{\omega^2 \mu}{T_0} - \frac{W \sin \phi_0}{2T_0}} \; s$$

which is a damped, oscillatory motion. This case, when $F_v = 0$, describes the motion of a heavy cable in a vacuum.

When $W \sin \phi_0$ is neglected

$$m_1 = -\frac{F_v V_0}{2T_0} + \left[\left(\frac{F_v V_0}{2T_0}\right)^2 - \left(\frac{\mu \omega^2 - i\omega F_v}{T_0}\right)\right]^{\frac{1}{2}}$$

$$m_2 = -\frac{F_v V_0}{2T_0} - \left[\left(\frac{F_v V_0}{2T_0}\right)^2 - \left(\frac{\mu \omega^2 - i\omega F_v}{T_0}\right)\right]^{\frac{1}{2}}$$

which can be written as

$$m_1 = -\frac{F_v V_0}{2T_0} + \sqrt[4]{b^2+c^2}\cos\frac{\psi}{2} + i\sqrt[4]{b^2+c^2}\sin\frac{\psi}{2}$$

$$m_2 = -\frac{F_v V_0}{2T_0} - \sqrt[4]{b^2+c^2}\cos\frac{\psi}{2} - i\sqrt[4]{b^2+c^2}\sin\frac{\psi}{2}$$

where

$$b = \left(\frac{F_v V_0}{2T_0}\right)^2 - \frac{\omega^2 \mu}{T_0} \qquad , \qquad c = \frac{\omega E}{T_0}$$

and

$$\tan \psi = c/b$$

Hence, the motion for this case is composed of an expotential component and an oscillatory component. Whether the motion is damped or amplified will depend on the sign of the real part of m_1 and m_2 , whereas the oscillatory component of the motion is given by the imaginary part of m_1 and m_2 .

The general case, when neither F_v or $W \sin \phi_0$ are zero, will be investigated for a particular example. Suppose, for example, that the cable-body system has the following characteristics:

¼" circular cable of weight, W , ...	0.10 #/ft
length of cable, s , ...	100 ft
towing velocity, C_0 , ...	22 kts
towline tension, T_0 , ...	600 #
angle of inclination, ϕ_0 , ...	25°
mass density/foot of cable, $\mu = W/g$, ...	0.0031 slugs/ft
density of water, ρ , ...	2.0 slugs/ft^3

When equation (7.0.8) is written as

$$m_1 = -a + \sqrt{b + ic}$$

$$m_2 = -a - \sqrt{b + ic}$$

the roots m_1 and m_2 can be expressed as the sum of a real part and an imaginary part in the following manner

$$b + ic = \sqrt{b^2 + c^2}\left[\frac{b}{\sqrt{b^2 + c^2}} + \frac{ic}{\sqrt{b^2 + c^2}}\right]$$

or

$$b + ic = \sqrt{b^2 + c^2}\; e^{i\psi}$$

where $\tan\psi = c/b$. Then

$$\sqrt{b + ic} = \sqrt[4]{b^2 + c^2}\; e^{i\frac{\psi}{2}}$$

therefore

$$m_1 = -a + \sqrt[4]{b^2+c^2}\,\cos\frac{\psi}{2} + i\sqrt[4]{b^2+c^2}\,\sin\frac{\psi}{2}$$

$$m_2 = -a - \sqrt[4]{b^2+c^2}\,\cos\frac{\psi}{2} - i\sqrt[4]{b^2+c^2}\,\sin\frac{\psi}{2}$$

where

$$a = \frac{F_v V_0 + W\sin\theta_0}{2T_0}$$

$$b = a^2 - \frac{\omega'^2\mu}{T_0}$$

$$c = \frac{\omega F_v}{T_0}$$

Evaluating a , b , c from the data given yields

$$a = 0.0221$$

$$b = 0.000485$$

$$c = 0.00131$$

Then

$$m_1 = 0.0307 - 0.0221 + i(0.0215)$$

$$m_2 = -0.0307 - 0.0221 - i(0.0215)$$

or

$$m_1 = 0.0086 + i(0.0215)$$

$$m_2 = -0.0528 - i(0.0215)$$

The solution for the disturbance velocity, equation (7.0.7), can now be written as

$$\bar{U} = A e^{0.0086 s} e^{i(0.0215)s} + B e^{-0.0528 s} e^{-i(0.0215)s}$$

Then the total U velocity, which was defined as

$$U = U_0 + \epsilon \bar{U} e^{i\omega t}$$

can be written as

$$U = U_0 + \epsilon \left[A e^{0.0086 s} e^{i(\omega t + 0.0215 s)} + B e^{-0.0528 s} e^{i(\omega t - 0.0215 s)} \right]$$

where U_0 is the steady-state normal velocity and ϵ is the wave height/wave length ratio encountered by the towing ship. From inspection of the above equation it is seen that the first term in the bracket describes the propagation of disturbances down the cable whereas the second term describes the propagation of disturbances up the cable. Hence, the disturbances traveling down the cable willbe slightly amplified due to the $e^{0.0086 s}$ term. However, disturbances traveling up the cable will be damped since the exponent is negative.

The velocity of propagation of these disturbances, when $\omega = 1.0$, is

$$k = \frac{\pm \omega}{0.0215} = \pm 46.5 \text{ ft/sec}$$

which is greater than the towing speed of 22 knots (37 ft/sec). Hence, the criterion that the disturbances are always damped when the towing speed is less than the velocity of propagation of the disturbance, as given in Reference 15, is not valid. Also, the velocity of propagation of the disturbances is much less than the value calculated for the corresponding cable in a vacuum, 46.5 ft/sec compared to $\sqrt{T_0/\mu} = 447.0$ ft/sec.

Even though a very simple example was presented, it illustrates the method and shows the type of information that can be obtained from an approximate solution.

SUMMARY

In the development of this paper some of the previous work on towing problems was discussed. In contrast to the sine-squared law for the normal hydrodynamic force acting on the cable which has been used previously, a new form for both the normal and tangential loading functions was proposed. These new loading functions, as well as the sine-squared law, were compared with experimental data. However, the exact form of the loading functions is not important for the mathematical developments in the rest of the paper. Therefore, if the proposed loading functions do not apply to a certain problem, an empirical relation could be substituted into the results of this investigation since the equations are in general form.

The equilibrium configuration is discussed and the cable functions are presented in a form suitable for numerical integration. In order to discuss the oscillatory motion of a cable-body system the general equations of motion are derived and a solution is attained by using the characteristic equations. For this solution it is not necessary to make assumptions about the magnitude of the forces or motions. From consideration of the characteristic equations several conclusions about the propagation of disturbances up and down the cable are made in section 4.2. The requirements of some problems are such that a detailed analysis of the configuration is not necessary, hence the general equations are linearized. The resulting linearized equations cannot, in general, be solved. Therefore, asymptotic equations, valid for small and large values of the perturbation frequency, are derived. Also, an example is presented to illustrate the method of applying the linearized equations.

By applying the results of this investigation the time and space behavior of cable-body systems can be studied. If the specific problem requires a detailed knowledge of the behavior of the system, the method of characteristics can be used. However, if an approximate solution is sufficient, the linearized equations or the asymptotic expansions can be employed.

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REFERENCES

1. McLeod, A.R., "On the Action of Wind on Flexible Cables, with Application to Cables Towed Below Aeroplanes and Balloon Cables", (Great Britain) Reports and Memorandum 554, Oct. 1918.

2. Glauert, H., "Heavy Flexible Cable for Towing a Heavy Body Below an Aeroplane", R. and M. 1592, Feb. 1934.

3. Thews, J.G., and L. Landweber, "On the Resistance of a Heavy Cable for Towing a Surface Float Behind a Ship", Experimental Model Basin Report 418, March 1936.

4. Thews, J.G., and L. Landweber, "The Tension in a Loop of Cable Towed Through a Fluid", EMB Report 422, June 1936.

5. ——— "Tension in a Cable Towing a Heavy Weight Through a Fluid", DTMB Report R-33, March 1941.

6. Landweber, L., and G. Grimminger, "Orientation and Position of a Heavy Body Suspended in a Uniform Current by a Flexible Cable", DTMB Report R-37, Aug. 1941.

7. Landweber, L., and M.H. Protter, "The Shape and Tension of a Light, Flexible Cable in a Uniform Stream", DTMB Report 533, Oct. 1944.

8. Pode, Leonard, "An Analysis of Cable and Housing Requirements for a Deep-Towed Body at High Speed", DTMB Report 661, Nov. 1948.

9. Pode, Leonard, "A Method of Determining Optimum Lengths of Towing Cables", DTMB Report 717, April 1950, (Supplement published in March 1956).

10. Pode, Leonard, "An Experimental Investigation of the Hydrodynamic Forces on Stranded Cable", DTMB Report 713, May 1950.

11. Pode, Leonard, "Tables for Computing the Equilibrium Configuration of a Flexible Cable in a Uniform Stream", DTMB Report 687, March 1951, (Supplement published in Sept. 1955).

12. Relf, E.F., and C.H. Powell, "Tests on Smooth and Stranded Wires Inclined to the Wind Direction, and a Comparison of Results on Stranded Wires in Air and Water", R. and M. 307, Jan. 1917.

13. Glauert, H., "The Stability of a Body Towed by a Light Wire", R. and M. 1312, Feb. 1930.

14. Phillips, W.H., "Stability of a Body Stabilized by Fins and Suspended from an Airplane", NACA Wartime Report L4D18, (new number L-28) April 1944.

15. Phillips, W.H., "Theoretical Analysis of Oscillations of a Towed Cable", NACA TN 1796, 1949.

16. Miles, J.W., "On the Motion of a Finite, Towed Cable", Douglas Aircraft Company Report No. SM-18593, Oct. 1954.

17. Bryant, L.W., Brown, W.S., and N.E. Sweeting, "Collected Researches on Kites and Towed Gliders", R. and M. 2[illegible]3, 1950.

18. O'Hara, F., "Extension of Glider Tow Cable Theory to Elastic Cables Subject to Air Forces of a Generalized Form", R. and M. 2334, 1950.

19. Den Hartog, J.P., Mechanical Vibrations, McGraw-Hill Book Company, Third Edition, 1947.

20. Prandtl, L., "Uber Flussigkeitsbewegung bei sehr kleiner Reibung", Reprint in Vier Abhandlungen zur Hydrodynamik und Aerodynamik, Ann Arbor, 1943

21. Schlicting, Herman, Boundary Layer Theory, McGraw-Hill Book Company, 1955.

22. Hoerner, S.F., Aerodynamic Drag, The Otterboin Press, 1951.

23. Squire, H.B., and A.D. Young, "The Calculation of the Profile Drag of Aerofoils", R. and M. 1838, 1938.

24. Gertler, M., "The Prediction of the Effective Horsepower of Ships by Methods in Use at the David Taylor Model Basin", DTMB Report 576, Dec. 1947.

25. Young, A.D., and T.B. Booth, "The Profile Drag of Yawed Wings of Infinite Span", The Aeronautical Quarterly, vol. III, Nov. 1951.

26. Sears, W.R., "Boundary Layers in Three-Dimensional Flow" Applied Mechanics Review, vol. 7, 1954.

27. Moore, F.K., "Three-Dimensional Boundary Layer Theory", Advances in Applied Mechanics, vol. IV, 1956.

28. Kemp, G., "Uber den Reibungswiderstand von Flachen Verschiedener Form", Proceedings of the First International Congress for Applied Mechanics, 1924.

29. Powell, C.H., "The Resistance of Inclined Struts", R. and M. 599, March 1919.

30. Lamb, H., Hydrodynamics, Dover Publication, Sixth Edition, 1945.

31. Courant, R., and Peter Lax, "On Nonlinear Partial Differential Equations with Two Independent Variables", Communications on Applied Mathematics, vol. II, No. 2-3, June-Sept. 1949.

32. Courant, R., Isaacson, E., and Mina Rees, "On the Solution of Nonlinear Hyperbolic Differential Equations by Finite Differences", Communications on Applied Mathematics, vol. V, No. 3, Aug. 1952.

33. Courant, R., and K.O. Friedrichs, <u>Supersonic Flow and Shock Waves</u>, Interscience Publishers, 1948.

34. Tamarkin, J.D., and W. Feller, <u>Partial Differential Equations</u>, Brown University, 1941.

35. Erdelyi, A., <u>Asymptotic Expansions</u>, Dover Publications, 1956.