AD NUMBER: AD0160118
LIMITATION CHANGES
TO: Approved for public release; distribution is unlimited.
FROM: Distribution authorized to U.S. Gov't. agencies and their contractors; Administrative/Operational Use; 15 Nov 1957. Other requests shall be referred to the Office of Naval Research, Washington, DC 20360.
AUTHORITY
ONR LTR, 28 JUL 1977

REPORT HAS BEEN DELIMITED
LEARED FOR PUBLIC RELEASE
DOD DIRECTIVE 5200.20 AND
STRICTIONS ARE IMPOSED UPON
SE AND DISCLOSURE

IBUTION STATEMENT A

VED FOR PUBLIC RELEASE;
BUTION UNLIMITED.

UNCLASSIFIED

AD 160118

Armed Services Technical Information Agency

ARLINGTON HALL STATION
ARLINGTON 12 VIRGINIA

FOR MICRO-CARD CONTROL ONLY

1 OF 3

UNCLASSIFIED

FC

OFFICE OF NAVAL RESEARCH

CONTRACT NONR-1866 (24)

NR-014-903

TECHNICAL MEMORANDUM
NO. 40

COUPLING COEFFICIENTS

BY

JOHN F. HERSH

NOVEMBER 15, 1957

ACOUSTICS RESEARCH LABORATORY
DEPT. OF ENGINEERING SCIENCES AND APPLIED PHYSICS
HARVARD UNIVERSITY - CAMBRIDGE, MASSACHUSETTS

Office of Naval Research

Contract Nonr 1866(24)

Technical Memorandum No. 40

COUPLING COEFFICIENTS

by

John F. Hersh

November 15, 1957

Abstract

A review of the literature on coupling reveals that coupling coefficients have not been given a general definition nor has their physical significance been well explained, particularly for systems with antireciprocal or gyroscopic coupling. A new definition of coupling coefficients is proposed in which the coupled systems are viewed as having a forward transmission path and a feedback path and the coefficients are defined as the values of the feedback return ratio with reference to the coupling element which are positive maximum with variations in load and frequency. This definition in terms of feedback makes possible the use of feedback theory to relate the coefficients to input impedance, critical frequencies, and the conditions for stability and physical realizability; this is illustrated by its application to several electrical and electromechanical systems, including examples of antireciprocal coupling. The theoretical limit of unity for the coefficients established by conditions of stability or physical realizability is verified by measured typical maximum values for several systems. New methods of measurement and values of the coefficients are described for the electrostatic and moving-coil transducers.

* * * * *

Acoustics Research Laboratory

Division of Engineering and Applied Physics

Harvard University, Cambridge, Massachusetts

TABLE OF CONTENTS

INTRODUCTION 1

A REVIEW OF COUPLING AND ITS COEFFICIENTS

The Definitions Summarized 4
The Definitions Criticized 23

A DEFINITION OF COUPLING COEFFICIENTS

The Coupled Systems 25
Coupling as Feedback 29
The Coupling Coefficients 35
The Consequences 39

APPLICATIONS OF COUPLING COEFFICIENTS

Reciprocal Systems

Inductive Coupling 49
Resistive Coupling 58
Electromechanical Coupling
The Electrostatic Transducer 60
The Piezoelectric Transducer 64

Antireciprocal Systems

The Moving-Coil Transducer 68
The Moving-Armature Transducer 78
The Magnetostriction Transducer 83

MEASUREMENTS OF COUPLING COEFFICIENTS

Methods of Measurement 87
The Electrostatic Transducer 89
The Moving-Coil Transducer 91
Other Coupled Systems 98
Summary of Measured Values 99

THE PROBLEM IN RETROSPECT AND PROSPECT

The Benefits of Feedback 101
The Burdens of Antisymmetry 105

LIST OF REFERENCES 110

LIST OF FIGURES

Figure		Page
2-1.	Equivalent T network of a crystal	16
2-2.	Two-terminal equivalent circuit of a crystal	16
2-3.	Two-terminal equivalent circuit of a magnetostriction transducer	23
3-1.	Equivalent T network of coupled systems	28
3-2.	Diagram of a simple feedback amplifier	29
3-3.	Feedback diagram of coupled circuits	30
3-4.	Signal-flow diagrams of coupled systems showing feedback loop	33
3-5.	Flow diagram for the definition of the return difference with reference to Z_{21}	34
3-6.	Equivalent network of coupled systems with impedances expressed as series R, L, and C	36
3-7.	Signal-flow diagram for determination of input impedance	40
4-1.	Electrical network with inductive coupling	49
4-2.	The feedback system equivalent to the circuit of Fig. 4-1	50
4-3.	Equivalent circuits from hybrid equations	52
4-4.	Inductively-coupled circuits with tuning capacitors at input and output	56
4-5.	The variation of input impedance with frequency in the tuned coupled circuits in Fig. 4-4	57
4-6.	Electrical network with resistive coupling	58
4-7.	Equivalent circuit for an electrostatic transducer	61
4-8.	Equivalent T network of a moving-coil transducer	69

Figure		Page
4-9.	Ideal-transformer equivalent network for a moving-coil transducer	75
4-10.	Equivalent T network for a moving-armature transducer	79
5-1.	Impedance and admittance diagrams for an experimental push-pull electrostatic loudspeaker	90
5-2.	Impedance and admittance diagrams for a dynamic or moving coil loudspeaker without baffle (W. E. 755A)	92
5-3.	A transformed equivalent circuit for the moving-coil transducer	93

COUPLING COEFFICIENTS

by

John F. Hersh

Acoustics Research Laboratory, Harvard University

Cambridge, Massachusetts

I

INTRODUCTION

The use of a coupling coefficient as a measure of the interaction between coupled systems has been familiar in physical science for more than a half-century, and it is natural to assume that the concept is by now well defined. After some thirty years of use, however, the definition was not beyond reproach in either clarity or correctness, as is indicated by G. W. O. Howe's editorial in Wireless Engineer (1932) beginning:

> Is it possible to give a simple, succinct definition--or, indeed, any definition--of the coupling between two circuits? We do not mean a definition of the method of calculating a numerical coefficient, but a definition of the physical conception which the word "coupling" is intended to convey. To be satisfactory the definition should be quite general and apply to the interaction between any two circuits, however constituted and however coupled. It is only when one tries to draw up such a definition that one realizes how vague and elusive the underlying conception really is. In textbooks and handbooks of radio-telegraphy one finds many formulae--some right and some wrong--for calculating coupling coefficients for a number of special cases, but in no case, as far as we are aware, is any attempt made to explain the exact physical significance of the coupling between two circuits. One is merely given the general impression that it is a measure of the interaction between two circuits.

To improve this situation Professor Howe then contributed a new definition of the coupling coefficient, and since that time a few others have followed his lead in the progression from general impression to general definition. In those ensuing 24 years the value of such a coupling coefficient has been confirmed beyond question not only by its continued application to electrical circuits but by its extension to electromechanical systems as well. It is surprising, therefore, to discover that a re-examination of the general definition of coupling finds the concept still vague and elusive.

This re-examination was the result of an attempt to define a coupling coefficient for the moving-coil transducer. Since coupling coefficients have found useful employment in other transducers, particularly the piezoelectric, it is another surprise to find that the extensive literature on transduction reveals no use of such a coefficient for the moving-coil transducer. There is every reason, however, to expect that there is such a coefficient for this coupled system similar in form and application to those of other systems. The only difficulty in a routine application of the conventional definitions to the moving-coil system is that this system is antisymmetrical or antireciprocal. In this respect it is similar only to gyroscopic coupling in a mechanical system and to the magnetostriction and moving-armature transducers; all the other systems for which coefficients are normally defined are symmetric or reciprocal systems.

The effect of this antireciprocity is deceptively simple — the transduction coefficients in the equations of motion of the coupled systems are equal in magnitude but opposite rather than equal in sign. But this simplicity conceals some fundamental difficulties which are far from simple, and these have been revealed by the attempt to define a coupling coefficient for such a system. The first of these is the physical significance of the coupling coefficient and the second is the physical significance of the antireciprocity. The definition of the coefficient for the moving-coil is not particularly difficult — indeed, a similar coefficient for the antireciprocal magnetostriction transducer has been used for several years. The physical justification of this definition, however, demands a basic understanding of both coupling and antireciprocity.

In the search for such understanding we join Professor Howe in asking for "a definition of the physical conception which the word 'coupling' is intended to convey." And we find, indeed, that "it is only when one tries to draw up such a definition that one realizes how vague and elusive the underlying conception really is." It is very easy here, as elsewhere, to mistake familiarity for comprehension. For every student of electric circuit theory knows what the coupling coefficient is: it is $k^2 = M^2/L_1L_2$. But what physical meaning does this have? It can be related to the flux linkages between the primary and the secondary, but why do we multiply the fraction of the primary flux that links the secondary by the fraction of the secondary flux that links the primary to obtain k^2? It can be related to the changes in resonant frequency resulting from the coupling of two tuned circuits, but why are the coupledfrequencies $(1 + k)$ and $(1 - k)$ times the uncoupled frequency? It can be related to the open and short-circuit input impedances, but why is the short-circuit impedance $(1 - k^2)$ times the open-circuit impedance? And why in this case are the possible resistive and capacitive elements in the system eliminated from the definition? Most of all, what is it that the coupling coefficient measures? The problem of defining this coefficient has much in common with the deceptive simplicity of defining the straight line.

This report will be concerned with the problem of a general definition of the coupling coefficient. To be general the definition must be applicable to mechanical, electrical, and electromechanical coupled systems, both tuned and untuned, reciprocal and antireciprocal. Such a definition should and, indeed, almost must involve in its generality the physical significance of the coefficient and the answer to the question about what it measures. The qualifications of the numerous definitions made during the long history of the coefficient will be examined in the light of these requirements, and their merits and demerits noted. With this information in hand an attempt will be made to define the coupling coefficient for all systems, to relate this definition to the other definitions, and to illustrate its application to several typical systems.

In so doing, what Professor Hunt has descriptively termed "the burdens of antisymmetry" will be added to the problems of definition. In order to determine how the antisymmetry of gyroscopic coupling affects the coupling coefficient defined for reciprocal or symmetric systems, the characteristics of antisymmetry will be investigated. The general definition of the coefficient will then be shown to be applicable to both types of systems by demonstrating that the physical significance of the coefficient as a measure of the stability limit is the same in two systems which are similar in all respects other than symmetry--the electrostatic and moving-armature transducers.

The general definition and its derivative definitions indicate several methods by which the coefficient can be measured, and this report will be concerned briefly with such measurements and with typical numerical values of the coefficient for various types of electro-mechanical systems. Experimental values of the coefficient for piezoelectric and magnetostriction transducers can be readily obtained from the literature, but for the electrostatic and moving-coil systems it has been necessary to make original determinations of such values.

In the problems of fundamental definition and physical significance the difficulties are more often philosophical than physical. Where, as in this case, the answers to particular problems are known but the general question is to be found, the logical path from answers to question may be confused by the multiplicity of starting points. It is certainly beset with pitfalls of semantics and of notation peculiar to particular systems--not to mention that of various languages and more variant investigators. To mark a few of the turns in this path and to by-pass some of the pitfalls are among the objectives of this report.

II
A REVIEW OF COUPLING AND ITS COEFFICIENTS

The Definitions Summarized

The history of the concept of coupling itself exhibits one of the characteristics of coupled systems, the mutual interaction of two related fields. The original interest in coupled systems was generated in the field of mechanics, but the resulting flow of information about mechanical vibrations quickly reacted on a newer branch of physics, electromagnetism. Here, in turn, the accumulation of knowledge about electrical circuits was fed back into the field of mechanics to augment the development of new interests there. A similar interchange of energy is characteristic of coupled physical systems, but fortunately in the interchange of knowledge the limitation of energy conservation does not require that the gain in one system be at the expense of loss in the other.

The state of knowledge about vibration problems in mechanics had been well advanced by the end of the 19th century both theoretically and experimentally by the work of men such as Lagrange,Kirchhoff, Kelvin, and Rayleigh. If we regard the second edition of Rayleigh's The Theory of Sound, published in 1894, as a summary of the field at that time, it is evident that the analysis of coupled systems was by then quite extensive, including such considerations as the effect of the coupling on resonant frequencies, the reaction upon the driving point, and the reciprocity of the effect. Although experimental investigations did not advance in step with the theoretical, the experiments of Warburg, Konig, and Rayleigh emphasized the effect of coupling in changing the natural frequencies of the coupled systems.

It is worthy of note, too, that even at this early date a connection was made between electrical theory and that of ordinary mechanics. The relation is evident in the works of Kelvin and Maxwell, and Rayleigh uses electrical vibrations frequently in his book as examples of the general theory. Of particular interest in regard to coupled circuits is the fact that Rayleigh includes, as an example taken from Maxwell's work, the reaction of a secondary circuit upon the primary when the two are coupled by mutual induction, showing the resulting reduction of inductance and increase of resistance in the primary. While it is evident that much was known about coupled systems by the end of the 19th century, it is also important to note here that no mention during this period has been found of a coupling coefficient as a measure of the interaction of the coupled systems.

It required a new experimental stimulus to advance the theory--the demonstration by Hertz in 1888 of the radiation of electromagnetic waves. Among the many resulting improvements in apparatus to generate such waves was the Tesla transformer, a tuned coupled circuit. And the efforts to improve this apparatus led to many re-examinations of the theory of coupled systems. Chief among these is the extensive and thorough work of Max Wien (1897) on the effects of coupling on resonant systems. In a later work by Wien (1915) the prior history of coupling is ably reviewed.

Since Wien's work is the basis for much that has followed and also introduces the coupling coefficient, it is worth considering in some detail.

Wien begins with a historical review, indicating the characteristics of coupled circuits revealed by the new electrical phenomena which led to the re-examination of the theory. His interest was aroused by the fact that the interaction of two undamped resonant electrical systems had been shown to result in the production of beats between the two natural periods produced by the coupling, a phenomenon that had seldom been encountered in acoustics in spite of the frequent application of similar systems. In order to make clear the basis of this phenomenon, Wien examined in more detail the theory of two coupled resonant systems which Rayleigh had outlined.

This analysis is concerned with two systems whose kinetic energy T, dissipation function F, and potential energy V can be represented by the equations

$$\begin{aligned} T &= \tfrac{1}{2} a_{11}\dot{x}_1^2 + \tfrac{1}{2} a_{22}\dot{x}_2^2 + a_{12}\dot{x}_1\dot{x}_2 \\ F &= \tfrac{1}{2} b_{11}\dot{x}_1^2 + \tfrac{1}{2} b_{22}\dot{x}_2^2 + b_{12}\dot{x}_1\dot{x}_2 \qquad (2\text{-}1) \\ V &= \tfrac{1}{2} c_{11}x_1^2 + \tfrac{1}{2} c_{22}x_2^2 + c_{12}x_1x_2, \end{aligned}$$

where the dot notation is used to indicate time derivatives. From the Lagrange equations the differential equations of motion for free oscillations of the system are found to be

$$\begin{aligned} a_{11}\ddot{x}_1 + b_{11}\dot{x}_1 + c_{11}x_1 + a_{12}\ddot{x}_2 + b_{12}\dot{x}_2 + c_{12}x_2 &= 0 \\ a_{22}\ddot{x}_2 + b_{22}\dot{x}_2 + c_{22}x_2 + a_{12}\ddot{x}_1 + b_{12}\dot{x}_1 + c_{12}x_1 &= 0. \end{aligned} \qquad (2\text{-}2)$$

These can be put in more convenient form by dividing the first by a_{11}, the second by a_{22}, and defining the following ratios: $2\delta_1 = b_{11}/a_{11}$, $2\delta_2 = b_{22}/a_{22}$, $\omega_1^2 = c_{11}/a_{11}$, $\omega_2^2 = c_{22}/a_{22}$, $\rho_1 = a_{12}/a_{11}$, $\rho_2 = a_{12}/a_{22}$, $\sigma_1 = b_{12}/b_{11}$, $\sigma_2 = b_{12}/b_{22}$, $\tau_1 = c_{12}/c_{11}$, $\tau_2 = c_{12}/c_{22}$. To avoid some confusion, Wien's notation for damping (h) and frequency (k) have here been changed to the more common δ and ω. The equations of motion now take the form

$$\ddot{x}_1 + 2\delta_1\dot{x}_1 + \omega_1^2 x_1 + \rho_1\ddot{x}_2 + 2\delta_1\,\sigma_1\dot{x}_2 + \omega_1^2\,\tau_1 x_2 = 0$$

$$\ddot{x}_2 + 2\delta_2\dot{x}_2 + \omega_2^2 x_2 + \rho_2\ddot{x}_1 + 2\delta_2\,\sigma_2\dot{x}_1 + \omega_2^2\,\tau_2 x_1 = 0 \qquad (2\text{-}3)$$

where the first three terms represent the motion of the uncoupled systems and the other terms represent the coupling effects. The three ratios ρ, σ, and τ are designated by Wien as the "coupling coefficients" for acceleration, friction, and force coupling or, in electrical terms, inductive, resistive, and capacitive coupling.

As Wien points out, in two electrical systems coupled by mutual inductance M, mutual resistance R_{12}, and mutual capacitance C_{12}, these coefficients have the form $\rho_1 = M/L_1$, $\sigma_1 = R_{12}/R_1$, $\tau_1 = C_1/C_{12}$. Therefore, Wien defined ρ_1, σ_1, and τ_1 in the following manner: "Through the oscillations of system 2 the number of lines of magnetic force, the current lines, and the lines of electric force in system 1 are altered. The coefficient ρ_1 is equal to this change in the lines of magnetic force divided by the number of lines in the uncoupled system 1. The coefficient σ_1 is equal to the change in the current lines divided by the number of current lines in the uncoupled system. The coefficient τ_1 is equal to the change in the lines of electric force divided by the number of lines in the uncoupled system."

We find here the first definition of coupling coefficients in terms of the relative magnitudes of the mutual coefficients a_{12}, b_{12}, c_{12}, compared to the coefficients of the uncoupled systems, a_{11}, a_{22}, b_{11}, b_{22}, c_{11}, c_{22}. Since the coefficients a,b,c, are proportional respectively to the kinetic energy, the dissipation, and the potential energy of portions of the system, these coupling coefficients are also energy ratios. It is not, however the ratios ρ, σ, and τ that are important in the analysis of coupled systems but rather the products of such ratios, e.g. $(\rho_1\,\rho_2)$, as Wien found in his analysis.

To illustrate this let us consider Wien's solution of the case of a simple coupled system without damping and having only force or capacitive coupling. When $\rho_1 = \rho_2 = \sigma_1 = \sigma_2 = 0$, the general equations (2.3) reduce to

$$\ddot{x}_1 + \omega_1^2 x_1 + \omega_1^2 \tau_1 x_2 = 0$$

$$\ddot{x}_2 + \omega_2^2 x_2 + \omega_2^2 \tau_2 x_1 = 0 . \qquad (2\text{-}4)$$

Solving this for x_1 or x_2 we obtain the linear fourth-order differential equation

$$\ddddot{x} + (\omega_1^2 + \omega_2^2)\ddot{x} + \omega_1^2\omega_2^2(1 - \tau_1\tau_2)x = 0. \qquad (2\text{-}5)$$

If we assume a solution of the form $x = e^{\mu t}$, the characteristic equation is

$$\mu^4 + (\omega_1^2 + \omega_2^2)\mu^2 + \omega_1^2\omega_2^2(1 - \tau_1\tau_2) = 0, \qquad (2\text{-}6)$$

and the solutions are

$$\mu^2 = -\tfrac{1}{2}\left[\omega_1^2 + \omega_2^2 \pm \sqrt{(\omega_1^2 + \omega_2^2)^2 - 4\omega_1^2\omega_2^2(1 - \tau_1\tau_2)}\ \right].$$

When the two systems without coupling are tuned to the same frequency, $\omega_1 = \omega_2 = \omega_o$, the natural frequencies with coupling become, if we let $\mu = j\omega$,

$$\omega' = \omega_o\sqrt{1 + \sqrt{\tau_1\tau_2}}$$

$$\omega'' = \omega_o\sqrt{1 - \sqrt{\tau_1\tau_2}} \qquad (2\text{-}7)$$

Since the Wien coupling coefficients appear here in the frequencies of the system as $\tau_1\tau_2$, it is convenient to define the root of this product as a coupling coefficient $\tau = \sqrt{\tau_1\tau_2}$. This, in fact, is done by Wien (1897) but without any reference to the new symbol as a coupling coefficient. However, another work of Wien's (1902) lists in the table of symbols the coefficients τ_1, τ_2, $\tau = \sqrt{\tau_1\tau_2}$ as "the coupling coefficients of the transmitter." Subsequent papers by other authors extending Wien's investigations continued the use of this coupling coefficient, and P. Drude (1904) refers to the familiar "magnetic coupling $k^2 = L_{12}L_{21}/L_{11}L_{22}$."

In contrast with this mathematical origin of the coupling coefficient in Wien's theoretical analysis, we find in a text book of this period by

J. Zenneck (1905, p. 578) an early explanation of coupling in more physical terms. Zenneck points out that whenever a primary system acts on a secondary the secondary must also react on the primary, and he describes this reaction in a system of coupled coils as follows: "The magnetic induction flux through the primary is proportional to the self-inductance coefficient L_1. The flux which the primary sends through the secondary is proportional to the mutual-inductance coefficient L_{21} of the two systems. As a result the flux and therefore the amplitude of oscillation in the secondary become greater in relation to the primary the greater L_{21} is in relation to L_1, i.e., the greater L_{21}/L_1 is. Likewise the flux through the primary due to the secondary becomes larger the larger is the ratio L_{12}/L_2. Altogether, thus, the reaction becomes more important the greater is $(L_{21}/L_1) \times (L_{12}/L_2)$. The root of this $k = \sqrt{L_{21}L_{12}/L_1L_2}$ or, if $L_{21} = L_{12}$, $k = L_{12}/\sqrt{L_1L_2}$ is denoted as the 'magnetic coupling coefficient' or simply the 'coupling coefficient' of the two systems."

In these quotations from Wien and Zenneck we have the three most common definitions of the coupling coefficient: first, in terms of the changes in the resonant frequencies of the tuned coupled system; second, in terms of the product of the ratios of energy in the coupling element to the energy in the primary and secondary circuit elements of the same type as the coupling element; third, in terms of the ratios of the flux linking the two circuits to the total primary and secondary fluxes. These definitions have been repeated with only minor variations in most of the other descriptions of coupling up until the present.

Texts prior to World War I with an emphasis on the use of coupled circuits in wireless telegraphy, such as G. W. Pierce's Principles of Wireless Telegraphy (1910) and Winkelmann's Handbuch der Physik (1908), contain essentially Wien's definition in terms of resonant frequencies. A change in emphasis in postwar texts to more general network theory produced some restatements of the previous definitions in different terms. For example, Alberti (1927) in Geiger and Scheel's Handbuch der Physik defines the coupling coefficient in the manner shown in Table 2-1.

Typical of other definitions are those of Hahnemann (1922) and Terman (1932). Hahnemann says, "In order to be able to give a general definition for the coupling factor for all possible oscillating networks and forms of coupling, we introduce into the individual elements of the network oscillating energy and designate the total energy in the first circuit including that in the coupling element with E_1 and the energy appearing in the coupling element alone with E_{K1}, and with E_2 and E_{K2} the corresponding energies of the second circuit. Then we have $k = \sqrt{E_{K1}E_{K2}/E_1E_2}$. This relation holds quite generally for all forms of oscillating networks and for all forms of inertia and force coupling." Terman puts the definition in terms of impedances in the form: "Any two circuits that are coupled by a common impedance have a coefficient of coupling that is equal to the ratio of the common impedance to the square root of the product of the total impedances of the same kind as the coupling impedances that are present in the two circuits. That is, $k = Zm/\sqrt{Z_1 Z_2}$."

It is not to be assumed, however, that all work on coupled systems subsequent to Wien made use of coupling coefficients. Numerous examples of analyses of coupling can be found which ignore the coefficient, among them works by L. Cohen (1909), Slater and Frank (1947), Morse (1936), and Den Hartog (1940). The predominant interest in electrical circuits does not mean, either, that mechanical coupling was entirely ignored. Hahnemann and Hecht (1920) returned to an analysis of mechanical systems and report that "in analogy to wireless telegraphy with electric waves and in agreement with the theory of coupled systems we designate with k^2 the expression $m_1 m_2/(m_1 + m_3)(m_2 + m_3)$ which represents the product of the ratios of the oscillating energy in the coupling terms to the total energy of each single system and call k the coupling coefficient of the two mechanical oscillating circuits."

Although well established by continued use over the period from 1897 to 1930, the definitions of coupling were not beyond reproach in either clarity

Table 2-1. Coupling Coefficients from Handbuch der Physick (1927)

$$k = \frac{e_k}{\sqrt{e_1 e_2}}$$

where e_k = voltage between points 2 and 3

e_1 = in inductive coupling, the voltage on C_1

e_1 = in capacitive coupling, the voltage on L_1

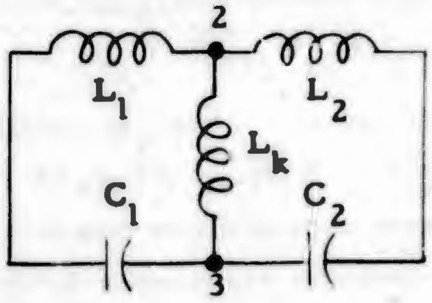
e_1 = in resistive coupling, the voltage on C_1 or L_1

e_2 = in inductive coupling, the voltage on C_2

e_2 = in capacitive coupling, the voltage on L_2

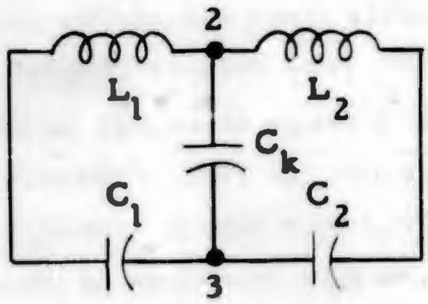
e_2 = in resistive coupling, the voltage on C_2 or L_2

Inductive Coupling



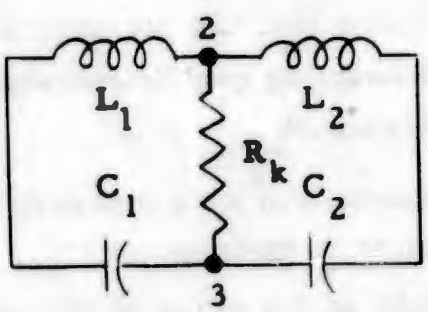
$$k = \frac{L_k}{\sqrt{(L_1 + L_k)(L_2 + L_k)}}$$

Capacitive Coupling



$$k = \sqrt{\frac{C_1 C_2}{(C_1 + C_k)(C_2 + C_k)}}$$

Resistive Coupling



$$k = \sqrt{\frac{R_k}{\sqrt{R_k^2 + (\omega L_1)^2}\sqrt{R_k^2 + (\omega L_2)^2}}}$$

or correctness. The deficiencies were ably illuminated by the critique of G. W. O. Howe (1932), whose opening paragraph was quoted in the Introduction. In this editorial Howe presents his general definition of coupling as "the relation between the possible rate of transfer of energy and the stored energy of the circuits. By the possible rate is meant the rate of energy transfer in the absence of all resistance other than that utilized for coupling, since, however closely the two circuits are coupled, the transfer of energy can be reduced to a very small value by increasing the resistance, and can be reduced to zero by increasing the resistance of one of the circuits to infinity, that is, by opening the circuit. The numerical value of the coupling is then $1/\omega$ or $1/\sqrt{\omega_1\omega_2}$ of the geometric mean of the ratios of the mutually induced e.m.f. to the maximum stored energy when unit current flows at the resonant frequency, first in one circuit and then in the other." In two circuits with different resonant frequencies ω_1 and ω_2, with induced e.m.f.'s E_{12} and E_{21} and stored energies W_1 and W_2, this definition takes the form $k^2 = (1/\omega_1\omega_2)(E_{12}E_{21}/W_1W_2)$.

Howe objects to the prevalent use of a coupling coefficient for resistive coupling of the form $k^2 = R_{12}R_{21}/R_1R_2$. This, he says, has "little real significance, for if two oscillatory circuits are entirely free from coupling except that due to a small common resistance R which is the only resistance in either circuit, the formula gives a coupling coefficient of unity, although the two circuits are in reality very loosely coupled. No formula for a coupling coefficient can neglect the storage of energy in the circuits, since the basic conception concerns not merely the rate of transfer of energy from one circuit to the other, but the ratio of this rate of transfer to the energy stored in the circuits." In addition to this criticism of some definitions of k, he comes to the interesting conclusion that "in the case of non-oscillatory circuits it is questionable whether a coupling coefficient should be employed; it certainly loses much of its significance."

Howe's idea of the coupling coefficient as a measure of the rate of transfer of energy finds support from H.H. Skilling (1937). As he states it, "the rate of transfer of energy depends on the value of the mutual parameter relative to the other parameters of the network. This relation is expressed as the coefficient of coupling." Particularly interesting in his discussion of coupling is this explicit statement of a characteristic that is implicit in other definitions of the coefficient, namely, that k is not a measure of efficiency: "It might seem, at first thought, that close coupling of the circuits would be necessary to transfer energy efficiently from one to the other. It is quite true that close coupling permits a faster transfer of energy, but it does not make the transfer any more complete."

In more recent works on circuit theory the coupling coefficient finds continued use and redefinition. The analysis of Tellegen (1947) is novel in that it introduces k through its appearance as a factor in the transfer admittance or impedance of the coupled circuits. In considering systems of small relative bandwidth in regions near resonance, Tellegen finds a real coupling coefficient for inductive and capacitive coupling but an imaginary k for resistive coupling. This imaginary k is shown to affect the damping of the two circuits in the same manner that the real k affects the resonant frequencies. And he agrees with Howe in relating the resistive k to the ratio of the coupling resistance to the reactances of primary and secondary, i.e., to the stored energy.

Another definition that is similar to Tellegen's in that k appears as a term in the relation of secondary voltage to primary voltage is found in Rocard (1949). A different definition of the resistive coupling coefficient, however, is used by Schmerwitz (1937). Using Wien's analysis he obtains a resistive coupling coefficient $\sigma = \sqrt{\sigma_1 \sigma_2}$ of the same form obtained in Eq. 2-7 for capacitive coupling. But in this case the coupling does not alter the resonant frequencies; it changes the damping constants of the two circuits to $(1 + \sigma)$ and $(1 - \sigma)$ times the damping without coupling, as Tellegen also found. The difference is that Schmerwitz's coefficient is equivalent to $k = R_{12}/\sqrt{R_1 R_2}$, which does not involve stored energy in the system.

LePage and Seely's General Network Analysis (1952) is notable for its complete treatment of transformer coupling from both the self- and mutual-inductance viewpoint and the mutual- and leakage-flux viewpoint. However, k is here introduced by multiplying together two alternate forms of the mutual inductance M in terms of the flux ratios $K_1 = \phi_{21}/\phi_{11}$ and $K_2 = \phi_{12}/\phi_{22}$ to obtain $M^2 = K_1 K_2 L_1 L_2$. In this, they note, "it is customary to define a new constant $k = \sqrt{K_1 K_2}$, which is termed the coefficient of coupling."

On the other hand, new physical significance for k is to be found in Guillemin's Introductory Circuit Theory (1953). For the case of two mutually coupled coils Guillemin shows that the requirement that the stored energy in the system be positive for all values of the coil currents leads to the condition that $L_{11}L_{22} - L_{12}^2 > 0$. If, then, $k^2 = L_{12}^2/L_{11}L_{22}$ be defined as the coupling coefficient, this condition can be expressed as $|k| < 1$. The limiting condition that k cannot exceed unity can also be explained in terms of flux linkages as the physical condition that all of the flux links all of the windings. But this derivation of the condition "does not lend itself to generalization, while the method based upon stored energy is readily extended to any number of coupled coils."

Among the recent reviews of mechanical coupling that of Richardson (1953,p. 3) is of interest because of what it suggests rather than because it provides any new definition of coupling. In discussing coupled vibrations Richardson begins, "So far we have considered what Rayleigh called an inexorable forcing system, but in practice the driven will react on the driver. Together they form a coupled system. When the two components have comparable mass the roles of driver and driven may be periodically interchanged as kinetic energy is bandied between them. The coupling coefficient σ may be regarded as determining the rate of this interchange or the magnitude of the "feedback". But no further use is made of the idea of feedback.

The accelerated research in the field of radio transmission occasioned by World War I led soon thereafter to commercial broadcasting and to the use of coupled electrical circuits in every home receiver. This was paralleled in another field that involved coupled systems. The connection between mechanical systems and electrical systems had long been a matter of scientific interest and, since the telephone developments of the 1870's, a matter of great commercial interest, too--indeed, there was before 1900 a coupled electromechanical system in every home telephone. But it was the military interest in electromechanical transducers for underwater echo-ranging that brought about the technological utilization of such a well-known but little-used principle of transduction as the piezoelectric effect, discovered by the Curies in 1880.

The rapidly increasing knowledge about piezoelectricity led to increasing use of it in electromechanical devices and to another major application as a frequency-controlling device. At about the same time other interest in electromechanical devices was stimulated by the need for improved phonograph recording and reproducing equipment. The design of such devices was considerably facilitated by the use of the analogy between mechanical and electrical systems, since the theory of electrical networks was well advanced by this time. The wealth of knowledge about electrical transmission systems now began to feed back into the mechanical through the use of electric-network analogs for the deductive analysis of mechanical and electromechanical systems. And the appearance of coupled electrical circuits as the analog of an electromechanical transducer led inevitably to the use of an electromechanical coupling coefficient as a measure of the coupling.

The inevitable, however, often approaches with reluctance, and such was the case with the electromechanical coupling coefficient. The equivalent electrical circuit for an electromechanical system was described by Butterworth (1915), and he concluded that "a vibrating system of one degree of freedom when set in motion by the interaction of a current on a magnetic field is shown to behave as a parallel combination of a capacity, conductance, inductance and when set in motion by the interaction of charged bodies on an electrostatic field it behaves as a series combination of inductance, resistance, and capacity." Equivalent circuits for electrodynamic and electromagnetic transducers were described by Hahnemann and Hecht in 1920 and for a piezoelectric crystal resonator by Van Dyke in 1925.

The increasing interest in the commercial applications of piezoelectric crystals promoted the further application of circuit theory to their equivalent circuits. An early result was the recognition of the importance of the ratio of the clamped capacitance of the crystal (C_0) to the equivalent motional capacitance (C_1) as a measure of the electromechanical activity. This was first noted by Dye (1926), and he defined it as the "piezoelectric ratio." The ratio is particularly useful since it is related to the bandwidth of the system and can conveniently be determined experimentally by the frequency separation between the resonance and antiresonance of the crystal.

The crystal has, thus, in its capacitance ratio a measure of bandwidth such as the tuned coupled circuit has in the coupling coefficient.

And at long last W. P. Mason (1934) derives a coupling coefficient for the crystal from its equivalent circuit and shows its relation to the capacitance or piezoelectric ratio. From the equations of motion of the coupled electromechanical system Mason derives an equivalent T network of the form shown in Fig. 2-1, with the coupling represented by a mutual capacitance C_{em}.

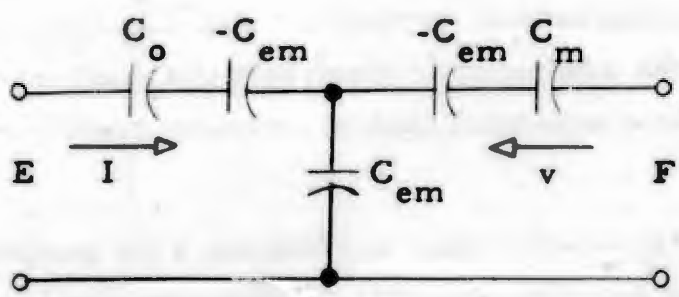


Fig. 2-1. Equivalent T network of a crystal.

As in the usual coupled electrical circuits, the coupling coefficient is defined by the equation $k^2 = C_o C_m / C_{em}^2$. Mason then shows that when the mechanical system is free to move, i. e., $F = 0$ and the mechanical terminals are shorted, this T network reduces to the familiar early equivalent form shown in Fig. 2 - 2 and that the capacitance ratio is related to k by the equation $C_o/C_1 = (1 - k^2)/k^2$. The relation between k and the

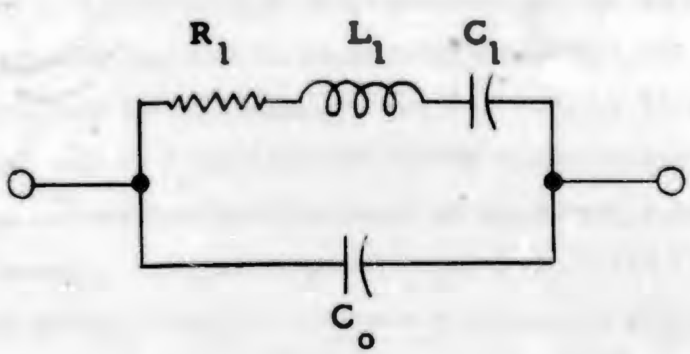


Fig. 2-2. Two-terminal equivalent circuit of a crystal.

frequencies of resonance (f_1) antiresonance (f_2) is found to be $k^2 = 1 - (f_1/f_2)^2$.

The physical meaning of this electromechanical coupling coefficient is described by Mason in his book Piezoelectric Crystals (1950, p. 4): "The difference between the free and the clamped constant is determined by the electromechanical coupling factor for that crystal. This is defined as the square root of the ratio of the energy stored in mechanical form, for a given type of displacement, to the total input electrical energy obtained from the input battery." This statement contains both an interpretation of k in terms of an energy ratio and another definition in terms of an energy ratio and another definition in terms of a difference between the free and clamped constants of the crystal. Such a definition had not appeared previously in either electrical or electromechanical systems, but, in Electromechanical Transducers and Wave Filters, Mason (1942) introduces k through this definition rather than through the equivalent T network described above. From the equations of motion he finds that the mechanical stiffness when the systems are coupled, i.e., the stiffness when the electrical terminals are shorted, is of the form $Y = Y_o(1 - D^2K/4\pi Y_o)$, and he defines the electromechanical coupling coefficient as $k = D\sqrt{k/4\pi Y_o}$. The effective value of the Young's modulus Y when the electrical terminals are shorted is thus lower than the modulus Y_o when the terminals are open by the factor $(1 - k^2)$.

Mason also defines a coupling coefficient for an electrostatic transducer in the second edition of Electromechanical Transducers and Wave Filters (1948), this time through the use of an equivalent T network with coupling through a mutual capacitance C_{em}. As in the crystal circuit of Fig. 2-1, the coefficient is here defined as $k^2 = C_oC_m/C_{em}^2$. And again Mason says, "The factor k^2 has the significance that it represents the portion of the total input electrical energy stored in mechanical form for a static or DC voltage." But here he adds, "The analogy with the definition for the coupling in a coil is obvious since here also the square of the coupling represents the ratio of the electromagnetic energy stored in the secondary to the total input electromagnetic energy for an applied DC current."

A different definition of the coupling coefficient in terms of energies is used by P. Vigoureux (1950 p. 58) and is produced by Skudrzyk (1954, p. 485). In this case the total energy in the coupled systems is considered to be the sum of the energies in the two systems without coupling and the mutual energy resulting from the coupling instead of the sum of the energies of a primary and secondary. Vigoureux expresses the definition as

$$k^2 = \frac{\text{square of mutual energy}}{\text{4 times product of individual energies}} \qquad (2\text{-}8)$$

and in Skudrzyk this becomes: "The relation of the spatial average of the mutual energy w_{em} to the doubled geometric mean of the spatial average of the mechanical and electrical potential energies is defined as the coupling factor, $k = w_{em} / \sqrt{4 w_e w_m}$." For the piezoelectric system considered by both authors, the total energy is $w = \frac{1}{2} fT^2 + dTE + \frac{1}{2} \epsilon E^2 = w_m + w_{em} + w_e$, where f is the elastic constant, d is the piezoelectric constant, and w_m, w_{em}, w_e are the mechanical, mutual, and electrical potential energies. For static deformation this results in a piezoelectric coupling coefficient $k = d/\sqrt{\epsilon f}$. Skudrzyk shows that k is related to the capacitance ratio through $k^2 = C_1/C_o = \frac{1}{2} C_1 V^2 / \frac{1}{2} C_o V^2$, and "the square of the coupling factor is thus equal to the ratio of the potential mechanical energy received from the crystal to the potential electrical energy: $k^2 = W_{pot}/W_{el}$. The potential electrical energy is determined by the blocked capacitance and the electrical voltage, the potential mechanical energy by the elasticity and the deformation of the crystal which appears in the equivalent circuit of the transducer through the capacitance C_1."

In his book on electroacoustics Fischer (1950, p. 26) defines a coupling coefficient for coupled electrical circuits and applies this to the equivalent circuits of the electromechanical systems. His definition is that of Hahnemann in terms of the ratios in each circuit of the energy in the coupling element to the total energy in that circuit in elements of the same type as the coupling. This is, of course, also Wien's definition, but Fischer introduces the product of the two ratios as the coupling coefficient through its appearance in the equation for the input impedance rather than through the equation for the coupled resonant frequencies used by Wien.

Fischer makes frequent but unusual use of the coefficient in that he applies it to circuits in which the elements are not independent of frequency and to circuits having elements which are not inherent in the transducer.

The recent work of Hecht (1954) on transducers also recognizes the value of the electromechanical coupling coefficient as a measure of transducer quality. But he raises some question about the use of the same term for electromechanical coupling as is used for the electrical coupling. "The word 'coupling factor' for this energy relation in electroacoustic transducers is also not a very happy choice, since the word 'coupling factor' in coupled electrical and coupled mechanical circuits is given to the relation of the energy in the coupling term to the total oscillating energy in both purely electrical or mechanical coupled circuits." He considers the electromechanical coupling factor to be that usually measured by the frequencies of maximum and minimum impedance, and "the coupling factor of a capacitive transducer is understood to mean the square root of the ratio of the mechanical deformation energy to the electrical energy of the condenser when the reference frequency is taken far below the resonance frequency, as in general it is in practice in such transducers." Hecht suggests that "one could call the coupling factor appearing in coupled circuits the pure electrical or mechanical coupling factor and that factor appearing in transducers the electromechanical coupling factor."

In the book Sonics by Hueter and Bolt (1955) are repeated the previous definitions of the piezoelectric coupling coefficient in terms of the frequencies of resonance and antiresonance and in terms of the capacitance ratio or the ratio of the energy stored mechanically to that stored electrically. An interesting feature, however, is Table 4.8 on p. 164, which compares the characteristics of electrostatic, piezoelectric, electrodynamic, and magnetostrictive transducers. The electromechanical coupling coefficient is here defined generally as $k^2 \approx \alpha^2 Z_o/X_o = \alpha'^2/X_o Z_o$, where Z_o is the clamped electrical impedance, X_o is the stiffness reactance, α is the transformation factor for ideal transformer coupling, and α' is the transformation factor for T coupling or the mutual coefficient in the equations represented by the equivalent T network. From this definition a coupling coefficient for each of the four types of transducers is obtained. For the moving-coil or electrodynamic transducer the definition takes the form

$$k^2 = (2\pi r N B_o)^2 / K_o L_o, \qquad (2\text{-}9)$$

where $2\pi rN$ is the length of the voice coil winding, B_o is the polarizing field, K_o is the diaphragm stiffness, and L_o is the voice-coil inductance. This is the only definition that has been found in the literature for the moving-coil system.

In Electroacoustics by F. V. Hunt (1954) the electromechanical coupling coefficient is defined both in terms of the change in input impedance resulting from coupling and in terms of the critical frequencies of the coupled systems. In the example of an electrostatic transducer the mechanical compliance when the electrical terminals are shorted is shown through equivalent circuits to be $c'_m = [(c_m)^{-1} + (-C_{em}{}^2/C_o)^{-1}]^{-1} = c_m/(1 - k^2)$, where c_m is the mechanical compliance without coupling. The coupling coefficient is thus defined explicitly for this transducer through $k^2 = c_m C_o / C_{em}{}^2$. More generally, the frequency spread between f_R and f_Y, the frequencies at which the motional impedance and motional admittance have maximum values, is identified as an index of how closely the electrical and mechanical meshes are coupled. "An effective coefficient of electromechanical coupling can be defined by the following pair of relations:

$$k_{eff}{}^2 = 1 - (f_R/f_Y)^2, \text{ for electromagnetic coupling}$$
$$(2\text{-}10)$$
$$k_{eff}{}^2 = 1 - (f_Y/f_R)^2, \text{ for electrostatic coupling."}$$

Essentially the same definition as that of Vigoureux is used by Bechmann (1955, p. 55) in his analysis of piezoelectric systems. But he expresses the total energy as the sum of the elastic energy W_1, the electrical energy W_2, and the piezoelectric energy W_{12}, and in so doing he divides the mutual energy W_M into two equal parts--the energy W_{12} in the mechanical system due to the electrical and W_{21} ($=W_{12}$) in the electrical system due to the mechanical. As the result, his definition of the electromechanical coupling coefficient is

$$k^2 = W_{12}{}^2 / W_1 W_2, \qquad (2\text{-}11)$$

and the factor 4 in Vigoureux's definition does not appear.

In applying this definition to the four possible forms of the equations of state of the sys , Bechmann distinguishes two different coefficients that result. The distinction is made by classifying the four equations into two groups on the basis of the variable being homogeneous or mixed. The variables are identified, as in thermodynamics, either as intensive parameters or generalized forces or as extensive parameters or generalized displacements. Two of the equations of state are, thus, homogeneous because the independent variables are extensive and the dependent intensive or vice versa; the other two are mixed because the independent and dependent variables are each a mixture of intensive and extensive parameters. The coupling coefficient definition can be applied to all four equations, but for the homogeneous set it yields the usual value (denoted as k_{hom}^2 by Bechmann) and for the mixed set another value (k_{mix}^2). The two k's are related by $k_{mix}^2 = k_{hom}^2/(1 - k_{hom}^2)$.

It is evident from its recurrent use that the electromechanical coupling coefficient has proved to be of value in the development of crystal transducers. It is not surprising, therefore, that the development of magnetostriction transducers found similar use for such a coefficient. The interest in magnetostriction transducers, however, lagged somewhat behind that in crystal transducers. Much as World War I was responsible for applications of the piezoelectric effect, World War II resulted in a renewed interest in the magnetostrictive effect for application to underwater transducers. Again paralleling the history of piezoelectricity, magnetostriction had long been familiar to scientists since the definitive work of Joule in 1842 but had found few practical applications other than those introduced by Pierce and his students after 1928 in the fields of oscillator control and ultrasonic vibrators.

The intensive wartime development of magnetostriction transducers led naturally to a theoretical analysis of this coupling which paralleled and embodied the existing theory of piezoelectric coupling. Previous work by Butterworth and Smith (1931) had established an equivalent circuit for the magnetostriction transducer and had employed the critical frequencies of this circuit in the same manner that the frequencies of resonance and antiresonance had been used for crystal transducers, but no use was made of a coupling coefficient. In a report of the war research on magnetostriction from the Harvard Underwater Sound Laboratory (1946), however, the coupling coefficient is introduced just as Mason introduced it for the crystal transducers. From the equations of state of the coupled systems the mechanical stiffness or Young's modulus with a constant magnetic field intensity (E^H) is shown to be related to the modulus with a constant flux density (E^B) by the equation $E^H = E^B (1 - 4\pi\lambda^2\mu/E^B)$. The factor $4\pi\lambda^2\mu/E^B = k^2$ is then defined as the square of the coupling coefficient. In considering the equivalent circuit as a band-pass filter, an effective coefficient of coupling is also defined as a measure of the width of the pass band. This is the definition in terms of f_R and f_Y, the frequencies of maximum motional impedance and admittance, later used by Hunt. This effective coupling coefficient differs from the coefficient first defined because of the inclusion of eddy-current effects, but the two can be considered equal at all frequencies below the characteristic frequency for eddy currents.

In the work of Sussman and Ehrlich (1950) the development of magnetostriction continues to follow the precedent established by piezoelectricity in that they define the coefficient in terms of energy by using the previous theory to show that "k^2 is the ratio of converted stored energy to input stored energy for the case of no losses or radiation." An energy definition is also used by van der Burgt (1953) in his analysis of magnetostriction in ferrites. He notes that "an important quantity is the magnetomechanical coupling coefficient k, the square of which is defined as the ratio between electromagnetic and elastic energy when only an alternating low-frequency stress is applied." From the constants in the magnetostriction equations of state, van der Burgt also defines the coupling coefficient through the equations

$$k^2 = 1 - (E^H/E^J) = 1 - (x^S/x^T), \qquad (2\text{-}12)$$

where E^H is the Young's modulus at constant field intensity or "free stiffness, E^J is the modulus at constant magnetization or "clamped" stiffness, x^S is the susceptibility at constant strain or "clamped" susceptibility, and x^T is the susceptibility at constant stress or "free" susceptibility.

The work of Woollett (1953) shows that the magnetostriction coupling coefficient can also be determined by the same resonance-antiresonance method commonly used with piezoelectric crystals. The k measured by this method is shown to be the same k measured by the impedance-circle and admittance-circle methods mentioned above when the dissipation is negligible. Woollett also defined an effective coupling coefficient in terms of the two-terminal equivalent circuit shown in Fig. 2-3, which represents a magnetostriction transducer with no internal dissipation and in the frequency region of a resonance. The coupling coefficient is here defined as $k_{eff}^2 = L_z/(L_b + L_z)$.

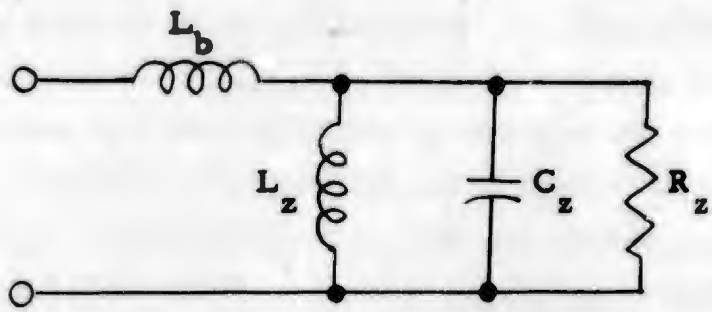


Fig. 2-3. Two-terminal equivalent circuit of a magnetostriction transducer. 4

Pigott and Kendig (1954) also relate the magnetostriction k to the frequencies ω_1 of maximum impedance and ω_o of maximum admittance, which are also respectively the resonant frequencies of the mechanical system with and without magnetostriction. The relation is shown to be $k^2 x_R = 1 - (\omega_1/\omega_o)^2$, where x_R is the real part of the complex eddy-current factor. As in Woollett's report, the measurement of these frequencies is shown to be a rapid method of determining k.

The Definitions Criticized

The preceding review of the history is intended to summarize rather than to criticize the important contributions to the definition of a coefficient of coupling. The object of this review was to find the general definition, the explanation of the physical significance of the coefficient sought by Howe in 1932. In this fifty-year history, however, many definitions have been found for many coupled systems, and the profusion has tended more to confuse than to clarify the concept of the coefficient as a measure of a physical characteristic common to all coupled systems. There is little question that almost all of the particular definitions are both consistent and correct. But there remains a need for a general definition from which all such definitions for particular coupled systems can be derived.

The recent contributions to the definition of an electromechanical coupling coefficient approach the generality desired. In particular, Mason's definition in terms of the ratio of the stored energy in the secondary to the total input energy has initial appeal in that it involves the basic concept of energy in the system and obviously limits the maximum k^2 to unity when all of the energy is in the secondary. It is found, however, that some of these concepts which are clear when attention is focused on a limited region of application tend to lose definition when the focus is broadened. One of the difficulties here is in defining "primary" and "secondary" in the coupled systems, and this is evident in the fact that where Mason uses the ratio of mechanical to total energy for the crystal transducer others define k^2 as the ratio of mechanical to electrical energy. Other difficulties with such apparently succinct definitions are to be found in the explanatory text that is omitted from the definition. To be sought in the omissions are the reasons why the secondary terminals are to be shorted before determining the energy, why the energy is to be determined for an applied DC current, why only elements of the same type as that used for coupling are to be considered.

The definitions of Vigoureux and Bechmann in terms of energy have the same promise of generality. But again certain simplifying operations on the system are assumed before the definition is applicable, and the physical significance is not apparent. In these definitions, as in the definitions in terms of inductance or capacitance ratios, e.g., $k^2 = (M/L_1)(M/L_2)$, the physical significance of the product of the two ratios is tantalizingly elusive, the definition being justified only by the appearance of this product in the expressions for the coupled resonance frequencies or for the input impedance with coupling.

This search for physical significance in the energy definitions encounters new obstacles in systems, such as magnetostriction transducers, where the coupling is antisymmetrical or antireciprocal. The energy definitions of k^2 in this case are apparently the same as those used for such symmetrical systems as the crystal transducers, but close examination of the equations to which they are applied reveals that these equations are symmetrical in mixed intensive and extensive independent variables instead of in the usual homogeneous variables. This difference is, perhaps, more obvious when the definitions are applied to the moving-coil transducer than when applied to the magnetostriction transducer because in the former the variables are commonly the familiar ones of force, voltage, velocity, and current. But for any of the antireciprocal transducers the energy definitions result in the relation of the potential energy in one of the coupled systems to the kinetic energy in the other. In the literature there is common use of "inverse" analogies for antireciprocal systems--with some doubt about which analogies are inverse--but little explicit recognition and no explanation of the necessity for changing the types of variable or the types of energy in defining coefficients for antireciprocal coupling.

Further need for clarification of the concept is evident in the literature in the disagreements in regard to the definition of the coefficient for resistive coupling and in the doubts about the applicability of the coefficient to all forms of coupled systems. Howe and Tellegen agree in defining k as the ratio of the coupling resistance to the square root of primary and secondary reactances; while Schmerwitz and others use the square root of the primary and secondary resistances. Howe also questions whether the coupling coefficient has much significance if the coupled systems are not tuned or resonant; Hecht questions whether the definition is the same for electrical as for electromechanical systems.

The answer to all these questions is to be found in a general definition of the coupling coefficient, applicable to the interaction between any two systems, however constituted and however coupled, and indicative of the physical significance of the coupling.

III

A DEFINITION OF COUPLING COEFFICIENTS

The Coupled Systems

The coupled systems here considered are those treated by Rayleigh (1877, I, § 82). Their motion takes place in the immediate neighborhood of a configuration of thoroughly stable equilibrium, and hence their kinetic and potential energies and the dissipation function can be expressed as the essentially positive homogeneous quadratic functions

$$T = \frac{1}{2} L_{11} \dot{q}_1^2 + \frac{1}{2} L_{22} \dot{q}_2^2 + \ldots\ldots + L_{12} \dot{q}_1 \dot{q}_2 + \ldots$$

$$V = \frac{1}{2} \frac{1}{C_{11}} q_1^2 + \frac{1}{2} \frac{1}{C_{22}} q_2^2 + \ldots\ldots + \frac{1}{C_{12}} q_1 q_2 + \ldots \qquad (3\text{-}1)$$

$$F = \frac{1}{2} R_{11} \dot{q}_1^2 + \frac{1}{2} R_{22} \dot{q}_2^2 + \ldots\ldots + R_{12} \dot{q}_1 \dot{q}_2 + \ldots$$

From these the equations of motion of the coupled systems can be obtained through Lagrange's equation

$$\frac{d}{dt}\left(\frac{d\,T}{d\,\dot{q}}\right) - \frac{d\,T}{d\,q} + \frac{d\,F}{d\,\dot{q}} + \frac{d\,V}{d\,q} = E, \qquad (3\text{-}2)$$

with a resulting system of equations of the form

$$E_1 = Z_{11} I_1 + Z_{12} I_2 + Z_{13} I_3 + \ldots\ldots$$

$$E_2 = Z_{21} I_1 + Z_{22} I_2 + Z_{23} I_3 + \ldots\ldots \qquad (3\text{-}3)$$

$$E_3 = Z_{31} I_1 + Z_{32} I_2 + Z_{33} I_3 + \ldots\ldots$$

....

In using the familiar impedance notation in these equations it has been assumed that the variation of the coordinate q with time is of the form $\dot{q} = Ie^{j\omega t}$. The generality is not limited by this assumption, since the results can be extended to other excitation functions by the use of Fourier's theorem. The notation used in all the equations is electrical because the systems of interest are predominantly electrical or electromechanical and all the systems have electrical equivalent circuits, but it is to be understood that mechanical systems can be represented by the same equations when voltage is replaced by force, current by velocity, inductance by mass, capacitance by compliance, and electrical resistance by mechanical resistance.

The coefficients L, C, R in the energy functions of Eqs. 3-1 may be constants, as in the usual electrical or mechanical systems, or they may be functions of the variables, as in the electromechanical systems. The systems with constant coefficients and many of those in which the coefficients vary are reciprocal systems in which $L_{mn} = L_{nm}$, $C_{mn} = C_{nm}$, $R_{mn} = R_{nm}$, as has been assumed in Eqs. 3-1. In these systems the impedance matrix of Eqs. 3-3 is symmetrical about the diagonal, since $Z_{mn} = Z_{nm}$, and such systems are termed symmetrical (Hunt, 1954, p. 105). On the other hand, the mechanical and electromechanical systems in which L (or its mechanical equivalent) varies with the coordinates in such fashion that terms of the first degree in the volocities appear in the kinetic energy function (T) have been shown by Le Corbeiller (1929) to have impedances with the antisymmetric relation $Z_{mn} = - Z_{nm}$. These systems with an antisymmetrical matrix are characterized as antireciprocal systems, and their coupling is said to be gyroscopic. The forces in such systems which are proportional to velocities but not due to dissipation are mentioned by Rayleigh (1877, I, 82), and Goldstein (1950, p. 19) shows that such forces may be included in the Lagrangian equations by the use of a generalized potential. These antireciprocal systems can still be represented by Eqs. 3-3 and will be included in the following general theory; the effects of the antireciprocity will be considered later.

In the Eqs. 3-3 there are as many equations as there are independent coordinates q or degrees of freedom of motion in the coupled systems. In each of these equations there is a term of the form $Z_{nn}I_n$ representing the energy in the n'th system resulting from the current I_n and also terms of the form $Z_{nm}I_m$ representing energy in the n'th system resulting from coupling to the current I_m. In general each system may be coupled to all the other systems. The effects of the coupling will be treated here, however, by considering only two systems at a time, that is by considering all the currents except the chosen I_m and I_n to be zero. Since the systems are linear, the effects of multiple coupling can then be determined by superposition of the results obtained from the paired systems.

The equations of motion, also called the equations of state or canonical equations of the system, for any two coupled systems may be written as

$$\begin{aligned} E_1 &= Z_{11}I_1 + Z_{12}I_2 \\ E_2 &= Z_{21}I_1 + Z_{22}I_2 \end{aligned} \qquad (3\text{-}4)$$

As the basic equations of a four-pole electrical network these are familiar in circuit theory and have been considered in detail by Feldtkeller (1937). For the reciprocal or symmetrical systems, in which $Z_{21} = Z_{12}$, they represent an electrical network of the "T" form shown in Fig. 3-1. Since

Eqs. 3-4 in electrical notation represent mechanical and electromechanical coupled systems as noted above, the circuit of Fig. 3-1 is "equivalent" also to such systems.

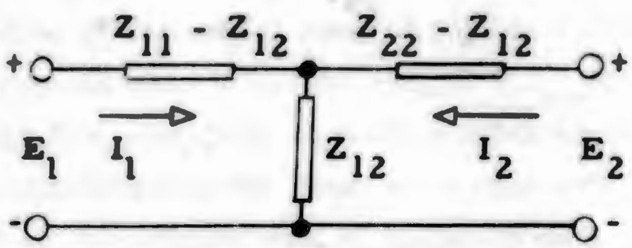


Fig. 3-1. Equivalent T Network of coupled systems.

It is possible, and often convenient, to express Eqs. 3-4 in three other possible forms obtained by using other of the four variables than I_1 I_2 as the independent variables. These other forms of the equations are:

$$
\begin{aligned}
I_1 &= Y_{11} E_1 + Y_{12} E_2 \\
I_2 &= Y_{21} E_1 + Y_{22} E_2
\end{aligned}
\qquad (3\text{-}5)
$$

$$
\begin{aligned}
I_1 &= h_{11} I_2 + h_{12} E_1 \\
E_2 &= h_{21} I_2 + h_{22} E_1
\end{aligned}
\qquad (3\text{-}6)
$$

$$
\begin{aligned}
I_2 &= h'_{11} I_1 + h'_{12} E_2 \\
E_1 &= h'_{21} I_1 + h'_{22} E_2
\end{aligned}
\qquad (3\text{-}7)
$$

The other two possible forms relating E_1 and I_1 to E_2 and I_2 and vice versa, are the so-called chain or transfer equations, useful in cascading networks but of little interest in coupling considerations. The Eq. 3-4 is often characterized as using impedance parameters, Eq. 3-5 as using admittance parameters, and Eqs. 3-6 and 3-7 as using series-parallel or hybrid parameters. The coefficients in these equations are, of course, related, and these relations are tabulated by Feldtkeller (1937, p. 90).

It is useful, also, to characterize these equations as thermodynamically homogeneous or mixed in the terminology of Bechmann (1955): Of the four variables in these equations the voltages, or generalized forces E_1 and E_2, are the intensive or mass-independent variables, the currents or generalized displacements I_1 and I_2 are the extensive or mass-dependent variables. Since the independent variables are extensive and the dependent intensive in Eqs. 3-4 and vice versa in Eqs. 3-5, these sets are called homogeneous. In Eqs. 3-6 and 3-7 the independent and dependent variables are both mixtures of intensive and extensive parameters, and these equations are called mixed.

Coupling as Feedback

In considering the interaction of two systems represented by Eqs. 3-4 and the circuit of Fig. 3-1 it is customary to regard the coupling as resulting from the impedance Z_{12} common to both systems. In the equations the coupling term $Z_{21} I_1$ is then a measure of the effect of system 1 on system 2, and the term $Z_{12} I_2$ is a measure of the simultaneous reaction of 2 on 1. If a driving voltage is applied to only one system, say system 1, this action and reaction can also be expressed by considering $Z_{21} I_1$ as a measure of the voltage or energy fed forward from 1 to 2 and $Z_{12} I_2$ as a measure of that fed back from 2 to 1. This suggests that the coupled systems may be viewed as a unilateral forward transmission system with feedback of some of the output energy to the input.

To pursue further this relation between coupling and feedback consider a simple amplifier with feedback, such as that shown in Fig. 3-2.

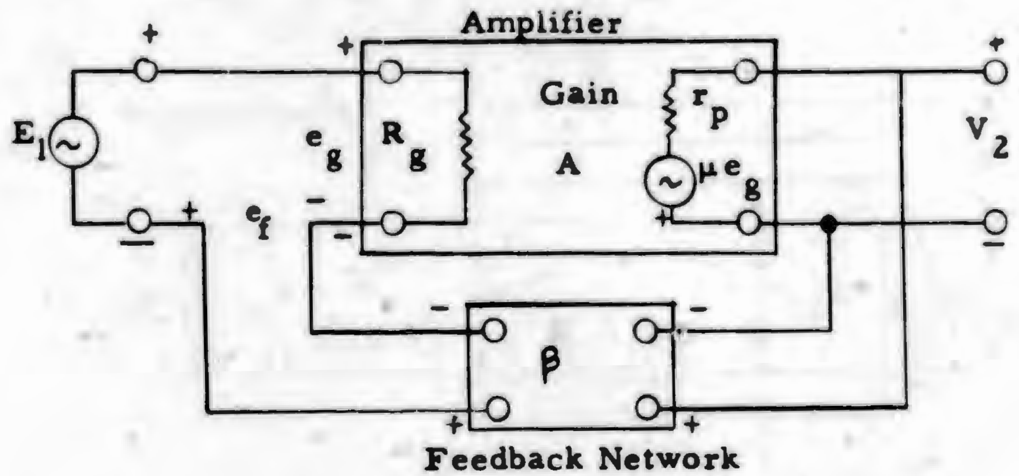


Fig. 3-2. Diagram of simple feedback amplifier.

If, as is often assumed, the feedback network has infinite input impedance, zero output impedance, and a voltage ratio β of output to input voltages, the equations for this amplifier with feedback and with the output terminals open as shown in Fig. 3-2 can be written as

$$\begin{aligned} E_1 &= R_g I_1 + \beta V_2 = e_g + e_f \\ 0 &= -\mu R_g I_1 + V_2 = -A e_g + V_2 . \end{aligned} \tag{3-8}$$

The measure of the feedback is usually taken as the ratio of the feedback voltage e_f to the grid voltage e_g, and this ratio $e_f/e_g = A\beta$ is called the feedback factor (Terman, 1947, p. 311). The Eqs. 3-8 have a form similar to that of the equations of motion of coupled systems. Consequently, a set of such equations as Eqs. 3-4 can be represented by a feedback diagram, as shown in Fig. 3-3. This is similar to Fig. 3-2 in that the effects of one system upon the other appear as voltages generated in the systems. The difference is that in the coupled systems these voltages are proportional to currents in the two systems rather than to other voltages, i. e., there is current rather than voltage feedback. This view of coupling in terms of voltages fed forward and backward is used by Terman (1947, p. 53) in his analysis of inductively-coupled circuits, but the voltages are there expressed in terms of impedances and currents in computing the effects of the coupling.

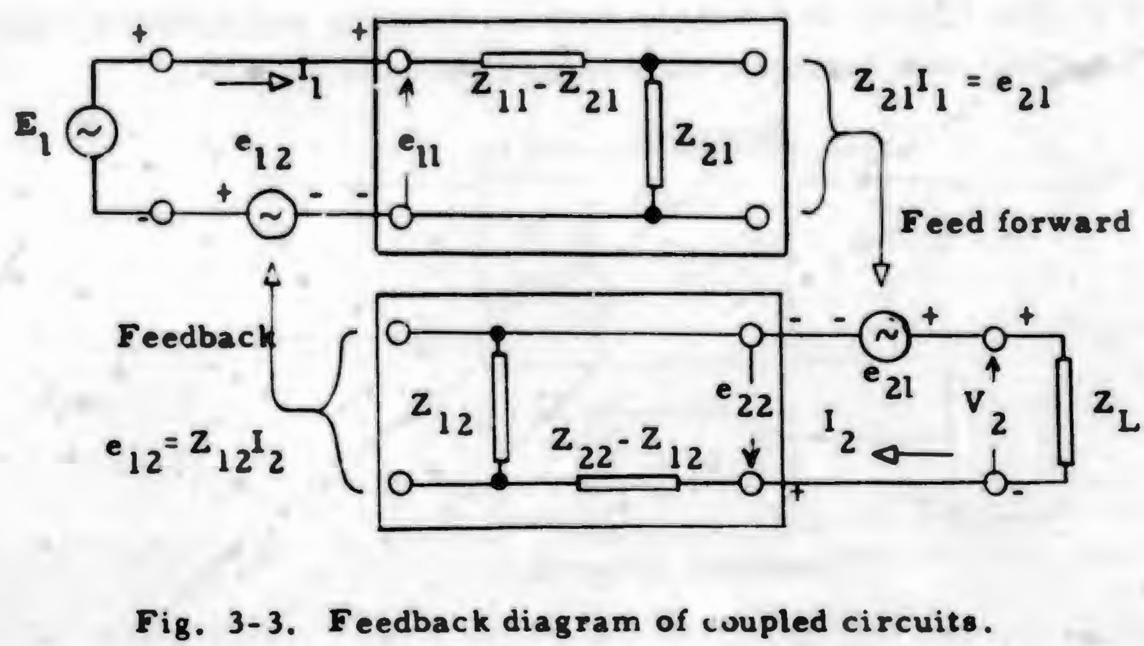


Fig. 3-3. Feedback diagram of coupled circuits.

To find for the coupled circuits a measure of the feedback similar to the feedback factor $A\beta$ above, consider a voltage e_{11} at the equivalent of the grid terminals of system 1 produced by a generator E_1 at the input terminals and a load Z_L (across the output terminals of system 2), as shown in Fig. 3-3. Current I_2 then flows in system 2, and there appears in system 1 a feedback voltage e_{12}, which is related to e_{11} by

$$e_{12} = \frac{Z_{12}}{Z_{22} + Z_L} e_{21} = \frac{Z_{12}}{Z_{22} + Z_L} \times \frac{Z_{21}}{Z_{11}} e_{11}. \qquad (3\text{-}9)$$

Hence, the equivalent of the feedback factor $A\beta = e_f/e_g$ is

$$\frac{e_{12}}{e_{11}} = \left(\frac{Z_{12}}{Z_{22} + Z_L} \right) \left(\frac{Z_{21}}{Z_{11}} \right), \qquad (3\text{-}10)$$

and this may be regarded as the feedback factor of the coupled circuits.

Such analysis of feedback in terms of a forward or A circuit and a backward or β circuit is useful only in particularly simple systems where the A and β circuits are distinguishable and independent entities. For a more general definition of feedback it is necessary to employ the mathematical definition used by Bode (1945, chap. iv). Feedback is there defined in the more general terms of a return difference F, corresponding to $(1 - A\beta)$, and a return ratio T, corresponding to the feedback factor $A\beta$. The return difference for any element w in a complete circuit is defined as the ratio of the values assumed by the circuit determinant when the specified element has its normal value and when the specified element vanishes, or

$$F_w = \frac{\Delta}{\Delta^0} \qquad (3\text{-}11)$$

To apply this definition to the equation of the coupled systems, the general Eqs. 3-4 will be re-written to correspond to the feedback diagram of Fig. 3-3, in which there is an applied e.m.f. from the generator E_1 at one pair of terminals but a load Z_L and no e.m.f. at the other terminals; these equations are

$$\begin{aligned} E_1 &= Z_{11} I_1 + Z_{12} I_2 \\ 0 &= Z_{21} I_1 + (Z_{22} + Z_L) I_2 \end{aligned} \qquad (3\text{-}12)$$

The return difference with reference to any element in the circuit can now be computed by using Eq. 3-11. In applying this definition to a coupling element, however, a distinction must first be made between bilateral and unilateral elements. Bode defines a bilateral element as one which is a constituent only of a self-impedance Z_{nn} and a unilateral element as one which is a constituent only of a mutual impedance Z_{mn}. This distinction is readily made for coupling elements such as the unilateral vacuum-tube considered by Bode. But in the usual coupled circuits composed of passive elements the coupling element in general appears in all the impedances, Z_{11}, Z_{12}, Z_{21}, and Z_{22}; an exception to this is a mutual inductance M as the coupling element. Many circuit elements which produce coupling are, therefore, neither unilateral nor bilateral as Bode defines the terms.

The problem that arises in finding the return difference with reference to a coupling element which is a component of both self and mutual impedances is that of defining the determinant Δ^o when the coupling element vanishes. A solution to this problem is to distinguish the simultaneous functions of such an element in providing the forward coupling, the feedback, and also energy storage or dissipation in the coupled systems. As a component of Z_{12} and Z_{21} the element may be regarded as unilateral, and as a component of Z_{11} and Z_{22} the same element may be regarded as bilateral. With this distinction, the return difference with reference to the forward coupling function of the element, through its appearance in Z_{21}, may be computed from Eq. 3-11 by considering Δ^o to be the determinant with this element zero in Z_{21} but not in Z_{11} or Z_{22}. This is the logical equivalent of opening the feed-forward link in Fig. 3-3 between Z_{21} and e_{21} without thereby altering the impedances of either system. The return difference with reference to the unilateral coupling impedance Z_{21} thus becomes

$$\underline{F}_{Z_{21}} = \frac{\Delta}{\Delta^o} = \frac{Z_{11}(Z_{22}+Z_L) - Z_{12}Z_{21}}{Z_{11}(Z_{22}+Z_L)}$$

$$= 1 - \underline{T} = 1 - \frac{Z_{12}Z_{21}}{Z_{11}(Z_{22}+Z_L)} \quad . \qquad (3\text{-}13)$$

Comparison of this with Eq. 3-10 shows that the return ratio $\underline{T}$ with reference to the coupling impedance is the same as the feedback factor for the coupled systems.

Coupling in terms of feedback can also be represented in a more general way visually as well as mathematically by using the new signal-flow diagrams of S. J. Mason (1953) instead of block diagrams. This method is very useful in giving graphic significance to Bode's general feedback theory, and it is well described by Truxal (1955, chap. ii). The fact that feedback can appear in the flow diagram of a simple ladder network is mentioned by Truxal (1955, p. 100), and his network is similar to that shown in Fig. 3-1 for the coupled systems. When the method of signal-flow analysis is applied to the coupled systems of Eqs. 3-12, a flow diagram can be drawn in the form shown in Fig. 3-4.

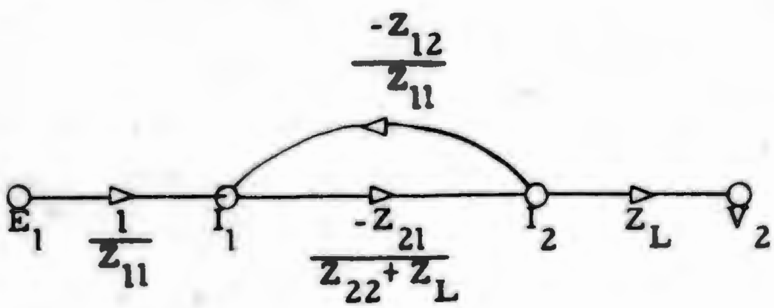


Fig. 3-4. Signal-flow diagrams of coupled systems showing feedback loop.

The feedback is here made evident in the flow diagram by the closed loop connecting I_1 and I_2, a result of the dependence of I_2 upon I_1, which in turn depends upon I_2.

Flow diagrams can also be used to define the return difference for any parameter in the circuit. To do this the diagram must be drawn so that the parameter appears in only one branch and as the transmittance of that branch. For the determination of the return difference with reference to Z_{21}, therefore, the diagram of Fig. 3-4 is not suitable; it may be redrawn, however, with Z_{21} as the transmittance of one branch, as shown in Fig. 3-5.

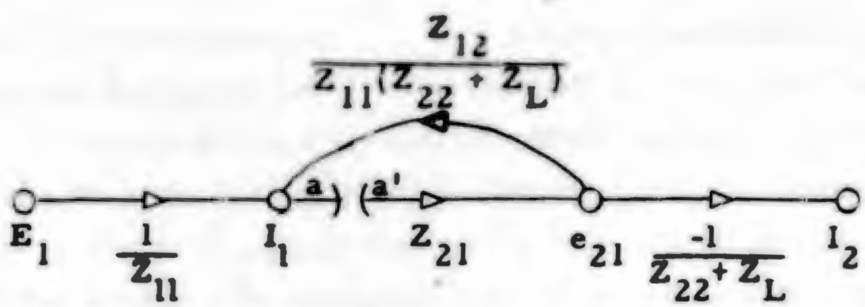


Fig. 3-5. Flow diagram for the definition of the return difference with reference to Z_{21}.

The return difference is then defined by breaking the branch containing Z_{21} at the points a and a' near its beginning. From a' a unit signal is then transmitted, and the signal returning to a is determined; all other inputs are assumed zero. The return difference is defined as the difference between the unit transmitted signal and the returned signal. In the diagram of Fig. 3-5 the returned signal is obviously the product of the transmissions, $[Z_{21}]\ [Z_{12}/Z_{11}(Z_{22}+Z_L)]$, and the return difference with reference to Z_{21} is

$$F_{Z_{21}} = 1 - \frac{Z_{12}Z_{21}}{Z_{11}(Z_{22}+Z_L)}, \qquad (3\text{-}14)$$

in agreement with Eq. 3-13 from Bode's definition.

It has been shown that the coupled systems can be viewed as a system with forward transmission from input to output and with feedback from output to input. The return difference with respect to the coupling element is a quantitative measure of this feedback. In expressing the characteristics of coupled systems, all of the effects of feedback on impedance, sensitivity, and stability analyzed by Bode and others can then be related to this return difference. Although the coupled systems have here been considered only in the form of the impedance Eqs. 3-4 because that form is the most familiar and leads also to the usual presentation of feedback in terms of voltages, it is equally possible to start with the other forms of Eqs. 3-5, 3-6, and 3-7 and to express the feedback in admittance or hybrid parameters with comparable results.

The Coupling Coefficients

The return difference with reference to Z_{21}, given by Eq. 3-14, has been shown to be a measure of the feedback in the coupled systems represented by Eqs. 3-4 and by Figs. 3-3 and 3-4. It will, however, be more convenient now to consider the return ratio, $\underline{T} = 1 - \underline{F}$, rather than the return difference $\underline{F}$, and for these systems it is

$$\underline{T}_{Z_{21}} = \frac{Z_{12} Z_{21}}{Z_{11}(Z_{22} + Z_L)} . \tag{3-15}$$

In general this $\underline{T}$ is a function of the circuit elements comprising the impedances Z_{11}, Z_{12}, Z_{21}, Z_{22} within the two terminal pairs of the system and of the frequency ω and the load impedance Z_L, which may be considered as variables external to the system. With the circuit elements regarded as fixed in value, the effects of the variables ω and Z_L upon the feedback measured by $\underline{T}_{Z_{21}}$ will be investigated.

To facilitate this investigation only the particular, but usual, case of reciprocal or symmetrical systems, in which $Z_{12} = Z_{21}$, will be considered at first; the effects of the antireciprocal relation $Z_{12} = -Z_{21}$ will be deferred. The systems can then be represented by the equivalent T network shown in Fig. 3-6, in which the impedances have been expressed as a series combination of R, L, and C. For this configuration of elements the return ratio can be expressed as an explicit function of ω and the elements R, L, C by the equation

$$\underline{T} = \frac{\left(R_{12} + j\omega L_{12} - j\frac{1}{\omega C_{12}}\right)^2}{\left(R_{11} + j\omega L_{11} - j\frac{1}{\omega C_{11}}\right)\left(R_{22} + j\omega L_{22} - j\frac{1}{\omega C_{22}} + R_L + j\omega L_L - j\frac{1}{\omega C_L}\right)} \tag{3-16}$$

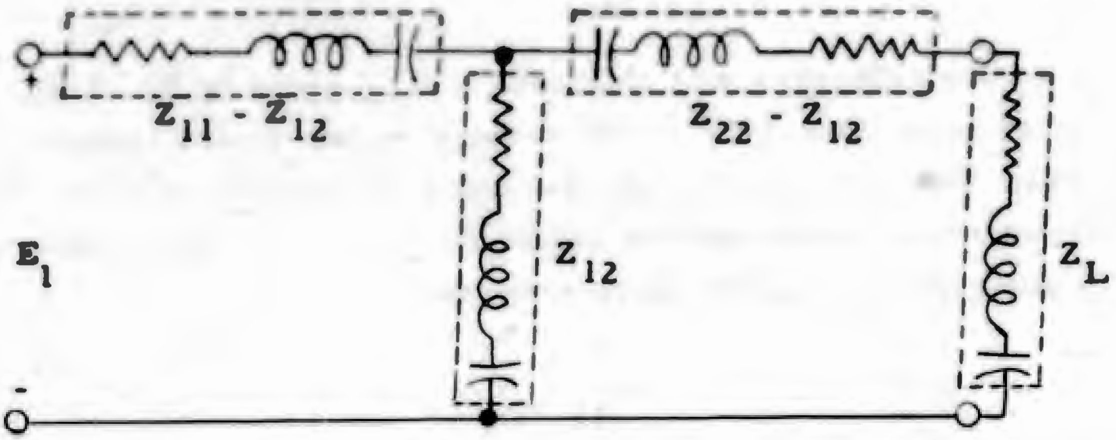


Fig. 3-6. Equivalent network of coupled systems with impedances expressed as series R, L, and C.

In general this complex T can be varied in both magnitude and phase by changes in ω and Z_L. Of particular interest, however, are the values of ω and Z_L which result in a feedback voltage e_{12} which is in phase with the input e_{11} and maximum in amplitude. For this maximum positive feedback the real part of T will be positive and maximum and the imaginary part zero. The values of ω and Z required for this can be found from Eq. 3-16 at the expense of considerable complex algebra. But they can be determined almost by inspection from the feedback diagram of Fig. 3-3. In this and in Eq. 3-10 the impedances appear as voltage dividers relating e_{21} to e_{11} and e_{12} to e_{21}. These voltages will be in phase only when the dividers consist of elements of the same type. When the impedances are series combinations of R, L, C this will be true as ω approaches infinity (the inductive reactances predominate), as ω approaches zero (the capacitive reactances predominate), and as ω approaches a resonance (the reactances vanish). The maximum feedback voltage, $e_{21} = Z_{21}/(Z_{22} + Z_L)$, will be obtained at all these frequencies only if Z_L is zero.

At the frequency limits of zero and infinity, the values of T which are real, positive, and maximum are found from Eq. 3-16 or from the voltage dividers to be

$$\lim_{\substack{\omega \to 0 \\ Z_L \to 0}} \underline{T} = \frac{C_{11} C_{22}}{C_{12}} \qquad (3\text{-}17)$$

$$\lim_{\substack{\omega \to \infty \\ Z_L \to 0}} \underline{T} = \frac{L_{12}^2}{L_{11} L_{22}} \qquad (3\text{-}18)$$

At a resonance frequency which makes all the reactances zero, i.e., $1/\omega_o^2 = L_{12}C_{12} = L_{11}C_{11} = L_{22}C_{22}$, the phase angle of both Z_{12}^2 and $Z_{11}Z_{22}$ is zero; in this case the coupling results from R_{12} alone, and the real, positive, maximum value of $\underline{T}$ is

$$\lim_{\substack{\omega \to \omega_o \\ Z_L \to 0}} \underline{T} = \frac{R_{12}^2}{R_{11} R_{22}} \qquad (3\text{-}19)$$

It is also possible for $\underline{T}$ to be real and positive at the frequencies which make the phase of Z_{12}^2 equal to that of $Z_{11}Z_{22}$ in Eq. 3-16 or the phase shifts in the dividers equal but opposite in sign. An example of this will be considered in the examination of resistive coupling in Chapter IV.

The maxima of positive feedback with respect to frequency and load measured by the maximum positive real values of $\underline{T}$ in Eqs. 3-17, 3-18, 3-19 may be regarded as a measure of the possible inherent coupling of the systems, that is, of the synchronous interaction of the systems determined solely by the elements regarded as internal constants of the coupled systems. For the series configuration of elements in Fig. 3-5 there are three values of this return ratio, corresponding to capacitive, inductive, and resistive coupling elements. In many coupled systems of practical interest only one such coupling element is present, and only one such ratio is needed to measure the coupling. Possible configurations of elements more complicated than the series R, L, C will not be examined here, but in such cases it should be possible to determine values of the maximum positive feedback by the same method.

It has been shown that the coupling of two systems may be defined as the transmission of energy forward from a driving system to a driven system and a simultaneous return or feedback of energy from the driven to the driver.

The feedback or return ratio is a measure of the loop transmission through the forward coupling and the feedback path and is therefore a quantitative measure of the mutual interaction of the systems. This ratio depends in general upon the internal constants and configuration of the systems and upon the external variables of load and driving frequency. The frequency and load can be varied to make the feedback positive and maximum; hence the return ratio real, positive, and maximum; and these values of the return ratio can be used as a measure of the inherent coupling of the systems. This leads to the following definition of the coupling coefficients in terms of feedback:

> The values of the return ratio which are real, positive and maximum under any variation of frequency and load are defined as the squares of the coupling coefficients of the coupled systems and denoted by the usual k^2.

This may be summarized in the following manner:
The equations of the coupled systems are:

$$\begin{aligned} E_1 &= Z_{11} I_1 + Z_{12} I_2 \\ 0 &= Z_{12} I_1 + (Z_{22} + Z_L) I_2 \end{aligned} \qquad (3\text{-}12)$$

The return ratio is:

$$\underline{T} = \underline{T}_r + j\underline{T}_i = \frac{Z_{12}^2}{Z_{11}(Z_{22} + Z_L)} \qquad (3\text{-}15)$$

The coupling coefficients are:

in general

$$k^2 = \underline{T}_{Z_{21}} \quad \text{for} \quad \begin{matrix} \omega \text{ such that } \underline{T}_i = 0 \\ Z_L \text{ such that } \underline{T}_r = \text{max.} \end{matrix} \qquad (3\text{-}20)$$

for capacitive coupling

$$k_C^2 = \lim_{\substack{\omega \to 0 \\ Z_L \to 0}} \underline{T} = \frac{C_{11} C_{22}}{C_{12}^2} \qquad (3\text{-}17)$$

for inductive coupling

$$k_L^2 = \lim_{\substack{\omega \to \infty \\ Z_L \to 0}} \underline{T} = \frac{L_{12}^2}{L_{11}L_{22}} \qquad (3\text{-}18)$$

for resistive coupling

$$k_R^2 = \lim_{\substack{\omega \to \omega_o \\ Z_L \to 0}} \underline{T} = \frac{R_{12}^2}{R_{11}R_{22}} \qquad (3\text{-}19)$$

The Consequences

The identification of the square of the coupling coefficient with a return ratio permits the use of all the accumulated information about feedback in the interpretation and application of the coefficient. It has already been shown that this feedback viewpoint gives phvsical significance to the concept of coupling and a quantitative measure in the form of the return ratio. The definition of the coupling coefficient as the return ratio when frequency and load make the feedback positive real and maximum also gives physical significance to the coefficient and at the same time justifies the appearance in the familiar definitions of the shorted secondary terminals and of only elements of the same type as that of the coupling. The relation to the loop transmission through the forward coupling and the feedback explains, too, the appearance in the definition of the product of two ratios, such as $k^2 = (L_{12}/L_{11})(L_{12}/_{22})$, as the two voltage divisions in the forward transmission and the feedback and justifies the designation of the return ratio as the square of k rather than as k.

As a return ratio the coupling coefficient should be related to all the familiar effects of feedback, such as changes in input impedance and gain and the production of oscillations or instability. These relations are most easily found in the many theorems on feedback in terms of the return difference in Bode (1945, chaps. v and vi). Of these theorems the two relating to impedance and gain are most important.

The first theorem concerns the effect which feedback may have upon the impedance or admittance measured between any two points of the circuit. This theorem, formulated by Blackman (1943), is stated by Truxal (1955, p. 131) in admittance form as the equation

$$\frac{Y}{(Y)_{n=0}} = \frac{\underline{F}_n' \text{ (with terminals open)}}{\underline{F}_n \text{ (with terminals shorted)}} \quad . \tag{3-21}$$

The admittance Y is that seen looking into any pair of terminals of the system with each source replaced by its internal impedance; $(Y)_{n=0}$ is the same admittance when a specified element $\underline{n}$ of the system is made zero. $\underline{F}_n$ is the return difference with reference to $\underline{n}$, and $\underline{F}_n'$ is the null return difference or the return difference evaluated under the condition that the input is adjusted to give zero output. Since the current into the terminals is to be regarded as the output and the voltage between the terminals as the input, $\underline{F}_n'$ is the return difference with reference to n with the terminals open and $\underline{F}_n$ with the terminals shorted.

In applying this theorem to find the input admittance of the coupled circuits a signal-flow diagram relating the input current to the input voltage is of value. This diagram is shown in Fig. 3-7, with the notation of Fig. 3-3 and with the use of the obvious but not trivial relations, $E_{in} = E_1$ and $I_{in} = I_1$. The return difference with reference to the coupling impedance Z_{21} as the element $\underline{n}$ can be obtained from this diagram by the method used in

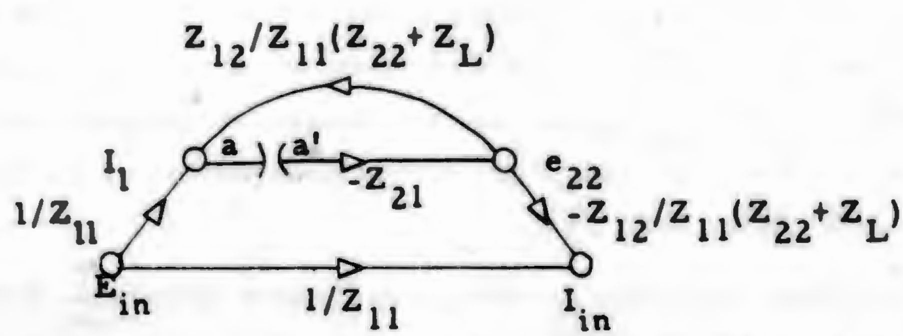


Fig. 3-7. Signal-flow diagram for determination of input impedance.

connection with the flow diagram of Fig. 3-4. For a unit signal at $\underline{a}'$ the returned signal at $\underline{a}$ with no input E_{in} is $Z_{21}Z_{12}/Z_{11}(Z_{22} + Z_L)$ and the return difference is thus $\underline{F} = 1 - Z_{21}Z_{12}/Z_{11}(Z_{22} + Z_L)$. To determine $\underline{F}'$, the unit signal is injected at $\underline{a}'$ together with an input E_{in} such that the resulting output flow at I_{in} is zero; the required input is $-Z_{12}Z_{21}/(Z_{22}+Z_L)$ times the unit signal at $\underline{a}'$. The resulting return at $\underline{a}$ from the two input signals is zero, and the null return difference is $\underline{F}' = 1$. Thus Eq. 3-21 becomes the admittance ratio

$$\frac{Y}{(Y)_{Z_{21}=0}} = \frac{1}{1 - Z_{12}Z_{21}/Z_{11}(Z_{22}+Z_L)}, \tag{3-22}$$

or, since $(Y)_{Z_{21}=0} = 1/Z_{11}$, the input impedance with coupling is related to that without coupling by

$$Z_{in} = Z_{11}\left[1 - \frac{Z_{12}Z_{21}}{Z_{11}(Z_{22}+Z_L)}\right] = Z_{11}(1 - \underline{T}_{Z_{21}}). \tag{3-23}$$

The definition of the coupling coefficient in terms of $\underline{T}_{Z_{21}}$ then makes it possible to use Eq. 3-23 to relate this coefficient to impedances at the input terminals. The coefficient is defined with the load impedance zero and with the frequency or circuit constituents such that only elements of the same type appear in the impedances. Under these conditions the input impedance with coupling Z_{in}^c and that without coupling Z_{in}^o can be related to k^2 through Eqs. 3-23 and 3-20 by the equation

$$Z_{in}^c = Z_{in}^o(1 - k^2). \tag{3-24}$$

Since the definition of k^2 requires that the load impedance be zero, the input impedance with coupling is that with the output terminals shorted. The input impedance without coupling, Z_{11}, is by definition that with $I_2 = 0$, and hence it is the input impedance with the output terminals open. The coupling coefficient can then be expressed in terms of these open- and short-circuit impedances as

$$k^2 = \frac{Z_{in}(\text{open}) - Z_{in}(\text{short})}{Z_{in}(\text{open})}, \tag{3-25}$$

in which the impedances are either all resistances or all reactances of the same sign.

The second general feedback theorem relates the gain of part of a circuit to a specific return difference. The equation for this theorem appears in Truxal (1955, p. 142) as

$$\frac{T_g}{T_{g_o}} = \frac{1}{F_{g_m}} \qquad (3\text{-}26)$$

In this T_g is the transmission from any point in the system to the grid of a specific tube; T_{g_o} is the same transmission with the tube dead or μ or g_m equal to zero; and F_{g_m} is the return difference with reference to the g_m of the tube. In applying this theorem to the coupled circuits the previous arbitrary, but useful, distinction of Z_{21} as the coupling impedance and Z_{12} as the feedback impedance will be made; and a comparison of Eqs. 3-12 of the coupled systems to Eqs. 3-8 of a feedback amplifier then reveals that Z_{21} occupies a position equivalent to that of the μ or g_m of the tube. The equivalent of the grid voltage is again the voltage e_{11} of Fig. 3-3.

The relation of the voltage e_{11} to the coupling coefficient can then be found through the theorem expressed by Eq. 3-26 and the definition of the coefficient as $F_{Z_{21}} = 1 - k^2$. The equivalent grid voltage e_{11} is thus found to be related to the input signal E_1 by

$$e_{11} = E_1/(1 - k^2), \qquad (3\text{-}27)$$

since $e_{11} = E_1$ when $Z_{21} = 0$.

The relation of gain to feedback is commonly made through the feedback factor and the equation $A_f = A_o/(1 - A_o\beta)$, where A_f is the gain with feedback and A_o is the gain without feedback. The use of $(1 - k^2)$ as the equivalent of $(1 - A\beta)$ for coupled systems suggests that a similar relation can be found for the gain in such systems. Since the output under the short-circuit condition for which k^2 is defined is the current I_2, the gain needs to be defined as the transfer admittance $T = I_2/E_1$. The gain without feedback is that with $Z_{12} = 0$ and is found from Eq. 3-12 with $Z_L = 0$ to be $T_o = -Z_{21}/Z_{11}Z_{22}$. With feedback the gain is $T_f = (Z_{12} - Z_{11}Z_{22}/Z_{21})^{-1}$, and the ratio of these gains or transmissions is

$$\frac{T_f}{T_o} = \frac{1}{1 - Z_{12}Z_{21}/Z_{11}Z_{22}} . \qquad (3\text{-}28)$$

Under the conditions for which k^2 is defined, this can be written

$$T_f = T_o / (1 - k^2). \qquad (3\text{-}29)$$

This relation appears to be a consequence of the theorem of Eq. 3-26, but the subtle distinction between bilateral and unilateral elements and between Z_{12} and Z_{21} must be considered here. The theorem involves a return difference with reference to a unilateral element such as the g_m of a tube and is generally applicable to the gain of only parts of the circuit, e.g., the transmission from any point to the grid. ... the relation of gain to feedback above, the transfer admittance was found to depend similarly upon a return difference, but in this case the reference was the feedback impedance Z_{12}, which was set equal to zero, rather than the coupling Z_{21}. The distinction is difficult, since in the coupled circuits the two are identical and the return ratio is the same for either. In the gain Eq. 3-28 the reference element Z_{12} has, in effect, been considered bilateral, and a third theorem of Bode (1945, p. 78), equivalent to Eq. 3-26 but for bilateral elements, is then applicable.

Another application of feedback theory to the interpretation of the coupling coefficient can be found in the relation of feedback to stability. The condition for the stability of feedback systems stated by Nyquist requires that the plot of the complex feedback factor $A\beta$ shall not enclose the point $(1 + j0)$. In less general terms this requirement can be derived from a gain equation such as Eq. 3-28, from a relation of feedback to input voltages such as Eq. 3-10, or from input impedance relations such as Eq. 3-23. In such equations it is evident that a return ratio or feedback factor which exceeds +1 implies a gain of infinity, a positive feedback voltage which exceeds the input, or a negative input impedance--all of which meansthe system is potentially unstable. As a return ratio, k^2 may be considered to be the ratio of the maximum energy returned to the source to the energy entering the system, so that when $k^2 = +1$ the system can supply its own input and oscillate. Thus, in systems which can be unstable, the coupling coefficient is given physical significance as a measure of the proximity of the coupled systems to the point of instability.

In passive systems to which the coupling coefficient is usually applied there is no possibility of such instability, since there are no internal sources of energy. The coupling coefficient of +1 still has significance, however, as the limit of physical realizability. As is shown by Bode (1945, chap. vii) and Hunt (1954, p. 94), the impedances in the Eqs. 3-4 of the coupled systems represent a physically realizable system if, and only if, the self-impedances are positive quantities and the three ratios of circuit elements defined as the coupling coefficients in Eqs. 3-17, 3-18, 3-19 are less than or equal to +1. This is also the condition for positive definiteness of the energy function mentioned by Guillemin (1953, p. 380). The definition of the coefficient as a return ratio makes this limit evident in that energy conservation requires that the maximum positive feedback voltage e_{12} cannot exceed the input e_{11}; hence in passive systems, with e_{12} maximum and in phase with e_{11},

$$e_{12}/e_{11} = \underline{T}_{Z_{21}} = k^2 \leq 1. \qquad (3\text{-}30)$$

The same physical limit may be found in the relation of e_{11} to E_1 in Eq. 3-27, of Z^c_{in} to Z^o_{in} in Eq. 3-24, and of the transfer admittances T_f to T_o in Eq. 3-29. The coupling coefficient is thus given physical significance in passive systems as a measure of the approach of the system to the limit of physical realizability.

Other physical significances for the coupling coefficient may also be found from its definition as a positive real ratio of the voltages e_{12} to e_{11} at the input of the coupled systems when the output is shorted. If, in the feedback diagram of Fig. 3-3, an input current I_1 is applied to the terminals of system 1 and the terminals of system 2 are shorted, then a positive real ratio of e_{12} to e_{11} means a power $e_{11}I_1$ entering the system and a power $e_{12}I_1$ being returned to the source. This suggests that the coupling coefficient, as the ratio of these powers, may be regarded as a ratio of reflected to incident energy at the input terminals when the output is terminated with a perfect reflector. A coupling coefficient of unity thus means perfect reflection at the input terminals; a coupling coefficient less than unity implies that some of the energy entering the coupled systems has been scattered during its transmission from the input to the perfect reflector at the output and then back again to the input terminals.

To further the investigation of the relation of such reflection and scattering to the coupling coefficient, the recent developments in the use of the scattering matrix as a powerful analytical tool in network theory may be utilized. The summary by Carlin (1956) of the characteristics of this matrix and its applications is a good reference for the theory needed in this investigation. In the scattering treatment of networks the variables at the network terminals are termed the reflected and incident voltages, v_r and v_i, and are defined as the linear combinations of the ordinary voltages and currents, V and I, at the terminals given by

$$V = \sqrt{r_o}\,(v_i + v_r) \tag{3-31a}$$

$$I = (1/\sqrt{r_o})(v_i - v_r) \ . \tag{3-31b}$$

In these equations r_o is an arbitrary normalizing number, the value of which is usually determined by convenience in the particular problem being considered. If a normalized voltage and current are defined as $v = V/\sqrt{r_o}$ and $i = I\sqrt{r_o}$, then the incident and reflected voltages are related to these by

$$v_i = \frac{1}{2}(v + i)$$
$$v_r = \frac{1}{2}(v - i). \tag{3-32}$$

It should be noted that v_i and v_r do not have the dimensions of true voltages, but are so defined that $1/2\ v_i v_i^*$, where v_i^* is the complex conjugate of v_i, is the average incident power and $\frac{1}{2} v_r v_r^*$ is the average reflected power.

The reflection factor or scattering coefficient at a terminal pair or port in the system is defined as

$$s = \frac{v_r}{v_i} = \frac{v - i}{v + i} = \frac{z - 1}{z + 1}, \tag{3-33}$$

where z is the normalized input impedance at the terminals and is related to the input impedance Z and to s by

$$z = \frac{v}{i} = \frac{V}{r_o I} = \frac{Z}{r_o} = \frac{1 + s}{1 - s} \tag{3-34}$$

For networks with multiple terminal pairs or ports, the incident and reflected voltages are related by the matrices

$$[v_r] = [S]\,[v_i]\,, \qquad (3\text{-}35)$$

where [S] is the matrix of the scattering parameters. It can be expressed in terms of the matrix [Z] of the normalized impedances z of the network by

$$[S] = [Z - I]\,[Z + I]^{-1}, \qquad (3\text{-}36)$$

where I is the unit matrix and the exponent -1 indicates the matrix inverse.

With these definitions and relations of the scattering variables in hand, the conditions at the input terminals of Fig. 3-3 can now be expressed in terms of incident and reflected voltages. From Eq. 3-34 the input impedance is related to the scattering coefficient at the input terminals, s_{in}, by

$$Z_{in} = r_o \frac{1 + s_{in}}{1 - s_{in}}\,. \qquad (3\text{-}37)$$

This input impedance is also expressed in Eq. 3-24 as $(1 - k^2)$ times the input impedance without coupling, $Z_{in}^{\,o} = Z_{11}$, when the output terminals are shorted. It is convenient, therefore, to define the normalizing factor r_o as Z_{11}, the input impedance without coupling or with the output terminals open. With $r_o = Z_{11}$ and with the output terminals shorted, the input impedance from Eqs. 3-24 and 3-37 is

$$Z_{in}' = Z_{11}\frac{1 + s_{in}'}{1 - s_{in}'} = Z_{11}(1 - k^2), \qquad (3\text{-}38)$$

so that

$$k^2 = 2s_{in}'/(1 - s_{in}'). \qquad (3\text{-}39)$$

The prime notation on Z_{in} and s_{in} here indicates that these quantities refer to input conditions with the output terminals shorted.

The input scattering coefficient s_{in} can be determined from the scattering matrix of the network by using the results of Carlin (1956,p. 92) for the ratio of reflected to incident voltages at the input of a 2-port terminated in a load of reflection factor s_2:

$$s_{in} = s_{11} + s_{12}s_{21}s_2/(1 - s_2 s_{22}). \qquad (3\text{-}40)$$

When the output termination is a short circuit, $Z_L = 0$, the reflection factor at the output is found from Eq. 3-34 to be $s_2 = -1$, the negative sign being the result of the reflected energy going into the output terminals. The input reflection factor for this short-circuit termination is then given by Eq. 3-40 as

$$s_{in}' = s_{11} - s_{12}s_{21}/(1 + s_{22}), \tag{3-41}$$

and this can be substituted into Eq. 3-39 to express k^2 in terms of the scattering parameters as

$$k^2 = \frac{-2[s_{11}(1 + s_{22}) - s_{12}s_{21}]}{(1 - s_{11})(1 + s_{22}) + s_{12}s_{21}} . \tag{3-42}$$

The significance of the relation of the coupling coefficient to the scattering coefficient is, however, more evident in Eq. 3-39 than in Eq. 3-42. As a result of the choice of Z_{11} as the normalizing impedance, when the input is driven by a source of internal impedance $r_o = Z_{11}$ the reflection at the input terminals will be zero when the output terminals are open. With this same source, when the output terminals are shorted the reflection at the input is related to k by Eq. 3-39. When $k^2 = 0$ the reflection factor $s_{in}' = 0$, and there is no reflection at the input even though there is a perfect reflector at the output terminals. The network with zero coupling may thus be considered to scatter all the incident energy during its transmission from input to the reflector at the output and back to the input. When $k^2 = 1$, on the other hand, the input reflection factor $s_{in}' = -1$, and all the incident energy is reflected. The network with unity coupling has, therefore, perfect transmission of the incident energy from input to the reflector at the output and back to the input without scattering. The coupling coefficient may thus be considered a measure of the scattering by the network when there is energy incident at the input and perfect reflection at the output terminals.

IV

APPLICATIONS OF COUPLING COEFFICIENTS

The definition of the coefficient of coupling in Eq. 3-20 as a maximum positive real return ratio with reference to the coupling element should be applicable to all systems -- mechanical, electrical, or electromechanical -- which can be represented by equations of the form of Eqs. 3-4 and by the T network shown in Fig. 3-1. For the three types of coupling element corresponding to energy storage in an electric field, energy storage in a magnetic field, and energy dissipation, the feedback definition was shown to lead to definitions in terms of the circuit elements given for capacitive coupling by Eq. 3-17, for inductive coupling by Eq. 3-18, and for resistive coupling by Eq. 3-19. The systems represented by these equations and these coefficients are, however, symmetrical or reciprocal systems, in which the mutual impedances are equal in sign and magnitude, i.e., $Z_{12} = Z_{21}$.

As has been mentioned before, it is also possible to have coupled systems in which $Z_{12} = -Z_{21}$. Such coupling, termed antisymmetrical or antireciprocal, occurs in mechanical systems when the coupling element is a gyroscope and in electromechanical systems when the coupling results from the interaction of a current and a magnetic field. The coupling element in such systems neither stores nor dissipates energy, hence can be considered as neither capacitance, inductance, nor resistance. Because of this and the antisymmetry of the equations, special consideration must be given to the application of the coupling coefficient to these systems.

The generality of the definition of the coupling coefficient will be demonstrated by its application to examples of the several types of coupled systems. It will first be applied to simple symmetrical systems to illustrate the important characteristics of the coefficient and its relation to the system parameters. For this purpose electrical circuits with inductive and resistive coupling elements and the electrostatic and piezoelectric transducers, with electromechanical coupling through an equivalent capacitance, will be considered. It will then be shown that when antisymmetrical systems, exemplified by the moving-coil, moving-armature, and magnetostriction transducers, are represented in symmetrical form, in general definition can be applied to these systems with no change in the physical significance of the coefficient.

Reciprocal Systems

Inductive Coupling.

As an example of coupled systems with symmetrical or reciprocal coupling a simple electrical system will be considered first. This is the two-mesh network of three inductances shown in Fig. 4-1. The coupling originates in

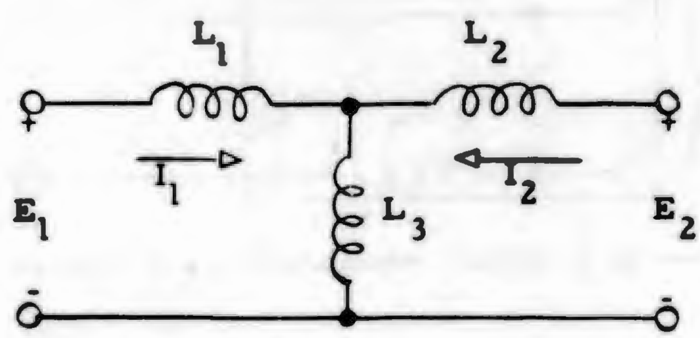


Fig. 4-1. Electrical network with inductive coupling.

in this case in the inductance L_3 common to the two meshes of the network. If an e.m.f. E_1 is applied to circuit 1, the flow of current I_1 produces a voltage $e_{21} = j\omega L_3 I_1$ in circuit 2, and the resulting flow of current I_2 produces a feedback voltage $e_{12} = j\omega L_3 I_2$ in circuit 1. Since this network is a particular case of the general Eqs. 3-12 and Fig. 3-5, the coupling coefficient is given by Eq. 3-18 for inductive coupling as

$$k^2 \equiv \frac{L_{12}^2}{L_{11}L_{22}} = \frac{L_3^2}{(L_1 + L_3)(L_2 + L_3)} . \qquad (4-1)$$

It is worthy of note that here, as in most systems, the coupling element L_3 appears as a component of $L_{11}(=L_1 + L_3)$ and $L_{22}(=L_2 + L_3)$.

Although the value of k^2 can be thus obtained from Eq. 3-18, it will be useful in this simple example to obtain the coefficient directly from the general definition in terms of feedback. The representation of the network of Fig. 4-1 as a feedback system in the form of Fig. 3-3 is shown in Fig. 4-2. In this form it is evident by inspection of the figure that the return ratio is

$$\frac{e_{12}}{e_{11}} = \frac{j\omega L_3}{j\omega(L_1 + L_3)} \times \frac{j\omega L_3}{j\omega(L_2 + L_3) + Z_L} . \tag{4-2}$$

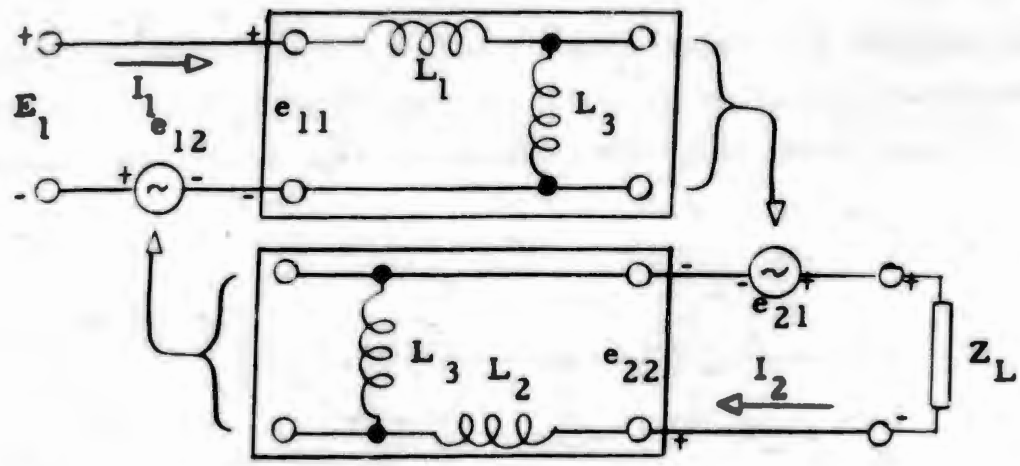


Fig. 4-2. The feedback system equivalent to the circuit of Fig. 4-1.

This ratio will be positive, real, and maximum with respect to variations in ω and Z_L for any ω and for $Z_L = 0$. Hence, the coupling coefficient is again that given by Eq. 4-1. It is also evident from Eq. 4-2 that k^2 cannot exceed +1 for any positive finite values of the inductances and that it approaches this value only as L_3 approaches infinity or as L_1 and L_2 approach zero.

The coupling coefficient has been determined here from the equations of the coupled systems in impedance form, that is, from the equations of motion in the form of Eqs. 3-4, or 3-12. As was mentioned in the discussion of Eqs. 3-4, there are other possible forms of these equations of motion. The circuit of Fig. 4-1 could also be represented by equations in admittance form, corresponding to Eqs. 3-5. In this case the feedback could be represented by current generators instead of voltage generators. Consideration of the positive feedback would lead again to the definition of the coupling coefficient obtained from the impedance equations, but this will not be demonstrated here. A consideration of the equations with hybrid parameters, as represented by

Eqs. 3-6 and 3-7, is worth some attention, however, because of the differences resulting from these equations.

The impedance equations for the circuit of Fig. 4-1 are

$$\begin{aligned} E_1 &= j\omega(L_1+L_3)\,I_1 + j\omega L_3\,I_2 \\ E_2 &= j\omega L_3\,I_1 + j\omega(L_2+L_3)\,I_2. \end{aligned} \tag{4-3}$$

When these are solved for E_1 and I_2 as dependent variables, the resulting hybrid equations, similar to Eqs. 3-7, are

$$\begin{aligned} E_1 &= j\omega[\,(L+L_3)-L_3^2/(L_2+L_3)]\,I_1 + [\,L_3/(L_2+L_3)]\,E_2 \\ I_2 &= [\,-L_3/(L_2+L_3)]\,I_1 + [\,1/j\omega(L_2+L_3)]\,E_2 \end{aligned} \tag{4-4}$$

The notable characteristics of these hybrid equations are that the matrix of hybrid parameters is antisymmetrical, that is, $h_{21} = L_3/(L_2+L_3) = -h_{12}$, and that this mutual parameter is a dimensionless ratio instead of an impedance or an admittance.

The antisymmetry of the mutual parameters appears in the feedback equivalent of the hybrid equations as a difference in the sign of the feedback voltage e_{12}. In these variables there is no feedback for $E_2 = 0$, but for $I_2 = 0$ the feedback voltage is $e_{12} = +[\,j\omega L_3^2/(L_2+L_3)]\,I_1$. In the corresponding symmetric Eqs. 4-3, however, the feedback voltage is $e_{12} = -[\,j\omega L_3^2/(L_2+L_3)]\,I_1$; and the negative sign here indicates that the source voltage E_1 is e_{11} minus e_{12}, as is shown in the feedback diagram of Fig. 4-2. Consequently, the positive e_{12} resulting from the antisymmetric equations signifies that E_1 is e_{11} plus e_{12}, so that the voltage e_{11} is less than E_1 when feedback is present; such feedback is negative. The return ratio with reference to this hybrid coupling parameter cannot, therefore, be a coupling coefficient, since the coefficient is defined for positive feedback. For this negative feedback, however, the return ratio can be related to the k^2 defined by Eq. 4-1 by

$$\frac{e_{12}}{e_{11}} = \frac{L_3^2}{(L_1 + L_3)(L_2 + L_3) - L_3^2} = \frac{k^2}{1 - k^2} \tag{4-5}$$

The appearance of the mutual parameter as a dimensionless ratio relating a voltage in circuit 1 to E_2 and a current in circuit 2 to I_1 suggests the use of the ideal transformer with a turns ratio $N = L_3/(L_2+L_3)$ to represent this coupling of the circuits. It is easy to verify that the network shown in Fig. 4-3 (a) using such a transformer is, indeed, equivalent to the hybrid Eqs. 4-4. The other set of hybrid equations, having E_2 and I_1 as dependent variables, has the similar equivalent shown in Fig. 4-3 (b).

These equivalent circuits with transformers lead to different interpretations of the coupling coefficient, which was defined for these systems by Eq. 4-1. In Fig. 4-3 (a),

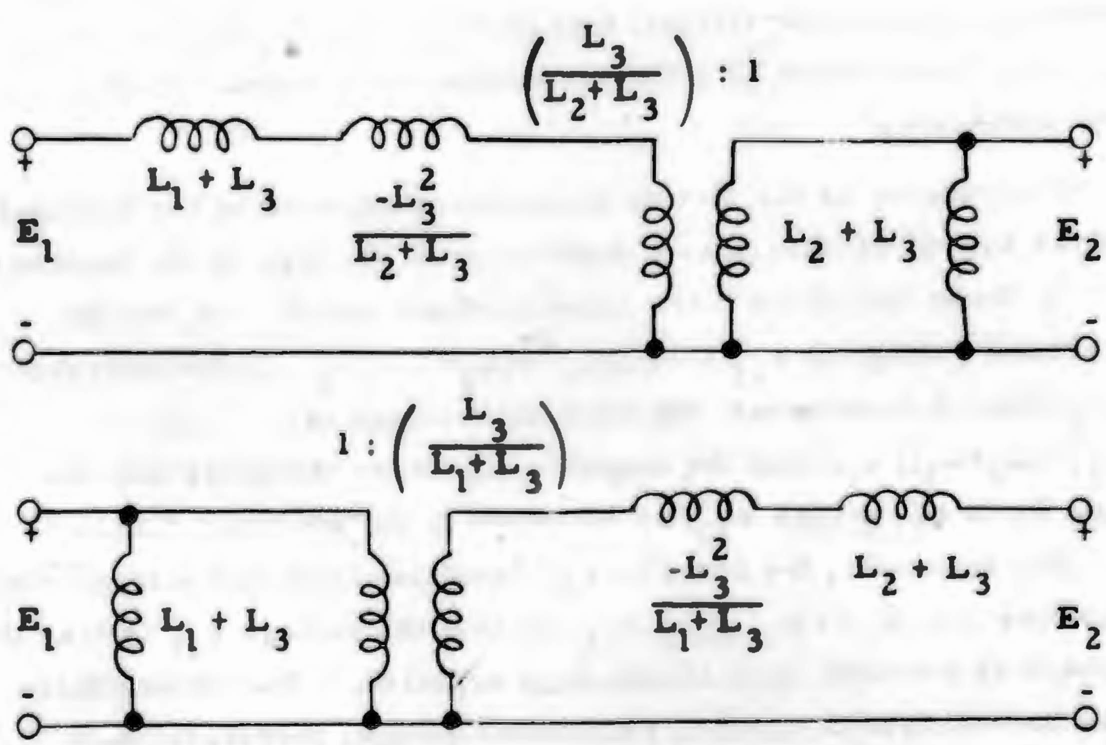


Fig. 4-3. Equivalent circuits from hybrid equations.

when the terminals of circuit 2 are shorted and an input is applied at the terminals of circuit 1, the input energy can be considered to be stored in the inductances $(L_1 + L_3)$ and $+L_3^2/(L_2 + L_3)$. The inductance $(L_1 + L_3)$ may here be considered as the "primary," since it is the input inductance with the terminals of circuit 2 open. The inductance $(L_2 + L_3)$ may similarly be regarded as the "secondary." When this inductance is moved through the transformer to the primary side, it becomes $L_3^2/(L_2 + L_3)$; hence the inductance $-L_3^2/(L_2 + L_3)$ may be regarded as the equivalent of the secondary inductance in the primary circuit, the negative sign indicating that the input inductance and stored energy is reduced by shorting the secondary terminals. Thus, if the input is provided by a constant-current source of magnitude I_1, the coupling coefficient can be expressed in terms of stored energies as

$$k^2 = \frac{L_3^2}{(L_1+L_3)(L_2+L_3)} = \frac{\frac{1}{2}\, L_3^2/(L_2 + L_3)\, I_1^2}{\frac{1}{2}(L_1 + L_3)\; I_1^2} \tag{4-6}$$

$$= \frac{\text{energy stored in the secondary}}{\text{energy stored in the primary}}$$

In the equivalent circuit of Fig. 4-3(b), however, the primary may again be considered to be $(L_1 + L_3) = L_p$, but the form of this circuit suggests that the "secondary" here be defined as $[(L_2 + L_3) - L_3^2/(L_1 + L_3)]$. When this secondary inductance is moved through the transformer to the primary side after the terminals of circuit 2 have been shorted it becomes a shunt inductance $L_s = [(L_2 + L_3) - L_3^2/(L_1 + L_3)]\,(L_1 + L_3)^2/L_3^2$ in parallel with L_p. With this interpretation of the secondary circuit and with the input provided by a constant-voltage generator of magnitude E_1, the coupling coefficient may be manipulated into another ratio of energies,

$$k^2 = \frac{L_3^2}{(L_1+L_3)(L_2+L_3)} = \frac{L_p}{L_s + L_p} = \frac{\frac{1}{2}\, 1/L_s\;\; E_1^2}{\frac{1}{2}\left(\frac{L_p L_s}{L_p+L_s}\right)^{-1} E_1^2}$$

$$= \frac{\text{energy stored in the secondary}}{\text{total stored energy}} \tag{4-7}$$

The difference in the two expressions of k^2 in terms of stored energies is, of course, in the different definitions of the "secondary." In Eq. 4-6 from Fig. 4-3(a), the secondary was defined as $(L_2 + L_3)$, which is the inductance seen from the terminals of circuit 2 when the terminals of circuit 1 are open; in this case the input was appropriately provided by a constant-current generator, which has infinite internal impedance. In Eq. 4-7 from Fig. 4-3(b), on the other hand, the secondary was defined as $[(L_2 + L_3) - L_3^2/(L_1 + L_3)]$, which is the inductance seen from the terminals of circuit 2 with the terminals of circuit 1 shorted; the input was here provided by a constant-voltage generator, which has zero internal impedance.

The form of the energy ratio expression of k^2 is seen, thus, to depend upon which of the input variables, E or I, is taken as independent of the coupling. If the current I is assumed constant, the series impedance form of the equivalent input circuit, Fig. 4-3(a), is appropriate, and the secondary is defined as the impedance viewed from the output terminals with the input open. If the voltage E is assumed constant, the shunt impedance form of the input circuit Fig. 4-3(b), is appropriate, and the secondary is defined as the impedance at the output terminals with the input shorted. The choice of the variable is theoretically arbitrary, but it is usually more convenient to use constant voltage for circuits with capacitive coupling and constant current for inductive coupling.

The general relation of the coupling coefficient to input impedances with and without the feedback measured by k^2 has been derived from a general feedback theorem and expressed in Eq. 3-23. In this simple example of inductive coupling it is easy to compute the input inductance at the terminals of mesh 1 with the terminals of mesh 2 shorted and, with the value of k^2 from Eq. 4-1, to show that this input inductance is related to k^2 by the equation

$$L_{in} = L_1 + \frac{L_2 L_3}{L_2 + L_3} = (L_1 + L_3)\left[1 - \frac{L_3^2}{(L_2 + L_3)(L_1 + L_3)}\right]$$

$$= (L_1 + L_3)(1 - k^2) \qquad (4\text{-}8)$$

Since $(L_2 + L_3)$ is the input inductance with the terminals of mesh 2 open, and hence with no feedback or coupling, this confirms the general relation expressed in Eq. 3-23: The input impedance with maximum positive feedback, hence with elements of the same type and with the load a short circuit, is $(1 - k^2)$ times the input impedance with no feedback or with the terminals open.

The relation of input impedance to k^2 for another particular termination than open or short circuit is also of interest and can be easily determined in this simple example of coupling. In many of the practical applications made of coupled systems one or both of the systems is resonant or tuned. As an example of this, consider, therefore, a capacitance C_2 across the terminals of mesh 2, as shown in Fig. 4-4. The impedance at the terminals of mesh 1 will then vary with frequency in a manner determined by the equation

$$Z_{in} = j\omega L_1 + j \frac{\omega L_3(\omega L_2 - 1/\omega C_2)}{\omega(L_2 + L_3) - 1/\omega C_2} . \qquad (4\text{-}9)$$

From this it is obvious that the input impedance becomes infinite at a frequency ω_A which makes the denominator zero, and this antiresonance frequency is

$$\omega_A^2 = 1/(L_2 + L_3)C_2 = \omega_o^2 , \qquad (4\text{-}10)$$

where ω_o is defined as the resonance frequency of the impedance Z_{22} in the impedance equations of the coupled systems.

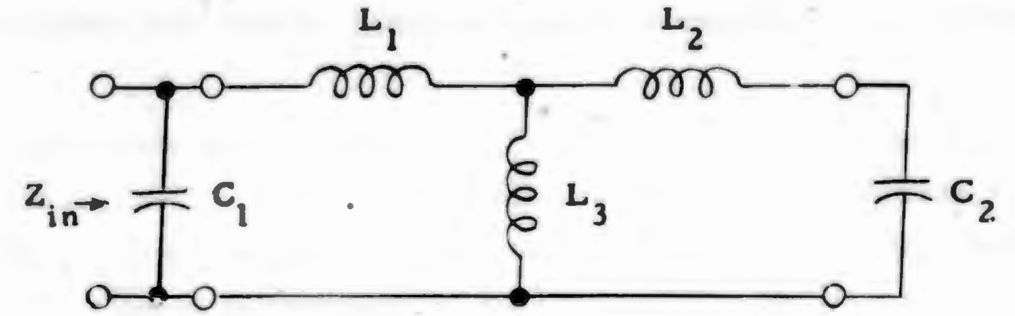
There is also a frequency at which the input impedance is zero, and this resonance frequency ω_R can be found from Eq. 4-9 by equating the first term to the negative of the second. It can also be derived almost by inspection from the equivalent circuit of Fig. 4-3(b) as the frequency of resonance of the series combination of C_2 and the inductances $L_2+L_3-L_3^2/(L_1+L_3)$. However, it will be convenient here and useful later to express the input impedance from Eq. 4-9 as a function of $\omega_o = \omega_A$ defined in Eq. 4-10 and of k^2 defined by Eq. 4-1. This function is

$$Z_{in} = j\omega(L_1 + L_3) [1 - k^2/(1 - \omega_o^2/\omega^2)] . \qquad (4\text{-}11)$$

From this it is evident that the resonance frequency ω_R for zero input impedance is that which makes $k^2 = (1 - \omega_o^2/\omega^2)$, or

$$\omega_R^2 = \omega_o^2/(1 - k^2) = \omega_A^2/(1 - k^2). \qquad (4\text{-}12)$$

The coupling coefficient can, therefore, be related to the frequencies of resonance and antiresonance at the input



-Fig. 4-4. Inductively coupled circuits with tuning capacitors at input and output.

terminals when the output is tuned by the equation

$$k^2 = \frac{\omega_R^2 - \omega_A^2}{\omega_R^2} = 1 - \left(\frac{\omega_A}{\omega_R}\right)^2. \qquad (4\text{-}13)$$

If, now, the input in Fig. 4-4 is also tuned by a capacitance C_1 across the input terminals, the poles and zeros of the impedance looking into these tuned coupled circuits can also be related to k^2. The general case if complicated, but the particular relation usually considered is that in which C_1 is adjusted to tune the impedance Z_{11} to the same frequency to which C_2 tunes Z_{22}. This frequency, as in Eq. 4-10, is $1/\omega_o^2 = (L_2 + L_3) C_2 = (L_1 + L_3) C_1$. Since C_1 is in parallel with the input admittance of Eq. 4-9, it is convenient here to consider the input admittance rather than the impedance, and with the aid of Eq. 4-11 this can be written in the simple form

$$Y_{in} = j\omega C_1 + 1/Z_{in} = j\omega C_1 - j/\omega(L_1 + L_3)\left[1 - k^2/(1 - \omega_o^2/\omega^2)\right]$$

$$= j\omega C_1 \left[1 - \frac{\omega_o^2/\omega^2}{1 - k^2/(1 - \omega_o^2/\omega^2}\right]. \qquad (4\text{-}14)$$

From this equation it is not difficult to show that the admittance is infinite for $\omega = 0$, for $\omega = \infty$, and for $\omega = \omega_R$, as defined by Eq. 4-12. The admittance is zero when the expression in brackets is zero, and this will occur when $k^2 = (1 - \omega_o^2/\omega^2)^2$. The antiresonance frequencies of the input are, therefore, given by

$$\omega_A^2 = \omega_o^2 / 1 \pm k ,$$

or $$\omega_A' = \omega_o / \sqrt{1 - k} \text{ and } \omega_A'' = \omega_o / \sqrt{1 + k} \quad . \qquad (4\text{-}15)$$

These are the natural frequencies of the tuned circuits when coupled, and they are often used in defining the coefficient of coupling. The relation of the critical frequencies in these tuned coupled circuits with only C_2 and with both C_1 and C_2 is shown in Fig. 4-5, where the variation of the input impedance with frequency is plotted.

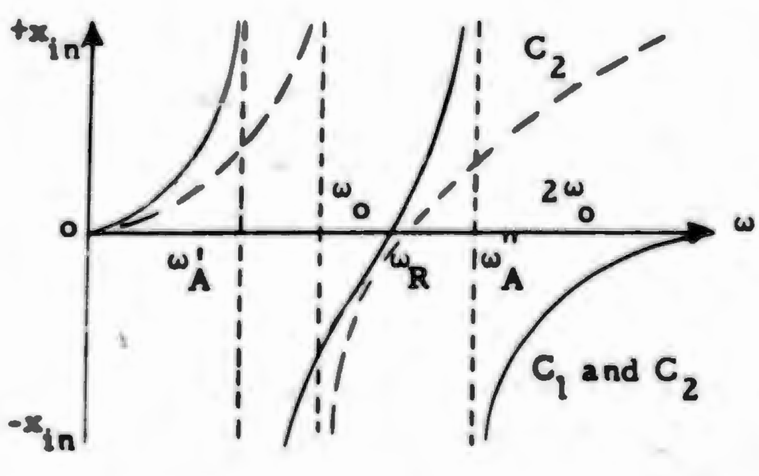


Fig. 4-5. The variation of input impedance with frequency in the tuned coupled circuits of Fig. 4-4.

A similar analysis of a T network of capacitances C_1, C_2, C_3 instead of the inductances L_1, L_2, L_3 in Fig. 4-4 shows relations between the coupling coefficient and the critical frequencies in tuned circuits with capacitive coupling which differ slightly from those obtained for inductive coupling. For the two meshes coupled by a common capacitance C_3 in such a T network the coupling coefficient from Eq. 3-17 is

$$k = \frac{C_1 C_2}{(C_1 + C_3)(C_2 + C_3)} \qquad (4-16)$$

With a tuning inductance across the terminals of mesh 2 the frequencies of resonance and antiresonance at the terminals of mesh 1 can be found by the method used above for inductive coupling. Corresponding to Eq. 4-13 for the network of inductances, the relation of k^2 to these critical frequencies for capacitive coupling is found to be

$$k^2 = 1 - \left(\frac{\omega_R}{\omega_A} \right)^2 . \qquad (4-17)$$

This differs from Eq. 4-13 in that the ratio of ω_R to ω_A has been inverted. The necessity for such inversion--and a quick check on the proper form of this ratio--is to be found in the requirement that k cannot exceed unity and in the recognition that with capacitive coupling the input reactance has a pole at zero frequency instead of the zero shown in Fig. 4-5 for the inductive coupling, so that the frequency of resonance must be lower than that of antiresonance. When both meshes of the network are tuned to the same frequency ω_o by inductances L_1 and L_2 across the terminals, the frequencies at which the input is antiresonant or the natural frequencies of the coupled circuits, corresponding to Eq. 4-15 for inductive coupling, are found to be for capacitive coupling

$$\omega_A' = \omega_o \sqrt{1 - k} \text{ and } \omega_A'' = \omega_o \sqrt{1 + k} . \qquad (4-18)$$

Resistive Coupling.

The characteristics of resistive coupling and its coefficient of coupling will be illustrated by the simple network of Fig. 4-6, in which the coupling results from a resistance R_3 common to the two meshes.

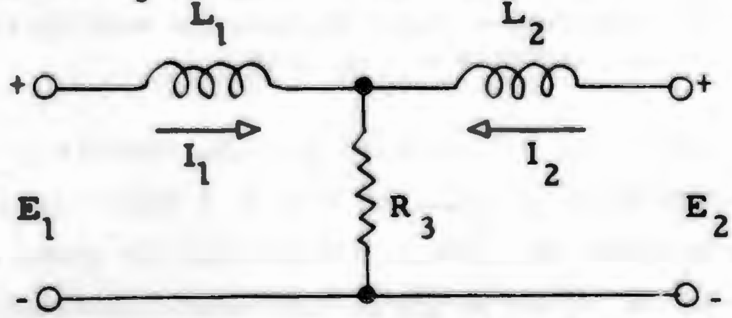


Fig. 4-6. Electrical network with resistive coupling.

The equations for these coupled systems are

$$E_1 = (R_3 + j\omega L_1) I_1 + R_3 I_2$$
$$E_2 = R_3 I_1 + (R_3 + j\omega L_2) I_2 . \qquad (4-19)$$

The general definition of the coupling coefficient is expressed for resistive coupling by Eq. 3-19, and in this example the coefficient becomes

$$k^2 = \frac{R_{12}^2}{R_{11}R_{22}} = \frac{R_3^2}{R_3R_3} = 1 . \qquad (4-20)$$

These circuits are, thus, perfectly coupled. The magnitude of the inductances does not enter at all in this measure of the coupling.

The significance of a coefficient which relates the coupling R to reactances, however, may be conveniently considered in this simple network, and it is worth some consideration because such a coefficient is frequently defined as a coupling coefficient. For this network the return ratio with reference to R_3 is found from Eq. 4-19 to be

$$\frac{e_{12}}{e_{11}} = \frac{R_3^2}{(R_3 + j\omega L_1)(R_3 + j\omega L_2)} . \qquad (4-21)$$

The feedback is, therefore, positive only when the frequency approaches zero and the reactances are negligible, and this is the condition for which this return ratio is defined as the coefficient of coupling. At frequencies for which the reactances are large compared to R_3, on the other hand, the return ratio becomes

$$\frac{e_{12}}{e_{11}} = \frac{-R_3^2}{\omega^2 L_1 L_2} \qquad (4-22)$$

but the negative sign identifies this as a negative feedback return ratio. Hence, the ratio of Eq. 4-22 is not a coupling coefficient as it is here defined. This ratio also differs from the coupling coefficients in being dependent upon frequency and in having a numerical value necessarily small compared to unity because of the assumption that R_3 was negligible compared to the reactances.

There is, however, a particular case of resistive coupling in which a ratio of resistance to reactance can be defined as a coupling coefficient. If in Fig. 4-6 the inductances L_1 and L_2 are tuned by series capacitances C_1 and C_2 to resonate at the frequencies given by $\omega_1^2 L_1 C_1 = 1$ and $\omega_2^2 L_2 C_2 = 1$, the return ratio can be expressed as

$$\frac{e_{12}}{e_{11}} = \frac{R_3^2}{\left[R_3 + j\omega L_1\left(1 - \frac{\omega_1^2}{\omega^2}\right)\right]\left[R_3 + j\omega L_2\left(1 - \frac{\omega_2^2}{\omega^2}\right)\right]} . \qquad (4\text{-}23)$$

When, in this case, the reactances are large compared to the resistance R_3, the ratio takes the simpler form

$$\frac{e_{12}}{e_{11}} = \frac{R_3^2}{L_1 L_2\left(\omega_1^2 + \omega_2^2 - \omega^2 - \frac{\omega_1^2 \omega_2^2}{\omega^2}\right)} . \qquad (4\text{-}24)$$

This has a maximum with respect to ω when $\omega^2 = \omega_1 \omega_2 = \omega_2^2$. At this frequency

$$\frac{e_{12}}{e_{11}} = \frac{R_3^2}{(\omega_1 - \omega_2)^2 L_1 L_2} , \qquad (4\text{-}25)$$

and this return ratio is positive, real, and maximum. It may, therefore, be defined as as a coupling coefficient for such resistance-coupled tuned circuits in the region between the resonant frequencies of the tuned circuits, provided also that the Q of the circuits is relatively high at such frequencies. Although in Eq. 4-25 the ratio appears to increase without limit as the resonant frequencies are adjusted to make $\omega_1 = \omega_2$, the assumption of negligible resistance above again restricts the ratio to values less than unity.

Electromechanical Coupling

The Electrostatic Transducer.

As an example of electromechanical symmetrical or reciprocal coupling the electrostatic transducer will be considered. In this device the coupling between the electrical and mechanical systems is a result of the change in force between two charged surfaces when the charge varies and the change in the potential between the surfaces when their separation varies. The equations of motion expressing this interaction are written by Hunt (1954, p. 179) as

$$\begin{aligned} E_1 &= Z_e \qquad I_1 + (q_o/j \quad {}_oS) \; v_1 \\ F_1 &= (q_o/j\omega \epsilon_o S) \, I_1 + \qquad z_m \qquad v_1 . \end{aligned} \qquad (4\text{-}26)$$

The mutual parameter or transduction coefficient in these equations is symmetric and may be written in the alternative forms

$$T_{em} = T_{me} = q_o/j\omega \epsilon_o S = E_o/j\omega(d + x_o) = 1/j\omega C_{em}, \qquad (4\text{-}27)$$

where q_o is the polarizing charge and E_o the polarizing potential on the plates of the transducer, S is the area of the plates, and $(d + x_o)$ is their separation.

In the final form of Eq. 4-27 the transduction coefficient is expressed as the reactance of an equivalent capacitance $C_{em} \equiv \epsilon_o S/q_o$. This makes it possible to draw an equivalent circuit of the transducer in which the electrical and mechanical meshes are coupled by the common capacitance C_{em}, as shown in Fig. 4-7.

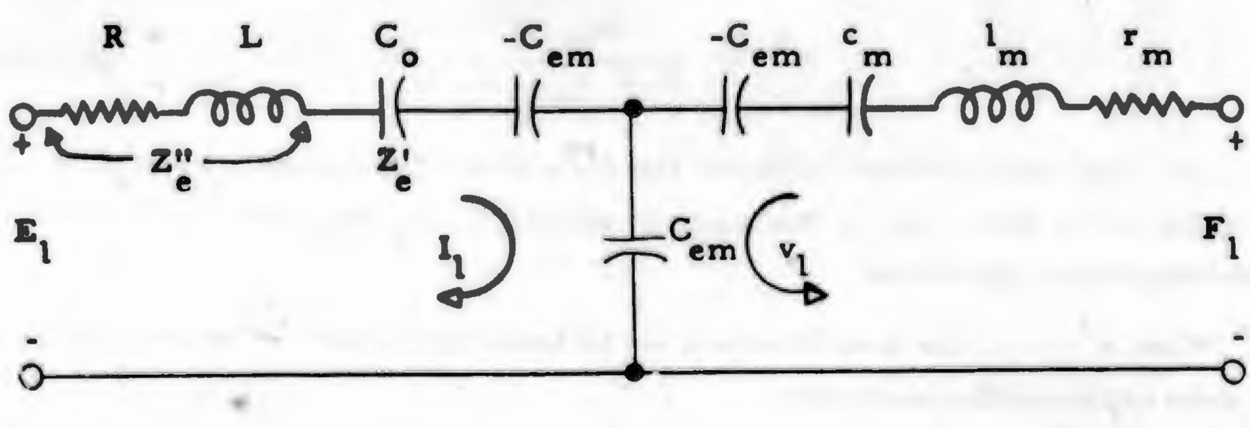


Fig. 4-7. Equivalent circuit for an electrostatic transducer.

Since the form of the equations and of the equivalent circuit of this transducer is the same as that of Eqs. 3-4 and Fig. 3-5 from which the coupling coefficient was defined, the definition can be applied to these systems without difficulty. Since the coupling is capacitive, the appropriate coefficient is that of Eq. 3-17, and, with the notation of Fig. 4-7 and Eq. 4-27, the coupling coefficient is

$$k^2 \equiv \frac{C_{22}C_{11}}{C_{12}^2} = \frac{C_o c_m}{C_{em}^2} = c_m C_o \left(\frac{E_o}{d + x_o}\right)^2 . \qquad (4\text{-}28)$$

This application of the definition of the coupling coefficient to electromechanical coupling can be justified by showing that the resulting electromechanical coupling coefficient has the same physical significance as the electrical coefficient. The value of k^2 in Eq. 4-28 must, therefore, be limited to +1 by either physical realizability or by stability. This limit is not obvious in any of the forms in Eq. 4-28, where, seemingly, the coefficient can be increased without limit by increasing the polarizing potential E_o. In fact, however, the displacement x_o is a function of E_o, the polarization, and the static capacitance $C_o = \epsilon_o S/(d + x_o)$ is, in turn, a function of x_o. The relation between x_o and E_o is shown by Hunt (1954,p. 182) to be

$$\frac{-x_o}{c_m} = E_o^2 \frac{\epsilon_o S}{2d^2} \left(1 + \frac{x_o}{d}\right)^{-2} . \qquad (4\text{-}29)$$

From this equation and the definition of C_o, the coupling coefficient can be expressed as the simple function of x_o and d,

$$k^2 = \frac{-2x_o}{d + x_o} . \qquad (4\text{-}30)$$

A negative sign appears here because the direction of x defined as positive is opposite to the direction of the force produced by E_o and the displacement x_o resulting from this force.

When $k^2 = +1$, the displacement x_o is found from Eq. 4-30 to be related to the separation of the plates by

$$-x_o = (1/3)d . \qquad (4\text{-}31)$$

But this is just the stability criterion shown by Hunt (1954, p. 183) to result from his analysis of the static balance between the elastic restoring force and the electrostatic force of attraction acting on the diaphragm of the transducer. When the polarizing potential is increased to the point at which the displacement exceeds 1/3 of the initial separation of the diaphragm and the fixed electrode, the attractive force exceeds the elastic force and the diaphragm collapses or "falls in."

This same stability condition appears in the relation of the mechanical input impedance to k^2. As was shown in connection with Eq. 3-23, the feedback measured by k^2 results in a reduction in input impedance when the output terminals are shorted. In Fig. 4-7, therefore, the effective input compliance c_m^s at the mechanical terminals with the electrical terminals shorted (the polarizing E_o being provided by a source with zero internal impedance) is related to k^2 and the open-circuit mechanical compliance c_m by

$$c_m^s = c_m/(1 - k^2) \quad . \qquad (4\text{-}32)$$

Here the limit of $k^2 = +1$ corresponds to an infinite input compliance or zero stiffness, which indicates again that the diaphragm equilibrium is unstable at this point.

The electromechanical coupling coefficient does, therefore, have significance as a measure of the approach of the system to the point of instability. The instability if possible in this case, although not in the electrical circuits previously considered, because the transducer has an internal energy source in the form of the polarizing charge or potential. This source appears only implicitly in the equivalent circuit of Fig. 4-7 through the dependence of the elements C_o and C_{em} upon the polarization.

It is appropriate to note here, also, that in Fig. 4-7 and in the coefficient related to that circuit the assumption has been made that all of the electrical field associated with the capacitance C_o is inherently associated with the electromechanical coupling. In general this is not necessarily true, and the electrical capacitance must be separated into a "stray" component C_o'' independent of the coupling and an "active" component C_o' inseparable from the coupling. This resolution of the general electrical impedance into two such components is described by Hunt (1954, pp. 133 and 205).

In the particular case of an electrostatic transducer the stray capacitance resulting, for example, from the leads and from any portion of the diaphragm which does not partake of the motion must be placed in a shunt position across the electrical input terminals of the equivalent circuit in Fig. 4-7. This shunt position is mandatory because the current through this stray capacitance is not a component of the current i_1 in Eq. 4-26 or Fig. 4-7 which produces a force in the mechanical system; the only effect of the stray current is to increase the input current from the source. The components of the equivalent T network are not altered by this stray capacitance across the input terminals nor are the equations of motion, so the coefficient of coupling defined from Fig. 4-7 and Eqs. 4-26 is independent of the stray component of capacitance. The significance of the coefficient as a measure of the stability limit is, therefore, also unaffected by the stray capacitance.

The presence of shunt stray capacitance across the electrical terminals, however, makes it physically impossible to have those terminals "open", except at zero frequency. Consequently, relations of k to the mechanical or electrical input impedance or to their critical frequencies which have assumed an open electrical termination are valid in theory only when the stray capacitance is removed from the circuit and in practice only when a correction is made for the modification of the terminal impedance by such capacitance. Components of stray impedance which are resistive or inductive do not, of course, appear in the coefficient defined for capacitive coupling. But, whether they occupy a shunt or series position in the equivalent circuit, they must, like the stray capacitance, be considered in any relations involving the terminal impedances.

The Piezoelectric Transducer.

Another example of symmetrical electromechanical coupling is the piezoelectric transducer. The coupling of the systems is here the result of a stress on certain crystalline materials producing a charge on the crystal surfaces and, conversely, a charge applied to the surfaces producing a strain in the material. This phenomenon of piezoelectricity and its second-order effect of electrostriction have found widespread application in electromechanical transducers and have been extensively treated in the literature. The equations of motion or equations of state of the coupled systems have been expressed in many forms, but the standard notation proposed by the Institute of Radio Engineers has been used by Haskins and Hickman (1950) and by Bechmann (1955) in their extensive treatments of these equations and will be used here.

These equations of state are expressed in the variables stress T, strain T, strain S, electric field intensity $\underline{E}$, and electric displacement D. As in the general equations of coupled systems given by Eqs. 3-4, 3-5, 3-6, 3-7, these equations can have several forms, depending on the choice of independent variables. The set that corresponds to Eq. 3-4 is

$$T = c^D S - h\,D$$
$$\underline{E} = -hS + \beta^s D \quad . \qquad (4\text{-}33)$$

The coefficients c, h, β are, in general, functions of the directions of stress and field, but only simple examples in which a single component can be used will be considered here. It will be presumed that more general cases of coupling between spatial components can be attacked by considering the components in pairs and treating the coupling between each pair in the manner described here. In these equations of state there is no problem of resolving time-phase components if the coefficients are assumed to be real quantities, since all the terms represent stored potential energy.

Since this set of equations has the same form as the equations of motion of other coupled systems, an equivalent circuit in terms of these variables could be drawn by the use of suitable analogies, and a positive feedback of stress, for example, could then be used in defining the coefficient of coupling. It is, however, more conventional, more convenient for comparison with other systems, and more consistent with the idea of electric-circuit equivalents to return to F, v, E, and I as the variables. The change in variables can be simply made by considering the piezoelectric material represented by Eqs. 4-33 to be the dielectric in a parallel-plate capacitor of area A and separation x_o. The stress can then be written T = F/A, the strain S = x/x_o, the field $\underline{E}$ = E/x_o, and the displacement D = q/A. Assuming the exponential time variation $e^{j\omega t}$ for the variables in relating q to I and S to v, Eqs. 4-33 can be written in the new variables and with impedance notation as

$$E = \frac{\beta^s x_o}{j\omega A} I - \frac{h}{j\omega} v$$

(4-34)

$$F = \frac{-h}{j\omega} I + \frac{Ac^D}{j\omega x_o} v$$

By defining the equivalent capacitances $C_o \equiv A/\beta^s x_o$, $c_m \equiv x_o/Ac^D$, and $C_{em} \equiv 1/h$, these equations become

$$E = 1/j\omega C_o \; I - 1/j\omega C_{em} \; v$$

(4-35)

$$F = -1/j\omega C_{em} I + 1/j\omega c_m \; v,$$

and they are now the same as Eqs. 4-26 for an electrostatic transducer having only capacitive reactances in its electrical and mechanical impedances. The equivalent circuit for a piezoelectric transducer is, therefore, that shown in Fig. 4-7 for the electrostatic, and the coupling coefficient is, as in Eq. 4-28,

$$k^2 = \frac{C_o c_m}{C_{em}^2} = \frac{h^2}{\beta^s c^D} . \qquad (4\text{-}36)$$

This coefficient for the piezoelectric transducer differs, however, from that for the electrostatic in that the limit of +1 is not established by an instability attainable in practice. The transduction coefficient $h/j\omega$ can be varied only by choosing crystal structures which produce different values of the piezoelectric constant h; in electrostrostrictive materials, such as barium titanate, this constant is a function of an external polarizing electric field, but an upper limit is still established by the internal structure of the material. In available piezoelectric materials the maximum values of the coefficient of coupling obtained are considerably less than unity, and there is no evidence of any mechanical instability similar to that in the electrostatic transducer.

To establish a theoretical limit for the coefficient determined by the collapse of the material as its mechanical stiffness decreases with increasing coupling, as was done in Eq. 4-32 for the electrostatic transducer, the constants C_o and C_{em} in Eqs. 4-35 have to be related to the polarization, as they were in the electrostatic. Such relations are not to be found in equations of state, such as Eqs. 4-35, which are applicable only to small increments in the variables about an operating point determined by the polarization. Additional information about the relation of the mechanical reaction to the total electrical field is required.

The requisite knowledge of the relation between mechanical deformation and polarization can be obtained from electrostrictive materials, such as barium titanate, which can be depolarized; and the deformation is found to be proportional to the square of the polarizing D field, as noted by Mason (1950, p. 289). This quadratic relation is the same as that used by Hunt in the relation of x_o to E_o in the electrostatic transducer, leading to Eq. 4-29 for the condition of balance between elastic and electrostatic forces. In the electrostrictive materials, therefore, the electric forces due to the polarization will similarly reduce the mechanical stiffness, and, as in the electrostatic transducer, when the coupling coefficient of Eq. 4-36 is unity, the polarization must be such that the electric forces in the material just cancel the internal mechanical forces and the material has no stiffness. The coupling coefficient has, thus, the limit of +1 imposed by the same condition of instability derived in the example of the electrostatic transducer.

Of the forms of the equations of state other than that of Eqs. 4-33 there is one, having T and $\underline{E}$ as the independent variables, which has mutual coefficients that are symmetric, as they are in Eqs. 4-33. When the definition of the coupling coefficient is applied to these equations, a coefficient equal to that in Eq. 4-36 will be obtained. The other equations, which have T and D or S and $\underline{E}$ as independent variables, will be antisymmetric in form, however, and, as was shown in connection with Eqs. 4-4 in hybrid parameters, this antisymmetry of the equations implies negative feedback. The return ratio for such negative feedback is not the coefficient of coupling but is simply related to it, as in Eq. 4-5. These antisymmetrical forms of the equations of state lead, therefore, to return ratios which are equal to $k^2/(1 - k^2)$ and not to the ratios which are defined here as coupling coefficients. These return ratios for negative feedback are the ones denoted by Bechmann (1955) as $k_{mix}{}^2$ because they come from the equations with mixed extensive and intensive variables, as described on page 21 ; the return ratios for positive feedback, which have been defined as coupling coefficients, are denoted as k_{hom}^2 because they come from the equations which are homogeneous in intensive and extensive variables.

The relation noted here in this example of the piezoelectric transducer between the form of the equations of motion or the equations of state and the definition of the coupling coefficient is typical of all coupled systems with symmetrical or reciprocal coupling. A general rule for such coupling is that the sets of equations in which the variables are thermodynamically homogeneous (as defined on p. 21) are those in which the feedback is positive and those to which the definition of the coupling coefficient as the return ratio is applicable. The sets of equations with mixed variables have negative feedback, and the return ratio is not the coupling coefficient but is $k^2/(1 - k^2)$.

A general rule for obtaining the coupling coefficient from the coefficeints in the equations of state of systems with reciprocal coupling can, therefore, be stated in the form of Table 4-1.

Antireciprocal Systems

The Moving-Coil Transducer

The most familiar and, in some respects, the simplest example of systems with antireciprocal coupling is the moving-coil or electrodynamic transducer, much used as the driving mechanism in loudspeakers and electrical measuring instruments. The coupling between electrical and mechanical systems here results from the force exerted by a current-carrying conductor in a magnetic field and from the e.m.f. developed by the motion of this conductor in the magnetic field. This interaction can be expressed by a pair of linear equations in the familiar form used for other coupled systems.

The usual form of the equations for these systems, used by Mason (1942, p. 188) and others, can be written as

$$E = Z_e I + (Bl)\,v$$
$$F = -(Bl)I + z_m\,v \,. \qquad (4\text{-}37)$$

Table 4-1 Relations of the coupling coefficient to the coefficients in the equations of state of systems with symmetrical coupling.

Systems with symmetrical or reciprocal coupling

such as

electrical systems coupled by resistance, capacitance, or inductance,

mechanical systems coupled by resistance, mass, or stiffness,

electromechanical systems coupled by an electric field, as in the electrostatic and piezoelectric transducers,

and having equations of state of the form

$$y_1 = a_{11}x_1 + a_{12}x_2$$

$$y_2 = a_{12}x_1 + a_{22}x_2 ,$$

in which the coefficients a_{nn} are coefficients or ratios of coefficients from the same energy function--kinetic, potential, or dissipation--

have a coupling coefficient related to the coefficients a_{nn} by the following ratios:

if the equations of state are homogeneous in extensive and intensive variables

$$k^2 = \frac{a_{12}a_{21}}{a_{11}a_{22}} ,$$

if the equations of state are mixed in extensive and intensive variables

$$k^2 = \frac{a_{12}\,a_{21}}{a_{11}\,a_{22} + a_{12}\,a_{21}} .$$

In these equations the mutual parameter or transduction coefficient (Bl) is the product of the polarizing magnetic field flux density and the length of the conductor in that field. In this case the two coefficients are equal in magnitude but opposite in sign; the relation can be written

$$T_{em} \equiv (E/v)_{I=0} = (Bl) = -(F/I)_{v=0} \equiv -T_{me} \quad . \qquad (4\text{-}38)$$

The matrix of the coefficients of Eq. 4-37 is, therefore, antisymmetric. One immediate consequence of this antisymmetry is that these equations cannot be represented by a simple T network of the form of Fig. 3-1, as the symmetrical equations were.

The advantages of such an equivalent network and of all the electric-network theory applicable to it are retained for the moving-coil and other systems with antireciprocal coupling through the introduction by Hunt (1954, p. 112) of a space operator $\underline{k}$ to express the intrinsic orthogonality of the electromechanical interactions. This operator or versor is so defined that the force produced by a current in a magnetic field can be written $F = (Bl)\underline{k}I$. The operator $\underline{k}$ indicates a rotation of the positive direction of the vector that follows it, +I, by 90° counterclockwise around the direction of the vector, +B, that precedes it in order to determine the positive direction of F. With this notation, the equations of the moving-coil transducer can be written (Hunt, 1954, p. 146) as

$$\begin{aligned} E &= Z_e \; I + (Bl)\,\underline{k}\; v \\ F &= (Bl)\,\underline{k}\; I + z_m \; v \quad . \end{aligned} \qquad (4\text{-}39)$$

The transduction coefficients are symmetrical in this form, and the usual equivalent T network can be drawn, as shown in Fig. 4-8.

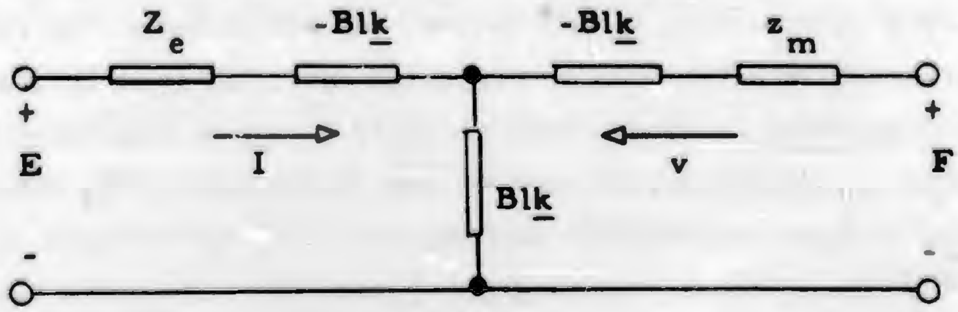


Fig. 4-8. Equivalent T network of a moving-coil transducer.

The determination of the coupling coefficient from the feedback, however, does not require an equivalent T network. A return ratio with reference to the coupling element can be determined from the equations of the coupled systems in the form of either Eqs. 4-37 or 4-39. This return ratio, as in the general expression of Eq. 3-15, will involve the mutual coefficients only as their product. If note is taken of the significance of applying the operator k twice to the same quantity, it will be evident that $\underline{k}^2 = -1$. The product of the coefficients is, therefore, negative in either set of terminals, say E, and a load, z_L, across the other terminals, the return ratio corresponding to Eq. 3-15 has the value

$$\underline{T} = \frac{e_{12}}{e_{11}} = \frac{-(Bl)^2}{Z_e(z_m + z_L)} . \tag{4-40}$$

If these impedances are expressed as series combinations of R, L, and C, the return ratio can be written as

$$\underline{T} = \frac{-(Bl)^2}{(R+j\omega L-j/\omega C)(r_m+r_L+j\omega l_m+j\omega l_L-j/\omega c_m-j/\omega c_L)}$$

(4-41)

$$= -(Bl)^2\left[R(r_m+r_L)+L\left(\frac{1}{c_m}+\frac{1}{c_L}\right)+\frac{(l_m+l_L)}{C}-\omega^2 L(l_m+l_L)\right.$$

$$\left.-\frac{1}{\omega C}\left(\frac{1}{\omega c_m}+\frac{1}{\omega c_L}\right)+jR(\ldots\ldots\ldots)+j(r_m+r_L)(\ldots\ldots\ldots)\right]^{-1}$$

An attempt to find for this ratio a positive, real, and maximum value when ω and z_L are varied, as was done for Eq. 3-16, encounters difficulty because of the negative sign in the numerator. Examination of Eq. 4-41 reveals that the frequency approaching either zero or infinity reduces the ratio to zero, and even if the reactive terms can be neglected the ratio is finite and independent of ω but negative. If the resistive terms can be neglected and only inductive reactances appear in one system and capacitive in the other or vice versa, the ratio is again independent of frequency and again negative. The conclusion is that the effect of antireciprocal coupling is to make the feedback which represents such coupling in the equations of the systems, such as

Eqs. 4-37 and 4-39, inherently negative.

This distinction of the feedback representing antireciprocal coupling as negative and that representing reciprocal coupling as positive can be supported by a simple physical argument. In feedback amplifiers a familiar effect of negative voltage feedback is an increase in the input impedance, while the effect of positive voltage feedback is a decrease in the impedance. A similar change in input impedance should also distinguish the sign of the feedback in these coupled systems. It has already been shown that an electrostatic transducer has its mechanical stiffness reduced--possibly even to zero--by the electromechanical coupling when its electrical terminals are shorted. The coupling in this case is reciprocal; the feedback is positive. When the electrical terminals of a moving-coil transducer are similarly shorted, however, the stiffness of the mechanical system is increased by the coupling--and this clamping effect of shorting the terminals is frequently utilized to protect the movements of moving-coil meters from damage during shipment. In this system the coupling is antisymmetrical; the feedback is negative.

The problem of applying the definition of the coupling coefficient to a system with antireciprocal coupling is, therefore, that of expressing the coupling as positive rather than negative feedback. This same problem has already been encountered--and solved--in connection with a reciprocal system whose equations were expressed in the hybrid parameters of Eq. 4-4. The coupled systems in that case were two electrical meshes with a common inductance, and there was no difficulty in defining a coupling coefficient in Eq. 4-1 from the impedance equations of the systems. However, when the same coupled systems were expressed in terms of hybrid parameters in Eq. 4-4, it was noted that these equations were antisymmetrical, that the mutual parameter was not an impedance or admittance, and that the resulting return ratio indicated negative feedback.

These are also characteristics of Eqs. 4-37 of the moving-coil transducer. The implication is, therefore, that this form of the equations corresponds to Eq. 4-4 in hybrid parameters and that another choice of the independent variables will result in equations which are symmetrical, as were the impedance Eqs. 4-3. It is easily demonstrated that this is true.

If Eqs. 4-37 are solved for I and F as independent variables, the resulting equations are

$$E = \left[Z_e + \frac{(Bl)^2}{z_m} \right] I + \frac{(Bl)}{z_m} F$$

$$v = \frac{(Bl)}{z_m} I + 1/z_m \; F \qquad (4\text{-}42)$$

The mutual parameters, $(Bl)/z_m$, in these equations are now symmetric and also have the time-phase characteristics of admittances. The definition of the coupling coefficient as a return ratio should, therefore, be applicable.

The coupling in these equations can be resistive, capacitive, or inductive if the mechanical impedance z_m in $(Bl)/z_m$ has resistive, inductive, or capacitive components. The appropriate coupling coefficient for each type of coupling can be found from Eqs. 3-17, 3-18, and 3-19. Of these three only the inductive coupling has much significance in this example, since only electrical inductance is inherently associated with the electromechanical coupling of the coil in a magnetic field; resistance and capacitance are not essential in the transduction. If, then, the electrical impedance is taken to be $Z_e = j\omega L_e$, it can be seen from Eq. 4-42 that the feedback can be positive at all frequencies only if $z_m = 1/j\omega c_m$. The coupling impedance $(Bl)/z_m$ is then inductive, and from Eq. 3-18 or from the positive return ratio obtained from Eq. 4-42 the coupling coefficient is

$$k^2 = \frac{(j\omega c_m Bl)^2}{(j\omega L_e + j\omega c_m B^2 l^2)(j\omega c_m)} = \frac{(Bl)^2}{L_e/c_m + (Bl)^2}. \qquad (4\text{-}43)$$

This definition of the coupling coefficient of the moving-coil transducer from symmetrical equations of systems in which the coupling is antisymmetrical can be justified if the coefficient can be shown to have the same physical significance and the same utility as the coefficients for symmetrical coupling. From the form of Eq. 4-43 it is obvious that for positive inductance and compliance the magnitude of k^2 cannot exceed +1. An increase in the polarizing field B can increase the coefficient, but only as B approaches infinity will k^2 approach unity. An increase in the length l of the conductor in the field by increasing the number of turns in the coil will also increase (Bl), but since the inductance L_e is proportional to the square of the number of turns this increase will have no effect upon k^2. The limit of unity is set, therefore, by physical realizability rather than by stability in this transducer.

The relation of k^2 to input impedances can be found from Eq. 4-37 for this transducer. With $Z_e = j\omega L_e$ and $z_m = 1/j\omega c_m$, the mechanical input impedance with the electrical terminals shorted is found to be

$$z_m^s = z_m + \frac{(Bl)^2}{Z_e} = \frac{1}{j\omega c_m} + \frac{(Bl)^2}{j\omega L_e} = \frac{1}{j\omega c_m^s}, \tag{4-44}$$

so that the short-circuit mechanical compliance is

$$c_m^s = (1 - k^2)\, c_m . \tag{4-45}$$

The coupling, therefore, reduces the mechanical compliance or increases the stiffness, and for $k^2 = +1$ the stiffness becomes infinite, the system extremely stable. This is to be compared with the electrostatic transducer, where the stiffness in Eq. 4-32 could be reduced to zero by perfect coupling, with resulting instability.

The electrical input impedance is similarly affected when the mechanical terminals are shorted. With reference to Fig. 4-8, shorting the mechanical terminals means the force F = 0 and the mechanical system is free to move with velocity v. From Eq. 4-37 the input inductance is found to be related to k by

$$L_e^s = L_e/(1 - k^2) . \tag{4-46}$$

Thus, with perfect coupling the input inductance becomes infinite. This increase in inductance may be compared to the familiar decrease in input current to an electric motor which is rotating freely without load.

The effect of the reciprocity of the coupling upon these relations of the coupling coefficient to input impedances is seen to be a simple inversion of the factor $(1-k^2)$ or an inversion of the open- and short-circuit impedances. In reciprocal systems, such as the electrostatic transducer, the electrical input impedances Z_o with the mechanical terminals of Fig. 4-7 open (or the system clamped) and Z_s with the terminals shorted (or the system free) are related by

$$Z_s = Z_o(1 - k^2)$$

or

$$k^2 = \frac{Z_o - Z_s}{Z_o} \tag{4-47}$$

With antireciprocal coupling, as in the moving-coil transducer, the similar relation with reference to the terminals of Fig. 4-8 is

$$Z_s = Z_o / (1 - k^2)$$

or

$$k^2 = \frac{Z_s - Z_o}{Z_s} \tag{4-48}$$

In both of these electromechanical systems, the analogy of force to voltage and velocity to current has been used. As a result, at the mechanical terminals of Figs. 4-7 and 4-8 "open" means zero velocity or a clamped mechanical system and "shorted" means zero force or a free mechanical system.

Other analogies are not only possible but prevalent. The symmetry of Eqs. 4-42, in which F is analogous to I and v to E, leads to frequent use of this mobility pairing for the sake of the equivalent electric network which can be drawn as the result of the symmetry. The resulting T or transformer equivalent network, such as is shown by Mason (1942, p. 190), uses current as the "through" quantity in the electrical mesh and force as its analog in the mechanical mesh. The mechanical terminals then have velocity as the "across" quantity or voltage analog, and in this equivalent network shorting these terminals means zero velocity or a clamped system. As a result, in the impedance relations of Eq. 4-48 the terms "open" and "short" would have to be interchanged. The terms "clamped" and "free", however, remain unambiguous in either analogy.

Mention should also be made of the other set of symmetrical equations for the moving-coil transducer, similar to Eqs. 4-42 but in which the independent variables are v and E. The variables have the same mobility pairing mentioned above, and the definition of k^2 is again that given by Eq. 4-43. This set of equations, however, requires an interchange of open-and short-circuit relations on the electrical side when compared to systems with reciprocal coupling.

Since the use of either the symmetrical equations with the mobility analog or of Hunt's k operator in the moving-coil system makes possible an equivalent T network, equivalent networks with ideal transformers of the form shown in Fig. 4-3 can also be used to represent this antisymmetrical coupling. The coupling coefficient can then be related to stored energies in these equivalents, as was done for the example of electrical circuits with symmetrical coupling in Eqs. 4-6 and 4-7. The effect of the antisymmetry on these relations can be found by considering such energy storage in the equivalent circuit of the moving-coil transducer shown in Fig. 4-9. This is one of the equivalents

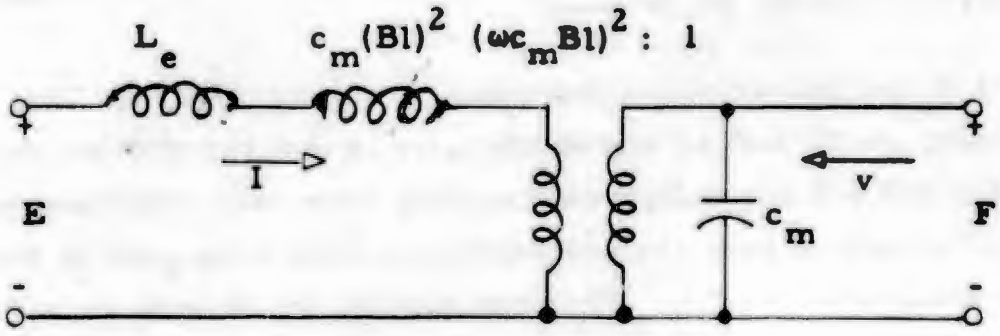


Fig. 4-9. Ideal-transformer equivalent network of a moving-coil transducer.

shown by Hunt (1954, p. 148), but here the mechanical impedance has been taken as $z_m = /j\omega c_m$ and the electrical as $Z_e = j\omega L_e$.

In this network, as in Fig. 4-3(a), the "primary" can be defined as

L_e and the "secondary" as c_m. The inductance $c_m(Bl)^2$ then represents the secondary impedance moved through the transformer into the primary circuit. The antireciprocity is evident here as a positive sign on the inductance in series with L_e on the primary side and a negative sign on the inductance which appears when c_m is moved through the transformer to the primary; this is to be compared to the opposite pairing of signs on the inductances in Fig. 4-3(a) for reciprocal coupling. In this equivalent network, when the electrical terminals are driven by a constant-current source I and the mechanical terminals are shorted, the coupling coefficient of Eq. 4-43 can be related to the inductance representing the primary and secondary by

$$k^2 = \frac{c_m(Bl)^2}{L_e + c_m(Bl)^2} = \frac{\frac{1}{2}c_m(Bl)^2 I^2}{\frac{1}{2}L_e I^2 + \frac{1}{2}c_m(Bl)^2 I^2} \qquad (4\text{-}49)$$

$$= \frac{\text{energy stored in secondary}}{\text{total stored energy}} .$$

As might be expected, a similar procedure with the other transformer equivalent corresponding to Fig. 4-5(b) and with the "secondary" defined as the mechanical input impedance with the electrical terminals shorted instead of open will result in the other relation

$$k^2 = \frac{\text{energy stored in secondary}}{\text{energy stored in primary}} . \qquad (4\text{-}50)$$

The effect of the antireciprocal coupling on these relations of the moving-coil coupling coefficient to stored energies is that the similar relations of Eqs. 4-6 and 4-7 for reciprocal coupling have been interchanged when the same definitions of primary and secondary have been used in the corresponding equivalent networks. This can also be expressed, as in the input impedance relations, as an interchange of open - and short-circuit conditions when the antireciprocal system is compared to the reciprocal. For example, if the "secondary" in Fig. 4-9 is defined as the mechanical impedance with shorted instead of open electrical terminals, the energy ratio of Eq. 4-50 will result; this is the same result obtained for the reciprocal system with the same equivalent network but with the other definition of the "secondary."

In a reciprocal system the coupling coefficient was shown to be related to the frequencies of resonance and antiresonance by Eqs. 4-13 and 4-15 when tuning reactances were placed across the terminals of one or both systems. The effect of antireciprocal coupling on these relations can now be determined by considering the input impedance of the moving-coil transducer under similar conditions. This input impedance will be determined for the simple case of $Z_e = j\omega L_e$ and $z_m = 1/j\omega c_m$ and, with reference to Fig. 4-8, with a mechanical inductance l_m connected across the mechanical terminals or, equivalently, with a mechanical impedance $z_m = j\omega l_m - j1/\omega c_m$ and the mechanical terminals shorted. From the circuit of Fig. 4-8 or Eq. 4-37 this input impedance can be calculated, and, when expressed in terms of k^2 defined by Eq. 4-43 and of the resonance frequency of the mechanical system, $\omega_o^2 = 1/_m c_m$, it is

$$Z_{in} = j\omega L_e + \frac{(Bl)^2}{j(\omega l_m - 1/\omega c_m)} = j\omega L_e \left[1 - \left(\frac{k^2}{1-k^2} \right) \frac{1}{(\omega^2/\omega_o^2 - 1)} \right] . \tag{4-51}$$

From this equation the input impedance is seen to be infinite at the antiresonance frequency $\omega_A = \omega_o$. The impedance is zero at a frequency ω_R such that the expression in brackets is zero, and this resonance frequency is

$$\omega_R^2 = \omega_o^2/(1 - k^2) = \omega_A^2/(1 - k^2) . \tag{4-52}$$

Comparison with Eq. 4-13 for symmetrical inductive coupling shows that the relation of k^2 to the frequencies of resonance and antiresonance is not changed by antireciprocal coupling.

In this example of the moving-coil transducer it has been shown that the coupling coefficient of this antireciprocal coupling resembles that for reciprocal coupling by having an upper limit of +1 but that its relation to input impedances and to stored energy ratios always differs in some simple but subtle way, such as an interchange of open- and short-circuit conditions on the terminations of the equivalent networks. For antireciprocal coupling the definition of the coupling coefficient in terms of positive feedback has been applied by the artifice of using a symmetrical form of the equations of the coupled systems in which force is analogous to current in producing positive voltage feedback. Justification for this has been found in the fact that the coefficient so defined still has physical significance as the limit of physical realizability and has useful relations to the effects of the coupling. It remains to be shown in another example of antireciprocal coupling that this coefficient also has significance as a measure of stability.

The Moving-Armature Transducer.

Another example of the antireciprocal coupling that is characteristic of the interaction of a current and a magnetic field is the moving-armature or magnetic transducer, an electromechanical device which in the telephone receiver and in magnetic relays has found more widespread application over a longer period of time than has any other form of electromechanical transducer. In this form of transduction the force exerted on an armature by a polarizing magnetic field is varied by modulation of the polarizing field with the field of a current-carrying coil, and the reaction of the mechanical system upon the electrical results from the motion of the armature changing the flux through that coil and inducing in it an e.m.f.

The electromechanical coupling equations for the moving-armature system are expressed by Hunt (1954, p. 227) in the form

$$\begin{aligned} E &= (Z_e'' + Z_e')\, I + T\, v \\ F &= T\, I + (z_m'' + T^2/Z_e')\, v. \end{aligned} \tag{4-53}$$

In these equations the antireciprocity is expressed by the operator $\underline{k}$, as in Eqs. 4-39 for the moving-coil transducer, and the transduction coefficient is defined as

$$T \equiv - \frac{N\phi_o \chi}{d + x}\, \underline{k}\,, \tag{4-54}$$

where N is the number of turns in the coil linking the magnetic circuit,

ϕ_o is the polarizing flux, χ is the complex hysteresis factor, and (d + x) is the length of the air gap between the armature and the pole-piece. The electrical impedance is written here as the sum of two components, of which Z_e'' includes all the ohmic resistance of the winding and the reactance associated with the leakage flux that plays no part in producing stress across the air gap and Z_e' is the reactance associated with all the air-gap flux and is thus inherently associated with any transduction that may arise. The mechanical impedance is also expressed as the sum of two components, of which z_m'' is the mechanical impedance of the armature system when the polarization is zero and $z_m' = T^2/Z_e' = \frac{1}{j\omega c_e}$ is the added mechanical impedance resulting from the application of a steady polarizing flux. An equivalent T network for Eqs. 4-53 is shown in Fig. 4-10.

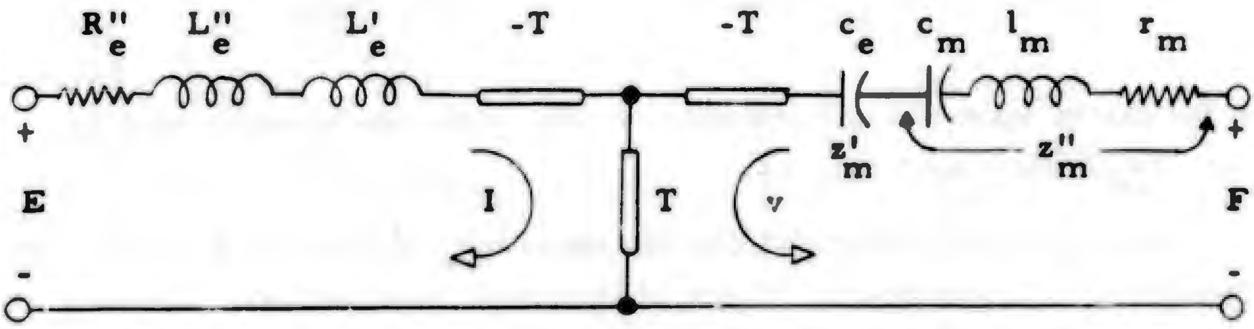


Fig. 4-10. Equivalent T network for a moving-armature transducer.

Since the coupling expressed by the transduction coefficient T in Eq. 4-53 is antireciprocal, the feedback in Eqs. 4-53 with I and v as independent variables is negative, as it was in the similar Eqs. 4-39 for the moving-coil transducer. But again a symmetrical form for the equations can be found by using I and F as the independent variables. With the abbreviated notation for mechanical impedance, $z_m = z_m'' + T^2/Z_e'$, these equations are

$$E = (Z_e - T^2/z_m)\, I + T/z_m\, F$$

$$v = \quad - T/z_m\, I + 1/z_m\, F. \qquad (4\text{-}55)$$

From these equations a positive return ratio can be obtained and coupling coefficients defined for the resistive, capacitive, and inductive components of z_m. As in the moving-coil transducer, however, only the inductive coupling is of interest, since it is the only one inherent in the transduction. For the same reason only the "active" component, L_e', of the electrical impedance will be included in a coupling coefficient which is to measure the inherent coupling. The component of z_m for inductive coupling is the capacitance of c_m and c_e in series; this will be denoted by c_m^o, since it is the capacitance with the electrical terminals open. From Eqs. 4-55 and from the definition in Eq. 3-18, the coupling coefficient for the moving-armature transducer is found to be

$$k^2 = \frac{(j\omega c_m^o T_o)^2}{(j\omega L_e' + j\omega c_m^o T_o^2)(j\omega c_m^o)} = \frac{c_m^o T_o^2}{L_e' + c_m^o T_o^2}, \tag{4-56}$$

when the factor upon which $\underline{k}$ operates in Eq. 4-54 has been denoted as $T_o = -N\phi_o \overline{\chi}/(d + x)$.

This coupling coefficient has the same form as that in Eq. 4-43 for the moving-coil transducer. There is, however, a significant difference concealed in the mechanical compliance c_m^o in Eq. 4-56. This compliance has been defined by Eq. 4-53 as the compliance with no current in the electrical circuit but with a polarizing magnetic field applied. It is related to the compliance c_m without polarization by the definition of z_m as $z_m \equiv z_m'' + T^2/Z_e'$; therefore, the compliance c_m can be related to c_m^o and to k^2 by

$$\frac{1}{c_m} = \frac{1}{c_m^o} + \frac{T_o^2}{L_e'} = \frac{1}{c_m^o}\left[1 + \frac{c_m^o T_o^2}{L_e'}\right] = \frac{1}{c_m^o(1 - k^2)}$$

or

$$c_m = c_m^o(1 - k^2). \tag{4-57}$$

If this equation is now used in Eq. 4-56, the coupling coefficient can be expressed in terms of the compliance c_m of the mechanical system when the polarization is zero by the equation

$$k^2 = \frac{c_m T_o^2}{L_e'} = \frac{c_m}{L_e'} \left(\frac{N\phi_o \overline{\chi}}{d + x} \right)^2 . \qquad (4\text{-}58)$$

In this form the coefficient of the moving-armature transducer resembles that of the electrostatic transducer in Eq. 4-28. And, as in that case, it appears that the magnitude of k^2 can be increased without limit by increasing the polarizing flux ϕ_o. Again, however, the limit of +1 is set by the condition that the system be stable. This limit is evident in Eq. 4-57, relating the compliance c_m^o with polarization to the compliance c_m without polarization. As the polarizing flux is increased, the coupling coefficient in Eq. 4-58 increases, but as k^2 approaches +1 the mechanical compliance c_m^o approaches infinity or the stiffness vanishes. The value $k^2 = +1$, therefore, indicates the point at which the mechanical system becomes unstable, just as it did in the electrostatic transducer.

The antireciprocity of the coupling in the moving-armature does, nevertheless, distinguish it from the reciprocal transduction of the electrostatic in this stability condition. The difference becomes evident in comparing the mechanical impedances in Eqs. 4-26 and 4-53 for the two transducers. In the electrostatic transducer the mechanical impedance is independent of the polarization when the current I is zero, hence when the electrical terminals are open; it becomes a function of the coupling and of the polarization, expressed by Eq. 4-32, when the electrical terminals are shorted. In the moving-armature transducer, on the other hand, the mechanical impedance in Eqs. 4-53 is a function of T, of the polarization, and, as in Eq. 4-57, of k^2 even when the current I is zero and the electrical terminals are open. The electrostatic is thus mechanically unstable for $k^2 = +1$ when the electrical terminals are shorted, but the moving-armature is unstable when the electrical terminals are open. The antireciprocity again produces an inversion of the open- and short-circuit conditions.

Since the electrostatic transducer is stable when its electrical terminals are open, the implication of this inversion by the antireciprocity is that the moving-armature should be similarly stable for short-circuited electrical terminals. From Eq. 4-53, with $E = 0$ and with $Z_e = j\omega L_e'$ and $z_m = 1/j\omega c_m^o$, the short-circuit mechanical compliance c_m^s is found, with the aid of Eq. 4-57, to be related to c_m by

$$\frac{1}{c_m^s} = \frac{1}{c_m^o} + \frac{T_o^2}{L_e'} = \frac{1}{c_m} - \frac{T_o^2}{L_e'} + \frac{T_o^2}{L_e'} = \frac{1}{c_m} \qquad (4\text{-}59)$$

Therefore, the compliance c_m^s with the electrical terminals shorted is equal to the compliance c_m without polarization and is independent of the coupling. The system must then be stable for any magnitude of the polarizing flux.

This stabilizing effect of antireciprocal coupling may be regarded as confirmation of the characterization of the feedback in such systems as negative. It was shown in the moving-coil system that the coupling equations indicated negative feedback and that the input impedances were increased by this feedback. The mechanical stiffness was there increased by the flow of induced current when the electrical terminals were shorted, but there is no negative stiffness in the moving-coil system because it is unique among polarized transducers in that the polarization has no effect upon the mechanical or electrical systems other than the coupling. In the moving-armature transducer, however, the polarization produces a negative stiffness $1/c_e = -T_o^2/L_e'$ when the current is zero, but when a short-circuit current is induced by mechanical motion the resulting negative feedback into the mechanical system adds an equal positive stiffness to cancel the negative stiffness. This cancellation is complete, however, only when all of the magnetic flux is active in producing stress across the air gap, and this has been assumed in the analysis of the moving-armature system by neglecting the leakage inductance L".

In general, as was noted in regard to the electrostatic transducer, there are "stray" components of the electrical impedance which are independent of the coupling. In all magnetic transducers the stray or leakage component of the inductance must be in series with the active component, since all of the current which is active in producing force in the mechanical system flows through both inductances. The leakage inductance L'', therefore, appears in the series arm of the equivalent T network of Fig. 4-10 and in the equations of motion of Eq. 4-53 and, hence, can appear in a coefficient of inductive coupling defined from that circuit or those equations. However, the coupling coefficient has been so defined as to include only the component of the inductance active in the transduction in order that the coefficient may be limited to unity by the stability requirement. A coefficient defined to include the leakage inductance would be smaller in magnitude than that which includes only the active inductance because of the increased magnitude of Z_{11} and instability could thus occur for values of the former coefficient which are less than unity. Such a coefficient has, therefore, little significance as a measure of intrinsic properties of the transduction. In the magnetic transducers the leakage inductance--and the stray resistive and capacitive components, both shunt and series, although not involved in the definition of the coefficient--modify the electrical terminal impedance, and correction must be made for such components in determining k from measurements at the terminals.

The Magnetostriction Transducer.

As the moving-armature transducer is similar to the electrostatic except for the antireciprocity of its coupling, so the magnetostriction transducer is the antireciprocal counterpart of the piezoelectric. The antireciprocal coupling is again the result of the interaction of a current and a magnetic field, but in the phenomena of piezomagnetism and its second-order effect of magnetostriction the current involved is the atomic Amperian currnet responsible for the magnetic characteristics of the material. The electromechanical coupling from piezomagnetism is produced by the change in strain in a piezomagnetic material when an external applied magnetic field is varied and the change in internal flux density when the stress on the material is varied. The coupling of two systems can be expressed by the usual pair of equations of state or of motion. In writing these equations the similarity of piezomagnetism to piezoelectricity has led to the proposal by Ehrlich (1952) of a nomenclature for piezomagnetism which, except for the variables used, resembles the standard forms for piezoelectricity. It will be convenient for purposes of comparison to use this notation here.

The piezomagnetic equations of state employ as variables the stress T, the strain S, the magnetic flux density B, and the magnetic field intensity H. One of the symmetrical forms of these equations, comparable to Eqs. 4-33 for the piezoelectric transducer, is

$$B = \mu^T H + d\,T$$

$$S = d\,H + s^H T \quad . \qquad (4\text{-}60)$$

In this form of the equations the mutual coefficients are symmetrical, and all the coefficients have the same time phase; the feedback is therefore positive, and the return ratio can be defined as the coupling coefficient, $k^2 = d^2/\mu^T s^H$, as in the comparable equations of the piezoelectric transducer.

It is instructive, however, to express these equations in the variables F, v, E, and I, as was done for the similar Eqs. 4-33 in the example of piezoelectric coupling. In this case the change of variables can be made conveniently by considering the magnetostrictive material represented by Eqs. 4-60 to be the core of a toroid of N turns, having a mean circumference x_o and cross-sectional area A. In an idealized situation with no leakage flux the stress can be expressed as $T = F/A$, the strain as $S = x/x_o$, the flux density as $B = (1/NA)\int E\,dt$, and the field intensity as $H = NI/x_o$. Assuming a time variation of the usual form, $e^{j\omega t}$, Eqs. 4-60 become in the new variables and with impedance coefficients

$$E = \frac{j\omega\mu^T N^2 A}{x_o} I + j\omega N d \quad F$$

$$v = j\omega N d \quad I + \frac{j\omega x_o s^H}{A} F \, . \qquad (4\text{-}61)$$

These equations are also symmetrical and have the same time phase in all terms, so the definition of the coupling coefficient again produces

$$k^2 = d^2/\mu^T s^H \, , \qquad (4\text{-}62)$$

which is independent of the constants N, x_o, and A of the teroidal coil.

As in the comparable coefficient of coupling for the piezoelectric transducer in Eq. 4-36, the upper limit of +1 for this coefficient is neither obvious in the relation of k^2 to the coefficients in the equations of state nor in the appearance of any instability in practice. The piezomagnetic constant h is a function of the polarizing magnetic field in magnetostriction transducers, but its saturation value is determined by the internal structure of the material; and the maximum values of k attained in available magnetostrictive materials are far from the upper limit.

A theoretical relation of the coupling coefficient for magnetostriction to the mechanical stiffness with and without polarization, such as Eq. 4-57 for the moving-armature transducer, to show that the stiffness vanishes as the coupling approaches unity requires information not available in equations of state such as Eqs. 4-60. In these equations the constant s^H is an implicit function of the polarization, whereas in Eqs. 4-55 for the moving-armature system the mechanical impedance z_m at constant current has been expressed in Eqs. 4-53 as $(z_m'' + T^2/Z_e')$, an explicit function of the impedance z_m'' with zero polarization and of the transduction coefficient related to the polarizing flux by Eq. 4-54. A similar relation of the compliance s^H to the compliance with zero polarization and to the polarizing flux requires a knowledge of the relation of the mechanical strain to the polarization.

In general, magnetic materials are characterized by the absence of simple analytic relations, but in magnetostrictive materials the mechanical strain is, to a good approximation, proportional to the square of the magnetization, as is described by Bozorth (1951, pp. 637 and 684). A similar quadratic relation in the moving-armature transducer analyzed by Hunt (1954, p. 222) led to the expression of the mechanical impedance in Eq. 4-53 as the impedance with zero polarization reduced by a negative stiffness resulting from the polarization. The stiffness of the magnetostrictive material will, therefore, also be related to the stiffness with zero polarization by Eq. 4-57 derived for the moving-armature example, and as k approaches unity the stiffness will vanish. The requirement that the material cannot collapse with increasing polarization to increase the coupling will thus establish for magnetostrictive coupling the usual limit of +1 for the coefficient of coupling.

Although the significance of the coupling coefficient as a stability measure is similar in both reciprocal and antireciprocal transducers, the antireciprocity of magnetostrictive coupling distinguishes its equations of state and its coefficient from those of reciprocal systems. A comparison of Eqs. 4-61 for magnetostriction with the similar Eqs. 4-34 for the reciprocal piezoelectric coupling reveals that, while both sets of equations are symmetrical in form, those for magnetostriction have as independent variables F and I--one intensive and the other extensive--while those for piezoelectricity have the variables v and I--both extensive. And reference to Eqs. 4-42 for the moving-coil and Eqs. 4-55 for the moving-armature transducers will show that the equations of symmetrical form in those examples also use F and I as independent variables. It is, therefore, a characteristic of systems with antireciprocal coupling that the symmetrical forms of the equations of state or motion are those in which the independent variables are mixed extensive and intensive thermodynamic variables.

Furthermore, an expression of Eqs. 4-61 with the extensive variables I and v as the independent variables will result in equations of antisymmetrical form, as were Eqs. 4-37 for the moving-coil transducer; the mutual coefficients will again not only have opposite signs but will differ in time phase from the electrical and mechanical impedance coefficients. As in the moving-coil example, equations of this form will lead to negative feedback and to a return ratio which is $k^2/(1 - k^2)$ rather than k^2. In systems with antireciprocal coupling, therefore, the equations which are antisymmetrical in form are those in which the variables are homogeneous.

The relations of the coefficient of coupling to the equations of state are summarized for systems with antireciprocal coupling in Table 4-2, as they were for reciprocal coupling in Table 4-1.

Table 4-2. Relations of the coupling coefficient to the coefficients in the equations of state of systems with antireciprocal coupling.

Systems with antisymmetrical or antireciprocal coupling

such as

mechanical systems coupled by a gyroscope

electromechanical systems coupled by a magnetic field, as in the moving-coil, moving-armature, and magnetostriction transducers,

and having equations of state of the form

$$y_1 = a_{11}x_1 + a_{12}x_2$$

$$y_2 = a_{21}x_1 + a_{22}x_2 ,$$

in which the coefficient a_{11} is a coefficient from a different energy function--kinetic or potential--from that of a_{22}

have a coupling coefficient related to the coefficients a_{nn} by the following ratios:

if the equations of state are mixed in extensive and intensive variables

$$k^2 = \frac{a_{12}\,a_{21}}{a_{11}\,a_{22}} ,$$

if the equations of state are homogeneous in extensive and intensive variables

$$k^2 = \frac{a_{12}\,a_{21}}{a_{11}\,a_{22} + a_{12}\,a_{21}} .$$

V

MEASUREMENTS OF COUPLING COEFFICIENTS

The numerical values of the coefficient of coupling of some typical coupled systems are of interest as confirmation of the theoretical limit of the magnitude of the coefficient to a maximum of unity by the general definition in terms of feedback on the basis of physical realizability in passive systems and on the basis of stability in active systems. Values of the coefficient for many systems are to be found in the literature, and for such systems reference will be made to typical maximum values and to the methods of measurement without detailed discussion. Comparative values for electrostatic and moving-coil transducers are not available, however, and it was necessary to determine these values from measurements made on electrostatic and electrodynamic loudspeakers. For these new measurements the method and results will be considered in more detail, but the emphasis will be more on order of magnitude than on precise values of the coefficients. The coupled systems will be compared only in respect to the approach of the maximum values of their coefficient to the theoretical limit, and no attempt will be made to evaluate the merits of various forms of transduction on the basis of the coefficients of coupling.

Methods of Measurement

The value of the coefficient of coupling, defined as a real, positive, maximum return ratio in Eqs. 3-17, 3-18, 3-19, and 3-20, can be determined directly from these defining equations for any pair of coupled systems having the requisite equations of state in the form of Eqs. 3-4 and the equivalent T network of Fig. 3-1. For this it is necessary not only that the magnitudes of the impedances Z_{11}, Z_{12}, and Z_{22} be known but that these impedances be separated into their component resistive, capacitive, and inductive elements or, in mechanical and electromechanical systems, into the equivalent mechanical elements. In the electromechanical systems a further separation of the impedances into active and stray components is required by the definition of the coefficient of electromechanical coupling in terms of only the active components. When all such components can be separated and evaluated, for example by electric-impedance bridge measurements, the value of the coupling coefficient can be computed from the defining equations.

In many practical coupled systems, particularly the electromechanical, the component elements are not accessible for direct measurement, and the coefficient must be evaluated from effects measurable at the two terminal pairs of the coupled systems. For this purpose use may be made of the relations of k to the terminal impedances and to the critical frequencies of such impedances which were derived as consequences of the general definition of the coefficient. For example, in Eq. 3-25 the coefficient was shown to be equal to the difference between the input impedances with the output terminals open and shorted divided by the input impedance with open output terminals. This simple relation presumes, however, that the impedances are either all resistances or all reactances of the same sign. The evaluation of k from this relation is thus possible only if it is known that the impedances in the coupled systems are all of the same type or that measurements can be made at a frequency such that the effects of all but one type are negligible. It is also necessary to know the effects of stray or leakage components upon the terminal impedances in order to establish whether the terminals can be effectively opened or shorted by external operations. In many coupled systems it is difficult to insure that the assumed conditions are satisfied because of stray components and for other reasons; for example, many mechanical systems cannot be effectively clamped to insure zero velocity because of the magnitude of the forces involved. The method of open- and short-circuit terminations, therefore, finds only limited application.

A relation that is more generally useful--and more generally used-- is the relation of k to the critical frequencies of the input impedance when the output system is tuned or resonant. This relation was derived for simple inductive coupling and is expressed in Eq. 4-13 in terms of the frequency of resonance ω_R and the frequency of antiresonance ω_A; for capacitive coupling the similar relation is found in Eq. 4-17. In such coupled systems with no stray components, k can be similarly related to the frequencies of maximum motional impedance ω_Z and of maximum motional admittance ω_Y, as in

Eqs. 2-10. For convenient reference these relations of k to the critical frequencies will be written here for inductive coupling as

$$k^2 = 1 - (\omega_A/\omega_R)^2 = 1 - (\omega_Z/\omega_Y)^2 \qquad (5\text{-}1)$$

and for capacitive coupling as

$$k^2 = 1 - (\omega_R/\omega_A)^2 = 1 - (\omega_Y/\omega_Z)^2 \qquad (5\text{-}2)$$

These equations in terms of critical frequencies are particularly useful in systems with electromechanical coupling since the mechanical system usually has a natural resonance. In using these relations to determine k, it is again necessary to know the effects of the stray components before the critical frequencies can be assumed to measure only the components related to the coupling coefficient. But in the coupled systems of greatest interest a simple measurement of the effective Q of the resonant system often suffices to establish the degree to which the stray components can be neglected. The critical-frequency method, therefore, finds wide application, particularly in the measurement of electromechanical coupling.

The Electrostatic Transducer

The theory of electrostatic coupling and its coefficient is developed in Electroacoustics by Hunt (1954, chap. vi). It is shown there that the electrostatic transducer has the equivalent T network shown in Fig. 4-7, so the coupling is capacitive and the coupling coefficient is related to the critical frequencies by Eq. 5-2. This simple relation presumes, however, that the stray components can be neglected. To determine whether the coefficient can be evaluated from the critical frequencies in any practical example of electrostatic coupling, the system must be analyzed in sufficient detail to separate the active and stray components. For this purpose the impedance and admittance diagrams, described by Hunt (1954, chap. iv), are useful.

The measurements of electric impedance required for such diagrams were made on an experimental push-pull electrostatic loudspeaker, designed and built by A. A. Janszen. In making these measurements, a routine application of conventional bridge techniques was found to be inadequate because the range of resonance frequencies was inconveniently low (in the neighborhood of 100 c/s) and the range of impedances was inconveniently high (1-100 megohms) and because the measurements had to be made with a relatively high polarizing voltage applied to the speaker. A bridge suitable for reliable measurements of these impedances was found to be a capacitance bridge with resistive ratio arms of 10 and 10K ohms and with decade standard resistors and capacitors in series in the known arm of the bridge. A General Radio Type P-508 Owen Bridge was readily converted to this form and was used for the measurements. To make these measurements reproducible, problems of stability in the loudspeaker unit arising from variations in the diaphragm polarizing voltage and temperature were solved with the indispensable cooperation of Mr. Janszen. The results of these measurements are shown in the impedance and admittance diagrams of Fig. 5-1. From these diagrams it can be determined by the methods described by Hunt (1954, chap. iv) that in this transducer the electrical and mechanical resistances are small compared to the reactive components (mechanical Q greater than 15 and higher electrical Q), but the stray component of the capacitance C_o is not negligible.

The effect of the stray capacitance is evident in the fact that the impedance diagram shows that the frequency of maximum motional impedance is not independent of the coupling and not equal to the frequency of mechanical resonance with zero polarization (ω_o), as would be the case if there were no stray components. An investigation of this change in frequency led to the inclusion of a stray component of capacitance in the equivalent circuit and to the derivation of the relation of the frequency of maximum motional impedance (ω_Z) to ω_o, to the coupling coefficient, and to the ratio of stray to total capacitance. This relation, shown in <u>Electroacoustics</u> (p. 206), is

$$\omega_Z^2 = \omega_o^2 \left(1 - k^2 \frac{C_s + \frac{1}{2} C_o''}{C_s + \frac{1}{2} C_o} \right) \qquad (5\text{-}3)$$

where $(C_s + \frac{1}{2} C_o'')$ is the stray component and $(C_s + \frac{1}{2} C_o)$ is the total capacitance. With the aid of this equation, the value of ω_o can be determined

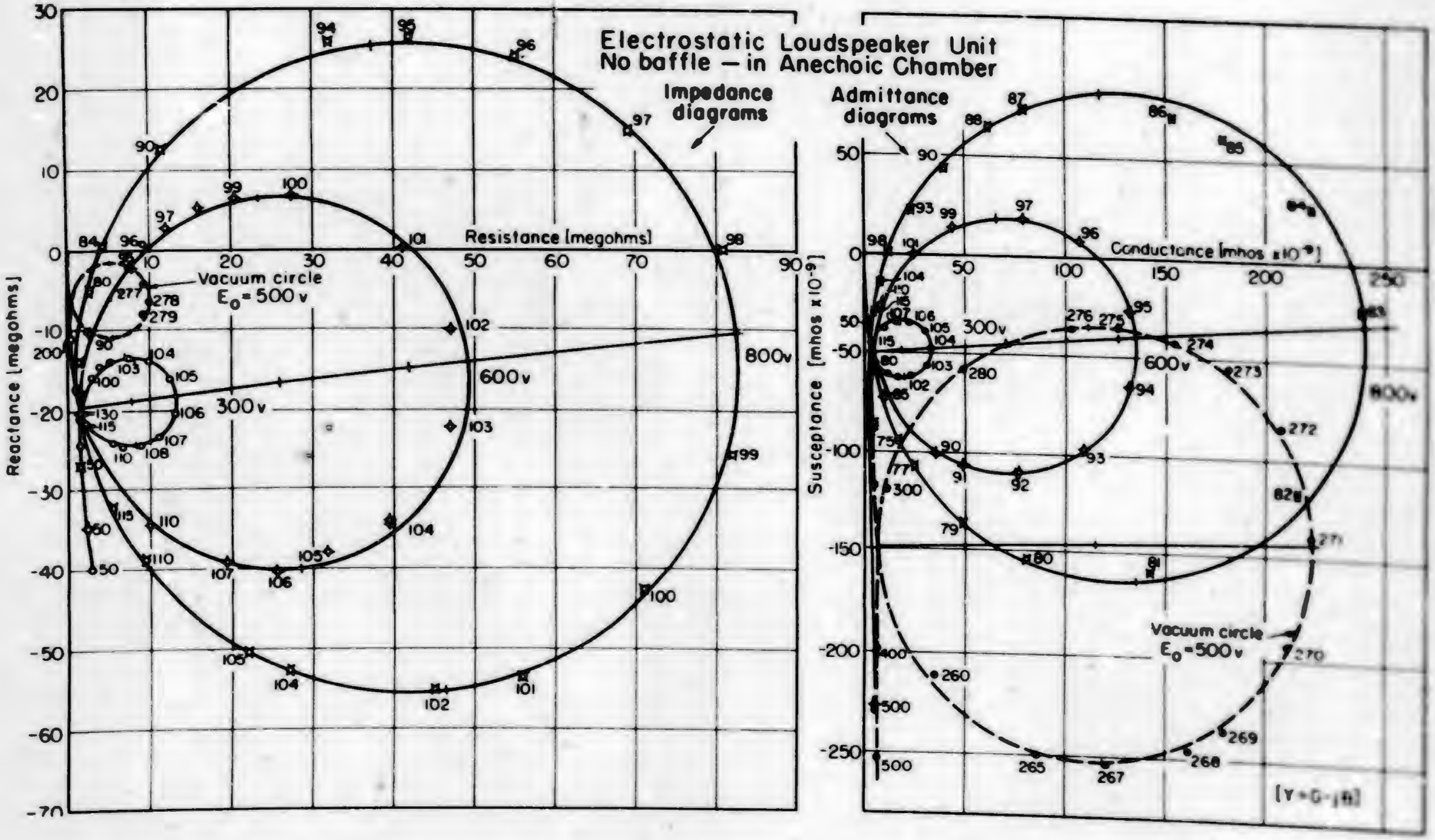


Fig. 5-1. Impedance and admittance diagrams for an experimental push-pull electrostatic loudspeaker.

from the impedance data by plotting the values of ω_Z^2 as a function of the square of the polarizing voltage E_o^2. The resulting straight line has an intercept with the $E_o = 0$ axis which determines ω_o, the frequency of maximum motional impedance when there is no stray component. The value of $f_o = 107$ c/s so obtained agrees well with an audible resonance of the unpolarized diaphragm when it was driven by a sound field of variable frequency.

With this correction for the stray component of the capacitance, the coupling coefficient can be obtained from the critical frequencies, $\omega_A = \omega_o$ and $\omega_R = \omega_Y$, and from Eq. 5-2, which can be written $k^2 = 1 - (\omega_Y/\omega_o)^2$. The resonance frequency or frequency of maximum motional admittance is independent of the stray capacitance and can be obtained directly from the admittance diagram. However, the erratic distribution of frequencies in the region of maximum motional admittance for $E_o = 800$ volts indicates error in the frequency measurements. An estimate of the correct value of f_Y can be made by plotting E_o^2 vs. ω_Y^2 from the other circles and extrapolating this linear relation to $E_o = 800$ volts; the correction is small, but the corrected value of $f_Y = 84$ c/s will be used in computing k. The values obtained for the coupling coefficient of this electrostatic loudspeaker with three values of the polarizing voltage are listed in Table 5-1.

Polarization	f_Y	f_o	$k = \sqrt{1 - (f_Y/f_o)^2}$
300 v.	104	107	0.23
600	94.6	107	0.46
800	84.	107	0.62

Table 5-1. Measured values of the coupling coefficient of an electrostatic loudspeaker.

The Moving-Coil Transducer

The theory of the electromechanical coupling in the moving-coil or electrodynamic transducer is also developed by Hunt (1954, chap. v), but the coupling coefficient is not considered there. This coefficient is, however, defined in Eq. 4-43 from Eqs. 4-42 and the equivalent T network of Fig. 4-8. Since the coupling is inductive in this case, the relation to the critical frequencies is that of Eq. 5-1, provided that the stray components can be neglected. To determine the characteristics of this type of transduction, impedance and admittance diagrams such as those used in analysing the electrostatic transducer are needed. The impedance of a Western Electric 755A Loudspeaker was, therefore, measured as a function of frequency with a Technology Instrument Corp. Type 310-A Z-Angle Meter, and the results are plotted in the impedance and admittance diagrams of Fig. 5-2.

This set of diagrams can be said to be, first of all, unfamiliar, as compared to the diagrams of moving-armature, piezoelectric, and magnetostriction transducers, which appear frequently in the literature; impedance-circle diagrams for a moving-coil speaker can be found, but the set of both impedance and admittance diagrams is rarely, if ever, to be seen. As compared to the familiar diagrams--and to those of the electrostatic, which are also rarely seen--, the moving-coil diagrams are also unusual in that the motional admittance decreases rather than increases the total admittance, so that the motional admittance circle lies to the left rather than to the right of the zero-frequency admittance.

One consequence of this is apparent when an attempt is made to use the relation of k to the frequencies of maximum motional impedance and admittance in Eq. 5-1. The frequencies are equal, and the coupling coefficient is zero! The absurdity of this leads to an immediate re-examination of the theory and to an explanation of the phenomenon. In examining the theory it is convenient to use, instead of the equivalent T network, a transformer equivalent similar to Fig. 4-9 but with the mechanical terminals shorted (F = 0) and the impedance of the resonant mechanical system brought through the transformer to the electrical side; such an equivalent circuit is shown in Fig. 5-3, in which the transformed mechanical components are denoted for convenience as $c_m(Bl)^2 = \ell$, $(Bl)^2/r_m = r$, and $l_m/(Bl)^2 = c$.

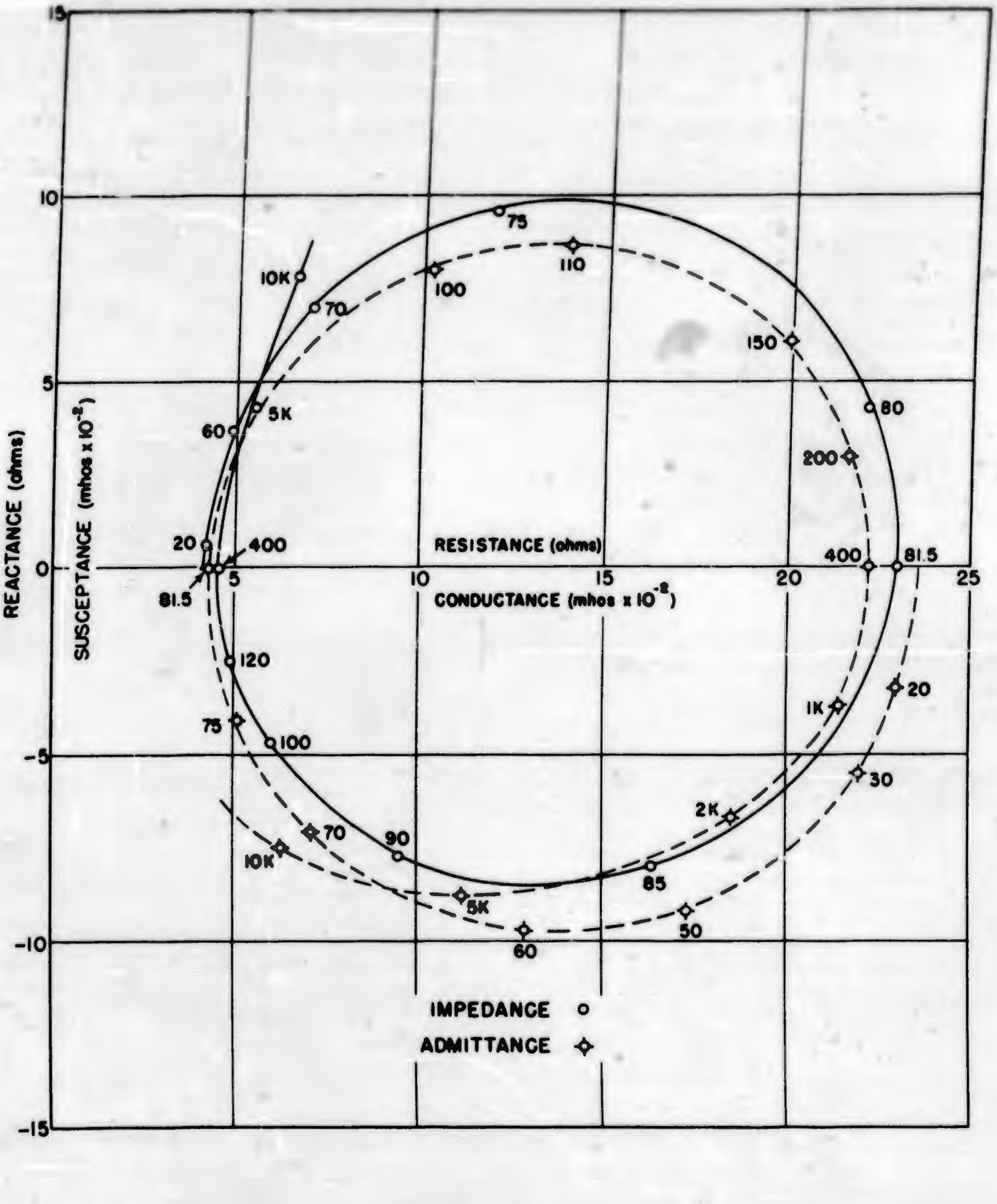


Fig. 5-2. Impedance and admittance diagrams for a dynamic or moving-coil loudspeaker without baffle (W.E. 755A). Admittance is defined as Y = G + jB.

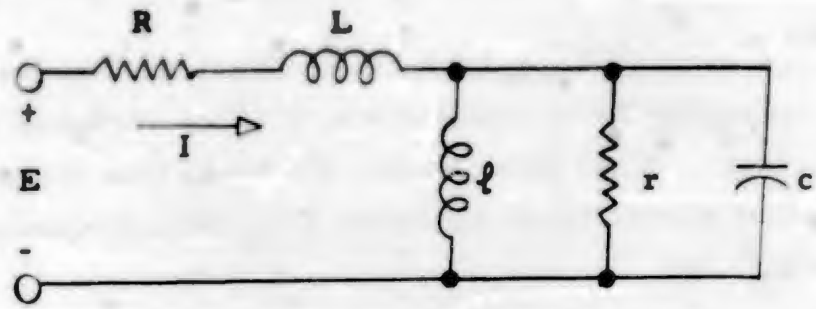


Fig. 5-3. A transformed equivalent circuit for the moving-coil transducer.

From this equivalent circuit it is obvious that both the total and the motional impedance are maximum at the antiresonance frequency of the parallel combination of ℓ and c. Hence the frequency of maximum motional impedance is $\omega_Z = 1/\sqrt{l_m c_m} = \omega_o$, the resonance frequency of the mechanical system. The total admittance must be a minimum at this antiresonance of the total impedance, but conclusions about the motional admittance cannot be made without resort to a few equations. For this purpose the total admittance of the circuit of Fig. 5-3 can be written in terms of the blocked admittance or the admittance with v = 0, $Y_b = (R + j\omega L)^{-1}$, as

$$Y_t = \frac{Y_b\left(\frac{1}{r} + \frac{1}{j\omega\ell} + j\omega c\right)}{Y_b + \frac{1}{r} + \frac{1}{j\omega\ell} + j\omega c} . \tag{5-4}$$

The motional admittance is then the total minus the blocked admittance, or

$$Y_{mot} = Y_t - Y_b = -Y_b^2\left(Y_b + \frac{1}{r} + \frac{1}{j\omega\ell} + j\omega c\right)^{-1} . \tag{5-5}$$

The frequency of maximum Y_{mot} can now be determined from Eq. 5-5. This can be done most conveniently by considering first the particular case in which the blocked electrical resistance is negligible, that is, $R \ll \omega L$ in the frequency range of interest. When R can be neglected, $Y_b = 1/j\omega L$, and Eq. 5-5 becomes

$$Y_{mot} = \left[\frac{\omega^2 L^2}{r} + j\omega L(\omega^2 Lc - \frac{L}{\ell} - 1)\right]^{-1} \quad (5\text{-}6)$$

The condition that the locus of Y_{mot} be a circle requires that the change in ωL with frequency be negligible in comparison with the variation in the second term in Eq. 5-6. When this is true, it is also true that the maximum Y_{mot} occurs when that second term vanishes, i.e., at a frequency ω_y related to $\omega_o^2 = 1/\ell c = \omega_Z^2$ by

$$\omega_Y^2 = \frac{1}{\ell c}\left[1 + \frac{\ell}{L}\right] = \omega_Z^2\left[\frac{L+\ell}{L}\right] \quad (5\text{-}7)$$

The definition of the moving-coil k in Eq. 4-43 can now be written in terms of the inductances ℓ and L as

$$k^2 = \frac{c_m(Bl)^2}{L_e + c_m(Bl)^2} = \frac{\ell}{L+\ell}; \quad (5\text{-}8)$$

and, with the aid of Eq. 5-7, k is related to ω_Z and ω_Y by

$$k^2 = 1 - (\omega_Z/\omega_Y)^2 . \quad (5\text{-}9)$$

When R, the stray component of the electrical impedance, is negligible, therefore, the coupling coefficient is related to the frequencies of maximum motional impedance and admittance by Eq. 5-9, which is the same as Eq. 5-1.

In the impedance data of Fig. 5-2, however, the resistance R is not negligible compared to the blocked reactance; on the contrary, R is considerably larger than ωL in the frequency region of the motional circle. In this case, the blocked admittance with $\omega L \ll R$ becomes $Y_b = 1/R$, and Eq. 5-5 becomes

$$Y_{mot} = -\left[R + \frac{R^2}{r} + R\left(\frac{1}{j\omega\ell} + j\omega c\right)\right]^{-1} \quad (5\text{-}10)$$

This has a maximum at the frequency $\omega_Y^2 = 1/\ell c = \omega_o^2$, which is also the frequency of maximum motional impedance ω_Z, so that in this case $\omega_Y = \omega_Z$. The appearance in the diagrams of Fig. 5-2 of the maxima of motional impedance and admittance at the same frequency and of a negative motional admittance is thus shown by Eq. 5-10 to be the result of the stray resistive component of electrical impedance being large compared to the active inductive component in this transducer.

In this example, therefore, the frequencies of maximum motional impedance and admittance are not a useful measure of the coupling coefficient. It is not safe to conclude, however, that the other critical frequencies cannot be so used, since a distinction must be made between ω_Z and ω_A and between ω_Y and ω_R whenever resistive components cannot be neglected. To make this distinction, consider the total impedance of the circuit of Fig. 5-3, which is

$$Z_t = (R + j\omega L) + \left(\frac{1}{r} + \frac{1}{j\omega \ell} + j\omega c\right)^{-1} . \tag{5-11}$$

If resonance and antiresonance are now defined as the conditions for zero phase angle in this impedance, these critical frequencies can be determined by the condition that the reactance of the first expression in Eq. 5-11 be equal and opposite to that of the second. When expressed in terms of the frequency $\omega_o^2 = 1/\ell c$, this equality becomes

$$\left(\frac{\omega^2}{\omega_o^2}\right)^2 - \left[2 + \frac{\ell}{L} - \frac{\omega_o^2 \ell^2}{r^2}\right]\left(\frac{\omega^2}{\omega_o^2}\right) + \frac{\ell}{L} + 1 = 0 \tag{5-12}$$

The troublesome stray R does not appear at all in this relation. And the mechanical r can be eliminated if the approximation can be made that $\omega_o^2 \ell^2/r^2 << 2 + (\ell/L)$. When expressed in terms of the mechanical $Q_m = r/\omega_o \ell$ and of k^2 from Eq. 5-8, this approximation becomes $Q_m >> (1 - k^2) / (2 - k^2)$, and this is easily satisfied even when the coupling is small. In fact, the requirement that the mechanical Q_m be high enough to produce an impedance circle will insure its satisfaction. With this approximation the two solutions of Eq. 5-12 take the simple form

$$\omega_R{}^2 = \omega_o{}^2 (1 + \frac{\ell}{L}) \text{ and } \omega_A = \omega_o . \qquad (5\text{-}13)$$

The coupling coefficient, related to the inductances by Eq. 5-8, can now be expressed in terms of these critical frequencies as

$$k^2 = 1 - (\omega_A/\omega_R)^2. \qquad (5\text{-}14)$$

This is again the relation derived in Eq. 4-13 for inductive coupling with no stray components, so that by defining ω_A and ω_R as the frequencies of zero phase in the total impedance the convenient relation of the coupling coefficient to critical frequencies of the impedance can still be used when the coupled systems have a large resistive stray component. Since the total admittance is the reciprocal of total impedance, the same critical frequencies of resonance and antiresonance will be found in the total admittance. From either the impedance or admittance diagram in Fig. 5-2, therefore, the critical frequencies are $f_R = 400$ c/s and $f_A = 81.5$ c/s. The coupling coefficient from Eq. 5-14 is

$$k = \left[1 - \left(\frac{81.5}{400}\right)^2\right]^{\frac{1}{2}} = 0.98 \qquad (5\text{-}15)$$

a surprisingly high value!

To confirm that the coefficient is, indeed, so close to its maximum value, it is not difficult to find values for the constants in Eq. 4-42 for this moving-coil system and to calculate k from the definition in Eq. 4-43. The inductance L_e with the mechanical system clamped can be determined easily by a bridge measurement at any frequency well above the mechanical resonance. The mechanical compliance c_m can be determined by the added-mass method of Kennelly and Pierce (1912); The mass of the moving system is increased by known increments and the resulting frequencies of mechanical resonance are determined from the frequencies of maximum motional impedance; a plot of the inverse squares of these frequencies against the added mass has as its slope the dynamic mechanical compliance. The force factor (Bl) can be determined by the current-balance method

(Ingerslev, 1953, p. 69): With the speaker in a horizontal position, a known mass is added to the voice coil, and the displaced coil is then returned to its initial position, determined by an electrical contact, by a measured current through the voice coil; the ratio of the gravitational force (mg) to the current (I) is the force factor (Bl).

For the Western Electric 755A Loudspeaker the values obtained by these methods were L_e = 0.23 millihenry, $c_m = 2.8 \times 10^{-4}$ meters/newton, and (Bl) = 3.2 newtons/ampere. With these values of the constants in Eq. 4-43, the coupling coefficient is

$$k = \left[\frac{c_m (Bl)^2}{L_e + c_m (Bl)^2} \right]^{\frac{1}{2}} = \left[\frac{2.87}{0.23 - 2.87} \right]^{\frac{1}{2}} = 0.96 \qquad (5\text{-}16)$$

Within the accuracy to be expected from these rough measurements, this confirms the value of the coefficient obtained in Eq. 5-15 from the critical frequencies.

To determine whether this high value of the coefficient found for a high-quality 7-inch loudspeaker was typical of moving-coil transducers, measurements were also made on other loudspeakers which might be expected to differ in their degree of coupling because of the size of their polarizing magnets. The critical frequencies determined from a 15-inch University C15W Super Woofer, having a very large magnet, indicate a coupling coefficient k = 0.98. And similar measurements on an inexpensive 2-inch loudspeaker with a very small magnet result in k = 0.92. It is evident from this that a high value of k is characteristic of these moving-coil systems and also that the coefficient is not very closely related to the quality of the loudspeaker as a reproducer of sound.

In all of these values of the coefficient no correction has been made for possible stray or leakage inductance; and it is quite likely that there is a leakage component because the length of the voice coil is usually made greater than the length of the region in which the polarizing flux is concentrated in order to keep the driving force of the speaker linear for large displacements. To determine the magnitude of this leakage, the polarizing magnetic field could be varied and the stray component computed from its relation to the polarizing field and the frequency of maximum motional admittance, as in the electrostatic the stray component of capacitance could be determined from Eq. 5-3 and the measured values of the frequencies of maximum motional impedance at different polarizing voltages. The necessary measurements have not yet been made because they are not essential in establishing the order of magnitude of the coupling coefficient, with which this report is most concerned. They would, however, be of interest in determining how closely the coefficient can be made to approach its limit when there is no leakage.

Other Coupled Systems

In systems with piezoelectric coupling the critical frequencies have long been used to evaluate the coupling coefficient. The methods and the results have been described in the books of Cady (1946) and of Mason (1950) and in the literature to which they refer. The necessary corrections of the measured results required by stray components are described in these references. It is necessary here to note only that another correction is sometimes required when the capacitance ratio, $r = (1 - k^2)/k^2$, at times defined as the inverse of this, is not distinguished from k^2. In numerical value, however, this causes small error, since the magnitude of k in this type of coupling is usually less than 0.3. Typical values of the coupling coefficient, tabulated by Hueter and Bolt (1955, p. 119) are 0.10 for X-cut quartz, 0.54 for Rochelle salt, 0.29 for ADP, and 0.5 for barium titanate.

In piezomagnetic coupling the coefficient has been evaluated both by measurements of the constant in the equations of state and by measurement of critical frequencies. Sussman and Ehrlich (1950) determined the coefficient from measured values of the permeability, Young's modulus, and the magnetostriction constant of the material. They report values of 0.14 for nickel and 0.17 for Hiperco. Woollett (1953) showed that the method of resonance- and antiresonance- frequency measurement commonly used for crystals is also applicable to magnetostriction transducers. The relation of the characteristic resonance and antiresonance frequencies in the electric impedance-versus-frequency curves was used by van der Burgt (1953) to find values of the coefficient for ferrites; he reports typical maximum values for nickel-cobalt ferrites of 0.24 and for lithium-cobalt ferrites of 0.10. Pigott and Kendig (1954) showed how k can be determined by a critical frequency method using the frequencies of maximum motional impedance and admittance. Hueter and Bolt (1955, p. 175) in a tabulation of coefficients give values as high as 0.31 for annealed nickel with 5100 gauss polarization, 0.12 for 45 Permalloy at 14,300 gauss, and 0.27 for Alfer at 11,500 gauss.

In coupled electrical or mechanical systems little use is made of a coefficient of coupling when the coupling is produced by a physical inductor, capacitor, or resistor--or their mechanical equivalents--common to two systems. It is most commonly applied to electric circuits in which the coupling results from a magnetic flux common to two inductors and the coupling element is a mutual inductance. In such circuits the inductances can be measured by bridge methods and the coefficient determined from its definition as $k^2 = M^2/L_1L_2$. The coupling between coils with air cores can be given a $k = 0.5$ only by special care in construction, but with iron cores it is not difficult to produce transformers with $k = 0.99$. Special toroidal-core transformers have been built to have a coefficient greater than 0.9997.

Summary of Measured Values

Typical maximum values of the coefficient of coupling which have been obtained in practical examples of coupled systems are listed in Table 5-2. Of particular interst in this tabulation are the two values which so closely approach the limit of unity, and these are the coefficients of the iron-core transformer and of the moving-coil transducer. These coupled systems have in common a coupling which is produced by a magnetic field and which is expressed in the equations of the systems by a mutual inductance. In the transformer the mutual or coupling factor is the mutual inductance of the coupled coils, and in the moving-coil transducer the coupling or transduction factor is the rate of change of the mutual inductance with respect to displacement of the moving coil, which is often expressed as the equivalent force factor (Bl). A result of such coupling through a mutual inductance is the fact that in both systems the coupling element does not appear in the impedances Z_{11} and Z_{22} of the systems without coupling. And with this type of coupling the electromechanical coupling coefficient resembles that of electrical systems in being limited to unity by physical realizability rather than by stability. In the other examples of electromechanical coupling the limit of the coefficient is set by stability, and the values of the coefficient obtained in practice have been considerably less than the maximum.

Table 5-2. Typical values of the maximum coupling coefficient commonly obtained in several examples of coupled systems.

Coupled System	Coupling Coefficient (k)
Electrical transformer	
air core	< 0.5
iron core	0.99
Moving-coil transducer (dynamic loudspeaker)	0.98
Electrostatic transducer (electrostatic loudspeaker)	0.62
Piezoelectric transducer	
X-cut quartz	0.10
Rochelle salt	0.54
ADP	0.29
Barium titanate	0.50
Magnetostriction transducer	
Annealed nickel	0.31
45 Permalloy	0.12
Alfer	0.27
Nickel-colbalt ferrites	0.24
Lithium-cobalt ferrites	0.10

UNCLASSIFIED

AD 160118

Armed Services Technical Information Agency

ARLINGTON HALL STATION
ARLINGTON 12 VIRGINIA

FOR
MICRO-CARD
CONTROL ONLY

3 OF 3

UNCLASSIFIED

VI

THE PROBLEM IN RETROSPHECT AND PROSPECT

The Benefits of Feedback

In this work bilateral coupling has been viewed as a combination of forward transmission and feedback, and the coefficients of coupling have been given a new definition in terms of the feedback by defining them as the values of the return ratio with reference to the coupling impedance which are real, positive, and maximum with respect to changes in the frequency and load. This definition has been shown to be applicable to examples of electrical and electromechanical systems having either reciprocal or antireciprocal coupling, and it has been shown to lead to the other definitions of the coefficients in common use.

The advantages of the new feedback definition are most apparent in its ability to provide answers to the questions raised by the critical examination of the previous definitions at the end of Chapter II and to the basic question of what property of the coupled systems is measured by the coupling coefficients. It gives emphasis to the fact that the coefficients are a measure of a loop transmission from input to output and back again to the input and not of just the forward transmission. And it explains thereby the appearance in the coefficients of the product of two ratios, corresponding to the forward transmission gain and the feedback-network attenuation in the usual feedback amplifier.

The necessity for shorting the output terminals in defining the coefficient is explained by the feedback definition in that it requires that the conditions external to the terminals be adjusted for maximum positive feedback; since the feedback is proportional to output current, it is maximum for zero load. The usual specification in other definitions for reciprocal coupling that all the circuit elements considered be of the same type is a consequence of the requirement in the feedback definition that the feedback measured by the coefficient be real and positive. This requirement is satisfied by networks containing elements of only one type — R, L or C — or, in networks containing series combinations of these elements, by the condition that the frequency approach zero for capacitive coupling, infinity for inductive coupling, and for resistive coupling a resonance frequency at which all reactances vanish. The feedback definition is here in agreement with those definitions of the coefficient of resistive coupling using a ratio of resistances rather than with those using a ratio of coupling resistance to reactance. But with resistive coupling the required positive feedback can also be obtained at a frequency for which the phase shift in the forward transmission is cancelled by an equal and opposite phase shift in the feedback; the coupling coefficient is in this case a ratio of resistance to reactance, as in the example of tuned coupled circuits in Chapter IV.

Although the feedback definition may be regarded as a definition in terms of energies, since it was shown to be simply related to the ratio of reflected to incident energy at the input terminals when the output was terminated in a perfect reflector, it avoids the difficulties of energy definitions in terms of primary, secondary, and mutual energies by being defined in terms of energies at the terminals. The terminals are readily identified as primary and secondary, input or output on the basis of operations made at the terminations, but the division of energy within the coupled systems required by other definitions can not be made without ambiguity and confusion even in simple systems.

Through its definition as a feedback factor or return ratio, the coupling coefficient is also related to the effects of feedback upon input impedance. This was shown to lead to some of the usual definitions of the coefficient in terms of changes in input impedance with and without coupling or with the output terminals shorted and open. When one or both of the coupled systems are resonant, the critical frequencies of the input impedance are also affected by the feedback, and the coefficient, which measures this feedback, was shown to have the simple relations to the critical frequencies which have often been used in defining the coefficient.

Perhaps the greatest advantage is to be found in the fact that a coupling coefficient defined as a maximum positive feedback factor becomes thereby a measure of either the physical realizability or the stability of the coupled systems. As was shown in the examples of coupled systems, the coefficient defined for passive systems has an upper limit of +1 imposed by the requirement that the circuit energy functions be positive definite; in feedback terms this limit is set by the fact that a positive feedback factor greater than unity requires that more energy be fed back to the source than enters the system, which is manifestly impossible in a passive system. The coefficient defined for active sysetsm, such as an electromechanical system with an energy source in the polarizing supply, was also limited to values less than +1 by the condition that the input impedances remain greater than zero; in feedback terms this limit is also imposed by the requirement that the voltage fed back cannot be of the right magnitude and phase to provide its own input or there will be "oscillation," i.e., motion in the system without an external applied force.

The problems of defining a coupling coefficient for systems with antireciprocal coupling arising from the antisymmetry of the equations of state were resolved by the definition of the coefficient in terms of feedback. The antisymmetry of the equations using the usual homogeneous independent variable, such as I and v, was shown to lead to feedback which is negative, so that the feedback definition of the coefficient cannot be applied to these equations. The equations using mixed independent variables, such as I and F, however, were found to be symmetric and to have a positive return ratio which can be defined as the coupling coefficient. As in the systems with reciprocal coupling, this definition in terms of positive feedback results in the coefficient being a measure of stability, as was shown in the example of the moving-armature transducer. In the particular case of the moving-coil transducer, the application of the feedback definition revealed that the amount of positive feedback, and hence the coupling coefficient, was limited by the requirement that the force factor (Bl) be finite; and thus in this example the limit was set by physical realizability rather than by the instability that appeared in other electromechanical transducers. The feedback definition shows also that in antireciprocal coupling positive feedback results only when the reactance of one system is opposite in sign to that of the other system, and it thus explains the appearance of the product of inductance and stiffness rather than of inductance and mass in the coefficients of antireciprocal systems, such as the moving-coil and moving-armature transducers.

The possible relations of the coupling coefficient to the effects of feedback have by no means been exhausted in this investigation. A further line of inquiry is suggested by the theorems derived by Bode (1945, pp. 76-78) relating return differences to transmissions. These theorems he characterizes as "the generalized Thevenin's theorem, applicable to unilateral as well as bilateral elements. In other words the return difference for a unilateral element plays the same role in determining the final response that the impedance relations at generator or receiver terminals would play in an ordinary calculation." Since the coupling coefficient has been defined as a return ratio for a coupling element regarded as effectively unilateral, the coefficient should embody the basic concepts involved in this generalized Thevenin's theorem. Some use of this has already been made in the application of the theorems relating feedback to input impedances, but all the implications have not been investigated.

In the introduction of the feedback definition of coupling coefficients the coupled systems were represented by an equivalent T network whose branches were composed of simple series combinations of R, L, and C. For this case of limited generality, the conditions for positive feedback led to the definition of three coefficients, corresponding to coupling through resistive, capacitive, and inductive elements. No attempt was made to obtain a single coefficient which would measure the "tightness of coupling" when the common branch of the T network contains more than one element, but a new approach to this problem is suggested by the feedback viewpoint. If the measure of the coupling is related to physical realizability or to the condition that the energy functions be positive definite, as are the coefficients here defined, the independence of the three energy functions will require three coefficients to measure coupling resulting from all three elements. But a useful coefficient might be found through further consideration of the complex return ratio and of the form of its locus in the complex plane instead of just the real, positive component considered here.

The feedback view of coupling suggested a relation of the coupling coefficient to the reflection coefficient at the terminals, and this relation was derived in Chapter III. It is possible, however, that a general definition of the coefficient could be based upon scattering instead of upon feedback. Since scattering parameters exist for every physical passive network, including systems which have no impedance matrix, a definition in terms of scattering is potentially more general than one related to impedance parameters. From the brief investigation made of the applications of the scattering matrix to this problem, it is not yet obvious that recouching the argument in terms of scattering would solve any of the problems in a more useful way, but further pursuit of this line of attack might be rewarding.

The Burdens of Antisymmetry

The application of the feedback definition of coupling coefficients to systems with antireciprocal coupling has required an investigation of the distinguishing characteristics of antireciprocity. The familiar characteristic of such coupling is the appearance in the equations of state or motion of mutual or transduction coefficients which are equal but opposite in sign, so that the matrix of coefficients is antisymmetrical. Since no equivalent T network of passive elements can be found for these antisymmetrical equations, it is customary to interchange one pair of dependent and independent variables, such as F and v, in order to obtain symmetrical equations and the ability to draw an equivalent T network for the coupled systems. This results in the mobility pairing of variables (E-v, I-F) in the equivalent networks for antireciprocal coupling instead of the impedance pairing (E-F, I-v) used for reciprocal coupling.

An alternative solution to the problem of obtaining an equivalent network is that proposed by Hunt (1954, chap. iii), using the space operator or versor k to obtain the required symmetry in the equations of antireciprocal systems with the impedance pairing of variables. The resulting equivalent network then has E paried with F and I paired with v, just as in the equivalent networks for reciprocal systems. A third method of dealing with the problem of equivalent networks is the introduction o new circuit element, such as the gyrator proposed by Tellegen (1948), to press the inverting property of antireciprocal coupling. These and other solutions to the problems introduced by antireciprocity have been summarized by Hunt (1945, p. 110).

Although an equivalent network can be obtained by various methods with either pairing of variables, the definition of the coupling coefficient in terms of feedback has served to emphaize in another way that a fundamental distinction must be made between reciprocal and antireciprocal coupling. It was found that when the equations of state are expressed with thermodynamically homogeneous dependent and independent variables, equivalent to the usual impedance pairing, the feedback resulting from the coupling is positive when the coupling is reciprocal but negative when the coupling is antireciprocal. On the other hand, when the equations are expressed with mixed variables, equivalent to the mobility pairing, the antireciprocal coupling results in positive and the reciprocal in negative feedback. A coupling coefficient defined in terms of positive feedback has the same significance as a measure of physical realizability and stability in both types of coupling, but it requires the use of mixed variables for antireciprocal coupling instead of the homogeneous variables used for reciprocal coupling. This difference appears also as an inversion of the terminal conditions for stability: a comparison of the electrostatic and moving-armature transducers, which are similar except for the symmetry of their coupling, revealed that the coupling could produce mechanical instability in the electrostatic when the voltage (intensive variable) was constant (shorted electrical terminals), but in the moving-armature transducer the instability appeared when the current (extensive variable) was constant (open electrical terminals).

The necessary distinction between the two types of coupling has been made in several ways--by a change in analogy, by the introduction of an operator, by the introduction of a new element. All of these are useful in fitting antireciprocal coupling to the tools of circuit analysis designed for reciprocal coupling, and some of them have been used in the solution of the coupling coefficient problem. They do not, however, indicate the source of the antireciprocity. The investigation of coupling coefficients has not required a knowledge of the source, but it has stimulated an interest in it. As a result, several paths to the source have been examined but not pursued at length since they diverged from the immediate problem.

The source of the differences between reciprocal and antireciprocal coupling has been found in analytic mechanics and in thermodynamics. The origins of antireciprocal coupling and its relation to the energy functions of the coupled systems have been very well described by Le Corbeiller (1929). He points out that the origin of the antisymmetric or gyroscopic coupling terms is to be found in the complementary acceleration which a point undergoes in relative movement and that such terms will appear whenever the movement of a system is related to a system of reference which is itself in motion. A familiar mechanical example of such relative motion is the gyroscope, and Le Corbeiller shows that the magnetic transducers have a similar moving reference system in their polarizing current. A consequence of this relative motion is the appearance of terms of the first degree in the velocities in the kinetic energy function (T) of the system, and whenever such terms are present the Lagrange equations lead to equations of motion with antisymmetric or gyroscopic coupling terms. To illustrate this, Le Corbeiller derives the equations of motion of the moving-coil, moving-armature, and other magnetic transducers as examples of systems in which the gyroscopic terms appear and of the electrostatic transducer as an example of a system which does not satisfy the necessary conditions for their appearance.

In thermodynamics the necessary conditions for reciprocity in coupled systems has been expressed in Onsager's principle of microscopic reversibility, developed and extended by Casimir (1946) and de Groot (1951). According to this principle, systems are reciprocal and have symmetric matrices of coefficients when the fundamental equations governing the motion of individual particles are symmetric with respect to past and future or, mathematically speaking, when the equations are invariant under a transformation $t \rightarrow (-t)$. It is shown, however, that the Lorentz force $(e\vec{v} \times \vec{B})$ produced by a magnetic field is not invariant under such a time transformation, and as a result the equations are antisymmetric and the coupling anti reciprocal whenever this force appears.

A lack of symmetry with respect to time has been evident in the equations of motion of the antireciprocal systems considered in Chapter IV. The equations of the moving-coil system, for example, were found to have a transduction coefficient (Bl) with a time phase different from that of the reactive mechanical and electrical impedances ($j\omega L$ and $1/j\omega c$) when the independent variables (I and v) were homogeneous. This difference is more evident in the differential equations of motion of this transducer, where in the electrical equation the voltage drop across the inductance is $L\,di/dt = Ld^2q/dt^2$ but that induced by the mechanical motion is $(Bl)v = (Bl)dx/dt$. In a system with reciprocal coupling, such as the electrostatic transducer, the equivalent voltages are q/C_o and x/C_{em}. In reciprocal coupling, therefore, the homogeneous variables q and x have the same relation to time, but in antireciprocal coupling they have different orders of time differentiation. A relation of the antireciprocity to the time variable was also made by de Goer (1927): He explained the antisymmetry of the equations of magnetic transducers as being a result of mechanical forces proportional to the current instead of to charge, the integral of the current, as in the electrostatic transducer; and he demonstrated that a force proportional to the derivative of the current results in reciprocal coupling by deriving the equations of a theoretical "magneto-electric" device having such a force-current relation. These appearances of asymmetry with respect to time in antireciprocal coupling suggest a relation to the reversibility with respect to time in Onsager's principle, but further investigation is required to establish the significance of this relation.

An interesting argument for the general reversibility in time of physical phenomena has been made by Blatt (1956). In a reply to comments on his article, Blatt (1957) states that in the antireciprocal systems, such as magnetic loudspeakers, which involve fixed magnetic fields the irreversibility is only approximate and "holds if, and only if, the reaction of the process back on the state of the magnet is ignored. In a fundamental analysis, the state of the magnet must be included in the description of the system under study. When this is done, the processes which seem to violate the reciprocity law are seen not to be really reciprocal processes at all!" The suggestion that the antireciprocity is connected with the reaction on the polarizing magnetic field has been investigated in a moving-coil system, but the results thus far have not been conclusive.

In his introduction of the gyrator as a new circuit element, Tellegen (1948) also looked for the origin of the reciprocity relation. He found that the reciprocity laws of thermodynamics suggest a source and lead to a relation of the reciprocity of the coupling to the choice of independent variables in the equations of state. This is very similar to the relation in this report of the reciprocity of the coupling to the choice between homogeneous and mixed variables in the equations. In his division of the variables into two classes, however, Tellegen must have relied upon a different criterion since he classified E and I as variables of the same type. Nevertheless, he agrees with the conclusions drawn here that for coupled electric circuits the equations using E and I as independent variables do not have the symmetry associated with reciprocity.

These brief considerations of the origins of antireciprocity have served to establish the conditions necessary for antireciprocal coupling, but they have not made it obvious why the coefficient of antireciprocal coupling must be defined from the equations in mixed variables in order to have the same relation to physical realizability and stability as the coefficient of reciprocal coupling. Further consideration of the reasons behind these necessary conditions should lead to a better understanding of the problem. The burdens of antisymmetry have been carried throughout this attack upon the coupling coefficient problem, but their future handling would be facilitated if some of the still loose ends could be more neatly tied together.

LIST OF REFERENCES

ALBERTI, E. 1927. "Elektrische Schwingungen," Vol. XV, Handbuch der Physik, ed. H. Geiger and K. Scheel. Berlin: Julius Springer.

BECHMANN, R. 1955. "Some Applications of the Linear Equations of State," Trans. Inst. Radio Engrs. PGUE-3, 43-62.

BLACKMAN, R. B. 1943. "Effect of Feedback on Impedance," Bell System Tech. J. 22, 269-277.

BLATT, J. M. 1956. "Time Reversal," Scientific American 195, 2, 107-114.

BLATT, J. M. 1957. Letters, Scientific American 196, 2, 12-16.

BODE, H. W. 1945. Network Analysis and Feedback Amplifier Design. New York: D. Van Nostrand Co., Inc.

BOZORTH, R. M. 1951. Ferromagnetism. New York: D. Van Nostrand Co., Inc.

BUTTERWORTH, S. 1915. "On Electrically-maintained Vibrations," Proc. Phys. Soc. (London) 27, 410-424.

BUTTERWORTH, S. and SMITH, F. D. 1931. "The Equivalent Circuit of the Magnetostriction Oscillator," Proc. Phys. Soc. (London) 43, 166-185.

CADY, W. G. 1956. Piezoelectricity. New York: McGraw-Hill Book Co., Inc.

CARLIN, H. J. 1956. "The Scattering Matrix in Network Theory," Trans. Inst. Radio Engrs. CT-3, 88-97.

CASIMIR, H. B. G. 1946. "On Onsager's Principle of Microscopic Reversibility," Philips Res. Rep. 1, 185-196.

COHEN, L. 1909. "Theory of Coupled Circuits," Bull. Bur. Standards 5, 511-541.

DE GOER, J. 1927. "Une Generalisation des Diagrammes de Kennelly," Ann. P. T. T. 16, 1009-1016.

DE GROOT, S. R. 1951. Thermodynamics of Irreversible Processes. New York: Interscience Publishers, Inc.

DEN HARTOG, J. P. 1940. Mechanical Vibrations. New York: McGraw-Hill Book Co., Inc.

DRUDE, P. 1904. "Über induktive Erregung zweier elektrischer Schwingungskreise mit Anwendung auf Perioden- und Dämpfungsmessung, Teslatransformatoren, und drahtlose Telegraphie," Ann. Physik 13, 512-561.

DYE, D. W. 1926. "The Piezo-Electric Quartz Resonator and Its Equivalent Electrical Circuit," Proc. Phys. Soc. (London) 38, 399-457.

EHRLICH, S. L. 1952. Proposal of Piezomagnetic Nomenclature for Magnetostrictive Materials, USL Report No. 173. New London, Conn.: U.S. Navy Underwater Sound Laboratory.

FELDTKELLER: R. 1937. Einführung in die Vierpoltheorie der elektrischen Nachrichtentechnik. Leipzig: S. Hirzel.

FISCHER, F. A. 1950. Grundzüge der Elektroakustik. Berlin: Fachverlag Schiele and Schon.

GOLDSTEIN, H. 1950. Classical Mechanics. Cambridge: Addison-Wesley Press, Inc.

GUILLEMIN, E. A. 1953. Introductory Circuit Theory. New York: John Wiley and Sons, Inc.

HAHNMANN, W. and HECHT, H. 1920. "Die Grundform des mechanisch-akustischen Schwingungskörpers. (Der Tonpilz)," Phys. Z. 21, 187-192.

HAHNEMANN, W. 1922. "Technische Anwendungen von Schwingungsvorgängen," Zeitschr. f. techn. Physik 3, 265-273.

HARVARD UNDERWATER SOUND LABORATORY. 1946. Magnetostrictive Transducers, Summary Technical Report of Division 6, N.D.R.C., Vol. 13. PB 77, 669. Washington: U.S. Department of Commerce, Office of Technical Services.

HASKINS, J. E. and HICKMAN, J. S. 1950. "A Derivation and Tabulation of the Piezoelectric Equations of State," J. Acous. Soc. Am. 22, 584-588.

HECHT, H. 1954. Die Elektroakustischen Wandler. Leipzig: Johann Ambrosius Barth.

HOWE, G. W. O. 1932. "Coupling and Coupling Coefficients," Wireless Engr. 9, 485-486.

HUETER, T. F. and BOLT, R. H. 1955. Sonics. New York: John Wiley and Sons, Inc.

HUNT, F. V. 1954. Electroacoustics. Cambridge: Harvard University Press.

INGERSLEV, F. 1953. Measurement of Linear and Non-Linear Distortion in Electrodynamic Loudspeakers. Copenhagen: I Kommission Hos Teknisk Forlag.

KENNELLY, A. E. and PIERCE, G. W. 1912. "The Impedance of Telephone Receivers as affected by the Motion of their Diaphragms," Proc. Am. Acad. Arts Sci. 48, 113-151.

LE CORBEILLER, P. 1929. "Origine des termes gyroscopiques," Ann. P. T. T. 18, 1-22.

LE PAGE, W. R. and SEELY, S. 1952. General Network Analysis. New York: McGraw-Hill Book Co., Inc.

MASON, S. J. 1953. "Feedback Theory--Some Properties of Signal Flow Graphs," Proc. Inst. Radio Engrs. 41, 1144-1156.

MASON, W. P. 1935. "An Electromechanical Representation of a Piezoelectric Crystal Used as a Transducer," Proc. Inst. Radio Engrs. 23, 1252-1263.

MASON, W. P. 1942-1948. Electromechanical Transducers and Wave Filters. 1st ed., 1942; 2nd ed., 1948. New York: D. Van Nostrand Co., Inc.

MASON, W. P. 1950. Piezoelectric Crystals and Their Applications to Ultrasonics. New York: D. Van Nostrand Co., Inc.

MORSE, P. M. 1936. Vibration and Sound. New York: McGraw-Hill Book Co., Inc.

PIERCE, G. W. 1910. Principles of Wireless Telegraphy. New York: McGraw-Hill Book Co., Inc.

PIGOTT, M. T. and KENDIG, P. M. 1954. "Rapid Method of Evaluating Magnetostrictive Materials for Electromechanical Transducers," J. Acoust. Soc. Am. 26, 974-976.

RAYLEIGH 1877. (John William Strutt, Third Baron). The Theory of Sound. London: Macmillan and Co., Ltd. 1st ed., Vol. I, 1877, Vol. II, 1878, 2nd ed., 1894; American reprint edition, New York: Dover Publications, 1945.

RICHARDSON, E. G. 1953. Technical Aspects of Sound, Vol. I. New York: Elsevier Publishing Co.

ROCARD, Y. 1949. Dynamique Generale des Vibrations. Paris: Masson et Cie.

SCHMERWITZ, G. 1937. "Das abweichende Verhalten der Dämpfungskopplung von den bekannten Kopplungserscheinungen," Ann. Physik 30, 209-223.

SKILLING, H. H. 1937. Transient Electric Circuits. New York: McGraw-Hill Book Co., Inc.

SKUDRZYK, E. 1954. Die Grundlagen der Akustik. Wien: Springer-Verlag.

SLATER, J. C. and FRANK, N. H. 1947. Mechanics. New York: McGraw-Hill Book Co., Inc.

SUSSMAN, H. and EHRLICH, S. L. 1950. "Evaluation of the Magnetostrictive Properties of Hiperco," J. Acoust. Soc. Am. 22, 499-506.

TELLEGEN, B. D. H. 1947. "Coupled Circuits," Philips Res. Rep. 2, 1-19.

TELLEGEN, B. D. H. 1948. "The Gyrator, A New Electric Network Element," Philips Res. Rep. 3, 81-101.

TERMAN, F. E. 1947. Radio Engineering. New York: McGraw-Hill Book Co., Inc.

TRUXAL, J. G. 1955. Automatic Feedback Control System Synthesis. New York: McGraw-Hill Book Co., Inc.

VAN DER BURGT, C. N. 1953. "Dynamical Physical Parameters of the Magnetostrictive Excitation of Extensional and Torsional Vibrations in Ferrites," Philips Res. Rep. 8, 91-132.

VIGOUREUX, P. and BOOTH, C. F. 1950. Quartz Vibrators. London: H. M. Stationery Office.

WIEN, M. 1897. "Über die Rückwirkung eines resonirenden Systems," Wied. Ann. Physik 61, 151-189.

WIEN, M. 1902. "Über die Verwendung der Resonanz bei der drahtlosen Telegraphie," Ann. Physik 8, 686-713.

WIEN, M. 1915. "Schwingungen gekoppelter Systeme, in Physik, ed. E. Warburg. Leipzig: B. G. Teubner.

WINKELMANN, A. 1908. (ed.) Handbuch der Physik. Leipzig: Johann Ambrosius Barth.

WOOLLETT, R. S. 1953. Electromechanical Evaluation of Magnetostrictive Cylinders by Resonance-Antiresonance Measurements, USL Research Report No. 225. New London, Conn.: U.S. Navy Underwater Sound Laboratory.

ZENNECK, J. 1905. Elektromagnetische Schwingungen und Drahtlose Telegraphie. Stuttgard: Ferdinand Enke.

DISTRIBUTION LIST

Acoustics

2 Chief of Naval Research — Code 411
Acoustics Branch
Washington 25, D. C.

6 Director
Naval Research Laboratory
Technical Information Officer
Washington 25, D. C.

1 Commanding Officer
Office of Naval Research Branch Office
1030 East Green Street
Pasadena 1, California

1 Commanding Officer
Office of Naval Research Branch Office
10th Floor
The John Crerar Library Building
86 East Randolph Street
Chicago 1, Illinois

1 Commanding Officer
Office of Naval Research Branch Office
495 Summer Street
Boston 10, Massachusetts

1 Commanding Officer
Office of Naval Research Branch Office
346 Broadway
New York 13, New York

2 Commanding Officer
Office of Naval Research
Navy No. 100, Fleet Post Office
New York, New York

1 Office of Technical Services
Department of Commerce
Washington, D. C.

5 Armed Services Technical Information Agency
Document Service Center
Knott Building
Dayton 2, Ohio

1 Commander
U. S. Naval Ordnance Laboratory
Sound Division
White Oak, Maryland

1 Dr. J. R. Smithson
Electrical Engineering Department
U. S. Naval Academy
Annapolis, Maryland

1 Director
U. S. Navy Underwater Sound
Reference Laboratory
P. O. Box 3629
Orlando, Florida

1 Commanding Officer and Director
U. S. Navy Electronics Laboratory
San Diego 52, California

1 Commanding Officer and Director
U. S. Navy Underwater Sound Laboratory
Fort Trumbull, New London, Connecticut

1 Commander
Naval Air Development Center
Johnsville, Pennsylvania

1 Chief, — Code 847
Bureau of Ships
Navy Department
Washington 25, D. C.

1 National Bureau of Standards
Sound Section
Washington, D. C.

1 Office of the Chief Signal Officer
Signal Corps
Pentagon
Washington 25, D. C.

1 David Taylor Model Basin
Carderock
Maryland

1 Superintendent
U. S. Navy Postgraduate School
Physics Department
Monterey, California
Attn: Professor L. E. Kinsler

1 Commanding Officer
Air Force Cambridge Research Laboratory
230 Albany Street
Cambridge 39, Massachusetts

1 National Science Foundation
1520 H Street, N. W.
Washington, D. C.

1 Chief, Office of Ordnance Research
Box CM, Duke Station
Durham, North Carolina

1 The Catholic University of America
Department of Physics
Washington 17, D. C.

1 Brown University
Department of Physics
Providence 12, Rhode Island

1 Western Reserve University
Department of Chemistry
Cleveland, Ohio
Attn: Dr. Ernest Yeager

1 Massachusetts Institute of Technology
Acoustics Laboratory
Cambridge 39, Massachusetts

1 Princeton University
Department of Electrical Engineering
Princeton, New Jersey

1 University of California
Department of Physics
Los Angeles, California

1 Chief, Physics Division
Office of Scientific Research
Hq., ARDC, P. O. Box 1395
Baltimore, Maryland

1 Commanding General
Wright-Patterson Air Force Base
Dayton, Ohio

1 Hudson Laboratories
Columbia University
145 Palisades Street
Dobbs Ferry, New York

1 Lamont Geological Laboratories
Columbia University
Torre Cliffs
Palisades, New York

1 Woods Hole Oceanographic Institution
Woods Hole
Massachusetts
Attn: Mr. Vine

1 Director
Defense Research Laboratory
University of Texas
Austin, Texas

1 Director
Ordnance Research Laboratory
The Pennsylvania State University
State College, Pennsylvania

1 Bell Telephone Laboratories
Murray Hill
New Jersey

1 Los Alamos Scientific Laboratory
P. O. Box 1663
Los Alamos, New Mexico
Attn: Dr. G. L. Campbell

1 U. S. Atomic Energy Commission
Library Section, Technical Information Branch
P. O. Box E
Oak Ridge, Tennessee
Attn: Dr. I. A. Warheit

1 U. S. Navy Postgraduate School
Monterey
California
Attn: L. E. Kinsler, H. Medwin

AD 160118

Armed Services Technical Information Agency

ARLINGTON HALL STATION
ARLINGTON 12 VIRGINIA

FOR
MICRO-CARD
CONTROL ONLY

3 OF 3

UNCLASSIFIED