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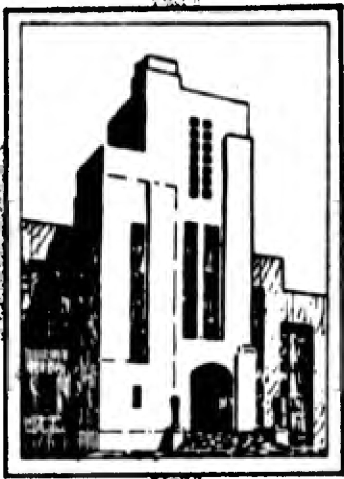

NAVY DEPARTMENT

DAVID TAYLOR MODEL BASIN

HYDROMECHANICS

o

AERODYNAMICS

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APPLIED MATHEMATICS

A VIBRATION MANUAL FOR ENGINEERS
(SECOND EDITION)

by

R.T. McGoldrick

STRUCTURAL MECHANICS LABORATORY

RESEARCH AND DEVELOPMENT REPORT

December 1957

Report R-189

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TABLE OF CONTENTS

Page

NOTATIONS

Symbol	Description	Units	Dimensions in Force-Length-Time System
A	Area of cross section	Square inches	l^2
a	Instantaneous acceleration	Inches per second^2	lt^{-2}
c	Linear damping force per unit velocity or Torsional damping torque per unit of angular velocity	Pound-seconds per inch Inch-pounds per radian per second	$fl^{-1}t$ flt
c_c	Critical damping constant, rectilinear or Torsional	Pound-seconds per inch or Inch-pounds per radian per second	$fl^{-1}t$ or flt
E	Young's modulus of elasticity	Pounds per square inch	fl^{-2}
e	Base of natural logarithms = 2.718 (nondimensional)	meric	
F	Magnification factor, ratio of the single amplitude produced by an exciting force $P_0 \sin \omega t$ to the static deflection that would be produced by the force P_0 (nondimensional)	Numeric	
F_c	Coulomb frictional resistance	Pounds	f
F_{res}	Magnification factor at resonance where frequency is defined as natural frequency without damping (nondimensional)	Numeric	
f	Frequency	Cycles per second	t^{-1}
f_n	Natural frequency with damping	Cycles per second	t^{-1}
G	Shear modulus of elasticity	Pounds per square inch	fl^{-2}
g	Acceleration of gravity	Inches per second^2	lt^{-2}
h	Thickness	Inches	l
I	Mass moment of inertia of a rigid body or Area moment of inertia with respect to the axis through the center of gravity	Pound-inch-seconds^2 Inches^4	flt^2 l^4
J	Polar moment of inertia of a sectional area	Inches^4	l^4

Symbol	Description	Units	Dimensions in Force-Length-Time System
k	Linear spring constant or Torsional spring constant	Pounds per inch Inch-pounds per radian	fl^{-1} fl
k'	Restoring force per unit length per unit displacement of a uniform elastic support of a beam	Pounds per inch^2	fl^{-2}
l	Length	Inches	l
M	Instantaneous value of exciting torque	Inch-pounds	fl
M_0	Maximum value of sinusoidal exciting torque	Inch-pounds	fl
m	Mass in ips units*	Pound-seconds^2 per inch	$fl^{-1}t^2$
m_e	Mass of an eccentric weight or of a reciprocating weight causing vibration	Pound-seconds^2 per inch	$fl^{-1}t^2$
P or $P(t)$	Instantaneous value of an exciting force	Pounds	f
P_0	Maximum value of a sinusoidal exciting force	Pounds	f
p	Natural circular frequency without damping, or the resonant circular frequency	Radians per second	t^{-1}
R	Radius of gyration	Inches	l
r	Eccentricity of a rotating mass causing vibration, or single amplitude of a reciprocating mass causing vibration, or radius of a circle	Inches	l
T	Tension	Pounds	f
t	Time	Seconds	t
v	Instantaneous velocity	Inches per second	lt^{-1}
W	Work per cycle in forced vibration	Inch-pounds	fl
w	Weight	Pounds	f
X	Maximum value of a sinusoidal displacement	Inches	l
x	Instantaneous value of displacement	Inches	l
y	Transverse displacement of bar or string	Inches	l

*The mass is determined by dividing the weight in pounds by the acceleration of gravity in inches per second squared; this is expressed as $w/386$, in pound-seconds squared per inch. This is referred to as the inch-pound-second (ips) unit; it has no other accepted name.

Symbol	Description	Units	Dimensions in Force-Length-Time System
Z	Ratio of the amplitude of a system of one degree of freedom under a driving force $m_e r\omega^2 \sin \omega t$ to the static deflection due to a force $m_e r p^2$ (nondimensional)	Numeric	
Δ	Static deflection	Inches	l
δ	Logarithmic decrement (nondimensional)	Numeric	
μ	Mass per unit length	Pound-seconds^2 per square inch	$fl^{-2}t^2$
ν	Poisson's ratio (nondimensional)	Numeric	
ρ	Weight density	Pounds per cubic inch	fl^{-3}
ϕ	Phase angle by which force leads displacement in forced vibration	Radians	
ω	Circular frequency. This is 2π times the frequency in cycles per second	Radians per second	t^{-1}
ω_{max}	Circular frequency at which maximum amplitude occurs	Radians per second	t^{-1}
ω_n	Natural circular frequency with damping	Radians per second	t^{-1}
$\approx$	This symbol means "is approximately equal to"		
$\ll$	This symbol means "is much less than"		

ABSTRACT

The ~~This~~ manual contains a collection of formulas useful to design engineers in their efforts to minimize trouble from mechanical vibration. The formulas conform with a notation based on the inch-pound-second (ips) system of units unless specifically stated otherwise.

INTRODUCTION

This edition of the vibration manual is a revision of TMB Report R-189 which was issued with the same title in 1944. That edition in turn was based on a collection of formulas which had been in use for a number of years in the Structural Mechanics Division of the David Taylor Model Basin and prior to that at the U.S. Experimental Model Basin . The first edition of TMB Report R-189 was published at the suggestion of CDR. J. Ormondroyd, USNR, who was officer in charge of the division at the time, and who made numerous constructive suggestions in guiding its preparation.

As in the case of the earlier edition the main purpose is to expedite calculations required by the practicing engineer and in no sense is this manual intended to be a substitute for a textbook on vibration.

CASE 1. LINEAR SYSTEM WITH ONE DEGREE OF FREEDOM

The ideal linear vibratory system of one degree of freedom is represented schematically in Figure 1. It consists of a mass m supported on frictionless and massless rollers attached to a spring k and to a dashpot c, both of which are in turn attached to an infinite mass. If a force P which is a function of time t acts on the mass, the differential equation

$$m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + kx = P(t) \qquad [1]$$

must be satisfied at all times. Here

- m is the mass in pound-seconds^2 per inch,
- k is the restoring force exerted by the spring per unit displacement, always acting in the direction opposite to the displacement, in pounds per inch,
- c is the viscous damping force per unit velocity, always acting in a direction opposite to that of the velocity, in pounds per inch per second,

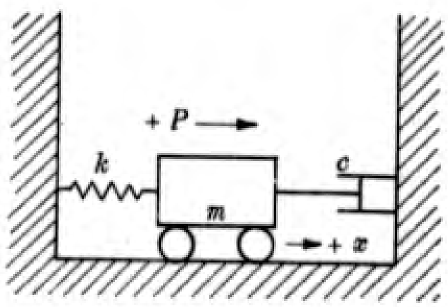


Figure 1 - Linear Vibratory System of One Degree of Freedom

$P(t)$ is the external force varying with time, considered positive to the right, in pounds,

t is the time in seconds, and

x is displacement from the ''at rest'' position, considered positive to the right, in inches.

If after an initial disturbance of the system shown in Figure 1 the external force ceases to act, the equation becomes

$$m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = 0 \qquad [2]$$

The form of the solution of this equation depends on the value of c in relation to the other constants of the system. If $c^2/4m^2 > k/m$ the mass m will not oscillate but will gradually return to its rest position. If $c^2/4m^2 < k/m$ there will result a decaying oscillation of circular frequency

$$\omega_n = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$

for which the corresponding frequency will be

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$

If

$$\frac{c^2}{4m^2} = \frac{k}{m}$$

this is the limiting case for no oscillation and the system is said to be critically damped or dead beat. This particular value of c is designated as c_c.

$$c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{mk} = 2mp = 2\frac{k}{p}$$

The solution of Equation [2] for less than critical damping is

$$x = e^{-\frac{c}{2m}t}\left[A \sin \omega_n t + B \cos \omega_n t\right] \qquad [3]$$

where A and B are arbitrary constants depending on the initial conditions. Figure 2 shows x as a function of t for this case.

The amplitude decreases by a definite percentage each cycle and the natural logarithm of the ratio of two successive amplitudes is called the logarithmic decrement $\delta = \log_e \frac{X_n}{X_{n+1}}$; or, for an interval of q cycles, $\delta = \frac{1}{q} \log_e \frac{X_n}{X_{n+q}}$

$$\delta = \frac{\pi c}{m\omega_n} = \frac{2\pi \frac{c}{c_c}}{\sqrt{1 - \left(\frac{c}{c_c}\right)^2}}$$

for small damping $\delta \approx 2\pi\,(c/c_c)$. If there is no damping, $c = 0$ and the natural circular frequency is $\sqrt{k/m}$. This is designated by p.

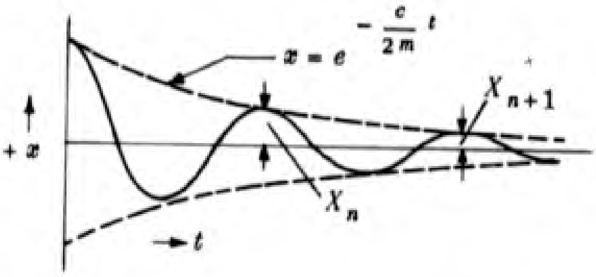


Figure 2 - Free Vibration with Less than Critical Damping

CASE 2. FORCED VIBRATION

If the driving force is sinusoidal $[P(t) = P_0 \sin \omega t]$, Equation [1] becomes

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = P_0 \sin \omega t \qquad [4]$$

The general solution of this equation is

$$x = e^{-\frac{c}{2m}t}\left[A \sin \omega_n t + B \cos \omega_n t\right] + \frac{P_0 \sin(\omega t - \phi)}{\sqrt{(c\omega)^2 + (k - m\omega^2)^2}} \qquad [5]$$

where A and B are arbitrary constants depending on the initial conditions. As before

$$\omega_n = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$

also

$$\phi = \arc\tan \frac{c\omega}{k - m\omega^2} \qquad [6]$$

The first term on the right-hand side of Equation [5] is called the transient term; it vanishes in time owing to the fact that the term $e^{-\frac{c}{2m}t}$ constantly diminishes. The second term is called the steady-state term and gives the amplitude of the forced vibration in terms of the system constants and the driving force.

The amplitude of the steady-state vibration is

$$X = \frac{P_0}{\sqrt{(c\omega)^2 + (k - m\omega^2)^2}} \qquad [7]$$

This is frequently expressed in the convenient form

$$X = \frac{\frac{P_0}{k}}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left[2\frac{c}{c_c}\frac{\omega}{p}\right]^2}} \qquad [8]$$

Here P_0/k is the displacement that would be produced by a static force P_0 and is frequently denoted by Δ. c/c_c is the ratio of the damping constant to the critical damping constant. If X/Δ, which is frequently called the magnification factor, is plotted against ω/p with c/c_c

as a parameter, the resonance curves of Figure 3 are obtained. The magnification factor at resonance [$\omega = p = \sqrt{k/m}$] for a system of a single degree of freedom with viscous damping is

$$F_{res} = \frac{X_{res}}{\Delta} = \frac{\pi}{\delta} = \frac{1}{2\frac{c}{c_c}}$$

Both the force and the displacement are sinusoids; the force leads the displacement by the phase angle ϕ. This angle depends on all the constants of the system as shown by Equation [6]. In Figure 4 the phase angle is plotted against ω/p with c/c_c as a parameter.

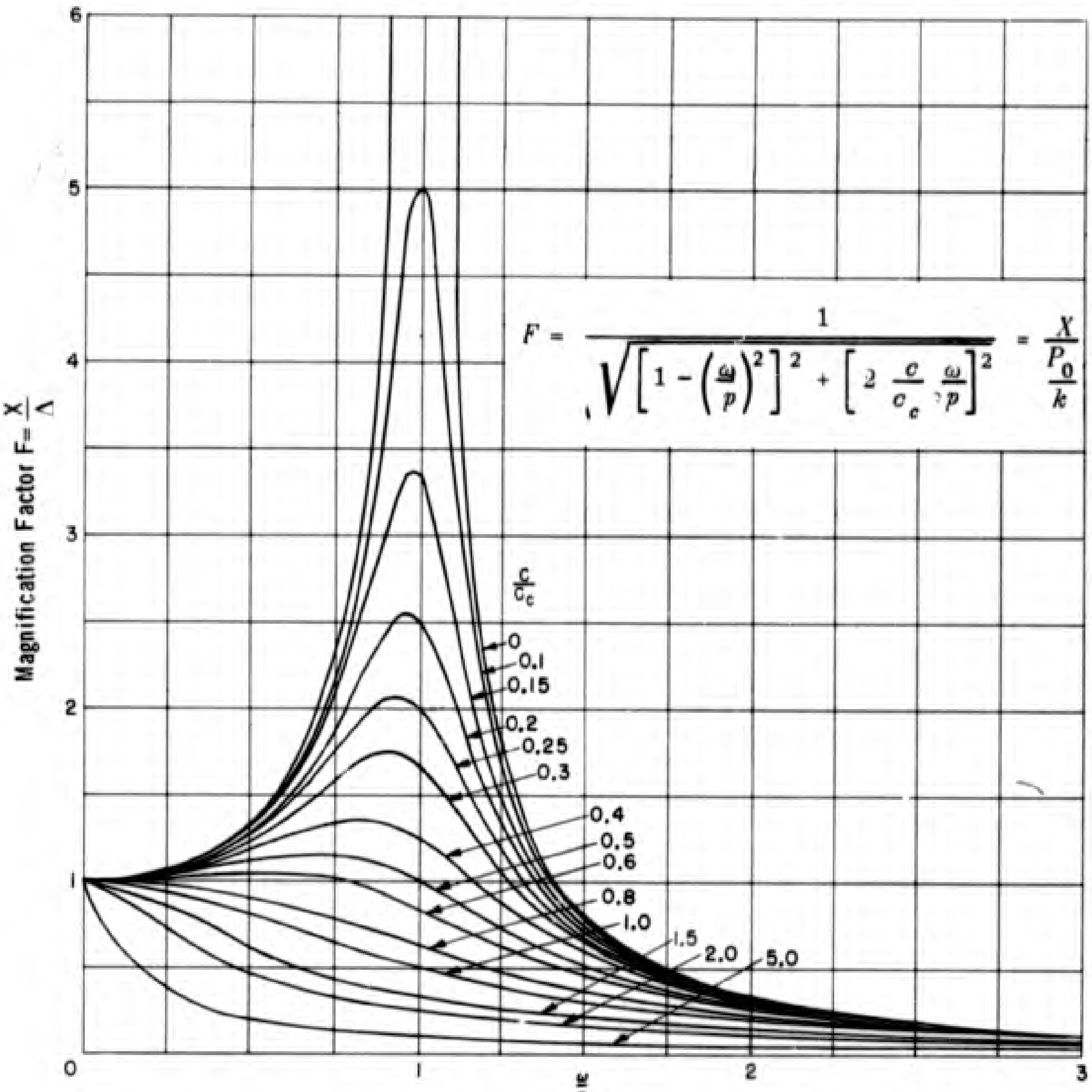


Figure 3 - Magnification Factor for a System of One Degree of Freedom with Viscous Damping Acted on by a Force $P_0 \sin \omega t$

F_{max} occurs at $\frac{\omega}{p} = \sqrt{1 - 2\left(\frac{c}{c_c}\right)^2}$.

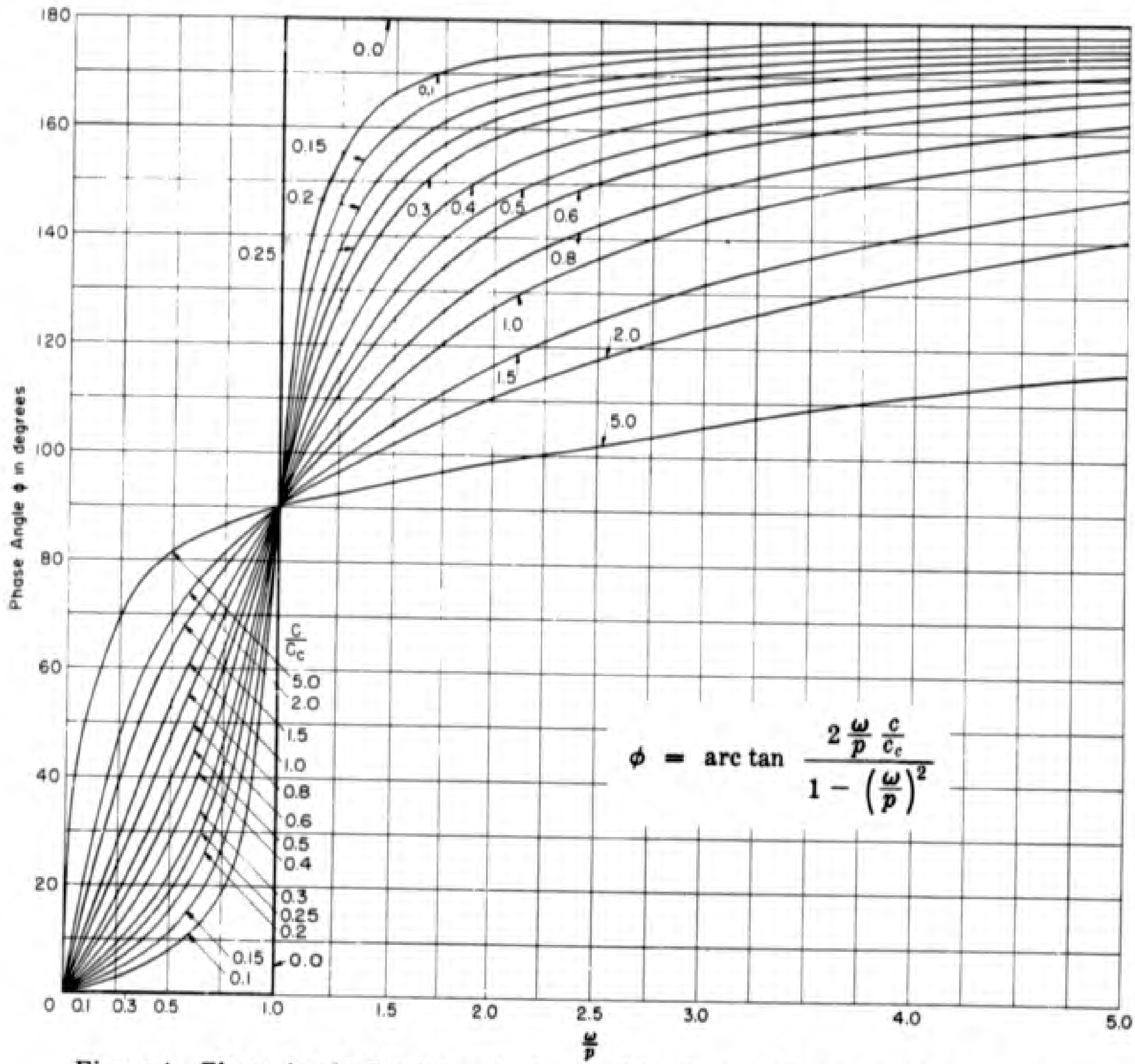


Figure 4 - Phase Angle Plotted against ω/p with c/c_c as a Parameter for a System of One Degree of Freedom Acted on by a Force $P_0 \sin \omega t$

CASE 3. SYSTEM HAVING A SINGLE DEGREE OF FREEDOM WITH THE DRIVING FORCE VARYING AS THE SQUARE OF THE FREQUENCY

Where the vibration is due to an unbalanced rotating member, the driving force varies as the square of the frequency. If m_e is the unbalanced mass, r its eccentricity, expressed as the distance from its center of gravity to the axis of rotation, and ω the angular velocity, the exciting force in a direction perpendicular to the axis of rotation becomes

$$P(t) = m_e r \omega^2 \sin \omega t$$

and the differential equation of motion is

$$m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = m_e r\omega^2 \sin \omega t \qquad [9]$$

The steady-state amplitude of the system under this condition is

$$X = \frac{m_e r\omega^2}{\sqrt{(c\omega)^2 + (k - m\omega^2)^2}} = \frac{\dfrac{m_e r\omega^2}{k}}{\sqrt{\left[1-\left(\dfrac{\omega}{p}\right)^2\right]^2 + \left[2\dfrac{c}{c_c}\dfrac{\omega}{p}\right]^2}} = \frac{r\dfrac{m_e}{m}\dfrac{\omega^2}{p^2}}{\sqrt{\left[1-\left(\dfrac{\omega}{p}\right)^2\right]^2 + \left[2\dfrac{c}{c_c}\dfrac{\omega}{p}\right]^2}} \qquad [10]$$

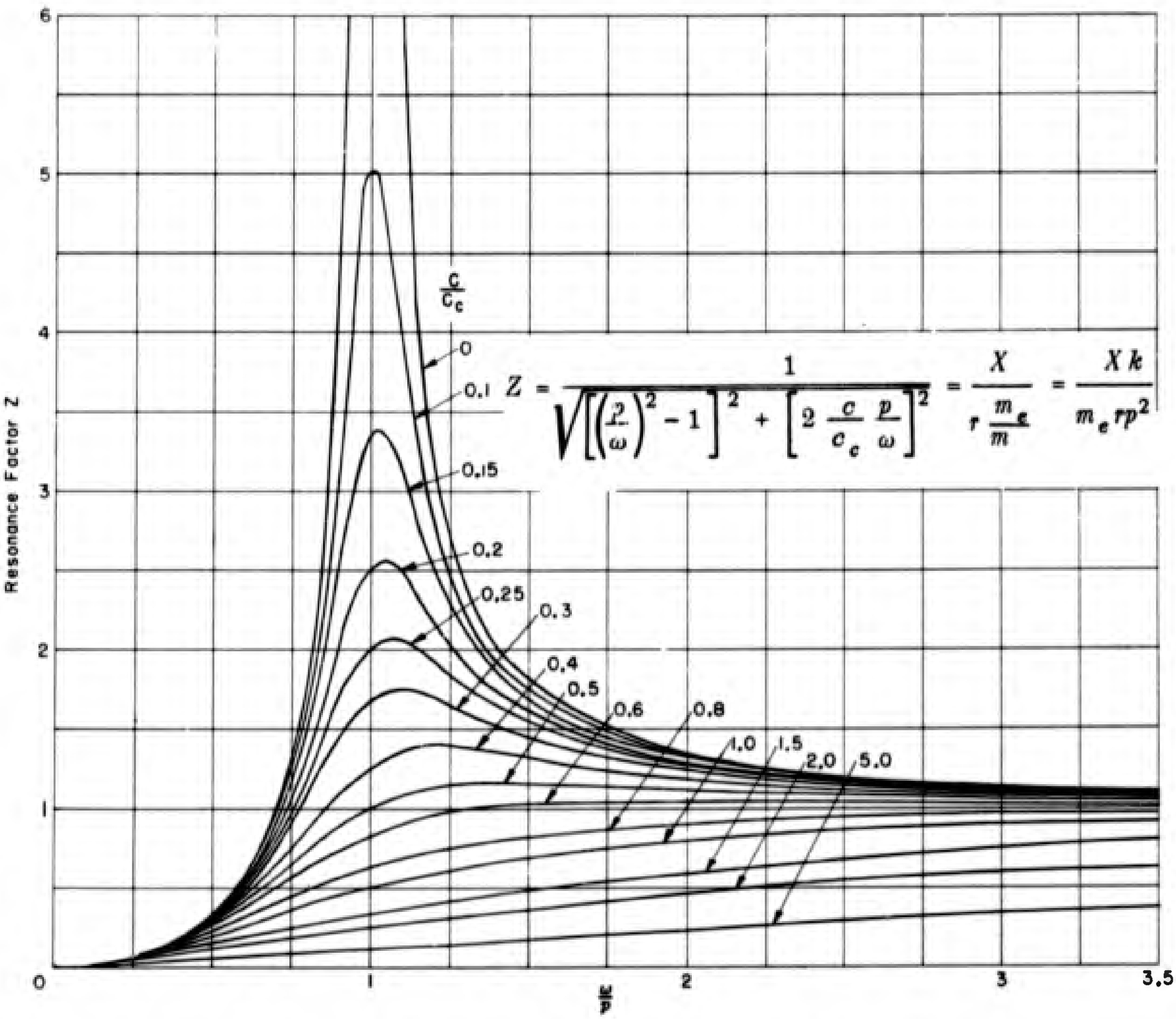


Figure 5 - Resonance Factor for a System of One Degree of Freedom with Viscous Damping Acted on by a Force $P = m_e r\omega^2 \sin \omega t$

This factor is based on the static deflection that would be produced by a constant force $m_e r p^2$.

$$Z_{max} \text{ occurs at } \frac{\omega}{p} = \frac{1}{\sqrt{1-2\left(\frac{c}{c_c}\right)^2}}$$

The phase angle by which the force leads the displacement is

$$\phi = \arc\tan \frac{c\omega}{k - m\omega^2}$$

Another convenient expression for X is

$$X = \frac{\dfrac{m_e r p^2}{k}}{\sqrt{\left[\left(\dfrac{p}{\omega}\right)^2 - 1\right]^2 + \left[2\dfrac{c}{c_c}\,\dfrac{p}{\omega}\right]^2}} \qquad [11]$$

where it should be noted that $m_e r p^2$ is the amplitude of the driving force at resonance, and the numerator in Equation [11] is the static deflection produced by the force $m_e r p^2$. The quantity

$$\frac{1}{\sqrt{\left[\left(\dfrac{p}{\omega}\right)^2 - 1\right]^2 + \left[2\dfrac{c}{c_c}\,\dfrac{p}{\omega}\right]^2}}$$

is designated by Z. In Figure 5, Z is shown as a function of ω/p for various values of c/c_c.

CASE 4. THE VIBRATION GENERATOR

The dynamic properties of structures are frequently studied by the use of vibration machines called vibration generators, as shown in Figure 6, in which the driving force can be controlled by the adjustment of eccentric weights. In these machines the eccentrics exist in pairs and rotate in opposite directions so that sinusoidal forces can be balanced in all directions except the desired one; see Figure 7. In the particular machine shown in Figure 6 each eccentric is represented by two weights attached to a disk, there being four disks and four eccentrics.

If m_e is the mass of all the eccentrics and r the eccentricity, the driving force on the structure is $m_e r\omega^2 \sin \omega t$, so that the amplitude is given by the solution of Equation [9].

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = m_e r\omega^2 \sin \omega t \qquad [9]$$

A test with a vibration generator will reveal not only the resonant frequencies of the structure but, from the form of the resonance curve obtained by plotting amplitude against frequency, estimates can be obtained of the damping characteristics of the structure as well as its effective mass and spring constant. By the effective constants of a structure are meant the values of the mass, spring constant, and damping constant of a simple linear system of one degree of freedom which would have the same amplitudes under the same driving forces.

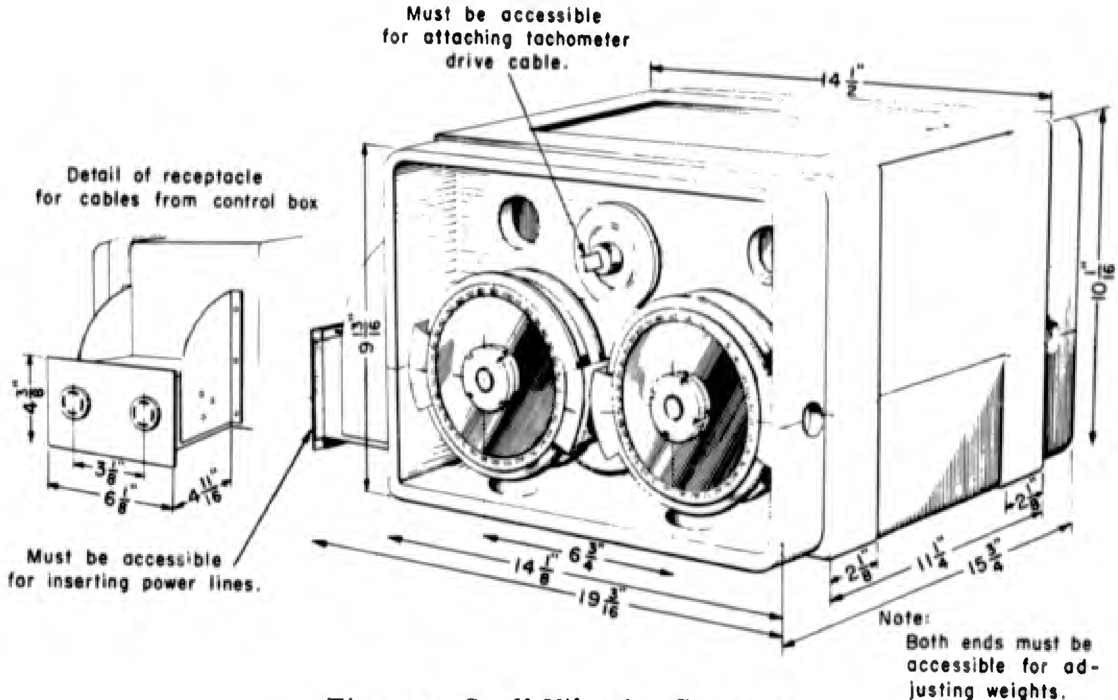


Figure 6 - Small Vibration Generator

The weight of this vibration generator is 140 pounds.

Thus, if Figure 8 is a resonance curve plotted from data taken in a test with a vibration generator, and f_1 and f_2 are two frequencies at which the amplitude bears the same ratio to the maximum amplitude, then

$$\frac{c}{c_c} \approx \frac{n}{4} \frac{f_2^2 - f_1^2}{f_{max}^2} \qquad [12]$$

where

$$n = \frac{\text{amplitude at } f_1 \text{ or } f_2}{\text{maximum amplitude}}$$

If $n \ll 1$ and $c/c_c \ll n$, the effective mass of the structure may be estimated from the approximate relation

$$m \approx \frac{2m_e r}{nX_{max}} \frac{f_{max}^2}{f_2^2 - f_1^2} \qquad [13]$$

The effective damping constant may be found by the approximate relation

$$c \approx \frac{2\pi f_{max} m_e r}{X_{max}} \qquad [14]$$

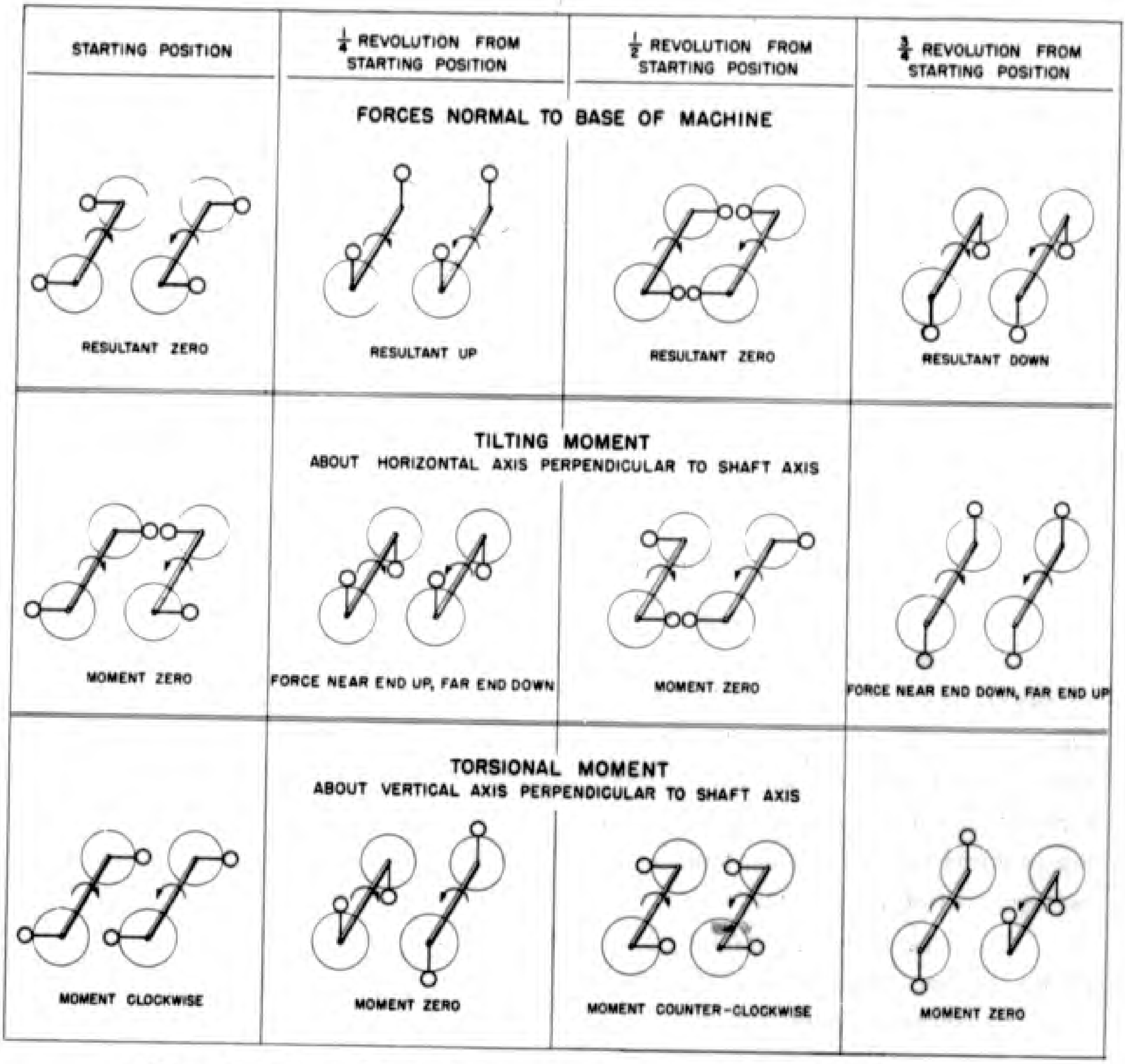


Figure 7 - Types of Cyclic or Periodic Forces and Moments Produced by Various Positions of Unbalanced Weights

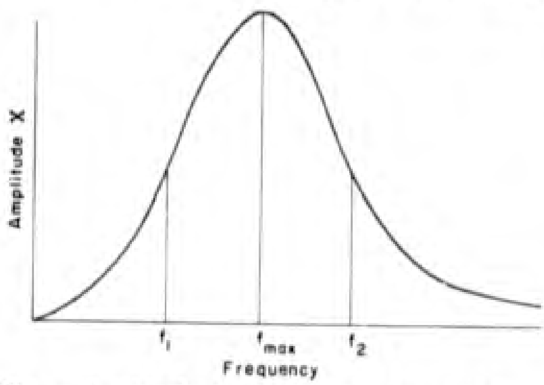


Figure 8 - Method of Deducing Damping from Resonance Curve Obtained from Test with Vibration Generator

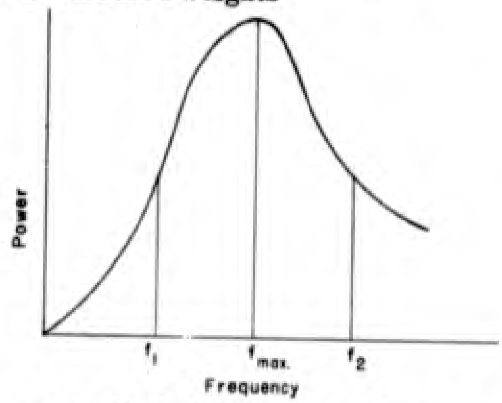


Figure 9 - Method of Deducing Logarithmic Decrement from Power Resonance Curve

The effective spring constant is obtained from the approximate expression

$$k \approx 8\pi^2 f_{max}^2 \frac{m_e r}{n X_{max}} \frac{f_{max}^2}{f_2^2 - f_1^2} \qquad [15]$$

If the power required to maintain the vibration can be estimated by measuring the power going into the vibration generator and correcting for the losses in the machine, then the logarithmic decrement can be estimated from the curve of power plotted on frequency, as shown in Figure 9, by the relation

$$\delta = \pi \frac{(f_2 - f_1)}{f_{max}}$$

where f_2 and f_1 are the two frequencies at which the power is one-half the maximum power.

CASE 5. THEORY OF VIBROGRAPHS, PALLOGRAPHS, AND ACCELEROMETERS

These instruments consist essentially of a frame or case inside of which is suspended a mass or element on springs. The case is assumed to be attached rigidly to the structure whose motion is to be measured. There is usually a damping device opposing the relative velocity between the element and the case. The displacement of the element relative to the case actuates a recording mechanism. The movement is usually multiplied before recording. For purposes of this analysis, however, it is not necessary to consider the multiplication of the movement of the element.

It is a fundamental principle of mechanics that the laws of motion relative to non-rotating moving axes are the same as those relative to fixed axes, provided an additional force is assumed to act on each mass, expressed by

force = minus the product of mass and instantaneous acceleration of axes

Hence if in Figure 10 x represents the absolute displacement of the structure, representing the moving axes, and x_1 the motion of the element relative to the case, that is, the recorded motion, we have for the differential equation of relative motion

$$m \frac{d^2x}{dt^2} + c \frac{dx_1}{dt} + kx_1 = -m \frac{d^2x}{dt^2} \qquad [16]$$

This is identical with Equation [1] if x_1 is substituted for x and $-m\ d^2x/dt^2$ for P. If the structure has a simple harmonic motion $X \sin \omega t$, then d^2x/dt^2 will be

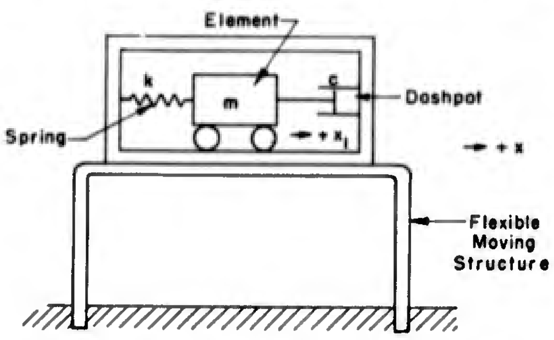


Figure 10 - Diagram Showing a Seismic Instrument on a Flexible Structure

$-X\omega^2 \sin \omega t$ and the differential equation for the relative motion of the element becomes

$$m \frac{d^2x_1}{dt^2} + c \frac{dx_1}{dt} + kx_1 = mX\omega^2 \sin \omega t \qquad [17]$$

This is identical with Equation [9] if x_1 is substituted for x and $mX\omega^2$ for $m_e r$. Hence from Equation [10]

$$X_1 = \frac{\dfrac{mX\omega^2}{k}}{\sqrt{\left[1 - \left(\dfrac{\omega}{p}\right)^2\right]^2 + \left[2 \dfrac{c}{c_c} \dfrac{\omega}{p}\right]^2}} \qquad [18]$$

Since $p^2 = k/m$, dividing both numerator and denominator by ω^2/p^2 gives

$$X_1 = \frac{X}{\sqrt{\left[\left(\dfrac{p}{\omega}\right)^2 - 1\right]^2 + \left[2 \dfrac{c}{c_c} \dfrac{p}{\omega}\right]^2}} \qquad [19]$$

Hence $X_1 = XZ$ where Z is defined in Figure 5 and the curves of this figure give X_1 as a function of ω/p for various values of relative damping. Figure 5 shows that an instrument such as indicated in Figure 10 will record, in fairly accurate fashion, steady-state vibrations at all frequencies above 3 times its undamped natural frequency for wide variations in damping ($Z \approx 1$). Furthermore if the damping can be controlled and set at about 0.6 critical, the instrument will record amplitudes accurately at frequencies down to about the natural frequency itself, i.e., Z is approximately unity even down to this frequency.

Equation [19] and Figure 5 also show that if the structure vibrates at frequencies lower than the natural frequency of the instrument

$$\frac{X_1}{X} \approx \frac{1}{\sqrt{\dfrac{p^4}{\omega^4}}} \approx \left(\frac{\omega}{p}\right)^2 \qquad [20]$$

For values of ω/p less than 1/2, X_1/X is parabolic for wide variations in damping (c/c_c) and hence is proportional to the acceleration in the vibration ($X\omega^2$). If the damping can be regulated at about 0.6 critical the parabolic law will be approximated up to $\omega/p = 1$.

CASE 6. TRANSIENT RESPONSE OF SEISMIC INSTRUMENTS

With reference to Figure 10, if the motion is not a steady-state vibration a transient term is obtained in the solution of Equation [16]. Fundamentally the instrument in all cases responds to acceleration, but the transient displacement of the structure may be obtained from the recorded displacement (x_1) as follows.

The basic equation of relative motion as given previously is

$$m \frac{d^2x_1}{dt^2} + c \frac{dx_1}{dt} + kx_1 = - m \frac{d^2x}{dt^2} \qquad [16]$$

Multiplying through by p^2/k gives

$$\frac{d^2x_1}{dt^2} + 2 \frac{c}{c_c} p \frac{dx_1}{dt} + p^2 x_1 = - \frac{d^2x}{dt^2} \qquad [21]$$

A double integration gives

$$x_1 + 2 \frac{c}{c_c} p \int_0^t x_1 dt + p^2 \int_0^t \int_0^t x_1 \, dt \, dt = - x \qquad [22]$$

where the constants of integration drop out if $x = \dot{x} = x_1 = \dot{x}_1 = 0$ when $t = 0$. Hence the displacement up to time t can be obtained by adding the recorded displacement and the first two integrals. The shorter the time interval the smaller the correction to the first term. Hence for short intervals the displacement is recorded directly.

CASE 7. TWO-BODY SYSTEM WITHOUT DAMPING

The system shown in Figure 11 has two degrees of freedom and two natural frequencies. In the mode of lower frequency both m_1 and m_2 oscillate in phase. In the mode of higher frequency, m_1 and m_2 oscillate in opposite phase. These two frequencies are obtained by solving for ω in the equation

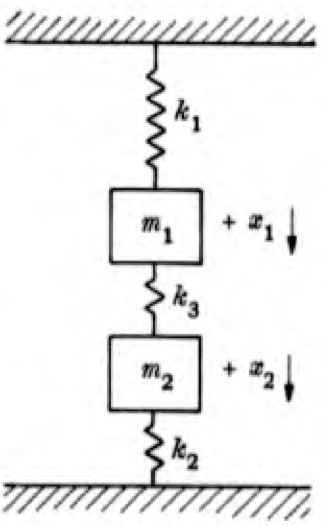


Figure 11 - Two-Body System without Damping, General Case

$$\omega^4 - \omega^2 \left(\frac{k_1 + k_3}{m_1} + \frac{k_2 + k_3}{m_2} \right) + \frac{k_1k_2 + k_2k_3 + k_1k_3}{m_1m_2} = 0 \qquad [23]$$

The ratio of the amplitudes in free vibration in one of normal modes is given by the equation

$$\frac{X_1}{X_2} = \frac{- k_3}{m_1\omega^2 - k_1 - k_3} = \frac{m_2\omega^2 - k_2 - k_3}{- k_3} \qquad [24]$$

CASE 8. VIBRATION-PROOF MOUNTING

If an engine or machine is flexibly mounted on a foundation the most general case of the vibratory system without damping is the same as shown in Figure 11 where

$k_1 = 0$,

m_1 is the mass of the machine,

k_3 is the spring constant of the flexible mounting,

m_2 is the mass of the foundation, and

k_2 is the spring constant of the connection between the foundation and the sub-structure that is assumed to be rigid.

During operation of the machine unbalance may be considered to exert a vertical force on m_1 equal to $m_e r\omega^2 \sin \omega t$, where $m_e r$ is the unbalance and ω is the angular velocity of the shaft. The system under forced vibration is therefore represented in Figure 12.

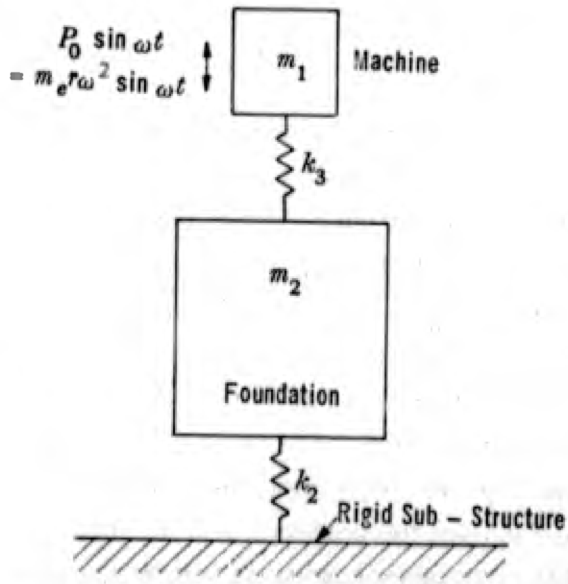


Figure 12 - Two-Body System Representing Flexible Mounting of Engine

The problem of designing the flexible mounting is to compute the optimum value of k_3. If the foundation is anchored to a rigid sub-structure, such as bedrock, k_2 approaches infinity and the system approaches one degree of freedom. In this case the ratio of the force (P_2) transmitted to m_2 to the force acting on m_1 is $P_2/m_e r\omega^2$. Moreover

$$\frac{P_2}{m_e r\omega^2} = \frac{k_3}{k_3 - m_1\omega^2} = \frac{1}{1 - \frac{\omega^2}{p_1^2}} \qquad [25]$$

where $p_1 = \sqrt{k_3/m_1}$. The right-hand term of Equation [25] is the same as F in Figure 3 for the case $c/c_c = 0$. Hence for a fixed engine speed in which ω is constant, the transmitted force can be made infinitesimal by making ω/p_1 very large (where $p_1 = \sqrt{k_3/m_1}$), which means making p_1 very small. Since m_1 is fixed this means making k_3 small or using soft springs. A value of k_3 making $p_1 = \omega$ would make conditions much worse than no spring at all.

Where k_2 is finite, the general case, the system has two natural frequencies at which the foundation will vibrate violently. Since the force transmitted to the sub-structure is the product of the amplitude of m_2 and k_2 the desired condition here is that X_2 be a minimum

$$X_2 = \frac{m_e r\omega^2 k_3}{(k_3 - m_1\omega^2)(k_3 + k_2 - m_2\omega^2) - k_3^2} \qquad [26]$$

By Equation [26] the value of k_3 required to hold X_2 to any desired amplitude can be estimated for any given unbalance and engine speed. The two frequencies to be avoided are obtained by solving Equation [23] with $k_1 = 0$.

CASE 9. VIBRATION NEUTRALIZER

In the system shown in Figure 13, m_2 and k_2 alone represent a one-body system subjected to a driving force $P_0 \sin \omega_1 t$. If there is attached to m_2 a small mass m_1 and the spring k_3 is such that $\sqrt{k_3/m_1} = \omega_1$, m_1 will vibrate at such an amplitude and in such phase as to neutralize the force $P_0 \sin \omega_1 t$ acting on m_2, which will come to rest. However, the two-body system thus produced has two resonant frequencies and if the frequency of P_0 is variable, conditions will be worse at speeds corresponding to these frequencies. These frequencies are found by solving Equation [23] with $k_1 = 0$.

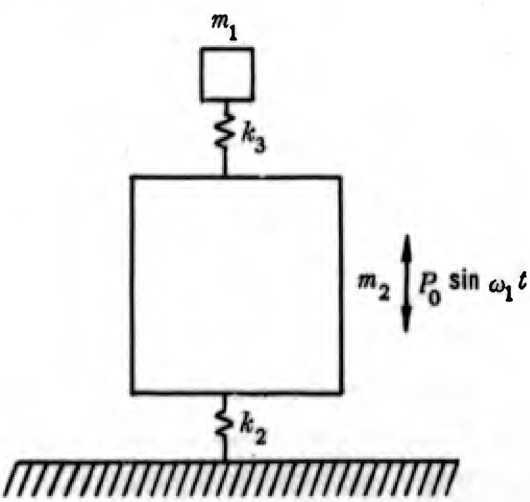


Figure 13 - Two-Body System Representing Application of Vibration Neutralizer

CASE 10. TORSIONAL SYSTEMS

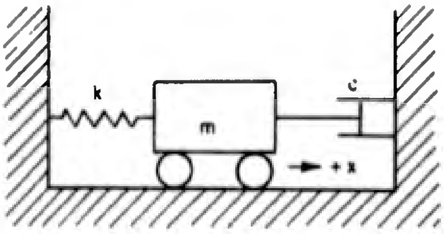
The same principles which apply to linear systems apply to torsional systems provided mass moments of inertia are substituted for masses, torques for forces, torsional spring constants for linear spring constants, and torsional damping constants for linear damping constants. The torsional and linear systems shown in Figure 14 are treated identically. Thus if an alternating torque is applied to the disk of mass moment of inertia I the steady-state oscillation is given by θ in Equation [27], Figure 14b. The torsional constant of a circular shaft of length l, one end of which is fixed, is $k = GJ/l$ where $J = \pi D^4/32$, and D is the diameter. A two-body torsional system is shown in Figure 15.

The two natural frequencies are obtained by solving for ω in the equation

$$\omega^4 - \omega^2\left(\frac{k_2}{I_1} + \frac{k_1 + k_2}{I_2}\right) + \frac{k_1 k_2}{I_1 I_2} = 0 \qquad [28]$$

In the mode of lower frequency the disks oscillate in phase, and in the mode of higher frequency they oscillate in opposite phase.

When two shafts are connected by a pair of gears their effective torsional inertias and stiffnesses are modified according to the ratio of the angular displacements of the gears. In

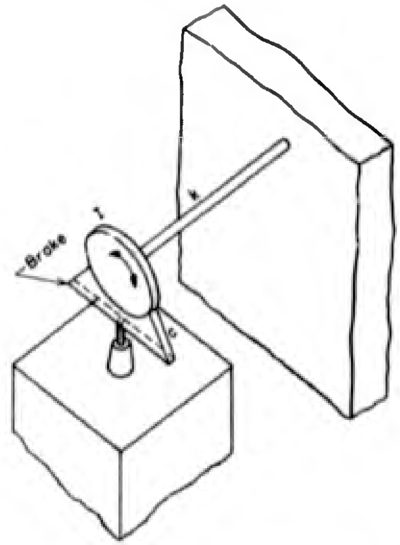


$$X = \frac{P_0 \sin(\omega t - \phi)}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

$$\phi = \text{arc tan } \frac{c\omega}{k - m\omega^2}$$

$$p = \sqrt{\frac{k}{m}}$$

Figure 14a - Linear System with Force $P_0 \sin \omega t$



$$\theta = \frac{M_0 \sin(\omega t - \phi)}{\sqrt{(k - I\omega^2)^2 + (c\omega)^2}} \qquad [27]$$

$$\phi = \text{arc tan } \frac{c\omega}{k - I\omega^2}$$

$$p = \sqrt{\frac{k}{I}}$$

Figure 14b - Torsional System with Moment $M_0 \sin \omega t$

Figure 14 - Comparsion of Linear and Torsional Systems of One Degree of Freedom

calculating the torsional natural frequencies the branch line may be treated as an extension of the main line, i.e., rotating at the same speed, provided its torsional inertias and stiffnesses are multiplied by the square of the gear ratio (taken greater than unity if the branch line rotates faster than the main line).

If in a line of shafting there are changes in cross section along the length, a length whose section has a polar moment of inertia of area J_1, may be treated as equivalent to a length of another size (J_2) provided that $J_2/l_2 = J_1/l_1$. This is based on equivalent torsional stiffnesses and is valid only when the inertia effect of the shaft is neglected in the calculation.

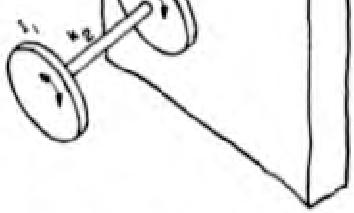


Figure 15 - Two-Body Torsional System

The fundamental torsional natural frequency of a uniform circular shaft (solid or hollow) having either both ends fixed or both ends free is given by the formula

$$\omega_1 = \frac{\pi}{l} \sqrt{\frac{Gg}{\rho}}$$

For steel

$$f_1 \approx \frac{6.4 \times 10^4}{l}$$

For either fixed-fixed or free-free end conditions the frequencies of the higher modes are related to the frequency of the fundamental mode by the series 1, 2, 3, etc. The frequencies depend only on the length and the end conditions and are independent of the diameter. For the free-free end conditions the fundamental mode has a mode in the middle of the shaft. For the fixed-fixed condition the fundamental mode has a loop in the middle of the shaft.

CASE 11. VIBRATION OF ELASTIC BODIES – FLEXURAL VIBRATIONS

The equations given here are based on the simple beam theory and are accurate only for beams having a length to depth ratio of the order of 10 or more. The effects of rotary inertia and shear deflection are neglected.

CASE 11A. UNIFORM BAR WITH FREE ENDS

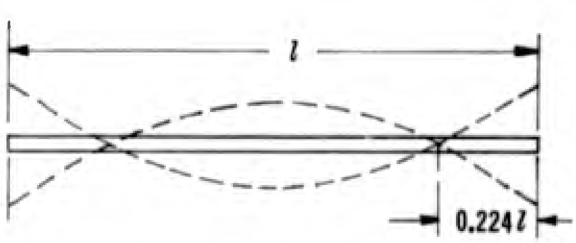


Figure 16 - Fundamental Mode of Vibration of Free-Free Uniform Bar

The fundamental mode of vibration or mode of lowest natural frequency, of a uniform beam with no restraints, has two nodes as shown in Figure 16. This is called a free-free beam. The fundamental frequency may be found from the equation

$$\omega_1 = \frac{22.4}{l^2}\sqrt{\frac{EI}{\mu}} = \frac{440R}{l^2}\sqrt{\frac{E}{\rho}} \qquad [29]$$

or

$$f_1 = \frac{70.1R}{l^2}\sqrt{\frac{E}{\rho}} \qquad [30]$$

For steel and aluminum $f_1 \approx 73 \times 10^4 R/l^2$. The equation giving the deflection of the normal elastic line in the fundamental mode is

$$y = \frac{1}{2.04}\left[-\sin 4.73x + 1.02\cos 4.73x - \sinh 4.73x + 1.02\cosh 4.73x\right]$$

This equation is in nondimensional form; x is a fraction of the length and hence varies from 0 to 1; y gives the relative amplitudes, where the value at the ends is unity. The value of y in the center is 0.608.

If the beam is divided into twenty parts the fundamental mode shape may be plotted with sufficient accuracy for practical purpose from the following table of approximate values.

Station	0 and 20	1 and 19	2 and 18	3 and 17	4 and 16	5 and 15	6 and 14	7 and 13	8 and 12	9 and 11	10
y	1.000	0.768	0.537	0.312	0.098	−0.099	−0.272	−0.414	−0.520	−0.586	−0.608

Other mode patterns may be obtained from the equation

$$y = \frac{1}{2.04}\left[-\sin \alpha_n x + 1.02 \cos \alpha_n x - \sinh \alpha_n x + 1.02 \cosh \alpha_n x\right]$$

where α_n is the characteristic number for the nth mode and is a root of the equation $\cos \alpha_n \cosh \alpha_n = 1$. The first three characteristic numbers for the free-free uniform beam are: 4.73; 7.853; and 10.996.

The frequencies of the higher modes of the free-free uniform bar are related to the frequency of the fundamental approximately as follows:

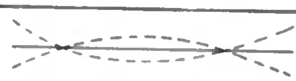
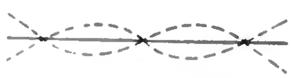
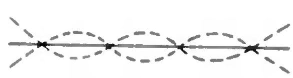
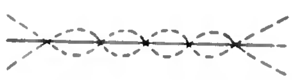
	Mode	Natural Circular Frequency
	2 nodes	$\omega_1 = \frac{22.4}{l^2}\sqrt{\frac{EI}{\mu}}$
	3 nodes	$\frac{5^2}{3^2}\,\omega_1$
	4 nodes	$\frac{7^2}{3^2}\,\omega_1$
	5 nodes	$\frac{9^2}{3^2}\,\omega_1$

Figure 17 - Amplitude Profiles

CASE 11B. UNIFORM BAR WITH SIMPLE SUPPORT AT ENDS

$$\omega_1 = \frac{\pi^2}{l^2}\sqrt{\frac{EI}{\mu}} = \frac{194\,R}{l^2}\sqrt{\frac{E}{\rho}} \qquad [31]$$

$$f_1 = \frac{30.9\,R}{l^2}\sqrt{\frac{E}{\rho}} \qquad [32]$$

For steel or aluminum

$$f_1 \approx 32 \times 10^4\,\frac{R}{l^2} \qquad [33]$$

The equation of the elastic line for the fundamental mode is

$$y = \sin\frac{\pi x}{l}$$

if the amplitude is taken as unity at the center.

Figure 18 - Fundamental Mode of Vibration for Uniform Bar with Simple Support at Ends

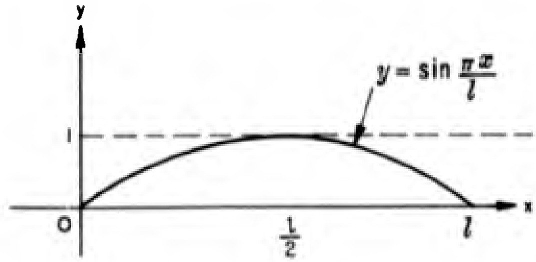


Figure 19 - Elastic Line for Uniform Bar with Simple Support Vibrating in Fundamental Mode

	Mode	Natural Frequency
	2 nodes	$\omega_1 = \frac{\pi^2}{l^2} \sqrt{\frac{EI}{\mu}}$
	3 nodes	$4\omega_1$
	4 nodes	$9\omega_1$
	5 nodes	$16\ \omega_1$

Figure 20 - Amplitude Profiles

The frequencies of the fundamental and higher modes of the uniform bar with simple support at the ends are shown in Figure 20.

The critical speeds and normal mode patterns for circular shafts carrying no attached members (so that gyroscopic effects can be neglected) are the same as the natural frequencies and modes of the uniform beam having the same type of support.

For a continuous shaft in uniformly spaced bearings offering simple support the lowest critical speed is given approximately by the formula for a simply supported beam having a length equal to the span between bearings, and the second mode has the frequency of a beam with clamped ends of the same length.

CASE 11C. UNIFORM CANTILEVER BEAM

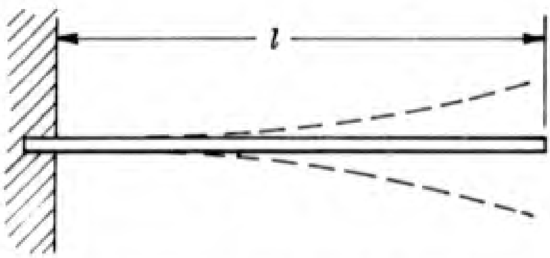


Figure 21 - Fundamental Mode of Vibration for Uniform Cantilever Beam

$$\omega_1 = \frac{0.36\,\pi^2}{l^2} \sqrt{\frac{EI}{\mu}} = \frac{69.8\,R}{l^2} \sqrt{\frac{E}{\rho}} \qquad [34]$$

$$f_1 = \frac{11.1\,R}{l^2} \sqrt{\frac{E}{\rho}} \qquad [35]$$

For steel or aluminum

$$f_1 \approx \frac{11.5 \times 10^4\,R}{l^2} \qquad [36]$$

The frequencies of the higher modes are related to the frequencies of the fundamental mode as follows:

$$\omega_2 = 6.27\ \omega_1 \qquad \omega_4 = 34.4\ \omega_1$$

$$\omega_3 = 17.6\ \omega_1 \qquad \omega_5 = 56.8\ \omega_1$$

CASE 11D. UNIFORM BEAM WITH CLAMPED ENDS

Frequencies are the same as for a free-free beam

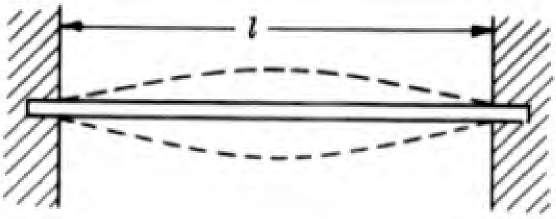


Figure 22 - Fundamental Mode of Vibration for Uniform Beam Clamped at Ends

$$\omega_1 = \frac{22.4}{l^2}\sqrt{\frac{EI}{\mu}} = \frac{440\,R}{l^2}\sqrt{\frac{E}{\rho}} \qquad [37]$$

or

$$f_1 = \frac{70.1\,R}{l^2}\sqrt{\frac{E}{\rho}} \qquad [38]$$

For steel or aluminum

$$f_1 \approx 73 \times 10^4\,\frac{R}{l^2} \qquad [39]$$

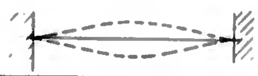
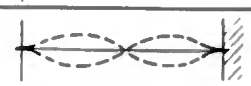
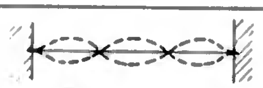
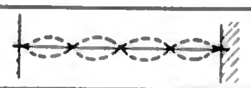
	Mode	Natural Frequency
	2 nodes	$\omega_1 = \frac{22.4}{l^2}\sqrt{\frac{EI}{\mu}}$
	3 nodes	$\frac{5^2}{3^2}\,\omega_1$
	4 nodes	$\frac{7^2}{3^2}\,\omega_1$
	5 nodes	$\frac{9^2}{3^2}\,\omega_1$

Figure 23 - Amplitude Profiles Showing Higher Modes of Clamped-Clamped Uniform Beam

CASE 11E. HINGED-FIXED UNIFORM BEAM

$$\omega_1 = \frac{15.4}{l^2}\sqrt{\frac{EI}{\mu}} = 303\,R\sqrt{\frac{E}{\rho}}$$

$$f_1 = 48.2\,R\sqrt{\frac{E}{\rho}}$$

For steel or aluminum

$$f_1 \approx \frac{49.5 \times 10^4\,R}{l^2}$$

The frequencies of the second and third modes are related to the fundamental frequencies as follows:

$$\omega_2 = 3.24\,\omega_1$$

$$\omega_3 = 6.76\,\omega_1$$

CASE 11F. UNIFORM CANTILEVER BEAM WITH CONCENTRATED MASS AT END

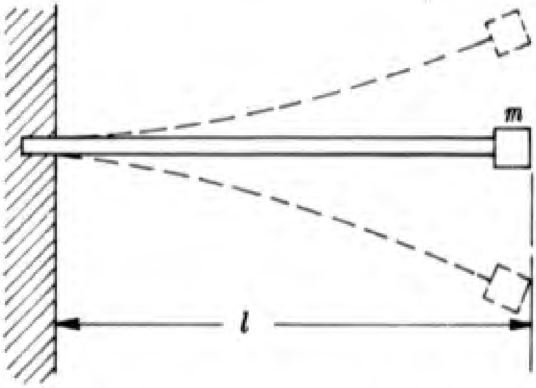


Figure 24 - Fundamental Mode of Vibration of Uniform Cantilever Beam with Concentrated Mass at End

$$f_1 = 0.28 \sqrt{\frac{EI}{\left(m + \frac{33}{140}\,\mu l\right) l^3}}$$

CASE 11 G. UNIFORM BEAM WITH SIMPLE SUPPORT AT ENDS AND CONCENTRATED MASS IN THE CENTER

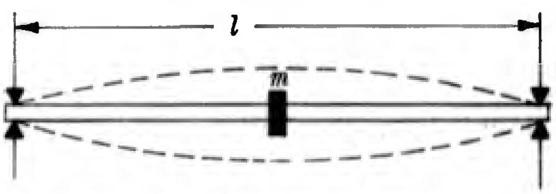


Figure 25 - Fundamental Mode of Vibration for Uniform Beam with Concentrated Mass at Center

The fundamental frequency is found from the approximate relation

$$f_1 = 1.1 \sqrt{\frac{EI}{\left(m + \frac{17}{35}\,\mu l\right) l^3}} \qquad [40]$$

CASE 11H. CIRCULAR PLATE OF RADIUS r AND THICKNESS h

Case 11H-1. Clamped at Boundary

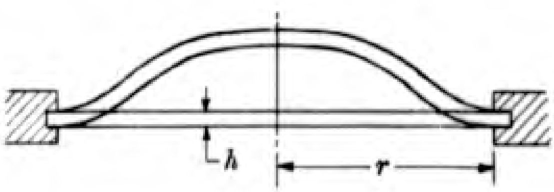


Figure 26 - Fundamental Mode of Vibration for Uniform Circular Plate Clamped at Edge

$$\omega_1 \approx \frac{58}{r^2} \sqrt{\frac{Eh^2}{\rho\,(1 - \nu^2)}}$$

$$f_1 \approx \frac{9.2}{r^2} \sqrt{\frac{Eh^2}{\rho\,(1 - \nu^2)}}$$

For steel

$$f_1 \approx 10^5 \frac{h}{r^2}$$

Case 11H-2. Simply Supported at Boundary

$$\omega_1 \approx \frac{29}{r^2}\sqrt{\frac{Eh^2}{\rho\,(1-\nu^2)}}$$

$$f_1 \approx \frac{4.62}{r^2}\sqrt{\frac{Eh^2}{\rho\,(1-\nu^2)}}$$

For steel

$$f_1 \approx \frac{10^5\,h}{2r^2}$$

CASE 11-I. RECTANGULAR PLATES

Case 11I-1. Rectangular Plate of Length a, Breadth b, and Thickness h, with Simple Support at Edges

$$\omega_1 = \pi^2 h\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\sqrt{\frac{gE}{12(1-\nu^2)\,\rho}}$$

$$f_1 = 8.91\,h\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\sqrt{\frac{E}{(1-\nu^2)\,\rho}}$$

$$f_1 \text{ for steel} \approx 9.7\times 10^4 h\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$

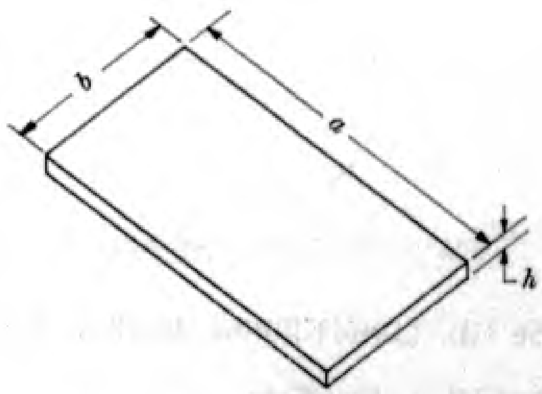


Figure 27 - Rectangular Plate with Simple Support at Edges

The support at the edges is not shown.

Frequencies of the higher modes are given by the general formula

$$f \approx 9.7\times 10^4 h\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

where m and n are integers depending on the number of nodal lines.

Case 11I-2. Rectangular Plate of Length a, Breadth b, and Thickness h, with Clamped Edges

$$f_1^{*} \text{ for steel} \approx 82{,}000\,\frac{h}{ab}\sqrt{7\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)+4}$$

---

* This formula was derived by Dr. G.E. Hudson, formerly of the Model Basin Staff.

CASE 11J. TRIANGULAR PLATE CLAMPED AT ONE EDGE

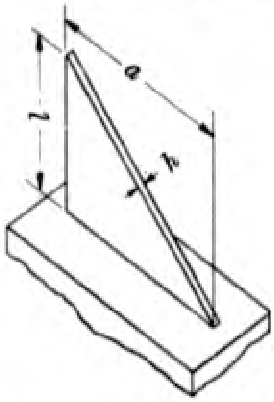


Figure 28 - Triangular Plate with One Edge Clamped

$$f_1 = 0.33 \frac{h}{l^2} \sqrt{\frac{E}{\rho}}$$

For steel

$$f_1 = 3.4 \times 10^3 \frac{h}{l^2}$$

CASE 11K. CIRCULAR MEMBRANE OF UNIFORM TENSION T POUNDS PER LINEAR INCH, MASS PER UNIT AREA μ_1, AND RADIUS r

Figure 29 - Circular Membrane

$$\omega_1 = 2.40 \sqrt{\frac{T}{\mu_1 r^2}}$$

$$f_1 = 0.38 \sqrt{\frac{T}{\mu_1 r^2}}$$

CASE 11L. LONGITUDINAL OR AXIAL VIBRATIONS OF UNIFORM BEAMS OR SHAFTS

Case 11L-1. Free Ends

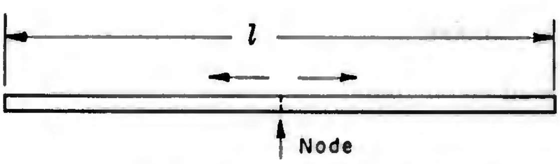


Figure 30 - Fundamental Mode of Longitudinal Vibration of Uniform Bar

The fundamental mode has a node in the middle

$$\omega_1 = \frac{\pi}{l} \sqrt{\frac{Eg}{\rho}} = \frac{61.6}{l} \sqrt{\frac{E}{\rho}} \qquad [41]$$

$$f_1 = \frac{1}{2l} \sqrt{\frac{Eg}{\rho}} = \frac{9.82}{l} \sqrt{\frac{E}{\rho}} \qquad [42]$$

For steel or aluminum

$$f_1 = \frac{10^5}{l}$$

The frequencies of the higher modes are related to the fundamental frequency by the series 1, 2, 3, etc. These frequencies depend only on the length and boundary conditions and are independent of the area of the section.

Case 11L-2. Fixed Ends

When both ends are fixed the frequencies are the same as for the case of free ends. However, in this case the fundamental mode has a loop in the middle of the beam or shaft.

CASE 11M. LATERAL VIBRATION OF UNIFORM STRING UNDER TENSION T POUNDS

$$\omega_1 = \frac{\pi}{l}\sqrt{\frac{T}{\mu}}$$

$$f_1 = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$$

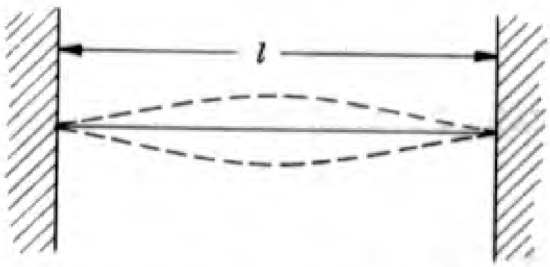


Figure 31 - Fundamental Mode of Lateral Vibration of Uniform String

	Mode	Natural Frequency
	2 nodes	$\omega_1 = \frac{\pi}{l}\sqrt{\frac{T}{\mu}}$
	3 nodes	$2\,\omega_1$
	4 nodes	$3\,\omega_1$
	5 nodes	$4\,\omega_1$

Figure 32 - Amplitude Profiles Showing Harmonics of Uniform String

CASE 12. MISCELLANEOUS

CASE 12A. SIMPLE HARMONIC VIBRATIONS

For simple harmonic vibrations the maximum acceleration in g's is approximately 1/10 f^2X where f is the frequency in cycles per second and X is the single amplitude in inches.

CASE 12B. SCHLICK'S EMPIRICAL FORMULA FOR TWO-NODED VERTICAL FLEXURAL FREQUENCY OF SHIP HULLS IN WATER

$$N = C\sqrt{\frac{I_1}{DL^3}}$$

Figure 33

where N is vibrations per minute,

C is Schlick's empirical constant, varying from 1.28×10^5 to 1.57×10^5,

I_1 is area moment of inertia of midship section in feet^2 inch^2 units,

D is displacement of the ship in tons (2240 pounds), and

L is overall length of the ship in feet.

CASE 12C. WORK PER CYCLE IN FORCED VIBRATION WITH VISCOUS DAMPING

$$W = \pi P_0 X \sin\phi$$

At resonance

$$W = \pi c \omega X^2$$

If the driving force is due to a rotating eccentric mass m_e having eccentricity r,

$$P_0 = m_e r \omega^2$$

and

$$W = \frac{\pi\omega^3 m_e^2 r^2}{c}$$

CASE 12D. POWER REQUIRED TO MAINTAIN A FORCED VIBRATION

$$\text{Horsepower} = \frac{P_0 X\omega \sin\phi}{13{,}200}$$

CASE 12E. NATURAL FREQUENCY IN TERMS OF STATIC DEFLECTION UNDER GRAVITY

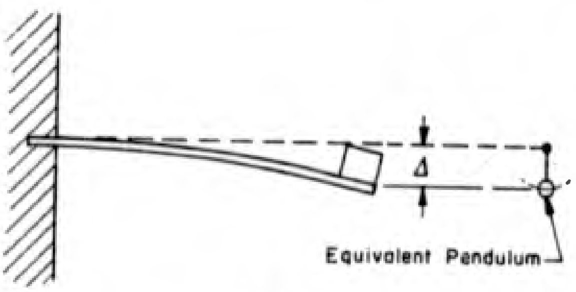


Figure 34

If a concentrated mass causes a static deflection Δ when placed on an elastic support of negligible mass, the natural frequency of the system is approximately

$$f \approx \frac{3.13}{\sqrt{\Delta}}$$

This frequency equals that of a simple pendulum whose length equals the static deflection Δ caused by the weight of the system itself.

CASE 12F. VISCOUS DAMPING CONSTANT EQUIVALENT TO COULOMB DAMPING

If a vibrating mass is subjected to a constant frictional resistance (coulomb friction) F_c, the equivalent viscous damping constant c, based on equal energy dissipation per cycle, is given by the equation

$$c = \frac{4F_c}{\pi \omega X}$$

CASE 12G. FREE CIRCULAR RING

The frequency of the radial mode of a free circular ring of mean radius r is given by the equation

$$f = \frac{1}{2\pi}\sqrt{\frac{Eg}{\rho r^2}}$$

For thin rings the fundamental flexural or "lobar" mode (inextensional and motion in the plane of the ring) is given by the equation

$$f = \frac{1}{2\pi}\sqrt{\frac{EgI}{\rho A r^4} \times \frac{36}{5}}$$

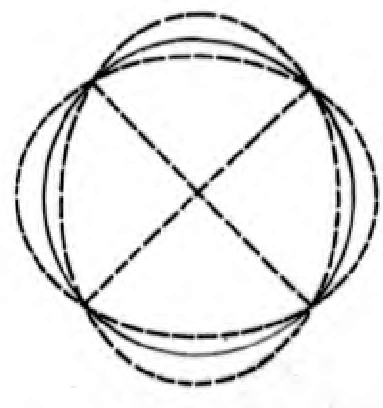


Figure 35 - Fundamental Flexural or "Lobar" Mode of Free Circular Ring

The general formula including the higher modes is

$$f_i = \frac{i}{2\pi}\sqrt{\frac{(i^2-1)^2}{1+i^2}}\sqrt{\frac{EgI}{\rho A r^4}}$$

where the i's are integers starting with 2. Thus, in this case the "fundamental" flexural mode corresponds to the case $i = 2$.

CASE 12H. SPRING CONSTANT OF HELICAL SPRING OF CIRCULAR CROSS SECTION

$$k = \frac{Gr^4}{4R^3 n}$$

where n is number of turns,

r is radius of cross section, and

R is radius of helix.

CASE 12I. COLUMNS - EFFECT OF AXIAL LOAD ON NATURAL FREQUENCY

$$f_1 = f_0\sqrt{1 - \frac{P}{P_{cr}}}$$

where P is the axial load,

P_{cr} is the Euler buckling load $\left(P_{cr} = \frac{\pi^2 EI}{l^2}\right)$, and

f_0 is the natural frequency for zero axial load. For the pin-ended column f_0 is given by f_1 of Case 11B. For the fixed-end column f_0 is given by f_1 of case 11D.

CASE 12J. UNIFORM FREE-FREE BEAM ON UNIFORM ELASTIC SUPPORT

In this case the flexural modes will have the same pattern as for the free-free beam without elastic support ($k' = 0$), but their frequencies will be increased in the ratio

$$\sqrt{1 + \frac{\omega_r^2}{\omega_n^2}} = \sqrt{1 + \frac{\frac{k'}{\mu}}{\omega_n^2}}$$

where ω_n is the circular frequency of the nth mode without elastic support and ω_r is the circular frequency of the rigid body translational mode in the direction of the elastic support.

CASE 12K. BURRILL'S EMPIRICAL FORMULA FOR TWO-NODED VERTICAL FLEXURAL FREQUENCY OF SHIP HULLS IN WATER

$$N = \frac{\phi}{\sqrt{\left(1 + \frac{B}{2d}\right)(1 + r)}} \times \sqrt{\frac{I}{\Delta L^3}}$$

where ϕ is an empirical coefficient given by Burrill as 24×10^5,

N is the frequency in cycles per minute

I is the effective moment of inertia of the midship section area in feet^4,

Δ is the displacement in tons,

L is the length between perpendiculars in feet,

B is the beam in feet,

d is the draft in feet, and

r is J. Lockwood Taylor's shear correction factor $\left[r = \frac{3.5D^2\,(3a^3 + 9a^2 + 6a + 1.2)}{L^2\,(3a + 1)}\right]$

where $a = \frac{B}{D}$ and D is the molded depth in feet.