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Free Boundary Problems for Parabolic Equations

by

Avner Friedman

Technical Report No. 28
Prepared under Contract Nonr-222(37)
(NR 041 157)

For
Office of Naval Research

Department of Mathematics
University of California
Berkeley 4, California

October 1958

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Free Boundary Problems for Parabolic Equations

Avner Friedman

Introduction. Free boundary problems for the heat equation have been considered for over a century (for references prior to 1929 see Brillouin [1]). In a few special cases the solutions have been found explicitly, but existence theorems for general cases started essentially about ten years ago. The well known one-dimensional Stefan problem concerning the melting of ice when heated at the boundary with a certain prescribed temperature was considered by Rubinstein [22] [23] [24] [25] [26] [27] and Dacev [2] [3] [4] [5] (see also Datzeff [6]). They used different methods whose basic features can be found in their papers [22] and [2] respectively.

Another Stefan problem where the heating at the boundary is given in terms of the flow of heat was considered by Evans [11] (see also [10]), J. Douglas and Gallie [7], Sestini [28] [29], Miranker [20], J. Douglas [8] and Kyner [19]. In the last two papers this problem (existence and uniqueness for all future times) was completely solved, regardless whether both ice and water or just ice alone are present at the beginning.

The papers of Dacev on the Stefan problem are of heuristic nature. Actually he did not give any precise proof to his assertions about existence (he did not prove at all uniqueness). It seems that such a proof should be quite involved. On the other hand except for a few incomplete details Rubinstein [22] proved existence and uniqueness, but only for small intervals of time. In [25] he stated that he can prove existence and uniqueness for any interval of time with the aid of certain lemmas on superparabolic functions.

A method to solve free boundary problems was developed systematically by Kolodner [18]. This method was applied by Miranker [20] to solve a Stefan problem mentioned above. It was earlier applied by Keller, Kolodner and Ritger [14], Kolodner [15] [16] and Kolodner and Ritger [17] to study the problems of evaporation and condensation of liquid drops.

In this paper we refine the method of Rubinstein [22] and apply it to solve Stefan problems for all future times and the problems of evaporation and condensation of drops for small intervals of time.

In Part I we are essentially concerned with the Stefan problem when the heating at the boundary is given in terms of the temperature and water is present already at the beginning. In § 1 we formulate the main result. In § 3 we reduce the original problem to the problem of finding a unique solution of a certain nonlinear integral equation. In proving this reduction we use an auxiliary lemma proved in § 2. In § 4 we prove that the ice-water line is an increasing function in time. In § 5 we prove the existence and uniqueness of the solution of the integral equation. Our method is a modification of the method of successive approximations (both on $v(t)$ and $s(t)$) used by Rubinstein [22] and is much more suitable for the purpose of continuation of the solution into the future. In § 6 we prove existence and uniqueness for all future times. The decisive step is an a priori estimate on the x-derivative of the solution $u(x,t)$ defined for $o \le t < t_0$; the bound being independent of t. In § 7 we mention that all the previous results hold for general parabolic equations with smooth coefficients. We conclude this section with a few open questions.

In Part II we consider the problems of evaporation and condensation of liquid drops. In § 1 we state the main results which we proceed to describe. If the initial vapor density on the surface of the drop is equal to the saturation density then in case of condensation ($\alpha < o$) existence and uniqueness are asserted for all future times and in case of condensation ($\alpha > o$) only as long as the radius of the drop is not "too small". Here α is a certain physical quantity which is assumed to be sufficiently small in absolute value. If there is a jump from the initial vapor density on the drop to the saturation density, then existence is asserted as above but uniqueness is asserted only under the extra condition that the drop does not start to grow "too fast". We finally state that the radius $s(t)$ of the drop in case of condensation grows at most like $A + Bt^{1/2}$, where A, B are appropriate constants.

The proof of the above results is given in §§ 2-5. In § 2 the problem is reduced to solving a certain nonlinear integral equation. This equation differs from the analogous equation of Part I in the fact that it contains terms which have a singularity like $t^{-1/2}$ at $t = o$. The main reason for dealing with that equation (rather than trying to eliminate the singularities by some integrations by parts) is that we prove (in §3) existence and uniqueness in an interval whose length depends on $|u(x,t)|$ and not on $|u_x(x,t)|$. The solution obtained however might have a singularity like $t^{-1/2}$ at $t = o$.

In § 4 we prove existence and uniqueness for large time-intervals, showing also that the above mentioned singularities of the solution are removable (Lemma 2). In § 5 we establish the above mentioned a priori bound on $s(t)$.

In Part III we consider the problem of dissolution of gas bubble in liquid. This problem is similar to the one of Part II with one exception: The law of conservation of mass at the free boundary leads to an ordinary differential equation which we cannot solve explicitly. Thus we first derive certain estimates on the solution of this equation without solving it explicitly, and then we can proceed by the method of Part II. The results are stated in Theorem 3. The results of Parts II, III can also be extended to general parabolic equation with smooth coefficients.

Part I: A Stefan Problem

1. Statement of the Main Result

In this Part I we shall essentially be concerned with the following Stefan type problem:

(1.1) $u_{xx} = u_t, \quad o < x < s(t), \; t > o$

(1.2) $u(o,t) = f(t), \; f(t) \geq o \quad t > o$

(1.3) $u(x,o) = \varphi(x), \; \varphi(x) \geq o \quad o < x \leq b, \; \varphi(b) = o, \; b > o$

(1.4) $u(s(t),t) = o, \; t > o$

(1.5) $u_x(s(t),t) = \dot{s}(t), \; t > o.$

$x = s(t)$ is the free boundary (for instance, the water-ice line) which is not known and has to be found together with $u(x,t)$. The conditions (1.2), (1.3), (1.4) are the usually given data whereas the additional condition (1.5) (the equation of heat balance) is a condition on the free boundary $x = s(t)$. The assumptions $f \geq o \quad \varphi \geq o$ result from the fact that the temperature of water is nonnegative.

<u>Definition</u>. We say that $u(x,t)$, $s(t)$ form a solution of the system (1.1)-(1.5) for all $t < \sigma$ $(o < \sigma \leq \infty)$ if $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ are continuous for $o < x < s(t)$, $o < t < \sigma$, if u and $\partial u/\partial x$ are continuous for $o \leq x \leq s(t)$, $t > o$, if u is continuous for $t = o$ $o < x \leq b$ and $o \leq \liminf u(x,t) \leq \limsup u(x,t) < \infty$ as $t \to o$, $x \to b$, if $s(t)$ $(o \leq t < \sigma)$ is continuously differentiable and, finally, if the equations (1.1)-(1.5) are satisfied.

The main result of this Part I is the following theorem.

Theorem 1. Assume that $f(t)$ $(o \leq t < \infty)$ and $\varphi(x)$ $(o \leq x \leq b)$ are continuously differentiable functions. Then there exists one and only one solution $u(x,t)$, $s(t)$ of the system (1.1)-(1.5) for all $t < \infty$. Furthermore, the function $x = s(t)$ is monotone nondecreasing in t.

The proof of Theorem 1 is given in §§ 2-6. In § 2 we prove an auxiliary lemma (stronger than a lemma stated and applied in [18]). In § 3 we prove that for any possible solution $s(t)$ is monotone nondecreasing in t. It is also shown that if either $\varphi(x) \not\equiv o$ or $f(t) \not\equiv o$ in some small interval $o < t < \varepsilon$ $(\varepsilon > o)$ then $s(t)$ is a strictly increasing function of t. In § 4 we reduce the original problem (1.1)-(1.5) to a problem of solving a certain nonlinear integral equation. In this proof we make use of the lemma proved in § 2. In § 5 we prove existence and uniqueness of solutions for the nonlinear integral equation for a small interval of time t. Finally in § 6 (using the results of §§ 3-5) we prove that the solution can be continued, and uniquely, to all future times. In § 7 we give various generalizations of Theorem 1.

2. Auxiliary Lemma

Lemma 1. Let $\rho(t)$ $(o \leq t \leq \sigma)$ be a continuous function and let $s(t)$ $(o \leq t \leq \sigma)$ satisfy the Lipschitz condition. Then for every $o < t \leq \sigma$

$$\lim_{x\to s(t)-0} \frac{\partial}{\partial x}\int_0^t \frac{\rho(\tau)}{2\pi^{1/2}(t-\tau)^{1/2}}\exp\{-\frac{(x-s(\tau))^2}{4(t-\tau)}\}d\tau = \frac{1}{2}\rho(t)$$

(2.1)

$$+\int_0^t \frac{\rho(\tau)}{2\pi^{1/2}(t-\tau)^{1/2}}\left[\frac{\partial}{\partial x}\exp\{-\frac{(x-s(\tau))^2}{4(t-\tau)}\}\right]_{x=s(t)} d\tau$$

Here, $x\to s(t)-0$ means $x\to s(t)$, $x < s(t)$.

Proof. We shall first prove that for any fixed δ $(\delta < t)$

$$I \equiv \int_{t-\delta}^t \frac{x-s(\tau)}{(t-\tau)^{3/2}}\exp\{-\frac{(x-s(\tau))^2}{4(t-\tau)}\}dt$$

(2.2)

$$-\int_{t-\delta}^t \frac{s(t)-s(\tau)}{(t-\tau)^{3/2}}\exp\{-\frac{(s(t)-s(\tau))^2}{4(t-\tau)}\}dt = -2\pi^{1/2}+O(\delta^{1/2})$$

as $x\to s(t)-0$, and $|O(\delta^{1/2})| \le A_0\delta^{1/2}$

where A_0 is a constant independent of x,t,δ. In the sequel we shall denote by A_i appropriate constants independent of x,t,δ.

To prove (2.2) we write $I = I_1 + I_2$ where

(2.3) $$I_1 = \int_{t-\delta}^t \frac{x-s(t)}{(t-\tau)^{3/2}}\exp\{-\frac{(x-s(\tau))^2}{4(t-\tau)}\}dt$$

(2.4) $$I_2 = \int_{t-\delta}^t \frac{s(t)-s(\tau)}{(t-\tau)^{3/2}}\left[\exp\{-\frac{(x-s(\tau))^2}{4(t-\tau)}\} - \exp\{-\frac{(s(t)-s(\tau))^2}{4(t-\tau)}\}\right]dt$$

Since $|s(t)-s(\tau)| \le A_1(t-\tau)$ we immediately get

(2.5) $$|I_2| \le A_1\int_{t-\delta}^t \frac{d\tau}{(t-\tau)^{1/2}} \le 2A_1\delta^{1/2}.$$

To evaluate I_1 denote

(2.6) $$J_1 = \int_{t-\delta}^{t} \frac{x-s(t)}{(t-\tau)^{3/2}} \exp\{-\frac{(x-s(t))^2}{4(t-\tau)}\} d\tau .$$

Then

(2.7) $$J_1-I_1= \int_{t-\delta}^{t} \frac{x-s(t)}{(t-\tau)^{3/2}}\exp\{-\frac{(x-s(t))^2}{4(t-\tau)}\}\left[1-\exp\{-\frac{(x-s(\tau))^2-(x-s(t))^2}{4(t-\tau)}\}\right].$$

The braces in the second exponent are bounded by

$$\frac{1}{4(t-\tau)}|s(t)-s(\tau)|(|x-s(t)|+|x-s(\tau)|) \leq A_2(|x-s(t)|+|s(t)-s(\tau)|).$$

Since we may assume that right side of the last inequality is smaller than 1, we conclude that the brackets on the right side of (2.7) are bounded by

$$A_3(|x-s(t)| + |s(t)-s(\tau)|).$$

Substituting in (2.7) we find that

(2.8) $$|J_1-I_1| \leq A_4 \int_{t-\delta}^{t} \frac{d\tau}{(t-\tau)^{1/2}} + A_4|x-s(t)| \int_{t-\delta}^{t} \frac{d\tau}{(t-\tau)^{1/2}} \leq A_5\delta^{1/2}.$$

Thus, in order to evaluate I_1 it is enough to evaluate J_1.

Substituting in the integral of (2.6)

$$z = \frac{t-\tau}{(x-s(t))^2}$$

and noting that $x-s(t)<o$, we get

(2.9) $$J_1=-\int_{0}^{\delta'} \frac{1}{z^{3/2}}\exp\{-\frac{1}{4z}\}dz \quad \text{where} \quad \delta' = \frac{\delta}{(x-s(t))^2} .$$

Hence, as $x \to s(t)$ $\delta' \to \infty$ and $J_1 \to -2\pi^{1/2}$.

Combining the last result with (2.8) and (2.5) and using the definition $I = I_1+I_2$ it follows that

(2.10) $$\limsup_{x\to s(t)-0} |I + 2\pi^{1/2}| \le A_6\,\delta^{1/2}$$

which is equivalent to the statement (2.2).

From the above evaluation of I_1 it also follows that

(2.11) $$|I_1| \le A_7 .$$

Having proved (2.2) we proceed to prove that

(2.12) $$K_0 \equiv \int_{t-\delta}^{t} \frac{|x-s(\tau)|}{(t-\tau)^{3/2}} \exp\{-\frac{(x-s(\tau))^2}{4(t-\tau)}\}d\tau \le A_8$$

(2.13) $$K_1 \equiv \int_{t-\delta}^{t} \frac{|s(t)-s(\tau)|}{(t-\tau)^{3/2}} \exp\{-\frac{(s(t)-s(\tau))^2}{4(t-\tau)}\}d\tau \le A_9 .$$

The proof of (2.13) follows immediately using the Lipschitz continuity of $s(t)$. The proof of (2.12) follows from the boundedness of I_1 (See (2.11)) and of

$$K_2 = \int_{t-\delta}^{t} \frac{|s(t)-s(\tau)|}{(t-\tau)^{3/2}} \exp\{-\frac{(x-s(\tau))^2}{4(t-\tau)}\}d\tau ,$$

noting that $K_0 \le |I|_1 + K_2$.

With the aid of (2.2), (2.12), (2.13) we proceed to prove (2.1). Denoting

(2.14) $$L_1' = \int_{t-\delta}^{t} \rho(\tau)\frac{x-s(\tau)}{(t-\tau)^{3/2}} \exp\{-\frac{(x-s(\tau))^2}{4(t-\tau)}\}d\tau - \int_{t-\delta}^{t} \rho(\tau)\frac{s(t)-s(\tau)}{(t-\tau)^{3/2}} \exp\{-\frac{(s(t)-s(\tau))^2}{4(t-\tau)}\}d\tau$$

we claim that

$$(2.15)\quad \limsup_{x\to s(t)-o} |L_1 + 2\pi^{1/2}\rho(t)| \le A_{10}(\delta^{1/2} + \underset{t-\delta\le\tau\le t}{\text{l.u.b.}}|\rho(t)-\rho(\tau)|).$$

Indeed, writing in (2.14) $\rho(\tau) = \rho(t)+(\rho(\tau)-\rho(t))$ and using (2.2), (2.12) and (2.13), the proof of (2.15) follows.

We next observe that the function

$$L_2 = \int_0^{t-\delta} \rho(\tau)\frac{x-s(\tau)}{(t-\tau)^{3/2}}\exp\{-\frac{(x-s(\tau))^2}{4(t-\tau)}\}d\tau$$

$$(2.16)\qquad - \int_0^{t-\delta} \rho(\tau)\frac{s(t)-s(\tau)}{(t-\tau)^{3/2}}\exp\{-\frac{(s(t)-s(\tau))^2}{4(t-\tau)}\}d\tau \qquad (\delta>o)$$

satisfies

$$(2.17)\qquad \lim_{x\to s(t)} L_2 = o .$$

Combining this remark with (2.15) we get

$$(2.18)\quad \limsup_{x\to s(t)-o} |(L_1+L_2)+2\pi^{1/2}\rho(t)| \le A_{10}(\delta^{1/2} + \underset{t-\delta\le\tau\le t}{\text{l.u.b.}}|\rho(t)-\rho(\tau)|).$$

Since the left side of (2.18) is independent of δ we conclude, on taking $\delta\to o$, that it is zero, and the proof of the lemma is completed.

3. Monotonicity of s(t)

Suppose $u(x,t)$, $s(t)$ form a solution of the system (1.1)-(1.5). By the maximum principle [20] $u(x,t)$ is nonnegative for $o<x<s(t)$, $t>o$. Since $u=o$ on $x = s(t)$, $u_x \le o$ and hence, by (1.5), $\dot{s}(t) \ge o$. This proves that $s(t)$ is nondecreasing in t.

We shall now prove that if either $\varphi(x) \not\equiv o$ or $f(t) \not\equiv o$ in some interval $o\le t\le \varepsilon$, then $s(t)$ is strictly increasing in t.

Indeed, in the contrary case there exist two points $t' > t''$ such that $s(t) \equiv s(t')$ for all $t' \leq t \leq t''$. Therefore, by (1.5), $u_x = \dot{s} = o$ on the straight segment $x = s(t)$ $t' \leq t \leq t''$. We can now use the strong maximum principle [21] to show that $u(x,t) > o$ for $o < x < s(t)$, $t > o$. Since $u(s(t),t) = o$ we can use [12; Theorem 2] to conclude that u_x is negative at the points $(s(t),t)$, $t' < t < t''$. This last conclusion however implies that $\dot{s}(t) > o$ for $t' < t < t''$ which is a contradiction.

4. The Integral Equation

We introduce Green's function for the half plane $x > o$

$$(4.1)\quad G(x,t;\xi,\tau) = \frac{1}{2\pi^{1/2}(t-\tau)^{1/2}}\exp\{-\frac{(x-\xi)^2}{4(t-\tau)}\} - \frac{1}{2\pi^{1/2}(t-\tau)^{1/2}}\exp\{-\frac{(x+\xi)^2}{4(t-\tau)}\}$$

Suppose $u(x,t)$, $s(t)$ form a solution of the system (1.1)-(1.5). Integrating Green's identity

$$\frac{\partial}{\partial\xi}(G\frac{\partial u}{\partial\xi} - u\frac{\partial G}{\partial\xi}) = \frac{\partial}{\partial\tau}(Gu)$$

over the domain $o < \xi < s(\tau)$ $\varepsilon < \tau < t-\varepsilon$ and letting $\varepsilon \longrightarrow o$ we get, on using (1.2), (1.3), (1.4)

$$(4.2)\quad u(x,t) = \int_o^t u_\xi(s(\tau),\tau)G(x,t;s(\tau),\tau)d\tau + \int_o^t f(\tau)G_\xi(x,t;o,\tau)d\tau + \int_o^b \varphi(\xi)G(x,t;\xi,o)d\xi = M_1 + M_2 + M_3$$

where M_i denotes the i^{-th} integral. Denoting

$$(4.3)\qquad v(\tau) = u_\xi(s(\tau),\tau)$$

we proceed to differentiate both sides of (4.2) with respect to x and let $x \to s(t)-o$. Using Lemma 1 we have

$$(4.4)\quad \lim_{x\to s(t)-o} \frac{\partial}{\partial x}M_1 = \frac{1}{2}v(t) + \int_0^t v(\tau)G_x(s(t),t;s(\tau),\tau)d\tau .$$

Here we used the fact that the second term of G is a regular function since $x + s(\tau) \geq b > o$ (since $s(t)$ is nondecreasing in t).

We proceed to evaluate $\partial M_i/\partial x$, $i = 2,3$. Denote by

$$(4.5)\quad N(x,t;\xi,\tau)= \frac{1}{2\pi^{1/2}(t-\tau)^{1/2}}\exp\{-\frac{(x-\xi)^2}{4(t-\tau)}\}+\frac{1}{2\pi^{1/2}(t-\tau)^{1/2}}\exp\{-\frac{(x+\xi)^2}{4(t-\tau)}\}$$

the Neumann function for the half space $x > o$. It satisfies $G_x = -N_\xi$. We now have

$$\frac{\partial M_2}{\partial x} = \int_0^t f(\tau)G_{x\xi}(x,t;o,\tau)d\tau = \int_0^t f(\tau)N_\tau(x,t;o,\tau)d\tau$$

(4.6)

$$= -f(o)N(x,t;o,o)- \int_0^t \dot{f}(\tau)N(x,t;o,\tau)d\tau .$$

Similarly

$$(4.7)\quad \frac{\partial M_3}{\partial x} = \int_0^b \varphi(\xi)G_x(x,t;\xi,o)d\xi=\varphi(o)N(x,t;o,o)+ \int_0^b \dot{\varphi}(\xi)N(x,t;\xi,o)d\tau$$

Using (4.4), (4.6), (4.7) we obtain on differentiating (4.2) with respect to x and taking $x \to s(t)-o$,

$$v(t) = 2[\varphi(o)-f(o)]N(s(t),t;o,o)+ 2\int_0^b \dot{\varphi}(\xi)N(s(t),t;\xi,o)d\xi$$

(4.8)

$$-2\int_0^t \dot{f}(\tau)N(s(t),t;o,\tau)d\tau +2\int_0^t v(\tau)G_x(s(t),t;s(\tau),\tau)d\tau$$

where, by (1.5) and (4.3),

$$s(t) = \ell - \int_0^t v(\tau)\,d\tau. \tag{4.9}$$

We have thus proved that for every solution u,s of the system (1.1)-(1.5) v (defined by (4.3)) must satisfy the integral equation (4.8), where s(t) is defined by (4.9).

Suppose conversely that for some $\sigma > 0$ v(t) ($0 \le t < \sigma$) is a continuous solution of the integral equation (4.8) where s(t) is defined by (4.9). We shall prove that u(x,t), s(t) (where u(x,t) is defined by (4.2) with $u_\xi(s(\tau),\tau)$ replaced by $v(\tau)$) form a solution of the system (1.1)-(1.5).

First of all one can easily verify (1.1)-(1.3). We now differentiate u(x,t) with respect to x and take $x \to s(t)-0$. Using Lemma 1, the previous calculations of $\partial M_i/\partial x$ ($i = 2,3$) and the integral equation (4.8) we find that $u_x(s(t),t) = v(t)$. Since by (4.9) $v(t) = -\dot{s}(t)$, (1.5) follows. We remark that u(x,t) and s(t) have all the regularity properties in order to form a solution in the sense of § 1. Thus it remains to prove that $u(s(t),t) = 0$.

To prove that, we integrate Green's identity (with G and u) in the domain $0 < \xi < s(\tau)$, $\varepsilon < \tau < t-\varepsilon$ and let $\varepsilon \to 0$. Comparing the integral representation for u(x,t) thus obtained with (4.2) we conclude that

$$\Psi(x,t) \equiv \int_0^t u(s(\tau),\tau)G(x,t;s(\tau),\tau)d\tau = 0 \text{ if } 0<x<s(t),\ 0<t<\sigma. \tag{4.10}$$

We claim that $\Psi(x,t) = 0$ if $s(t)<x<\infty$, $0<t<\sigma$. Indeed it vanishes on $t = 0$, $b \le x < \infty$, tends to zero as $x \to s(t)+0$

(by (4.10) and the continuity of $\psi(x,t)$ in the whole strip $o \le t < \sigma$), and tends to zero as $x \to \infty$ uniformly with respect to t, $o \le t < \sigma$. Hence, by the maximum principle, $\psi(x,t)=o$ for $s(t) < x < \infty$, $o \le t < \sigma$.

Applying Lemma 1 we now get

$$o = \left[\frac{\partial \psi}{\partial x}\right]_{x \to s(t)-o} - \left[\frac{\partial \psi}{\partial x}\right]_{x \to s(t)+o} = u(s(t),t) \qquad (o<t<\sigma).$$

We have thus proved that the problem (1.1)-(1.5) is equivalent to the problem of finding a continuous solution to the integral equation (4.8) where s(t) is defined by (4.9). In what follows we shall solve this nonlinear integral equation.

5. Existence and Uniqueness for Small Times

In this chapter we shall prove that the integral equation (4.8) has a unique solution for $o \le t < \sigma$, σ being a sufficiently small number. We denote by C_σ the Banach space of functions v(t) defined and continuous for $o \le t \le \sigma$ with the uniform norm, namely $||v|| = \ell.u.b.|v(t)|$. We denote by $C_{\sigma,M}$ the closed sphere $||v|| \le M$ of radius M in C_σ. On the set $C_{\sigma,M}$ we define a transformation w = Tv in the following way:

$$\begin{aligned} w(t) &= 2[\varphi(o)-f(o)]N(s(t),t;,o,o)+2\int_o^b \dot{\varphi}(\xi)N(s(t),t,\xi,o)d\xi \\ (5.1)\qquad & -2\int_o^t \dot{f}(\tau)N(s(t),t;o,\tau)d\tau +2\int_o^t v(\tau)G_x(s(t),t;s(\tau),t)d\tau \end{aligned}$$

$$(o \le t \le \sigma)$$

where

(5.2) $$s(t) = b + \int_0^t v(\tau)d\tau \ .$$

We shall prove a few properties of T.

5.1. T maps $C_{\sigma,M}$ into itself. From (5.2) it follows that

(5.3) $$|s(t) - s(\tau)| \leq M(t-\tau)\ .$$

In what follows we shall denote by B_i appropriate positive constants depending (and continuously so) only on b. We shall take σ to satisfy

(5.4) $$2M\sigma \leq b,\ \sigma \leq 1$$

so that

(5.5) $$\tfrac{1}{2}b \leq s(t) \leq \tfrac{3}{2}b \qquad (o \leq t \leq \sigma\ ).$$

With the aid of (5.3), (5.5) we easily find that

(5.6) $$\|w\| \leq 4\,\|\dot{\varphi}\| + B_1(\varphi(o) + f(o) + \|\dot{f}\|)\sigma + B_2M^2\sigma^{1/2}.$$

Defining

(5.7) $$M = 4\,\|\dot{\varphi}\| + 1$$

and taking σ to satisfy, besides (5.4),

(5.8) $$B_1(\varphi(o) + f(o) + \|\dot{f}\|)\sigma + B_2(4\,\|\dot{\varphi}\| + 1)^2\sigma^{1/2} \leq 1$$

we conclude that $\|w\| \leq M$, that is, T maps $C_{\sigma,M}$ into itself. Using (5.7) we can replace the conditions (5.4) by

(5.9) $$2(4\,\|\dot{\varphi}\|+1)\sigma \leq b,\ \ \sigma \leq 1$$

5.2. T is a contraction. Let $w = Tv$, $w' = Tv'$ and denote

(5.10) $$\|v - v'\| = \varepsilon$$

If $s'(t)$ corresponds to $v'(t)$ by (5.2), then we have the following inequalities:

(5.11) $$|s(t) - s'(t)| \leq t\,\varepsilon\,, \quad \|\dot{s}-\dot{s}'\| \leq \varepsilon\,.$$

Since $\|v\| \leq M$, $\|v'\| \leq M$ we also have

(5.12) $$|s(t)-s(\tau)| \leq M(t-\tau),\ |s'(t)-s'(\tau)| \leq M(t-\tau),\ \varepsilon \leq 2M.$$

Finally, taking σ to satisfy (5.9) we have

(5.13) $$\tfrac{1}{2}b \leq s(t) \leq \tfrac{3}{2}b,\ \tfrac{1}{2}b \leq s'(t) \leq \tfrac{3}{2}b \qquad 0 \leq t \leq \sigma\,.$$

With the aid of (5.11), (5.12), (5.13) we now proceed to prove that for small σ T is a contraction.

We write

(5.14) $$w - w' = V_1 + V_2 + V_3 + V_4$$

where

(5.15) $$V_1 = 2[\varphi(o)-f(o)][N(s(t),t;o,o) - N(s'(t),t;o,o)]$$

(5.16) $$V_2 = 2\int_0^b \dot{\varphi}(\xi)[N(s(t),t;\xi,o) - N(s'(t),t;\xi,o)]d\xi$$

(5.17) $$V_3 = -\int_0^t \dot{f}(\tau)[N(s(t),t;o,\tau) - N(s'(t),t;o,\tau)]d\tau$$

(5.18) $$-2\pi^{1/2}V_4 = \int_0^t v(\tau)\,\frac{s(t)-s(\tau)}{(t-\tau)^{3/2}}\,\exp\{-\frac{(s(t)-s(\tau))^2}{4(t-\tau)}\}d\tau - \int_0^t v'(\tau)\,\frac{s'(t)-s'(\tau)}{(t-\tau)^{3/2}}\,\exp\{-\frac{(s'(t)-s'(\tau))^2}{4(t-\tau)}\}d\tau.$$

Using the mean value theorem, (5.11) and (5.13) we easily get

(5.19) $$|V_1| \leq B_3[\varphi(o) + f(o)]\sigma\,\varepsilon\,.$$

To estimate V_2 we first estimate

(5.20) $$V_2' = \int_0^b \dot{\varphi}(\xi)\, \frac{1}{t^{1/2}}[\exp\{-\frac{(s(t)-\xi)^2}{4t}\} - \exp\{-\frac{(s'(t)-\xi)^2}{4t}\}]d\xi \,.$$

We may assume that $s'(t) > s(t)$ and divide the integral into three parts:

$$V_2' = \int_0^b = \int_0^{s(t)} + \int_{s(t)}^{s'(t)} + \int_{s'(t)}^{b} = Y_1 + Y_2 + Y_3$$

where it is understood that for instance if $s(t) < b < s'(t)$ then the last integral does not appear and the upper edge $s'(t)$ of the second integral is replaced by b. To estimate Y_1 we use the mean value theorem noting that if $s(t) < \tilde{s} < s'(t)$ then

$$\exp\{-\frac{(\tilde{s}-\xi)^2}{\alpha t}\} \le \exp\{-\frac{(s(t)-\xi)^2}{\alpha t}\} \qquad (\alpha > 0) \,.$$

We get

$$|Y_1| \le B_4 \|\dot{\varphi}\| \int_0^b \frac{|s(t)-s'(t)|}{t} \exp\{-\frac{(s(t)-\xi)}{8t}\}d\xi \le B_4 \|\dot{\varphi}\| \sigma^{1/2}\varepsilon \,,$$

where we made use of (5.11).

Y_2 can be estimated in similar manner. Finally,

$$|Y_3| \le |s'(t)-s(t)|\, \|\dot{\varphi}\|\, \frac{1}{t^{1/2}} \le \|\dot{\varphi}\| \sigma^{-1/2}\varepsilon \,.$$

Combining the estimates of the Y_i we get

(5.21) $$|V_2'| \le B_5 \|\dot{\varphi}\| \sigma^{-1/2}\varepsilon \,.$$

We can similarly estimate $2\pi^{1/2}V_2 - V_2'$ which is equal to the right side of (5.20) with ξ replaced by $-\xi$. We obtain

(5.22) $$|V_2| \le B_6 \|\dot{\varphi}\| \sigma^{1/2}\varepsilon \,.$$

To estimate V_3 we use the mean value theorem, (5.11) and

(5.13). We conclude that

$$|V_3| \leq B_7 \|\dot{f}\| \sigma \varepsilon . \tag{5.23}$$

To estimate V_4 we write

$$-2\pi^{1/2} V_4 = W_1 + W_2 + W_3 \tag{5.24}$$

where

$$W_1 = \int_0^t [v(\tau)-v'(\tau)] \frac{s(t)-s(\tau)}{(t-\tau)^{3/2}} \exp\{-\frac{(s(t)-s(\tau))^2}{4(t-\tau)}\} d\tau \tag{5.25}$$

$$W_2 = \int_0^t v'(\tau) [\frac{s(t)-s(\tau)}{(t-\tau)^{3/2}} - \frac{s'(t)-s'(\tau)}{(t-\tau)^{3/2}}] \exp\{-\frac{(s(t)-s(\tau))^2}{4(t-\tau)}\} d\tau \tag{5.26}$$

$$W_3 = \int_0^t v'(\tau) \frac{s'(t)-s'(\tau)}{(t-\tau)^{3/2}} \exp\{-\frac{(s(t)-s(\tau))^2}{4(t-\tau)}\} [1- \exp\{-\frac{(s'(t)-s'(\tau))^2-(s(t)-s(\tau))^2}{4(t-\tau)}\}] d\tau . \tag{5.27}$$

Using (5.10), (5.12) we get

$$|W_1| \leq M \varepsilon \int_0^t \frac{d\tau}{(t-\tau)^{1/2}} \leq B_8 M \sigma^{1/2} \varepsilon \tag{5.28}$$

To estimate W_2 we note that the brackets on the right side of (5.26) are equal to

$$\frac{[s(t)-s'(t)]-[s(\tau)-s'(\tau)]}{(t-\tau)^{3/2}} = \frac{\dot{s}(\tilde{\tau})-\dot{s}'(\tilde{\tau})}{(t-\tau)^{1/2}} \qquad (\tau < \tilde{\tau} < t).$$

Hence, using (5.11) we obtain

$$|W_2| \leq B_9 M \sigma^{1/2} \varepsilon . \tag{5.29}$$

To estimate W_3 we note that the second braces on the right side of (5.27) are bounded by

$$\frac{1}{4(t-\tau)}|[s'(t)-s(t)]-[s'(\tau)-s(\tau)]|[|s'(t)-s'(\tau)|+|s(t)-s(\tau)|]$$

(5.30)

$$\leq \frac{1}{2} M\varepsilon(t-\tau).$$

If we take σ to satisfy

(5.31) $$M^2\sigma \leq 1$$

then the right side of (5.30) is bounded by 1. We then conclude that the brackets on the right side of (5.27) are bounded by $B_{10}M\varepsilon(t-\tau)$. Making use of this last remark and of (5.12) and (5.31) we get

(5.32) $$|W_3| \leq B_{10}M^3\varepsilon\int_0^t(t-\tau)^{1/2}d\tau \leq \tfrac{2}{3}B_{10}M^3\sigma^{3/2}\varepsilon \leq \tfrac{2}{3}B_{10}M\sigma^{1/2}\varepsilon .$$

Combining the estimates (5.28), (5.29), (5.32) with (5.24) we get

(5.33) $$|V_4| \leq B_{11}M\sigma^{1/2}\varepsilon$$

Combining the estimates (5.19), (5.22), (5.23), (5.33) with (5.14) we get

(5.34) $$\frac{||w-w'||}{\varepsilon} \leq B_3[\varphi(o)+f(o)]\sigma+B_6||\dot{\varphi}||\sigma^{1/2}+B_7||\dot{f}||\sigma+B_{11}M\sigma^{1/2},$$

provided σ satisfies (5.31). In view of (5.7), (5.31) becomes

(5.35) $$(4||\dot{\varphi}||+1)^2\sigma \leq 1 .$$

Using (5.7) we conclude that if σ satisfies (5.35) and

(5.36) $$B_3[\varphi(o)+f(o)]\sigma+B_6||\dot{\varphi}||\sigma^{1/2}+B_7||\dot{f}||\sigma+B_{11}(4||\dot{\varphi}||+1)\sigma^{1/2}<1$$

then T is a contraction operator on $C_{\sigma,M}$.

5.3. Completion of the proof.

We have proved that if σ satisfies (5.8), (5.9), (5.35) and (5.36) then T is a contraction on $C_{\sigma,M}$ and maps $C_{\sigma,M}$ into itself. It is then well known that there exists a unique fixed point v(t) of T in $C_{\sigma,M}$.

Having proved the existence of a solution v(t) to the integral equation (4.8) (with s(t) defined by (4.9)) we proceed to prove uniqueness. Suppose $v'(t)$ $o \le t < \bar{\sigma}$ ($\bar{\sigma} \le \sigma$) is another solution and replace in the considerations of 5.1, 5.2 M by

$$M' = \max(M, \underset{o \le t \le \sigma}{\ell.u.b.} |v'(t)|).$$

Then instead of σ we get σ' ($\sigma' < \bar{\sigma}$) which satisfies the inequalities which guarantee that T is a contraction and that T maps $C_{\sigma',M'}$ into itself. Since v(t) and $v'(t)$ are both fixed points in $C_{\sigma',M'}$ of the transformation T we have $v(t) \equiv v'(t)$ ($o \le t \le \sigma'$).

Now let $\sigma_1 < \bar{\sigma}$ be any number such that $v(t) \equiv v'(t)$, $o \le t < \sigma_1$. Then $v(\sigma_1) = v'(\sigma_1)$. If we show that for some $\epsilon > o$ $v(t) \equiv v'(t)$, $o \le t < \sigma_1 + \epsilon$ then the proof of the uniqueness is completed.

Let u(x,t), u'(x,t) be the solutions of (1.1)-(1.5) corresponding to v(t), v'(t) respectively. If we apply Green's formula as in §4 we obtain integral equations for v(t) and v'(t) analogous to (4.8) in the domains $\sigma_1 \le t < \sigma$ and $\sigma_1 \le t < \bar{\sigma}$ respectively and with $\varphi(\xi)$ replaced by $u(\xi, \sigma_1)$ and $u'(\xi, \sigma_1)$ respectively. We recall that $u(\xi, \sigma_1) \equiv u'(\xi, \sigma_1)$. Proceeding as before (in the case $\sigma_1 = o$) we conclude that $v(t) \equiv v(t')$ for $\sigma_1 \le t < \sigma_1 + \epsilon$ for some $\epsilon > o$. We have thus completed the proof of existence and uniqueness of solutions for small times.

6. Existence and Uniqueness for All Times

We now apply the proof of §5 step by step. To show that such a process can be carried out so as to obtain a solution $u(x,t), s(t)$ (or $v(t)$) for all times we have to prove the following statement: <u>If the solution</u> $v(t)$ (<u>or</u> $u(x,t)$, $s(t)$) <u>exists and is unique for</u> $o \leq t < t_c$ <u>then it exists and is unique for</u> $o \leq t < t_o + \varepsilon$ <u>for some</u> $\varepsilon > o$.

We first prove existence. To do that we shall first prove that $v(t)$ $(o \leq t < t_o)$ is a bounded function. In this proof we shall make use of the fact proved in §3 that $s(t)$ is nondecreasing in t. Thus, $s(t) \geq b$. We shall also use the fact that $v(t) = -\dot{s}(t) \leq o$.

The method used to prove (4.8) can be used to prove that

$$v(t) = 2[u(o,t_o-\delta)-f(t_o-\delta)]N(s(t),t;o,t_o-\delta)$$

$$+ 2\int_o^{s(t_o-\delta)} u_\xi(\xi,t_o-\delta)N(s(t),t;\xi,t_o-\delta)d\xi \tag{6.1}$$

$$- 2\int_{t_o-\delta}^{t} \dot{f}(\tau)N(s(t),t;o,\tau)d\tau + 2\int_{t_o-\delta}^{t} v(\tau)G_x(s(t),t;s(\tau),\tau)d\tau$$

$$= \sum_{i=1}^{4} T_i$$

where $\delta > o$, $t < t_o$ and T_i denotes the i-th term on the right side of (6.1). Since $v(t) \leq o$ we only have to find a lower bound on $v(t)$. Denoting

$$\psi(t) = \underset{t_o-\delta<\tau<t}{\text{g.}\ell\text{.b.}}\ v(\tau) \tag{6.2}$$

we proceed to evaluate T_4.

$$2\pi^{1/2}T_4 = -\int_{t_o-\delta}^{t} v(\tau)\,\frac{s(t)-s(\tau)}{(t-\tau)^{3/2}}\exp\{-\frac{(s(t)-s(\tau))^2}{4(t-\tau)}\}d\tau$$

(6.3)

$$+\int_{t_o-\delta}^{t} v(\tau)\,\frac{s(t)+s(\tau)}{(t-\tau)^{3/2}}\exp\{-\frac{(s(t)+s(\tau))^2}{4(t-\tau)}\}d\tau = T_4' + T_4''.$$

Since $s(t)-s(\tau) \geq o$ and $v(\tau) \leq o$ we have

(6.4) $$T_4' \geq o.$$

Since $s(t) + s(\tau) \geq 2b$ we have

$$|T_4''| \leq |\Psi(t)|\int_{t_o-\delta}^{t} \frac{1}{t-\tau}\exp\{-\frac{b^2}{t-\tau}\}d\tau$$

(6.5)

$$\leq B_{12}|\Psi(t)|\delta \leq |\Psi(t)|$$

if

(6.6) $$\delta \leq \frac{1}{B_{12}}\ .$$

We now fix δ such that it satisfies (6.6). There exists a constant B' (depending on δ, but not on t) such that

(6.7) $|T_1 + T_2 + T_3| \leq B'$ for all $t_o-\delta \leq t<t_o$.

Combining (6.4), (6.5), (6.7) with (6.1) we conclude that

(6.8) $v(t) \geq \frac{1}{2}\Psi(t)-B'$ $(t_o-\delta \leq t < t_o)$.

Taking the g.ℓ.b. of both sides of (6.8) when $t_o-\delta \leq t \leq t'$ we easily get

$$\Psi(t') > -2B'$$

and the boundedness of $v(t)$ $(t_o-\delta \leq t < t_o)$ follows.

We now differentiate (4.2) with respect to x, and make use of Lemma 1, (2.12), the boundedness of v(t) and (4.6), (4.7). We then find that $u_x(x,t)$ is a bounded function for $o \leq x \leq s(t)$, $o \leq t < t_o$. u(x,t) is also bounded in this domain (by the maximum principle).

From the inequalities (5.8), (5.9), (5.35), (5.36) which restrict the length σ of the time for which a solution was proved to exist it is clear that if we proceed with the method of §5 but start from $t = t_o - \delta$ upward, then the inequalities restricting σ will be <u>independent</u> of δ. Thus, if we start with δ sufficiently small we shall get a solution $\overline{v}(t)$ for $t_o - \delta \leq t < t_o + \varepsilon$ for some $\varepsilon > o$. This solution coincides with v(t) for $t_o - \delta < t < t_o$, since we assumed uniqueness for $o \leq t < t_o$.

To prove that there exists only one solution v(t) for $o \leq t < t_o + \varepsilon$ we can use a reasoning similar to the one used in 5.3. Details will be omitted.

7. Generalizations

<u>7.1. A weaker definition of a solution.</u> In §1 we defined a solution of (1.1)-(1.5) as a pair of functions u(x,t), s(t) which satisfy the equations (1.1)-(1.5) and certain differentiability properties. Among these differentiability conditions only one may seem to be unnatural, namely, that $u_x(x,t)$ is continuous on x=o. We shall now prove that this condition is superfluous, that is, <u>if</u> f(t) <u>is continuously differentiable and if</u> u(x,t) <u>is solution of</u> (1.1), (1.2) <u>then</u> $u_x(x,t)$ <u>is continuous near and up to</u> x = o.

Introducing Green's function

$$K(x,t;\xi,\tau) = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi^{1/2}(t-\tau)^{1/2}} \exp\{- \frac{(x+4k\alpha-\xi)^2}{4(t-\tau)}\}$$

(7.1)

$$- \sum_{k=-\infty}^{\infty} \frac{1}{2\pi^{1/2}(t-\tau)^{1/2}} \exp\{- \frac{(-x+4k\alpha-\xi)^2}{4(t-\tau)}\}$$

in the rectangle $o \leq x \leq 2\alpha$, $o < \beta \leq t \leq \gamma$, we have the following representation formula for $u(x,t)$:

$$u(x,t) = \int_o^{2\alpha} u(\xi,\beta)K(x,t;\xi,\beta)d\xi + \int_\beta^t f(\tau)[\frac{\partial}{\partial\xi}K(x,t;\xi,\tau)]_{\xi=o} d\tau$$

(7.2)

$$- \int_\beta^t u(2\alpha,\tau)[\frac{\partial}{\partial\xi}K(x,t;\xi,\tau)]_{\xi=2\alpha} d\tau .$$

The proof of (7.2) follows by noting that the right side of (7.2) is a solution of the heat equation with the same boundary values as $u(x,t)$. We now differentiate (7.2) with respect to x, perform in the second integral integration by parts (compare with (4.6)) and then let $x \longrightarrow o$; the assertion then easily follows.

7.2. General parabolic equations. Theorem 1 can be generalized to parabolic equations

(7.3) $\qquad a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u = u_t$

provided $c(x,t) \leq o$ and the functions

$$\frac{\partial a}{\partial x}, \frac{\partial^2 a}{\partial x^2}, \frac{\partial a}{\partial t}, b, \frac{\partial b}{\partial x}, c$$

are Hölder continuous for $o \leq x < \infty$, $t \geq o$.

Indeed we have only to construct Green's function in the domain $x > o$, $t > o$ and then proceed as in the special case of the heat equation. To construct Green's function we apply the method of [13]. We first extend the coefficients of (7.3) into the whole strip $t \geq o$ by defining

$$a(-x) = a(x), \quad c(-x) = c(x) \quad \text{for} \quad x > o$$

$$b(-x) = -b(x) \quad \text{for} \quad x > o.$$

Then we can construct a fundamental solution $\Gamma(x,t;\xi,\tau)$ (by [9]) which will be continuous for $x \neq \xi$, $o \leq \xi < \infty$, $-\infty < x < \infty$ and continuously differentiable for $x \neq \xi$, $o < \xi < \infty$ $o < x < \infty$ and $o < \xi < \infty$, $-\infty < x < o$. We take for Green's function the function

$$\Gamma(x,t;\xi,\tau) - \Gamma(-x,t;\xi,\tau).$$

7.3. A general Stefan problem. We shall consider the case where the temperature of the ice is not constantly zero and is kept under control at $x = c > b$. We then have the following system

(7.4) $$\frac{\partial^2 u_1}{\partial x^2} = a_1^2 \frac{\partial u_1}{\partial t} \quad \text{for} \quad o < x < s(t),\ t > o$$

(7.5) $$\frac{\partial^2 u_2}{\partial x^2} = a_2^2 \frac{\partial u_2}{\partial t} \quad (a_2^2 \neq a_1^2) \quad \text{for } s(t) < x < c$$

(7.6) $u_1(o,t) = f(t)$, $f_1(t) \geq o$ for $t > o$

(7.7) $u_1(x,o) = \varphi_1(x)$, $\varphi_1(x) \geq o$ for $o < x \leq b$, $\varphi_1(b)=o$ $(b > o)$

(7.8) $u_2(x,o) = \varphi_2(x)$, $\varphi_2(x) \leq o$ for $b \leq x < c$, $\varphi_2(b) = o$ $(c > b)$

(7.9) $u_2(c,t) = f_2(t)$, $f_2(t) \leq o$ for $t > o$

(7.10) $u_1(s(t),t) = u_2(s(t),t) = o$ for $t \geq o$, $s(o) = b$

(7.11) $\dot{s}(t) = -k_1 \dfrac{\partial u_1(s(t),t)}{\partial x} + k_2 \dfrac{\partial u_2(s(t),t)}{\partial x}$ $(k_1 > o,\ k_2 > o)$.

With the aid of Green's functions G_1 for (7.4) (in the domain $x > o$, $t > o$) and G_2 for (7.5) (in the domain $x < c$, $t > o$) we can reduce the problem (7.4)-(7.11) to the problem of solving a system of two nonlinear integral equations which we symbolically write in the form

(7.12) $v_i(t) = I(s(t), v_1(t), v_2(t))$ $(i = 1,2)$

where s(t) is defined by

(7.13) $s(t) = -k_1 \int_o^t v_1(\tau)d\tau + k_2 \int_o^t v_2(\tau)d\tau + b$.

Using the method of §§ 5-6 we then can prove that the system (7.4)-(7.11) has a solution $u_1(x,t)$, $u_2(x,t)$, $s(t)$ for all values of t as long as the curve $x = s(t)$ does not intersect the line $x = c$. The curve $x = s(t)$ increases monotonically in t.

The previous method of proving Theorem 1 can also be modified to solve the system (1.1)-(1.5) with (1.2) replaced by

(1.2') $u_x(o,t) = f(t)$, $f(t) \leq o$ for $t > o$

and to solve the system (7.4)-(7.11) with (7.3) replaced by

(7.6') $\dfrac{\partial u_1(o,t)}{\partial x} = f_1(t)$, $f_1(t) \leq o$ for $t > o$

or with (7.9) replaced by

(7.9') $\dfrac{\partial u_2(c,t)}{\partial x} = f_2(t)$, $f_2(t) \geq o$ for $t > o$.

7.4. Some open problems. From the contractive character of the transformation T which appears in the proof of Theorem 1 it follows that the solution $v(t)$ (and hence $u(x,t)$, $s(t)$) can be calculated by successive approximations for a small range of time. Using this remark one can easily show that the solution changes continuously with the data f, φ. We raise the following question: If $f' \geq f$, $\varphi' \geq \varphi$ and s', s are the ice-water curves for the corresponding problems, is $s'(t) \geq s(t)$? In a few particular cases this question is answered by the affirmative, however we do not know whether the answer is yes for any f, φ, f', φ'. A similar question can be asked for other Stefan problems.

The case $b = o$ was excluded from our considerations. In that case Rubinstein [23] proved existence for small intervals of time under some additional restrictions on f. It remains to prove existence without these restrictions. Furthermore, the question of uniqueness is still open. We can prove very easily (using the law of conservation of energy, which is obtained by integrating the heat equation) that if u, s and u', s' are two solutions, then $s(t)$ and $s'(t)$ must intersect each other infinite number of times in any interval $o < t \leq \varepsilon$ $(\varepsilon > o)$. In this proof we only assume that $f(t)$ is a continuous function and therefore this result contains the result of [26].

Part II: Evaporation or Condensation of a Liquid Drop

1. Statement of the Main Result

A liquid in the presence of an undersaturated mixture containing its own vapor will evaporate, while if the mixture is supersaturated the vapor will condense. Assume that we have a 3-dimensional drop and that in the process of evaporation or condensation the drop will remain spherical due to some extraneous mechanism such as the surface tension and that the saturation density g is independent of the radius of the drop. Assume also that in the case of condensation, no new drops are created. Let ρ denote the density of the drop. We further assume that the initial density of the vapor is a function $c_0(x)$ of x, where x denotes the distance from the center of the drop, and that $c_0 = \lim c_0(x) \quad (x \to \infty)$ exists.

Denoting by D the coefficient of diffusion and making the substitutions

$$x \to x, \quad t \to \frac{t}{d}, \quad c(x,t) \to z(x,t) = \frac{c(x,t) - c_0}{g - c_0}$$

where c(x,t) is the vapor density at the point (x,t), we get for z(x,t) the following system of equations (see also [18]):

(1.1) $\quad \Delta z = z_t \quad$ for $\quad s(t) < x < \infty, \; t > 0$

(1.2) $\quad z(x,0) = \psi(x) \quad$ for $\quad x > s(0) = b \quad (b > 0)$

(1.3) $\quad z(s(t), t) = 1 \quad$ for $\quad t > 0$

(1.4) $\quad \alpha \, z_x(s(t), t) = \dot{s}(t) \quad$ for $\quad t > 0$

where $s(t)$ is the radius of the drop and

$$\alpha = \frac{g - c_o}{\rho - g} \quad , \qquad \psi(x) = \frac{c_o(x) - c_o}{g - c_o} \quad .$$

In what follows we shall need to assume the following additional conditions on $z(x,t)$ at infinity:

(1.5) $xz(x,t)$ and $\frac{\partial}{\partial x}[xz(x,t)]$ are bounded as $x \to \infty$, uniformly with respect to t in finite intervals.

Thus we shall have to assume that $x\psi(x)$ and $\frac{\partial}{\partial x}[x\psi(x)]$ satisfy the same boundedness conditions. This means that $c_o(x)$ tends to its limit c_o "sufficiently fast". We shall also assume below that $\psi(x) \leq 1$. This means that

$$\text{if } \alpha > o \quad (\text{or } c_o < g) \quad \text{then} \quad c_o(x) \leq g,$$

and similarly

$$\text{if } \alpha < o \quad (\text{or } c_o > g) \quad \text{then} \quad c_o(x) \leq g.$$ (Note that $\alpha > -1$.)

The first case is the case of an undersaturated mixture and the second case corresponds to a supersaturated mixture. It is thus seen that the assumption $\psi(x) \leq 1$ is natural.

Using the maximum principle [21] we easily conclude that $s_x(s(t),t) \leq o$. Therefore, by (1.4), if $\alpha > o$ then $\dot{s}(t) \leq o$ and the drop decreases (evaporation), while if $\alpha < o$ then $\dot{s}(t) \geq o$ and the drop grows (condensation). As in Part I we can prove that if $\psi(x) \not\equiv 1$ then $s(t)$ is strictly monotone in t.

Denoting $u(x,t) = xz(x,t)$ the system (1.1) - (1.5) takes the form

(1.6) $u_{xx} = u_t$ for $s(t) < x < \infty$, $t > o$

(1.7) $u(x,0) = \varphi(x)$ for $b < x < \infty$ $(\varphi(x) = s(o)\Psi(x) \leq s(o)$

(1.8) $u(s(t),t) = s(t)$ for $t < o$

(1.9) $\alpha u_x(s(t), t) = s(t)\dot{s}(t) + \alpha$ for $t > o$

(1.10) $u(x,t)$ and $u_x(x,t)$ are bounded as $x \to \infty$ uniformly with respect to t in finite intervals.

Definition. A pair of functions $u(x,t)$, $s(t)$ is called a solution of the system (1.6) - (1.10) for $t < \sigma$ $(\sigma \leq \infty)$ if they satisfy these equations, if u_{xx}, u_t are continuously differentiable for $s(t) < x < \infty$, $o < t < \sigma$, if u_x is continuously differentiable for $s(t) \leq x < \infty$, $o < t < \sigma$, if u is continuous for $s(t) \leq x < \infty$, $o < t < \sigma$ and $s(o) < x < \infty$, $t = o$ and $\varphi(b) \leq \liminf u \leq \limsup u \leq s(o)$ as $(x,t) \to (b,o)$, if $s(t)$ is continuous for $o \leq t < \sigma$ and, finally, if $\dot{s}(t)$ is continuous for $o < t < \sigma$ and

$\lim t^{1/2}\dot{s}(t)$ exists (or $\lim t^{1/2}u_x(s(t),t)$ exists) as $t \to o$.

If $\varphi(b) = b$ then we shall also demand that $\dot{s}(t)$ is continuous at $t = o$ and that $u(x,t)$ is continuous at (b,o).

We shall need the following assumption on $\varphi(x)$:

(Φ) The functions $\varphi(x)$, $\varphi'(x)$ are continuous and bounded for $b \leq x < \infty$ and $\int_b^\infty |\varphi(x)|dx$ exists.

We can now formulate the main result of this Part II.

Theorem 2. Assume that $\varphi(x)$ satisfies the condition (Φ). (i) If $\varphi(b) = b$ and $\alpha < 0$ and satisfies (3.50) then there exists a unique solution $u(x,t)$, $s(t)$ to the system (1.6)-(1.10) defined for all $0 \leq t < \infty$. Furthermore,

$$(1.11) \qquad s(t) \leq A + Bt^{1/2} \qquad (0 \leq t < \infty)$$

where A, B are constants depending only on α, b, φ. (ii) If $\varphi(b) = b$ and $\alpha > 0$ and satisfies (3.50) then there exists a solution $u(x,t)$, $s(t)$ as long as the radius $R = s(t)$ satisfies (4.6), (4.7). The solution is determined uniquely. (iii) If $\varphi(b) < b$ then the above results of (i), (ii)) about existence remain true. The solution is determined uniquely if we make the additional condition (3.54) (with s_0 replaced by s).

The proof of Theorem 2 is given in §§2-5. In §2 we transform the problem into a problem of solving a nonlinear integral equation. In §3 we solve the integral equation for small time-intervals by a method different in some important respects from the method of Part I. In §4 we repeat step by step the process of §3 and prove (using Lemma 2 and (1.11) that this process can be continued so as to obtain the assertions of Theorem 2. Finally, in §5 we prove the inequality (1.11).

2. The Integral Equation

Suppose $u(x,t)$, $s(t)$ form a solution of the system (1.6)-(1.10)

Using Green's identity with u(x,t) and

$$G(x,t;\xi,\tau) = \frac{1}{2\pi^{1/2}(t-\tau)^{1/2}} \exp\left\{-\frac{(x-\xi)^2}{4(t-\tau)}\right\} \tag{2.1}$$

in the domain $s(\tau) < x < \infty$, $\varepsilon < \tau < t-\varepsilon$ and using (1.6), (1.7), (1.8), (1.10), then letting $\varepsilon \to 0$, we obtain

$$u(x,t) = \int_b^\infty G(x,t;\xi,0)\,\varphi(\xi)\,d\xi - \int_0^t G_x(x,t;s(\tau),\tau)s(\tau)$$

$$- \int_0^t G(x,t;s(\tau),\tau)\,u_x(s(\tau),\tau)\,d\tau . \tag{2.2}$$

Denote

$$v(t) = u_x(s(t),t) \quad . \tag{2.3}$$

The x-derivative of the second integral on the right side of (2.2) is equal to

$$\int_0^t G_t(x,t;s(\tau),\tau)\,s(\tau)\,d\tau = -\int_0^t \Big[\frac{\partial}{\partial\tau} G(x,t;s(\tau),\tau)\quad s(\tau)$$

$$+\int_0^t G_\xi(x,t;s(\tau),\tau)\,\dot{s}(\tau)\,s(\tau)d\tau = G(x,t;b,0)\,b \tag{2.4}$$

$$+\int_0^t G(x,t;s(\tau),\tau)\,\dot{s}(\tau)d\tau + \int_0^t G_\xi(x,t;s(\tau),\tau)\,(\alpha v(\tau)-\alpha)d$$

where we made use of (1.9). Differentiating (2.2) with respect to x and making use of (2.4) and (1.9) we obtain, on letting $x \to s(t)+$ and using Lemma 1,

$$(1+\alpha)v(t) = -2bG(s(t),t;\ b,o) + \alpha + 2\int_b^\infty G_x(s(t),t;\xi,o)\varphi(\xi)d\xi$$

$$(2.5) \quad +2\alpha\int_o^t G(s(t),t;s(\tau),\tau)\frac{d\tau}{s(\tau)} - 2\alpha\int_o^t G_x(s(t),t;\ s(\tau),\tau)d\tau$$

$$-2\alpha\int_o^t G(s(t),t;\ s(\tau),\tau)\frac{v(\tau)}{s(\tau)}d\tau$$

$$-2(1-\alpha)\int_o^t G_x(s(t),t;s(\tau),\tau)\ v(\tau)d\tau.$$

If we integrate (1.9) and make use of (2.3) we find that

$$(2.6) \qquad s(t) = \left[b^2 - 2\alpha(t - \int_o^t v(\tau)d\tau)\right]^{1/2}.$$

We have thus proved that if $u(x,t)$, $s(t)$ form a solution of the system (1.6) - (1.10) for $o \leq t < \sigma$ then $v(t)$ (defined by (2.3)) is a solution of the nonlinear integral equation (2.5) (with $s(t)$ defined by (2.6)) for $o < t < \sigma$ and $t^{1/2}v(t)$ is a continuous function for $o \leq t < \sigma$. Conversely (as in Part I) one can show that if $v(t)$ is a solution of (2.5) (with $s(t)$ defined by (2.6)) for $o < t < \sigma$, and if $t^{1/2}v(t)$ is continuous for $o \leq t < \sigma$ then $u(x,t)$, $s(t)$ form a solution of (1.6) - (1.10) where u is defined by (2.2) with $u_x(s(t),t)$ replaced by $v(t)$.

The reason why we did not perform integration by parts in the third term on the right side of (2.5) is that we wish to define M and σ depending on $\varphi(\xi)$ and not on $\dot{\varphi}(\xi)$. The reason for that last wish is that the method of Part I to find an a priori bound on $v(t)$ or $u_x(x,t)$ fails in the present situation, and if we want to continue the solution to large intervals of time we have to find h, independent of $\dot{\varphi}(\xi)$.

However, the third term on the right side of (2.5) (as well as the first term) does not behave regularly at $t = o$. It behaves essentially like $t^{-1/2}$. We therefore define

(2.7) $$\bar{v}(t) = t^{1/2} v(t)$$

and transform the integral equation (2.5) into an integral equation in $\bar{v}(t)$.

3. Existence and Uniqueness for Small Times

Consider the transformation $\bar{w} = \bar{T}\,\bar{v}$ defined as follows:

(3.1)
$$\begin{aligned}(1+\alpha)\bar{w}(t) = &-2bt^{1/2}G(s(t),t;b,o) + \alpha t^{1/2} + \\ &+2\int_b^\infty t^{1/2}G_x(s(t),t;\xi,o)\,\varphi(\xi)\,d\xi \\ &+2\alpha\int_0^t t^{1/2}G(s(t),t;s(\tau),\tau)\frac{d\tau}{s(\tau)} \\ &-2\alpha\int_0^t t^{1/2}G_x(s(t),t;s(\tau),\tau)\,d\tau \\ &-2\alpha\int_0^t t^{1/2}G(s(t),t;s(\tau),\tau)\frac{\bar{v}(\tau)}{\tau^{1/2}s(\tau)}\,d\tau \\ &-2(1-\alpha)\int_0^t t^{1/2}G_x(s(t),t;s(\tau),\tau)\frac{\bar{v}(\tau)}{\tau^{1/2}}\,d\tau\end{aligned}$$

where

(3.2) $$s(t) = b^2 - 2\alpha\left(t - \int_0^t \frac{\bar{v}(\tau)}{\tau^{1/2}}\,d\tau\right)^{1/2}.$$

This transformation is defined on the Banach space. C_σ of functions $\bar{v}(t)$ continuous for $0 \leq t \leq \sigma$ and with the uniform norm $\| \bar{v} \| = \text{l.u.b.} |\bar{v}(t)|$. Denoting by $C_{\sigma,M}$ the subspace $\| \bar{v} \| \leq M$ we shall prove that for some positive M and σ $\bar{T}$ maps $C_{\sigma,M}$ into itself and is a contraction.

3.1 T maps $C_{\sigma,M}$ into itself. If we choose σ such that

$$2|\alpha|\sigma + 4|\alpha|\sigma^{1/2}M \leq \frac{b^2}{2} \tag{3.3}$$

then on using (3.2) and the inequality $\| \bar{v} \| \leq M$ we get

$$\frac{b}{2} \leq s(t) \leq \frac{3b}{2}, \quad t^{1/2}|\dot{s}(t)| \leq \frac{2|\alpha|}{b}\sigma^{1/2} + \frac{2|\alpha|}{b} M. \tag{3.4}$$

We shall also need the following inequality:

$$|s(t) - s(\tau)| \leq \frac{6|\alpha| M}{bt^{1/2}} (t-\tau) \tag{3.5}$$

provided

$$\sigma^{1/2} \leq M. \tag{3.6}$$

In what follows we shall assume that (3.6) holds. The proof of (3.5) follows from

$$s(t) - s(\tau) = \left| \int_\tau^t \dot{s}(\lambda) d\lambda \right| = \left| \int_\tau^t \left[\alpha \frac{\bar{v}(\lambda)}{\lambda^{1/2}} - \alpha\right] \frac{d\lambda}{s(\lambda)} \right|$$

$$\leq \frac{2|\alpha|M}{b} \int_\tau^t \frac{d\lambda}{\lambda^{1/2}} + \frac{2|\alpha|}{b}(t-\tau)$$

and

$$\int_\tau^t \frac{d\lambda}{\lambda^{1/2}} = 2(t^{1/2} - \tau^{1/2}) \leq \frac{2}{t^{1/2}}(t-\tau) \quad ,$$

where we made use of (3.4).

With the aid of (3.4), (3.5) it is now easy to estimate $\|\bar{w}\|$. We have

$$(1+\alpha)\,\|\bar{w}\| \le \frac{b}{\pi^{1/2}} + |\alpha|\sigma^{1/2} + 2\|\varphi\| + \frac{4|\alpha|}{\pi^{1/2}b}\sigma + \frac{6\alpha^2 M}{\pi^{1/2}b},$$

(3.7)

$$+ \frac{8|\alpha|M}{\pi^{1/2}b}\sigma^{1/2} + 12\,\frac{|\sigma|(1-\alpha)M^2}{\pi^{1/2}b}\,.$$

We define

(3.8) $$M = 2\,\frac{b + \|\varphi\|\pi^{1/2}}{\pi^{1/2}(1+\alpha)}$$

and assume that α satisfies

(3.9) $$\frac{48}{\pi}\;\frac{|\alpha|(1-\alpha)}{(1+\alpha)^2}\,(b + \|\varphi\|\pi^{1/2})^2 \le \vartheta\, b^2 \qquad (0<\vartheta<1)$$

for some ϑ. If σ satisfies

(3.10) $$\sigma^{1/2} + \frac{4}{\pi^{1/2}b}\sigma + \frac{6\alpha^2 M}{\pi^{1/2}b}\sigma^{1/2} + \frac{8|\alpha|M}{\pi^{1/2}b}\sigma^{1/2} \le \frac{(1-\vartheta)b}{\pi^{1/2}}$$

then it follows from (3.7) and (3.8), (3.9) that $\|\bar{w}\| \le M$, that is, $\bar{T}$ maps $C_{\sigma,M}$ into itself.

3.2. T is a contraction. In what follows we shall denote by A_i appropriate universal constants. Let $\bar{w} = \bar{T}\,\bar{v}$, $\bar{w}_o = \bar{T}\,\bar{v}_o$ and denote

(3.11) $$\|\bar{v} - \bar{v}_o\| = \varepsilon\,.$$

Since

(3.12) $$\|\bar{v}\| \le M, \quad \|\bar{v}_0\| \le M,$$

we find as in 3.1 that

(3.13) $$\frac{b}{2} \le s(t) \le \frac{3b}{2}, \quad \frac{b}{2} \le s_0(t) \le \frac{3b}{2}$$

(3.14) $$t^{1/2}|\dot{s}(t)| \le \frac{4|\alpha|M}{b}, \quad t^{1/2}|\dot{s}_0(t)| \le \frac{4|\alpha|M}{b}$$

(3.15) $$|s(t)-s(\tau)| \le \frac{6|\alpha|M}{bt^{1/2}}(t-\tau), \quad |s_0(t)-s_0(\tau)| \le \frac{6|\alpha|M}{bt^{1/2}}(t-\tau)$$

where $s_0(t)$ is defined by (3.2) with $\bar{v}$, s replaced by $\bar{v}_0$, s_0. In proving (3.13) we make use of (3.3) and in proving (3.14),(3.15) we make use also of (3.6).

Next we have

$$|s(t)-s_0(t)| = \frac{|s^2(t)-s_0^2(t)|}{s(t)+s_0(t)} \le \frac{2|\alpha|}{b} \int_0^t \frac{|\bar{v}(\tau)-\bar{v}_0(\tau)|}{\tau^{1/2}}\, d\tau .$$

Hence,

(3.16) $$|s(t)-s_0(t)| \le \frac{4|\alpha|}{b}\, \varepsilon\, t^{1/2}.$$

We shall also need the inequality

(3.17) $$|\dot{s}(t) - \dot{s}_0(t)| \le \frac{6|\alpha|\varepsilon}{bt^{1/2}} .$$

To prove (3.17) we first use (3.2) to conclude that

(3.18) $$|s(t)\dot{s}(t)-s_0(t)\dot{s}_0(t)| = \frac{|\alpha|}{t^{1/2}}\, |\bar{v}(t)-\bar{v}_0(t)| \le \frac{|\alpha|\varepsilon}{t^{1/2}} .$$

Next, using (3.14) and (3.16) we get

$$|s(t)\dot{s}(t)-s_o(t)\dot{s}(t)| = |\dot{s}(t)|\ |s(t)-s_o(t)| \le \frac{4|\alpha|M}{bt^{1/2}}\,\frac{4|\alpha|}{b}\,\varepsilon t^{1/2}$$

(3.19)
$$= 16\frac{\alpha^2 M\varepsilon}{b^2} \quad .$$

Combining (3.18), (3.19) and using (3.13) we conclude that

$$|\dot{s}(t)-\dot{s}_o(t)| \le \frac{2}{b}\left(\frac{|\alpha|\varepsilon}{t^{1/2}} + 16\,\frac{\alpha^2 M\varepsilon}{b^2}\right) \le \frac{6|\alpha|\varepsilon}{bt^{1/2}} \quad ,$$

where in the last inequality we made use of (3.3).

We now proceed to estimate $\bar{w} - \bar{w}_o$. We write

(3.20) $$(1+\alpha)\ (w-w_o) = \sum_{i=1} V_i$$

where

$$V_1 = -2bt^{1/2}\left[G(s(t),t;b,o) - G(s_o(t),t;b,o)\right]$$

$$V_2 = -\int_b^{\infty} t^{1/2}\left[\frac{s(t)-\xi}{t}\,G(s(t),t;\xi,o) - \frac{s_o(t)-\xi}{t}\,G(s_o(t),t;\xi,o)\right]\varphi(\xi)d$$

$$V_3 = 2\alpha\int_0^t t^{1/2}\left[G(s(t),t;s(\tau),\tau)\frac{1}{s(\tau)} - G(s_o(t),t;s_o(\tau),\tau)\frac{1}{s_o(\tau)}\right] d$$

$$V_4 = \alpha\int_0^t t^{1/2}\left[\frac{s(t)-s(\tau)}{t-\tau}\,G(s(t),t;s(\tau),\tau)\right.$$

$$\left. - \frac{s_o(t)-s_o(\tau)}{t-\tau}\,G(s_o(t),t;s_o(\tau),\tau)\right] d\tau$$

$$V_5=-2\alpha\int_0^t t^{1/2}\Big[G(s(t),t;s(\tau),\tau)\ \frac{\bar{v}(\tau)}{t^{1/2}s(\tau)}$$

$$-G(s_o(t),t;s_o(\tau),\tau)\ \frac{\bar{v}_o(\tau)}{\tau^{1/2}s_o(\tau)}\Big]\ d\tau$$

$$V_6=(1-\sigma)\int_0^t t^{1/2}\Big[\frac{s(t)-s(\tau)}{t-\tau}\ G(s(t),t;s(\tau),\tau)\ \frac{\bar{v}(\tau)}{\tau^{1/2}}$$

$$-\frac{s_o(t)-s_o(\tau)}{t-\tau}\ G(s_o(t),t;s_o(\tau),\tau)\ \frac{\bar{v}_o(\tau)}{\tau^{1/2}}\Big]\ d\tau.$$

To estimate V_1 we use the mean value theorem and (3.16). We get

(3.21) $$|V_1|\le 2\left(\frac{2}{\pi e}\right)^{1/2}|\alpha|\varepsilon\le|\alpha|\varepsilon.$$

To estimate V_2 we write $V_2 = V_2' + V_2''$ where

$$V_2' = -\int_b^\infty t^{1/2}\ \frac{s(t)-s_o(t)}{t}\ G(s(t),t;\xi,o)\varphi(\xi)\ d\xi$$

$$V_2'' = -\int_b^\infty t^{1/2}\ \frac{s_o(t)-\xi}{t}\ \big[G(s(t),t;\xi,o)-G(s_o(t),t;\xi,o)\ \varphi(\xi)$$

Using (3.16) we easily get

(3.22) $$|V_2'|\le\frac{4|\alpha|\ \|\varphi\|}{b}\ \varepsilon.$$

To estimate V_2'' we use the technique used in estimating V_2' in Part I, § 5. We divide the integral of V_2'' into three parts, namely,

$$(3.23) \quad -V_2'' = \int_b^\infty = \int_b^{s(t)} + \int_{s(t)}^{s_o(t)} + \int_{s_o(t)}^\infty = S_1 + S_2 + S_3$$

where without loss of generality we assumed that $b \leq s(t) \leq s_o(t)$. The estimation of S_2 is immediate. Indeed,

$$(3.24) \qquad |S_2| \leq |s_o(t)-s(t)|^2 \; \frac{1}{\pi^{1/2} t} \leq 32 \, \frac{\alpha^2 M}{b\,\pi^{1/2}} \, \varepsilon \; .$$

Here we made use of the inequalities

$$|s_o(t)-\xi| \leq |s_o(t)-s(t)|, \quad |G(x,t;\xi,o)| \leq \frac{1}{2\,\pi^{1/2} t^{1/2}}$$

and of (3.16).

To estimate S_3 we use the mean value theorem and the inequaliti

$$x\sqrt{2}\, e^{-x^2} \leq e^{-x^2/2} \quad , \; |\tilde{s}-\xi| \geq |s_o(t)-\xi|$$

where $s(t) < \tilde{s} < s_o(t)$. We get after some calculations

$$(3.25) \qquad |S_3| \leq \frac{8\sqrt{2}\,|\alpha|}{b} \, \|\varphi\| \varepsilon \; .$$

To estimate S_1 we write in the integrand of S_3

$$s_o(t) - \xi = [s_o(t) - s(t)] + [s(t) - \xi].$$

We then obtain two integrals. The one that corresponds to $[s(t) - \xi]$ can be estimated as S_3. The second integral is bounded by

$$\frac{4|\alpha|\varepsilon}{b} \int_b^\infty \left[G(s(t),t;\xi,o) + G(s_o(t),t;\xi,o)\right] |\varphi(\xi)| d\xi$$

$$\leq \frac{8|\alpha|\varepsilon}{b} \|\varphi\| .$$

Combining these last estimates with (3.25), (3.24) and (3.23) we get

$$(3.26) \qquad |V_2''| \leq 32\frac{\alpha^2 M}{b\pi^{1/2}}\varepsilon + (16 \frac{\sqrt{2}\,|\alpha|}{b} + 8 \frac{|\alpha|}{b}) \|\varphi\| \varepsilon .$$

Combining (3.26) with (3.22) we conclude that

$$(3.27) \qquad |V_2| \leq 32 \frac{\alpha^2 M}{b\pi^{1/2}}\varepsilon + 35 \frac{|\alpha| \, \|\varphi\|}{b} \varepsilon .$$

To estimate V_3 we write $V_3 = V_3' + V_3''$ where

$$V_3' = 2\alpha \int_0^t t^{1/2}\left[G(s(t),t;s(\tau),\tau) - G(s_o(t),t;s_o(\tau),\tau)\right] \frac{d\tau}{s(\tau)}$$

$$V_3'' = 2\alpha \int_0^t t^{1/2} G(s_o(t),t;s_o(\tau),\tau)\left[\frac{1}{s(\tau)} - \frac{1}{s_o(\tau)}\right] d\tau .$$

To estimate V_3' consider first the expression

$$(3.28) \qquad F \equiv 1 - \exp\left\{ - \frac{(s_o(t)-s_o(\tau))^2 - (s(t)-s(\tau))^2}{4(t-\tau)} \right\} .$$

The braces are bounded by

$$(3.29) \qquad \frac{1}{4(t-\tau)} \left[|s_o(t)-s(t)| + |s_o(\tau)-s(\tau)|\right]\left[|s_o(t)-s_o(\tau)| + |s(t)-s(\tau)|\right] \leq \frac{24\alpha^2 M\varepsilon}{b^2} ;$$

here we made use of (3.15), (3.16). For later purposes we shall assume that α satisfies

$$(3.30) \qquad 48\,\frac{\alpha^2 M^2}{b^2} \le \frac{1}{8} \quad .$$

Using (3.29) and the inequality $\mathcal{E} \le 2M$ we then conclude that

$$(3.31) \qquad |F| \le e^{1/2}\,\frac{24\,\alpha^2 M\,\mathcal{E}}{b^2} \quad .$$

Substituting (3.31) into V_3' we get

$$(3.32) \qquad |V_3'| = 2|\alpha|\,\left|\int_0^t t^{1/2}\,G(s(t),t;s(\tau),\tau)\frac{F}{s(\tau)}\,d\tau\right| \le A_1\frac{|\alpha|^3 M}{b^3}\sigma$$

To estimate V_3'' we first note that

$$(3.33) \quad \left|\frac{1}{s(\tau)} - \frac{1}{s_0(\tau)}\right| \le \frac{4}{b^2}\,|s(\tau)-s_0(\tau)| \le \frac{16|\alpha|\mathcal{E}}{b^3}\,\tau^{1/2} \quad .$$

Substituting (3.33) into V_3'' we get

$$(3.34) \qquad |V_3''| \le A_2\,\frac{\alpha^2}{b^3}\,\sigma^{3/2}\mathcal{E} \quad .$$

Combining (3.34) with (3.32) we conclude that

$$(3.35) \qquad |V_3| \le A_1\,\frac{|\alpha|^3 M}{b^3}\,\sigma\mathcal{E} + A_2\frac{\alpha^2}{b^3}\,\sigma^{3/2}\mathcal{E} \quad .$$

We proceed to estimate V_5. We write $V_5 = V_5' + V_5''$ where

$$V_5' = -2\alpha \int_0^t t^{1/2} \left[G(s(t),t;s(\tau),\tau) - G(s_0(t),t;s_0(\tau),\tau) \right] \frac{\bar{v}(\tau)}{\tau^{1/2} s(\tau)} \, d\tau$$

$$V_5'' = -2\alpha \int_0^t t^{1/2} \, G(s_0(t),t;s_0(\tau),\tau) \left[\frac{\bar{v}(\tau)}{s(\tau)} - \frac{\bar{v}_0(\tau)}{s(\tau)} \right] \frac{d\tau}{\tau^{1/2}} \, .$$

We can estimate V_5' by the method we used to estimate V_3'. Using (3.28), (3.31) we get

(3.36) $$|V_5'| \leq A_3 \frac{|\alpha|^3 M^2}{b^3} \sigma^{1/2} \varepsilon \, .$$

To estimate V_5'' we first note on using (3.13) and (3.33) that

(3.37) $$\left| \frac{\bar{v}(\tau)}{s(\tau)} - \frac{\bar{v}_0(\tau)}{s_0(\tau)} \right| \leq \frac{2\varepsilon}{b} + 16 \frac{|\alpha| M \varepsilon}{b^2} \tau^{1/2} .$$

Substituting (3.37) in V_5'' and using (3.3) we obtain

(3.38) $$|V_5''| \leq A_4 \frac{|\alpha|}{b^2} (1 + b) \sigma^{1/2} \varepsilon \, .$$

Combining (3.36) with (3.33) and using (3.30) we get

(3.39) $$|V_5| \leq A_5 \frac{|\alpha|}{b^2} (1 + b) \sigma^{1/2} \varepsilon \, .$$

To estimate V_6 we write

(3.40) $$V_6 = (1-\alpha) \left(V_6^{\,0} + V_6' + V_6'' \right)$$

where

$$V_6{}^o = \int_0^t t^{1/2} \frac{s(t)-s(\tau)}{t-\tau} G(s(t),t;s(\tau),\tau)\left[\bar{v}(\tau)-\bar{v}_o(\tau)\right] \frac{d\tau}{\tau^{1/2}}$$

$$V_6{}' = \int_0^t t^{1/2} \left[\frac{s(t)-s(\tau)}{t-\tau} - \frac{s_o(t)-s_o(\tau)}{t-\tau}\right] G(s(t),t;s(\tau),\tau) \frac{\bar{v}_o(\tau)}{\tau^{1/2}} d\tau$$

$$V_6{}'' = \int_0^t t^{1/2} \frac{s_o(t)-s_o(\tau)}{t-\tau} \left[G(s(t),t;s(\tau),\tau)-G(s_o(t),t;s_o(\tau),\tau\right] \frac{\bar{v}_o(\tau)}{\tau^{1/2}} d\tau.$$

Using (3.15) we easily get

$$(3.41) \qquad |V_6{}^o| \le \frac{12}{\pi^{1/2}} \frac{|\alpha|M}{b} \varepsilon .$$

To estimate $V_6{}'$ we divide it into two integrals

$$(3.42) \qquad V_6{}' = \int_0^t = \int_0^{t/2} + \int_{t/2}^t = N_1 + N_2.$$

Using (3.16) we get

$$(3.43) \qquad |N_1| \le \frac{16}{\pi^{1/2}} \frac{|\alpha|M}{b} \varepsilon .$$

Using the mean value theorem and (3.17) we get

$$(3.44) \qquad |N_2| \le \frac{6\sqrt{2}}{\pi^{1/2}} \frac{|\alpha|M}{b} \varepsilon .$$

Combining (3.44), (3.43) with (3.42) we find that

$$(3.45)\qquad |V_6'| \leq \frac{16 + 6\sqrt{2}}{\pi^{1/2}} \frac{|\alpha| M}{b} \varepsilon \ .$$

To complete the estimation of V_6 it remains to estimate V_6''. Using (3.15) and the inequality (3.31) we easily find that

$$(3.46)\qquad |V_6''| \leq 288 \left(\frac{\varepsilon}{\pi}\right)^{1/2} \frac{|\alpha|^3 M^3 \varepsilon}{b^3} \ .$$

Combining (3.46), (3.45), (3.41) with (3.40) we conclude that

$$(3.47)\qquad |V_6| \leq \left[26 \frac{|\alpha| M}{b} \varepsilon + 240 \frac{|\alpha|^3 M^3}{b^3} \varepsilon\right] (1-\alpha).$$

It finally remains to estimate V_4. From the similarity between the expressions V_4 and V_6 one can without any difficulty derive the following inequality:

$$(3.48)\qquad |V_4| \leq A_6 \left(\frac{\alpha^2}{b} \sigma^{1/2} \varepsilon + \frac{\alpha^4 M^2}{b^3} \sigma^{1/2} \varepsilon\right) .$$

Combining (3.48), (3.47), (3.39), (3.35), (3.27), (3.21) with (3.20) we get

$$(3.49)\qquad (1+\alpha) \frac{\| \bar{w} - \bar{w}_0 \|}{\varepsilon} \leq |\alpha| + 19 \frac{\alpha^2 M}{b} + 35 \frac{|\alpha| \, \|\varphi\|}{b} + \left(26 \frac{|\alpha| M}{b} + 240 \frac{|\alpha|^3 M^3}{b^3}\right) (1-\alpha) + A \sigma^{1/2}$$

where A is an appropriate continuous function of b, α, $\|\varphi\|$ if $b \neq 0$ (note that M is a function of b, $\|\varphi\|$, by (3.8)). We recall that in proving (3.49) we assumed that α satisfies (3.30).

<u>3.3 Conclusion.</u> It remains to choose α in such a way that (3.9) and (3.30) will be satisfied and such that $\bar{T}$ is a contraction. We remark here that the previous estimates are not very sharp and can easily be improved. However, we decided to sacrifice a precise evaluation of α for the sake of simplicity in the calculations. Proceeding with this point of view we shall now prove that if

(3.50) $$|\alpha| < \frac{1}{200}, \quad |\alpha| \frac{\|\gamma\|}{b} < \frac{1}{200}, \quad |\alpha| \frac{\|\gamma\|^2}{b^2} < \frac{1}{150}$$

then $\bar{T}$ maps $C_{\sigma, M}$ into itself and is a contraction.

Indeed using the definition of M in (3.8) we easily find that

(3.51) $$\frac{|\alpha| M}{b} < \frac{1}{50}$$

and, in particular, (3.30) holds. Using (3.51) and (3.50) we also find that

(3.52) $$\frac{\| \bar{w} - \bar{w}_0 \|}{\varepsilon} \leq \frac{3}{4} + A\tau^{1/2} .$$

Thus if τ satisfies (in addition to (3.3), (3.6), (3.10))

$$A\,\upsilon^{1/2} < \frac{1}{4}$$

then $\bar{T}$ is a contraction.

The condition (3.9) for some $\rho < 1$ is also easily verified on using the first and last inequalities of (3.50).

Having proved that $\bar{T}$ maps $C_{\sigma, M}$ into itself and is a contraction we conclude that there exists a function $\bar{v}(t)$ in $C_{\tau, M}$ such that $\bar{v} = \bar{T}\bar{v}$; $v(t) = t^{-1/2}\bar{v}(t)$ is then a solution of the integral equation (2.5).

If $v_o(t)$ is another solution of the integral equation (2.5) and if $\bar{v}_o(t) = t^{1/2}v_o(t)$ satisfies

(3.53) $$\lim_{t\to o} |\bar{v}_o(t)| < M$$

then $\bar{v}_o(t) \equiv \bar{v}(t)$ for $o \leq t < \sigma$ where σ is sufficiently small. Using the definition (3.8) of M and (1.9), it follows that if $v_o(t)$ is such that

(3.54) $$\lim_{t\to o} t^{1/2}|\dot{s}_o(t)| < \frac{2|\alpha|}{1+\alpha} \frac{b+\|\varphi\|\pi^{1/2}}{\pi^{1/2}b}$$

then $u_o(x,t) \equiv u(x,t)$, $s_o(t) \equiv s(t)$ for $o \leq t \leq \sigma$, where u_o, s_o is the solution of (1.6) - (1.10) corresponding to v_o. We have thus completed the discussion of the existence and uniqueness of solutions for small time-intervals.

4. Existence and Uniqueness for Large Time Intervals

As already remarked in § 2 the method used in Part I to prove existence and uniqueness for large intervals of time fails in the present case since we cannot find an appropriate a priori bound on $u_x(x,t)$ as in Part I. We shall therefore use a different method based on the inequalities derived in § 3 and which involves an a priori bound on $u(x,t)$.

We first discuss the problem of uniqueness. Suppose first that $\varphi(b) = b$ and perform integration by parts on the third integral on the right side of (2.5). We easily find that the sum of the first three terms on the right side of (2.5) is equal to

(4.1) $$\alpha + 2\int_b^\infty G(s(t),t;\xi,o)\,\dot{\varphi}(\xi)\,d\xi .$$

Thus the right side of (2.5) includes only terms which behave regularly at $t = o$. A comparison of the integral equation (2.5) with the corresponding integral equation for $v(t)$ in Part I shows that the methods and results of Part I § 5 remain valid with slight changes. Hence, there exists a unique solution for small intervals of time provided $\alpha > -1$. By a solution in this context we mean that it satisfies the system of equations (1.6) - (1.10) and all the regularity properties mentioned in § 1. (thus, $\dot{s}(t)$ is continuous at $t = o$ and $u(x,t)$ is continuous at (b,o)).

As in Part I we can also prove that the uniqueness of solutions for any interval of time follows from the uniqueness of solutions for small intervals of times. Thus, if $\varphi(b) = b$ then there exists at most one solution (in the sense of § 1) whereas if $\varphi(b) \neq b$ and two solutions coincide for some $t = \sigma > o$ then they coincide in the whole t-interval of their existence.

We proceed to discuss the question of existence of solutions for large intervals of time. We shall need a certain result which we state as a lemma.

Lemma 2. If $\varphi(b) = b$ and $\bar{v}(t)$ satisfies $\bar{v} = T\bar{v}$ for $o \leq t \leq \sigma$, then $v(t) = t^{-1/2}\bar{v}(t)$ satisfies (2.5) and is continuous for $o \leq t \leq$

Proof. As was proved in § 3 if $v_o(t)$ is another solution of (2.5) which satisfies (3.53), then $v(t) \equiv v_o(t)$ for all sufficiently small t. Now, as was mentioned above, if $\varphi(b) = b$ there exists a solution $v_o(t)$ of (2.5) continuous also at $t = o$ and it evidently satisfies (3.53). Hence, for $\bar{\sigma}$ sufficiently small $v(t) \equiv v_o(t)$ $(o < t \leq \bar{\sigma})$. This proves the continuity of $v(t)$ at $t = o$ and the proof of the lemma is thereby completed.

Suppose now that we have constructed a solution for all $o \leq t < t_o$ and assume that

$$(4.2) \qquad o < R \equiv \lim_{t \to t_o} s(t) < \infty .$$

Further assume (compare with (3.50)) that

$$(4.3) \qquad \frac{|\alpha|}{R} \limsup_{t \to t_o} |u(x,t)| < \frac{1}{200}$$

$$(4.4) \qquad \frac{|\alpha|}{R^2} \left[\limsup_{t \to t_o} |u(x,t)|\right]^2 < \frac{1}{150} \quad .$$

From the results of §3 it follows that there exists a positive number δ independent of t_o such that we can construct a solution $\overline{v}(t)$ to the integral equation $\overline{v} = \overline{T}\,\overline{v}$ defined by (3.1) but with $o, b, \varphi(\xi)$ replaced by $\overline{t}$, $s(\overline{t})$, $u(\xi, \overline{t})$, and that $\overline{v}(t)$ exists for $\overline{t} \leq t \leq \overline{t} + \delta$. Since $u(s(\overline{t}), \overline{t}) = s(\overline{t})$, we can use Lemma 2 to conclude that $v(t) = (t-\overline{t})^{-1/2}\overline{v}(t)$ is continuous at $t = \overline{t}$. From the uniqueness results proved above it follows (as in Part I) that the $v(t)$ just constructed coincides with the $v(t)$ whose existence was originally assumed for all $o \leq t < t_o$ in their common interval of existence. Since δ is independent of t_o, we have thus proved that the solution $v(t)$ (or $u(x,t)$, $s(t)$) can be continued above t_o to the interval $o \leq t < t_o + \delta$, as long as (4.2), (4.3), (4.4) are satisfied.

If $\alpha < o$ then $s(t)$ increases in t. Condition (4.2) then means that (s(t) remains bounded for bounded times. A quantitative result which includes this statement will be proved in § 5. It thus remains to consider the conditions (4.3), (4.4).

If $\alpha < 0$ then, by the maximum principle,

$$(4.5) \qquad |u(x,t)| \leq \max\,(s(t),\ \|\varphi\|)\ .$$

Substituting (4.5) into (4.3), (4.4) and noting that $s(t) \geq b$ we obtain the conditions

$$|\alpha|\ \max\,\left(1, \frac{\|\varphi\|}{b}\right) < \frac{1}{200}\ , \quad |\alpha|\ \frac{\|\varphi\|^2}{b^2} < \frac{1}{150}\ .$$

which coincide with the conditions (3.50).

If $\alpha > 0$ we get the conditions

$$(4.6) \qquad R > 200\ \alpha\ \max\,(b,\ \|\varphi\|)$$

$$(4.7) \qquad R^2 > 150\ \alpha\ \max\,(b^2,\ \|\varphi\|^2)\ .$$

We have thus completed the proof of existence and uniqueness of solutions.

Remark. Although the bounds on α can be improved, they are already quite satisfactory from the point of view of physics. Consider for illustration, the case of condensation $\alpha < 0$ and let $\varphi(x) \equiv 0$. Then (3.50) reduces to $|\alpha| < 1/200$, or (using the definition of α)

$$(4.8) \qquad c_0 - g < \frac{1}{200}\,(\rho - g)\ .$$

Take the case of water-steam. Then $\rho = 1$ and if the temperature is below 140°C then $g < 1.968/10^3$. Thus (4.8) roughly states that the density c_0 of the supersaturated vapor is not larger than two and a half times the saturation density g. If however this condition is violated new drops will be created too fast and the system (1.6) - (1.10) might not represent any more a reasonable approximation to the physical situation.

5. An a Priori Bound on s(t) for $\alpha < o$.

In this chapter we shall complete the proof of Theorem 2 by proving that if a solution $u(x,t)$, $s(t)$ is defined for $o \le t < t_o$ then there exist constants A, B depending only on α, b, $||\varphi||$ such that for $o \le t < t_o$

(5.1) $$s(t) \le A + Bt^{1/2} .$$

Only in this chapter we make use of the assumption that

(5.2) $$\int_b^\infty |\varphi(x)| \, dx \text{ is finite.}$$

We first remark that for every $t < t_o$

(5.3) $$\limsup_{x \to \infty} |u_x(x,t)| \le 2 ||\dot{\varphi}|| .$$

This follows by differentiating (2.2) with respect to x and then carrying out integration by parts in the second integral.

If we integrate the heat equation (1.6) over the domain $s(\tau) < \xi < K$, $o < \tau < t$ and make use of (1.8), (1.9), we get

(5.4) $$\int_0^t u_x(K,\tau)\, d\tau - \frac{1}{2\alpha} s^2(t) + \frac{1}{2\alpha} b^2 - t = \int_{s(t)}^K u(x,t)\, dx + \int_0^t s(\tau)\, \dot{s}(\tau)\, d\tau - \int_b^K \varphi(x)\, dx .$$

We proceed to estimate

$$(5.5)\qquad I = \int_{s(t)+H}^{K} |u(x,t)|\, dx \qquad\qquad (H = H_0 t^{1/2})$$

where H_0 is a constant to be determined later. Using the integral representation (2.2) of u(x,t) with u_x replaced according to (1.9) and noting that $s(t) \geq s(\tau)$, $x - s(\tau) \geq x - s(t) \geq H$, we get

$$2\pi^{1/2} I \leq \int_b^\infty \left[\int_0^\infty \frac{1}{t^{1/2}} \exp\left\{ - \frac{(x-\xi)^2}{4t} \right\} dx \right] |\varphi(\xi)|\, d\xi$$

$$+ s(t) \int_{s(t)+H}^{\infty} \left[\int_0^t \frac{1}{t-\tau} \exp\left\{ - \frac{(x-s(t))^2}{8(t-\tau)} \right\} d\tau \right] dx$$

$$(5.6)\qquad + \int_{s(t)+H}^{\infty} \left[\int_0^t \frac{1}{(t-\tau)^{1/2}} \exp\left\{ - \frac{(x-s(t))^2}{4(t-\tau)} \right\} d\tau \right] dx$$

$$+ \frac{1}{2|\alpha|} \int_{s(t)+H}^{\infty} \left| \int_0^t \frac{1}{(t-\tau)^{1/2}} \exp\left\{ - \frac{(x-s(\tau))^2}{4(t-\tau)} \right\} \frac{d}{d\tau} s^2(\tau)\, d\tau \right| dx$$

$$= \sum_{i=1}^{4} M_i$$

where M_i denotes the i-th term on the right side of (5.6). In what follows we shall denote by C_i appropriate positive constants depending on b, α and $\varphi(x)$.

Substituting $\rho = |x-\xi|/t^{1/2}$ in the interior integral of M_1 and using (5.1) we get

(5.7) $$M_1 \leq C_1 \quad .$$

To estimate M_2 we substitute $\rho = x - s(t)$ and note that $\rho \geq H$. We obtain

$$M_2 \leq s(t) \int_H^\infty \Big[\int_0^t \frac{\rho^2}{t-\tau} \exp\Big\{ -\frac{\rho^2}{8(t-\tau)}\Big\}\Big] \frac{d\rho}{\rho^2}$$

$$\leq C_2 s(t) \int_H^\infty t \exp\Big\{ -\frac{\rho^2}{9t}\Big\} \frac{d\rho}{\rho^2} \quad .$$

Substituting $\sigma = \rho/t^{1/2}$ and noting that $H/t^{1/2} = H_o$ we get, on taking $H_o \geq 1$,

(5.8) $$M_2 \leq C_3\ s(t)\ t^{1/2} \int_1^\infty \frac{1}{\sigma^2} \exp\Big\{ -\frac{\sigma^2}{9}\Big\}\, d\sigma \leq C_4 t^{1/2} s(t) \quad .$$

In a similar way we get

(5.9) $$M_3 \leq C_5 t .$$

To estimate M_4 we first perform integration by parts and thus obtain

$$M_4 \leq C_6 s^2(t) \int_{s(t)+H}^\infty \Big[\int_0^t \frac{1}{(t-\tau)^{3/2}} \exp\Big\{ -\frac{(x-s(t))^2}{5(t-\tau)}\Big\} d\tau\Big] dx + C_7$$

Substituting $\rho = x-s(t)$ we get

$$M_4 \leq C_8 s^2(t) \int_H^\infty t \exp\Big\{ -\frac{\rho^2}{5t}\Big\} \frac{d\rho}{\rho^3} + C_7 \quad .$$

Substituting $y = \rho/t^{1/2}$ and taking H_o to be such that

$$(5.10) \qquad C_8 \int_{H_o}^{\infty} \frac{1}{y^2} \exp\left\{ - \frac{y^2}{6} \right\} dy < \frac{1}{4|\alpha|} - \frac{1}{4}$$

we get

$$(5.11) \qquad M_4 \leq \left(\frac{1}{4|\alpha|} - \frac{1}{4}\right) s^2(t) + C_7 .$$

Combining (5.11), (5.9), (5.8), (5.7) with (5.6), we get

$$(5.12) \qquad 2\pi^{1/2} I \leq C_9 + C_5 t + C_4 t^{1/2} s(t) + \left(\frac{1}{4|\alpha|} - \frac{1}{4} \right) s^2(t)$$

We now note that by the maximum principle $|u(x,t)| \leq \max(\|\varphi\|, s(t))$. Hence,

$$(5.13) \qquad \int_{s(t)}^{s(t)+H} |u(x,t)|\, dx \leq H_o \|\varphi\| t^{1/2} + H_o t^{1/2} s(t).$$

The inequalities (5.12), (5.13) give the desired estimate on the first integral on the right side of (5.4). The second integral on the right side of (5.4) is equal to

$$\frac{1}{2} s^2(t) - \frac{1}{2} b^2 .$$

Using these results in (5.4), then taking $K \to \infty$ and using (5.3) we get the inequality

$$(5.14) \qquad s^2(t) \leq C_{10} + C_{11} t + C_{12} t^{1/2} s(t)$$

from which (5.1) follows.

Remark. From the above proof it follows that A, B depend only on b, α, $\|\varphi\|$, $\|\dot{\varphi}\|$ and the integral (5.2).

Formula (5.1) is physically quite satisfactory since it was experimentally verified that $s(t)\dot{s}(t)$ approximates a fixed number for large times.

Concluding Remark. Theorem 2 can be generalized to second order parabolic equations with smooth coefficients as in Part I. It can also be generalized to systems with (1.8) replaced by a more general equation of the form $u(s(t),t) = h(s(t))$ where h is a given function satisfying appropriate conditions.

Part III: Dissolution of Gas Bubble in Liquid

1. Statement of the Result

We shall denote every formula (n.m) of Part II by II(n.m). Let s(t) be the radius of a gas bubble submerged in a liquid and let c(x,t) denote the concentration of the gas dissolved in the liquid, where x is the distance from the center of the drop. If the effects of the surface tension are taken into consideration and if the concentration of the gas on the boundary of the bubble is 1 (which we may assume) then s(t) and u(x,t) = xc(x,t) satisfy the following system of equations (compare [18]):

(1.1) $$u_{xx} = u_t \quad \text{for} \quad s(t) < x < \infty \; , \; t>0$$

(1.2) $$u(x,0) = \varphi(x) \quad \text{for} \quad x>s(0) = b \qquad (b>0)$$

(1.3) $$u(s(t),t) = (1-k)s(t)+kb \quad \text{for } t>0$$

(1.4) $$\alpha u_x(s(t),t) = (s(t)+mb)\,\dot{s}(t) + \alpha b\left(\frac{1-k}{b} + \frac{k}{s(t)}\right) \quad \text{for } t>0.$$

Here k, m, α are constants and $k \geq 0$, $m \geq 0$, $\alpha > -1$, and $\varphi(x)/b$ is the given concentration of the gas at t=0. For simplicity we shall assume in the proof of Theorem 3 below that $k \leq 1$, however all the considerations remain true for any k. The system (1.1) - (1.4) reduces to the system II(1.6) - II(1.9) if k=m=0. Beside the system (1.1)-(1.4) we shall make the following condition on u:

(1.5) $u(x,t)$ and $u_x(x,t)$ remain bounded as $x \to \infty$ uniformly with respect to t in finite intervals.

We shall say that $u(x,t)$, $s(t)$ form a solution to the system (1.1)-(1.5) if they satisfy these equations and if, in addition, they satisfy the regularity properties mentioned in the definition of II §1.

Theorem 3. Assume that $\varphi(x)$ satisfies the condition II(Φ).
(i) If $\varphi(b) = b$ and α satisfies (2.33) where β is a certain positive constant depending only on k_0, m_0 if $k \leq k_0$, $m \leq m_0$, then there exists a unique solution $u(x,t)$, $s(t)$ to the system (1.1 - (1.5) defined in any interval $0 \leq t < \sigma$ for which $R = s(t)$ satisfies (2.32).
(ii) If $\varphi(b) < b$ then the above assertion about existence still holds. The solution is determined uniquely by the additional condition (2.35).
Remark. β can be taken independently of k,m for small k,m (for instance $k \leq 1$, $m \leq 1$) and then it is of order of magnitude of a few hundreds.

2. Proof of Theorem 3

With G defined by II(2.1) we have the following representation for u:

$$u(x,t) = \int_b^\infty G(x,t;\xi,o)\varphi(\xi)\,d\xi$$

(2.1) $$-\int_0^t G_x(x,t;s(\tau),\tau)\,[(1-k)s(\tau)+kb]\,d\tau$$

$$-\int_0^t G(x,t;s(\tau),\tau)\,u_x(s(\tau),\tau)\,d\tau.$$

Define

(2.3) $v(t) = u_x(s(t),t).$

If we differentiate the second term on the right side of (2.1) with respect to x, we get

$$-\int_0^t G_{xx}(x,t;s(\tau),\tau)\,[(1-k)s(\tau)+kb]\,d\tau$$

$$=\int_0^t \left[\frac{\partial}{\partial\tau}G(x,t;s(\tau),\tau)\right][(1-k)s(\tau)+kb]\,d\tau$$

$$-\int_0^t G_\xi(x,t;s(\tau),\tau)\,[(1-k)s(\tau)+kb]\,\dot{s}(\tau)\,d\tau$$

(2.4) $$= -bG(x,t;b,o)-(1-k)\int_0^t G(x,t;s(\tau),\tau)\,\dot{s}(\tau)\,d\tau$$

$$+\int_0^t G_x(x,t;s(\tau),\tau)\,[(1-k)s(\tau)+kb]\,\dot{s}(\tau)\,d\tau.$$

Differentiating (2.1) with respect to x, using (2.4) and (1.4), then letting $x \to s(t)+0$ and using Lemma 1, we get

$$
\begin{aligned}
\tfrac{1}{2} v(t) = & \int_b^{\infty} G_x (s(t), t; \xi, o) \varphi(\xi)\, d\xi - bG(s(t),t;b,o) \\
& -(1-k) \int_o^t G(s(t),t;s(\tau),\tau) \left[\frac{\alpha v(\tau)}{s(\tau)+mb} - \frac{\alpha b}{s(\tau)+mb} \left(\frac{1-k}{b} + \frac{k}{s(\tau)}\right) \right] d\tau \\
& - \tfrac{1}{2} \left[(1-k)s(t)+kb\right] \left[\frac{\alpha v(t)}{s(t)+mb} - \frac{\alpha b}{s(t)+mb} \left(\frac{1-k}{b} + \frac{k}{s(t)}\right) \right] \\
& + \int_o^t G_x(s(t),t;s(\tau),\tau) \left[(1-k)s(\tau)+kb\right] \left[\frac{\alpha v(\tau)}{s(\tau)+mb} - \frac{\alpha b}{s(\tau)+mb} \left(\frac{1-k}{b} + \frac{k}{s(\tau)}\right) \right] d\tau - \int_o^t G_x (s(t), t; s(\tau), \tau)\, v(\tau)\, d\tau
\end{aligned}
\tag{2.5}
$$

and $s(t)$ is defined by (1.4) with $u_x(s(t),t)$ replaced by $v(t)$ and $s(o) = b$.

Conversely one can show that if $s(t)$ is defined by (1.4) with $u_x = v$ and $s(o) = b$ and if $v(t)$ $(o<t\leq\sigma)$ is a continuous solution of (2.5) and $\lim t^{1/2}v(t)$ exists (as $t \to o$), then the $u(x,t)$ defined by (2.1) with u_x replaced by v forms together with $s(t)$ a solution to the system (1.1) - (1.5). It thus remains to consider the integral equation (2.5).

Denoting the right side of (2.5) by 1/2 Tv we conclude that v is a fixed point of the transformation $w = Tv$. Defining

$$\bar{v}(t) = t^{1/2}v(t) \quad , \quad \bar{w}(t) = t^{1/2}w(t)$$

this transformation takes the form $\overline{w} = \overline{T}\overline{v}$. We want to apply the methods of Part II to the present case. To do that, we have to prove some inequalities on s(t). We state them in a few lemmas.

The first lemma is an analogue to the inequality II (3.4) which holds for all σ satisfying II(3.3).

Lemma 3. Assume that $||\overline{v}|| = \underset{o \le t \le \sigma}{1.u.b.}|\overline{v}(t)| \le M$ and let s(t) be defined by (1.4) with $u_x = v$ and $s(o) = b$. If σ satisfies

(2.6) $$4|a|\sigma + 4|a|M\sigma^{1/2} < \frac{b^2}{8}$$

then

(2.7) $$\frac{b}{2} \le s(t) \le \frac{3b}{2} \qquad \text{for } o \le t \le \sigma.$$

Proof. Introducing the functions

(2.8) $$z(t) = s(t) + mb \quad , \quad y(t) = [z(t)]^2,$$

(1.4) (with $u_x = v$) and the condition $s(o) = b$ reduce to

(2.9)
$$y(t) + \frac{2akb}{(y(t))^{1/2} - mb} = \frac{2a}{t^{1/2}} v(t) - 2a(1-k),$$

$$y(o) = (1+m)^2 b^2.$$

A unique solution exists for all t $(o \le t \le \sigma)$ as long as $(y(t))^{1/2} - mb$ remains positive. For $\sigma > o$ sufficiently small we have

$$(2.10) \qquad \frac{b}{2} \leq (y(t))^{1/2} - mb \leq \frac{3b}{2} \qquad \text{if } 0 \leq t \leq \varepsilon .$$

Denoting

$$(2.11) \qquad \lambda(t) = \frac{2akb}{(y(t))^{1/2} - mb}$$

we then have

$$(2.12) \qquad |\lambda(t)| \leq 4|a|k .$$

Integrating both sides of (2.9) and using (2.12) we get

$$(2.13) \qquad |y(t) - (1+m)^2 b^2| \leq 4|a|t + 4|a| Mt^{1/2} .$$

It follows, on using (2.10), that

$$(2.14) \qquad |(y(t))^{1/2} - (1+m)b| = \frac{|y(t) - (1+m)^2 b^2|}{(y(t))^{1/2} + (1+m)b}$$

$$\leq \frac{4}{b^2} (4|a|t + 4|a|Mt^{1/2}) < \frac{1}{2}$$

if

$$(2.15) \qquad 4|a|\varepsilon + 4|a|M\varepsilon^{1/2} < \frac{b^2}{8} .$$

We have thus proved that (2.10) implies

$$(2.16) \qquad \frac{b}{2} < (y(t))^{1/2} - mb < \frac{3b}{2} \qquad \text{if } 0 \leq t \leq \varepsilon$$

provided ε satisfies (2.15). A simple argument about continuation now shows that (2.16) remains valid for all $0 \leq t \leq \sigma$ if σ satisfies

(2,6) Using the definitions (2,8) the proof of the lemma is completed.

The next lemma establishes an inequality analogous to II(3,5).

Lemma 4. Assume that $\|\bar{v}\| \leq M$ and let $s(t)$ be defined by (1.4) with u_x replaced by v and $s(o) = b$. If σ satisfies (2,6) and if

(2.17) $$4\sigma^{1/2} < M$$

then

(2.18) $$|s(t)-s(\tau)| \leq \frac{s|\alpha| M}{t^{1/2}b}(t-\tau) \qquad \text{for } o \leq \tau \leq t \leq \sigma .$$

Proof. Using (1.4) we have

$$s(t) - s(\tau) = \int_{\tau}^{t} \dot{s}(\lambda)\, d\lambda = \int_{\tau}^{t} \frac{\alpha\, \bar{v}(\lambda)}{\lambda^{1/2}(s(\lambda)+mb)}\, d\lambda$$

$$+ \int_{\tau}^{t} \frac{\alpha\, b}{s(\lambda)+mb}\left[\frac{1-k}{b} + \frac{k}{s(\lambda)}\right] d\lambda .$$

Using Lemma 3 we get

$$|s(t)-s(\tau)| \leq \frac{4|\alpha| M}{t^{1/2}b}(t-\tau) + \frac{4|\alpha|}{b}(t-\tau) \leq \frac{5|\alpha|\, M}{t^{1/2}b} ,$$

where we made use of (2.17).

With the aid of Lemmas 3, 4 we can proceed to prove that for some σ, M $\bar{T}$ maps $C_{\sigma,M}$ into itself. The final inequality is

(2.19) $$\|\bar{w}\| \leq \frac{b}{\pi^{1/2}} + 2\|\varphi\| + 4|\alpha| + 2|\alpha| M + 12(1+2|\alpha|)\frac{|\alpha| M^2}{\pi^{1/2}b} + B\sigma^{1/2}$$

where B is an appropriate function depending only on α, M, b ($b \neq o$).

We next turn to the proof that $\bar{T}$ is a contraction. The method of Part II can be applied if we prove appropriate analogues of the inequalities II(3.16), II(3.17). We shall first prove an analogue of II(3.16).

Lemma 5. Assume that δ satisfies (2.6), (2.17) and that

(2.20) $$16\frac{\delta}{b^2} < 1.$$

If $s(t)$, $s_o(t)$ are solutions of (1.4) with $u_x=s$, $u_x=s_o$ respectively and $s(o)= s_o(o)=b$ and if $\| v - v_o\| = \zeta$, then

(2.21) $$|s(t) - s_o(t)| \leq 12\frac{|\alpha|}{b}\zeta t^{1/2} .$$

Proof. Using the notation (2.8) and defining

(2.22) $$\eta(t) = y(t) - y_o(t)$$

where y_o, z_o correspond (via (2.5)) to s_o, we obtain, by (2.9),

(2.23) $$\dot{\eta}(t) + \mu(t)\eta(t) = \frac{2\alpha}{t^{1/2}}\left[\bar{v}(t) - \bar{v}_o(t)\right], \quad \eta(o)=o$$

where

(2.24) $$\mu(t) = -2\alpha kb\left\{\left[(y(t))^{1/2}-mb\right]\left[(y_o(t))^{1/2}-mb\right]\left[(y_o(t))^{1/2} + (y(t))^{1/2}\right]\right\}^{-1} .$$

Integrating (2.23) and using the inequality (see (2.10))

(2.25) $$|\mu(t)| \le \frac{8|\alpha|}{b^2}$$

we get

(2.26) $$|r(t)| \le 4|\alpha|\varepsilon t^{1/2} \exp\left\{16 \frac{|\alpha|\sigma}{b^2}\right\} \le 12|\alpha|\varepsilon t^{1/2},$$

where we made use of the inequalities (2.20) and $|\alpha| \le 1$. Using the definitions (2.22), (2.3) and using (2.26), (2.10) we find that

$$|s(t)-s_0(t)| = |z(t)-z_0(t)| = \frac{|y(t)-y_0(t)|}{(y(t))^{1/2}+(y_0(t))^{1/2}} \le 12\frac{|\alpha|\varepsilon}{b}t^{1/2},$$

and the proof of the lemma is completed.

We proceed to prove an analogue of II(3.17). In what follows we shall need the inequality

(2.27) $$|\dot{s}(t)| \le \frac{5|\alpha|M}{t^{1/2}b}$$

which follows from (2.18) on taking $\zeta \to t$.

Lemma 6. Under the assumptions of Lemma 5 the following inequality holds:

(2.28) $$|\dot{s}(t)-\dot{s}_0(t)| \le 20\frac{|\alpha|\varepsilon}{t^{1/2}b}.$$

Proof. From the proof of Lemma 5 we get

$$|\dot{y}(t)-\dot{y}_0(t)| \le |\mu(t)|\,|y(t)-y_0(t)| + \frac{2|\alpha|}{t^{1/2}}|\bar{v}(t)-\bar{v}_0(t)|$$

(2.29) $$\le \frac{8|\alpha|}{b^2}12|\alpha|\varepsilon t^{1/2} + \frac{2|\alpha|}{t^{1/2}}\varepsilon \le \frac{8|\alpha|\varepsilon}{t^{1/2}};$$

in the last inequality we made use of (2.20) and the inequality $|\alpha| \leq 1$. Using (2.8) we conclude from (2.29) that

$$(2.30) \qquad |(s(t)+mb)\,\dot{s}(t) - (s_o(t)+mb)\dot{s}_o(t)| \leq \frac{8|\alpha|\varepsilon}{t^{1/2}} .$$

We next find on using Lemma 5 and (2.27) that

$$(2.31) \qquad \begin{aligned} |(s(t)+mb)\,\dot{s}(t) - (s_o(t)+mb)\,\dot{s}(t)| &= |\dot{s}(t)|\;|s(t)-s_o(t)| \\ &\leq 60\frac{\alpha^2 M\varepsilon}{b^2} \leq \frac{2|\alpha|\varepsilon}{t^{1/2}} \end{aligned} ;$$

in the last inequality we made use of (2.6).

Combining (2.30) with (2.31) and using Lemma 3 we get

$$|\dot{s}(t)-\dot{s}_o(t) \leq \frac{1}{s_o(t)+mb} \;\frac{10|\alpha|\varepsilon}{t^{1/2}} \leq 20\,\frac{|\alpha|\varepsilon}{t^{1/2}b}$$

and the proof of the lemma is completed.

With the aid of Lemmas 3-6 we can now use the method of Part II to prove that T is a contraction, and that the (unique) solution can be continued as long as $R = s(t)$ satisfies

$$(2.32) \qquad R^2 > \beta\,|\alpha|\,\max\,(b^2, \|\varphi\|^2) ,$$

provided α satisfies:

$$(2.33) \qquad |\alpha| < \frac{1}{\beta} ,\; |\alpha|\,\frac{\|\varphi\|^2}{b^2} < \frac{1}{\beta} ,$$

where β is a constant depending on k_o, m_o where $k \leq k_o$, $m \leq m_o$. For reasonably small k, m (such as $k \leq 1$, $m \leq 1$) β is of order of magnitude of a few hundreds. We can also prove by the method of Part II that

(2.34) $s(t) \leq A + Bt^{1/2}$ $(A>0,\ B>0)$.

We finally remark that in case $\varphi(b) \neq b$ the condition

(2.35) $\lim_{t\to 0} t^{1/2}|\dot{s}(t)| < A|\alpha| \frac{b+\|\varphi\|}{b}$ $(A>0)$

implies uniqueness, as in Part II. Here A is an appropriate constant depending on k_0, m_0 where $k \leq k_0$, $m \leq m_0$.

Remark. Theorem 3 can be generalized to general parabolic equations with smooth coefficients. Furthermore, it can be extended to more general boundary conditions than (1.4) for which lemmas analogous to lemmas 3-6 can be proved. The boundary condition (1.3) can also be replaced by more general boundary conditions of the form $u(s(t),t) = h(s(t),b,k)$ where h is a given function.

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