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OPTIMAL INVENTORY DEPLETION

FC

By

PETER W. ZEHNA,

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TECHNICAL REPORT NO. 41

July 24, 1959

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OPTIMAL INVENTORY DEPLETION

By

Peter W. Zehna

1. Introduction and Summary

In recent years, a great deal of attention has been focused on inventory problems, particularly the problem of optimal policies for ordering stock. More recently, Derman and Klein [1], along with Lieberman [2] have investigated the problem of optimal policies for depleting stock. More specifically, an operation requires a stockpile of items to be issued according to some specified demand. The field life of an item is a known function, $L(S)$, of the age, S , of the item upon being issued. The problem is to determine the order of issue (issue policy) which maximizes the total field life of the stockpile. Such a policy is called optimal and it should be noted that optimality is defined only up to an equivalence, two policies being equivalent if they yield the same total field life.

In practice, one finds [3] that the two most commonly used policies are those of LIFO (last in, first out) the policy of always issuing the newest item when demanded, and FIFO (first in, first out) the policy of always issuing the oldest item when demanded. Under the assumption that demand occurs only when the item in use is completely exhausted, the authors in [1] and [2] determine conditions on $L(S)$ for which these two issue policies are optimal. Derman and Klein in [1] first prove the general inductive theorem that if $L(S)$ is a convex function and LIFO is optimal for a stockpile of size two then LIFO is optimal for any size greater than two, thereby

reducing the problem for convex functions to verifying the case $n = 2$, where n is the size of the stockpile. It is then shown that LIFO is optimal for $n = 2$ for the special cases $L(S) = \frac{a}{b+S}$ ($a > 0, b \geq 0$) and $L(S) = ce^{-kS}$ ($c > 0, k > 0$).

Lieberman in [2] concentrates on the FIFO policy. He first proves a general inductive theorem to the effect that if $L(S)$ is a differentiable function with derivative $L'(S) \geq -1$ for all S and LIFO is optimal for $n = 2$ then LIFO is optimal for all n . He then shows that if either $L(S)$ is convex or $L'(S) \geq -1$ and FIFO is optimal for $n = 2$ then FIFO is optimal for all n . Finally it is shown that if $L(S)$ is concave and monotone with $L'(S) \geq -1$ then FIFO is the optimal issue policy.

In both [1] and [2], the authors pose the problem of further characterizing the class of functions for which LIFO and FIFO will be optimal issue policies. The purpose of the present research is to examine and extend, where possible, some of the results found in [1] and [2], as well as to investigate other models which are stochastic in character. The findings are summarized below.

In Section 2, the Derman-Klein paper is analyzed. It is pointed out that an incorrect proof of their basic inductive theorem is given in [1]. It is shown that the policy of issuing just one item, the source of difficulty in their proof, may indeed be optimal. Furthermore it is proved that the hypothesis of their basic theorem cannot be satisfied unless the field life function, $L(S)$, is monotone. With the added assumption of monotonicity, a correct proof for the Derman-Klein basic theorem is then

given. It is then shown that if $L(S)$ is convex and $L'(S) < -1$ for all S then LIFO is optimal for all n . A condition yielding LIFO as an optimal policy for all n for a convex, twice differentiable function $L(S)$ is proved to be that $\frac{L''(S)}{L'(S)}$ is a non-decreasing function. This condition is found to encompass the special life functions found in [1] alluded to above and the theorem is applied to give an immediate generalization. Finally it is shown that LIFO is optimal for a concave field life $L(S)$ provided that $L'(S) < -1$ for all S .

Section 3 is devoted to an investigation of the FIFO policy under the same model assumed in Section 2. It is shown that Lieberman's condition that $L'(S) \geq -1$, yielding FIFO as optimal for $L(S)$ concave and monotone, may be extended to include non-monotone functions and that for concave functions, the condition is necessary as well as sufficient. It is also proved that whenever $L(S)$ is convex and $L'(S) \geq 1$ then FIFO is again optimal for all n .

An attempt is made in Section 4 to generalize the preceding results to the case where there is more than one source of demand for the items in the stockpile still under the deterministic model of a known field life function. It is shown, by example, that some of the "nicest" cases fail to generalize in a natural way so that no general statement can be made. However, it is proved that in the special case $L(S) = a + bS$ ($a > 0$, $-1 < b < 0$), so that $L'(S) > -1$, FIFO remains the optimal policy for two demand sources. Also, when $L(S)$ is either convex or concave with $L'(S) < -1$ for all S , LIFO is optimal for two sources.

Finally a general theorem is proved whereby whenever LIFO or FIFO is the optimal policy for two demand sources, the same will remain true for any finite number of sources greater than two.

In Section 5, the requirement that the field life function be known is relaxed and field life is allowed to be a random variable with known mean value. The concept of total field life is replaced by expected field life (called utility) and the inherent difficulties imposed by such a model are discussed. Nevertheless, some isolated results are obtained. Thus, whenever field life is distributed by a Γ -density with mean value $a + b S$ ($a > 0, b > 0$), FIFO is optimal for all n , while if the mean value is given by $L(S) = e^{-kS}$ ($k > 0$), LIFO is optimal for the case $n = 2$. In accordance with Derman and Klein in [1], demand is then made independent of field life so that it is assumed that an item is replaced in the field every t units of time. Under such a model, Derman and Klein have shown that if the mean value, $U(S)$, is convex, LIFO is optimal for all n . They further remark that FIFO is optimal whenever $U(S)$ is concave. An example is provided which shows that the latter statement is not always true. If it is assumed that all the arguments of the function $U(S)$ involved in the expression of total utility lie in the region of concavity then the statement is true and this corrected version is given. Moreover, the results are proved to remain true in the case of more than one demand source.

Finally, in the spirit of the remark made above, that LIFO and FIFO are the two most commonly used policies, the requirement of optimality

is suppressed in Section 6. A sequential model is defined whereby at the beginning of the operation, two items are in the stockpile. One of the two items is issued to begin the operation and immediately replaced by a new item, the next issue taking place when the field life of the item in use is exhausted. The operation then proceeds in stages, a new item always replacing the one issued and, moreover, it is agreed that either the LIFO or FIFO policy is followed at each stage. It is of interest to compare the relative merits of the two issuing schemes.

The ages of the two items at any time t are assumed to be random variables and thereby determine a stochastic process which is shown to have an imbedded Markov process under the two schemes of following LIFO and FIFO throughout. Field life is related, in a natural way, to the age of the newer item at any issue stage. If the field life, Z , of an item of age t at issue is assumed to be distributed by the density,

$$h(z,t) = \frac{(t+\beta)^{\alpha+1}}{\Gamma(\alpha+1)} z^{\alpha} e^{-z(t+\beta)} (z \geq 0) \text{ with parameters } \alpha > -1, \beta > 0,$$

the Markov process under each respective scheme is shown to have a unique stationary absolute probability distribution. The explicit solutions are displayed in each case and it is found that the calculation of moments of these distributions is not amenable to elementary calculus techniques. The special case $\alpha = 0, \beta = 1$ is then analyzed completely, moments being compared numerically to find that the LIFO scheme has greater stationary utility than that of FIFO.

2. Optimality of LIFO under Model I

2.1 Definition of Model I

Model I is defined by the following set of assumptions:

- (i) At the start of the operation, a stockpile has n items of ages $S_1, S_2, \dots, S_n$ where $S_i \neq S_j$ for $i \neq j$ and $S_i > 0$ for $i = 1, 2, \dots, n$.
- (ii) The field life of an item is a known, non-negative function $L(S)$ of the age S of the item upon being issued, where $S \geq 0$.
- (iii) A new item is issued only when the entire life of the preceeding one is ended.
- (iv) Items are issued successively until the entire stockpile is depleted and no new items are ever added.
- (v) $S_i \in \{S \mid L(S) > 0\}$ $i = 1, 2, \dots, n$.

The ages in (i) are called initial ages and (v) guards against beginning the operation with items which can yield no field life at the outset. The model is called deterministic in the sense that (ii) requires that $L(S)$ be known. In what follows, convexity and concavity of $L(S)$ play an important role. A real-valued function f with real domain $[a, b]$ is defined to be convex [concave] if, for every $x_1, x_2 \in [a, b]$ and real a_1, a_2 satisfying $a_1 \geq 0, a_2 \geq 0, a_1 + a_2 = 1$, it follows that $f(a_1 x_1 + a_2 x_2) \leq$

$$\leq a_1 f(x_1) + a_2 f(x_2) \quad [f(a_1 x_1 + a_2 x_2) \geq a_1 f(x_1) + a_2 f(x_2)] .$$

Several properties of convex [concave] functions which will be used are well known and may be found in many standard sources, [4] being mentioned as an excellent reference. Geometrically, a function is convex [concave] if for every pair of points x_1, x_2 in the domain of f , the line through $f(x_1)$ and $f(x_2)$ never lies below [above] the graph of f . It follows that f is continuous on (a,b) and if f is twice differentiable, f is convex [concave] if and only if $f''(x) \geq 0$ [$f''(x) \leq 0$] for all $x \in (a,b)$. Finally, if f is convex, α, β real numbers with $\alpha \leq \beta$ and $x_0, x_0 + \alpha, x_0 + \beta$ in the domain of f , then
$$\frac{f(x_0 + \alpha) - f(x_0)}{\alpha} \leq \frac{f(x_0 + \beta) - f(x_0)}{\beta}$$
 and, if $\alpha \geq 0$ and $x_1 < x_2$ with $x_1, x_2, x_1 + \alpha, x_2 + \alpha$ all in the domain of f ,
$$f, \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_2 + \alpha) - f(x_1 + \alpha)}{x_2 - x_1}$$
 with the reversed inequalities holding for f concave.

Before proceeding, it is convenient to introduce the notion of a truncation point. In writing explicit expressions for $L(S)$, defined for at least $S \geq 0$, one must guard against allowing L to be negative in accordance with assumption (11) in the model. Even the general assumption that $L(S)$ be concave and monotone decreasing has inherent in it the fact that if this is to hold for all $S \geq 0$, there then exists a finite S_0 for which $L(S_0) = 0$ and $L(S) \leq 0$ for all $S > S_0$. In such cases, it is necessary to re-define $L(S)$ to be identically zero for $S \geq S_0$ and S_0 is called a truncation point for L . In some cases it is convenient

to speak of a truncation point when no such finite S_0 need exist as, for example, when $L(S)$ is convex decreasing. In such cases S_0 may be taken as $+\infty$. In summary, then, if L is monotone decreasing on $[S', \infty)$ where $0 \leq S'$ and $L(S') > 0$, then $S_0 \leq \infty$ is a truncation point for L if and only if $S_0 = \inf \{S \in [S', \infty) \mid L(S) \leq 0\}$ and then it will always be understood that L is re-defined to be identically zero for $S \geq S_0$.

2.2 Modification of the Derman-Klein Theorem

Derman and Klein in [1] propose the theorem that if L is a convex function and LIFO is optimal whenever $n = 2$, then LIFO is optimal for all $n > 2$, thereby reducing the problem of determining LIFO as an optimal policy for convex L to that of verifying its truth for the simple case $n = 2$. However, the proof of the theorem relies heavily on the statement made that "obviously the policy of issuing but one item cannot be optimal." It will be seen presently, however, that this is not the case. In addition, the hypothesis cannot be satisfied unless L is a monotone function, as may be seen from the following theorems.

Theorem 2.1: Suppose $L(S)$ is a convex increasing function. Then LIFO cannot be optimal for $n = 2$.

Proof: Let $L^+(S) = \lim_{h \rightarrow 0^+} \frac{L(S+h) - L(S)}{h}$ be the right-hand derivative of L . Since L is convex, L^+ is defined for all $S \geq 0$ and, moreover,

L^+ is non-negative, non-decreasing with $L^+(S) > 0$ for $S > 0$. Let $0 < S < T$ be any pair of initial ages. If the FIFO policy is followed let Q_F denote the total field life so that $Q_F = L(T) + L(S + L(T))$; if the LIFO policy is followed, its total field life, say Q_L , will be given by $Q_L = L(S) + L(T + L(S))$.

Now suppose there exists $S_1 > 0 \ni L^+(S_1) \geq 1$ and let $f(S) = \frac{L(S) - L(S_1)}{S - S_1}$ for all $S > S_1$. Since L is convex, $f(S) \geq L^+(S_1)$ and $\lim_{S \rightarrow S_1} f(S) = L^+(S_1)$ so that $f(S) \geq 1$ implies $L(S) - L(S_1) \geq S - S_1$ or $S_1 + L(S) \geq S + L(S_1)$ for all $S > S_1$ whence $L(S_1 + L(S)) \geq L(S + L(S_1))$ and, since $L(S) > L(S_1)$, $L(S) + L(S_1 + L(S)) > L(S + L(S_1))$. Thus for this choice of S_1 and any $S_2 > S_1$, $Q_F > Q_L$ so that LIFO is not optimal.

On the other hand, suppose $L^+(S) < 1$ for all $S > 0$. Since $L^+(S)$ is non-decreasing and $L^+(S) > 0$, there exists $\alpha \ni 0 < \alpha \leq 1$ and $\lim_{S \rightarrow \infty} L^+(S) = \alpha$. Define $F(S) = L^+(S) + L^+(S)L^+(S + L(S)) - L^+(S + L(S))$ for $S > 0$. Then $\lim_{S \rightarrow \infty} F(S) = \alpha^2$ and $0 < \alpha^2 \leq 1$. But $\alpha^2 > 0$ means there is an $S_1 > 0 \ni F(S_1) > 0$. For this S_1 , let $G(S) = L(S) + L(S_1 + L(S)) - L(S_1) - L(S + L(S_1))$ for $S \geq S_1$.

Then,

$$G^+(S) = L^+(S) + L^+(S) L^+(S_1 + L(S)) - L^+(S + L(S_1)) \text{ and,}$$

$$G^+(S_1) = L^+(S_1) + L^+(S_1) L^+(S_1 + L(S_1)) - L^+(S_1 + L(S_1)) = F(S_1) > 0$$

which, together with $G(S_1) = 0$ implies the existence of $S_2 > S_1 \ni G(S_2) > 0$

and for this choice of initial ages $Q_F = G(S_2) + Q_L > Q_L$. Again LIFO cannot be optimal. q.e.d.

Theorem 2.2: Suppose $L(S)$ is a convex function and not monotone. Then, neither LIFO nor FIFO can be optimal for $n = 2$.

Proof: Since it is required that $L(S) \geq 0$ for all $S \geq 0$, and L is not monotone, $L(0) > 0$. Also, since L is continuous, there is an $S_0 \ni L(S)$ is strictly decreasing for $0 \leq S < S_0$ and non-decreasing for $S \geq S_0$ and there exists $S'_0 \geq S_0 \ni L(S)$ is again strictly increasing for $S > S'_0$. Moreover, $\lim_{S \rightarrow \infty} L(S) = +\infty$.

Now in the region $S \geq S_0$, L is convex increasing so that by Theorem 2.1, LIFO cannot be optimal here, hence in the region $S \geq 0$.

Choose $S_1 \in (0, S_0)$. Then $L(S) \rightarrow +\infty$ implies the existence of $S_2 > S_0 \ni L(S_2) = L(S_1)$. Since L is convex,

$$\frac{L(S_2) - L(S_1)}{S_2 - S_1} \leq \frac{L(S_2 + L(S_1)) - L(S_1 + L(S_1))}{S_2 - S_1} = \frac{L(S_2 + L(S_1)) - L(S_1 + L(S_2))}{S_2 - S_1}$$

so that $L(S_2) - L(S_1) \leq L(S_2 + L(S_1)) - L(S_1 + L(S_2))$ or,

$$Q_F = L(S_2) + L(S_1 + L(S_2)) \leq L(S_1) + L(S_2 + L(S_1)) = Q_L, \text{ i.e., FIFO}$$

is not optimal for this choice of S_1, S_2 for initial ages. q.e.d.

In the light of these two theorems, the basic theorem of [1] is revised

and proved as follows.

Theorem 2.3: If $L(S)$ is a convex monotone function and LIFO is optimal for $n = 2$, then LIFO is optimal for all $n > 2$.

Proof: Suppose LIFO is optimal for $n = 2$ and assume LIFO is optimal for $n = k > 2$. Let $n = k + 1$ and $0 < S_1 < S_2 < \dots < S_{k+1}$ be an arbitrary set of initial ages. For any issue policy, let S^* denote the age of the last item issued as in [1]. First observe that none of the $k!$ policies having $S^* = S_1$ can be optimal for, in any one of them, if S_1 denotes the initial age of the item issued next to last, then $S_1 > S_1$ and, by hypothesis, this policy could be improved by interchanging the order of issue of these last two items.

Now for $S^* \neq S_1$ let x denote the total field life obtained from the issue of the k preceding items. Then x can assume only the positive values $\{x_i\}_{i=1}^{k!}$ say and let $x^* = \max_1 \{x_i\}$. By the induction assumption, $x^* = L(S_1) + y$ where $y \geq 0$. As in [1], let $Q(x) = x + L(x + S^*)$ denote the total field life of all $k+1$ items for fixed S^* . Since L is now assumed to be monotone, two cases arise.

If L is non-decreasing, then so is Q and hence Q is maximized by $x = x^*$.

If L is non-increasing, then $S_1 < S^*$ implies $L(S_1) \geq L(S^*)$ and $x^* \geq L(S_1)$ so that $Q(x^*) = x^* + L(x^* + S^*) \geq x^* \geq L(S_1) \geq L(S^*) = Q(0)$. But for any $i \in \{1, 2, \dots, k!\}$, $x_i \leq x^*$ and Q is convex so that

$$\frac{Q(x_1) - Q(0)}{x_1} \leq \frac{Q(x^*) - Q(0)}{x^*} \leq \frac{Q(x^*) - Q(0)}{x_1}$$

since $Q(x^*) - Q(0) \geq 0$ and $0 < x_1 \leq x^*$. Then $Q(x_1) \leq Q(x^*)$ for $i=1,2,\dots,k!$.

Thus, in both cases, $Q(x)$ is maximized by $x = x^*$ and so, for fixed S^* , the optimal policy is obtained by using a LIFO order on the first k items issued by the inductive assumption.

Letting S^* vary over $S_2, S_3, \dots, S_{k+1}$, one obtains k policies and suppose the optimal among these is not the one where $S^* = S_{k+1}$, but $S^* = S_1$, say. Then, since x^* is a result of LIFO order, the item of age S_{k+1} was issued next to last and, by hypothesis, the policy could be improved by interchanging the order of issue of these last two items. Hence the optimal policy must be the one having $S^* = S_{k+1}$ which is precisely the LIFO policy with $n = k+1$. The theorem then follows by induction. q.e.d.

2.3 Optimality Conditions for LIFO

Having thus established the general Theorem 2.3, it is desirable to characterize the class of convex functions for which LIFO is optimal by verifying the case $n = 2$. A partial answer is given by the following two theorems.

Theorem 2.4: If $L(S)$ is a convex, differentiable function and $L'(S) < -1$ for all $S \geq 0$, then LIFO is optimal for $n \geq 2$.

Proof: In accordance with Section 2.1, L possesses a finite truncation point S_0 so that $L(S) = 0$ for all $S \geq S_0$. Still, $L(S)$ is convex for all $S \geq 0$. Moreover, for $S < S_0$, $L(S) \geq S_0 - S$ or $S + L(S) \geq S_0$, while

if $S \geq S_0$, $L(S) = 0$ and hence $S + L(S) \geq S_0$. Thus in all cases,
 $S + L(S) \geq S_0$ so that,

$$(2.3.1) \quad L(S + L(S)) = 0 \text{ for all } S \geq 0.$$

By Theorem 2.3, it suffices to verify the case $n = 2$. Suppose,
 then, that $0 < S_1 < S_2 < S_0$ are given initial ages and let Q_L, Q_F
 denote total field life under LIFO and FIFO respectively. Now
 $S_2 + L(S_1) > S_1 + L(S_1) \geq S_0$ so that

$$(2.3.2) \quad L(S_2 + L(S_1)) = 0 \text{ and } Q_L = L(S_1).$$

$$\begin{aligned} \text{Since } L \text{ is convex and } L(S_2) > 0, \quad \frac{L(S_2) - L(S_1)}{S_2 - S_1} &\leq \frac{L(S_2 + L(S_1)) - L(S_1 + L(S_2))}{S_2 - S_1} \\ &= - \frac{L(S_1 + L(S_2))}{S_2 - S_1} \text{ by (2.3.1) and } S_2 - S_1 > 0 \text{ implies} \end{aligned}$$

$$(2.3.3) \quad Q_F = L(S_2) + L(S_1 + L(S_2)) \leq L(S_1).$$

Combining (2.3.2) and (2.3.3), $Q_L \geq Q_F$ so that LIFO is optimal for
 $n = 2$ since the choice of S_1, S_2 was arbitrary. q.e.d.

Theorem 2.5: Suppose $L(S)$ is a twice differentiable, strictly
 convex function which is decreasing on $(0, \infty)$, with $\frac{L''(S)}{L'(S)}$ non-de-
 creasing. Then LIFO is optimal for $n \geq 2$.

Proof: Since L is convex decreasing, it suffices by Theorem 2.3 to verify the case $n = 2$. Suppose that $0 < S_1 < S_2$ are the given initial ages. For $S \geq S_1$, define

$$(2.3.4) \quad G(S) = L(S) + L(S_1 + L(S)) - L(S_1) - L(S + L(S_1)) .$$

Since $G(S_2) = Q_F - Q_L$, it suffices to show $G(S) \leq 0$ for all $S \geq S_1$.

$$(2.3.5) \quad \text{Let } H(S) = 1 + L'(S_1 + L(S)) - \frac{L'(S + L(S_1))}{L'(S)} \text{ for all } S \geq S_1 .$$

Then $H(S)$ is decreasing on $[S_1, \infty)$ which may be seen as follows.

$$H'(S) = L'(S)L''(S_1 + L(S)) - \frac{L'(S)L''(S + L(S_1)) - L''(S)L'(S + L(S_1))}{[L'(S)]^2}$$

and, since $\frac{L''(S)}{L'(S)}$ is non-decreasing, $\frac{L''(S + L(S_1))}{L'(S + L(S_1))} \geq \frac{L''(S)}{L'(S)}$ or ,

multiplying by the positive number $L'(S)L'(S + L(S_1))$,

$$(2.3.6) \quad L'(S)L''(S + L(S_1)) \geq L''(S)L'(S + L(S_1))$$

so that,

$$(2.3.7) \quad \frac{L'(S)L''(S + L(S_1)) - L''(S)L'(S + L(S_1))}{[L'(S)]^2} \geq 0 .$$

Applying (2.3.7) to $H'(S)$, since $L'(S)L''(S_1 + L(S)) < 0$, $H'(S) < 0$ for all $S \geq S_1$.

Since L is strictly convex and positive, it is clear that

$$(2.3.8) \quad \lim_{S \rightarrow \infty} L(S) = \alpha \geq 0 \quad \text{and} \quad \lim_{S \rightarrow \infty} L'(S) = 0 .$$

Moreover, since $\alpha \geq 0$,

$$(2.3.9) \quad \lim_{S \rightarrow \infty} G(S) = \alpha + L(S_1 + \alpha) - L(S_1) - \alpha \leq 0 .$$

Now $G(S_1) = 0$ and

$$G'(S) = L'(S) + L'(S)L'(S_1 + L(S)) - L'(S + L(S_1)) = L'(S) H(S) .$$

Suppose $H(S_1) < 0$. Since H is non-increasing, $H(S) < 0$ for all $S \geq S_1$ so that $G'(S) > 0$ or G is increasing on $[S_1, \infty)$ which, together with $G(S_1) = 0$ implies $\lim_{S \rightarrow \infty} G(S) > 0$ contrary to (2.3.9).

Thus $H(S_1) \geq 0$. If $H(S) \geq 0$ for all $S \geq S_1$, then $G'(S) \leq 0$ or G is decreasing on $[S_1, \infty)$ which, together with $G(S_1) = 0$, implies $G(S) \leq 0$ for all $S \geq S_1$. On the other hand, if it is not true that $H(S) \geq 0$ for all $S \geq S_1$ then there is $S_0 > S_1$ for which $H(S) \geq 0$ for all $S_1 \leq S \leq S_0$ and $H(S) < 0$ for $S > S_0$ since H is decreasing and continuous. Then $G'(S) \leq 0$ or G is non-increasing on $[S_1, S_0]$ while $G'(S) > 0$ or G is increasing on (S_0, ∞) which, together with $G(S_1) = 0$ and $\lim_{S \rightarrow \infty} G(S) \leq 0$, implies $G(S) \leq 0$ for all $S \geq S_1$.

Thus in all cases,

$$G(S) \leq 0 \quad \text{for all } S \geq S_1. \quad \text{q.e.d.}$$

While the attention thus far has been on convex functions due to the general Theorem 2.3, there is a special case of concave functions for which LIFO is the optimal policy as seen in the following theorem.

Theorem 2.6: Suppose $L(S)$ is a concave, differentiable function with $L'(S) < -1$. Then LIFO is optimal for $n \geq 2$.

Proof: Again there is a truncation point S_0 for L and the condition $L'(S) < -1$ is thus taken to hold on $(0, S_0)$. Also, if $0 < S_1 < S_2 < S_0$ are given initial ages,

$$\frac{L(S_1)}{S_1 - S_0} = \frac{L(S_1) - L(S_0)}{S_1 - S_0} < L'(S_1) < -1$$

so that $L(S_1) > S_0 - S_1$ or $S_1 + L(S_1) > S_0$ whence $L(S_1 + L(S_1)) = 0$.

Also $S_0 - S_1 < L(S_1)$ with L continuous and decreasing on $[S_1, S_0]$ implies the existence of $S'_0 \in (S_1, S_0)$ for which $L(S'_0) = S_0 - S_1$. For each $S \geq S_1$, define

$$(2.3.10) \quad G(S) = L(S) + L(S_1 + L(S)) - L(S_1)$$

and observe that

$$G'(S) = L'(S) + L'(S)L'(S_1 + L(S)) = L'(S)[1 + L'(S_1 + L(S))] .$$

Now, if $S_1 < S \leq S'_0$, $L(S) \geq L(S'_0) = S_0 - S_1$ so that $S_1 + L(S) \geq$

$S_1 + S_0 - S_1 = S_0$ and $L(S_1 + L(S)) = 0$. Then $G(S) = L(S) - L(S_1) < 0$.

If $S'_0 < S < S_0$, then $S_1 + L(S) < S_1 + L(S'_0) = S_0$. But $L'(S_1 + L(S)) < -1$

so that $1 + L'(S_1 + L(S)) < 0$ while $L'(S) < -1 < 0$ implies $G'(S) > 0$.

Also,

$$G(S'_0) = L(S'_0) + L(S_1 + L(S'_0)) - L(S_1) = S_0 - S_1 + L(S_1 + S_0 - S_1) - L(S_1) = (S_0 - S_1) - L(S_1) < 0$$

and

$$G(S_0) = L(S_0) + L(S_1 + L(S_0)) - L(S_1) = L(S_1) - L(S_1) = 0$$

so that G is non-decreasing on (S'_0, S_0) or $G(S) \leq 0$ for all $S \in (S'_0, S_0)$.

Thus,

$$(2.3.11) \quad G(S) < 0, \text{ or } L(S) + L(S_1 + L(S)) < L(S_1) \quad \text{for all } S_1 < S \leq S_0 .$$

Now $Q_L = L(S_1) + L(S_2 + L(S_1)) = L(S_1)$ since $S_2 + L(S_1) > S_1 + L(S_1) > S_0$,

and $Q_F - Q_L = G(S_2) \leq 0$ so that LIFO is optimal whenever $n = 2$.

Assume LIFO is optimal for $n = k-1 (k > 3)$ and suppose $n = k$. Let

$0 < S_1 < S_2 < \dots < S_k < S_0$ be given initial ages. An issue policy P is

just an ordering $S_{j_1} \prec S_{j_2} \prec \dots \prec S_{j_k}$ where $(j_1, \dots, j_k)$ is a permutation

of $(1, \dots, k)$ and $S_\alpha < S_\beta$ means the item of initial age S_α is issued prior to that of initial age S_β . The total field life $Q(P)$ of such a policy is then given by

$$(2.3.12) \quad Q(P) = L(S_{j_1}) + L(S_{j_2} + L(S_{j_1})) + \sum_{i=3}^k L(S_{j_i} + x_{i-1})$$

where

$$x_3 = L(S_{j_2} + L(S_{j_1})) \text{ and } x_{i+1} = x_i + L(S_{j_i} + L(S_{j_1}) + x_{i-1}) \quad i=3, 4, \dots, k-1.$$

Observe that $x_k \geq x_{k-1} \geq \dots \geq x_3 \geq 0$.

Let $j_1 = i$ be fixed where $i \in \{1, 2, \dots, k\}$. Of the $(k-1)!$ policies given by $S_i < S_{j_2} < \dots < S_{j_k}$, the induction hypothesis states that the optimal one is P_i where P_i is characterized by $j_2 < j_3 < \dots < j_k$. Then there are k such policies as i varies among which P_1 , given by $j_1 = 1, i = 2, \dots, k$ is the LIFO policy. Clearly, the optimal policy among $P_1, P_2, \dots, P_k$ is optimal for the problem at hand.

In accordance with (2.3.12), the total field life for LIFO is

$$Q(P_1) = L(S_1) + L(S_2 + L(S_1)) + \sum_{i=3}^k L(S_i + L(S_1) + x_{i-1}).$$

But, $S_2 + L(S_1) > S_1 + L(S_1) > S_0$ so that $x_3 = L(S_2 + L(S_1)) = 0$ and,

for $i \geq 3$,

$$S_1 + L(S_1) + x_1 \geq S_1 + L(S_1) > S_0$$

so that $L(S_1 + L(S_1) + x_1) = 0$. Thus $Q(P_1) = L(S_1)$.

For $m \neq 1$, the total field life of policy P_m is

$$Q(P_m) = L(S_m) + L(S_1 + L(S_m)) + \sum_{i=3}^k L(S_{j_1} + L(S_m) + x_i)$$

and, for $i \geq 3$,

$$S_{j_1} + L(S_m) + x_i > S_1 + L(S_m) + x_3 = [S_1 + L(S_m)] + L[S_1 + L(S_m)] > S_0$$

and $L(S_{j_1} + L(S_m) + x_i) = 0$. Hence, $Q(P_m) = L(S_m) + L(S_1 + L(S_m))$ and,

by (2.3.11), where S_1 was arbitrarily fixed, $Q(P_m) < L(S_1) = Q(P_1)$.

Then LIFO is optimal for $n = k$ and, by induction, for all $n \geq 2$. q.e.d.

2.4 Applications and Examples

Theorem 2.4 now provides a counter-example to the statement that the policy of issuing only one item cannot be optimal. For if L satisfies the conditions of the theorem, LIFO is optimal and, by (2.3.2) its total field life is given by the issue of a single item, the newest.

It is proposed in [1] that FIFO is an optimal policy to use whenever L is linear. An application of Theorem 2.4, however, shows this to be false.

Thus, let $L(S) = a - bS$ where $a > 0$, $b > 1$ so that L has truncation point $\frac{a}{b}$.

Now L is convex and $L'(S) = -b < -1$ for all $0 < S < \frac{a}{b}$ so that L satisfies the hypotheses of Theorem 2.4 and hence LIFO is an optimal policy. It is interesting to note that for $b = 1$ both LIFO and FIFO are optimal and according to Lieberman [2], FIFO is optimal for $b < 1$ so that $b = 1$ is a boundary case.

Derman and Klein have shown in [1] that LIFO is optimal for the two cases $L(S) = ce^{-kS}$ ($c, k > 0$) and $L(S) = \frac{a}{b+S}$ ($a > 0, b \geq 0$). Their proofs involve some rather cumbersome algebra. It is easily verified that both of these cases satisfy the conditions of Theorem 2.5. As a further application of that theorem, the latter case has an immediate generalization as follows. Let $L(S) = \frac{a}{(b+S)^\lambda}$ where $a > 0, b \geq 0$ and $\lambda \geq 0$. Then,

$$L'(S) = \frac{-a\lambda}{(b+S)^{\lambda+1}}, \quad L''(S) = \frac{a\lambda(\lambda+1)}{(b+S)^{\lambda+2}} \quad \text{whence} \quad \frac{L''(S)}{L'(S)} = -\frac{\lambda+1}{(b+S)} \quad \text{which}$$

is non-decreasing in S . Hence LIFO is optimal.

3. Optimality of FIFO under Model I

Lieberman has shown in [2] that the following general theorem is true. If, under Model I, $L(S)$ is differentiable with $L'(S) \geq -1$ and either LIFO or FIFO is optimal when $n = 2$ then LIFO or FIFO, respectively, is optimal for all $n \geq 2$. It is also shown (Theorem 2 of [2]) that if $L(S)$ is convex and FIFO is optimal for $n = 2$ then FIFO is optimal for all $n \geq 2$. However, by Theorem 2.2, if L is not monotone then FIFO cannot

be optimal for $n = 2$. If the added condition of monotonicity is imposed on $L(S)$, then Lieberman's Theorem 2 is, nevertheless, valid, the proof depending only on the fact that $Q(x)$, as defined in the proof of Theorem 2.3, is maximized by making x as large as possible.

A set of sufficient conditions is then given in [2] under which FIFO is optimal for all $n \geq 2$. Thus, Theorem 3 of [2] states that if $L(S)$ is a concave, monotone, differentiable function with $L'(S) \geq -1$, then FIFO is optimal for $n \geq 2$. It is a curious fact that, while monotonicity was so crucial when L is convex, the monotonicity requirement may be suppressed in this theorem and, moreover, if $L(S)$ is concave, the condition $L'(S) \geq -1$ is necessary as well as sufficient. These remarks are embodied in the next two theorems.

Theorem 3.1: Suppose $L(S)$ is a concave, differentiable function, not monotone, with $L'(S) \geq -1$. Then FIFO is optimal for $n \geq 2$.

Proof: Since L is not monotone and $L(0) \geq 0$, there is an $S^* > 0$ for which L is increasing on $(0, S^*)$ and non-increasing on (S^*, ∞) . Moreover, for some $S > S^*$, $L'(S) < 0$ (otherwise L would be monotone) and hence L possesses a truncation point $S_0 > S^*$. Let $0 < S_1 < S_2 < S_0$ be any given initial ages.

If $S_1 \geq S^*$, then FIFO is optimal by Theorem 3 of [2]. If $S_1 < S^*$, then, since L is continuous, $0 < L(S_1) < L(S^*)$ and $\lim_{S \rightarrow S_0} L(S) = 0$ so that there is an S'_1 such that $S^* < S'_1 < S_0$ and $L(S_1) = L(S'_1)$.

Several cases then arise depending upon S_2 .

(3.1) Suppose $S_1' \leq S_2 < S_0$ so that $L(S_2) \leq L(S_1)$.

Case I: $S_1 + L(S_2) < S_1'$. Then $L(S_1 + L(S_2)) \geq L(S_1)$ and $L(S_2) \geq L(S_2 + L(S_1))$, since L is non-increasing on (S^*, ∞) , so that $L(S_2) + L(S_1 + L(S_2)) \geq L(S_1) + L(S_2 + L(S_1))$ or FIFO is optimal.

Case II: $S_1 + L(S_2) \geq S_1'$. Then $S_1' \leq S_1 + L(S_2) \leq S_1 + L(S_1)$ and L is non-increasing so that $L(S_1 + L(S_2)) \geq L(S_1 + L(S_1))$ and

$$L(S_2) + L(S_1 + L(S_2)) \geq L(S_2) + L(S_1 + L(S_1)) \geq L(S_1) + L(S_2 + L(S_1)),$$

the latter inequality from the concavity of L . Again, FIFO is optimal.

(3.2) Suppose $S_1 < S_2 < S_1'$ so that $L(S_1) < L(S_2)$.

Case I: $S_1 + L(S_2) > S^*$ and $S_2 + L(S_1) < S_1 + L(S_2)$. Then,

$$\frac{L(S_2 + L(S_1)) - L(S_1 + L(S_2))}{(S_2 + L(S_1)) - (S_1 + L(S_2))} \geq L'(S_1 + L(S_2)) \geq -1 \text{ and } (S_2 + L(S_1)) - (S_1 + L(S_2)) < 0$$

so that

$$L(S_2 + L(S_1)) - L(S_1 + L(S_2)) \leq S_1 + L(S_2) - S_2 - L(S_1) < L(S_2) - L(S_1), \text{ i.e.,}$$

$L(S_2) + L(S_1 + L(S_2)) \geq L(S_1) + L(S_2 + L(S_1))$ or FIFO is optimal.

Case II: $S_1 + L(S_2) > S^*$ and $S_2 + L(S_1) \geq S_1 + L(S_2)$. Then,
 $L(S_2 + L(S_1)) \leq L(S_1 + L(S_2))$ since L is non-increasing on (S^*, ∞)
 while $L(S_2) > L(S_1)$ so that $L(S_2) + L(S_1 + L(S_2)) \geq L(S_1) + L(S_2 + L(S_1))$
 and FIFO is optimal.

Case III: $S_1 + L(S_2) \leq S^*$. Then $S_1 + L(S_1) < S_1 + L(S_2) \leq S^*$ so
 that $L(S_1 + L(S_1)) < L(S_1 + L(S_2))$ since L is increasing on $(0, S^*)$.
 Hence, $L(S_2) + L(S_1 + L(S_2)) > L(S_2) + L(S_1 + L(S_1)) \geq L(S_1) + L(S_2 + L(S_1))$,
 the latter from the concavity of L , and, again, FIFO is optimal.

Thus, in all cases, FIFO is optimal whenever $n = 2$ and hence for
 all n by Theorem 1 of [2]. q.e.d.

Theorem 3.2: Suppose $L(S)$ is a concave differentiable function.
 Then FIFO is an optimal policy for $n \geq 2$ if and only if $L'(S) \geq -1$.

Proof: Suppose FIFO is optimal. If L is monotone increasing, then
 $L'(S) \geq 0 > -1$. Otherwise, there is a finite truncation point S_0 .
 If $L(S^*) < -1$ for some $S^* < S_0$, then since L is concave, L' is
 decreasing so that $L'(S) < -1$ for all $S^* \leq S \leq S_0$. But LIFO is optimal
 on $[S^*, S_0]$ according to Theorem 2.6 which is a contradiction. Thus
 $L'(S) \geq -1$.

Conversely, if $L'(S) \geq -1$ then FIFO is optimal for $n \geq 2$ by applying Theorem 3.1, together with Theorem 3 of [2]. q.e.d.

The next theorem shows that FIFO need not be restricted to concave functions.

Theorem 3.3: Suppose $L(S)$ is a convex, differentiable function and $L'(S) \geq 1$ for all $S \geq 0$. Then FIFO is optimal for all $n \geq 2$.

Proof: Let $n = 2$ and $0 < S_1 < S_2$ be any given initial ages. Since L is convex and $L'(S) \geq 1$,

$$\frac{L(S_2) - L(S_1)}{S_2 - S_1} \geq 1 \text{ or } S_1 + L(S_2) \geq S_2 + L(S_1) \text{ and } L(S_1 + L(S_2)) \geq L(S_2 + L(S_1)).$$

But $L(S_2) \geq L(S_1)$ so that $L(S_2) + L(S_1 + L(S_2)) \geq L(S_1) + L(S_2 + L(S_1))$.

Hence FIFO is optimal for $n = 2$ and, by Theorem 2 of [2], is optimal for all $n > 2$. q.e.d.

The following example shows that it is not possible, however, to extend the preceding theorem to all convex increasing functions.

(3.3) Example: Let

$$L(S) = \begin{cases} \frac{1}{4} S & 0 < S < 4 \\ \frac{1}{2} S - 1 & 4 \leq S \end{cases}$$

and suppose $S_1 = \frac{7}{2}$, $S_2 = 4$. Then $L(S_1) = \frac{7}{8}$, $L(S_2) = 1$, $L(S_1 + L(S_2)) = \frac{5}{4}$

and $L(S_2 + L(S_1)) = \frac{23}{16}$. Then

$$L(S_2) + L(S_1 + L(S_2)) = 1 + \frac{5}{4} = 2.25 < 2.3125 = L(S_1) + L(S_2 + L(S_1)).$$

It is very easy to find, on the other hand, S_1 and S_2 for which the inequality is reversed, for example, $S_1 = 1, S_2 = 2$.

4. Model I with Multiple Demands

4.1 General Formulation and Counter-Example

In the preceeding sections, it was assumed that there was only one source of demand for the items in the stockpile. A natural generalization would appear to be the case where there is more than one source of demand. More specifically, suppose the same deterministic model as outlined in (2.1) holds. In addition, it will be assumed that there are v sources demanding items from the stockpile in accordance with those assumptions. Let the sources be labeled $M_1, M_2, \dots, M_v$.

It is assumed that there are $n > v$ items in the stockpile. The operation begins by issuing v items to $M_1, M_2, \dots, M_v$ and proceeding thereafter according to Model I. Clearly, it makes no difference which sources are labeled $M_1, M_2, \dots, M_v$. In other words, for any specified issue order (policy), the first v items issued may be freely interchanged at the outset within sources without affecting the total field life, i.e., there are $v!$ policies having the same total field life. Again, a policy will be optimal if its total field life is at least as

large as that of any other policy.

It would be highly desirable, under this model, to establish that if LIFO (FIFO) is optimal for $v = 1$, then LIFO (FIFO) will be optimal for arbitrary v . Unfortunately, this is not the case as may be seen from the following example where $v = 2$.

(4.1.1) Example: Let

$$L(s) = \begin{cases} 1 & 0 \leq s \leq 1 \\ 1 - \frac{1}{4}(s-1)^2 & 1 < s < 2 \\ \frac{7}{4} - \frac{1}{2}s & 2 \leq s < \frac{7}{2} \\ 0 & \frac{7}{2} \leq s \end{cases}$$

so that L has truncation point $\frac{7}{2}$ and

$$L'(s) = \begin{cases} 0 & 0 \leq s \leq 1 \\ \frac{1}{2} - \frac{1}{2}s & 1 < s < 2 \\ -\frac{1}{2} & 2 < s < \frac{7}{2} \end{cases}$$

Now in the region $[0, \frac{7}{2}]$, L is concave and differentiable in $(0, \frac{7}{2})$, indeed continuously so, with $L'(s) \geq -\frac{1}{2} > -1$. Hence $L(s)$ satisfies the conditions of Theorem 3 of [2] and FIFO is optimal for $v = 1$. But suppose $v = 2$ and the initial ages are $s_1 = .54$, $s_2 = .6$, $s_3 = 2.6$, $s_4 = 3.1$ and $s_5 = 3.3$. Letting Q_F denote the total field life of the FIFO policy,

$$Q_F = L(S_5) + L(S_4) + L(S_3 + L(S_5)) + L(S_2 + L(S_4)) + L(S_1 + L(S_5) + L(S_3 + L(S_5)))$$

$= 0.1 + 0.2 + 0.4 + 1 + .9996 = 2.6996$ as is easily verified by checking the demands after the initial issue of S_5 and S_4 to begin the operation. Now consider the policy of issuing S_5 and S_3 to begin with then following in the order S_4, S_2 and S_1 . Denoting the total field life of this policy by Q , it is readily verified that,

$$Q = L(S_5) + L(S_3) + L(S_4 + L(S_5)) + L(S_2 + L(S_5) + L(S_4 + L(S_5))) + L(S_1 + L(S_3))$$

$$= 0.1 + 0.45 + 0.15 + 1 + 1 = 2.70 .$$

Thus $Q > Q_F$ so that FIFO cannot be optimal.

It is worthy of note that under the conditions of Theorem 3 of [2] FIFO is optimal for $v = 2$ whenever $n = 3, 4$ as may be verified by sheer enumeration of cases. There is a special case, however, for which FIFO is optimal for the present problem as will be seen in the next sub-section.

4.2 Optimality Conditions for Two Sources

The following theorem gives a set of sufficient conditions for which FIFO will be the optimal policy. Its proof is facilitated by means of several lemmas given below.

Theorem 4.1: Let $a > 0$ and $-1 < b < 0$ be real numbers and $L(S) = a + bS$ (hence with truncation point $-\frac{a}{b}$). Then, for a stockpile

of n items ($n > 2$), FIFO is the optimal policy for two sources M_1, M_2 .

Let $0 < S_1 < S_2 < \dots < S_n < -\frac{a}{b}$ be arbitrary initial ages. For $i = 1, 2$, let S_{ij} , $j = 1, 2, \dots, k$ denote the initial age of the j^{th} item issued to M_i in any given issue policy. Then, the total field life, say Q_1 , contributed by M_1 for this arbitrary issue policy will be given by

$$(4.2.1) \quad Q_1 = L(S_{11}) + L(S_{12} + L(S_{11})) + L(S_{13} + L(S_{11}) + L(S_{12} + L(S_{11}))) + \dots$$

Now if all the arguments of L involved in Q_1 lie in the region $(0, -\frac{a}{b})$, it makes sense to use the identity $L(x+y) = L(x) + by$, obviously valid only when $(x+y) \in (0, -\frac{a}{b})$. When this is the case, the set $\{S_{ij}\}_{j=1}^k$ of initial ages will be said to satisfy Condition A throughout this section. The first lemma relates this condition to total field life.

Lemma 4.1: If the set $\{S_{ij}\}_{j=1}^k$ of initial ages satisfies Condition A, then

$$Q_1 = \sum_{j=1}^k L(S_{ij}) + b \sum_{j=1}^{k-1} L(S_{ij}) + b(b+1) \sum_{j=1}^{k-2} L(S_{ij}) + \dots + b(b+1)^{k-3} \sum_{j=1}^2 L(S_{ij}) + b(b+1)^{k-2} L(S_{11}).$$

Proof: The statement is trivial for $k = 1$. For $k = 2$, by definition, $Q_1 = L(S_{11}) + L(S_{12} + L(S_{11})) = L(S_{11}) + L(S_{12}) + b L(S_{11})$ in accordance with the assertion. Assume the lemma is true for $k = m > 2$ and suppose $k = m+1$. Again by definition, $Q_1 = Q_{1,m} + L(S_{1,m+1} + Q_{1,m})$ where $Q_{1,m}$ is the total

field life resulting from the ages $\{S_{1j}\}_{j=1}^m$. Since Condition A is

assumed, $L(S_{1,m+1} + Q_{1,m}) = L(S_{1,m+1}) + b Q_{1,m}$. In that case,

$Q_1 = L(S_{1,m+1}) + (b+1)Q_{1,m}$ and, by the inductive assumption,

$$(b+1)Q_{1,m} = \sum_{j=1}^m L(S_{1j}) + b \sum_{j=1}^m L(S_{1j}) + b(b+1) \sum_{j=1}^{m-1} L(S_{1j}) + \cdots + b(b+1)^{m-1} L(S_{11}),$$

and

$$Q_1 = \sum_{j=1}^{m+1} L(S_{1j}) + b \sum_{j=1}^m L(S_{1j}) + \cdots + b(b+1)^{m-1} L(S_{11})$$

which is the assertion for $k = m+1$. The lemma thus follows by induction.

q.e.d.

The next lemma gives a simple criterion in order that the ages satisfy Condition A which will be found useful in computing the total field life when FIFO is followed.

Lemma 4.2: The set $\{S_{1j}\}_{j=1}^k$ of initial ages satisfy Condition A if $S_{11} > S_{12} > \cdots > S_{1k}$.

Proof: First observe that if $x \in (0, -\frac{a}{b})$, then $x + L(x) < -\frac{a}{b}$. Consider the case $k = 2$ ($k=1$ having no meaning). Then $Q_1 = L(S_{11}) + L(S_{12} + L(S_{11}))$. Now S_{11} lies in the interval $(0, -\frac{a}{b})$ and $S_{12} + L(S_{11}) < S_{11} + L(S_{11}) < -\frac{a}{b}$ so that the assertion is true.

Assume the lemma is true for $k \leq m$ and suppose $k = m+1$. Then,

$$Q_1 = Q_{i,m-1} + L(S_{i,m} + Q_{i,m-1}) + L(S_{i,m+1} + Q_{i,m-1} + L(S_{i,m} + Q_{i,m-1}))$$

where $Q_{i,m-1}$ is the total field life resulting from $(S_{ij})_{j=1}^{m-1}$. By the inductive assumption, all arguments of L in the expression for $Q_{i,m-1}$ and $S_{i,m} + Q_{i,m-1}$ lie in $(0, -\frac{a}{b})$ and, moreover,

$$S_{i,m+1} + Q_{i,m-1} + L(S_{i,m} + Q_{i,m-1}) < S_{i,m} + Q_{i,m-1} + L(S_{i,m} + Q_{i,m-1}) < -\frac{a}{b}$$

which is the assertion for $k = m+1$.

The lemma thus follows by induction.

q.e.d.

The last two lemmas are concerned with the special policy of FIFO, the final one asserting, as one would intuitively surmise, that if FIFO is being considered, enlarging a stockpile can only increase the total field life.

Lemma 4.3: If the FIFO policy is followed on $(S_i)_{i=1}^n$, then the total field life, Q_n , of the policy is given by,

$$Q_n = \begin{cases} \sum_{i=1}^n L(S_i) + b \sum_{i=3}^n L(S_i) + b(b+1) \sum_{i=5}^n L(S_i) + \dots + b(b+1)^{\frac{n-4}{2}} \sum_{i=n-1}^n L(S_i) & \text{for } n \text{ even} \\ \sum_{i=1}^n L(S_i) + b \sum_{i=3}^n L(S_i) + \dots + b(b+1)^{\frac{n-5}{2}} \sum_{i=n-2}^n L(S_i) + b(b+1)^{\frac{n-3}{2}} L(S_n) & \text{for } n \text{ odd.} \end{cases}$$

Proof: First observe that, regardless of which items are issued M_1 in this policy, the corresponding sets of initial ages satisfy Condition A by lemma 4.2. Moreover, if $u \geq v \geq 0$ and $S \in (0, -\frac{a}{b})$, then

$$(4.2.2) \quad u + L(S+u) \geq v + L(S+v)$$

is a trivial consequence of the conditions on L .

Consider first the case where $n = 2N$ for some $N \geq 1$. Let $Q_{1,2k}$ be the total field life resulting from the issue of the first $2k$ items in source M_1 for $i = 1, 2$. Then, for $k = 1$, $Q_{1,2} = L(S_n)$, $Q_{2,2} = L(S_{n-1})$ and $Q_{1,2} < Q_{2,2}$.

Assume for $k = m > 1$, that M_1 receives items of initial ages $S_n, S_{n-2}, \dots, S_{n-2m+2}$ (hence M_2 receives those of ages $S_{n-1}, S_{n-3}, \dots, S_{n-2m+1}$) and that $Q_{2,2m-2} \leq Q_{1,2m} \leq Q_{2,2m}$. Then, $Q_{1,2m} \leq Q_{2,2m}$ implies $Q_{1,2m+2} = Q_{1,2m} + L(S_{n-2m} + Q_{1,2m}) \geq Q_{1,2m} + L(S_{n-2m+1} + Q_{1,2m})$ since $S_{n-2m} < S_{n-2m+1}$. But,

$$Q_{1,2m} + L(S_{n-2m+1} + Q_{1,2m}) \geq Q_{2,2m-2} + L(S_{n-2m+1} + Q_{2,2m-2}) = Q_{2,2m},$$

taking $u = Q_{1,2m}$ and $v = Q_{2,2m-2}$ in (4.2.2). Then, similarly,

$$Q_{1,2m+2} \geq Q_{2,2m} \text{ implies } Q_{2,2m+2} = Q_{2,2m} + L(S_{n-2m-1} + Q_{2,2m}) \geq$$

$Q_{1,2m} + L(S_{n-2m} + Q_{1,2m}) = Q_{1,2m+2}$. Thus M_1 receives items of initial ages $S_n, S_{n-2}, \dots, S_{n-2m+2}, S_{n-2m}$ and $Q_{2,2m} \leq Q_{1,2m+2} \leq Q_{2,2m+2}$. By induction (taking $k = N$), M_1 receives items of initial ages

$$S_n, S_{n-2}, \dots, S_2 \quad \text{and} \quad Q_{2,2N-2} \leq Q_{1,2N} \leq Q_{2,2N}.$$

If $n = 2N+1$, the preceding analysis applies to the first $2N = n-1$ items issued so that M_1 receives items of initial ages

$$S_n, S_{n-2}, \dots, S_3 \quad \text{and} \quad Q_{2,2N-2} \leq Q_{1,2N} \leq Q_{2,2N}$$

so that the last item, of initial age S_1 , is demanded by M_1 at time $Q_{1,2N}$ so that M_1 receives items $S_n, S_{n-2}, \dots, S_3, S_1$.

Applying lemma 4.1 when n is even, substitution yields,

$$(4.2.3) \quad Q_{1,n} = \sum_{i=1}^{\frac{n}{2}} L(S_{2i}) + b \sum_{i=2}^{\frac{n}{2}} L(S_{2i}) + b(b+1) \sum_{i=3}^{\frac{n}{2}} L(S_{2i}) + \dots + b(b+1)^{\frac{n}{2}-2} L(S_n),$$

and

$$Q_{2,n} = \sum_{i=1}^{\frac{n}{2}} L(S_{2i-1}) + b \sum_{i=2}^{\frac{n}{2}} L(S_{2i-1}) + b(b+1) \sum_{i=3}^{\frac{n}{2}} L(S_{2i-1}) + \dots + b(b+1)^{\frac{n}{2}-2} L(S_{n-1}).$$

Observe that in (4.2.3) the subscript 2i denotes the product and not the double subscript previously employed, a slight ambiguity which is immediately

resolved when the results are combined to give,

$$(4.2.4) \quad Q_{F_n} = Q_{1,n} + Q_{2,n} = \sum_{i=1}^n L(S_i) + b \sum_{i=3}^n L(S_i) + b(b+1) \sum_{i=5}^n L(S_i) + \dots + b(b+1)^{\frac{n}{2}-2} \sum_{i=n-1}^n L(S_i).$$

Similarly, applying lemma 4.1 and combining, for n odd,

$$Q_{1,n} = \sum_{i=1}^{\frac{n+1}{2}} L(S_{2i-1}) + b \sum_{i=2}^{\frac{n+1}{2}} L(S_{2i-1}) + \dots + b(b+1)^{\frac{n-5}{2}} \sum_{i=\frac{n-1}{2}}^{\frac{n+1}{2}} L(S_{2i-1}) + b(b+1)^{\frac{n-3}{2}} L(S_n)$$

$$(4.2.5) \quad Q_{2,n} = \sum_{i=1}^{\frac{n-1}{2}} L(S_{2i}) + b \sum_{i=2}^{\frac{n-1}{2}} L(S_{2i}) + \dots + b(b+1)^{\frac{n-7}{2}} \sum_{i=\frac{n-3}{2}}^{\frac{n-1}{2}} L(S_{2i}) + b(b+1)^{\frac{n-5}{2}} L(S_{n-1})$$

and

$$Q_{F_n} = \sum_{i=1}^n L(S_i) + b \sum_{i=3}^n L(S_i) + \dots + b(b+1)^{\frac{n-5}{2}} \sum_{i=n-2}^n L(S_i) + b(b+1)^{\frac{n-3}{2}} L(S_n)$$

and the lemma is proved.

q.e.d.

Lemma 4.4: Let $0 < S_1 < S_2 < \dots < S_N < -\frac{a}{b}$ be a set of given initial ages and Q_{F_N} the total field life obtained by issuing according to FIFO.

Let $M \geq 1$ additional items of initial ages $\{S_i^*\}_{i=1}^M$, $S_i^* \in (0, -\frac{a}{b})$ be given

and $Q_{F_{N+M}}$ the total field life obtained by issuing all $N+M$ items according

to FIFO. Then, $Q_{F_{N+M}} \geq Q_{F_N}$ for any N and M .

Proof: First consider the case where N is odd. Suppose $M = 1$.

Then three cases arise as follows.

(4.2.6) Case I: $S_1^* < S_1$. Then let $T_1 = S_1^*$, $T_j = S_{j+1}$ $j = 1, 2, \dots, N$.

Now $N+1$ is odd and, by lemma 4.3,

$$\begin{aligned}
 Q_{F_{N+1}} &= L(S_1^*) + \sum_{i=2}^{N+1} L(T_i) + b \sum_{i=3}^{N+1} L(T_i) + \dots + b(b+1) \sum_{i=N-2}^{N+1} L(T_i) + b(b+1) \sum_{i=N}^{N+1} L(T_i) \\
 &= L(S_1^*) + \sum_{i=1}^N L(S_i) + b \sum_{i=2}^N L(S_i) + \dots + b(b+1) \sum_{i=N-3}^N L(S_i) + b(b+1) \sum_{i=N-1}^N L(S_i) \\
 &= L(S_1^*) + \left[\sum_{i=1}^N L(S_i) + b \sum_{i=3}^N L(S_i) + \dots + b(b+1) \sum_{i=N-2}^N L(S_i) + b(b+1) \sum_{i=N}^N L(S_N) \right] \\
 &\quad + [b L(S_2) + \dots + b(b+1) \sum_{i=N-3}^N L(S_i) + b(b+1) \sum_{i=N-1}^N L(S_{N-1})] \\
 &\geq L(S_1^*) [1 + b + \dots + b(b+1) \sum_{i=N-3}^N 1 + b(b+1) \sum_{i=N-1}^N 1] + Q_{F_N} \text{ since } L(S_1^*) \geq L(S_j)
 \end{aligned}$$

$j = 2, 4, 6, \dots, N-3, N-1$, $b < 0$. Using the easily verified fact that

(4.2.7) $(b+1)^k = 1 + b + b(b+1) + \dots + b(b+1)^{k-1}$ for any integer k ,

$$Q_{F_{N+1}} \geq Q_{F_N} + (b+1) \sum_{i=1}^{N-1} L(S_i^*) \geq Q_{F_N} \text{ since } b+1 > 0.$$

(4.2.8) Case II: $S_N < S_1^*$. Then let $T_j = S_j$, $j = 1, 2, \dots, N$ and

$T_{N+1} = S_1^*$. By lemma 4.3,

$$\begin{aligned} Q_{F_{N+1}} &= \sum_{i=1}^{N+1} L(T_i) + b \sum_{i=3}^{N+1} L(T_i) + b(b+1) \sum_{i=5}^{N+1} L(T_i) + \dots + b(b+1)^{\frac{N-3}{2}} \sum_{i=N}^{N+1} L(T_i) \\ &= L(S_1^*) [1 + b + b(b+1) + \dots + b(b+1)^{\frac{N-3}{2}}] + \sum_{i=1}^N L(S_i) + b \sum_{i=3}^N L(S_i) + \dots + b(b+1)^{\frac{N-3}{2}} L(S_N) \\ &= b(b+1)^{\frac{N-1}{2}} L(S_1^*) + Q_{F_N} \geq Q_{F_N}. \end{aligned}$$

(4.2.9) Case III: There is integer $1 \in \{1, 2, \dots, N\}$ for which $S_1 \leq S_1^* \leq S_{1+1}$.

Then let $T_j = S_j$, $j = 1, \dots, i$; $T_{i+1} = S_1^*$ and $T_j = S_{j-1}$, $j = i+2, \dots, N+1$.

Then $T_1 < T_2 < \dots < T_{N+1}$, $N+1$ is even and $2k-1 \leq i+1 \leq 2k+1$ for some

$k \in \{1, 2, \dots, N\}$. Again, by lemma 4.3,

$$\begin{aligned} Q_{F_{N+1}} &= \sum_{j=1}^{N+1} L(T_j) + b \sum_{j=3}^{N+1} L(T_j) + \dots + b(b+1)^{k-2} \sum_{j=2k-1}^{N+1} L(T_j) + b(b+1)^{k-1} \sum_{j=2k+1}^{N+1} L(T_j) + \dots \\ &\quad + b(b+1)^{\frac{N-3}{2}} \sum_{j=N}^{N+1} L(T_j) \\ &= L(S_1^*) [1 + b + \dots + b(b+1)^{k-2}] + \sum_{j=1}^N L(S_j) + b \sum_{j=3}^N L(S_j) + \dots + b(b+1)^{k-2} \sum_{j=2k-1}^N L(S_j) \\ &\quad + \sum_{m=k-1}^{\frac{N-3}{2}} b(b+1)^m \sum_{j=2m+3}^{N+1} L(T_j). \end{aligned}$$

$$\begin{aligned} \text{But } \sum_{m=k-1}^{\frac{N-3}{2}} b(b+1)^m \sum_{j=2m+3}^{N+1} L(T_j) &= \sum_{m=k-1}^{\frac{N-3}{2}} b(b+1)^m \sum_{j=2m+2}^N L(S_j) \\ &= \sum_{m=k-1}^{\frac{N-3}{2}} b(b+1)^m L(S_{2m+2}) + \sum_{m=k-1}^{\frac{N-3}{2}} b(b+1)^m \sum_{j=2m+3}^N L(S_j) \end{aligned}$$

so that

$$\begin{aligned} Q_{F_{N+1}} &= L(S_1^*) [1+b+\dots+b(b+1)^{k-2}] + \sum_{m=k-1}^{\frac{N-3}{2}} b(b+1)^m L(S_{2m+2}) \\ &\quad + \left[\sum_{j=1}^N L(S_j) + b \sum_{j=3}^N L(S_j) + \dots + b(b+1)^{k-2} \sum_{j=2k-1}^N L(S_j) + b(b+1)^{k-1} \sum_{j=2k+1}^N L(S_j) + \dots \right. \\ &\quad \left. + b(b+1)^{\frac{N-3}{2}} L(S_N) \right] \end{aligned}$$

Now $2k-1 \leq i+1 \leq 2k+1$ implies $i+1 \leq 2k$ and hence

$$S_1^* \leq S_{i+1} < S_{2k} < S_{2k+3} < \dots < S_{N-1} \text{ so that } L(S_1^*) \geq L(S_{2m+2})$$

and, since

$$b(b+1)^m < 0, \quad b(b+1)^m L(S_{2m+2}) \geq b(b+1)^m L(S_1^*) \quad \text{all for } m=k-1, k, \dots, \frac{N-3}{2}.$$

Consequently,

$$\begin{aligned} Q_{F_{N+1}} &\geq L(S_1^*) [1+b+\dots+b(b+1)^{k-2} + b(b+1)^{k-1} + \dots + b(b+1)^{\frac{N-3}{2}}] + Q_{F_N} \\ &= L(S_1^*) (b+1)^{\frac{N-1}{2}} + Q_{F_N} \geq Q_{F_N}. \end{aligned}$$

Thus, in all cases, $Q_{F_{N+1}} \geq Q_{F_N}$ whenever N is odd. The proof for the case N even is, mutatis mutandis, the same.

Now suppose the lemma is true for $M > 1$ and consider adding $M + 1$ items of initial ages $\{S_i^*\}_{i=1}^{M+1}$. Ignoring S_{M+1}^* temporarily, the total field life of the remaining items, $Q_{F_{N+M}}$ satisfies $Q_{F_{N+M}} \geq Q_{F_N}$ by the inductive assumption. Then, adding S_{M+1}^* can only increase the total field life by the case $M = 1$, i.e., $Q_{F_{N+M+1}} \geq Q_{F_{N+M}} \geq Q_{F_N}$.

By induction, the lemma is proved. q.e.d.

With the aid of the above lemmas, Theorem 4.1 admits the following proof.

Proof of Theorem 4.1: Consider first the case where n is odd so that $n = 2N+1$ where $N > 1$. Let Q_n denote the total field life of the FIFO policy. For any other policy a certain number, say k , of items will be issued M_1 and, in accordance with lemma 4.1, the order of issue is specified by $\{S_{1j}\}_{j=1}^k$. Now, without loss of generality, it is supposed that $k \leq N$ for if $k > N$ then, because of the symmetric roles of M_1 and M_2 , there corresponds a case $k' < N$ yielding the same total field life. For this arbitrary policy, let Q_1 denote the field life contributed by M_1 , $i=1,2$, and $Q = Q_1 + Q_2$ the total field life of this arbitrary policy.

Suppose the sets of initial ages $\{S_{1j}\}_{j=1}^k$ and $\{S_{2j}\}_{j=1}^{n-k}$ both satisfy Condition A. By lemma 4.1,

$$Q_1 = \sum_{j=1}^k L(S_{1j}) + b \sum_{j=1}^{k-1} L(S_{1j}) + \dots + b(b+1)^{k-3} \sum_{j=1}^2 L(S_{1j}) + b(b+1)^{k-2} L(S_{11}), \text{ and}$$

(4.2.9)

$$Q_2 = \sum_{j=1}^{n-k} L(S_{2j}) + b \sum_{j=1}^{n-k-1} L(S_{2j}) + \dots + b(b+1)^{k-3} \sum_{j=1}^{n-2k+2} L(S_{2j}) + b(b+1)^{k-2} \sum_{j=1}^{n-2k+1} L(S_{2j})$$

$$+ b(b+1)^{k-1} \sum_{j=1}^{n-2k} L(S_{2j}) + b(b+1)^k \sum_{j=1}^{n-2k-1} L(S_{2j}) + \dots + b(b+1)^{N-1} \sum_{j=1}^{N-k+1} L(S_{2j})$$

$$+ b(b+1)^N \sum_{j=1}^{N-k} L(S_{2j}) + \dots + b(b+1)^{n-k-2} L(S_{21}) . \text{ Combining,}$$

$$Q = \sum_{j=1}^n L(S_j) + b \left[\sum_{j=1}^{k-1} L(S_{1j}) + \sum_{j=1}^{n-k-1} L(S_{2j}) \right] + \dots + b(b+1)^{k-2} \left[L(S_{11}) + \sum_{j=1}^{n-2k+1} L(S_{2j}) \right]$$

$$+ b(b+1)^{k-1} \sum_{j=1}^{n-2k} L(S_{2j}) + b(b+1)^k \sum_{j=1}^{n-2k-1} L(S_{2j}) + \dots + b(b+1)^{N-1} \sum_{j=1}^{N-k+1} L(S_{2j})$$

$$+ b(b+1)^N \sum_{j=1}^{N-k} L(S_{2j}) + \dots + b(b+1)^{n-k-2} L(S_{21}) . \text{ Finally,}$$

$$(4.2.10) \quad Q = \sum_{j=1}^n L(S_j) + \sum_{j=0}^{k-2} (b(b+1))^j \left[\sum_{i=1}^{k-j-1} L(S_{1i}) + \sum_{i=1}^{n-k-j-1} L(S_{2i}) \right]$$

$$+ \sum_{j=k-1}^{N-1} (b(b+1))^j \sum_{i=1}^{n-k-j-1} L(S_{2i}) + \sum_{j=N}^{n-k-2} b(b+1)^j \sum_{i=1}^{n-k-j-1} L(S_{2i}) .$$

By lemma 4.3,

$$Q_{F_n} = \sum_{i=1}^n L(S_i) + b \sum_{i=3}^n L(S_i) + \dots + b(b+1)^{k-2} \sum_{i=2k-1}^n L(S_i) + b(b+1)^{k-1} \sum_{i=2k+1}^n L(S_i) \\ + \dots + b(b+1)^{N-2} \sum_{i=n-2}^n L(S_i) + b(b+1)^{N-1} L(S_n) \quad \text{or,}$$

$$(4.2.11) \quad Q_{F_n} = \sum_{i=1}^n L(S_i) + \sum_{j=0}^{k-2} b(b+1)^j \sum_{i=2j+3}^n L(S_i) + \sum_{j=k-1}^{N-1} b(b+1)^j \sum_{i=2j+3}^n L(S_i).$$

Consequently, using (4.2.10) and (4.2.11),

$$(4.2.12) \quad Q_{F_n} - Q = \sum_{j=0}^{k-2} b(b+1)^j \left[\sum_{i=2j+3}^n L(S_i) - \sum_{i=1}^{n-k-1} L(S_{1i}) - \sum_{i=1}^{n-k-j-1} L(S_{2i}) \right] \\ + \sum_{j=k-1}^{N-1} b(b+1)^j \left[\sum_{i=2j+3}^n L(S_i) - \sum_{i=1}^{n-k-j-1} L(S_{2i}) \right] - \sum_{j=N}^{n-k-2} b(b+1)^j \sum_{i=1}^{n-k-j-1} L(S_{2i}).$$

Consider $j \in \{0, 1, \dots, k-2\}$. Since $\sum_{i=2j+3}^n L(S_i)$ is the smallest possible sum of the $L(S_i)$ having $n-2j-2$ such terms, $\sum_{i=2j+3}^n L(S_i) - \sum_{i=1}^{n-k-1} L(S_{1i})$ $- \sum_{i=1}^{n-k-j-1} L(S_{2i}) \leq 0$ and, since $b(b+1)^j < 0$, the first term on the right in (4.2.12) is non-negative.

If $j \in \{k-1, \dots, N-1\}$, then $0 \leq j-k+1 \leq N-k$ and

$$b(b+1)^j \left[\sum_{i=2j+3}^n L(S_1) - \sum_{i=1}^{n-k-j-1} L(S_{2i}) \right] = b(b+1)^j \left[\sum_{i=2j+3}^n L(S_1) - \sum_{i=1}^{n-2j-2} L(S_{2i}) \right] \\ - b(b+1)^j \sum_{i=n-2j-1}^{n-k-j-1} L(S_{2i}),$$

the first term of which is non-negative as above and the second term of which is non-negative since $b < 0$. Thus the second term on the right in (4.2.12) is non-negative.

Finally, if $j \in \{N, \dots, n-k-2\}$, $b(b+1)^j < 0$ and the last term on the right in (4.2.12) is non-negative. This shows that $Q_F \geq Q$ so that FIFO dominates. The proof for the case n even and the initial ages of an arbitrary policy satisfying Condition A is, mutatis mutandis, the same.

Now suppose n is arbitrary and consider any issue policy where the initial ages do not satisfy Condition A. Then there are exactly M items, say, that contribute nothing to the value of Q , the total field life, where $1 \leq M \leq n-2$. In other words, Q is the total field life based on the issue of $n-M$ items that do satisfy Condition A. Let Q_{n-M} denote the total field life that would be obtained had only these $n-M$ items been issued in FIFO order. Then, by the preceding part of the proof, $Q_F \geq Q$ and, by lemma 4.4, $Q_F \geq Q_{n-M} \geq Q$.

Thus, in all cases, $Q_{T_n} \geq Q$ so that FIFO is optimal. q.e.d.

A result similar to Theorem 4.1 holds for LIFO without the restriction to linear functions.

Theorem 4.2: Suppose $L(S)$ is either a convex or concave differentiable function with $L'(S) < -1$. Then LIFO is optimal for two sources M_1 and M_2 .

Proof: In both cases, as in Section 2, L has a finite truncation point S_0 . Moreover, as in the proofs of Theorems 2.4 and 2.6, $S + L(S) \geq S_0$ or $L(S + L(S)) = 0$ for $S \in (0, S_0)$. Let $0 < S_1 < S_2 < \dots < S_n < S_0$ be n given initial ages, where $n > 2$.

Consider the LIFO issue policy with total field life Q_L . By convention the item of initial age S_1 is issued M_1 and that of age S_2 to M_2 to start the operation. Now, for any $i \geq 3$ and $j = 1, 2$,

$$(4.2.13) \quad S_1 + L(S_j) \geq S_j + L(S_j) \geq S_0,$$

and so $L(S_1 + L(S_j)) = 0$. Thus, in the LIFO policy none of the items of initial ages $S_3, \dots, S_n$ contribute to Q_L or $Q_L = L(S_1) + L(S_2)$.

Now consider an arbitrary issue policy with total field life Q and, as in Theorem 4.1, let $\{S_{1j}\}_{j=1}^k$ and $\{S_{2j}\}_{j=1}^{n-k}$ be the initial ages of those items which would be issued M_1 and M_2 respectively, with respective

field life contributions Q_1 and Q_2 so that $Q = Q_1 + Q_2$. Let $\{T_{1j}\}_{j=1}^k$ be the set $\{S_{1j}\}_{j=1}^k$ ordered by $<$, i.e., $T_{1j} \in \{S_{1j}\}_{j=1}^k$ for $j = 1, 2, \dots, k$ and $T_{11} < T_{12} < \dots < T_{1k}$. Let $\{T_{2j}\}_{j=1}^{n-k}$ be similarly defined.

Ignoring M_2 , and issuing items $\{S_{1j}\}_{j=1}^k$ according to the order $\{T_{1j}\}_{j=1}^k$ (which is LIFO for M_1 with ages $\{S_{1j}\}_{j=1}^k$) would yield a field life, say Q'_1 , and, by Theorem 2.4, $Q'_1 \geq Q_1$. Defining Q'_2 similarly, $Q'_2 \geq Q_2$. But $T_{1j} \geq T_{11}$, $j = 2, \dots, k$ implies

$$T_{1j} + L(T_{11}) \geq T_{11} + L(T_{11}) \geq S_0 \quad \text{or} \quad L(T_{1j} + L(T_{11})) = 0 \quad j = 2, \dots, k,$$

so that $Q'_1 = L(T_{11})$ and, similarly, $Q'_2 = L(T_{21})$ and it follows that $Q = Q_1 + Q_2 \leq Q'_1 + Q'_2 = L(T_{11}) + L(T_{21})$. But,

$$L(S_1) + L(S_2) = \max_{\substack{i \neq j \\ 1, j \in \{1, \dots, n\}}} [L(S_i) + L(S_j)]$$

so that, $Q_L = L(S_1) + L(S_2) \geq L(T_{11}) + L(T_{21}) \geq Q$ and hence LIFO is optimal. q.e.d.

4.3 General Inductive Theorem

The characterization of the class of functions yielding LIFO and FIFO as optimal policies in 4.2 is admittedly far from satisfactory.

This section will be closed with a general theorem which extends theorems 4.1 and 4.2 from two to an arbitrary number, v , of demand sources.

Theorem 4.3: Suppose FIFO (LIFO) is optimal for $v = 1, 2$ demand sources. Then FIFO (LIFO) is optimal for $v > 2$ demand sources.

Proof: Assume FIFO is optimal when $v = 1, 2$ and let $M_1, M_2, \dots, M_v$, $v > 2$ be the demand sources with initial ages $0 < S_1 < S_2 < \dots < S_n$ where $n > v$. If the n items are issued according to FIFO then a certain number n_1 of items of initial ages $S_{11}, S_{12}, \dots, S_{1n_1}$ are issued M_1 , where the second subscript, as before, denotes the order of issue. For fixed $i \in \{1, 2, \dots, v\}$, $S_{11} > S_{12} > \dots > S_{1n_1}$ clearly so that the contribution to the total field life by M_1 alone cannot be strictly improved by issuing only these items in any other order since FIFO is optimal for $v = 1$.

Moreover, for two sources M_1 and M_j , the items issued these sources, taken in totality and ignoring the other sources, preserve the FIFO order (simply by deleting the items in the original FIFO order which were not issued M_1 and M_j). Hence the total field life of these two sources alone cannot be strictly improved since FIFO is optimal for $v = 2$. Thus, any policy not having at least these properties cannot be considered optimal since the total field life of such a policy could be improved (though possibly not strictly so) by ignoring the other sources and changing to FIFO order in the one source, or two sources, as the case may be.

Now consider any policy other than FIFO. Then there is an integer $i \in \{1, 2, \dots, n\}$ for which the item of initial age S_1 was issued prior to that of age S_{i+1} . Either these two items were issued the same source or two different sources. In either case the total field life of this policy can be improved by issuing in the opposite order and ignoring the other sources. By the above remarks, this policy cannot be optimal.

Thus FIFO yields a total field life at least as large as any other policy and hence is optimal.

The proof for LIFO is, mutatis mutandis, the same. q.e.d.

5. Model II - A Stochastic Version of Model I

5.1 Definition of Model II

The usefulness of the preceding results is, of course, dependent upon the exact knowledge of the field life function. This is rarely the case in practice and hence one must be content with knowing the general nature of $L(S)$ within limits, viz., concavity, convexity and derivatives, and apply the results as approximations. On the other hand, it is natural to introduce randomness into field life and, on the basis of assumptions as to the distributions of the resulting random variables, hope to determine optimal (in some sense) issue policies.

The simplest natural generalization of Model I to a stochastic model is Model II, defined by the following set of assumptions.

- (1) Assumptions (i),(iii) and (iv) of Model I given in (2.1)
(5.1) are to hold.
- (ii) The field life of an item is a non-negative random variable $X(S)$ dependent on the age, S , of the item upon being issued, where $S \geq 0$.

Under the above set of assumptions it is seen that field life $X(S)$ defines, as S ranges over its set of possible values, a non-negative stochastic process. If some distribution is imposed on the process, a mean value function will be thereby determined. The total field life, say Q_n , of a given issue policy is now the sum of n dependent random variables. As in [1] let $U_n = EQ_n$, the expected value of Q_n , to be called the utility of the issue policy. A policy which maximizes this utility will then be called optimal for the stockpile of n items.

Since Q_n is the sum of n dependent random variables, a natural plan of attack would seem to be that of iterating conditional expectations in order to compute the utility. It might then be possible to compare the utilities of all possible policies and select the optimal one. This general method is illustrated in the following section.

5.2 Some Optimal Conditions

The first theorem gives a set of sufficient conditions for FIFO to be optimal.

Theorem 5.1: Suppose for each $S \geq 0$, $X(S)$ has density

$$\frac{1}{[L(S)]^{\alpha+1} \Gamma(\alpha+1)} x^{\alpha} e^{-\frac{x}{L(S)}} \quad \text{for } x \geq 0$$

where $L(S) = a + bS$ with $a > 0$, $b > 0$ and $\alpha > -1$. Then FIFO is an optimal policy for $n \geq 2$.

Proof: Suppose $n = 2$ and $0 < S_1 < S_2$ are given initial ages.

Let Q_F and Q_L denote the total field life of the FIFO and LIFO policies, respectively, with $U_F = EQ_F$ and $U_L = EQ_L$ the corresponding utilities.

Now according to the model, $Q_F = X(S_2) + X(S_1 + X(S_2))$ and $Q_L = X(S_1) + X(S_2 + X(S_1))$. Let $Y_2 = X(S_2)$, $Y_1 = X(S_1 + X(S_2)) = X(S_1 + Y_2)$. Then, since the density assumed is the Γ -family with mean value $(\alpha+1)L(S)$, $EY_2 = (\alpha+1)L(S_2)$. Moreover,

$$E(Y_1|Y_2) = (\alpha+1)L(S_1 + Y_2) = (\alpha+1)L(S_1) + b(\alpha+1)Y_2 \quad \text{so that}$$

$$EY_1 = EE(Y_1|Y_2) = (\alpha+1)L(S_1) + b(\alpha+1)EY_2 = (\alpha+1)L(S_1) + b(\alpha+1)^2L(S_2).$$

Thus

$$U_F = EY_2 + EY_1 = b(\alpha+1)^2L(S_2) + (\alpha+1)L(S_1) + (\alpha+1)L(S_2). \quad \text{Similarly,}$$

$$U_L = b(\alpha+1)^2L(S_1) + (\alpha+1)L(S_1) + (\alpha+1)L(S_2) \quad \text{so that } U_F - U_L = b(\alpha+1)^2[L(S_2) - L(S_1)] \geq 0 \quad \text{since } S_1 < S_2 \text{ and } L \text{ is non-decreasing.}$$

Thus $U_F \geq U_L$ and FIFO is optimal for $n = 2$.

Assume FIFO is optimal for $n = k$ and suppose $n = k+1$. Let $0 < S_1 < S_2 < \dots < S_{k+1}$ be given initial ages and S^* denote a fixed member of $\{S_1, S_2, \dots, S_{k+1}\}$ to be the initial age of the last item issued. For any one of the $k!$ policies resulting from fixed S^* , let Q_k^* denote the total field life of the first k items issued with utility U_k^* . Then, the total field life, Q_{k+1}^* , of the stockpile is given by $Q_{k+1}^* = Q_k^* + X(S^* + Q_k^*)$ and has utility $U_{k+1}^* = U_k^* + E\{X(S^* + Q_k^*)\}$.

$$\text{Now } E\{X(S^* + Q_k^*) | Q_k^*\} = (\alpha + 1)L(S^* + Q_k^*) = (\alpha + 1)L(S^*) + b(\alpha + 1)Q_k^*$$

$$\text{and hence } E\{X(S^* + Q_k^*)\} = (\alpha + 1)L(S^*) + b(\alpha + 1)EQ_k^* = (\alpha + 1)L(S^*) + b(\alpha + 1)U_k^*.$$

Thus, $U_{k+1}^* = (\alpha + 1)L(S^*) + [1 + b(\alpha + 1)] U_k^*$, an increasing function of U_k^* . But by the induction assumption, U_k^* is maximized by issuing according to FIFO while U_{k+1}^* is maximized by making U_k^* as large as possible. From this point the proof is, mutatis mutandis, the same as that of Theorem 2.3. q.e.d.

In some practical situations it may happen that the stockpile consists of only two items as, for example, stocking heavy and/or expensive equipment. If this is the case, the next theorem gives a set of conditions for LIFO to be the optimal policy.

Theorem 5.2: Suppose for each $S \geq 0$, $X(S)$ has density

$$\frac{1}{[L(S)]^{\alpha+1} \Gamma(\alpha+1)} x^\alpha e^{-\frac{x}{L(S)}} \quad \text{for } x \geq 0$$

where $L(S) = e^{-kS}$ $k > 0$ and integer $\alpha > -1$. Then LIFO is optimal whenever $n=2$.

Proof: Let $0 < S_1 < S_2$ be given initial ages and observe that $E\{X(S)\} = (\alpha+1)L(S)$. As before denote total field life and utility by Q_L, U_L, Q_F, U_F for the respective policies of LIFO and FIFO. The fact that,

$$(5.2) \quad \int_0^{\infty} \omega^a e^{-b\omega} d\omega = \frac{\Gamma(a+1)}{b^{a+1}} \quad \text{for } a > -1, b > 0,$$

which is easily verified by the change of variable $z = b\omega$, will be useful.

Letting $Y_1 = X(S_1)$ and $Y_2 = X(S_2 + X(S_1)) = X(S_2 + Y_1)$, $Q_L = X(S_1) + X(S_2 + X(S_1)) = Y_1 + Y_2$ and hence $U_L = EY_1 + EY_2$. Now, since $L(x+y) = L(x)L(y)$ for $x \geq 0, y \geq 0$,

$$(5.3) \quad E(Y_2|Y_1) = (\alpha+1)L(S_2 + Y_1) = (\alpha+1)L(S_2)L(Y_1) \quad \text{so that}$$

$$EY_2 = EE(Y_2|Y_1) = (\alpha+1)L(S_2)E(L(Y_1)).$$

$$\text{But } E(L(Y_1)) = E(e^{-kY_1}) = \int_0^\infty \frac{e^{-ky_1} y_1^\alpha e^{-\frac{y_1}{L(S_1)}}}{[L(S_1)]^{\alpha+1} \Gamma(\alpha+1)} dy_1$$

$$= \frac{1}{[L(S_1)]^{\alpha+1} \Gamma(\alpha+1)} \int_0^\infty y_1^\alpha e^{-\left[\frac{1+kL(S_1)}{L(S_1)}\right] y_1} dy_1$$

$$= \frac{1}{[L(S_1)]^{\alpha+1} \Gamma(\alpha+1)} \cdot \frac{\Gamma(\alpha+1)[L(S_1)]^{\alpha+1}}{[1+kL(S_1)]^{\alpha+1}} = \frac{1}{[1+kL(S_1)]^{\alpha+1}}, \text{ applying (5.2).}$$

Substituting in (5.3), $EY_2 = \frac{(\alpha+1)L(S_2)}{[1+kL(S_1)]^{\alpha+1}}$ while $EY_1 = (\alpha+1)L(S_1)$ so

that $U_L = (\alpha+1) \left[L(S_1) + \frac{L(S_2)}{(1+kL(S_1))^{\alpha+1}} \right]$. Similarly,

(5.4)

$$U_F = (\alpha+1) \left[L(S_2) + \frac{L(S_1)}{(1+kL(S_2))^{\alpha+1}} \right]$$

Now define $F(S) = L(S_1) - L(S) + \frac{L(S)}{[1+kL(S_1)]^{\alpha+1}} - \frac{L(S_1)}{[1+kL(S)]^{\alpha+1}}$

and observe that $U_L - U_F = (\alpha+1)F(S_2)$. The theorem will be proved

then if it can be shown that $F(S) \geq 0$ whenever $S > S_1$.

But $F(S_1) = 0$ and, since $\lim_{S \rightarrow \infty} L(S) = 0$, $\lim_{S \rightarrow \infty} F(S) = 0$.

Also, since $L'(S) = -kL(S)$,

$$\begin{aligned} F'(S) &= -L'(S) + \frac{L'(S)}{[1+kL(S_1)]^{\alpha+1}} + \frac{L(S_1)(\alpha+1)[1+kL(S)]^{\alpha} \cdot kL'(S)}{[1+kL(S)]^{2\alpha+2}} \\ (5.5) \quad &= kL(S) - \frac{kL(S)}{[1+kL(S_1)]^{\alpha+1}} - \frac{L(S)L(S_1)(\alpha+1)k^2}{[1+kL(S)]^{\alpha+2}} \end{aligned}$$

= $kL(S)G(S)$, where

$$G(S) = 1 - \frac{1}{[1+kL(S_1)]^{\alpha+1}} - \frac{k(\alpha+1)L(S_1)}{[1+kL(S)]^{\alpha+2}}.$$

Now, $G'(S) = -\frac{k^3(\alpha+1)(\alpha+2)L(S_1)L(S)}{[1+kL(S)]^{\alpha+3}} < 0$ so that G is decreasing.

If $G(S) < 0$ for all $S > S_1$, $F'(S) = kL(S)G(S) < 0$ by (5.5) and F is decreasing on (S_1, ∞) which cannot be since $F(S_1) = 0 = \lim_{S \rightarrow \infty} F(S)$.

Similarly it cannot be true that $G(S) > 0$ for all $S > S_1$. Then,

$$\begin{aligned}
 G(S_1) &= 1 - \frac{1}{[1+kL(S_1)]^{\alpha+1}} - \frac{k(\alpha+1)L(S_1)}{[1+kL(S_1)]^{\alpha+2}} \\
 &= \frac{[1+kL(S_1)]^{\alpha+2} - [1+kL(S_1)] - k(\alpha+1)L(S_1)}{[1+kL(S_1)]^{\alpha+2}} \\
 (5.6) \quad &= \frac{[1+kL(S_1)]^{\alpha+2} - 1 - k(\alpha+2)L(S_1)}{[1+kL(S_1)]^{\alpha+2}} \\
 &= \frac{1+k(\alpha+2)L(S_1) + \sum_{j=2}^{\alpha+2} \binom{\alpha+2}{j} [kL(S_1)]^j - 1 - k(\alpha+2)L(S_1)}{[1+kL(S_1)]^{\alpha+2}} \\
 &= \frac{\sum_{j=2}^{\alpha+2} \binom{\alpha+2}{j} [kL(S_1)]^j}{[1+kL(S_1)]^{\alpha+2}} > 0.
 \end{aligned}$$

But, since $G(S_1) > 0$ and decreasing while it is not true that $G(S) > 0$ for all $S > S_1$, there exists $S_0 > S_1$ for which $G(S) \geq 0$ on $[S_1, S_0]$ and $G(S) < 0$ for all $S > S_0$. But then, by (5.5), $F'(S) \geq 0$ on $[S_1, S_0]$

and $F'(S) < 0$ on (S_0, ∞) or $F(S)$ is non-decreasing on $[S, S_0]$ and decreasing on (S_0, ∞) which, together with $F(S_1) = 0 = \lim_{S \rightarrow \infty} F(S)$ implies $F(S) \geq 0$ for all $S \geq S_1$. q.e.d.

It is freely admitted that it may be difficult to find applications of Theorem 5.1 because of the assumption of an increasing mean value function. On the other hand, Theorem 5.2 holds only for the case $n=2$. However, any attempt to impose decreasing mean value functions of the types given in Sections 3 and 4 for which optimality of the two policies considered was established, generally carries with it the necessity of truncating at a finite point S_0 whence it is necessary to define $X(S) \equiv 0$ for $S \geq S_0$ to preserve the non-negativity of the process. The difficulties caused before of such truncation are magnified here and appear insurmountable. Moreover, in any case, the extreme dependence of the present state of the process on the entire past results, upon attempting the iterative procedure beyond two steps, in expressions for which the corresponding conditional expectations are completely unwieldy. Attempts at a general inductive theorem to carry optimality from the case $n = 2$ to arbitrary n have only resulted in failure and this technique will be explored no further.

5.3 Modification to an Independent Demand Schedule.

The above-mentioned difficulties can, to a certain extent, be overcome by relaxing (iii) of (2.1) (which is to hold for Model II) in

accordance with [1] . Thus, instead of waiting to issue an item until the preceeding one has failed it will be assumed that there is a scheduling interval $t > 0$, an item is issued to begin the operation, and one is issued every t units of time thereafter until the stockpile is depleted. With this modification of Model II , field life $X(S)$ is still a non-negative stochastic process with non-negative mean value $U(S)$. An issue policy is again optimal if it maximizes the total utility.

Derman and Klein have shown (Theorem 2 of [1]) that if $U(S)$ is convex then LIFO is optimal regardless of the size of the stockpile. They remark that if $U(S)$ is concave, FIFO is an optimal policy. However, in the case of monotone decreasing $U(S)$ there may be a truncation point $S_0 < \infty$ for U (necessarily so if U is concave). Then one is forced to define $U(S) \equiv 0$ hence $X(S) \equiv 0$ for $S \geq S_0$ in order to comply with the model. Now in the convex case, $U(S)$, though so truncated is still convex and the optimality of LIFO is preserved as in Theorem 2 of [1]. In the concave case, however, $U(S)$ when so truncated is no longer concave on $(0, \infty)$. Then it may very well be that FIFO is no longer optimal even though the initial ages are restricted to $(0, S_0)$. That this is indeed the case is seen in the following example.

(5.7) Example: Let $U(S) = 2 - \frac{1}{2} S^2$ for $0 \leq S \leq 2$ with truncation point $S_0 = 2$. Suppose $S_1 = 0.1$, $S_2 = 1.6$ and $t = 1.5$. Then, in the notation previously adopted, $Q_L = X(0.1) + X(3.1)$, $U_L = U(0.1) + U(3.1) = 1.995$ while $Q_F = X(1.6) + X(1.6)$, $U_F = U(1.6) + U(1.6) = 2U(1.6) = 1.44$, since

$U(3.1) = 0$. Thus $U_F < U_L$ and hence FIFO is not optimal even for $n = 2$.

The simplest condition to impose on concave $U(S)$ in order that FIFO be optimal is one similar to Condition A of Section 4. This is embodied in the following theorem.

Theorem 5.3: Suppose $U(S)$ is a concave function with truncation point $S_0 \leq \infty$. If the initial ages $0 < S_1 < S_2 < \dots < S_n$ satisfy the condition $S_n + (n-1)t < S_0$, then FIFO is optimal.

Proof: For any $i \leq n$ and $k \leq n-1$, $S_i + kt \leq S_n + (n-1)t < S_0$ so that $S_i + kt \in (S | U(S) > 0)$. Thus any issue policy has the property that all arguments of U in the expression for total utility are in the region of concavity of U . Then the proof given for Theorem 2 in [1] applies with the reversal of inequalities for U concave and FIFO is optimal. q.e.d.

The condition $S_n + (n-1)t < S_0$ may be checked prior to the operation. If the condition is not satisfied, no general statement can be made as the following example shows.

(5.8) **Example:** Let $U(S)$ be defined as in (5.7) and $S_1 = 0.1$, $S_2 = 0.6$ with $t = 1.5$. Then $S_2 + t = 2.1 > 2$ so that the condition of Theorem 5.3 is not satisfied. Still, $U_F = U(.6) + U(1.6) = 2.54 > 1.95 = U(.1) + U(2.1) = U_L$.

Thus when the condition of Theorem 5.3 is not satisfied the choice of a policy depends on the relative initial ages and t , the scheduling interval.

An interesting situation results whenever $U(S) = a + bS$, $a > 0$ with truncation point $-\frac{a}{b}$ if $b < 0$ and $+\infty$ if $b \geq 0$. Suppose the

initial ages $0 < S_1 < S_2 < \dots < S_n < S_0$ satisfy the condition $S_n + (n-1)t < S_0$. In this case both Theorem 5.3 and Theorem 2 of [1] apply so that both LIFO and FIFO are optimal regardless of b . This is contrasted with the same case in the deterministic model where the value of b was critical. Indeed, even more is true in this case. Since, for any $1 \leq n$ and $k \leq n-1$, $S_1 + kt \leq S_n + (n-1)t < S_0$, it follows that $U(S_1 + kt) = U(S_1) + kbt$ and it is readily verified that the total utility of any issue policy whatever is given by $\sum_{i=1}^n U(S_i) + \frac{n(n-1)bt}{2}$. Thus all policies are optimal. It should be remarked that if $S_n + (n-1)t \geq S_0$, Theorem 5.3 no longer applies and one should then follow the LIFO policy since the utility is then at least as large as any other policy and may be larger.

Theorem 2 of [1] may be immediately extended to the case of more than one demand. Using the notation of Section 4, let M_i denote the i^{th} demand source, $i=1, \dots, v$. It is assumed that the present model holds, that v items are issued to begin the operation and v items are issued every t units of time thereafter until the stockpile is depleted.

Theorem 5.4: If $U(S)$ is a convex function and $n > v$, then LIFO is the optimal issue policy for arbitrary $v \geq 2$.

Proof: First consider the case for $v = 2$. If $n = 3$, there are three distinct policies having utilities,

$$\begin{aligned}
 U_L &= U(S_1) + U(S_2) + U(S_3 + t), \text{ the LIFO policy;} \\
 (5.9) \quad U_F &= U(S_3) + U(S_2) + U(S_1 + t), \text{ the FIFO policy; and} \\
 U &= U(S_1) + U(S_3) + U(S_2 + t).
 \end{aligned}$$

Now $U_F - U_L = [U(S_3) - U(S_1)] - [U(S_3 + t) - U(S_1 + t)] \leq 0$ by convexity, and $U - U_L = [U(S_3) - U(S_2)] - [U(S_3 + t) - U(S_2 + t)] \leq 0$ again, by convexity. Thus LIFO is optimal for $n = 3$.

Assume LIFO is optimal for all $n \leq m$ and suppose $n = m + 1$. For an arbitrary policy, let S_{1j} denote the initial age of the j^{th} item issued M_1 for $i = 1, 2$. For fixed $S_{11} = S_1$ and $S_{21} = S_j$, $i \neq j$, let $U(i, j)$ denote the total utility of any of the $(m-1)!$ policies having i, j fixed so that $U(i, j) = U(S_i) + U(S_j) + x_{1j}$ where x_{1j} is the total utility of the remaining $m-1$ items. Now at time t the stockpile consists of $m-1$ items and, by the induction assumption, x_{1j} is maximized by issuing these in LIFO order. Let $U^*(i, j)$ denote the maximum value of $U(i, j)$ so achieved. Then $U^*(1, 2)$ is the utility of the LIFO policy for $n = m + 1$.

Now suppose $S_1 = S_1$. If $S_j = S_2$ then $U(1, 2) \leq U^*(1, 2)$. If $S_j \neq S_2$ then $U(1, j) \leq U^*(1, j) = U(S_1) + U(S_j) + U(S_2 + t) + y_{1j}$ where y_{1j} is the remaining utility not accounted for by S_1, S_j and S_2 with LIFO followed after time t . Now consider changing S_j and S_2 in the issue order. This policy clearly has utility $U(1, 2) = U(S_1) + U(S_2) + U(S_j + t) + y_{1j}$ since y_{1j} is not affected by the interchange. Moreover,

$$U(1,2) \leq U^*(1,2) \text{ and } U^*(1,j) - U(1,2) = [U(S_j) - U(S_2)] - [U(S_j + t) - U(S_2 + t)] \leq 0$$

by convexity and $S_j > S_2$. Thus, $U^*(1,j) \leq U(1,2) \leq U^*(1,2)$.

Similarly if $S_j = S_1$ then $U^*(1,1) \leq U^*(1,2)$ from the symmetric roles of M_1 and M_2 .

Finally, if $S_1 \neq S_1 \neq S_j$ then $U^*(1,j) = U(S_1) + U(S_j) + U(S_1 + t) + y_{1j}$ and, interchanging S_1 and S_j yields the utility $U(1,j) = U(S_j) + U(S_1) + U(S_1 + t) + y_{1j}$. Again,

$$U^*(1,j) - U(1,j) = [U(S_1) - U(S_j)] - [U(S_1 + t) - U(S_j + t)] \leq 0.$$

Then $U(1,j) \leq U^*(1,j) \leq U(1,2) \leq U^*(1,2)$ from the preceding and in all cases $U^*(1,2)$ is the maximum utility. Hence LIFO is optimal for $n > 3$ by induction.

If $v > 2$ then LIFO is still optimal by Theorem 4.3 with the trivial modification of replacing $L(S)$ by $U(S)$ and field life by utility. q.e.d.

This section is closed with the remark that if $U(S)$ is concave with truncation point $S_0 \leq \infty$, it trivially follows that FIFO is optimal for $v \geq 2$ demand sources provided $S_n + [\frac{n-1}{v}]t < S_0$ where $[\lambda]$ is the greatest integer in λ . The proof would be identical to that of Theorem 5.4 with inequalities reversed for concavity, the condition insuring that all arguments of U involved lie in the region of concavity.

6. Model III - A Sequential Stochastic Model

6.1 Definition of Model III

As previously remarked, the most commonly used issue policies are those of LIFO and FIFO. It has been shown that in the stochastic model discussed, the question of an optimal policy admits no obvious answer. It may therefore be of practical interest to simply compare the relative merits of the two policies LIFO and FIFO. Even this simplification of the problem does not, however, overcome the inherent difficulties of Model II, viz., the extreme dependence on the entire past. It was also pointed out in Section 5 that a practical model might be one in which only two items are allowed in the stockpile. In terms of a long-range operation, it would then be practical to consider replacing an item upon being issued and thus maintain a stockpile of size two at all times. It would then be of interest to compare the overall effects of the LIFO policy versus that of FIFO.

More precisely, Model III is defined by the following set of assumptions:

- (i) The stockpile consists of two items having random ages X_0, Y_0 with $X_0 > Y_0 \geq 0$.
- (6.1) (ii) The operation begins by issuing one of the two items in (i) and replacing immediately by an item of age zero.
- (iii) Given that the item issued in (ii) was of age t , the amount of field life obtained from that item is a non-

negative random variable Z having a density function

$h(z,t)$ ($z \geq 0$) for $t \geq 0$.

(iv) The next issue takes place only when the item issued in

(iii) is exhausted, one item is issued and immediately

replaced by a new item (age zero) and the operation

continues by stages in this manner.

(v) At each stage of the operation, one of two decisions must

be made, namely, issuing the older (FIFO) or the newer

(LIFO) of the two items in stockpile. It is assumed that

either the newer item is issued at every stage, called

the LIFO scheme, or the older is issued at every stage,

called the FIFO scheme.

The problem, then, is to compare under Model III the relative merits

(with some suitable definition of merit) of the two schemes.

6.2 Imbedded Markov Process for LIFO Scheme

At any time t after the start of the operation, the two items

in the stockpile have random ages X_t and Y_t , say, with $X_t \geq Y_t$.

However, the only ages relevant to the problem are those at the

instant of demand for an item. It is then possible to determine an

imbedded Markov process as follows.

If the LIFO scheme is followed, then the item that was the older

of the two in stockpile at the beginning of the operation is never

issued and hence may be ignored. Then, at any stage N of the operation (a stage being determined from one demand time to the next), the age of the newer item in the stockpile, say Y_N , is a random variable. Moreover, Y_N is just the amount of field life contributed by the issue of the newer item in the stockpile at the $(N-1)$ -st stage (of age Y_{N-1}) since, at that time, the age of the present item was zero (it having been a replacement item then) and hence its present age is the amount of time the operation continued as a result of issuing an item of age Y_{N-1} . Thus Y_N is dependent upon Y_{N-1} but clearly is independent of any stage prior to the $(N-1)$ -st, or, $P[Y_N < y | Y_0, Y_1, \dots, Y_{N-1}] = P[Y_N < y | Y_{N-1}]$ for $y \geq 0$. Then the age, Y_N , of the newer item in the stockpile at stage N is a discrete parameter (N), continuous state Markov process with state space $X = [0, \infty)$. Moreover, in accordance with assumption (iii) of (6.1),

$$P[Y_N < y | Y_{N-1} = t] = \int_0^y h(z, t) dz$$

is clearly independent of N and defines a one-step, constant transition probability distribution.

Having thus found an imbedded Markov process, it would be desirable to find a unique stationary (absolute) probability distribution and compute the expected value under this distribution. The importance of such a discovery is, of course, the fact that if such a distribution exists then, regardless of the initial distribution imposed on Y_0 , if

a sufficient number of stages have occurred the age Y of the newer item in storage (hence field life) may be taken to be approximately distributed according to the stationary distribution. Thus the asymptotic nature of field life would be determined and the corresponding expected field life (utility), denoted $E_L Y$, will be called the stationary utility of the LIFO scheme.

The results to follow lean heavily on Chapter 5 of Doob [5]. The relevant parts of that chapter are here duplicated for the sake of continuity. The state space X has a Borel field $\mathcal{F}_X$ of subsets of X . The transition distribution from a state $x \in X$ to a state in $A \in \mathcal{F}_X$ is denoted $p(x, A)$, and when given by a density function, the corresponding density is denoted $p_0(x, y)$. If X is a Borel set in a Euclidean space and $\mathcal{F}_X$ the σ -field of Borel subsets of X (the present case) Doeblin's condition is said to be satisfied if there exists a finite measure ψ on $\mathcal{F}_X$ such that for some $v \geq 1$, $p^{(v)}(x, A) \leq \psi(A)$ uniformly in $x \in X$, where $p^{(v)}(x, \cdot)$ is the v -step transition which is found by the iterative formula, $p^{(v)}(x, A) = \int_A p^{(v-1)}(\xi, A) p(x, d\xi)$. A set $E \in \mathcal{F}_X$ is invariant if, for each $x \in E$, $p^{(n)}(x, E) = 1$ for all n . A set $E \in \mathcal{F}_X$ is ergodic if E is invariant and there is a probability measure π on $\mathcal{F}_X$ for which

$$\pi(E) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n p^{(m)}(x, A) = \pi(A)$$

for every $A \in \mathcal{F}_X$. Doob then shows that if Doeblin's condition is satisfied, then ergodic sets exist and the corresponding limits π are stationary absolute probability distributions. In particular, if there is only a single ergodic set (as will be the case here) then Theorem 5.7 of [5] states that there is a unique stationary absolute probability distribution, π , given by the solution of the integral equation $\int_X p(x,E)\pi(dx) = \pi(E)$.

In order to carry the analysis further, it is necessary to make some assumption concerning the distribution of field life Z . A quite general family of distributions suitable to the non-negative character of Z is the Γ -family of densities. Thus, it is further assumed

$$(6.2) \quad h(z,t) = \frac{(t+\beta)^{\alpha+1}}{\Gamma(\alpha+1)} z^{\alpha} e^{-z(t+\beta)} \quad \text{where } \alpha > -1, \beta > 0,$$

so that Z has mean value $\frac{\alpha+1}{t+\beta}$, a monotone decreasing function of t .

Now, under the LIFO scheme, the imbedded Markov process has, according to the above, a constant transition distribution function

$$p(x,y) = P[Y_N < y | Y_{N-1} = x] = \int_0^y h(z,x) dz$$

and so in this case there is a transition density $p_0(x,y) = \frac{dp}{dy} = h(y,x)$.

The following lemma shows Doeblin's condition is satisfied with $v=2$.

Lemma 6.1: There is a finite measure ψ on $\mathcal{F}_X$ (the Borel subsets of $X = [0, \infty)$) such that $p^{(2)}(x,A) \leq \psi(A)$, $A \in \mathcal{F}_X$ uniformly in x .

Proof: $p_0^{(2)}(x,y) = \int_0^\infty p_0(x,z)p_0(z,y)dz$

$$= \int_0^\infty \frac{(x+\beta)^{\alpha+1}}{\Gamma(\alpha+1)} z^\alpha e^{-z(x+\beta)} \frac{(z+\beta)^{\alpha+1}}{\Gamma(\alpha+1)} y^\alpha e^{-y(z+\beta)} dz$$

$$= \frac{(x+\beta)^{\alpha+1} y^\alpha e^{-\beta y}}{\Gamma^2(\alpha+1)} \int_0^\infty z^\alpha (z+\beta)^{\alpha+1} e^{-z(x+y+\beta)} dz$$

Applying the c_r -inequality to $(z+\beta)^{\alpha+1}$ whence $(z+\beta)^{\alpha+1} \leq c_{\alpha+1} z^{\alpha+1}$

+ $c_{\alpha+1} \beta^{\alpha+1}$, where $c_{\alpha+1} = 1$ or 2^α according as $\alpha \leq 0$ or $\alpha > 0$,

and using the fact mentioned in (5.2),

$$p_0^{(2)}(x,y) \leq \frac{(x+\beta)^{\alpha+1} y^\alpha e^{-\beta y} c_{\alpha+1}}{\Gamma^2(\alpha+1)} \left\{ \int_0^\infty z^{2\alpha+1} e^{-z(x+y+\beta)} dz + \beta^{\alpha+1} \int_0^\infty z^\alpha e^{-z(x+y+\beta)} dz \right\}$$

$$= \frac{(x+\beta)^{\alpha+1} y^\alpha e^{-\beta y} c_{\alpha+1}}{\Gamma^2(\alpha+1)} \left\{ \frac{\Gamma(2\alpha+2)}{(x+y+\beta)^{2\alpha+2}} + \frac{\beta^{\alpha+1} \Gamma(\alpha+1)}{(x+y+\beta)^{\alpha+1}} \right\}$$

$$\leq \left(\frac{c_{\alpha+1} \Gamma(2\alpha+1)}{\beta^{\alpha+1} \Gamma^2(\alpha+1)} + \frac{c_{\alpha+1} \beta^{\alpha+1}}{\Gamma(\alpha+1)} \right) y^\alpha e^{-\beta y}$$

since $(x+\beta)^{\alpha+1} \leq (x+y+\beta)^{\alpha+1}$ and $(x+y+\beta)^{\alpha+1} \geq \beta^{\alpha+1}$. Denoting

$$\frac{c_{\alpha+1} \Gamma(2\alpha+2)}{\beta^{\alpha+1} \Gamma^2(\alpha+1)} + \frac{c_{\alpha+1} \beta^{\alpha+1}}{\Gamma(\alpha+1)},$$

a constant independent of x , by K , $p_0^{(2)}(x,y) \leq Ky^{\alpha} e^{-\beta y}$.

But $\int_0^{\infty} y^{\alpha} e^{-\beta y} dy = \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} < \infty$. Hence, taking $\psi(A) = K \int_A y^{\alpha} e^{-\beta y} dy$,

ψ is a finite measure on X and $p^{(2)}(x,A) = \int_A p_0(x,y) dy \leq K \int_A y^{\alpha} e^{-\beta y} dy = \psi(A)$

for every $A \in \mathcal{F}_X$.

q.e.d.

Having established Doeblin's condition, it is clear that no set other than X itself may be invariant since, for any $x \in X$, $p_0(x,y) > 0$ for all $0 < y < \infty$. Thus if E is any proper subset of X and $x \in E$, $p(x, X-E) > 0$ so that E cannot be invariant. Hence there can be only one ergodic set and there exists a unique stationary absolute probability distribution, the solution of

$$(6.3) \quad F(y) = \int_0^{\infty} p(t,y) dF(t) \quad \text{or, if } dF(t) = f(t)dt,$$

$$(6.4) \quad f(y) = \int_0^{\infty} p_0(t,y) f(t) dt = \frac{y^{\alpha} e^{-\beta y}}{\Gamma(\alpha+1)} \int_0^{\infty} (t+\beta)^{\alpha+1} e^{-yt} f(t) dt.$$

But, for $f(t) = \frac{1}{k(\alpha, \beta)} \frac{t^{\alpha} e^{-\beta t}}{(t+\beta)^{\alpha+1}}$ where

$$k(\alpha, \beta) = \int_0^{\infty} \frac{t^{\alpha} e^{-\beta t}}{(t+\beta)^{\alpha+1}} dt \leq \frac{1}{\beta^{\alpha+1}} \int_0^{\infty} t^{\alpha} e^{-\beta t} dt < \infty ,$$

$$\frac{y^{\alpha} e^{-\beta y}}{\Gamma(\alpha+1)} \int_0^{\infty} (t+\beta)^{\alpha+1} e^{-yt} f(t) dt = \frac{y^{\alpha} e^{-\beta y}}{\Gamma(\alpha+1)} \cdot \frac{1}{k(\alpha, \beta)} \int_0^{\infty} t^{\alpha} e^{-t(y+\beta)} dy$$

$$= \frac{y^{\alpha} e^{-\beta y}}{\Gamma(\alpha+1)} \cdot \frac{1}{k(\alpha, \beta)} \cdot \frac{\Gamma(\alpha+1)}{(y+\beta)^{\alpha+1}}$$

$$= \frac{1}{k(\alpha, \beta)} \cdot \frac{y^{\alpha} e^{-\beta y}}{(y+\beta)^{\alpha+1}} = f(y) . \quad \text{Thus,}$$

the unique stationary density is

$$(6.5) \quad f(y) = \frac{1}{k(\alpha, \beta)} \frac{y^{\alpha} e^{-\beta y}}{(y+\beta)^{\alpha+1}} .$$

6.3 Imbedded Markov Process for FIFO Scheme

If the FIFO scheme is followed then the ages of both items in the stockpile are relevant since, at any stage (demand time) it is the older of the two items that is issued and the newer of the two items at this

stage becomes the older at the next stage. Let X_N and Y_N , as before, denote the ages of the older and newer items, respectively, at the N -th stage. At the $(N-1)$ -st stage, the item of age Y_N was of age zero so that its present age is the field life contributed by the issue of the item of age X_{N-1} . Once again, field life may be interpreted in terms of the age of the newer of the two items at any stage. Thus (X_N, Y_N) is a random vector dependent upon (X_{N-1}, Y_{N-1}) but clearly is independent of any stage prior to the $(N-1)$ -st, or

$$P[X_N < x, Y_N < y | (X_0, Y_0), \dots, (X_{N-1}, Y_{N-1})] = P[X_N < x, Y_N < y | (X_{N-1}, Y_{N-1})]$$

for all $x \geq y \geq 0$. Then the ages (X_N, Y_N) define a discrete parameter (N) , continuous state, two-dimensional Markov process with state space

$$X = \{(x, y) | x \geq y \geq 0\}.$$

The imbedded Markov process in the present case is further complicated by the fact that if $X_{N-1} = s$ and $Y_{N-1} = t$ with $s > t$, then a transition can only take place in one step to a point $(t+z, z)$ where z is the observed value of field life Z conditioned on the issue of an item of age s , i.e., an observation from the distribution $h(z, s)$ as specified by (iii) of (6.1) and, more specifically, by (6.2). The transition is however, clearly independent of the parameter N . Consequently, the one-step transition

distribution may be found as follows (cf. Fig. 1.)

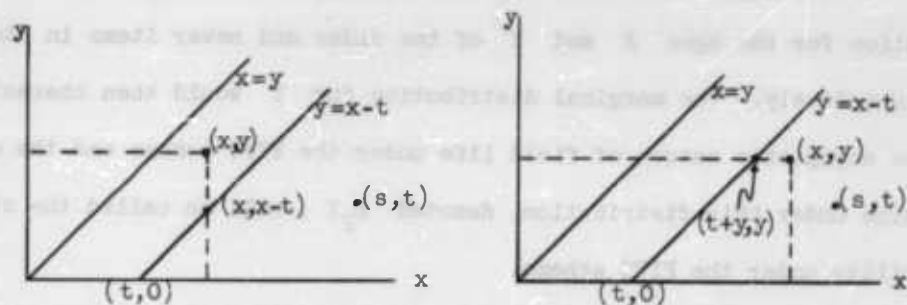


Fig. 1

Let (s, t) be fixed. If $x > t$ and $x - t < y < x$,

$$\begin{aligned} p[(s, t), (x, y)] &= P[X_N < x, Y_N < y | X_{N-1} = s, Y_{N-1} = t] = P[Z < x - t | s] \\ &= H(x - t, s) \quad \text{where} \quad H(x - t, s) = \int_0^{x-t} h(z, s) dz. \end{aligned}$$

On the other hand, if $x > t$ and $0 < y < x - t$,

$$p[(s, t), (x, y)] = P[Z < y | s] = H(y, s),$$

no other transitions being possible. Summarizing, then,

$$(6.6) \quad p[(s, t), (x, y)] = \begin{cases} H(x - t, s) & \text{if } x > t, x - t < y < x \\ H(y, s) & \text{if } x > t, 0 < y < x - t \\ 0 & \text{otherwise} \end{cases}$$

defines a (constant) one-step transition distribution function for the imbedded Markov process.

Once again it would be desirable to find a unique stationary distribution for the ages X and Y of the older and newer items in storage respectively. The marginal distribution for Y would then characterize the asymptotic nature of field life under the FIFO scheme and the expected value under this distribution, denoted $E_F Y$, will be called the stationary utility under the FIFO scheme.

To proceed it is necessary to verify Doeblin's condition and it is remarked that X is a two-dimensional Borel set and $\mathcal{F}_X$ is taken to be the set of Borel subsets of X . In the present case, the transition distribution possesses no bonafide two-dimensional density function since a transition from (s,t) is only possible to $(t+y,y)$ where $y \geq 0$, i.e., the measure $p[(s,t),d(u,v)]$ concentrates all its mass on a line. However, it is clear, and verified below, that the 2-step transition distribution possesses a density for, having made the transition to $(t+y,y)$ from (s,t) , the next transition takes place to $(y+z,z)$ where $z \geq 0$. The set of lines thereby determined as y ranges over its set of possible values spans X . More precisely, for $z \geq \omega \geq 0$,

$$\begin{aligned} p^{(2)}[(s,t),(z,\omega)] &= \int_X p[(u,v),(z,\omega)]p[(s,t),d(u,v)] \\ &= \int_L p[(u,v),(z,\omega)]p[(s,t),d(u,v)] , \quad \text{where} \end{aligned}$$

$L = \{(u,v) \in X | u = t+y, v = y, y \geq 0\}$, since $p[(s,t), d(u,v)]$ is zero otherwise.

But on L ,

$$p[(u,v), (z,\omega)] = p[(t+y,y), (z,\omega)] = \begin{cases} H(z-y, t+y) & \text{if } z-\omega \leq y \\ H(\omega, t+y) & \text{if } 0 \leq y < z-\omega \\ 0 & \text{otherwise} \end{cases}$$

and, using the natural isomorphism $(t+y, y) \leftrightarrow y, y \geq 0$,

$p[s, t], d(u, v)] = dH(y, s) = h(y, s)dy$ for t fixed and hence,

$$\begin{aligned} p^{(2)}[s, t], (z, \omega) &= \int_0^{\infty} p[(t+y, y), (z, \omega)] h(y, s) dy \\ &= \int_0^{z-\omega} H(\omega, t+y) h(y, s) dy + \int_{z-\omega}^{\infty} H(z-y, t+y) h(y, s) dy. \end{aligned}$$

Then, applying Leibnitz' rule,

$$\begin{aligned} \frac{\partial p^{(2)}}{\partial z} &= H(\omega, t+z-\omega) h(z-\omega, s) + \int_{z-\omega}^{\infty} h(z-y, t+y) h(y, s) dy - H(\omega, t+z-\omega) h(z-\omega, s) \\ &= \int_{z-\omega}^{\infty} h(z-y, t+y) h(y, s) dy \quad \text{and hence,} \end{aligned}$$

$$p_0^{(2)}[(s, t), (z, \omega)] = \frac{\partial^2 p^{(2)}}{\partial \omega \partial z} = h(\omega, t+z-\omega) h(z-\omega, s) \quad z \geq \omega, s \geq t.$$

It will be shown in the following lemmas that Doeblin's condition is satisfied with $v = 4$. Observe that,

$$\begin{aligned}
 p_0^{(4)}[(s,t),(x,y)] &= \int_0^\infty \int_v^\infty p_0^{(2)}[(s,t),(u,v)] p_0^{(2)}[(u,v),(x,y)] du dv \\
 &= \int_0^\infty \int_v^\infty h(v,t+u-v) h(u-v,s) h(y,v+x-y) h(x-y,u) du dv \\
 &= \int_0^\infty \int_0^\infty h(v,t+z) h(z,s) h(y,v+x-y) h(x-y,v+z) dz dv, \text{ using}
 \end{aligned}$$

the change of variable $z = u-v, v=v$ with Jacobian 1. Finally,

$$\begin{aligned}
 (6.7) \quad p_0^{(4)}[(s,t),(x,y)] &= \int_0^\infty \int_0^\infty \frac{(t+z+\beta)^{\alpha+1}}{\Gamma(\alpha+1)} v^\alpha e^{-v(t+z+\beta)} \frac{(s+\beta)^{\alpha+1}}{\Gamma(\alpha+1)} z^\alpha e^{-z(s+\beta)} \dots \\
 &\quad \frac{(v+x-y+\beta)^{\alpha+1}}{\Gamma(\alpha+1)} y^\alpha e^{-y(v+x-y+\beta)} \frac{(v+z+\beta)^{\alpha+1}}{\Gamma(\alpha+1)} (x-y)^\alpha e^{-(x-y)(z+v+\beta)} dz dv \\
 &= \frac{y^\alpha (x-y)^\alpha e^{-y(x-y)-x\beta}}{\Gamma^4(\alpha+1)} \int_0^\infty \int_0^\infty (s+\beta)^{\alpha+1} (t+z+\beta)^{\alpha+1} (v+x-y+\beta)^{\alpha+1} \dots \\
 &\quad (v+z+\beta)^{\alpha+1} v^\alpha z^\alpha e^{-v(t+x+\beta)-z(v+s+x-y+\beta)} dz dv.
 \end{aligned}$$

In the sequel, since finite bounds are of interest, K , with or without subscripts, is used generically throughout to denote a constant depending

only on α, β . Also, (5.2) will be freely used.

Lemma 6.2: Let $I = \frac{\Gamma^4(\alpha+1)p_0^{(4)}[(s,t),(x,y)]}{y^\alpha(x-y)^\alpha e^{-y(x-y)-x\beta}}$. Then $I \leq K_1 + K_2(x-y+\beta)^{\alpha+1}$.

Proof: $I = \int_0^\infty (v+x-y+\beta)^{\alpha+1} v^\alpha e^{-v(t+x+\beta)} I_1 dv$ where

$$I_1 = \int_0^\infty (s+\beta)^{\alpha+1} (t+z+\beta)^{\alpha+1} (v+z+\beta)^{\alpha+1} z^\alpha e^{-z(v+s+x-y+\beta)} dz. \text{ Applying}$$

the c_r -inequality to $(t+z+\beta)^{\alpha+1}$ and $(v+z+\beta)^{\alpha+1}$,

$$z^\alpha (t+z+\beta)^{\alpha+1} (v+z+\beta)^{\alpha+1} \leq c_{\alpha+1}^2 [z^{3\alpha+2} + (t+\beta)^{\alpha+1} z^{2\alpha+1} + (v+\beta)^{\alpha+1} z^{2\alpha+1} + (t+\beta)^{\alpha+1} (v+\beta)^{\alpha+1} z^\alpha].$$

$$\text{Then, } I_1 \leq K(s+\beta)^{\alpha+1} \int_0^\infty [z^{3\alpha+2} + (t+\beta)^{\alpha+1} z^{2\alpha+1} + (v+\beta)^{\alpha+1} z^{2\alpha+1} +$$

$$(t+\beta)^{\alpha+1} (v+\beta)^{\alpha+1} z^\alpha] e^{-z(v+s+x-y+\beta)} dz$$

$$= K(s+\beta)^{\alpha+1} \left[\frac{\Gamma(3\alpha+3)}{(v+s+x-y+\beta)^{3\alpha+3}} + \frac{(t+\beta)^{\alpha+1} \Gamma(2\alpha+2)}{(v+s+x-y+\beta)^{2\alpha+2}} + \right.$$

$$\left. \frac{(v+\beta)^{\alpha+1} \Gamma(2\alpha+2)}{(v+s+x-y+\beta)^{2\alpha+2}} + \frac{(t+\beta)^{\alpha+1} (v+\beta)^{\alpha+1} \Gamma(\alpha+1)}{(v+s+x-y+\beta)^{\alpha+1}} \right]$$

and, since $t+\beta \leq s + \beta \leq v+s+x-y+\beta$ while $(v+s+x-y+\beta)^m \geq \beta^m$ for $m > 0$,

$$\begin{aligned} I_1 &\leq \frac{K\Gamma(3\alpha+3)}{\beta^{2\alpha+2}} + K\Gamma(2\alpha+2) + K\Gamma(2\alpha+2)(t+\beta)^{\alpha+1}(v+\beta)^{\alpha+1} K\Gamma(\alpha+1) \\ &= K_1 + K_2(t+\beta)^{\alpha+1}(v+\beta)^{\alpha+1}. \end{aligned}$$

$$\text{Thus, } I \leq K_1 \int_0^\infty (v+x-y+\beta)^{\alpha+1} v^\alpha e^{-v(t+x+\beta)} dv +$$

$$K_2(t+\beta)^{\alpha+1} \int_0^\infty (v+x-y+\beta)^{\alpha+1}(v+\beta)^{\alpha+1} v^\alpha e^{-v(t+x+\beta)} dv.$$

Applying the c_r -inequality to $(v+x-y+\beta)^{\alpha+1}$, the term

$$\begin{aligned} I_3 &= \int_0^\infty (v+x-y+\beta)^{\alpha+1} v^\alpha e^{-v(t+x+\beta)} dv \leq \\ &K \int_0^\infty v^{2\alpha+1} e^{-v(t+x+\beta)} dv + K(x-y+\beta)^{\alpha+1} \int_0^\infty v^\alpha e^{-v(t+x+\beta)} dv \\ &= \frac{K\Gamma(2\alpha+2)}{(t+x+\beta)^{2\alpha+2}} + \frac{K(x-y+\beta)^{\alpha+1}\Gamma(\alpha+1)}{(t+x+\beta)^{\alpha+1}} \leq \frac{K\Gamma(2\alpha+2)}{\beta^{2\alpha+2}} + K\Gamma(\alpha+1) = K. \end{aligned}$$

Applying the c_r -inequality to $(v+\beta)^{\alpha+1}$ as well, the term

$$I_4 = (t+\beta)^{\alpha+1} \int_0^\infty (v+x-y+\beta)^{\alpha+1}(v+\beta)^{\alpha+1} v^\alpha e^{-v(t+x+\beta)} dv \leq$$

$$K(t+\beta)^{\alpha+1} \int_0^{\infty} [v^{3\alpha+2} + \beta^{2\alpha+1} v^{2\alpha+1} + (x-y+\beta)^{\alpha+1} v^{2\alpha+1} +$$

$$\beta^{\alpha+1} (x-y+\beta)^{\alpha+1} v^{\alpha}] e^{-v(t+x+\beta)} dv$$

$$= K(t+\beta)^{\alpha+1} \left[\frac{\Gamma(3\alpha+3)}{(t+x+\beta)^{3\alpha+3}} + \frac{\beta^{\alpha+1} \Gamma(2\alpha+2)}{(t+x+\beta)^{2\alpha+2}} + \frac{(x-y+\beta)^{\alpha+1} \Gamma(2\alpha+2)}{(t+x+\beta)^{2\alpha+2}} + \right. \\ \left. \frac{\beta^{\alpha+1} (x-y+\beta)^{\alpha+1} \Gamma(\alpha+1)}{(t+x+\beta)^{\alpha+1}} \right]$$

$$\leq \frac{K\Gamma(3\alpha+3)}{\beta^{2\alpha+2}} + \frac{K\beta^{\alpha+1} \Gamma(2\alpha+2)}{\beta^{\alpha+1}} + K\Gamma(2\alpha+2) + K\beta^{\alpha+1} \Gamma(\alpha+1) (x-y+\beta)^{\alpha+1}$$

$$= K_1 + K_2 (x-y+\beta)^{\alpha+1}.$$

$$\text{Hence, } I \leq K_1 I_3 + K_2 I_4 \leq K_1 + K_2 (x-y+\beta)^{\alpha+1}.$$

q.e.d.

Lemma 6.3: There is a finite measure ψ on $\mathcal{F}_X$ such that

$$p^{(4)}[(s,t), A] \leq \psi(A), A \in \mathcal{F}_X, \text{ uniformly in } s, t.$$

Proof: From (6.7) and applying lemma 6.2,

$$p_0^{(4)}[(s,t), (x,y)] = \frac{y^{\alpha} (x-y)^{\alpha} e^{-y(x-y)-x\beta}}{\Gamma^4(\alpha+1)} \quad I \leq$$

$$[K_1 + K_2(x-y+\beta)^{\alpha+1}]y^{\alpha}(x-y)^{\alpha}e^{-y(x-y)-x\beta} \quad x \geq y \geq 0.$$

$$\text{Letting } g(x,y) = [K_1 + K_2(x-y+\beta)^{\alpha+1}]y^{\alpha}(x-y)^{\alpha}e^{-y(x-y)-x\beta},$$

$$\begin{aligned} \int_0^{\infty} \int_y^{\infty} g(x,y) dx dy &= \int_0^{\infty} \int_0^{\infty} g(z+y,y) dz dy \\ &= K_1 \int_0^{\infty} y^{\alpha} e^{-\beta y} \int_0^{\infty} z^{\alpha} e^{-z(y+\beta)} dz dy + K_2 \int_0^{\infty} y^{\alpha} e^{-\beta y} \int_0^{\infty} z^{\alpha} (z+\beta)^{\alpha+1} e^{-z(y+\beta)} dz dy \\ &\leq K_1 \int_0^{\infty} \frac{y^{\alpha} e^{-\beta y}}{(y+\beta)^{\alpha+1}} dy + K_2 \int_0^{\infty} \frac{y^{\alpha} e^{-\beta y}}{(y+\beta)^{2\alpha+2}} dy \text{ (using } c_r\text{-inequality on } (z+\beta)^{\alpha+1}) \\ &\leq K \int_0^{\infty} e^{-\beta y} dy \left(\text{since } \frac{y^{\alpha}}{(y+\beta)^{\alpha+1}} \leq \frac{1}{\beta^{\alpha+1}} \text{ and } \frac{y^{\alpha}}{(y+\beta)^{2\alpha+2}} \leq \frac{1}{\beta^{\alpha+2}} \right) \\ &= \frac{K}{\beta} = K < \infty. \end{aligned}$$

Hence, the measure $\psi(A) = \iint_A g(x,y) dx dy$ is finite and independent of s, t .

$$\text{Moreover, } p^{(4)}[(s,t), A] = \iint_A p_0^{(4)}[(s,t), (x,y)] dx dy \leq \iint_A g(x,y) dx dy = \psi(A)$$

for every $A \in \mathcal{G}_X$ uniformly in s, t .

q.e.d.

Having thus established Doeblin's condition it is once more clear that there can be but one ergodic set. For given any proper subset

E of X , $E \in \mathcal{F}_X$, it is always possible to find a point $(s,t) \in E$ for which $p[(s,t), X-E] > 0$. Thus only X can be invariant. Then there exists a unique stationary absolute probability distribution, the solution of

$$(6.8) \quad F(x,y) = \iint_X p[(s,t), (x,y)] dF(s,t) \quad \text{or, if} \quad dF(s,t) = f(s,t) ds dt,$$

$$(6.9) \quad F(x,y) = \iint_X p[(s,t), (x,y)] f(s,t) ds dt$$

$$= \int_{x-y}^x \int_t^{\infty} H(x-t,s) f(s,t) ds dt + \int_0^{x-y} \int_t^{\infty} H(y,s) f(s,t) ds dt.$$

Then,

$$\frac{\partial F}{\partial y} = \int_{x-y}^{\infty} H(y,s) f(s, x-y) ds + \int_0^{x-y} \int_t^{\infty} h(y,s) f(s,t) ds - \int_{x-y}^{\infty} H(y,s) f(s, x-y) ds$$

$$= \int_0^{x-y} \int_t^{\infty} h(y,s) f(s,t) ds, \quad \text{whence}$$

$$(6.10) \quad f(x,y) = \frac{\partial^2 F}{\partial x \partial y} = \int_{x-y}^{\infty} h(y,s) f(s, x-y) ds \quad \text{or, letting } t=x-y, u=s-t,$$

$$(6.11) \quad f(t+y, y) = \int_0^{\infty} h(y, u+t) f(u+t, t) du$$

$$= \int_0^{\infty} \frac{(u+t+\beta)^{\alpha+1}}{\Gamma(\alpha+1)} y^{\alpha} e^{-y(u+t+\beta)} f(u+t, t) du =$$

$$\frac{y^\alpha e^{-y(t+\beta)}}{\Gamma(\alpha+1)} \int_0^\infty (u+t+\beta)^{\alpha+1} e^{-yu} f(u+t, t) du.$$

But if $f(u+t, t) = \frac{(ut)^\alpha e^{-(ut+u\beta+t\beta)}}{c(\alpha, \beta)(u+t+\beta)^{\alpha+1}}$, where $c(\alpha, \beta) = \int_0^\infty \int_0^\infty f(u+t, t) du dt$,

$$\frac{y^\alpha e^{-y(t+\beta)}}{\Gamma(\alpha+1)} \int_0^\infty (u+t+\beta)^{\alpha+1} e^{-yu} f(u+t, t) du = \frac{y^\alpha e^{-y(t+\beta)}}{c(\alpha, \beta)\Gamma(\alpha+1)} \cdot t^\alpha e^{-t\beta} \int_0^\infty u^\alpha e^{-u(t+y+\beta)} du$$

$$= \frac{y^\alpha e^{-y(t+\beta)-t\beta} t^\alpha}{c(\alpha, \beta)\Gamma(\alpha+1)} \cdot \frac{\Gamma(\alpha+1)}{(t+y+\beta)^{\alpha+1}} = \frac{(ty)^\alpha e^{-(ty+t\beta+y\beta)}}{c(\alpha, \beta)(t+y+\beta)^{\alpha+1}} = f(t+y, y)$$

so that the given density is the unique stationary distribution which may be expressed as,

$$(6.12) \quad f(x, y) = \frac{y^\alpha (x-y)^\alpha e^{-(x\beta+xy-y^2)}}{c(\alpha, \beta)(x+\beta)^{\alpha+1}} \quad x \geq y \geq 0.$$

6.4 Comparison of Utilities under Stationary Distributions.

Having found unique stationary distributions, a natural criterion for judging the relative merits of the two schemes would be the stationary utilities. Thus in the notation adopted the FIFO (LIFO) scheme is preferred to the LIFO (FIFO) scheme if $E_F Y \geq E_L Y$ ($E_L Y \geq E_F Y$). Unfortunately the

general solutions (6.5) and (6.12) are not amenable to standard calculus forms and each case should be treated individually, by numerical methods if necessary, before making a decision. This section is closed with a complete analysis of the case $\alpha = 0, \beta = 1$.

According to (6.5), $k = k(0,1) = \int_0^{\infty} \frac{e^{-y}}{y+1} dy = e \int_1^{\infty} \frac{e^{-u}}{u} du = eE_1(1)$ where $E_1(x) = \int_x^{\infty} \frac{e^{-u}}{u} du$ is the exponential integral function, tabulated in [6] for $x > 0$. In particular, $E_1(1) = 0.219383934^+$ and so $k = 0.596340674^+$. Now $E_L Y = \frac{1}{k} \int_0^{\infty} \frac{ye^{-y}}{y+1} = \frac{1}{k} \int_0^{\infty} e^{-y} dy - \frac{1}{k} \int_0^{\infty} \frac{e^{-y}}{y+1} dy = \frac{1}{k} - 1$ and, with $k < .596341$, $E_L Y > .67689291$.

In the FIFO case, (6.12) yields $c=c(0,1) = \int_0^{\infty} e^{y^2} \int_y^{\infty} \frac{e^{-x(y+1)}}{x+1} dx dy$ and, by the change of variable $u = (x+1)(y+1)$, for each $y > 0$,

$$(6.13) \quad \int_y^{\infty} \frac{e^{-x(y+1)}}{x+1} dx = e^{y+1} \int_{(y+1)^2}^{\infty} \frac{e^{-u}}{u} du = e^{y+1} E_1[(y+1)^2]$$

so that,

$$(6.14) \quad c = \int_0^{\infty} e^{y^2+y+1} E_1[(y+1)^2] dy \quad \text{and}$$

$$(6.15) \quad E_F Y = \frac{1}{c} \int_0^{\infty} ye^{y^2+y+1} E_1[(y+1)^2] dy \\ = \frac{1}{2c} \int_0^{\infty} (2y+1)e^{y^2+y+1} E_1[(y+1)^2] dy - \frac{1}{2c} \int_0^{\infty} e^{y^2+y+1} E_1[(y+1)^2] dy.$$

But the last integral in (6.15) is just c and the first may be integrated by parts taking $u = E_1[(y+1)^2]$ whence $du = -\frac{2e^{-(y+1)^2}}{y+1}$ and

$dv = (2y+1)e^{y^2+y+1}$ so that $v = e^{y^2+y+1}$ to yield

$$\int_0^{\infty} (2y+1)e^{y^2+y+1} E_1[(y+1)^2] dy = e^{y^2+y+1} E_1[(y+1)^2] \Big|_0^{\infty} + 2 \int_0^{\infty} \frac{e^{-y}}{y+1} dy.$$

Applying L'Hospital's rule,

$$\lim_{y \rightarrow \infty} \frac{E_1[(y+1)^2]}{e^{-(y^2+y+1)}} = \lim_{y \rightarrow \infty} \frac{-2e^{-(y+1)^2}}{-(2y+1)e^{-(y^2+y+1)}} = \lim_{y \rightarrow \infty} \frac{2}{(y+1)(2y+1)e^y} = 0$$

so that

$$\int_0^{\infty} (2y+1)e^{y^2+y+1} E_1[(y+1)^2] dy = -e E_1(1) + 2k = -k + 2k = k.$$

Thus,

$$(6.16) \quad E_F Y = \frac{k}{2c} - \frac{1}{2} = \frac{k-c}{2c}.$$

By applying successive approximations a bound for c may be determined as follows. Returning to (6.12),

$$c = \int_0^{\infty} e^{y^2} \int_y^{\infty} \frac{e^{-x(y+1)}}{x+1} dx dy.$$

For fixed $y > 0$, the function $\varphi(x) = \frac{y+1}{x+1} e^{\frac{x-y}{y+1}}$ has the properties,

$$\varphi(y)=1, \quad \varphi'(x) = \frac{e^{\frac{x-y}{y+1}}}{(x+1)^2} (x-y) > 0 \quad \text{for } x > y, \text{ i.e. } \varphi(x) \text{ increases on}$$

$[y, \infty)$ so that $\frac{1}{x+1} > e^{-\frac{(x-y)}{y+1}}$. Thus,

$$\int_0^{\infty} \frac{e^{-x(y+1)}}{x+1} dx > \frac{e^{\frac{y}{y+1}}}{y+1} \int_0^{\infty} e^{-x[\frac{(y+1)^2+1}{y+1}]} dx = \frac{e^{\frac{y}{y+1}}}{(y+1)^2+1} \left[-e^{-x[\frac{(y+1)^2+1}{y+1}]} \right]_y^{\infty}$$

$$= \frac{e^{-y(y+1)}}{(y+1)^2+1} \quad \text{and therefore,}$$

$$\begin{aligned} (6.17) \quad c &> \int_0^{\infty} \frac{e^{y^2} e^{-y(y+1)}}{y^2+2y+2} dy = \int_0^{\infty} \frac{e^{-y}}{y^2+2y+2} dy \\ &= \int_0^{\frac{1}{2}} \frac{e^{-y}}{y^2+2y+2} dy + \int_{\frac{1}{2}}^{\infty} \frac{e^{-y}}{y^2+2y+2} dy. \end{aligned}$$

The same type of approximation is again employed. Thus, on $[0, \frac{1}{2}]$,

the function $\varphi(y) = \frac{2e^y}{y^2+2y+2}$ has the properties,

$$\varphi(0)=1, \quad \varphi'(y) = \frac{2e^y y^2}{(y^2+2y+2)^2} > 0 \quad \text{for } y > 0 \text{ so that } \varphi(y) > 1$$

$$\text{or } \frac{1}{y^2+2y+2} > \frac{1}{2} e^{-y}.$$

Similarly, on $(\frac{1}{2}, \infty)$ the function $\psi(y) = \frac{\frac{13}{4} e^{\frac{12}{13}(y-\frac{1}{2})}}{y^2+2y+2}$ has the

$$\text{properties } \psi(\frac{1}{2}) = 1, \psi'(y) = \frac{\frac{13}{4} e^{\frac{12}{13}(y-\frac{1}{2})}}{(y^2+2y+2)^2} (\frac{12}{13} y^2 - \frac{1}{13}(2y+2)) > 0 \text{ for } y > \frac{1}{2},$$

since $\frac{12}{13} y^2 - \frac{1}{13}(2y+2)$ is zero for $y = \frac{1}{2}$ and increases on $(\frac{1}{2}, \infty)$.

$$\text{Thus } \psi(y) > 1 \text{ or } \frac{1}{y^2+2y+2} > \frac{4}{13} e^{-\frac{12}{13}(y-\frac{1}{2})} \text{ for } y > \frac{1}{2}.$$

Applying these results to (6.17),

$$\int_0^{\frac{1}{2}} \frac{e^{-y}}{y^2+2y+2} dy > \frac{1}{2} \int_0^{\frac{1}{2}} e^{-2y} dy = \frac{1}{4} [-e^{-2y}]_0^{\frac{1}{2}} = \frac{1}{4} - \frac{e^{-1}}{4} \text{ and}$$

(6.18)

$$\int_{\frac{1}{2}}^{\infty} \frac{e^{-y}}{y^2+2y+2} dy > \frac{4}{13} e^{\frac{6}{13}} \int_{\frac{1}{2}}^{\infty} e^{-\frac{25}{13}y} dy = \frac{4}{25} e^{\frac{6}{13}} \left[-e^{-\frac{25}{13}y} \right]_{\frac{1}{2}}^{\infty} = \frac{4}{25} e^{-\frac{1}{2}}$$

$$\text{so that } c > \frac{1}{4} - \frac{e^{-1}}{4} + \frac{4}{25} e^{-\frac{1}{2}} = .25 - .09198^- + .09704^+$$

$$> .25 - .09198 + .09704 = .25506 > .255.$$

Applying (6.18) to (6.16) ,

$$k-c < .597-.255 = .342, 2c > .51 \text{ and } \frac{1}{2c} < 1.961 .$$

Then $E_F Y = \frac{k-c}{2c} < .671 < .676 < E_L Y$, and the stationary utility of the LIFO scheme dominates that of FIFO .

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