

UNCLASSIFIED

AD NUMBER

AD200695

LIMITATION CHANGES

TO:

Approved for public release; distribution is unlimited.

FROM:

Distribution authorized to U.S. Gov't. agencies only; Administrative/Operational Use; APR 1958. Other requests shall be referred to Office of Naval Research, Arlington, VA 22203.

AUTHORITY

ONR ltr 9 Nov 1977

THIS PAGE IS UNCLASSIFIED

THIS REPORT HAS BEEN DELIMITED
AND CLEARED FOR PUBLIC RELEASE
UNDER DOD DIRECTIVE 5200.20 AND
NO RESTRICTIONS ARE IMPOSED UPON
ITS USE AND DISCLOSURE.

DISTRIBUTION STATEMENT A

APPROVED FOR PUBLIC RELEASE,
DISTRIBUTION UNLIMITED.

UNCLASSIFIED

A 200695

Armed Services Technical Information Agency

**ARLINGTON HALL STATION
ARLINGTON 12 VIRGINIA**

**FOR
MICRO-CARD
CONTROL ONLY**

1 OF 1

NOTICE: WHEN GOVERNMENT OR OTHER DRAWINGS, SPECIFICATIONS OR OTHER DATA ARE USED FOR ANY PURPOSE OTHER THAN IN CONNECTION WITH A DEFINITELY RELATED GOVERNMENT PROCUREMENT OPERATION, THE U. S. GOVERNMENT THEREBY INCURS NO RESPONSIBILITY, NOR ANY OBLIGATION WHATSOEVER; AND THE FACT THAT THE GOVERNMENT MAY HAVE FORMULATED, FURNISHED, OR IN ANY WAY SUPPLIED THE SAID DRAWINGS, SPECIFICATIONS, OR OTHER DATA IS NOT TO BE REGARDED BY IMPLICATION OR OTHERWISE AS IN ANY MANNER LICENSING THE HOLDER OR ANY OTHER PERSON OR CORPORATION, OR CONVEYING ANY RIGHTS OR PERMISSION TO MANUFACTURE, USE OR SELL ANY PATENTED INVENTION THAT MAY IN ANY WAY BE RELATED THERETO.

UNCLASSIFIED

AD No 200695

ASTIA FILE COPY

Columbia University
in the City of New York

DEPARTMENT OF CIVIL ENGINEERING
AND ENGINEERING MECHANICS



FILE COPY

Return to

ASTIA

ARLINGTON HALL STATION

ARLINGTON 12, VIRGINIA

Attn: T1555

EXTENSIONAL VIBRATIONS OF ELASTIC PLATES

by

R. D. MINDLIN and M. A. MEDICK

Report
to

Office of Naval Research, Washington, D. C.
U. S. Army Signal Engineering Laboratories, Fort Monmouth, N. J.

Office of Naval Research Project NR-064-388
Contract N6nr-200(00)
Technical Report No. 28
CU-36-58-ONR-266(09)-CE

Department of the Army Project 3-99-11-022
Signal Corps Contract DA-36-039 SC-72831
Interim Report
CU-7-58-SC-72831-CE

April 1958

Reproduction in whole or in part is permitted for any purpose of the United States Government.

ASTIA
NOV 1 9 1958

**Columbia University
in the City of New York**

**DEPARTMENT OF CIVIL ENGINEERING
AND ENGINEERING MECHANICS**



**EXTENSIONAL VIBRATIONS OF
ELASTIC PLATES**

by

R. D. MINDLIN and M. A. MEDICK

Report
to

Office of Naval Research, Washington, D. C.
U. S. Army Signal Engineering Laboratories, Fort Monmouth, N. J.

Office of Naval Research Project NR-064-388
Contract Nonr-266(09)
Technical Report No. 28
CU-36-58-ONR-266(09)-CE

Department of the Army Project 3-99-11-022
Signal Corps Contract DA-36-039 SC-72831
Interim Report
CU-7-58-SC-72831-CE

April 1958

Reproduction in whole or in part is permitted for any purpose of the United States Government.

Extensional Vibrations of Elastic Plates¹

by R. D. Mindlin² and M. A. Medick³

A system of approximate, two-dimensional equations of extensional motion of isotropic, elastic plates is derived. The equations take into account the coupling between extensional, symmetric thickness-stretch and symmetric thickness-shear modes and also include two face-shear modes. The spectrum of frequencies for real, imaginary and complex wave-numbers in an infinite plate is explored in detail and compared with the corresponding solution of the three-dimensional equations.

¹ This investigation was supported by the Office of Naval Research and the U. S. Army Signal Engineering Laboratories.

² Professor of Civil Engineering, Columbia University, New York, N. Y.

³ Senior Staff Scientist, Avco Research and Advanced Development Division, Avco Manufacturing Co., Lawrence, Mass. Formerly Research Assistant, Department of Civil Engineering and Engineering Mechanics, Columbia University, New York, N. Y.

Introduction

In a previous paper [1]⁴ a derivation was given of approximate equations of motion that take into account the coupling between face and thickness modes of extensional vibration of elastic plates. The thickness motion considered was that of the lowest thickness-stretch mode, in which the displacement is normal to the middle plane of the plate and the middle plane is the nodal plane. The frequency of this mode depends on Poisson's ratio (ν): the frequency increasing with increasing ν . When $\nu > 1/3$ the frequency of the lowest thickness-stretch mode is higher than the frequency, independent of ν , of the lowest, symmetric thickness-shear mode. In the latter the displacement is also unidirectional but it is parallel to the middle plane of the plate and there are two nodal planes symmetrically disposed with respect to the middle plane. In the range of Poisson's ratios commonly encountered, both thickness modes can couple with the face modes and with each other. This circumstance has a marked influence on phase and group velocities of waves and on the frequencies and shapes of high-frequency vibrational modes. In the present paper a derivation is given of approximate equations of motion which include the effects of both thickness-modes.

The approximate, two-dimensional equations are deduced from the three-dimensional equations of elasticity by a procedure based on the series expansion methods of Poisson [2] and Cauchy [3] and the integral method of Kirchhoff [4]. A detailed exposition of the procedure, using

⁴ Numbers in brackets refer to the list of references at the end of the paper.

a power series, and its application to approximations of orders zero and one are given elsewhere [5]. In approximations of the second and higher orders, awkward mathematical forms are encountered due to the lack of orthogonality of the terms of a power series. At the suggestion of W. Prager, an expansion in a series of Legendre polynomials was studied. In that case, although similar awkward forms appear as a result of the more complicated formula for the derivative, they do not occur, for the most part, until the terms of the third order are reached. Hence, the expansion in a series of Legendre polynomials is a convenient one on which to base the present second order approximation. The method of derivation and the resulting equations (with the inertia terms omitted) are closely related to E. Reissner's theory [6] of three-dimensional corrections for the equations of generalized plane stress.

The expansion in a series of Legendre polynomials of the thickness-coordinate, followed by an integration across the thickness, converts the three-dimensional equations of elasticity into an infinite series of two-dimensional equations which then are truncated to produce the approximate equations. The full series expressions of displacement, strain, stress, energies and equations of motion, in conjunction with an understanding of Rayleigh's [7] exact solution of the problem of vibrations of an infinite plate, are of aid in deciding what to include in various orders of approximation and in understanding the implications of what is discarded and what retained. The series expressions and Rayleigh's exact solution also supply both the motivation and the necessary data for making adjustments, of terms that are left after the truncation, in addition to an

adjustment of the type made by Poisson in establishing the zero-order equations. The adjustment analogous to Poisson's serves to uncouple the modes retained from the higher ones without seriously affecting the behavior of the lower ones. The additional adjustments are made to improve the match between the frequency spectra of an infinite plate as obtained from the approximate and exact equations. This is accomplished by the introduction of coefficients analogous to the shear coefficient in the Timoshenko beam equations [8] and the analogous equations for plates [9]. In the latter paper [9] it was shown how the shear coefficient may be chosen to effect a perfect match in one or another part of the spectrum depending upon the frequency range and mode of greatest interest in a particular application of the approximate equations. In the present case four such coefficients are introduced and, due to the complexity of the spectrum, several possible combinations of matching points present themselves. The choice is made here to do all the matching at zero wave number. The range of wave numbers and frequencies over which the match remains good is a measure of both the usefulness of the approximate equations and their range of applicability. In this range, solutions of the approximate equations, in the case of finite plates, may be expected to be reliable inasmuch as these solutions are composed essentially of combinations of the modes and overtones of the infinite plate.

When both the symmetric thickness-shear and thickness-stretch deformations are taken into account, important properties of the frequency spectrum contained in the exact theory are reproduced in the resulting approximate equations of the second order, whereas they do not

appear in the previous approximation of the first order [1]. These properties of the exact frequency spectrum include the imaginary loop discovered by Aggarwal and Shaw [10]; the anomalous behavior of the second and third branches with variation of Poisson's ratio [5]; the frequency minimum of the second branch [5] with its associated zero group-velocity at a non-zero wave-number and phase and group velocities of opposite sign at smaller wave-numbers, as described by Tolstoy and Usdin [11]; and, finally, a pair of complex branches which account for edge vibrations observed in experiments.

Expansion in Infinite Series

We refer the plate to rectangular coordinates x_i ($i = 1, 2, 3$) with x_1 and x_3 in the middle plane and the faces at $x_2 = \pm b$. The components of displacement u_j ($j = 1, 2, 3$) are expressed as

$$u_j = \sum_{n=0}^{\infty} P_n(\eta) u_j^{(n)}(x_1, x_3, t), \quad (1)$$

where $\eta = x_2/b$, the $P_n(\eta)$ are the Legendre polynomials

$$P_0(\eta) = 1, \quad P_1(\eta) = \eta, \quad P_2(\eta) = (3\eta^2 - 1)/2, \dots$$

$$\dots P_n(\eta) = \frac{1}{2^n n!} \frac{d^n (\eta^2 - 1)^n}{d\eta^n}, \dots$$

and the $u_j^{(n)}$, it is to be noted, are functions of the coordinates x_1 , x_3 and the time, t , only. The $u_j^{(n)}$ are the amplitudes of polynomial distributions of displacements across the thickness of the plate. For

convenience, however, they will be referred to as displacements of order n or, simply, as displacements.

Stress-Equations of Motion. The series expression for the displacement is substituted in the equation

$$\int_V (\tau_{ij,i} - \rho \ddot{u}_j) \delta u_j dV = 0, \quad (2)$$

which is obtained from the variational equation of motion (Reference [12], p. 167). In Equation (2) the integration is over the volume, V , of the plate; the τ_{ij} are the components of stress; and the summation convention for repeated indices is employed, as are the comma notation for differentiation with respect to the coordinates x_i and the dot notation for differentiation with respect to time.

When the integration with respect to η , from -1 to $+1$, is performed in Equation (2), the result is

$$\int_A \sum_{n=0}^{\infty} (b \tau_{ij,i}^{(n)} - \sum_{m=1,3,\dots}^n D_{mn} \tau_{2j}^{(n-m)} + F_j^{(n)} - \rho b C_n \ddot{u}_j^{(n)}) \delta u_j^{(n)} dA = 0, \quad (3)$$

where A is the area of the plate and

$$\tau_{ij}^{(n)} = \int_{-1}^1 P_n(\eta) \tau_{ij} d\eta, \quad F_j^{(n)} = [P_n(\eta) \tau_{2j}]_{-1}^1, \quad (4)$$

$$D_{mn} = 2(n-m)+1, \quad C_n = 2/(2n+1).$$

The $\tau_{ij}^{(n)}$ and $F_j^{(n)}$ are defined as the n th-order components of stress and face-traction, respectively; while the constants D_{mn} and C_n arise from the operations

$$\frac{dP_n}{d\eta} = \sum_{m=1,3,\dots}^n D_{mn} P_{n-m}, \quad \int_{-1}^1 P_m P_n d\eta = \begin{cases} 0, & m \neq n, \\ C_n, & m = n. \end{cases} \quad (5)$$

The appearance of $F_j^{(n)}$ and $\tau_{2j}^{(n-m)}$, in Equation (3), follows from an integration, by parts, of the terms in Equation (2) that contain $\partial/\partial x_2$.

Since Equation (3) must hold for all A and arbitrary $\delta u_j^{(n)}$, the quantity in parentheses must vanish and we arrive at the stress-equations of motion of order n :

$$b\tau_{ij,i}^{(n)} - \sum_{m=1,3,\dots}^n D_{mn} \tau_{2j}^{(n-m)} + F_j^{(n)} = \rho b C_n \ddot{u}_j^{(n)}. \quad (6)$$

In the analogous equations of motion obtained from an expansion in power series (Reference [5], page 3.04), an infinite series appears on the right hand side and no series appears on the left hand side. However, the series in Equation (6) contributes more than one term only for $n > 2$.

Strain. In the three-dimensional theory, the components of strain, ϵ_{ij} , are expressed in terms of the components of displacement by

$$2\epsilon_{ij} = u_{i,j} + u_{j,i}. \quad (7)$$

Inserting the series expansion from Equation (1) and using the formula for the derivative from Equation (5), we find

$$2 \epsilon_{ij} = \sum_{n=0}^{\infty} \sum_{m=1,2,\dots}^n [(u_{i,j}^{(n)} + u_{j,i}^{(n)}) P_n + (\delta_{2j} u_i^{(n)} + \delta_{2i} u_j^{(n)}) b^{-1} D_{mn} P_{n-m}], \quad (8)$$

where δ_{2j} is the Kronecker symbol δ_{ij} with $i=2$.

In order to define components of strain of order n , the summand in Equation (8) must be expressed as the product of P_n and a function independent of x_2 . Considering the double sum as a triangular array and interchanging the order of summation of columns and rows, we find

$$\epsilon_{ij} = \sum_{n=0}^{\infty} P_n \epsilon_{ij}^{(n)} \quad (9)$$

where the $\epsilon_{ij}^{(n)}$, defined as the components of strain of order n , are given by

$$2 \epsilon_{ij}^{(n)} = u_{i,j}^{(n)} + u_{j,i}^{(n)} + b^{-1} (2n+1) \sum_{m=1,2,\dots}^n (\delta_{2j} u_i^{(m+n)} + \delta_{2i} u_j^{(m+n)}), \quad (10)$$

In the analogous expression obtained by a power series expansion (Reference [5], p. 3.08), there is no sum over m . However, the additional terms in Equation (10) do not appear until $m=3$.

Stress-Strain Relations. The relations between the $\tau_{ij}^{(n)}$ and the $\epsilon_{ij}^{(n)}$ may be obtained by inserting the three-dimensional expressions

$$\tau_{ij} = c_{ijkl} \epsilon_{kl} = c_{ijkl} \sum_{n=0}^{\infty} P_n \epsilon_{kl}^{(n)} \quad (11)$$

in the first of Equations (4). After performing the integration, we find

$$\tau_{ij}^{(n)} = C_n c_{ijkl} \epsilon_{kl}^{(n)}. \quad (12)$$

For later use, it is convenient to have Equation (12) expressed in the reduced indicial notation, in which double indices ranging from 1 to 3 are replaced by single indices, ranging from 1 to 6, as follows:

$$\begin{aligned} \tau_1 &= \tau_{11}, & \tau_4 &= \tau_{23}, & \epsilon_1 &= \epsilon_{11}, & \epsilon_4 &= 2\epsilon_{23}, \\ \tau_2 &= \tau_{22}, & \tau_5 &= \tau_{31}, & \epsilon_2 &= \epsilon_{22}, & \epsilon_5 &= 2\epsilon_{31}, \\ \tau_3 &= \tau_{33}, & \tau_6 &= \tau_{12}, & \epsilon_3 &= \epsilon_{33}, & \epsilon_6 &= 2\epsilon_{12}. \end{aligned}$$

Then Equation (12) becomes

$$\tau_p^{(n)} = C_n c_{pq} \epsilon_q^{(n)}; \quad p, q = 1, 2, \dots, 6, \quad c_{pq} = c_{qp}. \quad (13)$$

Energy Densities. Using the strain-energy-density, U , given in the three-dimensional theory by

$$2U = c_{ijkl} \epsilon_{ij} \epsilon_{kl} = c_{pq} \epsilon_p \epsilon_q, \quad (14)$$

we define a plate-strain-energy-density

$$\bar{U} = \int_{-1}^1 U d\eta \quad (15)$$

and find, with the aid of Equations (14), (9) and the second of (5),

$$2\bar{U} = c_{pq} \sum_{n=0}^{\infty} C_n \epsilon_p^{(n)} \epsilon_q^{(n)}. \quad (16)$$

We also note that

$$2\bar{U} = \sum_{n=0}^{\infty} \tau_p^{(n)} \epsilon_p^{(n)}, \quad (17)$$

$$\tau_p^{(n)} = \partial \bar{U} / \partial \epsilon_p^{(n)}. \quad (18)$$

Similarly, using the kinetic energy-density, K , as given in the three-dimensional theory by

$$2K = \rho \dot{u}_j \dot{u}_j, \quad (19)$$

we define a kinetic energy-density of the plate by

$$2\bar{K} = 2 \int_{-1}^1 K d\eta = \rho \sum_{n=0}^{\infty} C_n \dot{u}_j^{(n)} \dot{u}_j^{(n)}. \quad (20)$$

Extensional Vibrations of Isotropic Plates

When the plate is isotropic, motions symmetric (extensional) and antisymmetric (flexural), with respect to the middle plane, may be considered separately. In the case of the former, only those components of displacement $u_j^{(n)}$ are retained for which $j+n$ is odd. As a result, only those components of stress $\tau_{ij}^{(n)}$ and strain $\epsilon_{ij}^{(n)}$ appear for which $i+j+n$ is even. In the single index notation, $\tau_p^{(n)}$, $\epsilon_p^{(n)}$, the terms which appear are $p+n$ odd for p odd and

$p+n$ even for p even.

The stress-equations of motion (6) then are

$$\frac{\partial \tau_1^{(n)}}{\partial x_1} + \frac{\partial \tau_2^{(n)}}{\partial x_2} + \frac{F_1^{(n)}}{b} = 2\rho \frac{\partial^2 u_1^{(n)}}{\partial t^2}$$

$$\frac{\partial \tau_1^{(n)}}{\partial x_1} + \frac{\partial \tau_2^{(n)}}{\partial x_2} + \frac{F_2^{(n)}}{b} = 2\rho \frac{\partial^2 u_2^{(n)}}{\partial t^2}$$

$$\frac{\partial \tau_1^{(n)}}{\partial x_1} + \frac{\partial \tau_2^{(n)}}{\partial x_2} - \frac{\tau_1^{(n)}}{b} + \frac{F_2^{(n)}}{b} = \frac{2\rho}{3} \frac{\partial^2 u_1^{(n)}}{\partial t^2}$$

(21)

$$\frac{\partial \tau_1^{(n)}}{\partial x_1} + \frac{\partial \tau_2^{(n)}}{\partial x_2} - \frac{3\tau_1^{(n)}}{b} + \frac{F_1^{(n)}}{b} = \frac{2\rho}{5} \frac{\partial^2 u_1^{(n)}}{\partial t^2}$$

$$\frac{\partial \tau_1^{(n)}}{\partial x_1} + \frac{\partial \tau_2^{(n)}}{\partial x_2} - \frac{3\tau_2^{(n)}}{b} + \frac{F_2^{(n)}}{b} = \frac{2\rho}{5} \frac{\partial^2 u_2^{(n)}}{\partial t^2}$$

$$\frac{\partial \tau_1^{(n)}}{\partial x_1} + \frac{\partial \tau_2^{(n)}}{\partial x_2} - \frac{5\tau_1^{(n)}}{b} - \frac{\tau_2^{(n)}}{b} + \frac{F_2^{(n)}}{b} = \frac{2\rho}{7} \frac{\partial^2 u_1^{(n)}}{\partial t^2}$$

$$\vdots \qquad \qquad \qquad \vdots$$

The components of strain that remain, in Equations (10), are

$$\begin{aligned}
 \epsilon_1^{(0)} &= \partial u_1^{(0)} / \partial x_1 \\
 \epsilon_2^{(0)} &= (u_2^{(0)} + u_2^{(3)} + \dots) / b \\
 \epsilon_3^{(0)} &= \partial u_3^{(0)} / \partial x_3 \\
 \epsilon_5^{(0)} &= \partial u_3^{(0)} / \partial x_1 + \partial u_1^{(0)} / \partial x_3 \\
 \epsilon_4^{(1)} &= \partial u_2^{(1)} / \partial x_3 + 3(u_3^{(2)} + u_3^{(4)} + \dots) / b \\
 \epsilon_6^{(1)} &= \partial u_2^{(1)} / \partial x_1 + 3(u_1^{(2)} + u_1^{(4)} + \dots) / b \\
 \epsilon_1^{(2)} &= \partial u_1^{(2)} / \partial x_1 \\
 \epsilon_2^{(2)} &= 5(u_2^{(3)} + u_2^{(5)} + \dots) / b \\
 \epsilon_3^{(2)} &= \partial u_3^{(2)} / \partial x_3 \\
 \epsilon_5^{(2)} &= \partial u_3^{(2)} / \partial x_1 + \partial u_1^{(2)} / \partial x_3 \\
 \epsilon_4^{(3)} &= \partial u_2^{(3)} / \partial x_3 + 7(u_3^{(4)} + u_3^{(6)} + \dots) / b \\
 \epsilon_6^{(3)} &= \partial u_2^{(3)} / \partial x_1 + 7(u_1^{(4)} + u_1^{(6)} + \dots) / b \\
 \vdots & \quad \quad \quad \vdots
 \end{aligned} \tag{22}$$

The stress-strain relations (13) reduce to

$$\begin{aligned}
 \tau_1^{(0)} &= 2[(\lambda + 2\mu)\epsilon_1^{(0)} + \lambda(\epsilon_2^{(0)} + \epsilon_3^{(0)})] \\
 \tau_2^{(0)} &= 2[(\lambda + 2\mu)\epsilon_2^{(0)} + \lambda(\epsilon_1^{(0)} + \epsilon_3^{(0)})] \\
 \tau_3^{(0)} &= 2[(\lambda + 2\mu)\epsilon_3^{(0)} + \lambda(\epsilon_1^{(0)} + \epsilon_2^{(0)})] \\
 \tau_5^{(0)} &= 2\mu\epsilon_5^{(0)} \\
 \tau_4^{(1)} &= 2\mu\epsilon_4^{(1)}/3 \\
 \tau_6^{(1)} &= 2\mu\epsilon_6^{(1)}/3 \\
 \tau_1^{(2)} &= 2[(\lambda + 2\mu)\epsilon_1^{(2)} + \lambda(\epsilon_2^{(2)} + \epsilon_3^{(2)})]/5 \\
 \tau_2^{(2)} &= 2[(\lambda + 2\mu)\epsilon_2^{(2)} + \lambda(\epsilon_1^{(2)} + \epsilon_3^{(2)})]/5 \\
 \tau_3^{(2)} &= 2[(\lambda + 2\mu)\epsilon_3^{(2)} + \lambda(\epsilon_1^{(2)} + \epsilon_2^{(2)})]/5 \\
 \tau_5^{(2)} &= 2\mu\epsilon_5^{(2)}/5 \\
 \tau_4^{(3)} &= 2\mu\epsilon_4^{(3)}/7 \\
 \tau_6^{(3)} &= 2\mu\epsilon_6^{(3)}/7 \\
 \vdots & \quad \quad \quad \vdots
 \end{aligned} \tag{23}$$

where λ and μ are Lamé's constants.

The strain-energy-density, in the form given in Equation (17), becomes

$$\begin{aligned}
 2\bar{U} = & \tau_1^{(0)} \epsilon_1^{(0)} + \tau_3^{(0)} \epsilon_3^{(0)} + \tau_5^{(0)} \epsilon_5^{(0)} \\
 & + \tau_2^{(1)} \epsilon_2^{(1)} + \tau_4^{(1)} \epsilon_4^{(1)} + \tau_6^{(1)} \epsilon_6^{(1)} \\
 & + \tau_1^{(2)} \epsilon_1^{(2)} + \tau_3^{(2)} \epsilon_3^{(2)} + \tau_5^{(2)} \epsilon_5^{(2)} \\
 & + \tau_2^{(3)} \epsilon_2^{(3)} + \tau_4^{(3)} \epsilon_4^{(3)} + \tau_6^{(3)} \epsilon_6^{(3)} \\
 & + \dots
 \end{aligned} \tag{24}$$

and, finally, the kinetic energy-density (20) is

$$\begin{aligned}
 \bar{K} = & \rho (\dot{u}_1^{(0)} \dot{u}_1^{(0)} + \dot{u}_3^{(0)} \dot{u}_3^{(0)} + \frac{1}{3} \dot{u}_2^{(1)} \dot{u}_2^{(1)} \\
 & + \frac{1}{5} \dot{u}_1^{(2)} \dot{u}_1^{(2)} + \frac{1}{5} \dot{u}_3^{(2)} \dot{u}_3^{(2)} + \frac{1}{7} \dot{u}_2^{(3)} \dot{u}_2^{(3)} + \dots).
 \end{aligned} \tag{25}$$

Truncation of Series

We begin by setting

$$\begin{aligned}
 u_1^{(n)} = 0, \quad u_3^{(n)} = 0, \quad n > 2, \\
 u_2^{(n)} = 0, \quad n > 3.
 \end{aligned} \tag{26}$$

This leaves only the components $u_1^{(0)}$, $u_3^{(0)}$, $u_2^{(1)}$, $u_1^{(2)}$, $u_3^{(2)}$ and $u_2^{(3)}$, as illustrated in Fig. 1. The terms $u_1^{(0)}$ and $u_3^{(0)}$ are the amplitudes of uniform distributions that represent the thickness

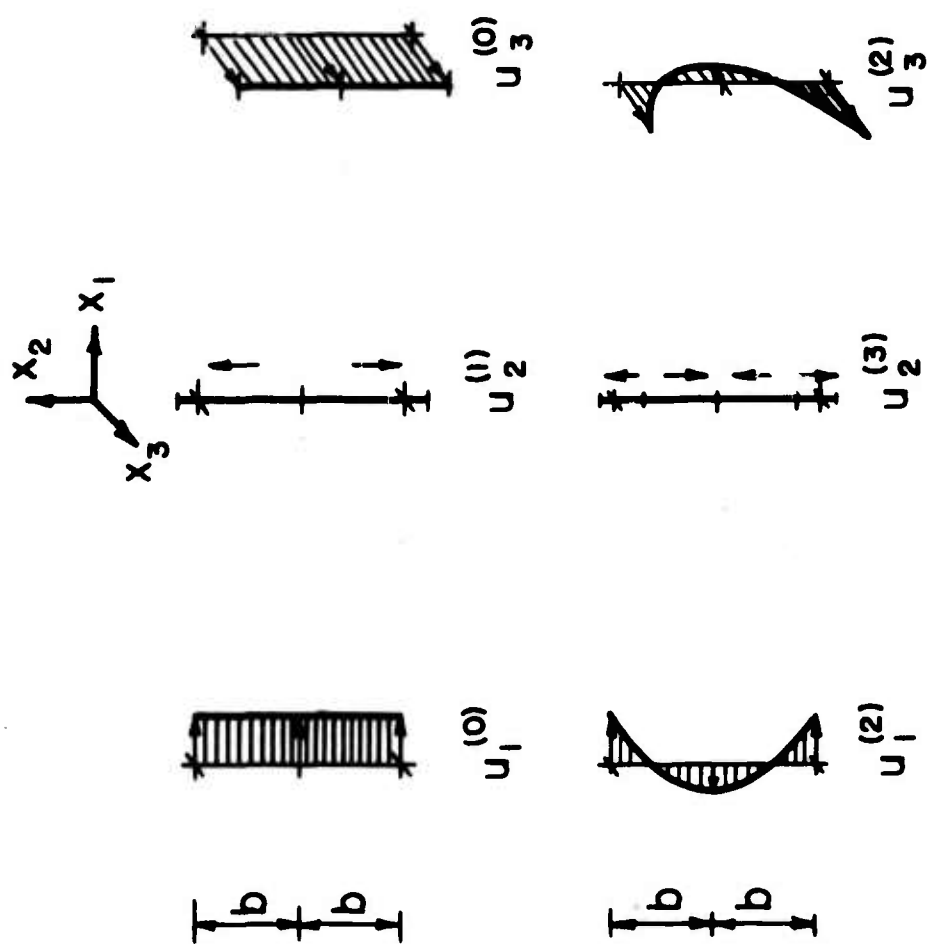


Fig. 1 - Components of displacement

distributions of displacements which occur in low frequency extensional and shear motions in the plane of the plate; $u_2^{(1)}$ is the amplitude of a linear distribution of displacement which is an approximation to the exact sinusoidal distribution in the lowest, symmetric, thickness-stretch mode; $u_1^{(2)}$ and $u_3^{(2)}$ are the amplitudes of quadratic distributions of displacements which are approximations to the sinusoidal distributions in the lowest, symmetric, thickness-shear mode and the face-shear mode of the same order.

The last "displacement" retained ($u_2^{(3)}$) is the amplitude of a cubic distribution which is an approximation to the sinusoidal distribution of displacement in the second, symmetric, thickness-stretch mode. This is a mode of higher order than is to be included in the approximate equations of the second order which we seek. However, $u_2^{(3)}$ produces the second order strain $\epsilon_2^{(2)}$ (see the eighth of Equations (22)) which, in turn, appears in the second order stress-strain relations (see the seventh, eighth and ninth of Equations (23)). In order to permit the alternating expansion and contraction, through the thickness, which should accompany the stresses $\tau_1^{(2)}$ and $\tau_3^{(2)}$ (by coupling through Poisson's ratio) and at the same time avoid coupling with the undesired higher mode, the thickness stress $\tau_2^{(2)}$ and the velocity $\dot{u}_2^{(3)}$ are set equal to zero. The eighth of Equations (23) is then used to express $\epsilon_2^{(2)}$ in terms of $\epsilon_1^{(2)}$ and $\epsilon_3^{(2)}$:

$$\epsilon_2^{(2)} = -\lambda(\epsilon_1^{(2)} + \epsilon_3^{(2)})/(\lambda + 2\mu) \quad (27)$$

and this result is substituted in the expressions for $\tau_1^{(2)}$ and $\tau_3^{(2)}$ to obtain

$$\tau_1^{(2)} = 2E'(\epsilon_1^{(2)} + \nu\epsilon_3^{(2)})/5, \quad (28)$$

$$\tau_3^{(2)} = 2E'(\epsilon_3^{(2)} + \nu\epsilon_1^{(2)})/5,$$

where $\nu = \lambda/2(\lambda + \mu)$ is Poisson's ratio and

$$E' = 4\mu(\lambda + \mu)/(\lambda + 2\mu) = E/(1 - \nu^2), \quad (29)$$

in which E is Young's modulus. In addition, the contributions of the stresses $\tau_4^{(2)}$ and $\tau_6^{(2)}$ to the strain energy (see Equation (24)) are neglected in order to destroy coupling with the unwanted higher mode when the displacements vary with x_1 and x_3 .

At this stage, the stress-equations of motion end with the fifth of Equations (21); the strain-displacement relations end with $\epsilon_5^{(2)}$ in Equations (22); the stress-strain relations end with $\tau_5^{(2)}$ in Equations (23), with the expressions for $\tau_1^{(2)}$, $\tau_2^{(2)}$ and $\tau_3^{(2)}$ replaced by Equations (27) and (28); the strain-energy-density ends with the third line in Equation (24); and the kinetic energy-density ends with the fifth term in Equation (25).

Up to this point, the process of truncating the series expressions and adjusting the remaining terms is similar to the one employed by Poisson [2] to obtain the zero-order equations of extensional motions of isotropic plates (Reference [12], p. 497); the main difference being that here the process is carried on at a level two orders higher. In

Poisson's equations only the zero-order displacements, $u_1^{(0)}$ and $u_3^{(0)}$ survive. These are the amplitudes of uniform distributions, across the thickness of the plate, which are good approximations to the nearly uniform distributions found in the lowest extensional mode of the exact theory (at long wave-lengths) and are exact for the lowest face-shear mode at all wave-lengths. The additional terms $u_2^{(0)}$, $u_1^{(1)}$ and $u_3^{(1)}$, which are now included, are the amplitudes of first and second degree polynomials and these are not good approximations to the distributions in the thickness-stretch and thickness-shear modes of the exact theory. For example, at infinite wave-length along the plate, the exact distributions are sinusoidal across the thickness. It is advisable, therefore, to introduce additional adjustments to compensate, as well as possible, for the omission of the polynomials of higher degrees.

The incorrect distributions of displacements affect the frequencies mainly through the thickness-strains and velocities. Accordingly, we make the substitutions

$$\begin{aligned} \kappa_1 \epsilon_2^{(0)} & \text{ for } \epsilon_2^{(0)}, & \kappa_3 \dot{u}_2^{(0)} & \text{ for } \dot{u}_2^{(0)}, \\ \kappa_2 \epsilon_4^{(1)} & \text{ for } \epsilon_4^{(1)}, & \kappa_4 \dot{u}_1^{(2)} & \text{ for } \dot{u}_1^{(2)}, \\ \kappa_2 \epsilon_6^{(1)} & \text{ for } \epsilon_6^{(1)}, & \kappa_4 \dot{u}_3^{(2)} & \text{ for } \dot{u}_3^{(2)}. \end{aligned}$$

in the strain-energy and kinetic energy densities, so that the coefficients κ_r ($r=1,2,3,4$) will be available for appropriate adjustments of the equations.

Finally, as a matter of expediency, we omit the term $u_2^{(3)}$ from

the strain $\epsilon_2^{(0)}$. This term complicates the equations and may be shown to have little influence on the long wave-length end of the spectrum.

Second Order Approximation

As a result of the truncations and adjustments described in the preceding section, the variables and equations of the second order approximation are

Kinetic Energy-Density:

$$\begin{aligned} \bar{K}^{(2)} = \rho [& \dot{u}_1^{(0)} \dot{u}_1^{(0)} + \dot{u}_3^{(0)} \dot{u}_3^{(0)} + \frac{1}{3} \kappa_3^2 \dot{u}_2^{(1)} \dot{u}_2^{(1)} \\ & + \frac{1}{3} \kappa_4^2 (\dot{u}_1^{(1)} \dot{u}_1^{(1)} + \dot{u}_3^{(1)} \dot{u}_3^{(1)})] \end{aligned} \quad (30)$$

Strain-Energy-Density:

$$\begin{aligned} \bar{U}^{(2)} = & (\lambda + 2\mu) (\epsilon_1^{(0)} \epsilon_1^{(0)} + \kappa_1^2 \epsilon_2^{(0)} \epsilon_2^{(0)} + \epsilon_3^{(0)} \epsilon_3^{(0)}) \\ & + 2\lambda (\kappa_1 \epsilon_2^{(0)} \epsilon_3^{(0)} + \epsilon_3^{(0)} \epsilon_1^{(0)} + \kappa_1 \epsilon_1^{(0)} \epsilon_2^{(0)}) + \mu \epsilon_3^{(0)} \epsilon_3^{(0)} \\ & + \mu \kappa_2^2 (\epsilon_4^{(1)} \epsilon_4^{(1)} + \epsilon_6^{(1)} \epsilon_6^{(1)}) / 3 \\ & + E' (\epsilon_1^{(1)} \epsilon_1^{(1)} + \epsilon_3^{(1)} \epsilon_3^{(1)} + 2\nu \epsilon_1^{(1)} \epsilon_3^{(1)}) / 5 + \mu \epsilon_5^{(1)} \epsilon_5^{(1)} / 5 \end{aligned} \quad (31)$$

Stress-Strain Relations ($\tau_p^{(n)} = \partial \bar{U}^{(n)} / \partial \epsilon_p^{(n)}$):

$$\begin{aligned}
 \tau_1^{(n)} &= 2[(\lambda + 2\mu)\epsilon_1^{(n)} + \lambda(\kappa_1\epsilon_2^{(n)} + \epsilon_3^{(n)})] \\
 \tau_2^{(n)} &= 2[\kappa_1^2(\lambda + 2\mu)\epsilon_2^{(n)} + \lambda\kappa_1(\epsilon_3^{(n)} + \epsilon_1^{(n)})] \\
 \tau_3^{(n)} &= 2[(\lambda + 2\mu)\epsilon_3^{(n)} + \lambda(\epsilon_1^{(n)} + \kappa_1\epsilon_2^{(n)})] \\
 \tau_s^{(n)} &= 2\mu\epsilon_s^{(n)} \\
 \tau_4^{(n)} &= 2\mu\kappa_2^2\epsilon_4^{(n)}/3 \\
 \tau_6^{(n)} &= 2\mu\kappa_2^2\epsilon_6^{(n)}/3 \\
 \tau_1^{(2)} &= 2E'(\epsilon_1^{(2)} + \nu\epsilon_3^{(2)})/5 \\
 \tau_3^{(2)} &= 2E'(\epsilon_3^{(2)} + \nu\epsilon_1^{(2)})/5 \\
 \tau_s^{(2)} &= 2\mu\epsilon_s^{(2)}/5
 \end{aligned} \tag{32}$$

(The components of stress are illustrated in Fig. 2).

Strain-Displacement Relations:

$$\begin{aligned}
 \epsilon_1^{(n)} &= \partial u_1^{(n)} / \partial x_1 \\
 \epsilon_2^{(n)} &= u_2^{(n)} / b \\
 \epsilon_3^{(n)} &= \partial u_3^{(n)} / \partial x_3 \\
 \epsilon_s^{(n)} &= \partial u_3^{(n)} / \partial x_1 + \partial u_1^{(n)} / \partial x_3 \\
 \epsilon_4^{(n)} &= 3u_3^{(n)} / b + \partial u_2^{(n)} / \partial x_3 \\
 \epsilon_6^{(n)} &= \partial u_2^{(n)} / \partial x_1 + 3u_1^{(n)} / b \\
 \epsilon_1^{(2)} &= \partial u_1^{(2)} / \partial x_1 \\
 \epsilon_3^{(2)} &= \partial u_3^{(2)} / \partial x_3 \\
 \epsilon_s^{(2)} &= \partial u_3^{(2)} / \partial x_1 + \partial u_1^{(2)} / \partial x_3
 \end{aligned}$$

(The components of strain are illustrated in Fig. 3).

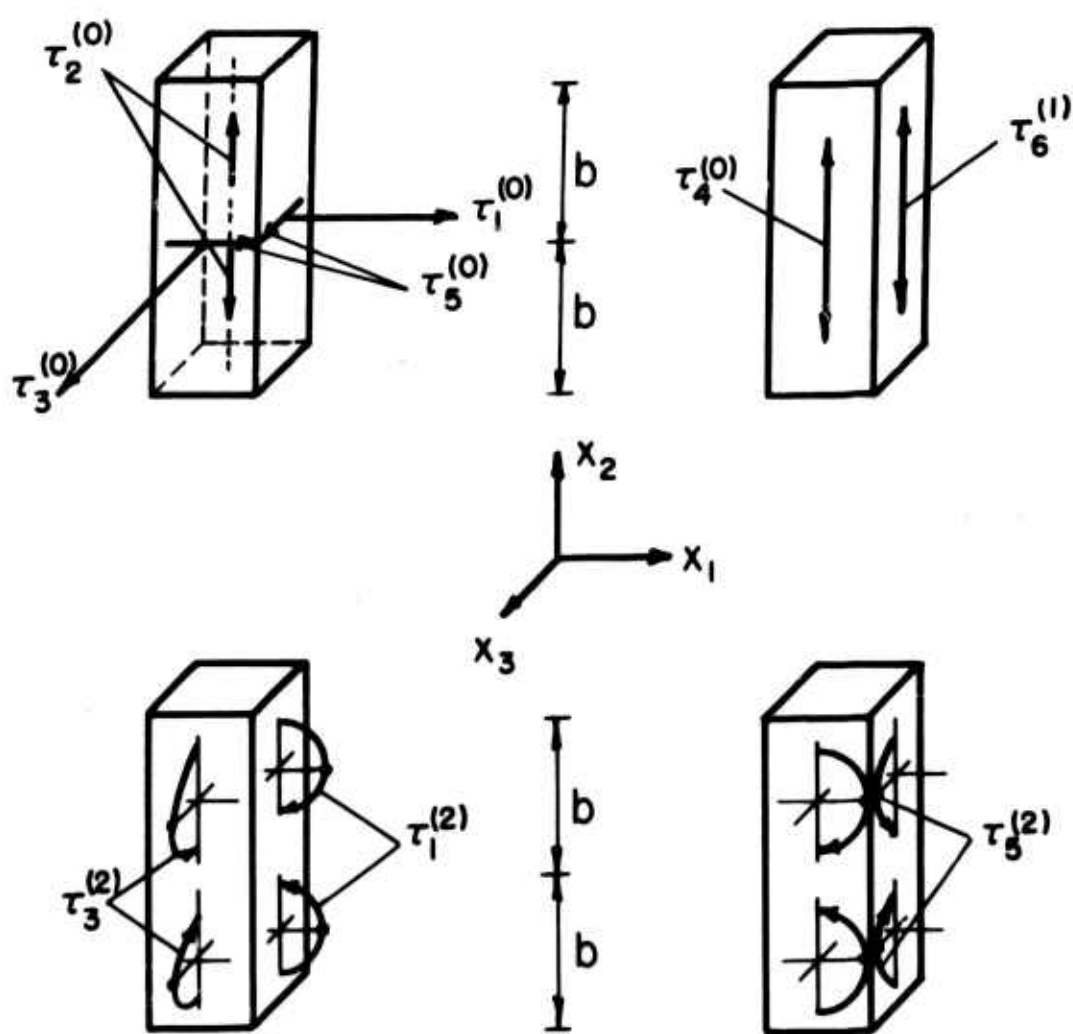


Fig. 2 - Components of stress

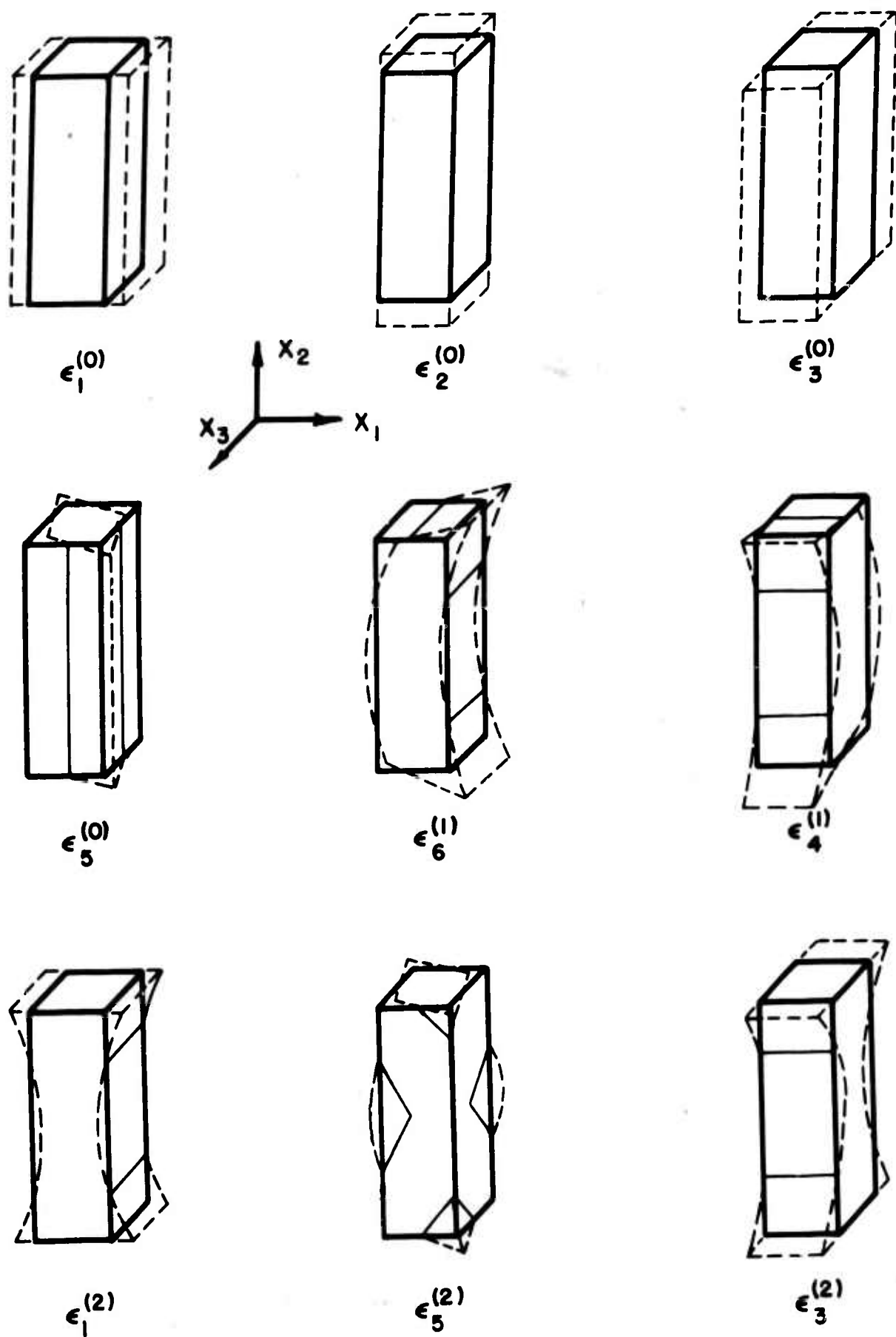


Fig. 5 - Components of strain

Stress-Equations of Motion:

$$\begin{aligned}
\frac{\partial \tau_1^{(0)}}{\partial x_1} + \frac{\partial \tau_2^{(0)}}{\partial x_3} + \frac{F_1^{(0)}}{b} &= 2\rho \frac{\partial^2 u_1^{(0)}}{\partial t^2} \\
\frac{\partial \tau_2^{(0)}}{\partial x_1} + \frac{\partial \tau_3^{(0)}}{\partial x_3} + \frac{F_2^{(0)}}{b} &= 2\rho \frac{\partial^2 u_2^{(0)}}{\partial t^2} \\
\frac{\partial \tau_6^{(1)}}{\partial x_1} + \frac{\partial \tau_4^{(1)}}{\partial x_3} - \frac{\tau_2^{(0)}}{b} + \frac{F_2^{(1)}}{b} &= \frac{2\rho \kappa_3^2}{3} \frac{\partial^2 u_2^{(1)}}{\partial t^2} \\
\frac{\partial \tau_1^{(1)}}{\partial x_1} + \frac{\partial \tau_5^{(1)}}{\partial x_3} - \frac{3\tau_6^{(1)}}{b} + \frac{F_1^{(1)}}{b} &= \frac{2\rho \kappa_4^2}{5} \frac{\partial^2 u_1^{(1)}}{\partial t^2} \\
\frac{\partial \tau_5^{(2)}}{\partial x_1} + \frac{\partial \tau_3^{(2)}}{\partial x_3} - \frac{3\tau_4^{(1)}}{b} + \frac{F_2^{(2)}}{b} &= \frac{2\rho \kappa_4^2}{5} \frac{\partial^2 u_3^{(2)}}{\partial t^2}
\end{aligned} \tag{34}$$

Displacement-Equations of Motion (obtained by substituting (33) in (32)

and the result in (34)):

$$\begin{aligned}
\mu \nabla^2 u_1^{(0)} + (\lambda + \mu) \frac{\partial e_0}{\partial x_1} + \frac{\lambda \kappa_1}{b} \frac{\partial u_2^{(0)}}{\partial x_1} + \frac{F_1^{(0)}}{2b} &= \rho \frac{\partial^2 u_1^{(0)}}{\partial t^2} \\
\mu \nabla^2 u_3^{(0)} + (\lambda + \mu) \frac{\partial e_0}{\partial x_3} + \frac{\lambda \kappa_1}{b} \frac{\partial u_2^{(0)}}{\partial x_3} + \frac{F_2^{(0)}}{2b} &= \rho \frac{\partial^2 u_3^{(0)}}{\partial t^2} \\
\mu \kappa_2^2 \nabla^2 u_2^{(1)} - \frac{3\lambda \kappa_1 e_0}{b} - \frac{3\kappa_1^2 (\lambda + 2\mu) u_2^{(0)}}{b^2} \\
+ \frac{3\mu \kappa_2^2 e_0}{b} + \frac{3F_2^{(1)}}{2b} &= \rho \kappa_3^2 \frac{\partial^2 u_2^{(1)}}{\partial t^2} \\
\frac{E'}{2} \left[(1-\nu) \nabla^2 u_1^{(1)} + (1+\nu) \frac{\partial e_2}{\partial x_1} \right] \\
- \frac{5\mu \kappa_2^2}{b} \left(\frac{\partial u_2^{(1)}}{\partial x_1} + \frac{3u_1^{(1)}}{b} \right) + \frac{5F_1^{(1)}}{2b} &= \rho \kappa_4^2 \frac{\partial^2 u_1^{(1)}}{\partial t^2} \\
\frac{E'}{2} \left[(1-\nu) \nabla^2 u_3^{(1)} + (1+\nu) \frac{\partial e_2}{\partial x_3} \right] \\
- \frac{5\mu \kappa_2^2}{b} \left(\frac{\partial u_2^{(1)}}{\partial x_3} + \frac{3u_3^{(1)}}{b} \right) + \frac{5F_2^{(1)}}{2b} &= \rho \kappa_4^2 \frac{\partial^2 u_3^{(1)}}{\partial t^2}
\end{aligned} \tag{35}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}, \quad e_0 = \frac{\partial u_1^{(0)}}{\partial x_1} + \frac{\partial u_2^{(0)}}{\partial x_3}, \quad e_2 = \frac{\partial u_1^{(2)}}{\partial x_1} + \frac{\partial u_3^{(2)}}{\partial x_3}.$$

The Coefficients κ_i

The coefficients κ_i are determined from a comparison of properties of the solutions of Equations (35), for the case of straight-crested waves in an infinite plate, with the corresponding properties of the analogous solution of the three-dimensional equations. Taking the wave-normal in the direction of x_1 , we find that Equations (35) separate into a group of three coupled equations on $u_1^{(n)}$, $u_2^{(n)}$ and $u_3^{(n)}$ and two independent equations: one on $u_1^{(n)}$ and one on $u_3^{(n)}$. The coupled equations govern three "compressional modes" and the remaining two equations govern "face-shear" modes. To consider, first, the group of three, we set $F_j^{(n)} = 0$ and

$$\begin{aligned} u_1^{(n)} &= A \sin \xi x_1 e^{i\omega t}, & u_3^{(n)} &= 0, \\ u_2^{(n)} &= B \cos \xi x_1 e^{i\omega t}, \\ u_1^{(2)} &= C \sin \xi x_1 e^{i\omega t}, & u_3^{(2)} &= 0 \end{aligned} \tag{36}$$

in Equations (35) and obtain the secular equation

$$\begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & a_{23} \\ 0 & a_{23} & a_{33} \end{vmatrix} = 0 \tag{37}$$

where

$$a_{11} = k^2 z^2 - \Omega^2$$

$$a_{22} = \kappa_2^2 z^2/3 + 4\kappa_1^2 k^2/\pi^2 - \kappa_3^2 \Omega^2/3$$

$$a_{33} = E' z^2/5\mu + 12\kappa_2^2/\pi^2 - \kappa_4^2 \Omega^2/5$$

$$a_{12} = 2\kappa_1(k^2 - 2)z/\pi$$

$$a_{23} = -2\kappa_2^2 z/\pi$$

and

$$z = 2\xi b/\pi, \quad \Omega = \omega/\omega_s, \quad \omega_s = \pi(\mu/\rho)^{1/2}/2b,$$

(38)

$$k^2 = (\lambda + 2\mu)/\mu = 2(1-\nu)/(1-2\nu).$$

We also obtain the three sets of amplitude ratios

$$A_i : B_i : C_i = 1 : a_i/z_i : a_i/b_i, \quad i = 1, 2, 3$$

(39)

where

$$a_i = \pi(\Omega^2 - k^2 z_i^2)/2\kappa_1(k^2 - 2)$$

$$b_i = \frac{3}{2\pi} \left(\frac{\pi^2 \kappa_4^2 \Omega^2}{15 \kappa_2^2} - 4 \right) - \frac{2\pi(k^2 - 1)z_i^2}{5 \kappa_1^2 k^2}$$

and the z_i are the roots of Equation (37).

It may be seen that z , which is proportional to the wave-number ξ , is the ratio of the thickness of the plate to the real half-wave-length along the plate; κ is the ratio of the velocity of dilatational waves to the velocity of equivoluminal waves in an infinite medium; and Ω is the ratio of the frequency to the frequency of the lowest anti-symmetric thickness-shear mode.

The relation between Ω and z , in Equation (57), should match, as closely as possible, the corresponding relation obtained from the three-dimensional equations. The match is improved, within the framework of the present approximation, by choosing appropriate values for the coefficients κ_i ; but, since the κ_i are constants, a perfect match can be made only at one value of z for each of them. Now, large enough z corresponds to frequencies high enough to enter the range of modes that have not been included in the approximate equations. In a plate vibrating at such high frequencies under practical (i.e., not mixed) edge conditions, the high modes would, in general, couple with the ones of lower order. Thus the applicability of the approximate equations to bounded plates is limited to frequencies below the lowest frequency of the lowest neglected mode. There is, then, little advantage to be gained in matching the approximate and exact solutions at short wave-lengths (large z) at the expense of a good match at long wave-lengths. In fact, we go to the extreme and do all of the matching in the neighborhood of $z = 0$ primarily because of the intricate behavior of the exact solution at long wave-lengths and also because this choice results in a reasonably good match out to as short wave-lengths as the frequency limitation permits.

When $z = 0$, Equation (37) has the three roots

$$\Omega^2 = 0, \quad 12\kappa_1^2 k^2 / \pi^2 \kappa_3^2, \quad 60\kappa_2^2 / \pi^2 \kappa_4^2 \quad (40)$$

corresponding to the amplitude ratios

$$A : B : C = 1 : 0 : 0, \quad 0 : 1 : 0, \quad 0 : 0 : 1$$

respectively. It may be seen that, at infinite wave-length, the three "compressional modes" uncouple to form an extensional, a thickness-stretch and a thickness-shear mode. The three limiting frequencies are to be compared with the exact values

$$\Omega^2 = 0, \quad k^2, \quad 4 \quad (41)$$

of the cut-off frequencies of the extensional, first thickness stretch and first symmetric thickness shear modes in Rayleigh's exact solution [7]. Hence, we set

$$\kappa_1 / \kappa_3 = \pi / \sqrt{12}, \quad \kappa_2 / \kappa_4 = \pi / \sqrt{15} \quad (42)$$

and these relations between pairs of κ_i are used in what follows.

We proceed to examine the roots of Equation (37) in the neighborhood of the roots given in Equations (40) or (41) (which are now the same).

First, for $z \ll 1$, $\Omega \ll 1$, we find

$$\Omega^2 = 4z^2(k^2 - 1)/k^2, \quad (43)$$

i.e., phase velocity $[\epsilon/\rho(1-\nu^2)]^{1/2}$. This is exactly the result obtained from the three-dimensional theory for the lowest mode at long wave-lengths.

Next, for $z \ll 1$, $\Omega = k + \epsilon$, $\epsilon \rightarrow 0$, Equation (37) gives

$$\frac{d\Omega}{dz} = \frac{z}{k^2-4} \left(\frac{\pi^2 k \kappa_2^2}{12 \kappa_1^2} + \frac{(k^2-2)^2(k^2-4)}{k^3} \right) \quad (44a)$$

and in the limit, as $k \rightarrow 2$ and $z \rightarrow 0$,

$$\frac{d\Omega}{dz} = \pm \frac{\pi \kappa_2}{4 \kappa_1 \sqrt{3}}. \quad (45a)$$

Finally, for $z \ll 1$, $\Omega = 2 + \epsilon$, $\epsilon \rightarrow 0$, we find

$$\frac{d\Omega}{dz} = \frac{\pi^2 z}{6(k^2-4)} \left(\frac{4(k^2-1)(k^2-4)}{5 k^2 \kappa_2^2} - \frac{\kappa_2^2}{\kappa_1^2} \right) \quad (46a)$$

and in the limit, as $k \rightarrow 2$ and $z \rightarrow 0$,

$$\frac{d\Omega}{dz} = \mp \frac{\pi \kappa_2}{4 \kappa_1 \sqrt{3}}. \quad (47a)$$

Equations (44a) to (47a) are to be compared with the exact values (Reference [5], p. 2.43) given in the following correspondingly numbered equations:

$$\frac{d\Omega}{dz} = \frac{4kz}{\pi} \left(\frac{\pi}{4} + \frac{4}{k^3} \cot \frac{\pi k}{2} \right) \quad (44b)$$

$$\frac{d\Omega}{dz} = \pm \frac{2}{\pi} \quad (45b)$$

$$\frac{d\Omega}{dz} = \frac{z}{\pi} \left(\frac{\pi}{2} - \frac{4}{k} \tan \frac{\pi}{k} \right) \quad (46b)$$

$$\frac{d\Omega}{dz} = \mp \frac{2}{\pi} \quad (47b)$$

Equating (44a) to (44b) and (46a) to (46b), we find

$$\frac{\kappa_2^2}{\kappa_1^2} = \frac{48(k^2-4)}{\pi^3 k^4} \left[\pi(k^2-1) + 4k \cot \frac{\pi k}{2} \right], \quad (48)$$

$$\frac{1}{\kappa_1^2} = \frac{5k^2 \kappa_2^2}{4\kappa_1^2(k^2-1)(k^2-4)} \left[\frac{\kappa_2^2}{\kappa_1^2} + \frac{6(k^2-4)}{\pi^3} \left(\frac{\pi}{2} - \frac{4}{k} \tan \frac{\pi}{k} \right) \right].$$

We also note, from Equations (48), that

$$\lim_{k \rightarrow 2} \frac{\kappa_2^2}{\kappa_1^2} = \frac{192}{\pi^4}, \quad (49)$$

$$\lim_{k \rightarrow 2} \frac{1}{\kappa_1^2} = \frac{3840(\pi^2-6)}{\pi^6}.$$

Equations (42) and (48) give the values of the four coefficients κ_i in terms of Poisson's ratio. As a result of these relations, all of the ordinates, slopes and curvatures of the curves Ω vs. z ,

characterizing the frequency spectrum of the compressional modes of the approximate theory, are identical with those of the exact solution at $z = 0$. Equations (48) and, in fact, all of the equations of this second-order approximation should be used only for Poisson's ratios less than about 7/16. Above that value the frequency of the thickness-stretch mode is so high in the spectrum that, in the exact theory, coupling with the second symmetric thickness-shear mode becomes important and that mode is not included in the present approximation.

Turning, now, to the two face-shear modes, we set $F_j^{(n)} = 0$,

$$\begin{aligned} u_1^{(0)} &= u_2^{(0)} = u_3^{(0)} = 0, \\ u_3^{(0)} &= A_1 \cos \xi x, e^{i\omega t}, \\ u_3^{(2)} &= A_2 \cos \xi x, e^{i\omega t}, \end{aligned} \tag{50}$$

in Equations (35), and find that $u_3^{(0)}$ and $u_3^{(2)}$ are not coupled and have frequencies given by

$$\Omega^2 = z^2, \quad 4 + z^2/\kappa_2^2 \tag{51}$$

respectively. These are to be compared with the frequencies

$$\Omega^2 = z^2, \quad 4 + z^2 \tag{52}$$

of the face-shear modes of orders zero and two obtained from the exact equations. It may be seen that the zero-order face-shear mode is reproduced exactly, in the approximate equations, for all wave-lengths.

The second order face-shear mode has the correct ordinate (Ω) and slope ($d\Omega/dz$) at $z=0$ but it does not have the correct curvature because of the presence of the coefficient κ_4 . By choosing $\kappa_4=1$, this discrepancy could be eliminated, but only at the expense of incorrect behavior of one of the coupled modes.

Frequency Spectrum

The spectrum of frequencies of the five possible modes of vibration of an infinite plate, which the approximate theory contains, is given by the five independent roots of Equations (37) and (51). We consider, first, the three compressional modes.

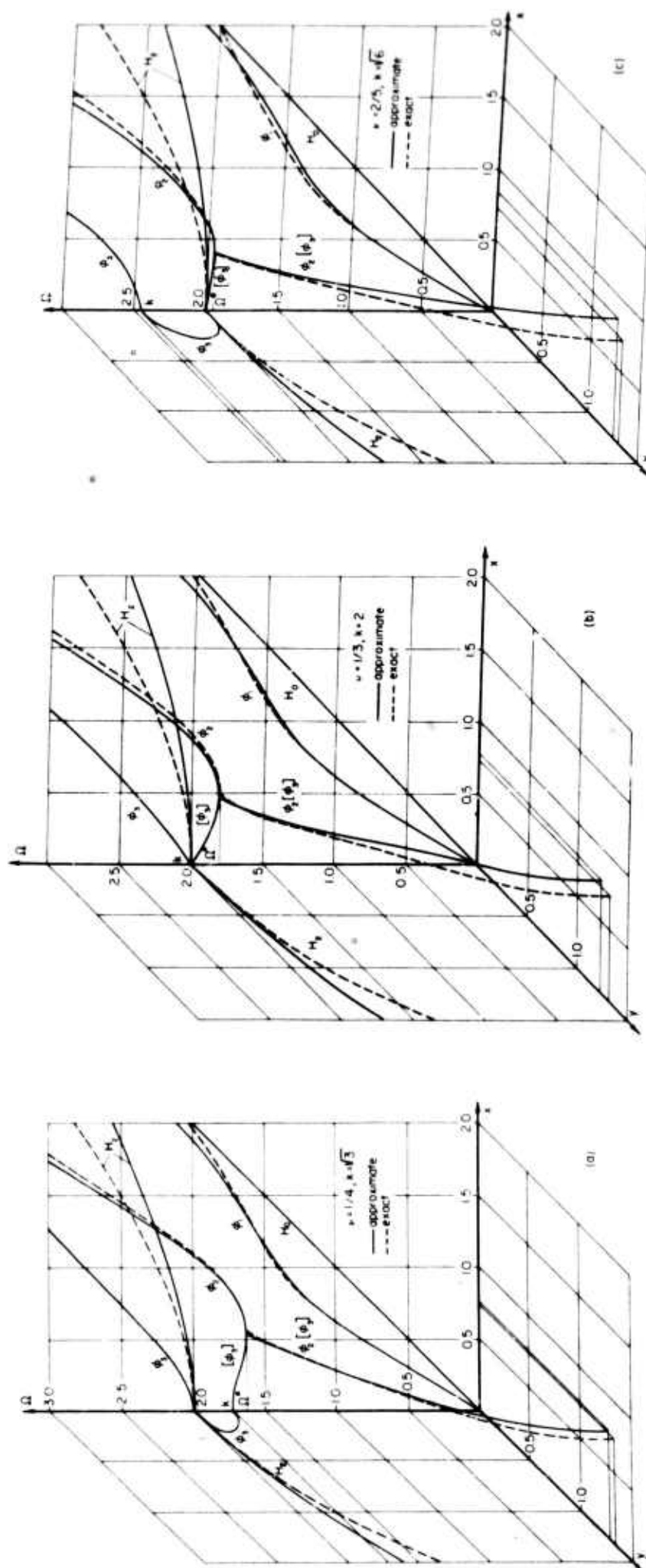
Equations (37) relate Ω^2 and z^2 but only real, positive values of Ω have physical significance. For real, positive Ω , the roots z^2 may be three real positive or two real positive and one real negative or one real positive and two conjugate complex. Hence the roots

$$z = x + iy \quad (53)$$

may be three real or two real and one imaginary or one real and two conjugate complex. In addition, the character of the spectrum is different according as Poisson's ratio is less than, equal to or greater than $1/3$.

We consider the case $\nu < 1/3$ (i.e., $k < 2$) first. Then, when $\Omega > 2$, the three roots of Equation (37) are real ($y = 0$). They are illustrated in Fig. 4a (for $\nu = 1/4$, i.e., $k^2 = 3$) by the three curves (full lines) marked ϕ_1, ϕ_2, ϕ_3 in the plane $y = 0$ and the region $\Omega > 2$. As Ω drops below $\Omega = 2$, the largest

Fig. 4 - Frequency spectrum of an infinite plate. Comparison of approximate and exact branches for three values of Poisson's ratio.



root, ϕ_1 , approaches zero with asymptotic behavior given by Equation (45); the intermediate root, ϕ_2 , approaches a minimum, Ω^* , below which there are no real roots other than ϕ_1 ; the smallest root, ϕ_3 , approaches a minimum in the real plane $y=0$ at $\Omega=2$, $x=0$ with asymptotic behavior given by Equation (46a) or (46b) which are the same in view of Equations (48). Continuing with the root ϕ_3 , as Ω drops below $\Omega=2$, the root is imaginary ($x=0$), again with asymptotic behavior given by Equation (46a) or (46b) in the neighborhood of $\Omega=2$. As Ω approaches k , from above, the root ϕ_3 forms a loop in the imaginary plane ($x=0$) and approaches $\Omega=k$, $y=0$ with asymptotic behavior given by Equation (44a) or (44b). As Ω continues to drop below $\Omega=k$, the root ϕ_3 becomes real, with behavior in the neighborhood of $\Omega=k$, $x=0$ again given by (44a) but now x is negative. Upon further diminution of Ω , the root ϕ_3 approaches a minimum at $\Omega=\Omega^*$ and negative x . This portion of ϕ_3 (i.e., between $\Omega=k$ and $\Omega=\Omega^*$) is identified in Fig. 4a by $[\phi_3]$, where the brackets indicate that it is the reflection in the plane $x=0$ that is shown. Finally, when $\Omega < \Omega^*$, the roots ϕ_2 and ϕ_3 are conjugate complex ($z = \pm x + iy$). One of them is shown, in Fig. 4a, as the curve marked ϕ_2 ; this curve is also $[\phi_3]$, i.e., the reflection of the conjugate root ϕ_3 in the plane $x=0$.

As Poisson's ratio approaches $1/3$ from below, the frequency of the thickness-stretch mode increases, approaching the frequency of the thickness shear mode; i.e., at $z=0$ the intercept $\Omega=k$ approaches $\Omega=2$. Conjointly, the curvature of ϕ_3 at $\Omega=k$, $z=0$

approaches negative infinity in the plane $y = 0$ and positive infinity in the plane $x = 0$ while both slopes remain zero: all in accordance with Equation (44a) or (44b). At the same time, the curvature of ϕ , at $\Omega = 2$, $z = 0$ approaches positive infinity in the plane $y = 0$ and negative infinity in the plane $x = 0$ while the slopes remain zero: all in accordance with Equation (46a) or (46b). Meanwhile, the imaginary loop shrinks toward the point $\Omega = 2$, $y = 0$.

At $\nu = 1/3$, the thickness-stretch and thickness-shear modes have the same frequency and the slopes of the two branches of ϕ , become $2/\pi$ in accordance with Equations (45a) and (47a). This situation is illustrated in Fig. 4b. Here, again, the portions marked [ϕ] are the reflections, in the plane $x = 0$ of the actual branches.

When $\nu > 1/3$ the thickness-stretch mode has a frequency higher than that of the thickness-shear mode. The spectrum (illustrated in Fig. 4c for $\nu = 2/5$) has now undergone an important change in that the imaginary branch loops up, from $\Omega = 2$, instead of down.

Since only powers of z^2 occur in Equation (35), there is another set of physically significant roots given by the reflections of the curves of Fig. 4 in the planes $x = 0$ and $y = 0$.

Turning, now, to the face-shear modes, the first root in Equation (51) yields the straight line marked H_0 in Fig. 4. The second face-shear mode gives the roots marked H_2 in the figures; these roots are real for $\Omega > 2$ and imaginary for $\Omega < 2$. As before, there is an additional set of roots given by the reflections of the curves H_0 and H_2 in the planes $x = 0$ and $y = 0$.

The spectra of the corresponding five modes, as computed from Rayleigh's solution of the exact equations, are also shown⁵ in Figs. 4a, b and c (as dashed lines). The importance of the introduction of the coefficients κ_i and their definitions, in terms of Poisson's ratio, is apparent. Without these coefficients, the extraordinarily complicated behavior of the branches of the exact frequency spectrum at long wavelengths would not be reproduced in the approximate equations. In the whole range of frequencies and wave numbers depicted, the approximate spectrum is reasonably close to the exact one; the poorest representation occurring in portions of the complex branches and the spectrum of the second face-shear mode. On the whole, fair results may be expected from solutions of the approximate equations in the case of finite plates.

The shapes of modes in the various ranges of frequency can be anticipated from the real, imaginary or complex character of the roots. In rectangular coordinates, for example, real roots correspond to trigonometric mode-shapes; imaginary roots to exponential or hyperbolic mode-shapes; and complex roots correspond to modes whose shapes are given by products of trigonometric and exponential or hyperbolic functions. A striking example of the latter is to be found in the experiments with circular disks by Shaw [13].

The phase velocities (v) and group velocities (v_g) can be visualized readily from Figs. 4 inasmuch as

⁵ The complex roots were kindly supplied by Dr. Morio Onoe.

$$\begin{aligned}
 v &= \frac{\omega}{\xi} = \frac{\Omega}{x} \sqrt{\frac{\mu}{\rho}}, \\
 v_g &= \frac{d\omega}{d\xi} = \frac{d\Omega}{dx} \sqrt{\frac{\mu}{\rho}}.
 \end{aligned}
 \tag{53}$$

For example, the phenomenon of phase and group velocities of opposite sign, noticed by Tolstoy and Usdin [11] in Rayleigh's solution, is represented by the branch $[\phi_3]$ in the real plane. Also, the minimum at Ω^* represents zero group velocity and non-zero phase velocity. As the wave-length approaches zero ($x \rightarrow \infty$) the frequencies rise beyond the range of applicability of the equations and the asymptotic behavior of the velocities of the three compressional modes are

$$v = v_g = \frac{\kappa_2}{\kappa_1} \sqrt{\frac{\mu}{\rho}}, \sqrt{\frac{\lambda+2\mu}{\rho}}, \frac{1}{\kappa_2} \sqrt{\frac{E}{\rho(1-\nu^2)}}.
 \tag{54}$$

According to the exact theory, the first of these should be the velocity of Rayleigh surface waves and the second and third should be the velocity of equivoluminal waves.

Additional Results

a. The equations of compatibility may be obtained by eliminating the displacements from Equations (52), with the result:

$$\begin{aligned}
\frac{\partial^2 \epsilon_1^{(0)}}{\partial x_3^2} + \frac{\partial^2 \epsilon_3^{(0)}}{\partial x_1^2} &= \frac{\partial^2 \epsilon_5^{(0)}}{\partial x_1 \partial x_3} \\
\frac{\partial^2 \epsilon_2^{(0)}}{\partial x_1^2} + \frac{3 \epsilon_1^{(2)}}{b^2} &= \frac{1}{b} \frac{\partial \epsilon_6^{(1)}}{\partial x_1} \\
\frac{3 \epsilon_2^{(2)}}{b^2} + \frac{\partial^2 \epsilon_3^{(2)}}{\partial x_3^2} &= \frac{1}{b} \frac{\partial \epsilon_6^{(1)}}{\partial x_3} \\
\frac{\partial^2 \epsilon_1^{(2)}}{\partial x_3^2} + \frac{\partial^2 \epsilon_3^{(2)}}{\partial x_1^2} &= \frac{\partial^2 \epsilon_5^{(2)}}{\partial x_1 \partial x_3} \\
\frac{3}{b} \frac{\partial \epsilon_1^{(2)}}{\partial x_3} &= \frac{1}{2} \frac{\partial}{\partial x_1} \left(-\frac{\partial \epsilon_4^{(1)}}{\partial x_1} + \frac{3 \epsilon_5^{(2)}}{b} + \frac{\partial \epsilon_6^{(1)}}{\partial x_3} \right) \\
\frac{\partial^2 \epsilon_2^{(2)}}{\partial x_1 \partial x_3} &= \frac{1}{2b} \left(-\frac{3 \epsilon_5^{(2)}}{b} + \frac{\partial \epsilon_6^{(1)}}{\partial x_3} + \frac{\partial \epsilon_4^{(1)}}{\partial x_1} \right) \\
\frac{3}{b} \frac{\partial \epsilon_3^{(2)}}{\partial x_1} &= \frac{1}{2} \frac{\partial}{\partial x_3} \left(-\frac{\partial \epsilon_4^{(1)}}{\partial x_3} + \frac{\partial \epsilon_5^{(2)}}{\partial x_1} + \frac{3 \epsilon_5^{(2)}}{b} \right)
\end{aligned} \tag{55}$$

The first of (55) is the usual compatibility equation of generalized plane stress and the remaining six equations correspond to the ordinary six compatibility equations; the main differences being that here the components $\epsilon_p^{(n)}$ have a factor 3 and the operator $\partial/\partial x_2$ is replaced by $1/b$.

b. It may also be shown that there are nine dislocations when the strains and their derivatives are continuous. Three of them are the two translational and one rotational dislocations of generalized plane stress. Of the remaining six, three are translational (in the displacements $u_1^{(1)}, u_2^{(1)}, u_3^{(1)}$) and three are rotational, in the component of rotation

$$\frac{1}{2} \left(\frac{3 u_2^{(1)}}{b} - \frac{\partial u_3^{(1)}}{\partial x_3} \right), \frac{1}{2} \left(\frac{\partial u_3^{(1)}}{\partial x_1} - \frac{\partial u_1^{(1)}}{\partial x_3} \right), \frac{1}{2} \left(\frac{\partial u_2^{(1)}}{\partial x_1} - \frac{3 u_1^{(1)}}{b} \right). \tag{56}$$

c. A theorem of uniqueness of solutions of the approximate equations of motion may be established along the lines of Neumann's theorem ([11], p. 176 and [3]). This leads to the following conditions sufficient for a unique solution (in the absence of discontinuities and singularities):

- i. Initial values of $u_1^{(0)}$, $u_2^{(0)}$, $u_3^{(0)}$, $u_1^{(1)}$ and $u_2^{(1)}$ throughout the plate.
- ii. Initial values of $\dot{u}_1^{(0)}$, $\dot{u}_2^{(0)}$, $\dot{u}_3^{(0)}$, $\dot{u}_1^{(1)}$ and $\dot{u}_2^{(1)}$ throughout the plate.
- iii. One member of each of the five products $F_1^{(0)} u_1^{(0)}$, $F_2^{(0)} u_2^{(0)}$, $F_3^{(0)} u_3^{(0)}$, $F_1^{(1)} u_1^{(1)}$, $F_2^{(1)} u_2^{(1)}$ at each point of the plate, i.e., five surface conditions.
- iv. At each point on the edge of the plate (normal n tangent s) one member of each of the five products $\tau_{nn}^{(0)} u_n^{(0)}$, $\tau_{ns}^{(0)} u_s^{(0)}$, $\tau_{ns}^{(1)} u_s^{(1)}$, $\tau_{nn}^{(1)} u_n^{(1)}$, $\tau_{ns}^{(1)} u_s^{(1)}$, i.e., five edge conditions.

The requirement that the strain-energy-density $\bar{U}^{(0)}$, Equation (31), be positive definite is satisfied by the addition of the requirements $\kappa_1 > 0$, $\kappa_2 > 0$ to the usual requirements $3\lambda + 2\mu > 0$, $\mu > 0$.

d. In the case of steady vibrations, the displacements may be expressed conveniently in terms of potentials that satisfy Poisson equations. Omitting a factor $e^{i\omega t}$ the results are

$$u_1^{(0)} = \frac{\partial \phi_1}{\partial x_1} + \frac{\partial \phi_2}{\partial x_1} + \frac{\partial \phi_3}{\partial x_1} - \frac{\partial H_0}{\partial x_3}$$

$$u_3^{(0)} = \frac{\partial \phi_1}{\partial x_3} + \frac{\partial \phi_2}{\partial x_3} + \frac{\partial \phi_3}{\partial x_3} + \frac{\partial H_0}{\partial x_1}$$

$$u_2^{(0)} = \alpha_1 \phi_1 + \alpha_2 \phi_2 + \alpha_3 \phi_3 \quad (57)$$

$$u_1^{(2)} = \beta_1 \frac{\partial \phi_1}{\partial x_1} + \beta_2 \frac{\partial \phi_2}{\partial x_1} + \beta_3 \frac{\partial \phi_3}{\partial x_1} - \frac{\partial H_2}{\partial x_3}$$

$$u_3^{(2)} = \beta_1 \frac{\partial \phi_1}{\partial x_3} + \beta_2 \frac{\partial \phi_2}{\partial x_3} + \beta_3 \frac{\partial \phi_3}{\partial x_3} + \frac{\partial H_2}{\partial x_1}$$

$$\mu \nabla^2 H_0 + \rho \omega^2 H_0 = 0$$

$$\mu \nabla^2 H_2 + (\kappa_1^2 \rho \omega^2 - 15 b^2 \kappa_2^2 \mu) H_2 = 0 \quad (58)$$

$$\nabla^2 \phi_i + \xi_i^2 \phi_i = 0, \quad i = 1, 2, 3$$

where

$$\alpha_i = b [(\lambda + 2\mu) \xi_i^2 - \rho \omega^2] / \kappa_1 \lambda \quad (59)$$

$$\beta_i = \alpha_i b / \left[\frac{3}{4} (\Omega^2 - 4) - \frac{E' b^2 \xi_i^2}{5 \mu \kappa_2^2} \right]$$

and the ξ_i^2 are, again, the roots of Equation (57). It may be seen that H_0 and H_2 are the potentials of the two face-shear modes and the ϕ_i are the potentials of the three compressional modes.

e. The tensor, and hence invariant, characters of the quantities that appear in the second order equations are, for the most part, apparent.

For example, in the displacement equations of motion (55), if the first and second equations are regarded as the rectangular components of a vector and the fourth and fifth equations the rectangular components of another vector, the only differential operators that appear are the gradient, divergence, Laplacian and $\partial^2/\partial t^2$: all invariants. The dependent variables are the scalar $u_2^{(1)}$ and the two vectors

$$\underline{u}^{(1)} = u_1^{(1)} \underline{k}_1 + u_3^{(1)} \underline{k}_3 = u_\alpha^{(1)} \underline{k}_\alpha + u_\gamma^{(1)} \underline{k}_\gamma, \quad (60)$$

$$\underline{u}^{(2)} = u_1^{(2)} \underline{k}_1 + u_3^{(2)} \underline{k}_3 = u_\alpha^{(2)} \underline{k}_\alpha + u_\gamma^{(2)} \underline{k}_\gamma,$$

while the gradient operator is

$$\underline{\nabla} = \underline{k}_1 \frac{\partial}{\partial x_1} + \underline{k}_3 \frac{\partial}{\partial x_3} = \underline{k}_\alpha \frac{\partial}{\partial s_\alpha} + \underline{k}_\gamma \frac{\partial}{\partial s_\gamma} \quad (61)$$

where $\underline{k}_1, \underline{k}_3$ and $\underline{k}_\alpha, \underline{k}_\gamma$ are unit vectors in the rectangular directions 1, 3 and the orthogonal curvilinear directions α, γ respectively: all in the plane of the plate.

The appropriate strain tensors and their expression in terms of vector displacements are not quite as apparent. We define another vector displacement

$$\underline{u}' = \underline{u}^{(1)} + \underline{k}_2 u_2^{(1)} \quad (62)$$

and a gradient operator

$$\underline{\nabla}' = \underline{\nabla} + \underline{k}_2 / b \quad (63)$$

where $\underline{k}_2$ is a unit vector normal to the plane of the plate. Then the two tensors

$$\begin{aligned}\underline{\xi}^{(a)} &= \frac{1}{2} (\underline{\nabla} \underline{u}^{(a)} + \underline{u}^{(a)} \underline{\nabla}) , \\ \underline{\xi}' &= \frac{1}{2} (\underline{\nabla}' \underline{u}' + \underline{u}' \underline{\nabla}')\end{aligned}\tag{64}$$

constitute the strain. The seven equations of compatibility, for example, become

$$\begin{aligned}\underline{\nabla} \times \underline{\xi}^{(a)} \times \underline{\nabla} &= 0 , \\ \underline{\nabla}' \times \underline{\xi}' \times \underline{\nabla}' &= 0 .\end{aligned}\tag{65}$$

References

1. T. R. Kane and R. D. Mindlin, "High-frequency extensional vibrations of plates," *Journal of Applied Mechanics*, Vol. 23 (Trans. ASME, Vol. 78) 1956, pp. A-277-283.
2. S. D. Poisson, "Mémoire sur l'équilibre et le mouvement des corps élastiques," *Mém. Acad. Sci., Ser. 2*, Vol. 8, Paris 1829, pp. 357-570.
3. A. L. Cauchy, "Sur l'équilibre et le mouvement d'une plaque solide," *Exercices de Mathématique*, Vol. 3, 1828, pp. 328-355.
4. G. Kirchhoff, "Über das Gleichgewicht und die Bewegung einer elastischen Scheibe," *Crelles Journal*, Vol. 40, 1850, pp. 51-88.
5. R. D. Mindlin, "An introduction to the mathematical theory of vibrations of elastic plates," *Signal Corps Engineering Laboratories*, Fort Monmouth, New Jersey, 1955.
6. E. Reissner, "On the calculation of three-dimensional corrections for the two-dimensional theory of plane stress," *Proceedings of the Fifteenth Semi-annual Eastern Photoelasticity Conference*, Addison-Wesley Press, Cambridge, Massachusetts, 1942.
7. Lord Rayleigh, "On the free vibrations of an infinite plate of homogeneous isotropic elastic matter," *Proc. London Math. Soc.*, Vol. 20, 1889, pp. 225-234.
8. S. Timoshenko, "On the correction for shear of the differential equation for transverse vibrations of prismatic bars," *Philosophical Magazine*, Ser. 6, Vol. 41, 1921, pp. 744-746.

9. R. D. Mindlin, "Influence of rotatory inertia and shear on flexural vibrations of isotropic, elastic plates," *Journal of Applied Mechanics*, Vol. 18 (Trans. ASME, Vol. 73) 1951, pp. 31-38.
10. R. R. Aggarwal and E. A. G. Shaw, "Axially symmetrical vibrations of a finite isotropic disk. IV," *Journ. Acoust. Soc. Am.*, Vol. 26, 1954, pp. 341-342.
11. I. Tolstoy and E. Usdin, "Wave propagation in elastic plates: low and high mode dispersion," *Journ. Acoust. Soc. Am.*, Vol. 29, 1957, pp. 37-42.
12. A. E. H. Love, "A Treatise on the Mathematical Theory of Elasticity," Cambridge University Press, London, 1927.
13. E. A. G. Shaw, "On the resonant vibrations of thick barium titanate disks," *Journ. Acoust. Soc. Am.*, Vol. 28, 1956, pp. 38-50.