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APPLIED MATHEMATICS AND STATISTICS LABORATORY

STANFORD UNIVERSITY
CALIFORNIA

SCATTERING OF SURFACE WAVES ON AN IDEAL FLUID

By

HAROLD LEVINE

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TECHNICAL REPORT NO. 78
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§ 1. INTRODUCTION.

Problems relating to the generation, propagation and scattering of surface waves on a fluid have received considerable attention. The bulk of the theory deals with gravity waves in perfect fluids, and is appropriate to infinitesimal wave height. In this framework, Havelock (1917, 1928), Kotchin (1939, 1940) and Ursell (1953) discuss a number of two and three dimensional problems where the surface waves are generated by prescribed motion of floating or submerged bodies. A group of related studies on gravity wave motion in perfect fluids, due to Martin, Moyce, Penney, Price and Thornhill (1952), was inspired by atomic bomb tests, and features an attempt to improve on the infinitesimal theory for stationary and space periodic waves; there is also an account of the diffraction produced by a semi-infinite breakwater inclined at any angle to a parallel harmonic train of infinitesimal waves in water of uniform depth.

Using a variety of techniques, Dean (1948), Heins (1948), John (1948, 1949, 1950), Lewy (1946), Stoker (1947) and Ursell (1947) solve problems in which infinitesimal time harmonic surface waves are incident on fixed or free obstacles located in the fluid. Thus, Ursell obtains the reflection and transmission coefficients for deep water waves scattered by a rigid (and infinitely wide) plane barrier extending vertically downwards from the surface to an arbitrary depth, when the wave crests are parallel to the plane of the barrier and the problem is of two dimensional nature. The formulation hinges on a one dimensional integral equation for the normal fluid velocity in the plane of, and below the barrier, and by adjoining the derivative of this equation in the same plane, an explicit solution is effected. Stoker treats the problem of deep water waves incident at an arbitrary angle on a vertical cliff which is of infinite extension, both parallel to and downwards from the free surface. After separating a known factor of the potential function, which contains the dependence on the coordinate parallel to the shore, the two dimensional residual function satisfies a modified potential equation and is subject to the mixed boundary conditions on the free surface and at the cliff. Following a device of Lewy, the residual function is obtained on solving a partial differential equation, whose inhomogeneous term represents a function which also obeys the modified potential equation and has zero boundary values, thus advancing a simpler primary calculation. For explicit realization of the latter function a singularity has to be introduced in the finite part of the plane, and this is placed at the shore line, or intersection of the free surface and cliff, although the condition for infinitesimal waves nearby is then violated. Effecting the integration suitably, Stoker is able to construct progressive wave solutions which exhibit a parallel wave front at large distance from the shore line. John also employs the Lewy-Stoker method to study a generalization of the problem solved by Ursell, namely with the barrier inclined to the vertical at any one of a discrete set of angles, zero included. The auxiliary harmonic function with null value at the free surface, obtained on differentiation of the potential function, is continued analytically across the latter surface and defined throughout the plane, with a cut at the trace of the barrier and its reflection in the surface.

The present investigation is aimed at extending the integral equation and Lewy-Stoker methods to cope with a wider class of problems, such as those where more than one characteristic length is involved. A complex Fourier transform analysis of the typically singular integral equation is used to advantage, and also the possibility of exact solution on approximation to the kernel. Two features distinguish our calculations from those of Lewy and Stoker. The first pertains to the disposition of singularities (in derivatives) of the velocity potential, which is linked with the occurence of singularities in the fluid velocity at immersed sharp obstacle edges. The second refers to the use of conformal mapping for construction of the auxiliary function with zero boundary values on the free and obstacle surfaces.

A plan of this paper is as follows: First, the problem of scattering of straight crested surface waves at a plane vertical barrier is resolved by the integral equation method ~~(§ 2)~~ in the manner of Ursell; the same problem is discussed by a modified Lewy-Stoker (mapping) technique ~~(§ 3)~~, and then by an integral equation-transform analysis ~~(§ 4)~~. The latter method is next applied to obtain an approximate solution in the case of oblique incidence for a deep barrier ~~(§ 5)~~, and the mapping technique is utilized (§ 6) to derive a complete solution for two parallel barriers at an arbitrary separation, which problem does not appear capable of resolution by integral equation procedures. Finally, a correction factor is obtained (~~§ 7~~ 5) for the effect of finite fluid depth, with a single barrier.

§ 2. INTEGRAL EQUATION METHOD I.

Consider a fluid of semi-infinite extent, with x and z axes in the plane of the undisturbed surface, and the y axis directed vertically downwards, in which there is a rigid vertical barrier, with the trace $x = 0$, $0 < y < 1$, at any value of z (Figure 1). A steady state situation is contemplated, wherein a regular train of waves incident normally on the barrier from the left $(x < 0)$ is partially reflected and partially transmitted. If the fluid is incompressible and inviscid, and performs irrotational, time harmonic motion, there exists

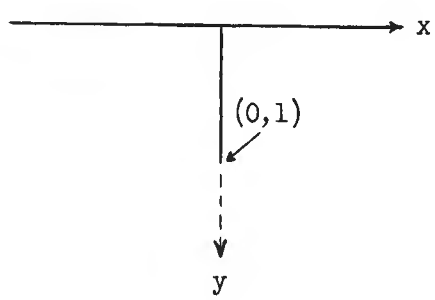

Figure 1.

a velocity potential

$$\Phi(x,y,t) = \varphi(x,y)e^{-i\omega t} ,$$

such that

$$\Phi_{tt} = - \omega^2 \Phi , \qquad (2.1)$$

and

$$\nabla^2\varphi = \varphi_{xx} + \varphi_{yy} = 0 \,, \qquad y > 0 \;. \tag{2.2}$$

When the wave height is infinitesimal, the free surface condition expressing (to first order) constancy of pressure and equality of surface and fluid velocities, takes the form

$$\Phi_{tt} - g\,\Phi_y = 0 \,, \qquad y = 0 \tag{2.3}$$

or

$$(\partial_y + \kappa)\varphi = 0 \,, \qquad y = 0 \tag{2.4}$$

where

$$\kappa = \omega^2/g \tag{2.5}$$

and g denotes the acceleration of gravity. At the rigid barrier, the normal fluid velocity must vanish, whence

$$\partial_x \varphi = 0 \,, \quad x = 0 \,, \quad 0 < y < 1 \;. \tag{2.6}$$

In the absence of a barrier, the equations (2) and (4) admit the surface wave solution

$$\varphi^{inc} = e^{i\kappa x - \kappa y} \,, \tag{2.7}$$

which may be designated as the incident wave function. The problem is now to solve (2), (4) and (6) with the asymptotic requirements

$$\left.\begin{aligned}\varphi &\sim e^{i\kappa x-\kappa y} + R\, e^{-i\kappa x-\kappa y}\ , && x \to -\infty \\ &\sim T\, e^{i\kappa x-\kappa y}\ , && x \to +\infty\end{aligned}\right\} \qquad (2.8)$$

where R and T denote the reflection and transmission coefficients for the surface waves. The terms omitted in (8) are ordinary potential functions, which tend to zero uniformly throughout the distant regions of the fluid.

The effect of the barrier on an incident wave train can be formulated in different ways according to the choice of fundamental solution (Green's function) of the differential equation

$$(\partial_x^2 + \partial_y^2) G(x,y;x',y') = -\,\delta(x-x')\delta(y-y')\ , \qquad (2.9)$$

where the inhomogeneous term contains a product of Dirac delta functions. If the barrier is regarded as an obstacle in the otherwise uniform fluid medium, the free surface condition

$$(\partial_y + \kappa) G = 0\ , \qquad y = 0 \qquad (2.10)$$

may be conveniently appended. An explicit representation of the function satisfying (9), (10) and also the outgoing wave (radiation) condition is

$$G(x,y;x',y') = -\frac{1}{2\pi}\log(R/R^{*}) + i\, e^{i\kappa|x-x'|}\, e^{-\kappa(y+y')}$$

$$+ \frac{1}{\pi}\int_0^\infty \frac{d\zeta}{\zeta^2+\kappa^2}\, e^{-\zeta|x-x'|}(\zeta\cos\zeta(y+y') - \kappa\sin\zeta(y+y')) \;,$$

$$-\infty < x,x' < \infty \;, \qquad 0 < y,y' < \infty \tag{2.11}$$

with $R^2 = (x-x')^2 + (y-y')^2$, and $R^{*2} = (x-x')^2 + (y+y')^2$. In (11), the second term refers to the surface wave mode of excitation of the fluid.

The application of Green's integral theorem to the functions $\varphi(x,y)$ and $G(x,y;x',y')$ results in an expression

$$\varphi(x,y) = \varphi^{inc}(x,y) - \int_0^1 f(y')\partial_{x'}\, G(x,y;0,y')dy' \tag{2.12}$$

for the velocity potential at any point of the fluid, where the secondary contribution arises from the discontinuity function

$$f(y) = \varphi(-0,y) - \varphi(+0,y) \;, \qquad 0 < y < 1 \tag{2.13}$$

at the barrier. Applying the boundary condition (6),

$$\partial_x\, \varphi^{inc}(0,y) = \lim_{x\to 0} \partial_x \int_0^1 f(y')\partial_{x'} G(x,y;0,y')dy' \;, \qquad 0 < y < 1 , \tag{2.14}$$

and this can be expressed as an integro-differential equation,

$$i\kappa\, e^{-\kappa y} = \frac{d^2}{dy^2}\int_0^1 f(y')G(0,y;0,y')dy' \ , \qquad 0 < y < 1 \ . \tag{2.15}$$

If the surface $x = 0$, $y > 1$ (all z) below the barrier is regarded as the common boundary for the domains $x \gtrless 0$, then by defining Green's functions $G_{\pm}$ therein, according to the equations (9), (10) and the additional condition

$$\partial_x G_{\pm} = 0 \ , \qquad x = 0$$

it follows that $G_{\pm}(x,y;0,y') = 2G(x,y;0,y')$, whence the representations

$$\varphi(x,y) = 2\int_1^\infty v(y')G(x,y;0,y')dy' \ , \qquad x > 0 \tag{2.16}$$

and

$$\varphi(x,y) = \varphi^{inc}(x,y) + \varphi^{inc}(-x,y) - 2\int_1^\infty v(y')G(x,y;0,y')dy' \ , \ x < 0 \tag{2.17}$$

are obtained; here

$$v(y) = -\,\partial_x\varphi(0,y) \tag{2.18}$$

is the normal fluid velocity below the barrier.

The requirement of continuous variation in the velocity potential when traversing the surface of separation leads to an integral equation

$$e^{-\kappa y} = 2\int_1^\infty v(y')G(0,y;0,y')dy' \ , \qquad y > 1 \tag{2.19}$$

for the normal velocity.

When $x \to \pm \infty$, and the Green's function is represented asymptotically by its surface wave component, $i e^{i\kappa|x-x'|} e^{-\kappa(y+y')}$, it follows from (12) that

$$\varphi(x,y) \sim \varphi^{inc}(x,y) \mp \kappa\, e^{\pm i\kappa x-\kappa y} \int_0^1 f(y)e^{-\kappa y}dy \quad ,$$

whence the reflection and transmission coefficients R,T take the forms

$$R = 1 - T = \kappa \int_0^1 f(y)e^{-\kappa y}dy \quad ; \tag{2.20}$$

similarly, from (16), (17),

$$R = 1 - T = 1-2i \int_1^\infty v(y)e^{-\kappa y}dy \quad . \tag{2.21}$$

Ursell has given a method for solving the integral equation (19), which is based on simplification of the kernel; the procedure is illustrated here with reference to the integro-differential equation (15). Two successive integrations of (15) yield

$$\frac{d}{dy} \int_0^1 f(y')G(0,y;0,y')dy' = - i\, e^{-\kappa y} + A \quad , \tag{2.22}$$

and

$$\int_0^1 f(y')G(0,y;0,y')dy' = \frac{i}{\kappa}\, e^{-\kappa y} + Ay + B \quad , \tag{2.23}$$

whence

$$\int_0^1 f(y')[\kappa G(0,y;0,y')+ \frac{d}{dy} G(0,y;0,y')]dy'= A(1+\kappa y)+ \kappa B\ , \quad 0<y<1\ . \tag{2.24}$$

The operator $(\frac{d}{dy} + \kappa)$ applied to $G(0,y;0,y')$ results in a simpler expression than that of the latter function alone, and takes (24) into the form

$$\int_0^1 f(y')[\kappa \log \left|\frac{y-y'}{y+y'}\right| + \frac{1}{y-y'} + \frac{1}{y+y'}]dy' = C_1 y + C_2 , \quad 0<y<1 \qquad (2.25)$$

where $C_1 = -2\pi\kappa A$, $C_2 = -2\pi A - 2\pi\kappa B$. Define

$$F(y) = \int_y^1 f(y)dy \quad , \qquad (2.26)$$

with $dF/dy = -f(y)$, $F(1) = 0$, and then

$$\int_0^1 f(y')\log \left|\frac{y-y'}{y+y'}\right|dy' = -\int_0^1 \log \left|\frac{y-y'}{y+y'}\right|dF(y')$$

$$= -\int_0^1 F(y')[\frac{1}{y-y'} + \frac{1}{y+y'}]dy' \quad ,$$

so that (25) may be written

$$\int_0^1 \{f(y')-\kappa F(y')\}[\frac{1}{y-y'} + \frac{1}{y+y'}]dy' = C_1 y + C_2 , \qquad 0<y<1 . \quad (2.27)$$

The constant $C_2 = 0$, as inferred by choosing $y = 0$, and thus a principal value integral equation

$$\int_0^1 \mathcal{F}(y') \frac{dy'}{y^2-y'^2} = \frac{1}{2} C_1 \quad , \quad 0 < y < 1 , \quad \mathcal{F}(y) = f(y) - \kappa F(y) \qquad (2.28)$$

of a type familiar in airfoil theory, is obtained. The solution of (28) which vanishes at $y = 1$ is found to be (Schmeidler, (1950))

$$\mathcal{F}(y) = \frac{C_1}{\pi}\sqrt{1-y^2} \quad , \tag{2.29}$$

whence

$$\frac{df}{dy} + \kappa f = -\frac{C_1 y}{\pi\sqrt{1-y^2}}$$

and

$$f(y) = \frac{C_1}{\pi} e^{-\kappa y} \int_y^1 \frac{\zeta\, e^{\kappa\zeta}}{\sqrt{1-\zeta^2}}\, d\zeta = \frac{C_1}{\pi} e^{-\kappa y} P(y) \quad , \tag{2.30}$$

say.

There remains only the specification of C_1 , which calls for substitution of (30) into the original integro-differential equation (15), that can be written as

$$\frac{d^2}{dy^2}\int_0^1 f(y')\left\{i\, e^{-\kappa(y+y')} + \frac{1}{\pi}\int_0^\infty d\zeta\, \frac{(\zeta \cos \zeta y - \kappa \sin \zeta y)(\zeta \cos \zeta y' - \kappa \sin \zeta y')}{\zeta(\zeta^2+\kappa^2)}\right\}dy' = i\kappa\, e^{-\kappa y} \,,$$

$$0 < y < 1 \,. \tag{2.31}$$

Now

$$\int_0^1 f(y)e^{-\kappa y}dy = \frac{C_1}{-2\pi\kappa}\int_0^1 P(y)d\, e^{-2\kappa y} = \frac{C_1}{2\kappa} I_1(\kappa) \,, \tag{2.32}$$

where I_1 denotes the cylinder function

$$I_1(x) = \frac{1}{i} J_1(ix) \quad . \tag{2.33}$$

If

$$Q_1 = \int_0^1 e^{-\kappa y} P(y) \cos \zeta y \, dy \ ,$$

$$Q_2 = \int_0^1 e^{-\kappa y} P(y) \sin \zeta y \, dy \ ,$$

then integration by parts in Q_1 reveals that

$$\zeta Q_1 - \kappa Q_2 = \frac{\pi}{2} J_1(\zeta) \ ,$$

and therefore

$$\int_0^1 f(y)(\zeta \cos \zeta y - \kappa \sin \zeta y) dy = \frac{C_1}{\pi} (\zeta Q_1 - \kappa Q_2) = \frac{C_1}{2} J_1(\zeta) \ . \tag{2.34}$$

A few preliminary manipulations suffice to verify that

$$\int_0^\infty \frac{d\zeta}{\zeta^2 + \kappa^2} \frac{J_1(\zeta)}{\zeta} (\zeta \cos \zeta y - \kappa \sin \zeta y) = - \frac{1}{\kappa} e^{-\kappa y} K_1(\kappa) + \frac{1}{\kappa^2} (1 - \kappa y) \ , \tag{2.35}$$

where K_1 denotes the cylinder function

$$K_1(x) = - \frac{\pi}{2} H_1^{(1)}(ix) \ . \tag{2.36}$$

Substituting the results of (32), (34) and (35) into (31), the constant C_1 turns out to be

$$C_1 = \frac{2\pi}{\pi I_1(\kappa) + i K_1(\kappa)} \ , \tag{2.37}$$

and thus, by (20), (32),

$$R = \frac{\pi I_1(\kappa)}{\pi I_1(\kappa) + iK_1(\kappa)} = 1 - T \quad . \tag{2.38}$$

The reflection and transmission coefficients given in (38) are modified for a barrier of depth d by the replacement of κd for κ ; these are the results of Ursell.

§ 3. CONFORMAL MAPPING METHOD*/.

Let us define

$$\psi(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} + \kappa\right)\varphi(x,y) \quad ; \tag{3.1}$$

then by (2.2),

$$\nabla^2 \psi = 0 \quad , \tag{3.2}$$

and the boundary conditions (2.4), (2.6) imply

$$\begin{aligned} \psi(x,0) &= 0 \\ \psi(0,y) &= 0 \,, \qquad 0 < y < 1 \quad . \end{aligned} \tag{3.3}$$

The asymptotic terms of (2.8) are eliminated by the differentiation in (1), so that

$$\psi(x,y) \to 0 \qquad \text{as} \qquad r = \sqrt{x^2 + y^2} \to \infty \quad . \tag{3.4}$$

*/ This calculation was described at the international symposium on the behavior of ships in a seaway, in Wageningen, Holland, Sept. 1957.

ψ is singular at the edge $x = 0$, $y = 1$ of the barrier. The velocity potential φ varies with the square root of the distance from the edge; hence φ_x has an inverse square root singularity and φ_{xy} has an inverse $\frac{3}{2}$ power singularity there.

With $z = x + iy$, the conformal transformation

$$\zeta = \sqrt{1+z^2} \tag{3.5}$$

(taking the branch of the square root for which $\zeta \sim z$ at ∞) maps the region of the fluid onto the upper half of the ζ-plane. By (2)-(4), ψ is a harmonic function of ζ in the upper half-plane which is zero on the real axis and at ∞, with a singularity at $\zeta = 0$, namely the image of the edge $z = i$ under (5). From the behavior of ψ near the edge in the xy-plane, its behavior near $\zeta = 0$ may be deduced. The most general harmonic function which has this behavior and satisfies the boundary conditions is of the form (bar denotes complex conjugate)

$$C_1 \operatorname{Im} \frac{1}{\bar{\zeta}} + C_2 \operatorname{Im} \frac{1}{\bar{\zeta}^3} ,$$

where C_1 and C_2 are constants. By (5) ,

$$\psi = C_1 \operatorname{Im} \frac{1}{(1+\bar{z}^2)^{1/2}} + C_2 \operatorname{Im} \frac{1}{(1+\bar{z}^2)^{3/2}} \tag{3.6}$$

for some C_1 , C_2 .

Equation (2.8) shows that

$$\left(\frac{\partial}{\partial y} + \kappa\right)\varphi \to 0 \qquad \text{as} \qquad x \to \pm\infty \ ,$$

and thus, integrating (1) with respect to x separately for $x \gtrless 0$,

$$\left(\frac{\partial}{\partial y} + \kappa\right)\varphi = \int_{\pm\infty}^{x} \psi(\xi,y)d\xi \qquad\qquad (x \gtrless 0)$$

or

$$\frac{\partial}{\partial y}\left(e^{\kappa y}\,\varphi\right) = e^{\kappa y}\int_{\pm\infty}^{x} \psi(\xi,y)d\xi \tag{3.7}$$

For $y > 1$, the left hand side of (7) must be continuous across $x = 0$, and hence

$$\int_{-\infty}^{\infty} \psi(\xi,y)d\xi = 0 \ , \qquad y > 1 \tag{3.8}$$

which reduces by (6) to

$$C_1 = 0 \ . \tag{3.9}$$

The integral of (7) with respect to y is

$$\varphi(x,y) = e^{-\kappa y}\, f_{\pm}(x) + e^{-\kappa y}\int_{0}^{y} e^{\kappa\eta}\, d\eta \int_{\pm\infty}^{x} \psi(\xi,\eta)d\xi \ , \qquad x \gtrless 0 \ . \tag{3.10}$$

Integration by parts and the application of (2) simplify this to

$$\varphi = e^{-\kappa y}[f_{\pm}(x) + \frac{1}{\kappa^2}\int_{\pm\infty}^{x} \psi_y(\xi,0)d\xi - \frac{1}{\kappa^2}\int_0^y e^{\kappa\eta}\psi_x(x,\eta)d\eta] - \frac{1}{\kappa^2}\int_{\pm\infty}^{x} \psi_y(\xi,y)d\xi \ . \quad (3.11)$$

Now

$$\nabla^2\varphi = e^{-\kappa y}[(\frac{d^2}{dx^2} + \kappa^2)f_{\pm}(x) + \int_{\pm\infty}^{x} \psi_y(\xi,0)d\xi] = 0 \quad , \quad (3.12)$$

by (2.2), which provides the ordinary differential equations

$$(\frac{d^2}{dx^2} + \kappa^2)f_{\pm}(x) = -\int_{\pm\infty}^{x} \psi_y(\xi,0)d\xi \quad (3.13)$$

for $f_{\pm}(x)$. From (6) and (9),

$$\psi_y(x,0) = \pm\frac{3C_2\,x}{(1+x^2)^{5/2}} \quad , \quad x \gtrless 0 \quad (3.14)$$

whence, evaluating the inhomogeneous terms of (13),

$$(\frac{d^2}{dx^2} + \kappa^2)f_{\pm}(x) = \frac{\pm C_2}{(1+x^2)^{3/2}} \ . \quad (3.15)$$

In the general solution of these equations,

$$f_{\pm}(x) = C_{3\pm}\,e^{i\kappa x} + C_{4\pm}\,e^{-i\kappa x} \pm \frac{C_2}{\kappa}\int_{\pm\infty}^{x} \frac{\sin\kappa(x-\xi)}{(1+\xi^2)^{3/2}}d\xi \quad , \quad (3.16)$$

the constants of integration may be determined by observing that (11) implies a common asymptotic form of φ and $e^{-\kappa y} f_{\pm}(x)$ as $x \to \pm\infty$. Recalling (2.8), the appropriate choice of constants is such that

$$f_{\pm}(x) = \left\{ \begin{matrix} T\, e^{i\kappa x} \\ \\ e^{i\kappa x} + R\, e^{-i\kappa x} \end{matrix} \right\} \pm \frac{C}{\kappa} \int_{\pm\infty}^{x} \frac{\sin \kappa(x-\xi)}{(1+\xi^2)^{3/2}}\, d\xi \tag{3.17}$$

(where $C = C_2$).

C, T and R must still be determined. One equation relating them may be obtained by imposing on (11) the condition that $\varphi(x,y)$ exhibit a continuous behavior across $x = 0$ for $y > 1$:

$$f_+(0) - f_-(0) + \frac{1}{\kappa^2}\left(\int_{+\infty}^{0} - \int_{-\infty}^{0}\right)\psi_y(\xi,0)\,d\xi - \frac{1}{\kappa^2}\int_0^{1+} e^{\kappa\eta}[\psi_x(0+,\eta) - \psi_x(0-,\eta)]\,d\eta = 0 .$$

Employing (17), (6) and (9) , this may be simplified to

$$T - 1 - R + \frac{2C}{\kappa}\int_0^{\infty} \frac{\sin \kappa\,\xi}{(1+\xi^2)^{3/2}}\,d\xi - 2C\int_0^1 \frac{\eta\, e^{\kappa\eta}}{\sqrt{1-\eta^2}}\,d\eta = 0 \quad ,$$

or

$$T - 1 - R - 2\,C\,\pi\, I_1(\kappa) = 0 \quad . \tag{3.18}$$

The remaining condition is that the normal derivative of $\varphi(x,y)$ vanish at the barrier. By (11), for $0 < y < 1$,

$$\varphi_x(0\pm, y) = e^{-\kappa y}[f'_\pm(0) - \frac{1}{\kappa^2}\int_0^y e^{-\kappa\eta}\,\psi_{xx}(0\pm,\eta)d\eta]$$

$$= e^{-\kappa y}\, f'_\pm(0) \quad .$$

Hence,

$$f'_\pm(0) = \left\{\begin{matrix} i\kappa T \\ \\ i\kappa(1-R) \end{matrix}\right\} \pm C \int_{\pm\infty}^{0} \frac{\cos \kappa \xi}{(1+\xi^2)^{3/2}}\, d\xi = 0 \quad ,$$

or

$$T = 1 - R = \frac{C}{i\kappa}\int_0^\infty \frac{\cos \kappa \xi}{(1+\xi^2)^{3/2}}\, d\xi = - i\, C\, K_1(\kappa) \quad . \tag{3.19}$$

Solving (18) and (19) for R and T again gives the result (2.38).

§ 4. INTEGRAL EQUATION METHOD II.

A different approach to the integral equation for the normal fluid velocity in the plane of, and below the barrier, involves preliminary Fourier transformation. By analysis of the functional (transform) relation, a new integral equation for the transform of the velocity is established, which has some advantages over the original integral equation. Thus, it will be shown in this section that the transform integral equation can be brought to a form where the kernel is a difference function, and solved explicitly. In the next section, the inhomogeneous nature of the integral equation is exploited to obtain an approximate solution for the case of oblique incidence, with improved accuracy as the depth of the barrier increases.

The Green's function representation (2.11) is convenient for the identification of propagating and non-propagating motions along the surface (x direction), and is less convenient for exhibiting the dependence on the normal thereto (y direction). An alternative representation may be derived, where, in the interest of Fourier transformation, the differential equation (2.9) is replaced by

$$(\partial_x^2 + \partial_y^2 - \epsilon^2)G(x,y;x',y') = -\,\delta(x-x')\delta(y-y') \quad , \qquad \epsilon > 0 \ . \tag{4.1}$$

The solution of (1), subject to the boundary condition (2.10) can be displayed in the forms ($y,y' > 0$)

$$G(x,y;x',y') = \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{i\zeta(x-x')}[e^{-\eta(y_> - y_<)} + \frac{\eta+\kappa}{\eta-\kappa}\, e^{-\eta(y+y')}]\,\frac{d\zeta}{\eta} \ , \tag{4.2}$$

$$= \frac{i}{4} H_0^{(1)}(i\,\epsilon R) + \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{i\zeta(x-x')}\,\frac{\eta+\kappa}{\eta-\kappa}\, e^{-\eta(y+y')}\,\frac{d\zeta}{\eta} \ , \tag{4.3}$$

$$= \frac{i}{4}\,[H_0^{(1)}(i\,\epsilon R) - H_0^{(1)}(i\,\epsilon R^*)] + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\zeta(x-x')}\,\frac{e^{-\eta(y+y')}}{\eta-\kappa}\, d\zeta \ , \tag{4.4}$$

where $\eta = +\sqrt{\zeta^2 + \epsilon^2}$, and R, R^* are as defined in (2.11). If $\epsilon \to 0$ in (4.4), so that

$$H_0^{(1)}(i\,\epsilon R) - H_0^{(1)}(i\,\epsilon R^*) \to \frac{2i}{\pi}\log\frac{R}{R^*} \ ,$$

and

$$\int_{-\infty}^{\infty} e^{i\zeta(x-x')} \frac{e^{-\eta(y+y')}}{\eta-\kappa} d\zeta = 2\int_0^{\infty} \cos\zeta(x-x') \frac{e^{-\zeta(y+y')}}{\zeta-\kappa} d\zeta ,$$

with the path of integration indented below the point $\zeta = \kappa$, then the form (2.11) is easily recovered.

Let the domain of the integral equation (2.19) for the normal velocity be altered to $\bar{y} > 0$ by the displacements, $y = \bar{y} + 1$, $y' = \bar{y}' + 1$. Then

$$e^{-\kappa(\bar{y}+1)} = 2\int_0^{\infty} v(\bar{y}')K(\bar{y},\bar{y}')d\bar{y}' , \qquad \bar{y} > 0 \tag{4.5}$$

where $K(\bar{y},\bar{y}') = G(0,\bar{y}+1;0,\bar{y}'+1)$

$$= \frac{i}{4} H_0^{(1)}(i\epsilon|\bar{y}-\bar{y}'|) + \frac{1}{2\pi}\int_{\epsilon}^{\infty} \frac{\xi+\kappa}{\xi-\kappa} e^{-\xi(\bar{y}+\bar{y}'+2)} \frac{d\xi}{\sqrt{\xi^2-\epsilon^2}} , \tag{4.6}$$

by (3). If the integral equation (5) is extended to the infinite domain $-\infty < \bar{y} < \infty$, in accordance with the scheme

$$\begin{Bmatrix} 0 \\ u(\bar{y}) \end{Bmatrix} + \begin{Bmatrix} e^{-\kappa(\bar{y}+1)} \\ 0 \end{Bmatrix} = 2\int_{-\infty}^{\infty} \begin{Bmatrix} v(\bar{y}') \\ 0 \end{Bmatrix} K(\bar{y},\bar{y}')d\bar{y}' , \qquad \begin{Bmatrix} \bar{y} > 0 \\ < 0 \end{Bmatrix} \tag{4.7}$$

the result of multiplying by $e^{-i\zeta\bar{y}}$ and integrating with respect to $\bar{y}$ is

$$\bar{u}(\zeta) - \frac{ie^{-\kappa}}{\zeta-i\kappa} = \frac{\bar{v}(\zeta)}{\sqrt{\zeta^2+\epsilon^2}} + \frac{1}{\pi i}\int_{\epsilon}^{\infty} \frac{\xi+\kappa}{\xi-\kappa} \frac{\bar{v}(-i\xi)\, e^{-2\xi}}{(\zeta-i\xi)\sqrt{\xi^2-\epsilon^2}} d\xi . \tag{4.8}$$

Here

$$\bar{u}(\zeta) = \int_{-\infty}^{0} e^{-i\zeta\bar{y}}\, u(\bar{y})d\bar{y} \quad ,$$

and

$$\bar{v}(\zeta) = \int_{0}^{\infty} e^{-i\zeta\bar{y}}\, v(\bar{y})d\bar{y} \quad . \qquad (4.9)$$

The function $\bar{u}(\zeta)$ will be eliminated by bringing equation (8) into the form

$$F_1(\zeta) = F_2(\zeta) \quad , \qquad (4.10)$$

where $F_1(\zeta)$ is analytic for $\operatorname{Im} \zeta > - \epsilon$, $F_2(\zeta)$ is analytic for $\operatorname{Im} \zeta < \epsilon$, and each function approaches zero as $\zeta \to \infty$ in its half plane of regularity. From this it follows that F_1 and F_2 are two representations of an entire function, which is zero by virtue of its asymptotic behavior. Hence, (8) may be replaced by

$$F_2(\zeta) = 0 \quad . \qquad (4.11)$$

To accomplish the indicated reduction, first multiply (8) by $\sqrt{\epsilon - i\zeta}$. Then the first terms of the left and right members are analytic for $\operatorname{Im} \zeta > - \epsilon$ and $\operatorname{Im} \zeta < \epsilon$, respectively. The remaining terms are each readily broken into two parts, which are analytic in one of the half planes. The result is

$$\bar{u}(\zeta)\sqrt{\epsilon-i\zeta} - \frac{ie^{-\kappa}(\sqrt{\epsilon-i\zeta}-\sqrt{\epsilon+\kappa})}{\zeta-i\kappa} - \frac{1}{\pi i}\int_{\epsilon}^{\infty}\frac{\xi+\kappa}{\xi-\kappa}\,\frac{\sqrt{\epsilon-i\zeta}-\sqrt{\epsilon+\xi}}{\zeta-i\xi}\,\frac{\bar{v}(-i\xi)e^{-2\xi}}{\sqrt{\xi^2-\epsilon^2}}\,d\xi$$

$$= \frac{\bar{v}(\zeta)}{\sqrt{\epsilon+i\zeta}} + \frac{ie^{-\kappa}\sqrt{\epsilon+\kappa}}{\zeta-i\kappa} + \frac{1}{\pi i}\int_{\epsilon}^{\infty}\frac{\xi+\kappa}{\xi-\kappa}\,\frac{\bar{v}(-i\xi)e^{-2\xi}}{(\zeta-i\xi)\sqrt{\xi-\epsilon}}\,d\xi \quad ,$$

an equation of the form (10). Hence, as in (11), the right side of this equation is zero. Let $\zeta = -it$, and then

$$\frac{\bar{v}(-it)}{\sqrt{t+\epsilon}} = \frac{e^{-\kappa}\sqrt{\kappa+\epsilon}}{t+\kappa} - \frac{1}{\pi}\int_{\epsilon}^{\infty}\frac{\xi+\kappa}{\xi-\kappa}\,\frac{\bar{v}(-i\xi)e^{-2\xi}}{(t+\xi)\sqrt{\xi-\epsilon}}\,d\xi \quad , \tag{4.12}$$

which is valid for $\operatorname{Re} t > -\epsilon$. If $\epsilon \to 0$,

$$\frac{\bar{v}(-it)}{\sqrt{t}} = \frac{e^{-\kappa}\sqrt{\kappa}}{t+\kappa} - \frac{1}{\pi}\int_{0}^{\infty}\frac{\xi+\kappa}{\xi-\kappa}\,\frac{\bar{v}(-i\xi)e^{-2\xi}}{(t+\xi)\sqrt{\xi}}\,d\xi \quad . \tag{4.12'}$$

Assume that at $\bar{y} = 0$, $v(\bar{y}) \sim A\bar{y}^{-1/2}$. Then, for $\operatorname{Re} t > 0$, $\bar{v}(-it) \sim A\sqrt{\pi}\,t^{-1/2}$ as $t \to \infty$. Let

$$\frac{t+\kappa}{\sqrt{t}}\,\bar{v}(-it) = A\sqrt{\pi}\,[1 + w(t)] \quad , \tag{4.13}$$

and thus $w(t)$ is analytic for $\operatorname{Re} t > 0$ and approaches zero as $t \to \infty$. Hence,

$$\bar{w}(s) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} w(t)e^{st}\,dt \tag{4.14}$$

(where α is a positive constant) is defined for real positive s . w is the Laplace transform of $\bar{w}$:

$$w(t) = \int_0^\infty \bar{w}(s)e^{-st}\,ds \ . \tag{4.15}$$

To find the integral equation satisfied by $\bar{w}$, first eliminate $\bar{v}$ from (12') by (13); this yields

$$A\sqrt{\pi}\,[1+w(t)] = e^{-\kappa}\sqrt{\kappa} - \frac{A}{\sqrt{\pi}}(t+\kappa)\int_0^\infty \frac{1+w(\xi)}{(\xi-\kappa)(t+\xi)}e^{-2\xi}\,d\xi \ . \tag{4.16}$$

Taking the limit $t \to \infty$ in (16) gives a condition which will be satisfied later,

$$A\sqrt{\pi} = e^{-\kappa}\sqrt{\kappa} - \frac{A}{\sqrt{\pi}}\int_0^\infty \frac{1+w(\xi)}{\xi-\kappa}e^{-2\xi}\,d\xi \ ; \tag{4.17}$$

meanwhile, (16) can be simplified to

$$w(t) = \frac{1}{\pi}\int_0^\infty \frac{1+w(\xi)}{t+\xi}e^{-2\xi}\,d\xi \ . \tag{4.18}$$

Multiply by e^{st} and integrate over t as in (14); then

$$\bar{w}(s) = \frac{1}{\pi}\int_0^\infty e^{-(s+2)\xi}[1+\int_0^\infty \bar{w}(u)e^{-\xi u}\,du]d\xi \ ,$$

or

$$\bar{w}(s) = \frac{1}{\pi(s+2)} + \frac{1}{\pi}\int_0^\infty \frac{\bar{w}(u)du}{u+s+2} \ . \tag{4.19}$$

Equation (19) may be transformed into an equation with a difference kernel. Let $s = e^x - 1$, and

$$p(x) = e^{x/2}\, \bar{w}(e^x - 1) \quad . \tag{4.20}$$

Then (19) implies

$$p(x) = \frac{1}{2\pi} \operatorname{sech} \frac{x}{2} + \frac{1}{2\pi} \int_0^\infty \operatorname{sech} \frac{x-t}{2}\, p(t)dt \quad . \tag{4.21}$$

If it be assumed that $v(\bar{y})$ approaches zero sufficiently rapidly as $\bar{y} \to \infty$, then $\bar{v}(0)$ is finite, and from (13), (14),

$$\bar{w}(s) = O(s^{-1/2}) , \qquad s \to \infty$$

so that $p(x)$ is bounded.

Let (20) be extended to an integral equation on $(-\infty, \infty)$, according to the scheme

$$\begin{Bmatrix} 0 \\ q(x) \end{Bmatrix} + \begin{Bmatrix} p(x) \\ 0 \end{Bmatrix} = \frac{1}{2\pi} \operatorname{sech} \frac{x}{2} + \frac{1}{2\pi} \int_{-\infty}^\infty \operatorname{sech} \frac{x-t}{2} \begin{Bmatrix} p(t) \\ 0 \end{Bmatrix} dt \ , \quad \begin{Bmatrix} x > 0 \\ < 0 \end{Bmatrix} \tag{4.22}$$

whence

$$q(x) = O(e^{x/2}) , \qquad x \to -\infty \ .$$

The integration of (22) with respect to x , after multiplication by $e^{-\zeta x}$, can thus be effected, if $0 < \operatorname{Re} \zeta < \frac{1}{2}$, and yields

$$\bar{p}(\zeta) + \bar{q}(\zeta) = \frac{1}{\cos\pi\zeta} + \frac{\bar{p}(\zeta)}{\cos\pi\zeta} ,$$

or

$$[1 + \bar{p}(\zeta)](1 - \frac{1}{\cos\pi\zeta}) = 1 - \bar{q}(\zeta) , \qquad 0 < \operatorname{Re}\zeta < \frac{1}{2} \tag{4.23}$$

where

$$\left.\begin{aligned} \bar{p}(\zeta) &= \int_0^\infty e^{-\zeta x}\, p(x)dx \\ \bar{q}(\zeta) &= \int_{-\infty}^0 e^{-\zeta x}\, q(x)dx \end{aligned}\right\} . \tag{4.24}$$

The second factor on the left in equation (23) may be written as the ratio of two functions analytic in the appropriate half planes by applying the basic formula

$$\sin\pi z = \frac{\pi}{\Gamma(z)\Gamma(1-z)}$$

to obtain

$$1 - \frac{1}{\cos\pi\zeta} = - \frac{\pi\zeta^2}{2} \frac{\Gamma(\frac{1}{2}+\zeta)\Gamma(\frac{1}{2}-\zeta)}{\Gamma(1+\frac{\zeta}{2})^2\Gamma(1-\frac{\zeta}{2})^2} .$$

Using this in (23),

$$[1 + \bar{p}(\zeta)] \frac{2^{-\zeta}\zeta^2\,\Gamma(\frac{1}{2}+\zeta)}{\Gamma(1+\frac{\zeta}{2})^2} = - \frac{2}{\pi} \frac{2^{-\zeta}\,\Gamma(1-\frac{\zeta}{2})^2}{\Gamma(\frac{1}{2}-\zeta)} [1 - \bar{q}(\zeta)] . \tag{4.25}$$

The left side of (25) is analytic for $\operatorname{Re}\zeta > 0$, and the right side is analytic for $\operatorname{Re}\zeta < \frac{1}{2}$; thanks to the factor $2^{-\zeta}$, both are $O(|\zeta|)$

as $\zeta \to \infty$ in the respective half planes. Hence, the entire function represented by the two sides of (25) is linear, and

$$[1 + \bar{p}(\zeta)] \frac{2^{-\zeta} \zeta^2 \Gamma(\frac{1}{2} + \zeta)}{\Gamma(1 + \frac{\zeta}{2})^2} = B\zeta + C \ , \tag{4.26}$$

for suitable constants B, C . The constants are determined through the limits $\zeta \to 0$, $\zeta \to \infty$, with the latter requiring asymptotic forms of the Γ - functions. It turns out that

$$B = \sqrt{2/\pi} \ , \qquad C = 0$$

and consequently

$$\bar{p}(\zeta) = -1 + \sqrt{\frac{2}{\pi}} \ \frac{2^{\zeta}}{\zeta} \ \frac{\Gamma(1 + \frac{\zeta}{2})^2}{\Gamma(\frac{1}{2} + \zeta)} \ . \tag{4.27}$$

This result must be transformed back to a formula for $\bar{v}$.

The procedure begins with inversion of (24) ,

$$\begin{aligned} p(x) &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \bar{p}(\zeta) e^{x\zeta} \, d\zeta \\ &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} [-1 + \sqrt{\frac{2}{\pi}} \ \frac{2^{\zeta}}{\zeta} \ \frac{\Gamma(1 + \frac{\zeta}{2})^2}{\Gamma(\frac{1}{2} + \zeta)}] e^{x\zeta} \, d\zeta \ , \end{aligned} \tag{4.28}$$

where a is a positive constant. To secure an integral with improved convergence properties, note that

$$p(x) = \frac{d^2}{dx^2} q(x) \quad , \tag{4.29}$$

if

$$q(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} [- 1 + \sqrt{\frac{2}{\pi}} \frac{2^\zeta}{\zeta} \frac{\Gamma(1 + \frac{\zeta}{2})^2}{\Gamma(\frac{1}{2} + \zeta)}] \frac{e^{x\zeta}}{\zeta^2} d\zeta \quad . \tag{4.30}$$

This may be reduced to an integral of elementary functions by the integral representations

$$B(\frac{1}{2}\zeta, \frac{1}{2}\zeta) = \frac{\Gamma(\frac{\zeta}{2})^2}{\Gamma(\zeta)} = 2^{2-\zeta} \int_0^1 \frac{y^{\zeta-1}\,dy}{\sqrt{1-y^2}} \quad ,$$

$$B(\frac{1}{2}, \zeta) = \frac{\Gamma(\frac{1}{2})\Gamma(\zeta)}{\Gamma(\frac{1}{2} + \zeta)} = \int_0^1 \frac{z^{\zeta-1}\,dz}{\sqrt{1-z}} \quad ,$$

where B denotes the beta function. Utilizing these in (30) and integrating the first term yields

$$q(x) = - x + \frac{1}{\sqrt{2}\,\pi^2\, i} \int_{a-i\infty}^{a+i\infty} \frac{e^{x\zeta}}{\zeta} d\zeta \int_0^1 \frac{y^{\zeta-1}\,dy}{\sqrt{1-y^2}} \int_0^1 \frac{z^{\zeta-1}\,dz}{\sqrt{1-z}} \quad .$$

The ζ integration may be carried out by closing the path of integration at ∞, in a way which depends on whether $e^x yz \gtrless 1$. The result is

$$q(x) = - x + \frac{\sqrt{2}}{\pi} \int_{e^{-x}}^{1} \frac{dy}{y\sqrt{1-y^2}} \int_{\frac{1}{ye^x}}^{1} \frac{dz}{z\sqrt{1-z}} \quad . \tag{4.31}$$

Differentiating once removes the z integral, to yield

$$q'(x) = -1 + \frac{\sqrt{2}}{\pi} \int_{e^{-x}}^{1} \frac{dy}{\sqrt{y(1-y^2)(y-e^{-x})}} \; . \tag{4.32}$$

The integrand of (32) is an analytic function in the upper half plane which is real along the real axis on the intervals $(-1,0)$ and $(e^{-x},1)$. Contour integration around the upper half plane shows that the sum of the integrals over these two intervals is zero. Hence, the integral in (32) may be replaced by an integral on $(-1,0)$. Replacing y by $-y$ in the latter integral,

$$q'(x) = -1 + \frac{\sqrt{2}}{\pi} \int_{0}^{1} \frac{dy}{\sqrt{y(1-y^2)(y+e^{-x})}} \; .$$

This may be differentiated directly, and by (29),

$$p(x) = q''(x) = \frac{e^{-x}}{\sqrt{2}\,\pi} \int_{0}^{1} \frac{dy}{\sqrt{y(1-y^2)(y+e^{-x})^3}} \; , \tag{4.33}$$

$$= -\frac{1}{2} e^{x/2} F'\left(\frac{3}{2}, -\frac{1}{2}, 1, \frac{1-e^{x}}{2}\right) \; , \tag{4.34}$$

where F' is the derivative of the hypergeometric function with respect to its last argument. The result (34) may be verified in several ways, for example, by expanding the integral of (33) in powers of $(1-e^{x})/2$.

From (15) and (20),

$$\bar{w}(s) = -\frac{1}{2} F'\left(\frac{3}{2}, -\frac{1}{2}, 1, -\frac{s}{2}\right) \; ,$$

and

$$w(t) = -\frac{1}{2}\int_0^\infty e^{-st}\, F'(\tfrac{3}{2}, -\tfrac{1}{2}, 1, -\tfrac{s}{2})ds$$

$$= -1 + t\int_0^\infty e^{-st}\, F(\tfrac{3}{2}, -\tfrac{1}{2}, 1, -\tfrac{s}{2})ds \quad , \qquad (4.35)$$

on integration by parts. Differentiation under the integral sign, followed by integration by parts and application of the hypergeometric differential equation

$$[(s^2 + 2s)\frac{d^2}{ds^2} - 4(s+1)\frac{d}{ds} - 3]F(\tfrac{3}{2}, -\tfrac{1}{2}, 1, -\tfrac{s}{2}) = 0 \quad ,$$

shows that $1 + w(t)$ is a solution of the equation

$$(t^2\frac{d^2}{dt^2} - 2t^2\frac{d}{dt} - \tfrac{3}{4})z(t) = 0 \quad .$$

Hence, $1 + w(t)$ is a linear combination of the solutions

$$\sqrt{t}\, e^t\, I_1(t) \quad , \qquad \sqrt{t}\, e^t\, K_1(t) \quad ,$$

where I_1 , K_1 are the modified Bessel functions. From (35),

$$1 + w(t) \sim 1 \ , \quad t \to +\infty$$

and therefore

$$1+w(t) = \sqrt{\frac{2t}{\pi}}\, e^{t}\, K_1(t) \quad . \tag{4.36}$$

The constant A may now be determined from equation (17):

$$A\sqrt{\pi} = e^{-\kappa}\sqrt{\kappa} - \frac{A\sqrt{2}}{\pi}\int_0^\infty \frac{\sqrt{\xi}\, e^{-\xi} K_1(\xi)}{\xi-\kappa}\, d\xi \quad . \tag{4.37}$$

Using the integral representation

$$K_1(\xi) = \int_1^\infty \frac{u\, e^{-\xi u}}{\sqrt{u^2-1}}\, du \quad ,$$

the integral in (37) can be written

$$\int_1^\infty \frac{u\, du}{\sqrt{u^2-1}} \int_0^\infty \frac{e^{-(u+1)\xi}\sqrt{\xi}}{\xi-\kappa}\, d\xi \quad .$$

Make the change of variable $\xi = x/(1+u)$ in the ξ integral and reverse the order of integration. Then the integration with respect to u is elementary, and as a result

$$\int_0^\infty \frac{\sqrt{\xi}\, e^{-\xi} K_1(\xi)}{\xi-\kappa}\, d\xi = \pi\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\left(-\frac{1}{\sqrt{2}} + \frac{1-x/\kappa}{\sqrt{2-x/\kappa}}\right)dx$$

$$= -\pi\sqrt{\frac{\pi}{2}} + \pi\int_0^\infty \frac{e^{-x}(1-x/\kappa)}{\sqrt{x(2-x/\kappa)}}\, dx \quad . \tag{4.38}$$

In the remaining integral over x, shift the path of integration up to the ray from the origin to ∞ through κ, and perform the substitution $x = \kappa(1+t)$. For $t > 1$, the argument of the square root is $\pi/2$. Thus (38) becomes

$$\int_0^\infty \frac{\sqrt{\xi}\, e^{-\xi} K_1(\xi)}{\xi-\kappa}\, d\xi = -\pi\sqrt{\frac{\pi}{2}} - \pi\sqrt{\kappa}\, e^{-\kappa} \int_{-1}^{1} \frac{t\, e^{-\kappa t}}{\sqrt{1-t^2}}\, dt$$

$$+ i\pi\sqrt{\kappa}\, e^{-\kappa} \int_1^\infty \frac{t\, e^{-\kappa t}}{\sqrt{t^2-1}}\, dt \quad . \qquad (4.39)$$

The two integrals on the right of (39) are $-\pi I_1(\kappa)$ and $K_1(\kappa)$. Finally, then

$$\int_0^\infty \frac{\sqrt{\xi}\, e^{-\xi} K_1(\xi)}{\xi-\kappa}\, d\xi = -\pi\sqrt{\frac{\pi}{2}} + \pi^2\sqrt{\kappa}\, e^{-\kappa} I_1(\kappa) + i\pi\sqrt{\kappa}\, e^{-\kappa} K_1(\kappa) \quad ,$$

and (37) reduces to

$$A\sqrt{\pi} = e^{-\kappa}\sqrt{\kappa} - A\sqrt{2}\left[-\sqrt{\frac{\pi}{2}} + \pi\sqrt{\kappa}\, e^{-\kappa} I_1(\kappa) + i\sqrt{\kappa}\, e^{-\kappa} K_1(\kappa)\right] \quad ,$$

or

$$A = \frac{1}{\sqrt{2}\,[\pi I_1(\kappa) + i K_1(\kappa)]} \quad . \qquad (4.40)$$

The function $\bar{v}$ is obtained from (13), (36) and (40), namely

$$\bar{v}(-it) = \frac{1}{\pi I_1(\kappa) + i K_1(\kappa)} \frac{t e^t K_1(t)}{t + \kappa} . \tag{4.41}$$

By (2.21) and (9),

$$R = 1 - T = 1 - 2i e^{-\kappa} \bar{v}(-i\kappa) \tag{4.42}$$

$$= \frac{\pi I_1(\kappa)}{\pi I_1(\kappa) + i K_1(\kappa)} , \tag{4.43}$$

which agrees with (2.38).

§ 5. INTEGRAL EQUATION APPROXIMATION METHOD.

Equation (4.42) shows that the reflection and transmission coefficients are directly related to the velocity transform $\bar{v}$, with argument $-i\kappa$; the transform is determined by an integral equation, which takes the forms (4.12), (4.12') according as the parameter ϵ is different from or equal to zero. It is noteworthy that the integral equation evidences an inhomogeneous character, and moreover that the inhomogeneous member can be identified with an approximation to the original integral equation (4.5), wherein the kernel is given by the first term of (4.6), namely the Green's function of a totally unbounded region, omitting a free surface boundary condition. Thus, an iterative procedure for calculation of the velocity transform may be expected to yield useful information when the lower edge of the barrier is deeply submerged. In order to assess the significance of the results, consider the approximate form of the reflection coefficient (2.38) when κ is large. By employing the asymptotic expansions for the cylinder functions,

$$R = 1 - \frac{i\,K_1(\kappa)}{\pi\,I_1(\kappa) + i\,K_1(\kappa)} \tag{5.1}$$

$$\simeq 1 - i\,e^{-2\kappa}\left(1 + \frac{3}{4\kappa} + \cdots\right), \qquad \kappa \gg 1 . \tag{5.2}$$

Using only the inhomogeneous term of (4.12'), $\bar{v}(-i\kappa) = e^{-\kappa}/2$, and it follows from (4.42) that the corresponding reflection coefficient agrees with the first two terms of (2). A first iteration in (4.12') yields

$$\bar{v}(-i\kappa) = \frac{1}{2}\,e^{-\kappa}\left[1 - 2\,\frac{\kappa}{\pi}\int_0^\infty \frac{e^{-2\xi}}{\xi^2 - \kappa^2}\,d\xi\right], \tag{5.3}$$

whence

$$R = 1 - i\,e^{-2\kappa}\left(1 + \frac{1}{\pi\kappa} + \cdots\right), \tag{5.4}$$

and there is a discrepancy with (2) in numerical coefficient of the term in $1/\kappa$. Further iterations supply contributions to reduce the gap between $\frac{1}{\pi}$ and $\frac{3}{4}$, though the convergence is rather slow. This aspect diminishes the utility of the iteration procedure, but the concern is only slight, since it has been shown in the preceding section that the transform integral equation can otherwise be solved explicitly.

The procedure is restored to importance, however, when the problem involves oblique wave incidence, and the transform integral equation has the form (4.12), which cannot be solved directly.

Let the incident wave normal make an angle α with respect to the x - direction, so that the time independent part of the velocity potential is

$$\varphi^{inc} = e^{i\kappa(x \cos \alpha + z \sin \alpha) - \kappa y} . \tag{5.5}$$

Then, a rigid vertical barrier in the plane $x = 0$, which extends from $z = -\infty$ to $z = +\infty$, between $y = 0$ and $y = 1$, will scatter the incident wave, without altering the z dependent factor. The complete velocity potential can therefore be represented by

$$\varphi(x,y,z) = \psi(x,y)e^{i\kappa z \sin \alpha} , \tag{5.6}$$

where

$$\psi_{xx} + \psi_{yy} - (\kappa \sin \alpha)^2 \psi = 0 , \qquad y > 0 \tag{5.7}$$

$$(\partial_y + \kappa)\psi = 0 , \qquad y = 0 \tag{5.8}$$

$$\partial_x \psi = 0 , \qquad x = 0, \quad 0 < y < 1 \tag{5.9}$$

and

$$\psi \sim e^{i\kappa x \cos\alpha - \kappa y} + R\, e^{-i\kappa x \cos\alpha - \kappa y}\,, \qquad x \to -\infty$$

$$\sim T\, e^{i\kappa x \cos\alpha - \kappa y}\,, \qquad x \to +\infty \quad . \tag{5.10}$$

An integral equation reformulation of this boundary value problem is carried out as in the particular case of normal incidence $(\alpha = 0)$, except that the Green's function pertains to the differential equation (4.1), and the explicit parameter $\epsilon = \kappa \sin\alpha$. With such adaptation, equations (4.42) and (4.12) serve to characterize the reflection coefficient R, namely

$$R = 1 - 2i\, e^{-\kappa}\, \bar{v}(-i\kappa)\,, \tag{5.11}$$

where

$$\frac{\bar{v}(-it)}{\sqrt{t+\kappa\sin\alpha}} = \frac{e^{-\kappa}\sqrt{\kappa(1+\sin\alpha)}}{t+\kappa} - \frac{1}{\pi}\int_{\kappa\sin\alpha}^{\infty} \frac{\xi+\kappa}{\xi-\kappa}\, \frac{\bar{v}(-i\xi)e^{-2\xi}}{(t+\xi)\sqrt{\xi-\kappa\sin\alpha}}\, d\xi \ . \tag{5.12}$$

One iteration of the integral equation (12) yields

$$\bar{v}(-i\kappa) = \frac{1+\sin\alpha}{2}\, e^{-\kappa}\{1 - \frac{2\kappa}{\pi}\int_{\kappa\sin\alpha}^{\infty} \frac{e^{-2\xi}}{\xi^2-\kappa^2}\sqrt{\frac{\xi+\kappa\sin\alpha}{\xi-\kappa\sin\alpha}}\, d\xi\}$$

$$\simeq \frac{1+\sin\alpha}{2}\, e^{-\kappa}\{1 + \frac{2}{\sqrt{\pi\kappa}}\, \frac{\sqrt{\sin\alpha}}{\cos^2\alpha}\, e^{-2\kappa\sin\alpha} + \cdots\}\,, \quad \kappa\sin\alpha \gg 1 \tag{5.13}$$

and thus

$$R = 1 - i(1+ \sin\alpha)e^{-2\kappa}\left(1+ \frac{2}{\sqrt{\pi\kappa}}\ \frac{\sqrt{\sin\alpha}}{\cos^2\alpha}\, e^{-2\kappa\sin\alpha} + \cdots\right)\ ,\quad \kappa\sin\alpha \gg 1\ . \tag{5.14}$$

In contrast with (4), the second term in the parentheses above is of an exponential order relative to the first, and subsequent iteration will generate distinct lower order terms.

§ 6. TWO PARALLEL BARRIERS.

The conformal mapping method may be applied to a two dimensional flow with any number of vertical barriers, of various depths and separations, but the details become somewhat cumbersome. The simplest case after that of § 2 is two barriers of the same depth, which will now be treated.

Let the barriers lie in the planes $x = a$ and $x = -a (a > 0)$, each occupying the strip $0 < y < 1$. The trace of the flow region in the xy - plane is the image of the upper half of the complex ζ - plane under a Schwarz - Christoffel transformation, which involves an elliptic integral for this geometry.

In the following, z will denote the complex variable $x + iy$. Normalize the mapping from ζ to z by the conditions

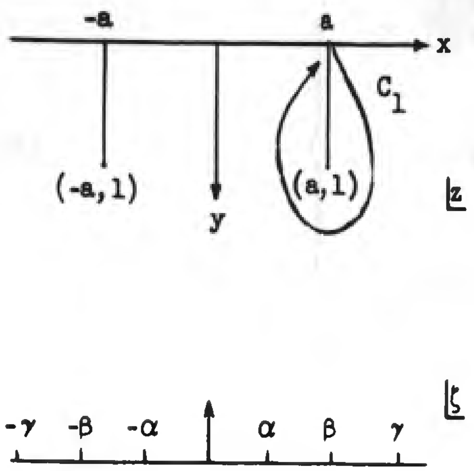

Figure 2.

$$z(0) = 0 \ ,$$

$$z(\infty) = \infty \ ,$$

and $$z'(0) = 1 \ .$$

Then there are six symmetrically located points on the real ζ-axis which are mapped into the corners of the image region. The points are so named that (Figure 2)

$$\begin{array}{ll} \alpha \to a - , & -\alpha \to -a + \\ \beta \to a + i \ , & -\beta \to -a + i \ , \\ \gamma \to a + & -\gamma \to -a - \ . \end{array} \tag{6.1}$$

In terms of these quantities

$$z = \frac{\alpha\gamma}{\beta^2} \int_0^{\zeta} \frac{\beta^2 - t^2}{\sqrt{(\alpha^2-t^2)(\gamma^2-t^2)}} \, dt \ , \tag{6.2}$$

where $\operatorname{Im} t > 0$ along the path of integration, and that branch of the square root is used which is positive at $t = 0$.

α, β and γ are determined by (1). Setting $\zeta = \alpha, \beta, \gamma$ in (2) provides the characteristic system of equations, expressed in terms of standard elliptic integrals:

$$\left.\begin{aligned} \frac{\beta}{\gamma} &= \sqrt{\frac{E(k)}{K(k)}} \ , \\ \alpha[K(k)E(k,\vartheta) - E(k)F(k,\vartheta)] &= E(k) \ , \\ \frac{\pi}{2}\alpha &= a\,E(k) \ , \end{aligned}\right\} \qquad (6.3)$$

where

$$\left.\begin{aligned} k &= \sqrt{1 - \alpha^2/\gamma^2} \ , \\ \vartheta &= \sin^{-1}\frac{1}{k}\sqrt{1-\beta^2/\gamma^2} \ . \end{aligned}\right\} \qquad (6.4)$$

The usual notation for the complete and incomplete elliptic integrals of the first and second kinds has been employed.

The potential φ is a solution of Laplace's equation in the half plane $y > 0$ with cuts $x = \pm a$, $0 < y \leq 1$, and satisfies the boundary conditions

$$\left.\begin{aligned} (\partial_y + \kappa)\varphi &= 0 \ , \qquad y = 0 \\ \partial_x \varphi &= 0 \ , \qquad x = \pm a \ , \quad 0 < y < 1 \ . \end{aligned}\right\} \qquad (6.5)$$

The asymptotic behavior of φ is again given by (2.8). From this it follows that

$$\psi = \partial_x(\partial_y + \kappa)\varphi \qquad (6.6)$$

is a harmonic function in the cut half plane which approaches zero at ∞ and is zero on the boundary, except for singularities at $z = \pm a + i$, the edges of the barriers. Near these points,

$$\psi = O(|z \mp a - i|^{-3/2}) \quad .$$

Transforming to the ζ - plane by (2), ψ is a harmonic function in $\operatorname{Im} \zeta > 0$, which is zero on the real axis and at ∞. It has poles at $\zeta = \pm \beta$, where

$$\psi = O(|\zeta \mp \beta|^{-3}) \quad .$$

If the constants A, B, C and D are properly chosen,

$$\psi = A \operatorname{Im} \frac{1}{\zeta-\beta} + B \operatorname{Im} \frac{1}{\zeta+\beta} + C \operatorname{Im}\left[\frac{1}{(\zeta-\beta)^3} + \frac{p}{(\zeta-\beta)^2}\right] + D \operatorname{Im}\left[\frac{1}{(\zeta+\beta)^3} - \frac{p}{(\zeta+\beta)^2}\right] \tag{6.7}$$

with

$$p = \frac{\alpha^2\gamma^2 + \beta^2(\alpha^2 + \gamma^2) - 3\beta^2}{2\beta(\gamma^2 - \beta^2)(\beta^2 - \alpha^2)} \quad . \tag{6.8}$$

The expressions in brackets following C and D are proportional to the leading terms in the expansions of $(z - a - i)^{-3/2}$ about $a + i$, and $(z + a - i)^{-3/2}$ about $-a + i$, respectively. These combinations must be used to avoid logarithmic terms in φ.

In the following, $\psi(x,y)$ will denote the function in the xy - plane, determined by (7) and (2) . It is convenient to perform the operations of taking real and imaginary parts in a formal way, treating A, B, C and D as real numbers. If this is done,

$$\psi(x,y) = \operatorname{Im} f(z) \quad , \tag{6.9}$$

where

$$f(z) = \frac{A}{\zeta-\beta} + \frac{B}{\zeta+\beta} + C\left[\frac{1}{(\zeta-\beta)^3} + \frac{p}{(\zeta-\beta)^2}\right] + D\left[\frac{1}{(\zeta+\beta)^3} \quad \frac{p}{(\zeta+\beta)^2}\right] \; , \tag{6.10}$$

z and ζ being related by (2).

Integrate (6) separately in each of the regions $x > a$, $x < -a$, and $-a < x < a$. This yields

$$\varphi(x,y) = e^{-\kappa y} \int_0^y e^{\kappa v}\, dv \int_{\pm\infty}^{x} \psi(u,v)du + e^{-\kappa y} \lambda_{\pm}(x) \quad \begin{vmatrix} x > a \\ x < -a \end{vmatrix} \tag{6.11}$$

and

$$\varphi(x,y) = e^{-\kappa y} \int_0^y e^{\kappa v}\, dv \int_0^x \psi(u,v)du + e^{-\kappa y}\mu(x) + \nu(y) \; , \; -a < x < a \; . \tag{6.12}$$

The functions $\lambda_{\pm}(x)$, $\mu(x)$ and $\nu(y)$ enter as constants of integration. They satisfy ordinary differential equations which are obtained from the relation $\nabla^2\varphi = 0$. Employing (11) ,

$$\left(\frac{d^2}{dx^2} + \kappa^2\right)\lambda_{\pm}(x) = - \int_{\pm\infty}^{x} \psi_y(u,0)du$$

or, by (9),

$$\left(\frac{d^2}{dx^2} + \kappa^2\right)\lambda_{\pm}(x) = - f(x) , \quad \begin{vmatrix} x > a \\ x < - a \end{vmatrix} . \tag{6.13}$$

From (12),

$$\left(\frac{d^2}{dx^2} + \kappa^2\right)\mu(x) + \int_0^x \psi_y(u,0)du + \int_0^y e^{\kappa v} \psi_x(0,v)dv + e^{\kappa y} \nu''(y) = 0 , \quad -a < x < a ,$$

and, using (9) again to eliminate ψ,

$$\left(\frac{d^2}{dx^2} + \kappa^2\right)\mu(x) + f(x) - e^{\kappa y} \operatorname{Re} f(iy) + \kappa \int_0^y e^{\kappa v} \operatorname{Re} f(iv)dv + e^{\kappa y} \nu''(y) = 0 .$$

According to (12), $\mu(x)$ is only determined up to an additive constant. This constant may be fixed by placing

$$\left(\frac{d^2}{dx^2} + \kappa^2\right)\mu(x) + f(x) = 0 , \quad - a < x < a \tag{6.14}$$

and then

$$\frac{d^2}{dy^2} \nu(y) - \operatorname{Re} f(iy) + \kappa e^{-\kappa y} \int_0^y e^{\kappa v} \operatorname{Re} f(iv)dv = 0 . \tag{6.15}$$

The functions $\lambda_{\pm}(x)$ are uniquely determined by (13) and (2.8), which imply

$$\lambda_{\pm}(x) = -\frac{1}{\kappa}\int_{\pm\infty}^{x} \sin\kappa(x-u)f(u)du + \begin{Bmatrix} T\,e^{i\kappa x} \\ e^{i\kappa x} + R\,e^{-i\kappa x} \end{Bmatrix} , \quad \begin{vmatrix} x > a \\ x < -a \end{vmatrix} . \tag{6.16}$$

The second of the conditions (5) ,

$$\partial_x\varphi = 0 , \quad x = \pm a , \quad 0 < y < 1 ,$$

when applied to (11), (12) yield

$$\lambda'_{\pm}(\pm a) = 0 , \quad \mu'(\pm a) = 0 \quad ; \tag{6.17}$$

consequently, the relationship of R and T with f may be obtained by differentiating (16), and setting $x = \pm a$:

$$\left.\begin{aligned} T &= -\frac{1}{i\kappa}e^{-i\kappa a}\int_a^{\infty}\cos\kappa(a-x)f(x)dx , \\ R &= e^{-2i\kappa a} - \frac{1}{i\kappa}e^{-i\kappa a}\int_{-\infty}^{-a}\cos\kappa(a+x)f(x)dx . \end{aligned}\right\} \tag{6.18}$$

(17) also furnishes boundary conditions for (14), whose solution is

$$\mu(x) = -\frac{1}{\kappa}\int_0^x \sin\kappa(x-u)f(u)du - \frac{\cos\kappa(a+x)}{\kappa\sin 2\kappa a}\int_0^a \cos\kappa(a-u)f(u)du$$

$$- \frac{\cos\kappa(a-x)}{\kappa\sin 2\kappa a}\int_{-a}^{0}\cos\kappa(a+u)f(u)du , \quad -a < x < a \tag{6.19}$$

if $2\kappa a$ is not a multiple of π.

The first part of (5),

$$(\partial_y + \kappa)\varphi = 0 , \qquad y = 0$$

is automatically satisfied for $|x| > a$. On $|x| < a$, applying this condition to (12) yields

$$\nu'(0) + \kappa\, \nu(0) = 0 \quad .$$

The solution of (15) subject to this condition is

$$\nu(y) = \frac{1}{\kappa} \int_0^y (1 - e^{-\kappa y + \kappa v}) \operatorname{Re} f(iv)dv + A_0(1 - \kappa y) \quad , \tag{6.20}$$

where A_0 is a constant to be determined. By (2),

$$\zeta \sim \frac{\beta^2}{\alpha\gamma} z \; , \qquad z \to \infty \quad . \tag{6.21}$$

Using this in (10),

$$\operatorname{Re} f(iy) = O(\frac{1}{y^2}) \; , \qquad y \to \infty \quad .$$

Hence, the integral in (20) approaches a finite limit as $y \to \infty$. For $\varphi(x,y)$ to approach zero as $y \to \infty$, it is necessary that $\nu(y) \to 0$. To make $\nu(y)$ bounded at ∞, A_0 must be zero, and then (20) becomes

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$$v(y) = \frac{1}{\kappa} \int_0^y (1 - e^{-\kappa y + \kappa v}) \mathrm{Re}\, f(iv)dv \quad . \tag{6.22}$$

Moreover, $v(\infty) = 0$ if the additional condition

$$\int_0^\infty \mathrm{Re}\, f(iy)dy = 0 \tag{6.23}$$

is satisfied. This may be written out in detail as an integral in the ζ - plane, by (2) and (10). With $\eta = \mathrm{Im}\,\zeta$, it becomes

$$\int_0^\infty \left\{ (B - A)\beta + (C - D) \left[\frac{3\beta\eta^2 - \beta^3}{(\eta^2 + \beta^2)^2} - p\, \frac{\eta^2 - \beta^2}{\eta^2 + \beta^2} \right] \right\} \frac{d\eta}{\sqrt{(\eta^2 + \alpha^2)(\eta^2 + \gamma^2)}} = 0 \ .$$

The value of p is such that after an integration by parts the preceding relation can be expressed in terms of complete elliptic integrals as

$$A - B = \frac{C - D}{(\gamma^2 - \beta^2)(\beta^2 - \alpha^2)} \left[2\gamma^2 \, \frac{E(k)}{K(k)} - \frac{(\gamma^2 + \beta^2)(\beta^2 + \alpha^2)}{2\beta^2} \right] ,$$

where k is given by (4), and thus, from (3) and (8),

$$A - B = - \frac{p}{\beta} (C - D) \quad . \tag{6.24}$$

Additional equations to determine the coefficients of $f(z)$ will be obtained by equating the values given to φ by (11) and (12) on each of the lines $x = \pm a (y > 1)$. Integrating by parts in each equation and applying (9), (22) and (23) brings them to the more useful forms

$$\varphi(x,y) = \frac{1}{\kappa}\int_{\pm\infty}^{x} \psi(u,y)du - \frac{1}{\kappa} e^{-\kappa y} \int_0^y e^{\kappa v} \operatorname{Re} f(x+iv)dv + e^{-\kappa y} \lambda_{\pm}(x) \;, \quad \begin{pmatrix} x > a \\ x < -a \end{pmatrix} \tag{6.25}$$

and

$$\varphi(x,y) = \frac{1}{\kappa}\int_0^x \psi(u,y)du - \frac{1}{\kappa} e^{-\kappa y}\int_0^y e^{\kappa v} \operatorname{Re} f(x+iv)dv - \frac{1}{\kappa}\int_y^\infty \operatorname{Re} f(iv)dv + e^{-\kappa y} \mu(x) \;,$$

$$-a < x < a \;. \tag{6.26}$$

Let $x \to a$ in (26) and the first equation of (25), with $y > 1$, and equate their difference to zero, whence

$$-\frac{1}{\kappa}\int_0^\infty \psi(u,y)du + \frac{1}{\kappa}\int_y^\infty \operatorname{Re} f(iv)dv - \frac{e^{-\kappa y}}{\kappa} \lim_{\epsilon \to 0+} \int_0^y e^{\kappa v} \operatorname{Re}[f(a+\epsilon+iv) - f(a-\epsilon+iv)]dv$$

$$+ e^{-\kappa y}[\lambda_+(a) - \mu(a)] = 0 \;, \qquad y > 1 \;. \tag{6.27}$$

The limit in the third term is constant for all $y > 1$, since as $\epsilon \to 0$ the integrand approaches zero uniformly for $1 < y_0 < v < y$. Thus the first two terms taken together must approach zero as $y \to \infty$. These terms combine to

$$\frac{1}{\kappa}\int_{+\infty+iy}^{i\infty} \operatorname{Im}(f(z)dz) \;.$$

By shifting the path of integration far away from the origin and using (21) and (10) ,

$$\frac{1}{\kappa}\int_{+\infty+iy}^{i\infty} \mathrm{Im}(f(z)dz) = \frac{1}{\kappa}\frac{\beta^2}{\alpha\gamma}\int_{+\infty+iy}^{i\infty} \mathrm{Im}(f(z)d\zeta)$$

$$= \frac{1}{\kappa}\frac{\beta^2}{\alpha\gamma}\frac{\pi}{2}(A+B) \quad .$$

Hence,

$$A + B = 0 \quad . \tag{6.28}$$

Now the sum of the first two terms in (27) is zero, so that

$$\lambda_+(a) - \mu(a) = \frac{1}{\kappa}\lim_{\epsilon\to 0+}\int_0^y e^{\kappa v}\,\mathrm{Re}[f(a+\epsilon+iv) - f(a-\epsilon+iv)]dv \quad . \tag{6.29}$$

The limit may not be taken under the integral sign because of the poor behavior of the integrand at $v = 1$. Let us rewrite the integral as follows:

$$\int_0^y e^{\kappa v}\,\mathrm{Re}[f(a+\epsilon+iv) - f(a-\epsilon+iv)]dv$$

$$= \int_0^y (e^{\kappa v}-e^{\kappa})\mathrm{Re}[f(a+\epsilon+iv)-f(a-\epsilon+iv)]dv + e^{\kappa}\,\mathrm{Im}\left[\int_{a+\epsilon}^{a+\epsilon+iy} f(z)dz + \int_{a-\epsilon+iy}^{a-\epsilon} f(z)dz\right] \quad . \tag{6.30}$$

In the first integral on the right the limit $\epsilon\to 0$ may be taken inside, for the integrand is $O(|v-1|^{-1/2})$. The remaining integrals are independent of path, so that they may be carried out along curves which bend around $z = a + i$ when $\epsilon = 0$. Inserting (30) in (29) reduces it to

$$\lambda_+(a) - \mu(a) = \frac{1}{\kappa}\int_0^1 (e^{\kappa v} - e^{\kappa})[f(a+0+iv) - f(a-0+iv)]dv + \frac{1}{\kappa} e^{\kappa} \operatorname{Im} \int_{C_1} f(z)dz , \quad (6.31)$$

where C_1 is a curve from a+ to a- in the cut half plane occupied by the fluid (Figure 2). A direct computation in the ζ - plane shows that by virtue of (24) and (28),

$$\int_{C_1} f(z)dz = 0 . \quad (6.32)$$

The first integral on the right in (31) can be rewritten as an integral over C_1 . For convenience later, another multiple of the integral in (32) is included, giving

$$\lambda_+(a) - \mu(a) = \frac{1}{i\kappa}\int_{C_1} (e^{-i\kappa z + i\kappa a} - 2 \cosh \kappa) f(z)dz .$$

The left side of this equation may be expressed in terms of integrals of f by (16), (18) and (19). After some simplification, the equation becomes

$$\frac{e^{-2i\kappa a}}{\sin 2\kappa a}\int_0^a \cos \kappa(a-x)f(x)dx + \frac{1}{\sin 2\kappa a}\int_{-a}^0 \cos \kappa(a+x)f(x)dx$$

$$- e^{-i\kappa a}\int_0^\infty e^{-\kappa y} f(iy)dy + 2i\int_{C_1} [\cos \kappa(z-a) - \cosh \kappa]f(z)dz = 0 . \quad (6.33)$$

A similar relation may be obtained by requiring continuity of φ across $x = -a$, $y > 1$. The incident part of φ gives a term not involving f . This equation is most conveniently written in terms of the extension of $f(z)$ to an analytic function in the whole z - plane with vertical cuts from $a+i$ to $a-i$ and from $-a+i$ to $-a-i$. The equation corresponding to (33) is

$$\frac{e^{-2i\kappa a}}{\sin 2\kappa a}\int_0^a \cos \kappa(a-x)f(-x)dx + \frac{1}{\sin 2\kappa a}\int_{-a}^0 \cos \kappa(a+x)f(-x)dx$$

$$- e^{-i\kappa a}\int_0^\infty e^{-\kappa y} f(-iy)dy + 2i\int_{C_1}[\cos \kappa(z-a) - \cosh \kappa]f(-z)dz = -2\kappa\, e^{-i\kappa a} . \tag{6.34}$$

Note that the left sides of (33) and (34) are the same except for replacement of $f(z)$ by $f(-z)$ in the latter. The sum and difference of these equations are accordingly equations for the even and odd parts of $f(z)$. By (10), (24) and (28), if

$$\left.\begin{aligned} f_1(z) &= \frac{3\beta\zeta^2 + \beta^3}{(\zeta^2-\beta^2)^3} + \frac{2p\,\beta^2}{(\zeta^2-\beta^2)^2} , \\ &\text{and} \\ f_2(z) &= \frac{\zeta^3 + 3\beta^2\zeta}{(\zeta^2-\beta^2)^3} + \frac{2p\,\beta\,\zeta}{(\zeta^2-\beta^2)^2} , \end{aligned}\right\} \tag{6.35}$$

then

$$f(z) = (C-D)f_1(z) + (C+D)f_2(z) \quad . \tag{6.36}$$

This is the decomposition of f into even and odd functions. Using the symmetry of f_1 and f_2 to effect simplifications, the equations which follow from (33) and (34) are

$$\left.\begin{aligned} (C-D)\left[\frac{e^{-i\kappa a}}{\sin \kappa a} I_1(1) - e^{-i\kappa a} I_2(1) + 2i\, I_3(1)\right] &= -\kappa\, e^{-i\kappa a} \ , \\ (C+D)\left[\frac{e^{-i\kappa a}}{i \cos \kappa a} I_1(2) - e^{-i\kappa a} I_2(2) + 2i\, I_3(2)\right] &= +\kappa\, e^{-i\kappa a} \ , \end{aligned}\right\} \tag{6.37}$$

where

$$\left.\begin{aligned} I_1(j) &= \int_0^a \cos \kappa(a-x) f_j(x)dx \\ I_2(j) &= \int_0^\infty e^{-\kappa y} f_j(iy)dy \\ I_3(j) &= \int_{C_1} [\cos \kappa(a-z) - \cosh \kappa] f_j(z)dz \end{aligned}\right\} \quad j = 1,2 \ . \tag{6.38}$$

By (18),

$$T = -\frac{1}{i\kappa} e^{-i\kappa a} \operatorname{Re} \int_a^\infty e^{-i\kappa a + i\kappa x} f(x)dx \quad ,$$

and when the integration path is replaced by a combination of those in (38),

$$T = -\frac{1}{i\kappa} e^{-i\kappa a} \{ -\int_0^a \cos\kappa\,(a-x) f(x)dx + \operatorname{Re} i\, e^{-i\kappa a} \int_0^\infty e^{-\kappa y} f(iy)dy$$

$$+ \operatorname{Re} \int_{C_1} e^{-i\kappa a + i\kappa z} f(z)dz \} \quad . \tag{6.39}$$

Equation (32) allows the last integral to be put in the form

$$\int_{C_1} [e^{-i\kappa a + i\kappa z} - e^{-\kappa}] f(z)dz \quad ,$$

and now C_1 may be contracted to the barrier, where $\operatorname{Im} f(z) = 0$. Hence, the last term of (39) vanishes. By (36) and (38),

$$T = \frac{1}{i\kappa} e^{-i\kappa a} \{ (C-D)[I_1(1) - \sin\kappa a\, I_2(1)] + (C+D)[I_1(2) - i\cos\kappa a\, I_2(2)] \} \quad . \tag{6.40}$$

The elimination of $C \mp D$ by (37) yields

$$T = i\, e^{-i\kappa a} \left\{ \frac{\sin\kappa a [I_1(1) - \sin\kappa a\, I_2(1)]}{I_1(1) - \sin\kappa a\, I_2(1) + 2i\, e^{i\kappa a} \sin\kappa a\, I_3(1)} \right.$$

$$\left. - \frac{i\cos\kappa a [I_1(2) - i\cos\kappa a\, I_2(2)]}{I_1(2) - i\cos\kappa a\, I_2(2) - 2\, e^{i\kappa a} \cos\kappa a\, I_3(2)} \right\} \tag{6.41}$$

and similarly

$$R = e^{-2i\kappa a} + i e^{-i\kappa a}\left\{\frac{\sin \kappa a[I_1(1) - \sin \kappa a\ I_2(1)]}{I_1(1) - \sin \kappa a\ I_2(1) + 2i\ e^{i\kappa a} \sin \kappa a\ I_3(1)}\right.$$

$$\left. + \frac{i \cos \kappa a[I_1(2) - i \cos \kappa a\ I_2(2)]}{I_1(2) - i \cos \kappa a\ I_2(2) - 2\ e^{i\kappa a} \cos \kappa a\ I_3(2)}\right\} . \quad (6.42)$$

The six integrals in (41) and (42) can be evaluated numerically. In the ζ-plane they are integrals over the intervals $(0,\alpha)$, $(0, i\,\infty)$ and (α,γ) . The trigonometric and exponential factors in the integrals depend on values of x or y which are given by the following formulas, deduced from (2):

$$\left.\begin{array}{ll}
I_1:\ x = \alpha[\frac{\gamma^2}{\beta^2} E(\frac{\alpha}{\gamma}, \sin^{-1} \frac{\zeta}{\alpha}) - (\frac{\gamma^2}{\beta^2} - 1) F(\frac{\alpha}{\gamma}, \sin^{-1} \frac{\zeta}{\alpha})] , & 0 < \zeta < \alpha \\
I_2:\ y = \frac{\alpha\gamma}{\beta^2} \eta \sqrt{\frac{\eta^2+\gamma^2}{\eta^2+\alpha^2}} + \alpha\, F(k, \tan^{-1} \frac{\eta}{\alpha}) - \alpha \frac{\gamma^2}{\beta^2} E(k, \tan^{-1} \frac{\eta}{\alpha}) , & 0 < \eta < \infty \\
\qquad \text{where} \quad \eta = \operatorname{Im} \zeta = -i\zeta & \\
I_3:\ y = \alpha\left[\frac{\gamma^2}{\beta^2} E(k, \sin^{-1} \sqrt{\frac{\gamma^2-\zeta^2}{\gamma^2-\alpha^2}}) - F(k, \sin^{-1} \sqrt{\frac{\gamma^2-\zeta^2}{\gamma^2-\alpha^2}})\right] , & \alpha < \zeta < \gamma
\end{array}\right\} \quad (6.43)$$

For computation, C_1 is contracted to the barrier, reducing $\cos \kappa(z-a)$ to $\cosh \kappa y$.

The case in which $2\kappa a$ is a multiple of π calls for separate consideration, as (19), (33), (34) and (37) require modification. The final formulas (41) and (42) remain valid, and also equations (35) and (38) which define the integrals $I_1(1), \cdots, I_3(2)$. Corresponding fractions drop out in (41) and (42), so that R and T satisfy a linear relation similar to that of (2.20). If $2\kappa a = n\pi$,

$$(-1)^n R = 1 - T \quad . \tag{6.44}$$

(41) and (42) may be put into another form which, while less suited to numerical calculations, is more convenient for other purposes. Notice that the combinations $I_1(1) - \sin \kappa a \, I_2(1)$ and $I_1(2) - i \cos \kappa a \, I_2(2)$ which appear in (41) and (42) are each the real part of a single complex integral from $i\infty$ to $a-$. These may be replaced by integrals from $a+$ to $+\infty$. If

$$K_j(\kappa, a) = \frac{2}{\kappa} \int_a^\infty \cos \kappa(x-a) f_j(x) dx \, , \qquad j = 1,2 \tag{6.45}$$

then

and

$$\left.\begin{aligned} I_1(1) - \sin \kappa a \, I_2(1) &= -\tfrac{1}{2} \kappa \, K_1(\kappa, a) \\ I_1(2) - i \cos \kappa a \, I_2(2) &= -\tfrac{1}{2} \kappa \, K_2(\kappa, a) \end{aligned}\right\} \quad . \tag{6.46}$$

Define also

$$I_j(\kappa,a) = \frac{2i}{\pi\kappa}\int_{C_1} \cos\kappa(z-a) f_j(z)dz = \frac{2i}{\pi\kappa} I_3(j) \;, \qquad (6.47)$$

and the scattering coefficients T, R take the forms

$$\left.\begin{aligned} T &= i\,e^{-i\kappa a}\left[\frac{\sin\kappa a\, K_1(\kappa,a)}{K_1(\kappa,a)-2\pi\, e^{i\kappa a}\sin\kappa a\, I_1(\kappa,a)} - \frac{i\cos\kappa a\, K_2(\kappa,a)}{K_2(\kappa,a)-2\pi i\, e^{i\kappa a}\cos\kappa a\, I_2(\kappa,a)}\right] \\ R &= e^{-2i\kappa a} + i\,e^{-i\kappa a}\left[\frac{\sin\kappa a\, K_1(\kappa,a)}{K_1(\kappa,a)-2\pi\, e^{i\kappa a}\sin\kappa a\, I_1(\kappa,a)} + \frac{i\cos\kappa a\, K_2(\kappa,a)}{K_2(\kappa,a)-2\pi i\, e^{i\kappa a}\cos\kappa a\, I_2(\kappa,a)}\right] \end{aligned}\right\} (6.48)$$

The quantities $K_j(\kappa,a)$ and $I_j(\kappa,a)$ approach the modified Bessel functions $K_1(\kappa)$ and $I_1(\kappa)$, respectively, when $a \to \infty$. To obtain their limiting forms, observe that (3) gives the following information for large a :

$$\left.\begin{aligned} \beta &= a + \frac{5}{4a} + O(\frac{1}{a^3}) \;, \\ \left.\begin{matrix}\gamma-\beta\\ \beta-\alpha\end{matrix}\right\} &= 1 + \frac{7}{8a^2} + O(\frac{1}{a^3}) \;. \end{aligned}\right. \qquad (6.49)$$

These relations allow (2) to be expanded in inverse powers of a, for z on C_1 or (a,∞). The result may be solved for ζ, to yield

$$\zeta = \beta + \sqrt{1+(z-a)^2}\left\{1+ \frac{7}{8a^2} + \frac{1}{8a^2}\, \frac{(z-a)^2}{\sqrt{1+(z-a)^2}\,[2a+\sqrt{1+(z-a)^2}]} + 0(\frac{1}{a^3})\right\} .$$

If this is substituted in (35), (45) and (47) may be reduced to

$$\left.\begin{aligned} K_j(\kappa,a) &= K_1(\kappa)[1 - \frac{21}{8a^2} + 0(\frac{1}{a^3})] , \\ I_j(\kappa,a) &= I_1(\kappa)[1 - \frac{21}{8a^2} + 0(\frac{1}{a^3})] , \end{aligned}\right\} \qquad j = 1,2 \tag{6.50}$$

assuming that κ is not small. Substituting (50) in (48),

$$T = \frac{(K_1(\kappa))^2 + 0(\frac{1}{a^3})K_1(\kappa)I_1(\kappa)}{[K_1(\kappa)-2\pi\, e^{i\kappa a} \sin \kappa a\; I_1(\kappa)][K_1(\kappa)-2\pi i\, e^{i\kappa a} \cos \kappa a\; I_1(\kappa)]} , \tag{6.51}$$

and

$$R = e^{-2i\kappa a} + \frac{2\pi \sin 2\kappa a\; K_1(\kappa)I_1(\kappa) - e^{2i\kappa a}(K_1(\kappa))^2}{[K_1(\kappa)-2\pi\, e^{i\kappa a} \sin \kappa a\, I_1(\kappa)][K_1(\kappa)-2\pi i\, e^{i\kappa a} \cos \kappa a\, I_1(\kappa)]}[1 + 0(\frac{1}{a^3})] . \tag{6.52}$$

This expression for R can be reduced still further, when κ is large, by noting that

$$\frac{K_1(\kappa)}{I_1(\kappa)} = 0(e^{-2\kappa}) , \qquad \kappa \to \infty ;$$

the result is

$$R = e^{-2i\kappa a}\left\{1 + \frac{C}{\pi i}\,\frac{K_1(\kappa)}{I_1(\kappa)}\,[1 + O(\frac{1}{a^3}) + O(e^{-2\kappa})]\right\} \quad , \qquad (6.53)$$

where

$$C = \begin{cases} 1\,, & \sin 2\kappa a \neq 0 \\ \frac{1}{2}\,, & \sin 2\kappa a = 0 \quad . \end{cases} \qquad (6.54)$$

The behavior of T for large κ and a is obscured by the error term in (51).

A direct expansion of (48) for large κ may also be performed. If the path of integration in the formula (45) for $K_j(\kappa,a)$ is shifted to a vertical line from $a+$ to $i\infty$, then the dominant terms in both $I_j(\kappa,a)$ and $K_j(\kappa,a)$ stem from the parts of the integrals in the neighborhood of $z = a+i$. Here

$$\zeta = \beta + \sqrt{ir[z-(a+i)]} + \frac{1}{3}\,ipr\,[z-(a+i)] + \cdots , \qquad (6.55)$$

with

$$r = \frac{\beta}{\alpha\gamma}\sqrt{(\gamma^2 - \beta^2)(\beta^2 - \alpha^2)} \quad . \qquad (6.56)$$

Applying this expansion to (35),

$$\left.\begin{aligned} \left.\begin{matrix} f_1(z) \\ f_2(z) \end{matrix}\right\} &= \frac{1}{2\{ir[z-(a+i)]\}^{3/2}} + O\left(\frac{1}{|z-(a+i)|^{1/2}}\right) , \\ f_2(z)-f_1(z) &= \frac{p}{2\beta\sqrt{ir[z-(a+i)]}} + O(1) \quad . \end{aligned}\right\} \qquad (6.57)$$

Now, by (45) and (47)

$$
\left.
\begin{aligned}
I_j(\kappa,a) &= \frac{2}{\sqrt{\pi r^3}}\,\frac{e^{\kappa}}{\sqrt{\kappa}}\,[1 + O(\tfrac{1}{\kappa})]\ , \\
K_j(\kappa,a) &= 2\sqrt{\frac{\pi}{r^3}}\,\frac{e^{-\kappa}}{\sqrt{\kappa}}\,[1 + O(\tfrac{1}{\kappa})]\ , \\
I_2(\kappa,a) - I_1(\kappa,a) &= -\frac{p}{\beta\sqrt{\pi r}}\,\frac{e^{\kappa}}{\kappa^{3/2}}\,[1 + O(\tfrac{1}{\kappa})]\ , \\
K_2(\kappa,a) - K_1(\kappa,a) &= \frac{p}{\beta}\sqrt{\frac{\pi}{r}}\,\frac{e^{-\kappa}}{\kappa^{3/2}}\,[1 + O(\tfrac{1}{\kappa})]\ .
\end{aligned}
\right\} \qquad (6.58)
$$

The last two expressions are needed for T if $\sin 2\kappa a \neq 0$. Then T can be written as

$$
T = \frac{e^{-2i\kappa a}}{2\pi i}\;\frac{K_1(\kappa,a)[I_2(\kappa,a) - I_1(\kappa,a)] - I_1(\kappa,a)[K_2(\kappa,a) - K_1(\kappa,a)]}{I_1(\kappa,a)I_2(\kappa,a)} + O(e^{-4\kappa})\ .
$$

For all other cases, only the first two lines of (58) are needed. The result of applying (58) to (48) is

$$
R = \begin{cases}
e^{-2i\kappa a}[1 - i\,e^{-2\kappa} + O(\frac{e^{-2\kappa}}{\kappa})]\ , & \sin 2\kappa a \neq 0 \\[2ex]
e^{-2i\kappa a}[1 - \frac{1}{2}\,i\,e^{-2\kappa} + O(\frac{e^{-2\kappa}}{\kappa})], & \sin 2\kappa a = 0
\end{cases} \qquad (6.59)
$$

$$T = \begin{cases} \frac{1}{2} i\, e^{-2i\kappa a} \frac{pr}{\beta} \frac{e^{-2\kappa}}{\kappa} [1 + O(\frac{1}{\kappa})] , & \sin 2\kappa a \neq 0 \\ \frac{1}{2} i\, e^{-2\kappa} [1 + O(\frac{1}{\kappa})] , & \sin 2\kappa a = 0 . \end{cases} \tag{6.60}$$

Note that if $\sin 2\kappa a \neq 0$, the value of R given by (59) and the more accurate value for large a given by (53) agree with the result (5.2) for the single barrier, except for the factor $e^{-2i\kappa a}$, which is due to the displacement of the first barrier from $x = 0$. This is to be expected, since the amplitude of the waves below $y = 1$ decreases exponentially with increasing κ .

§ 7. A CORRECTION FOR FINITE DEPTH OF THE FLUID.

According to (5.2), the reflection coefficient of a single barrier with depth d takes the form

$$R \simeq 1 - i\, e^{-2\kappa d} + O(\frac{e^{-2\kappa d}}{\kappa d}) , \tag{7.1}$$

when $\kappa d \gg 1$. The contribution, $- i\, e^{-2\kappa d}$, stems from an approximation to the integral equation for the normal fluid velocity below the barrier in which the kernel or Green's function is that appropriate to a totally unbounded region. This suggests a corresponding approximation in the case of finite fluid depth $(h > d$, Figure 3) , with a rigid bottom,

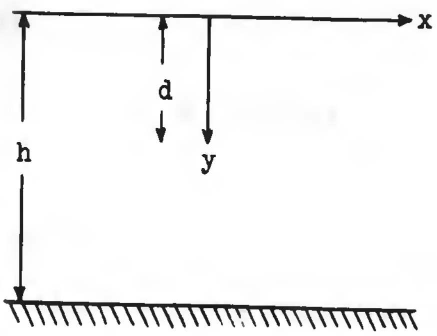

Figure 3.

namely use of the Green's function for a semi-infinite region, with the boundary condition of vanishing normal derivative at the bottom.

The two dimensional, time periodic free modes of a fluid with free surface at $y = 0$ and a rigid bottom at $y = h$ are described by a velocity potential $\varphi(x,y)e^{-i\omega t}$, such that

$$\varphi_{xx} + \varphi_{yy} = 0 , \qquad 0 < y < h \tag{7.2}$$

$$(\partial_y + \kappa)\varphi = 0 , \qquad y = 0 , \qquad \kappa = \omega^2/g \tag{7.3}$$

and

$$\partial_y \varphi = 0 , \qquad y = h . \tag{7.4}$$

A solution of (2) which satisfies (4) is

$$\varphi(x,y) = \frac{\cosh \bar{k}(h-y)}{\cosh \bar{k}h} e^{i\bar{k}x} , \tag{7.5}$$

and from the free surface condition (3),

$$\bar{k}/\kappa = \coth \bar{k}h . \tag{7.6}$$

The latter relation determines one real positive value of $\bar{k} = k(>\kappa)$, the wave number of a mode which propagates along the surface; since $k \to \kappa$ when $h \to \infty$, the limiting form of (5) is $e^{i\kappa x - \kappa y}$, or the surface wave potential on a fluid of infinite depth. All other values of $\bar{k}$ that satisfy (6) are purely imaginary, and correspond to non-propagating modes.

In terms of a Green's function $G(x,y;x',y')$ with the specifications

$$(\partial_x^2 + \partial_y^2)G(x,y;x',y') = -\delta(x-x')\delta(y-y'), \quad -\infty < x < \infty,\ 0<y<h \qquad (7.7)$$

$$(\partial_y + \kappa)G = 0, \qquad y = 0 \qquad (7.8)$$

and

$$\partial_y G = 0, \qquad y = h \qquad (7.9)$$

the following representations of the velocity potential $\varphi(x,y)$ apply to the left and right of the barrier in Figure 3, when a propagating mode is incident from the left (compare (2.16), (2.17)):

$$\varphi(x,y) = \frac{\cosh k(h-y)}{\cosh kh}(e^{ikx} + e^{-ikx}) - 2\int_d^h v(y')\,G(x,y;0,y')\,dy', \quad x < 0 \qquad (7.10)$$

$$\varphi(x,y) = 2\int_d^h v(y')G(x,y;0,y')dy', \qquad x > 0. \qquad (7.11)$$

Matching these expressions in the aperture $x = 0$, $d < y < h$, provides an integral equation for the normal velocity $v(y)$ there, namely

$$\frac{\cosh k(h-y)}{\cosh kh} = 2 \int_d^h v(y')G(0,y;0,y')dy' \ , \quad d < y < h \ . \tag{7.12}$$

The propagating mode part of the Green's function,

$$\frac{i\, e^{ik|x-x'|}}{kh + \frac{1}{2}\sinh 2kh} \cosh k(h-y)\cosh k(h-y') \quad ,$$

is solely responsible for the asymptotic behavior of this function when $|x-x'| \to \infty$, and thus, according to (10), the reflection coefficient takes the form

$$R = 1 - \frac{4i \cosh kh}{\sinh 2kh + 2kh} \int_d^h v(y)\cosh k(h-y)dy \quad . \tag{7.13}$$

Assuming that $kd \gg 1$, let us approximate to the kernel of the integral equation (12) by a Green's function which satisfies (7) and (9), only, viz:

$$G = -\frac{1}{2\pi}\log[(x-x')^2+(y-y')^2]^{1/2} - \frac{1}{2\pi}\log[(x-x')^2+(2h-y-y')^2]^{1/2} \ . \tag{7.14}$$

Then the integral equation becomes

$$\int_d^h v(y')[\log|y-y'| + \log(2h-y-y')]dy = -\pi \frac{\cosh k(h-y)}{\cosh kh}, \quad d < y < h, \quad (7.15)$$

or

$$\frac{1}{\pi}\int_0^\delta V(\xi')\log(|\xi^2 - \xi'^2|)d\xi' = -\frac{\cosh k\xi}{\cosh kh}, \quad 0 < \xi < \delta \quad (7.16)$$

after the transformations

$$\xi = h - y, \qquad \xi' = h - y', \qquad v(y) \to V(\xi), \quad (7.17)$$

and denoting

$$\delta = h - d \quad (7.18)$$

as the aperture width. A further change of variable.

$$\xi = \delta \cos\frac{\vartheta}{2}, \qquad \xi^2 = \frac{\delta^2}{2}(1 + \cos\vartheta) \quad (7.19)$$

enables (16) to be written in the form

$$\frac{1}{\pi}\int_0^\pi \log(2|\cos\vartheta - \cos\vartheta'|)[V \sin\frac{\vartheta'}{2}]d\vartheta'$$

$$= -\frac{2}{\pi}\sum_1^\infty \int_0^\pi \frac{\cos n\vartheta \cos n\vartheta'}{n}[V \sin\frac{\vartheta'}{2}]d\vartheta'$$

$$= -\frac{2}{\delta}\,\frac{\cosh(k\delta\cos\vartheta/2)}{\cosh kh} + A, \qquad 0 < \vartheta < \pi \quad (7.20)$$

where A is constant. By virtue of the orthogonality of the functions $\cos n\vartheta$ on the interval $(0,\pi)$,

$$V \sin\frac{\vartheta}{2} = \frac{2}{\pi}\sum_{1}^{\infty}\int_{0}^{\pi} n \cos n\vartheta \cos n\vartheta' \left\{\frac{2}{\delta}\ \frac{\cosh(k\delta \cos \vartheta'/2)}{\cosh kh} + A\right\} d\vartheta' + B \ . \quad (7.21)$$

The constant A evidently makes no contribution, and since the normal velocity tends to zero as $k \to \infty$, the constant B must equal zero.

Now,

$$R = 1 - \frac{4i \cosh kh}{\sinh 2kh + 2kh} \cdot \frac{\delta}{2}\int_{0}^{\pi}[V \sin\frac{\vartheta}{2}]\cosh(k\delta \cos\frac{\vartheta}{2})d\vartheta \ , \quad (7.22)$$

and substituting from (21), it follows that

$$R = 1 - \frac{8\pi i}{\sinh 2kh + 2kh}\sum_{1}^{\infty} n(I_{2n}(k\delta))^2$$

$$\simeq 1 - 16\pi i\, e^{-2k(d+\delta)}\sum_{1}^{\infty} n(I_{2n}(k\delta))^2 \ , \qquad kd \gg 1 \quad (7.23)$$

where I_{2n} denotes the cylinder function of order $2n$. A comparison of (1) and (23) reveals the correction factor for finite depth of the fluid (note that the wave number correction $k - \kappa \sim 2\kappa\, e^{-2\kappa h}$ is exponentially small),

$$\Delta = 16\pi\, e^{-2k\delta}\sum_{1}^{\infty} n(I_{2n}(k\delta))^2 \ . \quad (7.24)$$

The sum in (24) can be evaluated explicitly (Watson, 1944), with the result

$$\sum_{1}^{\infty} n(I_{2n}(z))^2 = \frac{z^2}{8}\,[(I_1(z))^2 - I_0(z)I_2(z)]\ , \tag{7.25}$$

and therefore, replacing also k by κ ,

$$\begin{aligned}\Delta &= 2\pi(\kappa\delta)^2\, e^{-2\kappa\delta}[(I_1(\kappa\delta))^2 - I_0(\kappa\delta)I_2(\kappa\delta)]\\ &= O(\kappa\delta)^4\ , \qquad \kappa\delta \to 0\\ &\simeq 1\ , \qquad \kappa\delta \to \infty\ .\end{aligned} \tag{7.26}$$

The quantity Δ rises from zero to unity without oscillation as $\kappa\delta$ increases.

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