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by

Kenneth J. Arrow and Leonid Hurwicz[*]

1. Uniqueness

1.1 In an earlier paper dealing with problems of stability, we asserted that the competitive equilibrium is unique if the excess demand functions satisfy the weak axiom of revealed preference ([4], p. 534). This assertion is correct if the equilibrium set is assumed to consist of isolated points, but examples can be constructed (see 1.5 below) where there is a multiplicity of (non-isolated) equilibrium points. In general, one is only entitled to say that the equilibrium set is _convex_. Hence, in Theorem 2 of [4], the term "unique" should be replaced by "convex," with the assertion concerning stability remaining unchanged. Thus we obtain[1]

Theorem 1. If the aggregate excess demand functions satisfy the weak axiom of revealed preference, then (a) the instantaneous adjustment process is stable (whether or not there is a numéraire), and (b) the set of equilibrium points is convex.

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[1] We use the notations and definitions of [4]. Unlike [4], we do not ignore "corner" equilibria (where $\bar{x}_r < 0$ with $\bar{P}_r = 0$).

Proof. (a) As essentially shown in [4], the weak axiom of revealed preference implies that,

(1) $$\sum_{k=0}^{m} \bar{P}_k x_k > 0 \quad \text{unless} \quad x_k \leq 0 \quad \text{for all} \quad k = 0, \ldots, m,$$

where $x_k = F_k(P)$, the excess demand for the kth proof at price vector P, and $\bar{P}$ is any equilibrium price vector. Let,

(2) $$V_{\bar{P}} = \sum_{k=0}^{m} (P_k - \bar{P}_k)^2/2 = D_{\bar{P}}^{\,2}/2,$$

where $D_{\bar{P}}$ is the (Euclidean) distance from the given price vector $P(t)$ on the solution (of the differential system (3) below) to any equilibrium price vector $\bar{P}$. If there is no numéraire, the dynamic system defining the motion of the prices is,

(3) $$dP_k/dt = \begin{cases} 0 & \text{if } P_k = 0, \ x_k < 0 \\ x_k & \text{otherwise.} \end{cases}$$

(The condition in the first line prevents prices from becoming negative.) Let,

(4) $$T = \{k: P_k = 0, \ x_k < 0\},$$

$\tilde{T}$ its complement. Then, along the path,

(5) $$dV_{\bar{P}}/dt = \sum_{k=0}^{m} (P_k - \bar{P}_k) \cdot dP_k/dt = \sum_{k \in \tilde{T}} (P_k - \bar{P}_k) x_k$$

$$= \sum_{k=0}^{m} (P_k - \bar{P}_k) x_k - \sum_{k \in T} (P_k - \bar{P}_k) x_k$$

$$= \sum_{k=0}^{m} P_k x_k - \sum_{k=0}^{m} \bar{P}_k x_k - \sum_{k \in T} P_k x_k + \sum_{k \in T} \bar{P}_k x_k.$$

The first term vanishes by Walras' Law. By definition of T, $P_k = 0$,
$x_k < 0$ for all $k \in T$; therefore, the third term vanishes and the fourth
is non-positive. From (1), it follows that

(6) $$dV_{\overline{P}}/dt < 0 \quad \text{unless} \quad P(t) \text{ is an equilibrium point.}$$

If the equilibrium price vector $\overline{P}$ were known to be unique, then it
would follow by Lyapunov's so-called second method[2] that the solution
$\psi(t; P^0)$ must converge to $\overline{P}$.

But even if there is a possibility of non-uniqueness of equilibrium,
we may conclude that (i) the solution path is bounded (since its distance
from any given equilibrium point is non-increasing), and (ii) any limit
point of the path must itself be an equilibrium point[3]. Given (i) and
(ii), by reasoning analogous to that of [1][4], we then conclude that con-
vergence to some equilibrium point will take place, which (by definition)
means that the system is globally stable.

We have so far assumed that there is no numéraire, so that the dynamics
are described by (3). If commodity 0 is the numéraire, then (3) con-
tinues to hold for $k \geq 1$, while,

$$P_0 \equiv \overline{P}_0 ,$$

where $\overline{P}_0$ is a positive number (1 if the units are chosen properly). Then
(5) becomes,

---

[2] See, for example, [3], p. 124.

[3] This can be proved by the method of [3].

[4] The proof of convergence in Lemma 6 of [1], pp. 97-8, (following eq.(4))
goes over except that the Euclidean norm replaces that based on the maximal
component of the vector.

$$dV_{\bar{P}}/dt = \sum_{\substack{k \in \tilde{T} \\ k \neq 0}} (P_k - \bar{P}_k)x_k$$

$$= \sum_{k=0}^{m} P_k x_k - \sum_{k=0}^{m} \bar{P}_k x_k - \sum_{k \in T} P_k x_k + \sum_{k \in T} \bar{P}_k x_k - (P_0 - \bar{P}_0)x_0 .$$

Since $P_0 = \bar{P}_0$, the last term vanishes; the first four terms are the same as before, so that (6) again holds, and we have global stability of the system.

(b) Let $\bar{P}^\alpha (\alpha = 1,2, \ldots)$ be two equilibria price vectors and P^* a convex combination of them, say $P^* = \lambda \bar{P}^1 + (1-\lambda) P^2,\ 0 < \lambda < 1.$ Consider the solution $\psi(t; P^*)$ starting from P^*. and suppose that P^* is not an equilibrium point. Then, for both $\alpha = 1$ and $\alpha = 2,$ by (6), the distance $D[\psi(t; P^*), \bar{P}^\alpha]$ from the moving point to the equilibrium point $\bar{P}^\alpha$ is less than the distance $D[P^*, \bar{P}^\alpha]$ from P^* to $\bar{P}^\alpha$ if $t > 0.$ Hence the sum $D[\psi(t; P^*), \bar{P}^1] + D[\psi(t; P^*) \bar{P}^2]$ of the distances from the moving point to the two equilibria must be less than the sum of the distances $D[P^*, \bar{P}^1] + D[P^*, \bar{P}^2]$ from P^* to the two equilibria. But P^* lies on a straight line segment between $\bar{P}^1$ and $\bar{P}^2$, and hence must minimize the sum of distances from any point to the two equilibria. This contradiction shows that P^* must be an equilibrium point and hence that any convex combination of equilibria must itself be equilibrium.

1.2. Remark. The "dynamic" approach used in the preceding proof of the convexity of the set of equilibrium points is natural to use in connection with investigations of stability and has the merit of applicability in more general situations. However, for the case of weak revealed

preference, a direct "static" proof can also be given. Let again
$P^* = \lambda \bar{P}^1 + (1-\lambda) \bar{P}^2$, $0 < \lambda < 1$, and denote by x_h^*, $h = 0,1, \ldots, m$, the
excess demand at P^* for the kth good. Suppose P^* is not an equilibrium
point. Then, by (1),

$$(3) \qquad \sum_{h=0}^{m} \bar{P}_h^{\alpha} x_h^* > 0 \qquad \text{for} \quad \alpha = 1,2, \ldots,.$$

Hence

$$(4) \qquad \sum_{k=0}^{m} P_k^* x_k^* = \sum_{k=0}^{m} [\lambda \bar{P}_k^1 + (1-\lambda) \bar{P}_k^2] x_k^*$$

$$= \lambda \sum_{k=0}^{m} \bar{P}_k^1 x_k^* + (1-\lambda) \sum_{k=0}^{m} \bar{P}_k^2 x_k^*$$

$$> 0$$

which contradicts the Walras law requirement that $\sum_{k=0}^{m} P_k^* x_k^* = 0$.

1.3. It can be noted that the proof of Theorem 1 made use of the weak
axiom of revealed preference only to establish (1). That is, it was only
used to compare the excess demands at an equilibrium price vector with that
at a disequilibrium price vector. We may therefore strengthen Theorem 1 to

Theorem 2. If the aggregate excess demand functions $F_k(p)$ satisfy
the condition that

$$\sum_{k=0}^{m} \bar{P}_k F_k(p) > 0$$

whenever $\bar{P}$ is an equilibrium and P is not, then the instantaneous adjust-
ment process is stable (whether or not there is a numéraire), and the set of
equilibrium points is convex.

1.4. In [1], Lemma 5, it was shown that the hypothesis of Theorem 2 was valid if all commodities were strong gross substitutes $(\partial F_r/\partial p_s > 0$ for $r \neq s)$. In this case, the last part of the theorem is uninteresting because the equilibrium is in fact unique. However, a modification of the proof shows that the lemma is still valid for weak gross substitutes $(\partial F_r/\partial \underline{p}_s \geqq 0$ for $r \neq s)$ (see [2], Theorem 1). Hence,

Corollary: If all commodities are weak gross substitutes, the instantaneous adjustment process is stable and the set of equilibria is convex.

[Related results are to be found in unpublished papers by Uzawa, McKenzie, and Morishima.]

1.5. Finally, it may be useful to give two examples of non-unique equilibrium in which the weak axiom of revealed preference holds for the aggregate excess demand functions. This will certainly be true if there is only one individual in the market or, more generally, if the aggregate excess demand function could be that of a single utility-maximizing individual. In the following examples, E is the initial endowment, I the indifference curve through E , and P^1 and P^2 two possible budget lines through E such that the individual will in fact demand E; hence the negative of the slope of either line is an equilibrium price ratio.
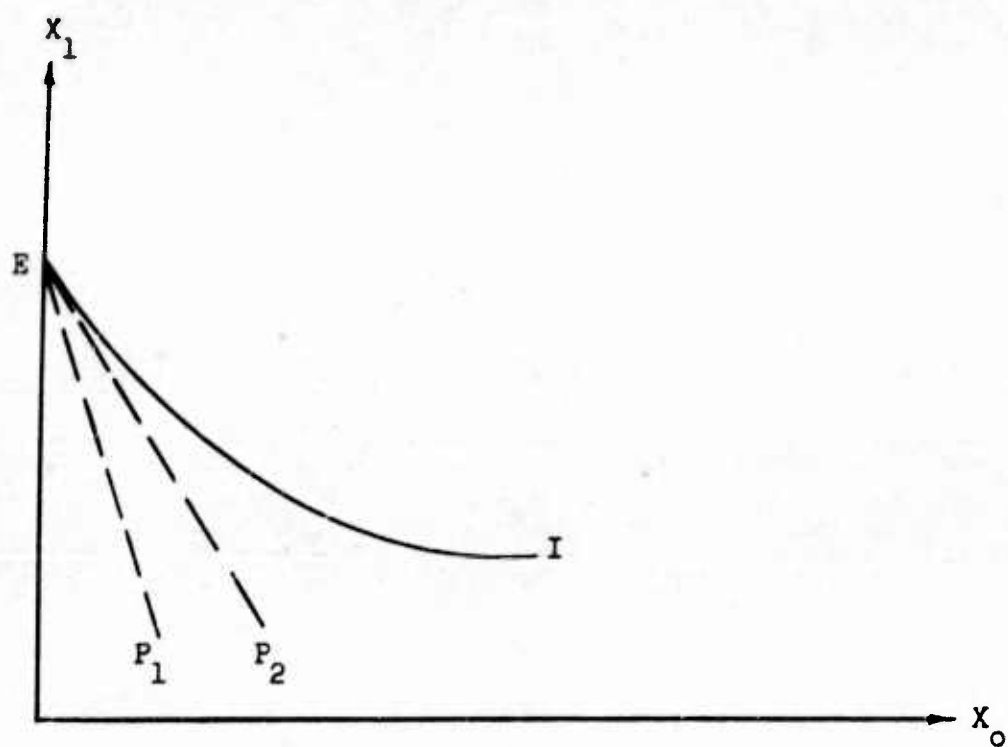
X_1

E

P_1 P_2
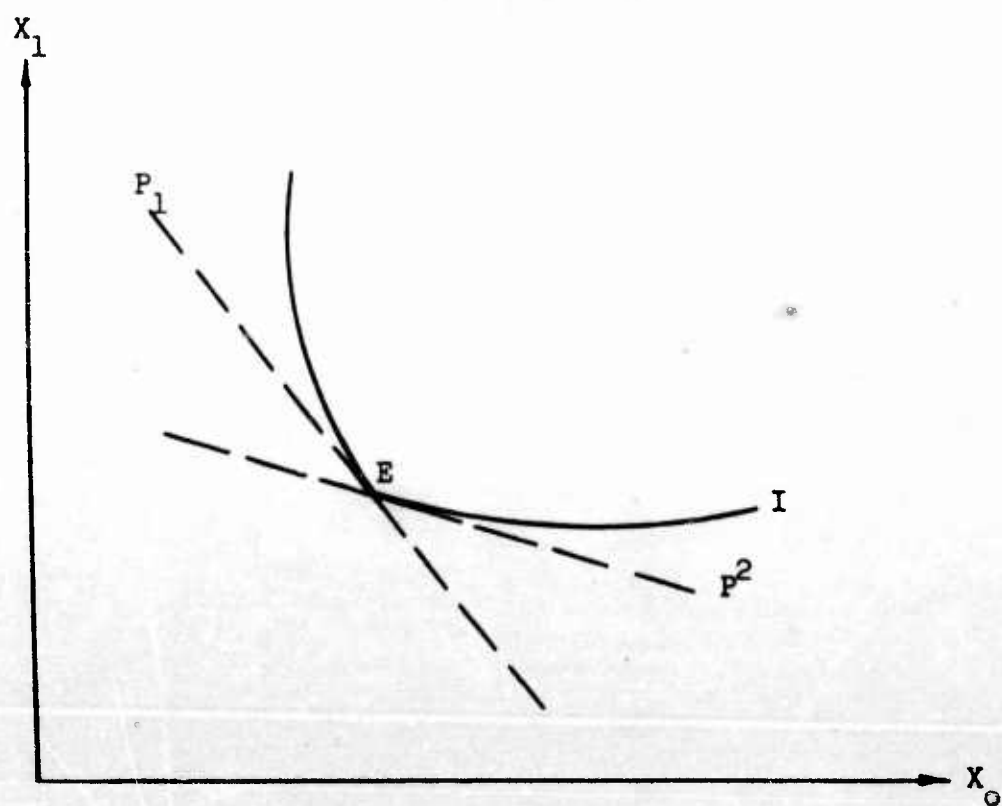
I

X_o

Figure 1

X_1

P_1

E

I

P^2

X_o

Figure 2

In the first example, the possibility of multiple equilibrium rests on a (commodity space) corner maximization of utility. [Note this is not the same as a (price space) corner equilibrium of the market since supply equals demand, and both prices can be positive.] It requires that there exist no initial endowment of commodity 1, a condition which is somewhat peculiar for a pure exchange economy but which is natural enough in a production economy. The second example requires a kink in the indifference curve, again a condition very possible in an economy with production but which is inconsistent with the smooth indifference curves usually postulated for a pure exchange economy.

2. Existence

A clarification is also in order concerning a statement ([4], p. 527) that the continuity and single-valuedness of the excess demand functions (with positive homogeneity of degree zero and Walras' law also assumed) imply the existence of a competitive equilibrium. This statement is correct if continuity is understood in the usual sense, so that the function is finite-valued everywhere. However, it is frequently convenient to permit the excess demand for a good to tend to infinity as its price tends to zero. This will be true of any commodity which always has a positive marginal utility (cf. [4], Theorem 7); in [1], Lemma 1, infinite excess demands are shown to be a con-sequence of strong gross substitutability.

In such cases, the concept of continuity may be broadened to permit an infinite value for $F_k(P)$ and continuity defined by the condition

$\lim_{n \to \infty} F_k(P^n) = \infty$ for any sequence $\{P^n\}$ such that $P^n \to P$ (cf.[4], foot-note 36, p. 541). With this definition, it is true that continuity, single-valuedness and the boundedness from below of the excess demand function

(together with homogeneity and Walras' Law) imply the existence of competitive equilibrium. Boundedness from below is the case, for example, under pure trade (absence of production) where the excess supply cannot be higher than the initial endowment.

The following example shows that the condition of boundedness from below cannot be despensed with.[5] Let $m = 1$ (i.e., only one commodity other than numéraire), $p = \dfrac{P_1}{P_0}$ (the price of the non-numéraire good in terms of numéraire), the excess demand for numeraire $x_0 = -p$, the excess demand for the non-numeraire good $x_1 = +1$ for all $\underline{p}$. Then all the conditions other than boundedness from below are satisfied, yet equilibrium obviously does not exist since the excess demand for the non-numéraire good is always positive.

3. Inferior Goods and Giffen's Paradox

The need for the following correction is [4] has been pointed out to us by Robert Mundell. On p. 542 we note that, "'Giffen's paradox' must be absent (a good cannot be 'inferior') at a unique equilibrium point in the case of two goods." The statement is correct except for the assertion in parentheses which, of course, is not equivalent to the presence of Giffen's paradox. The following example illustrates this point.

Let the economy consist of one individual whose utility function for the two points x_0, x_1 is given by the admittedly formidable looking function

$$U(x_0, x_1) = \varphi_0(x_0) + \varphi_1(x_1)$$

---

[5] It may, of course, be replaced by other conditions.

where

$$\varphi_0(x_0) = \begin{cases} -\ln(2-x_0) & \text{for } x_0 \le 1 \\\\ \dfrac{2}{\sqrt{3}}\left(\arctan \dfrac{2x_0 - 3}{\sqrt{3}} - \arctan \dfrac{-1}{\sqrt{3}}\right) & \text{for } x_0 > 1 \end{cases}$$

and

$$\varphi_1(x_1) = \frac{\ln(2 + 2x_1)}{2}.$$

Here, x_0, x_1, are <u>excess</u> demands; $x_k = X_k - X_k^0$ $(k = 0, 1)$, where X_k is the quantity demanded and X_k^0 the initial endowment. Hence, at equilibrium, $x_k = 0$ $(k = 0, 1)$. (It is assumed that $X_1^0 < 1$.)

It may be verified that the marginal utilities are positive and the utility function is quasi-concave. With commodity 0 as numéraire and p the price of commodity 1, the budget constraint is

$$X_0 + pX_1 = M$$

where M is income, or $x_0 + px_1 = M^*$, where $M^* = M - X_0^0 - pX_1^0$. When utility is maximized subject to this constraint, the excess demand (when $x_0 \le 1$) is given by

$$x_0 = 2(M^* + p - 1)$$
$$x_1 = \frac{2-M^*}{p} - 2.$$

With $M^* = 0$, at equilibrium (which is at $x_0 = x_1 = 0$ so that $\bar{p} = 1$), commodity 1 is an inferior good, since $\dfrac{\partial x_1}{\partial M} = \dfrac{\partial x_1}{\partial M^*}\bigg|_{p=\bar{p}=1} = -1 < 0$. On the other hand, holding M constant,

and evaluating at $p = \bar{p} = 1$, we have

$$\frac{\partial X_1}{\partial p} = \frac{\partial x_1}{\partial p} = -\frac{1}{p^2}(2 - M^*) - \frac{1}{p}\frac{\partial M^*}{\partial p}$$

$$= -2 + X_1^0 < 0 \quad ,$$

and

$$\frac{\partial X_0}{\partial(\frac{1}{p})} = \frac{\partial x_0}{\partial(\frac{1}{p})} = -2(1 - X_1^0) < 0 \quad ;$$

hence the Giffen paradox is absent, as was to be expected.

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