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A THEORY OF THE SHOCKWAVE PRODUCED BY A POINT EXPLOSION

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1 DECEMBER 1957



U. S. NAVAL ORDNANCE LABORATORY

WHITE OAK, MARYLAND

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NAVORD Report 4182

A THEORY OF THE SHOCKWAVE
PRODUCED BY A POINT EXPLOSION

by

Hans G. Snay

ABSTRACT: Approximate integrals of the spherical blast equations with variable isentropic exponents are derived. The distributions of velocity, density and pressure within the sphere of disturbance are expressed in polynomials. These are used to calculate the "reduced energy" which is closely related to the relationship between shockwave peak pressure and distance. For very large distances, the well known asymptotic behavior of weak shockwaves is obtained.

EXPLOSIONS RESEARCH DEPARTMENT
U. S. NAVAL ORDNANCE LABORATORY
WHITE OAK, MARYLAND

1 December 1957

This report gives a detailed solution of the problem of the shockwave produced by a point explosion in water. This new approach can be applied to other media and is generally valid over a very wide range. It is the belief of the author that the methods described represent a considerable advance in the theoretical treatment of blast phenomena. The work was done under Task NO 701-267/76001/01040, and contributes to the solution of Key Problem No. 20, Chapter II, "Key Problems in Explosives Research and Development, Part II", NAVORD Report 4299 (SECRET Restricted Data).

The author wishes to thank Mr. J. F. Butler, formerly in the Underwater Explosions Division, for his help in the preparation of this report.

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A THEORY OF THE SHOCKWAVE PRODUCED BY A POINT EXPLOSION

CHAPTER I INTRODUCTION

The theoretical treatment of shockwaves caused by conventional explosions is complicated by the presence of two media: the reaction products of the explosive and the ambient medium into which the shock is transmitted. Assumption of an infinitely small "explosive charge" which delivers a finite energy removes the difficulty of the two-medium problem and permits study of the shockwave propagation through a homogeneous medium. Such an analysis is applicable to atomic explosions or to electric discharges of high energy in small spark gaps.

Several attempts have been made to attack this problem [a] - [e]*; the best known is Sir Geoffrey Taylor's now classical treatise "The Formation of a Blast Wave by a Very Intense Explosion" [a]. Recently Lockwood Taylor [b] has shown that the solution of Taylor's differential equations can be given in closed form. These theories apply to very high shock pressures only, i.e., to the initial phase of the blast.

The present paper describes an approach to the general problem which also holds for lower pressures. Particular consideration was given to underwater explosions. Since the thermodynamic data for water cannot be expressed in simple equations, once a large pressure range is considered, the calculations were set up for the use of tabulated thermodynamic data. The methods of calculation which were developed can be handled with ordinary desk computing machines.

*Such letters refer to the list of references at the end of this report.

CHAPTER II THE TRANSFORMATION OF THE HYDRODYNAMIC DIFFERENTIAL EQUATIONS

2.1 The Hydrodynamic Equations. Even for strong energy discharges, radiation effects are unimportant under water, because the mean free paths of photons and electrons are small in media of high density. Also, it is well known that viscosity and heat conduction do not affect the fluid motion we are considering here except at shock fronts. There, these effects are implicitly accounted for by the Rankine-Hugoniot shock relations. Thus, for regimes behind shock fronts the energy is transmitted by pressure forces only and each particle has constant entropy as long as it is not traversed by a further shock. This situation is described by the well-known hydrodynamic equations. For the case of spherical symmetry, these are:

$$(2.1) \quad u_t + u u_r + \frac{1}{\rho} p_r = 0,$$

$$(2.2) \quad \rho_t + u \rho_r + \rho u_r + \frac{2\rho u}{r} = 0,$$

$$(2.3) \quad S_t + u S_r = 0.$$

Introducing the isentropic exponent

$$(2.4) \quad \gamma = \gamma(p, \rho) = \left(\frac{\partial \ln p}{\partial \ln \rho} \right)_s,$$

(2.2) and (2.3) can be combined as follows:

$$(2.2a) \quad \frac{1}{\rho} (p_t + u p_r) = \frac{1}{\rho} (\rho_t + u \rho_r) = \gamma (u_r + \frac{2u}{r}).$$

The symbols are defined in the table at the end of this paper. It should be noted that p designates the excess pressure above the static pressure P_0 . This convention affects γ which is here defined somewhat differently from the common isentropic exponent, γ^* . The latter is related to γ as follows:

$$(2.5) \quad \gamma^* = \left(\frac{\partial \ln(p + P_0)}{\partial \ln p} \right)_s = \frac{p}{p + P_0} \gamma$$

At high pressures, there is no difference between these two magnitudes but for $p \rightarrow 0$, $\gamma \rightarrow \infty$, whereas γ^* remains finite.

2.2 The Rankine-Hugoniot Conditions. For a shock advancing into an undisturbed medium, the following conditions must be observed:

$$(2.6) \quad u_1^2 = p_1 \frac{p_1 - p_0}{\rho_1 \rho_0},$$

$$(2.7) \quad U = u_1 \frac{\rho_1 - \rho_0}{\rho_1 - \rho_0},$$

$$(2.8) \quad E_1 - E_0 = \frac{p_1 + p_0}{2} \frac{\rho_1 - \rho_0}{\rho_1 \rho_0}.$$

(The subscript 1 designates the state just behind the shock front; zero refers to the undisturbed state).

2.3 Thermodynamic Data. Equation (2.8) must be evaluated using the thermodynamic data of the fluid considered. The evaluation is simplified by the introduction of the "reduced internal energy"

$$(2.9) \quad J = J(p, \rho) = \frac{E - E_0}{p} \rho$$

which will also be used in other connections in this paper.

J is a dimensionless magnitude related to the heat capacity. (For an ideal gas at high temperature and pressure, $J = c_v/R$, where c_v is the heat capacity at constant volume and R is the gas constant). The relation between J and γ is:

$$(2.10) \quad \gamma = \frac{1}{J} \frac{p + P_0}{p} + 1 - \left(\frac{\partial \ln J}{\partial \ln p} \right)_s.$$

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This relation holds generally for any type of fluid. For constant heat capacity, the last term vanishes and one obtains a relation which is widely used in the literature.

Once $J(p, \rho)$ is known, one can readily determine the shock pressure p , as a function of the density ρ , and the magnitude

$$(2.4a) \quad \gamma_{RH} = \gamma_{RH}(\rho_1) = \frac{d \ln p_1}{d \ln \rho_1}$$

This is analogous to the isentropic exponent (2.4) except that the differential quotient is taken along the Rankine-Hugoniot adiabetic instead of the isentropic. According to the Rankine-Hugoniot conditions, $1/\gamma_{RH}$ vanishes for infinitely high pressures whereas $1/\gamma$ remains finite.

2.4 Change from Free Boundary Conditions to Fixed Boundary Conditions

Equations (2.1) to (2.3) constitute a free boundary problem since the boundary conditions must be fulfilled along an unknown line, - namely, the radius-time curve of the sphere bounded by the shock front. The hydrodynamic equations will now be transformed in such a way that three partial differential equations are obtained whose boundary conditions must be fulfilled at a fixed and known place. This is possible by adding one more equation, namely a relationship for the position of the shock front.

The transformation is readily performed by the introduction of the following reduced variables:

$$(2.11) \quad \text{Reduced velocity: } \varphi = \frac{u}{u_1}$$

$$(2.12) \quad \text{Reduced density: } \chi = \frac{\rho}{\rho_1}$$

$$(2.13) \quad \text{Reduced pressure: } \psi = \frac{p}{p_1}$$

$$(2.14) \quad \text{Reduced distance: } \xi = \frac{r}{r_1}$$

All magnitudes with the subscript 1 refer to the shock front and are functions of time only; in particular, we note that $dr_1/dt = U$.

With the use of these variables and the Rankine-Hugoniot relations (2.6) and (2.7), the hydrodynamic equations take the form:

$$(2.15) \quad \varphi_\xi = \frac{\varphi}{\xi} \frac{\gamma_1 \frac{1}{\rho_1} \frac{p_0}{A-p_0} + \frac{r_1}{\rho_1 u_1} (u_1 \varphi)_t}{\frac{p_1}{A-p_0} - \varphi/\xi},$$

$$(2.16) \quad \chi_\xi = \frac{\chi}{\xi} \frac{\varphi_\xi + \frac{2\varphi}{\xi} + \frac{r_1}{u_1 \rho_1 \chi} (p_1 \chi)_t}{\frac{p_1}{A-p_0} - \varphi/\xi},$$

$$(2.17) \quad \psi_\xi = \frac{\psi}{\xi} \frac{\gamma_1 (\varphi_\xi + \frac{2\varphi}{\xi}) + \frac{r_1}{u_1 \rho_1 \psi} (p_1 \psi)_t}{\frac{p_1}{A-p_0} - \varphi/\xi}.$$

We introduce the abbreviations:

$$(2.18) \quad H_1 = \frac{r_1}{u_1} \frac{d \ln p_1}{dt},$$

$$(2.19) \quad L_1 = \frac{r_1}{u_1} \frac{d \ln u_1}{dt} = \frac{H_1}{2} \left(1 + \frac{1}{\gamma_{RH}} \frac{p_0}{p_1 - p_0} \right),$$

$$(2.20) \quad G_1 = \frac{r_1}{u_1} \frac{d \ln \rho_1}{dt} = H_1 / \gamma_{RH}.$$

Since γ_{RH} is known as a function of ρ , only H_1 remains to be determined. (This will be done in the next chapter).

Using G , we can eliminate the time t from the partial differential equations and obtain equations with ξ and ρ , as independent variables, where ρ takes the place of t :

$$(2.21) \quad \varphi_{\xi} = \frac{\varphi}{\xi} \frac{\frac{\varphi_{\xi}}{\varphi} \frac{\rho_0}{\rho_1 - \rho_0} + L_1 + G_1 \rho_1 (\ln \varphi)_{\rho_1}}{\frac{\rho_1}{\rho_1 - \rho_0} - \frac{\varphi}{\xi}},$$

$$(2.22) \quad \chi_{\xi} = \frac{\chi}{\xi} \frac{\varphi_{\xi} + \frac{\varphi_{\xi}}{\xi} + G_1 (1 + \rho_1 (\ln \chi)_{\rho_1})}{\frac{\rho_1}{\rho_1 - \rho_0} - \frac{\varphi}{\xi}},$$

$$(2.23) \quad \psi_{\xi} = \frac{\psi}{\xi} \frac{\delta(\varphi_{\xi} + \frac{\varphi_{\xi}}{\xi}) + H_1 + G_1 \rho_1 (\ln \psi)_{\rho_1}}{\frac{\rho_1}{\rho_1 - \rho_0} - \frac{\varphi}{\xi}}.$$

2.5 Boundary Conditions. The transformation of the partial differential equations has simplified the boundary conditions. These are:

$$(2.24) \quad \left. \begin{array}{l} \varphi = 1 \\ \chi = 1 \\ \psi = 1 \end{array} \right\} \text{ for } \xi = 1.$$

At the center of the sphere, $\xi = 0$, the velocity must vanish. Hence,

$$(2.25) \quad \varphi = 0 \quad \text{for } \xi = 0.$$

Using (2.1), it can be deduced that

$$(2.26) \quad \psi_{\xi} = 0 \quad \text{for } \xi = 0,$$

if χ and $\frac{\partial \varphi}{\partial \xi}$ are finite or zero at $\xi = 0$.

The partial differential equations (2.21) through (2.23) have some unique properties when combined with the boundary

conditions (2.24). The derivatives with respect to ρ vanish at $\xi = 1$. This means that the derivatives of φ , χ and ψ with respect to ξ at $\xi = 1$ are functions of ρ and H_1 only, provided the thermodynamic properties of the medium, expressed by J and J_{RH} , are given. This also holds for the higher derivatives. The consequences of this behavior will be discussed in the next chapter after the relation for H_1 has been found.

CHAPTER III GROSS PROPERTIES OF THE SPHERE OF DISTURBANCE

The shockwave which is emitted from the point explosion is spherically symmetric. We call the region which is enclosed by the shock front the "sphere of disturbance".

3.1 Average Density Relation. If we neglect the presence of the device which discharges the energy, the medium inside the sphere of disturbance is the same as that outside the sphere. Since the ambient medium remains undisturbed until the shock front arrives, the average density within the sphere must be that of the undisturbed medium, or

$$(3.1) \quad 3\rho_1 \int_0^1 \chi \xi^2 d\xi = \rho_0$$

This relation is consistent with the hydrodynamic equation for the conservation of mass. It makes a simple statement about the behavior of χ . Similar relations for φ and ψ do not exist.

3.2 The Energy Integral. In order to derive an expression for H_1 , we consider the total energy of the sphere bounded by the shock front. As this sphere expands, its energy in excess of that of the undisturbed medium must remain constant and equal to the total energy Q of the initial discharge.

The total energy increment of the sphere is

$$(3.2) \quad Q = 4\pi \int_0^r \left(\frac{\rho u^2}{2} + \rho E \right) r^2 dr - 4\pi \int_0^r \rho_0 E_0 r^2 dr,$$

where the first term is the sum of the kinetic and internal energies inside the sphere and the last term is the total energy of the undisturbed fluid contained in a sphere of equal size.

Introducing the dimensionless variables φ , χ , ψ and the magnitude J , (2.9), we obtain

$$(3.3) \quad Q = \frac{4\pi}{3} r_1^3 \frac{\rho_1 u_1^2}{2} \left[3 \int_0^1 (\chi \varphi^2 + \frac{2J}{\rho_1 u_1^2} \rho_1 \psi) \xi^2 d\xi - \frac{2E_0}{\rho_1 u_1^2} (\rho_0 - 3\rho_1 \int_0^1 \chi \xi^2 d\xi) \right].$$

The last term vanishes due to (3.1). The first integral is abbreviated, using (2.6), by

$$(3.4) \quad \eta_1 = 3 \int_0^1 (\chi \varphi^2 - \frac{2J\rho_0}{\rho_1 - \rho_0} \psi) \xi^2 d\xi, \quad J = J(\rho_1, \chi, \psi).$$

This magnitude represents the reduced energy of the sphere of disturbance. The total energy is

$$(3.5) \quad Q = \frac{4\pi}{3} r_1^3 \rho_1 \frac{\rho_1 - \rho_0}{2\rho_0} \eta_1(\rho_1).$$

Equation (3.5) provides a convenient relationship between the shock pressure and the distance r_1 for any given energy yield Q . Hence, the primary task is to find η_1 as a function of ρ_1 .

Setting $\frac{dQ}{dt} = 0$, we obtain after rearrangement

$$(3.6) \quad \frac{\eta_1}{u_1} \frac{d \ln p_1}{dt} = H_1 = \frac{-3 \frac{p_1}{p_1 - p_0}}{1 + \frac{1}{\beta_{RH}} \left[\frac{p_1}{p_1 - p_0} + \frac{d \ln \eta_1}{d \ln p_1} \right]}$$

$$= \frac{-3 \frac{p_1}{p_1 - p_0}}{1 + \frac{1}{\beta_{RH}} \frac{p_1}{p_1 - p_0} [1 + \beta]}$$

where

$$\beta = \frac{d \ln \eta_1}{d \ln \frac{p_1}{p_1 - p_0}}$$

(3.6) together with (3.4) gives the desired relationship between H_1 and p_1 . The solution of our problem must satisfy the three partial differential equations (2.21) through (2.23) as well as the integro-differential equations (3.6) and (3.4).

It is now possible to see the significance of the transformation (2.15) to (2.17). We assume that η and β_{RH} are known. As shown in the previous chapter, the first and higher derivatives of φ , χ and ψ with respect to ξ , at $\xi = 1$, functions of ρ , and β only. If φ , χ and ψ were analytic functions, they could be expanded around $\xi = 1$, in series. Introducing these into the integral for η , (3.4), and using (3.6), we obtain an ordinary, first order differential equation for η . Thus, our transformation and the expansion around $\xi = 1$ reduces our problem to one ordinary differential equation. Once $\eta(\rho)$, or $\beta(\rho)$, is known, the problem is solved.

In this paper we will not discuss whether or not this type of solution is possible. For practical calculations, expansions of this kind are usually very inconvenient if a large number of terms is involved. However, there remains the highly significant point that the behavior of our three reduced variables near $\xi = 1$ gives the greatest contribution

to η , (due to the factor ξ^2 which occurs under the integral) and that this behavior can be calculated without solving the partial differential equations.

Closely related to H , is a magnitude of great practical interest in any shockwave problem, namely, the inclination of the pressure-distance curve in a logarithmic p - r plot:

$$(3.7) \quad \frac{d \ln p}{d \ln r} = \frac{\xi}{\eta} \frac{d \ln p}{d \xi} = \frac{p - p_0}{p} H = \frac{-3}{1 + \frac{p}{p_0} \frac{1}{\eta} (1 + \beta)}$$

This shows that for $1/\eta = 0$, the peak pressure is inversely proportional to the third power of the distance. This holds for very high pressures. For large distances r and low pressures, the wave assumes a nearly acoustic behavior and $d \ln p / d \ln r$ approaches unity, as will be shown in Chapter VII. For a finite η and negative β , the pressure may decrease even more rapidly than with the cube of the distance.

3.3 The Dissipated Energy. At the shock front mechanical energy is constantly dissipated and converted into irreversible energy. According to the second law of thermodynamics, the increase of dissipated energy is

$$(3.8) \quad \begin{aligned} \frac{dQ_{\text{diss}}}{dr} &= 4\pi r^2 \rho_0 \int_{S(0)}^{S(r)} T''(S) dS \\ &= 4\pi r^2 \rho_0 h \end{aligned}$$

$T''(S)$ is the lowest permanent temperature which can occur in the system. For liquids, this temperature corresponds closely to that which the fluid attains after passage of the shock-wave and return to zero excess pressure.

The integral in (3.8) is commonly called the dissipated enthalpy increment h . If the isentropic p - ρ relationship is known, h can be conveniently determined from the difference of the enthalpy increments at the shock front, ΔH , and that

of an isentropic expansion to zero excess pressure:

$$(3.9) \quad h = \Delta H - \int_0^{p_1} \frac{dp}{\rho(p, S=\text{const})}$$

$$\Delta H = \frac{p_1 + p_0}{2} \left(\frac{1}{\rho_0} + \frac{1}{\rho_1} \right)$$

This relation shows that h is a shock parameter like U or M , and, therefore, is a function of ρ_1 only. Behind shock fronts, h is constant along any isentropic line.

The amount of dissipated energy which is within the sphere of disturbance can be found by integrating (3.8) along the shock front. Introducing η_{Diss} defined by

$$(3.10) \quad Q_{Diss} = \frac{4\pi}{3} r_1^3 p_1 \frac{p_1 - p_0}{2\rho_0} \eta_{Diss},$$

the following expression is found:

$$(3.11) \quad \frac{d \ln p_1}{d \ln r_1} = -3 \frac{1 - \frac{2\rho_0^2 h(p_1)}{\eta_{Diss} p_1 (p_1 - p_0)}}{1 + \frac{1}{\rho_1^2} \left[\frac{p_1}{p_1 - p_0} - \frac{d \ln \eta_{Diss}}{d \ln p_1} \right]}$$

This and (3.7) establish a relationship between η and η_{Diss} . Either one of these equations can be used to determine H . It is convenient to use η for high values of ρ_1 , whereas, (3.11) will be used for the case where ρ_1 approaches ρ_0 , i.e. for very low shock wave pressures.

CHAPTER IV THREE IMPLICIT INTEGRALS

In this chapter we will derive implicit integrals of the three partial differential equations (2.21) through (2.23). These integrals contain derivatives with respect to ρ_1 and are therefore not complete solutions of our problem. But they give the initial distribution of our solution and provide an understanding of the behavior of the general solution.

Further, they can be used for iterations to improve approximate solutions.

4.1 Integral Obtained From Energy Considerations. Consider a sphere lying concentrically within the sphere of disturbance. The inner sphere does the

$$\text{work} = 4\pi \int_{t_1}^t (\rho + P_0) r'^2 u \, dt$$

on the surrounding medium while it expands. It contains the

$$\text{energy} = 4\pi \int_0^{r'} \left(\frac{\rho u^2}{2} + E\rho \right) r^2 dr.$$

In these relations, r' denotes the radius of the inner sphere. According to the assumption made in paragraph 2.1, the change of energy of the sphere is equal to the rate of mechanical work done on the surrounding medium:

$$(4.1a) \quad - \frac{d}{dt} \int_0^{r'} \left\{ \frac{\rho u^2}{2} + E\rho \right\} r^2 dr = r'^2 (\rho + P_0) u$$

or, since $dr'/dt = u$

$$(4.1b) \quad \int_0^{r'} \frac{\partial}{\partial t} \left\{ \left[\frac{\rho u^2}{2} + \rho(E-E_0) \right] r^2 \right\} dr + r'^2 u \left\{ \frac{\rho u^2}{2} + \rho(E-E_0) + \rho + P_0 \right\} = 0.$$

In (4.1b) E has been replaced by $E-E_0$ since, by virtue of (2.2)

$$E_0 \left(\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial r^2 \rho u}{\partial r} \right) = 0.$$

The partial derivative in the integral of (4.1b) refers to constant r . If we change to the variable ξ , we must note that

$$\begin{aligned} \left(\frac{\partial}{\partial t} \right)_r &= \left(\frac{\partial}{\partial t} \right)_\xi - \frac{u}{r} \xi \left(\frac{\partial}{\partial \xi} \right)_\xi \\ &= \left(\frac{\partial}{\partial t} \right)_\xi - \frac{u}{r} \frac{\rho}{\rho - \rho_0} \xi \left(\frac{\partial}{\partial \xi} \right)_\xi. \end{aligned}$$

Introduction of the reduced variables yields

$$(4.2) \quad \frac{2}{\mu_1^2 \rho_1 r_1^2} \int_0^{\xi'} \left(\frac{\partial}{\partial t} \left\{ \frac{\mu_1^2 \rho_1 r_1^3}{2} [\chi \psi^2 + \frac{2\rho_0}{\rho_1 - \rho_0} J \psi] \xi^2 \right\} \right)_{\xi} d\xi +$$

$$\xi'^2 \psi \left\{ \chi \psi^2 + \psi \frac{2\rho_0}{\rho_1 - \rho_0} [J+1 + \frac{R_0}{\rho_0}] \right\} - \xi'^3 \frac{\rho_1}{\rho_1 - \rho_0} \left\{ \chi \psi^2 + \psi \frac{2\rho_0 J}{\rho_1 - \rho_0} \right\} = 0$$

Introducing further

$$(4.3) \quad \eta = \eta(\xi', \rho_1) = \int_0^{\xi'} (\chi \psi^2 + \frac{2\rho_0}{\rho_1 - \rho_0} J \psi) \xi^2 d\xi$$

and making use of (2.19), (2.20) and (3.6), we obtain:

$$(4.4) \quad \text{first term of (4.2)} = \frac{1}{\mu_1} \frac{1}{3} \frac{\partial \eta}{\partial t} + \frac{2}{3} \frac{1}{\mu_1} \frac{d}{dt} \left\{ \ln \mu_1^2 \rho_1 r_1^3 \right\}$$

$$= \frac{2}{3} \left[G_1 \frac{\partial \ln \eta}{\partial \ln \rho_1} + 2L_1 + G_1 + 3 \frac{U}{\mu_1} \right]$$

$$= \frac{4}{3} G_1 \left[\frac{\partial \ln \eta}{\partial \ln \rho_1} - \frac{d \ln \eta}{d \ln \rho_1} \right].$$

Rearrangement of (4.2) and (4.4) gives finally

$$(4.5) \quad \frac{\psi}{\chi \psi^2} = \frac{\frac{\rho_1}{\rho_1 - \rho_0} - \psi/\xi}{\frac{2\rho_0 J}{\rho_1 - \rho_0} \left[\left(1 + \frac{1}{J} + \frac{1}{J} \frac{\rho_0}{\rho_1}\right) \xi - \frac{\rho_1}{\rho_1 - \rho_0} \right] - \frac{G_1 \eta}{3 \xi^3} \frac{\partial \ln \eta}{\partial \ln \rho_1}}$$

In this equation the prime is omitted on ξ , since it now represents the independent variable.

4.2 Integral for the Reduced Velocity. By combination of (2.21) and (2.23) we obtain

$$(4.6) \quad \frac{\partial \ln \psi}{\partial \ln \xi} \left\{ \frac{\rho_1}{\rho_1 - \rho_0} - \frac{\psi}{\xi} - \frac{\psi}{\xi^2 \chi} \frac{\frac{\rho_1}{\rho_1 - \rho_0} - \psi/\xi}{\frac{\rho_1}{\rho_1 - \rho_0} - \psi/\xi} \right\}$$

$$= L_1 + G_1 \frac{\partial \ln \psi}{\partial \ln \rho_1} + \frac{\psi}{\xi \chi} \frac{\frac{\rho_1}{\rho_1 - \rho_0} - \psi/\xi}{\frac{\rho_1}{\rho_1 - \rho_0} - \psi/\xi} \left[\frac{2\chi \psi}{\xi} + H_1 + G_1 \frac{\partial \ln \psi}{\partial \ln \rho_1} \right].$$

From (4.5),

$$(4.7) \quad \frac{\psi}{\beta \chi \psi} \frac{\frac{\rho_0}{\rho_1 - \rho_0}}{\frac{\rho_1}{A - \rho_0} - \psi/\xi} = \frac{\psi/\xi}{2(J+1)\psi/\xi - \frac{2J}{\rho_1 - \rho_0} \rho_1 + \Gamma}$$

$$= \frac{\gamma}{g\gamma - q},$$

where we have used the following abbreviations:

$$(4.8) \quad \begin{aligned} \gamma &= \psi/\xi \\ g &= 2(J+1) \\ q &= \frac{2\rho_1}{A - \rho_0} J - \Gamma \\ \Gamma &= \frac{A - \rho_0}{\rho_0} \frac{\gamma}{\psi \xi^2} \frac{G_1}{3} \frac{\partial \ln \gamma_1/\gamma}{\partial \ln \rho_1} \end{aligned}$$

Furthermore, we abbreviate

$$(4.9) \quad \begin{aligned} L &= L_1 + G_1 \frac{\partial \ln \psi}{\partial \ln \rho_1} \\ H &= H_1 + G_1 \frac{\partial \ln \psi}{\partial \ln \rho_1} \\ G &= G_1 + G_1 \frac{\partial \ln \chi}{\partial \ln \rho_1} \end{aligned} \quad \mu = \frac{\rho_1}{\rho_1 - \rho_0}$$

(4.6) then takes the form:

$$(4.10) \quad - \frac{\partial \ln \psi}{\partial \ln \xi} \left[(g + \gamma) \gamma^2 - (\mu g + q) \gamma + \mu q \right]$$

$$= (3\gamma + g) \gamma^2 + (H + gL - g\mu - q) \gamma - (L - \mu) q.$$

Integration gives:

$$(4.11) \quad \ln \xi = - \int \frac{(g + \gamma) \gamma^2 - (\mu g + q) \gamma + \mu q}{(3\gamma + g) \gamma^2 + (H + gL - g\mu - q) \gamma - (L - \mu) q} \frac{d\gamma}{\gamma} + \text{const.}$$

If the coefficients of γ in (4.11) are constants, the integral can be expressed in closed form:

$$\begin{aligned}
 \ln \xi = & \frac{K_0 \delta_1^2 + K_1 \delta_1 + K_2}{\delta_1 (\delta_1 - \delta_2)} \ln \left(\frac{\eta/\xi - \delta_1}{1 - \delta_1} \right) \\
 (4.12) \quad & + \frac{K_0 \delta_2^2 + K_1 \delta_2 + K_2}{\delta_2 (\delta_2 - \delta_1)} \ln \left(\frac{\eta/\xi - \delta_2}{1 - \delta_2} \right) + \frac{K_2}{\delta_1 \delta_2} \ln \frac{\eta}{\xi},
 \end{aligned}$$

where

$$\begin{aligned}
 K_0 &= -(g + \gamma)/(3\gamma + g) \\
 (4.13) \quad K_1 &= (\mu g + g)/(3\gamma + g) \\
 K_2 &= -\mu g/(3\gamma + g)
 \end{aligned}$$

and where δ_1, δ_2 are the roots of the quadratic equation in the denominator of the integrand.

4.3 An Integral Involving the Reduced Density and Pressure.

From the differential equations (2.22) and (2.23) one obtains, using the abbreviation (4.9),

$$(4.14) \quad \frac{\partial \ln(\chi \xi^{-1-\gamma})}{\partial \xi} = -\ln \chi \frac{\partial \chi}{\partial \xi} + \frac{\frac{\partial p}{\partial \xi} + \frac{2p}{\xi} + H - G}{\xi}$$

Integration yields

$$(4.15) \quad \chi^{\gamma-1} = \xi^3 \left(\frac{p_1}{p_0} - \frac{\eta}{\xi} \frac{p_1 - p_0}{p_0} \right) \exp \left\{ - \int_{\xi}^{\delta_1} \ln \chi d\xi + \int_{\xi}^1 \frac{\frac{3p_1}{p_0 - p_0} + H - (2-\gamma)G}{\frac{p_1}{p_0 - p_0} - \eta/\xi} \frac{d\xi}{\xi} \right\}$$

It is interesting to note that both integrals remain finite for all values of ξ from zero to unity.

For $\gamma_{AH} = 0$ and constant γ the expressions (4.5), (4.12) and (4.15) correspond to the Lockwood Taylor solutions of the strong point blast wave [b].

CHAPTER V
THE TAYLOR CASE

5.1 The Initial Distribution of the General Solution. G. I. Taylor has described a special case which applies to the early phase where the shock pressure is very high [a]. Corresponding solutions of the hydrodynamic equations are called "progressive waves" by Courant and Friedrichs[f]. The essential feature of this solution is that the velocity, pressure and density distributions inside the sphere do not change shape as the shockwave advances; only the scale changes. In our coordinates, this means that the φ - ξ , χ - ξ and ψ - ξ -curves remain the same; they do not change with time nor with ρ , which replaces time in our analysis.

A glance at equations (2.21) to (2.23) shows immediately that this can be true only if ρ is constant. Constancy of ρ is also required in Taylor's treatment and is justified for high pressures where, according to the Rankine-Hugoniot conditions, ρ becomes independent of ρ , i.e., $1/\gamma_{RH} = 0$. This reduces the partial differential equations to ordinary differential equations.

Since $1/\gamma_{RH} = 0$ for infinitely high pressure, the Taylor solution provides the initial distribution from which the general solution of the partial differential equations develops. Thus, we assume that we have Taylor distributions from the very beginning and that these remain stationary as long as ρ does not change.

(Actual explosions do not have such distributions from the very beginning. However, it is safe to assume that the distributions of pressure, velocity and density will quickly converge to the Taylor distributions or the subsequent distributions calculated on the basis of an initial Taylor solution.)

5.2 The Taylor Solution. For a true point explosion both the pressure and the temperature are infinitely high initially, whereas the density is finite. Under such circumstances the medium will be completely ionized, i.e., all the atoms of the medium will be completely stripped of their orbital electrons. Even at relatively high densities such a medium behaves like an ideal monatomic gas. Effects not present in the ideal gas theory are those associated with the electrostatic forces between atoms and the radiation pressure. We neglect both of these in this analysis. Then γ is constant throughout the sphere and equal to $5/3$, the value for an ideal monatomic gas. Furthermore, we have the following well known relations which hold for ideal gases with constant heat capacities:

$$\begin{aligned}
 \frac{p_1}{p_0} &= \frac{\gamma+1}{\gamma-1} \\
 \gamma &= \frac{p_1 + p_0}{p_1 - p_0} = 1 + \frac{1}{\gamma} \\
 (5.1) \quad \frac{p_1}{p_1 - p_0} &= \frac{\gamma+1}{2} \\
 J &= \frac{p_1 - p_0}{2p_0}
 \end{aligned}$$

With $\gamma = 5/3$, we have $p_1 = 4p_0$. This is the highest value of p_1 which can occur in our problem. For the Taylor case, we have

$$\begin{aligned}
 1/\gamma_H &= 0 \\
 H_1 &= \frac{-3p_1}{p_1 - p_0} \\
 (5.2) \quad G_1 &= 0 \\
 L_1 &= \frac{H_1}{2} \\
 \Gamma &= 0
 \end{aligned}$$

The integral (4.5) becomes:

$$(5.3) \quad \frac{\gamma}{\chi \psi^2} = \frac{\frac{\rho_1}{\rho_1 - \rho_0} - \gamma/\delta}{\gamma \psi/\delta - \frac{\rho_1}{\rho_1 - \rho_0}}$$

The equation for the reduced velocity is also simplified since the coefficients of γ in (4.11) are constants. With

$$(5.4) \quad g = \gamma \frac{\rho_1 - \rho_0}{\rho_0} = \frac{\rho_1 + \rho_0}{\rho_0}$$

and $g = \rho_1/\rho_0$,

we obtain from (4.13),

$$(5.5) \quad \begin{aligned} K_0 &= -\frac{\rho_1}{\rho_1 + 2\rho_0} \\ K_1 &= \frac{2\rho_1}{\rho_1 + 2\rho_0} \frac{1}{\delta} \frac{\rho_1}{\rho_1 - \rho_0} \\ K_2 &= -\frac{\rho_1}{\rho_1 + 2\rho_0} \frac{1}{\delta} \frac{\rho_1}{\rho_1 - \rho_0} \end{aligned}$$

The roots of the quadratic are

$$(5.6) \quad \begin{aligned} \delta_1 &= \frac{\rho_1}{\rho_1 + \rho_0} = \frac{1}{\delta} \frac{\rho_1}{\rho_1 - \rho_0} \\ \delta_2 &= \frac{5}{2} \frac{\rho_1}{\rho_1 + 2\rho_0} \end{aligned}$$

If we examine (2.23) at $\gamma = 0$, we find, since $\frac{\partial \psi}{\partial \gamma}$ vanishes,

$$(5.7) \quad \left(\frac{\partial \psi}{\partial \gamma} \right)_{\gamma=0} = -\frac{H_1}{\delta \gamma} = \frac{1}{\delta} \frac{\rho_1}{\rho_1 - \rho_0} = \delta_1$$

We will frequently deal with this magnitude and use the abbreviation

$$(5.8) \quad C_0 = \delta_1$$

The equation for the reduced velocity then becomes:

$$\begin{aligned}
 \ln \xi &= n_1 \ln \frac{\psi/\xi - C_0}{1 - C_0} + n_2 \ln \frac{\psi/\xi - d_1}{1 - d_1} - \frac{2}{5} \ln \frac{\psi}{\xi} \\
 n_1 &= \frac{2\rho_0}{3A + \rho_0} = \left(\frac{3}{j-1} + 2\right)^{-1} \\
 n_2 &= \frac{-5\rho_1 \frac{P_1 + P_0}{A + 2\rho_0} - \frac{8}{5} \rho_0 + \frac{16}{5} \rho_1}{3\rho_1 + \rho_0}
 \end{aligned}
 \tag{5.9}$$

Finally, we obtain, from (4.15), the simple expression

$$\chi^{\delta-1} = \psi \xi^3 \left[\frac{\rho_1}{\rho_0} - \frac{\psi}{\xi} \frac{\rho_1 - \rho_0}{\rho_0} \right]
 \tag{5.10}$$

Combination of (5.3) and (5.10) permits the computation of ψ and χ , once ψ/ξ has been found from (5.9). All three functions are equal to unity for $\xi = 1$. ψ and χ vanish at $\xi = 0$, whereas ψ remains finite. It is of interest to study the behavior of these three functions at $\xi = 0$.

From (5.9), it can be deduced that

$$\lim_{\xi \rightarrow 0} [\ln(\frac{\psi}{\xi} - C_0)] = \ln C_1 \xi^{\frac{1}{n_1}},
 \tag{5.11}$$

where

$$C_1 = (1 - C_0) \left[\frac{C_0 - d_1}{1 - d_1} \right]^{\frac{n_2}{n_1}} C_0^{\frac{2}{5n_1}}.
 \tag{5.12}$$

Hence, we see that, for $\xi = 0$ in the Taylor case, the behavior of ψ is

$$\psi = C_0 \xi + C_1 \xi^{\left(\frac{3}{j-1} + 3\right)} + \dots
 \tag{5.13}$$

or, for $j = 5/3$,

$$\psi = 0.8 \xi + 0.229 \xi^{7.5} + \dots
 \tag{5.13a}$$

Combining (5.3) and (5.10), we obtain for the reduced pressure ψ ,

$$(5.14) \quad \lim_{\xi \rightarrow 0} \psi = \psi_0 - \left[\frac{\gamma C_1 / C_0^2}{\frac{P_1}{P_0} - \frac{P_1 - P_0}{P_0} C_0} \right]^{\frac{\gamma-1}{2-\gamma}} \left[\frac{P_1}{P_0} - \frac{P_1 - P_0}{P_0} C_0 \right]^{-\frac{1}{2-\gamma}}$$

This shows that ψ_0 is finite. ($\psi_0 = 0.3060$ for $\gamma = 5/3$).

Putting (5.13) into (2.23), we obtain for small values of ξ :

$$(5.15) \quad \psi = A_0 + A_1 \xi^{\left(\frac{3}{\gamma-1} + 1\right)} + \dots$$

where

$$(5.16) \quad A_1 = A_0 \frac{\gamma C_1}{\frac{P_1}{P_0} - \frac{P_1 - P_0}{P_0} C_0} \frac{\frac{3}{\gamma-1} + 5}{\frac{3}{\gamma-1} + 2}$$

Hence, for $\gamma = 5/3$, we have near $\xi = 0$,

$$(5.17) \quad \psi = 0.3060 + 0.3198 \xi^{5.5} + \dots$$

For the reduced density, we obtain directly from (4.15):

$$(5.18) \quad \lim_{\xi \rightarrow 0} \chi = B_0 \xi^{\frac{3}{\gamma-1}} + \dots$$

with

$$(5.19) \quad B_0 = A_0^{\frac{1}{\gamma-1}} \left[\frac{P_1}{P_0} - \frac{P_1 - P_0}{P_0} C_0 \right]^{\frac{1}{\gamma-1}} = 0.34258 \quad \text{for } \gamma = 5/3.$$

5.3 The Primakov Case. For $\gamma = 7$, the Taylor solution takes on an interesting form, first found by Primakov [g]. In this case, $P_1 = \frac{4}{3} P_0$ and $C_0 = 1$. C_1 , A_0 , A_1 , and B_0 vanish. The solution of (2.21) to (2.23) is

$$(5.20) \quad \begin{aligned} \psi &= \xi \\ \chi &= \xi \\ \psi &= \xi^2 \end{aligned} \quad \text{for } \gamma = 7.$$

Of course, this solution fulfills the average density condition (3.1). The Primakov case is often mentioned in connection with point explosions under water, because an isentropic pressure-density relationship using the exponent 7 gives a good representation of the behavior of water. This, however, holds for moderately high pressures only. This is evident if we consider the highest density, which is $1.333\rho_0$ in the Primakov case but $4\rho_0$ for a completely dissociated and ionized medium.

CHAPTER VI METHOD OF ATTACK FOR THE GENERAL CASE

6.1 The Determination of H_1 . In order to perform calculations using the transformed differential equations (2.21) to (2.23), it is necessary that H_1 be known as a function of ρ_1 .

The proposed method assumes, as an initial step, several arbitrary values for $\frac{d\ln \eta_1}{d\ln \rho_1}$, or β , such as 1, 0, -1, -2. Then H_1 , L , and G , can be computed as functions of ρ_1 , since $\delta_{RH}(\rho_1)$ is known. Next, φ , χ and μ must be determined from the partial differential equations by one of the methods described below. Subsequently η_1 is calculated using (3.4). When η_1 is plotted versus $(\rho_1 - \rho_0)/\rho_0$ in logarithmic scale, the inclination of the derived η_1 curve must coincide with the value of β which was used to calculate this particular curve. Such a curve is readily drawn in the same way as is the graphical solution of differential equations by the method of isoclines. Figure 1 illustrates the procedure. Usually, the first step gives satisfactory accuracy. The procedure can be repeated assuming more closely spaced values of β , until the desired accuracy is obtained.

This method is practical only if η_1 can be determined from computations which are not too cumbersome. This is a formidable task, if exact solutions are desired, especially for

variable γ and J . However, it is possible to devise approximations which simplify the computations considerably without too great loss of accuracy.

6.2 Differential Quotients for $f = 1$. The reduced functions φ , χ and ψ have the value one at $f = 1$. Hence, the derivations with respect to ρ , vanish in (2.21) to (2.23), and we obtain the following simple relations:

$$(6.1) \quad \varphi' = \left(\frac{\partial \varphi}{\partial f} \right) = \frac{2\gamma_1 + H_1 + L_1}{\frac{\rho_0}{\rho_1 - \rho_0} - \gamma_1},$$

$$(6.2) \quad \chi' = \left(\frac{\partial \chi}{\partial f} \right) = \frac{\rho_1 - \rho_0}{\rho_0} (\varphi' + 2 + G_1),$$

$$(6.3) \quad \psi' = \left(\frac{\partial \psi}{\partial f} \right) = \frac{\rho_1 - \rho_0}{\rho_0} [\gamma_1 (\varphi' + 2) + H_1].$$

The second derivatives are:

$$(6.4) \quad \begin{aligned} \varphi'' = \left(\frac{\partial^2 \varphi}{\partial f^2} \right) = & \left\{ 2\gamma_1 (\varphi' - 1) + \frac{\rho_0}{\rho_1 - \rho_0} \varphi' (\varphi' - 1) \right. \\ & + (L_1 - \frac{\varphi' \rho_0}{\rho_1 - \rho_0}) (1 + \chi' - \varphi' - \varphi' \frac{\rho_1 - 2\rho_0}{\rho_0} + \frac{\rho_1 - \rho_0}{\rho_0}) \\ & + \gamma_1 \left[\left(\frac{\partial \ln \gamma}{\partial \ln \rho} \right)_{\rho=\rho_0} \varphi' + \left(\frac{\partial \ln \gamma}{\partial \ln \rho} \right)_{\rho=\rho_0} \chi' \right] (\varphi' + 2) \\ & \left. + G_1 \left(\frac{d\varphi'}{d \ln \rho_0} + \frac{d\chi'}{d \ln \rho_0} \right) + \varphi' + 2 \right\} \frac{1}{\gamma_1 - \frac{\rho_0}{\rho_1 - \rho_0}}, \end{aligned}$$

$$(6.5) \quad \chi'' = \left(\frac{\partial^2 \chi}{\partial f^2} \right) = \frac{\rho_1 - \rho_0}{\rho_0} \left[\varphi'' + 2(\varphi' - 1) + \chi' \varphi' + G_1 \frac{d\chi'}{d \ln \rho_0} \right],$$

$$(6.6) \quad \begin{aligned} \psi'' = \left(\frac{\partial^2 \psi}{\partial f^2} \right) = & \frac{\rho_1 - \rho_0}{\rho_0} \left[\gamma_1 (\varphi'' + \varphi' - 4) + (\gamma_1 + 1) \varphi' \varphi' \right. \\ & \left. + (2\gamma_1 + H_1 - 1) \varphi' - H_1 + G_1 \frac{d\psi'}{d \ln \rho_0} + \left(\frac{\partial \chi}{\partial f} \right) (\varphi' + 2) \right]. \end{aligned}$$

These and the higher order derivatives are completely determined as functions of ρ , once H , or β and the thermodynamic data of the medium are given. The case $\gamma = \beta/(\beta - 1)$ must be excluded. This situation is approached for $\rho \rightarrow \rho_0$ and is discussed later in Chapter VII.

6.3 The Behavior Near the Center of the Disturbance. The Taylor solution shows that the density is zero at the center of the explosion but the pressure is finite. If we consider (2.9a) for the center ($\mu = 0$), we have

$$(6.7) \quad \frac{dp}{d\rho} = \gamma \frac{p}{\rho}$$

For a constant $\gamma = \gamma_0$, this becomes the familiar isentropic

$$(6.8) \quad \begin{aligned} p &= C \rho^{\gamma_0} \\ T &= C' \rho^{\frac{\gamma_0 - 1}{\gamma_0}} \end{aligned}$$

The isentropic constants, C and C' , are infinitely large at the center. This is a consequence of the assumption of a point explosion, where at the beginning the pressure and temperature are infinitely high; it holds not only for the Taylor solution but also for the subsequent stages. Therefore, the density remains zero and the temperature infinitely high at the center as long as the basic assumptions of our calculations, in particular the absence of radiative energy transport, are valid. These are clearly not fulfilled at the center since high temperature and low density necessarily mean strong radiation. For underwater explosions, this inconsistency with the physical picture is restricted to the almost empty spaces adjoining the center. At some distance from the center, radiation effects will become small because of the increased density. Actually, the hot center will cool down very rapidly due to radiation, equalizing the temperature in the core. This temperature may be too low for appreciable radiation to the surrounding medium, but still high enough to cause dissociation. This may indicate

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that our basic equations are acceptable approximations and that $\delta_0 = 5/3$. This is the same value as for the initial distribution; thus, near the center χ is constant with ρ , or with time.

From (4.15) it can be concluded that

$$(6.9) \quad \lim_{\xi \rightarrow 0} \chi = B(\rho_0) \xi^m$$

with

$$(6.10) \quad m = \frac{3}{\delta_0 - 1} = 4.5.$$

Inserting (6.9) into the partial differential equations (2.21) to (2.23) one finds that the same expansions as in the Taylor case, (5.13) and (5.15), satisfy these equations near $\xi = 0$, namely

$$(6.11) \quad \psi = C_0 \xi + C_1 \xi^{m+3} + C_2 \xi^{2m+5} + \dots$$

$$(6.12) \quad \chi = B_0 \xi^m + B_1 \xi^{2m+2} + \dots$$

$$(6.13) \quad \psi = A_0 + A_1 \xi^{m+2} + A_2 \xi^{2m+4} + \dots$$

This expansion breaks down if A_0 becomes zero as illustrated by the Primakov case. Therefore, it is of interest to examine the conditions for which A_0 may vanish. We set

$$(6.14) \quad \lim_{\xi \rightarrow 0} \psi = A \xi^n, \quad n > 0.$$

Then, according to (4.15)

$$(6.15) \quad \lim_{\xi \rightarrow 0} \chi = B^* \xi^{\frac{3+n}{\delta_0-1}}$$

Putting this into (2.21),

$$(6.16) \quad \lim_{\xi \rightarrow 0} \frac{\partial \psi}{\partial \xi} = \frac{\frac{\gamma A}{\xi^2} \frac{\rho_0}{\rho_1 - \rho_0} \xi^{\frac{\gamma-2}{\gamma-1} - 2 - \frac{1}{\gamma-1}} + L_1 \frac{\psi}{\xi} + \frac{G_1}{\xi} \frac{\partial \psi}{\partial \ln \rho_1}}{\frac{\rho_1}{A - \rho_0} - \frac{\psi}{\xi}}$$

shows that for $\gamma < 2$, $A_0 = 0$ can only occur if we permit a singularity in ψ at the center. This may occur when standing waves within the sphere are reflected at the center but can be excluded for any other condition. Thus the expansions (6.11) to (6.13) hold for $\gamma < 2$.

The coefficients A, B, C in (6.11) to (6.14) are functions of ρ , and are determined by ordinary first order differential equations. The first four of these are:

$$(6.17) \quad 3\gamma_0 C_0 + H_1 + G_1 \frac{d \ln A_0}{d \ln \rho_1} = 0$$

$$(6.18) \quad (m+2) A_1 \left[\frac{\rho_1}{\rho_1 - \rho_0} - C_0 \right] - \gamma_0 (m+5) C_1 A_0 - G_1 A_0 \frac{d A_1}{d \ln \rho_1} = 0$$

$$(6.19) \quad m \left(\frac{\rho_1}{\rho_1 - \rho_0} - C_0 \right) - 3C_0 - G_1 - G_1 \frac{d \ln B_0}{d \ln \rho_1} = 0$$

$$(6.20) \quad \frac{\rho_0}{\rho_1 - \rho_0} \frac{m+2}{C_0 B_0} A_1 + L_1 - \frac{\rho_1}{\rho_1 - \rho_0} - C_0 + G_1 \frac{d \ln C_0}{d \ln \rho_1} = 0$$

These are four equations in five unknowns, namely A_0, A_1, B_0, C_0, C_1 . Adding one more differential equation always brings in one more unknown. Hence, this expansion around $\xi = 0$ remains undetermined, as it was in the Taylor case. An additional expression which relates the expansion around $\xi = 0$ with the complete solutions, is required.

Several such expressions can be set up. The most convenient one, equation (6.25), will be discussed in the next paragraph. Another interesting relationship can be found by combination of (6.17) and (6.19) and by integration from the initial density ρ_0 to ρ_1 :

$$(6.21) \quad \frac{A_0(\rho_1)}{A_0(\rho_0)} = \frac{\rho_1 - \rho_0}{\rho_0 - \rho_1} \frac{\eta_1(\rho_1)}{\eta_1(\rho_0)} \left(\frac{\rho_1 B_0(\rho_1)}{\rho_0 B_0(\rho_0)} \right)^{2/5}$$

This integral requires the knowledge of η , and, therefore, is less convenient to use than equation (6.25).

6.4 Polynomials for the Reduced Velocity, Density and Pressure. Since the behavior of the solutions near $\xi = 1$ and $\xi = 0$ is known, one may construct approximate distributions for φ , χ and ψ by merging these solutions in the intermediate range. One of the simplest ways to do this is to use polynomials in ξ , with exponents in terms of m , which satisfy the partial differential equation near $\xi = 0$. The coefficients are determined in such a way that the polynomials satisfy the boundary conditions and have, up to a certain order, the correct behavior near $\xi = 1$ and $\xi = 0$.

If only first order terms are used, we have four-term polynomials for χ and ψ and a three-term polynomial for φ :

$$(6.22) \quad \varphi = \sum_{i=0}^2 C_i \xi^{1+i(m+2)}$$

$$(6.23) \quad \chi = \sum_{i=0}^3 B_i \xi^{m+i(m+2)}$$

$$(6.24) \quad \psi = \sum_{i=0}^3 A_i \xi^{i(m+2)}$$

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There are eleven coefficients to be determined for which we have the following conditions:

- (a) Three boundary conditions at $\xi = 1$ $\begin{cases} \varphi_1 = 1 \\ \chi_1 = 1 \\ \psi_1 = 1 \end{cases}$
- (b) Three differential quotients at $\xi = 1$ $\begin{cases} \varphi'_1 & (6.1) \\ \chi'_1 & (6.2) \\ \psi'_1 & (6.3) \end{cases}$
- (c) The average density condition (3.1)
- (d) Three ordinary differential equations involving A_0, B_0, C_0, B_1 : (6.17), (6.19), (6.20)
- (e) One expression which relates the expansion around $\xi = 0$ to the entire solution, preferably a condition for A_0 .

The last condition (e) can be found from (2.21). Solving for ψ_ξ yields

$$(6.25) \quad \rho_0 \psi_\xi = \rho_1 \psi_\xi \chi^\xi - (\rho_1 - \rho_0) \psi_\xi \varphi \chi - (\rho_1 - \rho_0) L_1 \varphi \chi - (\rho_1 - \rho_0) \chi G_{\rho_1} \varphi_{\rho_1}$$

The integration from $\xi = 0$ to $\xi = 1$ can be readily performed with the use of the polynomials (6.22) to (6.24).

With

$$(6.26) \quad \beta_j = \sum_{i=0}^3 \frac{B_i}{j+i}$$

we obtain

$$(6.27) \quad \begin{aligned} \psi_0 = A_0 = & 1 - C_0 (\beta_1 - 2\beta_2 + \beta_3) - (14 - \varphi'_1) \frac{\beta_2 - \beta_3}{6.5} + \beta_3 \\ & + \frac{\rho_1}{\rho_0} \left\{ 13C_0 (\beta_2 - \beta_3) - (14 - \varphi'_1) (\beta_2 - 2\beta_3) + 13\beta_3 \right\} \\ & + \frac{\rho_1 - \rho_0}{\rho_1} \left\{ C_0^2 \beta_1 + 8.5 C_0 C_1 \beta_2 + (7.5 C_1^2 + 15 C_0 C_2) \beta_3 \right. \\ & \left. + 21.5 C_1 C_2 \beta_4 + 14 C_2 \beta_5 \right\} + \frac{\rho_1 - \rho_0}{\rho_1} G_1 \left\{ \frac{dC_0}{d\ln \rho_1} (\beta_1 - 2\beta_2 + \beta_3) - \frac{d\varphi'_1}{d\ln \rho_1} \frac{\beta_2 - \beta_3}{6.5} \right\} \end{aligned}$$

(6.27) is an ordinary differential equation for C_0 as a function of ρ . Therefore, we have three simultaneous ordinary differential equations for

$$A_0 - - - - (6.17)$$

$$B_0 - - - - (6.19)$$

$$C_0 - - - - (6.27)$$

and one algebraic equation for A_1 , namely the combination of (6.20) and (6.27). The rest of the coefficients are then determined by

$$(6.28) \quad A_2 = 3 - 3A_0 - 2A_1 - \frac{4'}{6.5}$$

$$(6.29) \quad A_3 = 1 - A_0 - A_1 - A_2$$

$$(6.30) \quad B_1 = -5.6 B_0 + \frac{30.568}{\rho_1} - 7.3728 + 0.1656 X_1'$$

$$(6.31) \quad B_2 = 8.2 B_0 - \frac{61.136}{\rho_1} + 18.4379 - 0.4852 X_1'$$

$$(6.32) \quad B_3 = -3.6 B_0 + \frac{30.568}{\rho_1} - 10.0651 + 0.3195 X_1'$$

$$(6.33) \quad C_1 = \frac{14}{6.5} - 2C_0 - \frac{4'}{6.5}$$

$$(6.34) \quad C_2 = 1 - C_0 - C_1$$

The accuracy of these polynomials was tested for the Taylor case where the exact solution is known. The result was very encouraging; for instance, $\frac{4}{\rho}$ was given accurately by the approximation (6.27) up to the fourth decimal place. This does not necessarily mean that this approximation will be equally good for the general case, but it shows that this treatment gives a first order approximation which can later be improved by iteration.

6.5 Iteration for Improving the Approximate Solutions. The transformation of the hydrodynamic equations (2.21) to (2.23) gives the solution in a form which is relatively insensitive to one of the independent variables, namely ρ . This suggests the use of the implicit integrals (4.8), (4.11) and (4.15) for iterations. Once good approximations for φ , χ and ψ are known, these approximations can be improved by calculating approximate $J(\xi)$ and $\Gamma(\xi)$ for (4.8) and g, q, L and H for (4.10) and (4.11). Solving these for φ , χ and ψ would yield improved approximations.

J. F. Butler, NOL, has derived other integrals which can be used for iterations and which need only χ for the initial approximate solution. These may be useful because the average density conditions together with the derivatives $\chi', \chi'', \dots$ lead to good approximations of χ without much numerical effort.

Solving (2.22) for φ and integrating from $\xi = 1$ to ξ gives

$$(6.35) \quad \varphi = \frac{1}{\chi \xi^2} \left[1 + \frac{P_1 - P_0}{P_1 - P_0} (\chi \xi^2 - 1) + \left(\frac{3P_1}{P_1 - P_0} + G_1 \right) \int_{\xi}^1 \chi \xi^2 d\xi \right. \\ \left. + G_1 \int_{\xi}^1 \frac{\partial \chi}{\partial \ln \rho} \xi^2 d\xi \right].$$

The reduced pressure can be obtained from (2.21)

$$(6.36) \quad \frac{\partial \varphi}{\partial \xi} = \chi \varphi \frac{P_1 - P_0}{P_0} \left[\frac{\xi}{\varphi} \frac{\partial \varphi}{\partial \xi} \left(\frac{P_1}{P_1 - P_0} - \frac{\varphi}{\xi} \right) - L_1 - G_1 \frac{\partial \ln \varphi}{\partial \ln P_1} \right].$$

Numerical integration of this equation yields $\varphi(\xi, P_1)$. Using the results obtained for φ and ψ , (2.23) can be used to improve φ by integrating the following relation:

$$(6.37) \quad \frac{\partial \varphi}{\partial \xi} = - \frac{\varphi}{\xi} \left[2 + \frac{\xi \psi'}{\psi} \right] - \frac{1}{\xi} \left[H_1 + G_1 \frac{\partial \ln \psi}{\partial \ln P_1} - \frac{P_1}{P_1 - P_0} \frac{\xi \psi'}{\psi} \right].$$

From here on (4.5) and (6.36) can be used to improve the functions ψ and χ further.

CHAPTER VII THE LOW PRESSURE RANGE

When ρ/ρ_0 approaches unity it becomes more and more difficult to evaluate the integral η , because J increases rapidly with decreasing ρ , and ξ . For values of ρ/ρ_0 close to unity, it is possible to derive an asymptotic relationship which is useful in solving the problem in the range of low pressures. The first step in doing this is to obtain simple expressions for the thermodynamic data of water which permit a representation of the equation for η , in closed form.

7.1 Thermodynamic Data for Water at Low Pressures. For low amplitudes, the Hugoniot adiabat coincides with the isentropic; therefore,

$$(7.1) \quad \lim_{\rho \rightarrow \rho_0} \gamma_{RH} = \gamma,$$

If the sound velocity changes linearly with pressure,

$$(7.2) \quad c = c_0 (1 + \epsilon p)$$

the following relations can be derived:

$$(7.3) \quad \frac{\rho - \rho_0}{\rho_0} = \frac{\rho}{\rho_0 c_0^2} \frac{1}{1 + \epsilon p}$$

$$(7.4) \quad \gamma \frac{\rho - \rho_0}{\rho} = 1 + \epsilon p$$

$$(7.5) \quad \gamma \frac{\rho - \rho_0}{\rho_0} = 1 + \left(\epsilon + \frac{1}{\rho_0 c_0^2} \right) p$$

Strictly speaking, ϵ refers to the change of the sound velocity with pressure along an isentropic. For low pressures, however, it is permissible to use values of ϵ which are obtained by measuring sound velocities as functions of pressure at constant temperature, namely $\epsilon = 0.108$. This holds for

sea water as well as fresh water at all temperatures of interest. For low pressures (7.3) to (7.5) yield the following first-order approximations:

$$(7.6) \quad \lim_{p_1 \rightarrow p_0} \gamma_{RH} \frac{p_1 - p_0}{p_1} = \lim_{p_1 \rightarrow p_0} \gamma_1 \frac{p_1 - p_0}{p_1} = 1 + \alpha_1 x$$

$$(7.7) \quad \lim_{p_1 \rightarrow p_0} \gamma_1 \frac{p_1 - p_0}{p_0} = 1 + \alpha_2 x$$

where $x = (p_1 - p_0)/p_0$

$$\alpha_1 = \epsilon p_0 c_0^2 = 2.376 \quad \begin{cases} \text{for fresh water at } 20^\circ \text{ C} \\ \text{for sea water at } 8^\circ \text{ C} \end{cases}$$

and $\alpha_2 = 1 + \alpha_1$

Using these approximations, we obtain from (3.9) for the dissipated enthalpy increment

$$(7.8) \quad \lim_{p_1 \rightarrow p_0} p_0 h = \frac{\alpha_2}{6} p_1 x^2$$

7.2 Behavior of the Shockwave Peak Pressure. As the sphere of disturbance increases in size the shockwave peak pressure decreases and the wave finally behaves nearly like an acoustic wave. It is well known that weak shockwaves never attain an exactly acoustic behavior as far as the pressure decay with distance is concerned i.e., the pressure does not exactly decrease inversely proportional to distance. For instance, Kirkwood and Bethe have found the asymptotic behavior of weak shockwaves to be as follows

$$(7.9) \quad p_1 = \frac{r_0}{r_1} \frac{C}{(\ln r_1/r_0)^{1/2}}$$

r_0 = reference length, C = constant

This gives

$$(7.10) \quad \lim_{p_1 \rightarrow 0} \frac{d \ln p_1}{d \ln r_1} = - \left(1 + \frac{1}{2} \frac{1}{\ln r_1 / r_0} \right)$$

Our expression for $\frac{d \ln p_1}{d \ln r_1}$, (3.7) becomes with (7.6),

$$(7.11) \quad \lim_{p_1 \rightarrow 0} \frac{d \ln p_1}{d \ln r_1} = \frac{-3}{2 + \beta - \alpha_1 (1 + \beta)}$$

$$= - \left(1 + \frac{1 - \beta}{3} + \alpha_1 x \frac{(1 + \beta)}{3} \dots \right)$$

where

$$\beta = d \ln \eta_1 / d \ln x$$

comparison of these two equations shows that, for low pressures, β must slowly (logarithmically) approach unity. This means that η_1 tends to zero, when p_1 approaches p_0 .

7.3 Expressions for φ_1' , χ_1' and η_1' at Low Pressures. Using (7.6) and (7.7), we obtain the following limiting equations for H_1/δ_1 and L_1 :

$$(7.12) \quad \lim_{p_1 \rightarrow p_0} \frac{H_1}{\delta_1} = \lim_{p_1 \rightarrow p_0} \left\{ \frac{1}{\delta_1} \frac{p_1}{p_1 - p_0} \frac{-3}{1 + \frac{1}{\delta_{RM}} \frac{p_1}{p_1 - p_0} (1 + \beta)} \right\}$$

$$= -3 \frac{1 - \alpha_1 x}{2 + \beta - \alpha_1 x (1 + \beta)}$$

$$= -\frac{3}{2 + \beta} \left(1 - \frac{\alpha_1 x}{2 + \beta} \dots \right)$$

$$(7.13) \quad \lim_{p_1 \rightarrow p_0} L_1 = H_1 \left(1 - \frac{\alpha_2}{2} x \right)$$

Introduction of these expressions into the equation for φ_1' yields

$$\begin{aligned}
 \lim_{\rho_1 \rightarrow \rho_0} \varphi_1' &= \lim_{\rho_1 \rightarrow \rho_0} \left\{ 2 \frac{1 + \frac{H_1}{\rho_1} \left[\frac{1}{2} + \frac{L_1}{2H_1} \right]}{\frac{1}{\rho_1} \frac{\rho_0}{\rho_1 - \rho_0} - 1} \right\} \\
 (7.14) \quad &= \frac{2}{a_2} \frac{1 - 3(1 - \frac{a_1 x}{2+\beta} - \frac{a_2 x}{2}) / (2+\beta)}{-x} \\
 &= \frac{2}{a_2 x} \frac{1-\beta}{2+\beta} - \frac{6}{a_2} \frac{1}{2+\beta} \left(\frac{a_1}{2+\beta} + \frac{a_2}{4} \right).
 \end{aligned}$$

Correspondingly, we obtain for φ_1' :

$$\begin{aligned}
 \lim_{\rho_1 \rightarrow \rho_0} \varphi_1' &= \lim_{\rho_1 \rightarrow \rho_0} \left\{ \gamma_1 \frac{\rho_1 - \rho_0}{\rho_0} (\varphi_1' + 2) + H_1 \frac{\rho_1 - \rho_0}{\rho_0} \right\} \\
 (7.15) \quad &= (1 + a_2 x) \left[\lim_{\rho_1 \rightarrow \rho_0} \varphi_1' + 2 \right] - \frac{3 \rho_1 / \rho_0}{2 + \beta - a_2 x} \\
 &= \lim_{\rho_1 \rightarrow \rho_0} \varphi_1' + 2 + 2 \frac{1-\beta}{2+\beta} - \frac{3 \rho_1 / \rho_0}{2+\beta} \dots \\
 &= \lim_{\rho_1 \rightarrow \rho_0} \varphi_1' + \frac{3}{2+\beta} + \dots
 \end{aligned}$$

for the derivative of the reduced density, we find

$$(7.16) \quad \lim_{\rho_1 \rightarrow \rho_0} \chi_1' = \lim_{\rho_1 \rightarrow \rho_0} \left\{ \frac{\rho_1 - \rho_0}{\rho_0} (\varphi_1' + 2 + G_1) \right\}$$

7.4 The Shockwave Energy and its Relation to the Total Energy.

When the shockwave has propagated to large distances, the energy distribution within the sphere of disturbance may be visualized as follows: The core of the sphere is still very hot. Although pressure and density are low, the region near the center has a substantial amount of internal energy. In the case of underwater explosions this energy is associated with the pulsations of the bubble. Near the surface of the

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sphere, i.e., near the shock front, one finds the kinetic and potential energy of the shockwave as well as an increment of internal energy which stems from the energy dissipation at the shockfront. Thus, we may distinguish three energy terms: bubble energy, shockwave energy and dissipated energy. In the following, we deal with the shockwave energy.

The potential energy (reversible internal energy) of the shockwave is

$$(7.17) \quad V = 4\pi \int_0^r (E_s \rho - E_0 \rho_0) r^2 dr$$

where

$$(7.18) \quad E_s = - \int_{v_0}^v P(v, s = \text{const}) dv + E_0$$

For low pressures,

$$(7.19) \quad \lim_{P_1 \rightarrow 0} \{E_s - E_0\} \sim P_1 \frac{\Delta v}{2} = \frac{P_1 + P_0}{2} (v_0 - v_1)$$

If we introduce this expression into the integral (7.18) several terms cancel because of the average density relation (3.1).

The terms left are

$$(7.20) \quad \begin{aligned} \lim_{P_1 \rightarrow P_0} V &= 4\pi r_1^3 \int_0^1 \frac{P_1}{2} \frac{P_1 - P_0}{P_0} \xi^2 d\xi \\ &= 4\pi r_1^3 \frac{P_1}{2} \frac{P_1 - P_0}{P_0} \int_0^1 \psi \frac{P_1 - P_0}{P_1 - P_0} \xi^2 d\xi \\ &= 4\pi r_1^3 \frac{P_1}{2} \frac{P_1 - P_0}{P_0} \int_0^1 \psi^2 \xi^2 d\xi \end{aligned}$$

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where (7.3) has been used.

Defining a reduced shockwave energy η_s in an exactly analogous way to η_i , (3.5) and η_{diss} , (3.10), namely:

$$(7.21) \quad \eta_s = \frac{Q_s}{\frac{4\pi}{3} r_1^3 p_1 \frac{p_1 - p_0}{2 p_0}}$$

we obtain

$$(7.22) \quad \lim_{p_1 \rightarrow p_0} \eta_s = 3 \int_0^1 \varphi^2 \chi \xi d\xi + 3 \int_0^1 \varphi^2 \xi^2 d\xi$$

In this expression, functions for φ , χ and ψ must be used which refer to the shockwave. Suitable functions applicable to the low pressure range are

$$(7.23) \quad \begin{aligned} \varphi &= \xi^{n_1} & n_1 &= \varphi'_1 \\ \chi &= \xi^{n_2} & n_2 &= \chi'_1 \\ \psi &= \xi^{n_3} & n_3 &= \psi'_1 \end{aligned}$$

This yields

$$(7.24) \quad \lim_{p_1 \rightarrow p_0} \eta_s = \eta_{s0} = \frac{3}{2\varphi'_1 + \chi'_1 + 3} + \frac{3}{2\psi'_1 + 3}$$

or, for very low pressures, neglecting 3 and χ'_1 in comparison with $2\varphi'_1$ and $2\psi'_1$:

$$(7.25) \quad \eta_{s0} = \frac{3}{\psi'_1} = \frac{3}{2} a_2 \times \frac{2+\beta}{1-\beta}$$

It is interesting to note that this result can be also obtained from

$$(7.26) \quad Q_s = 4\pi r_1^2 \int_{t=t_0}^{\infty} p u dt$$

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by assuming $p(t)/p_0$ and $u(t)/u_1$ are exponential functions, which is a fairly good approximation for any shockwave when used in (7.26).

Equation (7.25) is useful only if an inter-relationship between

and

$$\beta = \frac{d \ln \eta_1}{d \ln p_1}$$

$$\beta_{s0} = \frac{d \ln \eta_{s0}}{d \ln p_1}$$

is found. For this purpose, we study the change of shockwave energy with the distance r_1 . At the shock front, mechanical energy is dissipated due to the irreversible processes associated with the discontinuous rise of pressure, velocity and density, as discussed in Chapter III.

Hence

$$(7.27) \quad \frac{dQ_s}{dr_1} = - \frac{dQ_{s0}}{dr_1} = - 4\pi r_1^2 \rho_0 h$$

Comparison with (3.11), shows immediately that

$$(7.28) \quad \frac{d \ln p_1}{d \ln p_0} = -3 \frac{1 + \frac{2\rho_0 h}{p_1 x} \frac{1}{\eta_s}}{1 + \frac{1}{\gamma_{sh}} \frac{p_1}{p_1 - p_0} (1 + \beta_s)}$$

Combination of (7.26) and (3.7) yields, after rearrangement,

$$(7.29) \quad \eta_s = \frac{2h\rho_0}{p_1 x} \frac{1 + \frac{1}{\gamma_{sh}} \frac{p_1}{p_1 - p_0} (1 + \beta)}{\frac{1}{\gamma_{sh}} \frac{p_1}{p_1 - p_0} (1 + \beta_s)}$$

or for very small values of x

$$(7.30) \quad \eta_{s0} = \frac{2h\rho_0}{p_1 x} \frac{2 + \beta}{\beta_{s0} - \beta} = \frac{Q_2}{3} x \frac{2 + \beta}{\beta_{s0} - \beta}$$

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Combination with (7.25) gives the following differential equation for η_{so} :

$$(7.31) \quad \eta_{so} = \frac{3a_2x}{2} \frac{4/3 + \beta_{so}}{1 - \beta_{so}} .$$

$$\beta_{so} = d\ln \eta_{so} / d\ln X .$$

The solution of (7.31) is

$$(7.32) \quad \eta_{so} = \frac{3a_2x}{2} \left[\frac{7}{3} \ln \frac{1}{x} - \ln c \frac{\eta_{so}}{x} \right] .$$

c is the integration constant.

Introduction of (7.32) into (7.25) yields

$$(7.33) \quad \lim_{\rho_i \rightarrow \rho_0} \beta = 1 - \frac{3}{1 + \chi}$$

with

$$(7.34) \quad \chi e^{\chi} = \left(\frac{c_1}{x} \right)^{1/3} .$$

A rough approximation for β is

$$(7.35) \quad \lim_{\rho_i \rightarrow \rho_0} \beta \sim 1 - \frac{9/7}{\ln \frac{c_1}{x}} .$$

With (7.11), we obtain

$$(7.36) \quad \lim_{\rho_i \rightarrow \rho_0} \frac{d\ln \rho_i}{d\ln r_i} = - \left(1 + \frac{3}{7} \frac{1}{\ln c_1/x} \right)$$

This expression resembles the asymptotic expression of Kirkwood and Bethe (7.10). For the actual calculation, approximation (7.35) is not necessary. It is almost as easy to use (7.33).

7.5 The Intermediate Pressure Range. The relationships derived in Chapter VII are suitable for large values of ρ , i.e., for the high pressure range. Numerical calculations have shown that it is difficult to extend this method to such pressures, where the above discussed "low-pressure" relations become valid. It is possible to bridge this gap by means of the "peak approximation". A suitable method is described in reference [h]. There, two ordinary differential equations are given which relate the shockwave peak pressure, time factor and shape factor with distance. The time factor is given by

$$(7.37) \quad \alpha = - \frac{a_0}{c_0} \frac{p_t}{p} = \frac{a_0}{r_1} \frac{U}{c_0} \left[\frac{3}{1 + \frac{p_t}{p_0} [1 + \beta]} + \psi' \right]$$

and the shape factor by

$$(7.38) \quad \frac{d\psi}{d^2} = \frac{p}{p_t^2} \frac{dp_t}{dt}$$

or, expressed in the magnitudes occurring in this report:

$$(7.39) \quad \begin{aligned} \frac{d\psi}{d^2} = & \frac{a_0}{r_1 a} \left\{ ([\psi' + 2] \gamma + \psi') \left(\frac{u_1}{c_0} - \frac{Q_{108}}{c_0} \right) \right\} \\ & + \left(\frac{a_0}{r_1 a} \right)^2 \left\{ \frac{u_1^2}{c_0^2} \frac{p_t}{p_0 - p_0} [\psi' - \psi'^2 + \psi'' + \gamma(\psi' - \psi'^2 + \psi'')] \right. \\ & \quad \left. + \left(\frac{\partial \chi}{\partial \xi} \right)_{p_1} (\psi' + 2) \right\} \\ & - \frac{u_1^2}{c_0^2} G_1 \left(\frac{\partial \psi'}{\partial \ln r_1} + \gamma \frac{\partial \psi'}{\partial \ln p_1} + \frac{\partial \chi}{\partial \ln p_1} (\psi' + 2) \right) \\ & - \frac{Q_{107}}{c_0^2} ([\psi' + 2] \gamma + \psi') \end{aligned}$$

In these equations a_0 and c_0 are reference values which cancel in the process of the calculation. Q_0 may be chosen as an arbitrary distance, e.g., the shock radius at the end of the high pressure range, and c_0 as the sound velocity of the

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undisturbed water. Q_{107} and Q_{108} are derived and explained in reference [h]. (7.37) and (7.39) as well as the peak pressure calculated at the end of the high pressure range, provide the initial conditions for the peak-approximation method. The shape factor must now be estimated as a function of distance. This is not difficult, since this magnitude is close to unity for moderate and low pressures. A function starting at the calculated values in the high pressure region and approaching unity is usually sufficiently accurate. Integration of the differential equations for peak pressure and time factor given in reference [h] makes it possible to obtain β for such low values of pressure that (7.33) becomes valid. This determines the integration constant in (7.34). β can be conveniently expressed in the terms of the functions P_{11} and P_{12} which occur in the peak-approximation theory:

$$(7.40) \quad \beta = \gamma_{RH} \frac{P_1 - P_0}{P_1} \left[\frac{3P_1}{P_{11} + P_{12} \frac{dr_1}{dr_0}} - 1 \right] - 1.$$

This provides the equations for the calculation of the shockwave parameters, beginning at the extremely high pressures occurring shortly after the explosion down to the low pressures of a nearly acoustic wave.

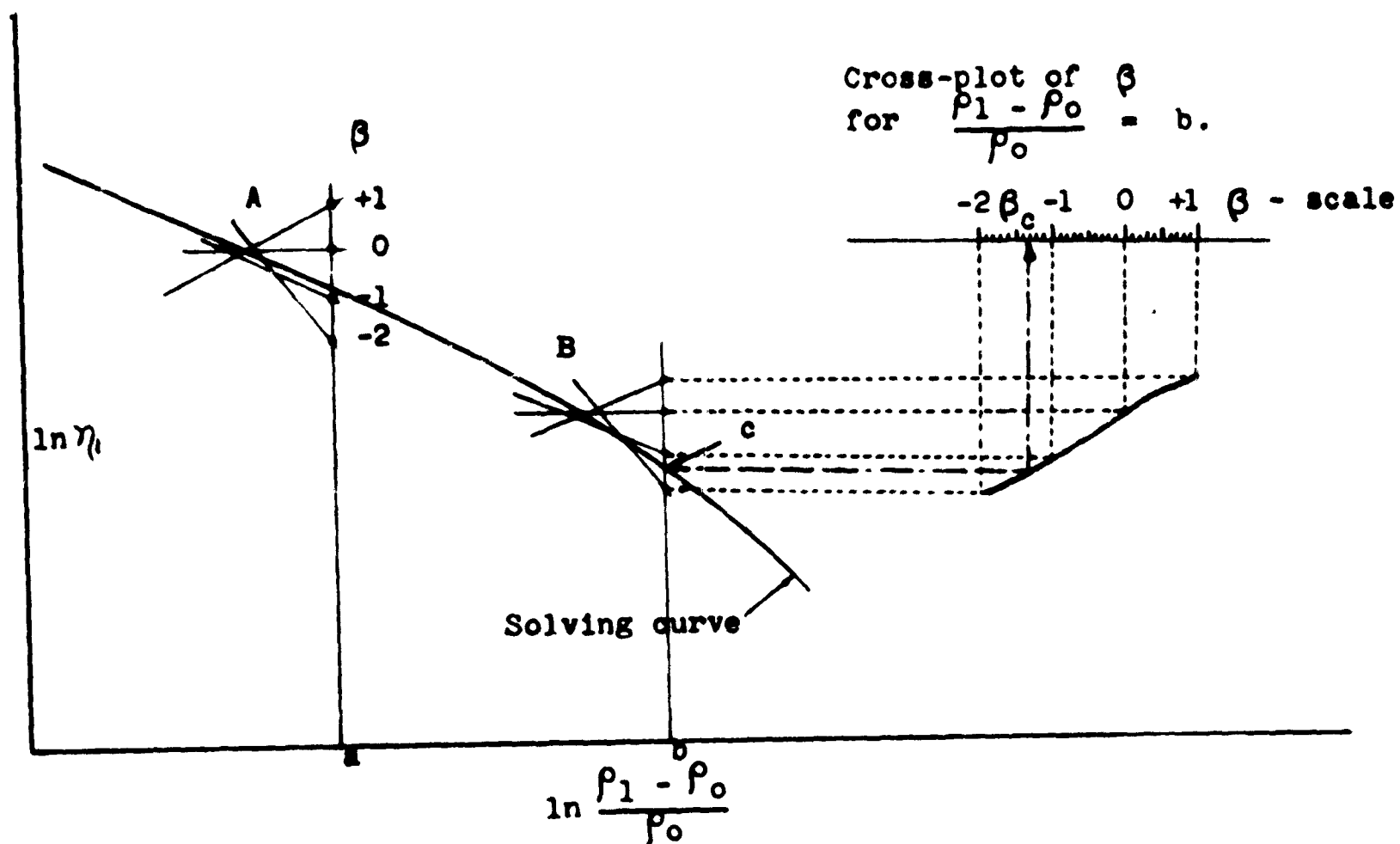


Figure 1. Determination of $\beta = \frac{d \ln \eta_i}{d \ln \frac{p_1 - p_0}{p_0}}$

For each value of p_1 several values of β are assumed, such as $\beta = +1, 0, -1, -2$, etc. For each of these values of β , η_i is calculated using equation 1.3 and $\ln \eta_i$ is plotted versus $\ln \frac{p_1 - p_0}{p_0}$. The correct β is the slope of the solving curve in this plot. The solving curve is obtained by interpolation among the calculated data. Examples of this data for two values of p_1 are shown at a and b in the above sketch. As a first approximation, a curve is drawn from the known initial value of η_i at $p_1 = 4p_0$ through the regions of intersections A and B. The approximate curve is then adjusted at each p_1 , until the slope of the curve is equal to the interpolated value of β . This interpolation is readily accomplished with the aid of a crossplot such as the one shown. For example, the inclination of the solving curve at the point c must be equal to β_c read from the β scale of the crossplot.

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LIST OF SYMBOLS

- a, a_2 Abbreviations defined by (7.6) and (7.7).
- $A_0, A_1, \dots$ Coefficients of the expansion of ψ
- $B_0, B_1, \dots$ Coefficients of the expansion of χ
- c Sound velocity
- c_0 Sound velocity of the undisturbed medium
- $C_0, C_1, \dots$ Coefficients of the expansion of φ
- e 2.71828....
- E Internal energy
- g Abbreviation defined by (4.8)
- G Abbreviation defined by (4.13a)
- G Decay factor of shock front density (2.20)
- h Dissipated enthalpy increment
- ΔH Enthalpy increment
- H Abbreviation defined by (4.9)
- H Decay factor of shockwave peak pressure (2.18)
- J Reduced internal energy (2.5)
- K_0, K_1, K_2 Abbreviations defined by (4.13)
- L Abbreviation defined by (4.9)
- L Decay factor of partial velocity behind shock front (2.19)
- $m = 3/(\gamma_0 - 1)$
- n_1, n_2 Abbreviations defined by (5.9)
- p Excess pressure
- p Shockwave peak pressure
- P_0 Absolute pressure in the undisturbed medium
- q Abbreviation defined by (4.8)
- Q Energy yield
- Q_{ds} Dissipated energy
- Q_B Bubble energy
- Q_s Shockwave energy
- r Radius
- S Entropy

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t	Time
T	Temperature
T^*	Lowest temperature of the system
u	Particle velocity
u_1	Particle velocity right behind shock front
U	Propagation velocity of shock front
v	Specific volume = $1/\rho$
x	$= (p_1 - p_0)/p_0$
y	$= \varphi/\xi$
β	$= d\ln \eta / d\ln \{(p_1 - p_0)/p_0\}$
β_s	$= d\ln \eta_s / d\ln \{(p_1 - p_0)/p_0\}$
γ	Adiabatic exponent as defined by (2.4)
γ^*	True adiabatic exponent; see (2.5)
γ_0	Adiabatic exponent right behind the shock front
γ_c	Adiabatic exponent at the center of the disturbance
γ_{RH}	Rankine-Hugoniot exponent
δ, δ_2	Roots of denominator in (4.11)
ϵ	Coefficient of change of sound velocity with pressure
η	Reduced energy
η_1	Total reduced energy of the sphere of disturbance
η_{diss}	Reduced dissipated energy
η_s	Reduced shockwave energy
μ	$= (p_1 - p_0)/p_1$
ξ	Reduced radius
π	3.14159....
ρ	Density
ρ_1	Density right behind shock front
ρ_{1T}	Density right behind shock front at the initial condition (Taylor Solution)
ρ_0	Density of the undisturbed medium
φ	Reduced particle velocity
χ	Reduced density

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ψ Reduced pressure
 ψ_0 Reduced pressure at the center of the disturbance

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