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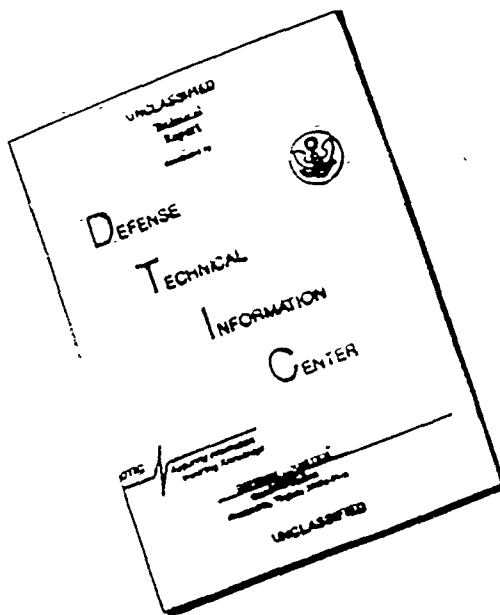
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ONE-SIDED INEQUALITIES OF THE CHEBYSHEV TYPE

by

ALBERT W. MARSHALL AND INGRAM OLKIN

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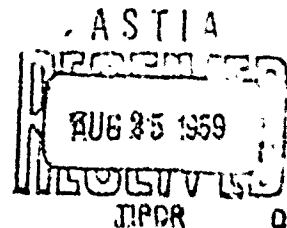
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APPLIED MATHEMATICS AND STATISTICS LABORATORY

STANFORD UNIVERSITY

STANFORD, CALIFORNIA



PART I

A one-sided inequality of the Chebyshev type,  
Albert W. Marshall and Ingram Olkin

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## PREFACE

In many applications one is frequently confronted with the problem of obtaining a bound for the probability that a random vector falls in a certain region. Alternatively, the bound may be specified and one may want to define the limits of the region. This report is concerned with the theoretical development of two such bounds, and we now give several examples which illustrate the use of the results.

### 1. MULTIPLE DECISION PROCEDURES; SLIPPAGE PROBLEMS.

Several authors have considered these problems, e.g., Bechhofer, Dunnett, and Sobel; Paulson; Karlin and Truax. For this situation we have a random vector  $t = (t_1, \dots, t_k)$ , where each  $t_i$  has a student's  $t$ -distribution. However, the  $t_i$  are correlated because the estimate of variance is the same in each  $t_i$ . In choosing the population with the largest mean or in deciding whether the mean in one of several populations has slipped, the relevant statistic is  $\max_{1 \leq j \leq k} t_j$ . Thus one requires information about  $P \{ \max_{1 \leq j \leq k} t_j \geq c \} \leq \alpha$ . In this case one may desire the value of  $\alpha$  if  $c$  is given, or the value of  $c$  if  $\alpha$  is given.

Since the joint distribution of the  $t_i$  is known, one might hope to obtain more accurate bounds for  $\max_{1 \leq j \leq k} t_j$ . However the distribution of  $\max_{1 \leq j \leq k} t_j$  is not easily determined so that Chebyshev type bounds can be used as an approximation.

Some typical problems where this arises are:

- (i) The determination of which of several types of alloys will produce the highest mean tensile strength,
- (ii) the determination of which of several drugs produces the greatest mean effect,
- (iii) the determination of which of several lots of projectiles will produce the longest mean range.

## 2. ANALYSIS OF VARIANCE.

A parallel situation to 1 prevails in some types of analysis of variance problems. Here we have a random vector  $F = (F_1, \dots, F_k)$ , where each  $F_1$  has an F-distribution, but the  $F_1$  are correlated. Again the relevant statistic is  $\max_{1 \leq j \leq k} F_j$ . Some tables for small values of  $k$  have been given by Ramachandran.

## 3. PSYCHOLOGICAL TESTING.

In certain testing problems there may be a number of tests which in some sense increase in difficulty. For example, an ordering of difficulty might be addition, multiplication, calculus. Following a model of Guttman, the score on test one is given by  $y_1$ , the score on test two by  $y_1 + y_2$ , and the score on test  $j$  by  $y_1 + \dots + y_j$ . Thus the increase in difficulty from test  $j$  to test  $j+1$  is represented by  $y_{j+1}$ . Given various assumptions on the  $y$ 's, one is interested in some personnel assignment problems in finding a bound for the  $P\left\{ \max_{1 \leq j \leq k} (y_1 + \dots + y_j) \geq c \right\}$ .

A ONE-SIDED INEQUALITY OF THE CHEBYSHEV TYPE

by

Albert W. Marshall and Ingram Olkin<sup>1/</sup>

0. Summary: Let  $x = (x_1, \dots, x_k)$  be a random vector with  $Ex_i = 0$  and either (i)  $Ex_i^2 = \sigma^2$ ,  $Ex_i x_j = \sigma^2 \rho$  ( $i \neq j$ ),  $i, j = 1, \dots, k$ , or (ii)  $Ex_i^2 = \sigma_i^2$ ,  $i = 1, \dots, k$ . In each case a sharp upper bound (only involving the specified moments) for  $P(x_1 \geq 1 \text{ or } \dots \text{ or } x_k \geq 1)$  is obtained. A formulation for more general one-sided bounds is also discussed.

1. Introduction: Let  $x$  be a random variable with mean 0 and variance  $\sigma^2$ . Then according to Chebyshev's inequality,

$$(1.1) \quad P(|x| \geq 1) \leq \sigma^2;$$

the corresponding one-sided inequality

$$(1.2) \quad P(x \geq 1) \leq \sigma^2 / (\sigma^2 + 1)$$

is also known (see e.g. [2, p.198]). Both inequalities are sharp.

A generalization of inequality (1.1) was obtained by Olkin and Pratt [1] for  $P(|x_1| \geq 1 \text{ or } \dots \text{ or } |x_k| \geq 1)$  in the case  $Ex_i = 0$ ,  $Ex_i^2 = \sigma^2$ ,  $Ex_i x_j = \sigma^2 \rho$  ( $i \neq j$ ),  $i, j = 1, \dots, k$ . We are concerned here with the corresponding bound for  $P(x \in T) = P(x_1 \geq 1 \text{ or } \dots \text{ or } x_k \geq 1)$ .

<sup>1/</sup> On leave of absence from Michigan State University. This research was sponsored in part by the Office of Ordnance Research.

Consider a real-valued function  $f(x) \equiv f(x_1, \dots, x_k)$  such that

$$(1.3) \quad f(x) \geq 0 \quad \text{for all } x, \quad f(x) \geq 1 \quad \text{for } x \in T;$$

then

$$(1.4) \quad E f(x) \geq \int_{\{x \in T\}} f(x) dP \geq P\{x \in T\}.$$

Since we wish the bound to be a function of the covariance matrix

$\Sigma = (\sigma_{ij})$ , we choose

$$(1.5) \quad f(x) = (x-a) A (x-a)',$$

where  $a = (a_1, \dots, a_k)$ ,  $A = (a_{ij}) : k \times k$ . A "best" bound is one which minimizes

$$(1.6) \quad E f(x) = \text{tr } A(\Sigma + a'a),$$

subject to (1.3). Let  $z = (x-a) D_{1-a}^{-1}$ , where  $D_{1-a} = \text{diag}(1-a_1, \dots, 1-a_k)$

and let  $A^* = D_{1-a}^{-1} A D_{1-a}$ . Since  $f(a) = 0$ , (1.3) implies that  $a \notin T$ ,

i.e.  $a_i < 1$ ,  $i = 1, \dots, k$ . Thus (1.3) becomes

$$(1.7) \quad z^* A^* z \geq 0 \quad \text{for all } z, \quad z^* A^* z \geq 1 \quad \text{if some } z_j \geq 1.$$

By the results of [1] the bound (1.6) will be minimized by a positive

definite matrix  $A^*$  (or in turn  $A$ ) for which  $B = (A^*)^{-1}$  has ones on the main diagonal. Thus the problem has been transformed to that of minimizing

$$(1.8) \quad \text{tr } A(\Sigma + a'a) = \text{tr } B^{-1} D_{1-a}^{-1} (\Sigma + a'a) D_{1-a}^{-1}$$

for  $a_i < 1$  and  $B = (b_{ij})$  positive definite with  $b_{ii} = 1$ ,  $i = 1, \dots, k$ . Any such  $a$  and  $B$  will yield a valid bound which in general will not be sharp.

The problem of finding a sharp bound has not been solved for general covariance matrices even when  $k = 2$ . We give a solution for the case that  $\sigma_{ii} = \sigma^2$ ,  $\sigma_{ij} = \sigma^2 \rho$  ( $i \neq j$ ), and an example which attains equality in Sections 2 and 3. In Section 4 we give a sharp bound which only involves the variances.

We will frequently encounter a class  $\mathcal{D}$  of positive definite matrices  $\Delta = (\delta_{ij})$  with  $\delta_{ii} = 1$ ,  $\delta_{ij} = \delta$  ( $i \neq j$ ). Then  $\Delta$  can be written as

$$(1.9) \quad \Delta = (1-\delta) I + \delta e'e,$$

where  $e = (1, \dots, 1)$ . Such a matrix may be transformed to diagonal form by an orthogonal matrix  $\Gamma$ , whose first row is  $e/\sqrt{k}$ , i.e.

$$(1.10) \quad \begin{aligned} \Gamma \Delta \Gamma' &= (1-\delta) I + \delta \Gamma e'e \Gamma' \\ &= (1-\delta) I + \delta (\sqrt{k}, 0, \dots, 0)' (\sqrt{k}, 0, \dots, 0) \\ &= \text{diag } (1+(k-1)\delta, 1-\delta, \dots, 1-\delta). \end{aligned}$$

Hence  $\Delta$  is positive definite if and only if  $(k-1)^{-1} < \delta < 1$ .

2. Derivation of the bound: If  $\Sigma \in \mathcal{D}$ , then because of symmetry we suspect that the minimizing  $B = (1-b)I + be'e \in \mathcal{D}$  and  $a = \alpha e$ . In any event, we may find the best bound obtainable from such  $B$  and  $\alpha$ , and an example of sharpness would justify the choice.

By using (1.10) and assuming  $a = \alpha e$ ,  $B \in \mathcal{D}$ , the bound (1.8) can be written in the form

$$\begin{aligned} H(\alpha, b) &= (1-\alpha)^{-2} \text{tr } B^{-1}(\Sigma + \alpha^2 e'e) \\ &= (1-\alpha)^{-2} \text{tr } (\Gamma B \Gamma')^{-1} (\Gamma \Sigma \Gamma' + \alpha^2 \Gamma e'e \Gamma') . \end{aligned}$$

Since  $\Gamma B \Gamma' = \text{diag}(1+(k-1)b, 1-b, \dots, 1-b)$ ,  $\Gamma \Sigma \Gamma' = \sigma^2 \text{diag}(1+(k-1)\rho, 1-\rho, \dots, 1-\rho)$ , and  $\Gamma e' = (\sqrt{k}, 0, \dots, 0)$ , so that

$$\begin{aligned} (2.1) \quad H(\alpha, b) &= \frac{1}{(1-\alpha)^2} \left[ \frac{\sigma^2 + \sigma^2 \rho(k-1) + \alpha^2 k}{1+(k-1)b} + \frac{\sigma^2(k-1)(1-\rho)}{1-b} \right] \\ &= \frac{k[\alpha^2 + \sigma^2 + b(\sigma^2 t - \alpha^2)]}{(1-\alpha)^2(1-b)[1+(k-1)b]} , \end{aligned}$$

where

$$(2.2) \quad t = (k-1)(1-\rho) - 1 .$$

The solution of  $\partial H/\partial \alpha = 0$  for  $\alpha$  is given by

$$(2.3) \quad \alpha_0 = -\sigma^2(1+bt)/(1-b),$$

from which

$$(2.4) \quad H(\alpha_0, b) = k \sigma^2(1+bt)[1+(k-1)b]^{-1} [1+\sigma^2-b(1-\sigma^2t)]^{-1}.$$

The equation  $\partial H(\alpha_0, b)/\partial b = 0$  becomes

$$(2.5) \quad b^2 t(1-\sigma^2 t) + 2b(1-\sigma^2 t) - (\sigma^2 + \rho) = 0,$$

and has roots

$$(2.6) \quad b = -\frac{1}{t} \pm \frac{\sqrt{1+\rho}}{t\sqrt{1-\sigma^2 t}} = -\frac{1}{t} \pm \frac{\sqrt{(1+t)(k-1-t)}}{t\sqrt{(k-1)(1-\sigma^2 t)}}.$$

Using (1.10), we see that the term  $1+\rho = (1-\rho)[1+(k-1)\rho] > 0$

and we assume that  $1-\sigma^2 t > 0$ , so that the roots are real.

B is positive definite if and only if  $-(k-1)^{-1} < b < 1$ , i.e.

$$(2.7) \quad \frac{k-1-t}{t(k-1)} \leq \pm \frac{\sqrt{(1+t)(k-1-t)}}{t\sqrt{(k-1)(1-\sigma^2 t)}} \leq \frac{1+t}{t}.$$

If  $t > 0$  ( $t < 0$ ), then the LHS and RHS are positive (negative),

and the lower sign is impossible. We denote by  $b_0$  the root with the positive sign, and by  $B_0$  the corresponding matrix.

Lemma:  $B_0$  is positive definite if and only if

$$(2.8) \quad k \geq \sigma^2(k-1)(1+t).$$

Proof: The condition that  $B_0$  be positive definite is equivalent to the inequality (2.7) with the plus sign. The second inequality is (2.8), and the first inequality always holds. ||

An example of sharpness will show that  $b_0$  leads to a minimum. We assume (2.8) and evaluate  $H(\alpha_0, b_0)$  by showing (using (2.6)) that

$$(2.9) \quad H(\alpha_0, b_0) = \frac{k\sigma^2(1+b_0 t)}{[1+(k-1)b_0][1+\sigma^2-b_0(1-\sigma^2 t)]}$$
$$= \frac{k\sigma^2 t}{[k-2+t\sigma^2+\sigma^2(k-1)]-2b_0(k-1)(1-\sigma^2 t)}.$$

Then substituting for  $b_0$ , we obtain

$$(2.10) \quad H(\alpha_0, b_0) = k\sigma^2 t / (u-2\sqrt{v}),$$

where  $u = t^2\sigma^2 + t[k-2-(k-1)\sigma^2] + 2(k-1)$ , and  $v = (1+t)(k-1-t)(k-1)(1-\sigma^2 t)$ .

Rationalization of the denominator in (2.10) and simplification yields

$$(2.11) \quad H(\alpha_0, b_0) = \frac{k\sigma^2(u+2\sqrt{v})}{[t\sigma^2 - k\sigma^2(k-1)]^2} .$$

After substituting for  $t$  from (2.2) we obtain the theorem.

Theorem: Let  $x$  be a random vector with  $Ex_1=0, Ex_1^2=\sigma^2, Ex_1x_j=\sigma^2\rho(1 \neq j)$  .

If (i)  $1-\sigma^2t > 0$ , (ii)  $k \geq \sigma^2(k-1)(1+t)$ , then

$$(2.12) \quad P \equiv P\{x_1 \geq 1 \text{ or } \dots \text{ or } x_k \geq 1\} \leq H(\alpha_0, b_0) \\ = \frac{k\sigma^2\{\sqrt{[1+(k-1)\rho][1+\sigma^2-\sigma^2(k-1)(1-\rho)]} + (k-1)\sqrt{1-\rho}\}^2}{\{k + \sigma^2[1+(k-1)\rho]\}^2} ;$$

otherwise  $P \leq 1$  .

For the special case  $\rho = 1$  , the inequality reduces to (1.2), and for  $\rho = -1/(k-1)$  , the bound is  $(k-1)\sigma^2$  , which reduces to (1.1) for  $k = 2$  .

It should be noted that the bound  $H(\alpha_0, b_0) \leq 1$  is equivalent to

$$(\sigma^2(k-1)\sqrt{(1-\rho)[1+(k-1)\rho]} - k\sqrt{1+\sigma^2-\sigma^2(k-1)(1-\rho)})^2 \geq 0$$

3. Sharpness: We show sharpness of (2.12) by exhibiting an example which achieves equality whenever the conditions (i) and (ii) of the theorem are satisfied. For cases that the theorem provides only the trivial bound unity, we give examples when  $k = 2$  .

Let  $z$  be a random vector with the following distribution:

$$P\{z = b^{(i)}\} = p/k, \quad i = 1, \dots, k,$$

$$P\{z = 0\} = 1-p,$$

where  $b^{(i)}$  is the  $i$ -th row of  $B_0$ . If  $x = (1 - \alpha_0)z + \alpha_0 e$  satisfies the conditions of the theorem, then

$$(3.1) \quad E(z) = -\alpha_0 e / (1 - \alpha_0) = [1 + (k-1)b_0] p e / k,$$

$$(3.2) \quad E(z'z) = (\Sigma + \alpha_0^2 e'e) / (1 - \alpha_0)^2 = p B_0^2 / k.$$

Substituting for  $\alpha_0$  from (2.3) in (3.1) and solving for  $p$ , we obtain  $p = H(\alpha_0, b_0)$ , where  $H(\alpha_0, b_0)$  is given by (2.9). Because of the special form of  $\Sigma$ , the matrix equation (3.2) is equivalent to the two equations

$$(3.3) \quad [(1 - b_0)^2 + 2b_0(1 - b_0) + b_0^2 z] p/k = (\sigma^2 + \alpha_0^2) / (1 - \alpha_0)^2,$$

$$(3.4) \quad [2b_0(1 - b_0) + b_0^2 k] p/k = (\sigma^2_0 + \alpha_0^2) / (1 - \alpha_0)^2.$$

Substitution of  $p$  and  $\alpha_0$  from (2.3) in (3.3) and in (3.3) minus (3.4) yields (2.5) with  $b = b_0$  in each case. Hence (3.1) and (3.2) are satisfied when  $p = H(\alpha_0, b_0)$ , that is, when  $p$  is given by the bound of (2.12). Since  $P\{z_1 \geq 1 \text{ for some } i\} = p$ , and  $z_1 \geq 1$  if and only if  $x_1 \geq 1$ , it follows that  $x = (1 - \alpha_0)z + \alpha_0 e$  achieves equality in (2.12).

Now suppose that  $k = 2$ , in which case conditions (i) and (ii) become  $1 + \sigma^2 \rho \geq 0$ , and  $2 \geq \sigma^2(1 - \rho)$ , respectively.

If  $1 + \sigma^2 \rho < 0$ , then a distribution having the prescribed moments and achieving the bound of one is:

$$P((1, -c)) = P((-c, 1)) = p_1/2,$$

$$P((c, -c)) = P((-c, c)) = p_2/2,$$

$$P((1, 1)) = 1 - p_1 - p_2,$$

where  $p_1 = 2\sigma^2(1+\rho)/(c^2-1)$ ,  $p_2 = (1+\sigma^2\rho)/(1-c^2)$ ,  $c = \frac{1}{2} \left\{ \sigma^2(1+\rho) + \sqrt{[\sigma^4(1+\rho)^2 + 4\sigma^2]} \right\}$ .

The condition  $1 + \sigma^2 \rho < 0$  implies that  $\sigma^2 > 1$  and  $c > 1$ . Hence  $0 \leq p_1, p_2$ ,  $p_1 + p_2 \leq 1$ .

If  $2 < \sigma^2(1 - \rho)$  and  $1 + \sigma^2 \rho > 0$ , then a distribution with the moments as prescribed in the theorem and achieving the bound of one is:

$$P((1, -c)) = P((-c, 1)) = p/2, \quad P((d, d)) = 1 - p,$$

where

$$p = \frac{2d}{2d+c-1} = \frac{2\sigma^2(1-\rho)}{(1+c)^2}, \quad c = -\frac{1 + \sqrt{(1-\rho^2)(1+\sigma^2\rho)}}{\rho},$$

( $c = \sigma^2/2$  if  $\rho = 0$ ). The condition  $2 < \sigma^2(1 - \rho)$  implies that  $c > 1$ , which in turn implies that  $p < 1$ . It also implies  $1 + c < \sigma^2(1 - \rho)$  which is equivalent to  $d = p(c-1)/2(1-p) > 1$ .

If  $1 + \sigma^2 \rho = 0$ , then the above distribution with  $d = 1$ ,  $c = \sigma^2$ ,  $p = 2/(1 + \sigma^2)$  is the required example.

4. An inequality involving variances. If  $x_1, \dots, x_k$  are random variables with  $Ex_1 = 0$ ,  $Ex_1^2 = \sigma_1^2$ ,  $i = 1, \dots, k$ , then by (1.1)

$$(4.1) \quad P(|x_1| \geq 1 \text{ or } \dots \text{ or } |x_k| \geq 1) \leq \sum_1^k P(|x_j| \geq 1) \leq \sum_1^k \sigma_j^2.$$

This inequality was proved to be sharp in [1], and the unique distribution attaining equality has zero covariances.

The corresponding one-sided inequality, by (1.2), is

$$(4.2) \quad P(x_1 \geq 1 \text{ or } \dots \text{ or } x_k \geq 1) \leq \sum_1^k P(x_j \geq 1) \leq \sum_1^k \sigma_j^2 / (1 + \sigma_j^2).$$

If the bound is  $\leq 1$ , the unique distribution attaining equality is

$$P((1, -\sigma_2^2, -\sigma_3^2, \dots, -\sigma_k^2)) = \sigma_1^2 / (1 + \sigma_1^2),$$

$$P((-\sigma_1^2, 1, -\sigma_3^2, \dots, -\sigma_k^2)) = \sigma_2^2 / (1 + \sigma_2^2),$$

$$P((-\sigma_1^2, -\sigma_2^2, \dots, -\sigma_{k-1}^2, 1)) = \sigma_k^2 / (1 + \sigma_k^2),$$

$$P((-\sigma_1^2, -\sigma_2^2, \dots, -\sigma_{k-1}^2, -\sigma_k^2)) = 1 - \sum_1^k \sigma_j^2 / (1 + \sigma_j^2).$$

Uniqueness follows by an argument similar to that used in [1]. We note that

in this case the covariances  $Ex_1 x_j = -\sigma_1^2 \sigma_j^2$  are not zero.

An alternative proof of (4.2) following the procedures of Section 1 is to choose  $B = I$  in (1.8) and minimize  $\text{tr } D_{1-a}^{-2}(\Sigma + a'a)$  with respect to  $a < 1$ . The minimizing  $a_j = -\sigma_j^2$ .

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A ONE-SIDED ANALOG OF KOLMOGOROV'S INEQUALITY

by

Albert W. Marshall

1. Introduction and summary.

It is well-known (see e.g. [1] p. 198) that for every positive  $\epsilon$  and every square integrable random variable  $X$  with zero expectation,

$$P(X \geq \epsilon) \leq E(X^2)/[\epsilon^2 + E(X^2)].$$

In this paper an inequality is obtained that generalizes this in the same way that Kolmogorov's inequality generalizes Chebyshev's inequality. The inequality is proved in Section 2 and an example is given to show that equality can be achieved. The possibility of obtaining greater generality is also discussed. In Section 3 an extension to continuous parameter martingales is obtained, and a condition under which equality can be achieved is given.

2. The inequality.

Theorem 2.1: Let  $X_1, X_2, \dots, X_n$  be random variables with  $E(X_1) = 0$ ,  $E(X_i | \mathcal{F}_{i-1}) = 0$  a.e.,  $i = 2, 3, \dots, n$ , and  $E(X_i^2) = \sigma_i^2 < \infty$ ,  $i = 1, 2, \dots, n$ . Then for every positive  $\epsilon$ ,

$$P \left( \max_{1 \leq k \leq n} (X_1 + X_2 + \dots + X_k) \geq \epsilon \right) \leq \frac{\sigma_1^2 + \dots + \sigma_n^2}{\epsilon^2 + \sigma_1^2 + \dots + \sigma_n^2}.$$

$$s_n = \sum_{i=1}^n \sigma_i^2 .$$

Proof: Let  $F(x) = F(x_1, x_2, \dots, x_n) = \frac{(\epsilon \sum_{i=1}^n x_i + \frac{s_n}{n})^2}{(\epsilon^2 + s_n)^2}$ , and let

$$A_k = \{X_1 + X_2 + \dots + X_i < \epsilon, \quad i = 1, 2, \dots, k-1, \quad X_1 + X_2 + \dots + X_k \geq \epsilon\}$$

$k = 1, 2, \dots, n$ . Then

$$\begin{aligned} \int F(x) dP &\geq \sum_{k=1}^n \int_{A_k} F(x) dP = \\ &= \frac{1}{(\epsilon^2 + s_n)^2} \sum_{k=1}^n \int_{A_k} [(\epsilon \sum_{i=1}^k X_i + s_n)^2 + 2(\epsilon \sum_{i=1}^k X_i + s_n)(\epsilon \sum_{i=k+1}^n X_i) + (\epsilon \sum_{i=k+1}^n X_i)^2] dP \\ &\geq \frac{1}{(\epsilon^2 + s_n)^2} \sum_{k=1}^n \int_{A_k} (\epsilon \sum_{i=1}^k X_i + s_n)^2 dP \geq \sum_{k=1}^n P(A_k) = P(\max_{1 \leq i \leq n} (X_1 + \dots + X_i) \geq \epsilon) . \end{aligned}$$

Since  $\int F(x) dP = s_n / (\epsilon^2 + s_n)$ , the proof is complete..

To show that equality can be achieved in (2.1), let  $s_k = \sum_{i=1}^k \sigma_i^2$ ,

$k = 1, 2, \dots, n$ , and let  $Z = (Z_1, Z_2, \dots, Z_n)$  be a random variable having the following distribution:

$$P\{Z = (\epsilon, 0, \dots, 0)\} = \frac{\sigma_1^2}{\epsilon^2 + s_1} = p_1 ,$$

$$P\{Z = \frac{1}{\epsilon}(-\sigma_1^2, -\sigma_2^2, \dots, -\sigma_k^2, \epsilon^2 + s_{k-1}, 0, \dots, 0)\} = \frac{\epsilon^2 \sigma_k^2}{(\epsilon^2 + s_{k-1})(\epsilon^2 + s_k)} = p_k, \quad k = 2, 3, \dots, n$$

$$P\{Z = \frac{1}{\epsilon}(-\sigma_1^2, -\sigma_2^2, \dots, -\sigma_n^2)\} = \frac{\epsilon^2}{\epsilon^2 + s_n} .$$

It is easily verified by induction on  $j$  that

$$(2.2) \quad \sum_{k=1}^j p_k = 1 - \epsilon^2 / (\epsilon^2 + s_j), \quad j = 1, 2, \dots, n,$$

so that this is a valid probability distribution. Let us check that the random variables  $Z_1, Z_2, \dots, Z_n$  satisfy the conditions of Theorem 2.1.

Trivially  $E(Z_1) = 0$ . For  $1 < j \leq n$ ,  $E(Z_j | Z_{j-1} \neq -\sigma_{j-1}^2/\epsilon) = 0$  a.e.

since  $Z_{j-1} \neq -\sigma_{j-1}^2/\epsilon$  implies  $Z_j = 0$  with probability one. Using

(2.2), it follows that  $E(Z_j | Z_{j-1} = -\sigma_j^2/\epsilon)$  is a.e. proportional to

$$p_j(\epsilon^2 + s_{j-1})/\epsilon - (1 - \sum_{k=1}^n p_k)\sigma_j^2/\epsilon = 0. \text{ Hence } E(Z_j | Z_{j-1}) = 0 \text{ a.e.,}$$

and it is clear that  $E(Z_j | Z_{j-1}) = E(Z_j | Z_1, Z_2, \dots, Z_{j-1})$  a.e. Again

$$\text{using (2.2), it follows that } E(Z_j^2) = p_j(\epsilon^2 + s_{j-1})^2/\epsilon^2 + (1 - \sum_{k=1}^j p_k)\sigma_j^4/\epsilon^2,$$

$j = 1, 2, \dots, n$ . Thus the random variable  $Z$  satisfies the conditions of

Theorem 2.2; furthermore, equality holds in (1.1) whenever  $X = (X_1, X_2, \dots, X_n) = Z$

a.e.

It is natural now to ask what the best upper bound is for

$$P\left\{ \max_{1 \leq i \leq n} \frac{1}{\epsilon_i} (X_1 + X_2 + \dots + X_i) \geq 1 \right\} \text{ under the conditions of Theorem 2.1}$$

(where  $\epsilon_i > 0$ ,  $i = 1, 2, \dots, n$ ). Unfortunately this bound has no simple

expression even for small  $n$ , and is not easily obtained. It is given here

only for  $n = 2$ .

Theorem 2.2: Let  $X_1$  and  $X_2$  be random variables with  $E(X_1) = 0$ ,

$E(X_2 | X_1) = 0$  a.e.,  $E(X_1^2) = \sigma_1^2 < \infty$ ,  $i = 1, 2$ . Then if  $\epsilon_1 > 0$

and  $\epsilon_2 > 0$ ,

$$(2.3) \quad P\{X_1 \geq \epsilon_1 \text{ or } X_1 + X_2 \geq \epsilon_2\} \leq \frac{\sigma_2^2 + \alpha_1^2 (\alpha_2, \alpha_1)^2}{\sigma_2^2 + \alpha_2^2 \alpha_1^2},$$

where  $\alpha_i = \sigma_i^2 + \eta_i \eta_i$ ,  $i = 1, 2$ , and  $\eta_1 = \min(\epsilon_1, \epsilon_2)$ ,  $\eta_2 = \epsilon_2$ .

Proof: Let  $F(x_1, x_2) = c_1 F_1^2(x_1) + c_2 F_2^2(x_1 + x_2)$  where

$$c_1 = \frac{\eta_1^2}{\alpha_1^2} - \frac{\eta_1^2 (\alpha_2 + \sigma_2^2)^2}{(\alpha_2^2 + \sigma_2^2 \alpha_1^2)^2}, \quad c_2 = \frac{\eta_1^2 \alpha_2^2}{(\alpha_2^2 + \sigma_2^2 \alpha_1^2)^2},$$

$$F_1(x) = \left(x + \frac{\sigma_1^2}{\eta_1}\right), \quad F_2(x) = \left(x + \frac{\sigma_1^2}{\eta_1} + \frac{\sigma_2^2 \alpha_1}{\eta_2 \alpha_2}\right),$$

and let  $A_1 = \{X_1 \geq \eta_1\}$ ,  $A_2 = \{X_1 < \eta_1, X_1 + X_2 \geq \eta_2\}$ . Since  $\alpha_2 \geq \alpha_1 > 0$ , it follows that  $c_1 \geq 0$ , and

$$\begin{aligned} \int_{A_1} F(x_1, x_2) dP &= \int_{A_1} [c_1 F_1^2(x_1) + c_2 F_2^2(x_1) + c_2 X_2 F_2(x_1) + c_2 X_2^2] dP \\ &\geq \int_{A_1} [c_1 F_1^2(x_1) + c_2 F_2^2(x_1)] dP \geq \int_{A_1} [c_1 F_1^2(\eta_1) + c_2 F_2^2(\eta_1)] dP = P(A_1), \end{aligned}$$

$$\int_{A_2} F(x_1, x_2) dP \geq \int_{A_2} c_2 F_2^2(x_1 + x_2) dP \geq \int_{A_2} c_2 F_2^2(\eta_2) dP = P(A_2).$$

Thus  $\int F(x_1, x_2) dP \geq P(A_1) + P(A_2) = P\{X_1 \geq \eta_1 \text{ or } X_1 + X_2 \geq \eta_2\}$

$\geq P\{X_1 \geq \epsilon_1 \text{ or } X_1 + X_2 \geq \epsilon_2\}$ . It is straightforward to verify that

upon integrating the function  $F(X_1, X_2)$ , one obtains the bound given in (2.3), and this completes the proof.

To show that equality can be achieved in (2.3), let  $(Z_1, Z_2)$  have the following joint distribution:

$$P\{(Z_1, Z_2) = (\eta_1, 0)\} = \sigma_1^2 / \alpha_1$$

$$P\{(Z_1, Z_2) = (-\frac{\sigma_1^2}{\eta_1}, \frac{\alpha_2}{\eta_1})\} = \frac{\eta_1^2 \sigma_2^2}{\alpha_1 \sigma_2^2 + \alpha_2^2},$$

$$P\{(Z_1, Z_2) = (-\frac{\sigma_1^2}{\eta_1}, -\frac{\sigma_2^2 \alpha_1}{\eta_1 \alpha_2})\} = \frac{\eta_1^2 \alpha_2^2}{\alpha_1 (\alpha_1 \sigma_2^2 + \alpha_2^2)}.$$

It is easily verified that  $(Z_1, Z_2)$  satisfies the conditions of Theorem 2.2. Furthermore,

$$P\{(Z_1, Z_2) = (\eta_1, 0)\} + P\{(Z_1, Z_2) = (-\frac{\sigma_1^2}{\eta_1}, \frac{\alpha_2}{\eta_1})\} = \frac{\sigma_2^2 + \sigma_1^2 (\alpha_2 / \alpha_1)^2}{\sigma_2^2 + (\alpha_2^2 / \alpha_1)}$$

$$= P\{Z_1 \geq \eta_1 \text{ or } Z_1 + Z_2 \geq \eta_2\} = P\{Z_1 \geq \epsilon_1 \text{ or } Z_1 + Z_2 \geq \epsilon_1 + \epsilon_2\},$$

so that equality holds in (2.3) whenever  $(X_1, X_2) = (Z_1, Z_2)$  a.e.

Several inequalities follow from (2.3) simply by a change of variables. The corollaries below are given to illustrate the possibilities.

Corollary 2.3: Let  $X_1$  and  $X_2$  be random variables with  $E(X_1) = a$ ,

$E(X_2|X_1) = bX_1 + c$  a.e. (where  $b \neq -1$ ), and  $\text{Var}(X_i) = \sigma_i^2 < \infty$ ,  $i = 1, 2$ .  
 Then if  $\epsilon_1 - a > 0$  and  $[\epsilon_2 - \delta(a + ab + c)]/|b + 1| > 0$  where  $\delta = \text{sign}(b + 1)$ ,

$$(2.4) \quad P\{X_1 \geq \epsilon_1 \text{ or } \delta(X_0 + X_2) \geq \epsilon_2\} \leq \frac{\sigma_2^2 - b^2\sigma_1^2 + \alpha_1^2[(b+1)\alpha_2/\alpha_1]^2}{\sigma_2^2 - b^2\sigma_1^2 + [(b+1)^2\alpha_2^2/\alpha_1]}$$

where  $\alpha_i = \sigma_i^2 + \eta_i\eta_i$ ,  $i = 1, 2$ , and  $\eta_2 = [\epsilon_2 - \delta(a + ab + c)]/|b + 1|$ ,  
 $\eta_1 = \min(\epsilon_1 - a, \eta_2)$ .

Proof: This follows from Theorem 2.2 by making the change of variables

$$X_1' = X_1 + a, \quad X_2' = bX_1 + (b+1)X_2 + ab + c, \quad \epsilon_1' = \epsilon_1 + a, \quad \epsilon_2' = \epsilon_2|b+1| + \delta(a+ab+c)$$

and dropping the primes.

Note that by taking  $a = b = c = 0$  in this corollary, one obtains Theorem 2.2.

Corollary 2.4: Let  $X_1$  and  $X_2$  be random variables such that  $E(X_i) = \mu_i$ ,  $\text{Var}(X_i) = \sigma_i^2 < \infty$ ,  $i = 1, 2$ ,  $\text{Cov}(X_1, X_2) = \sigma_{12} \neq 0$ , and suppose that the regression of  $X_2$  on  $X_1$  is linear. Then if  $\epsilon_1 - \mu_1 > 0$  and  $(\delta\epsilon_2 - \sigma_1^2\mu_2)/\sigma_{12} > 0$  where  $\delta = \text{sign } \sigma_{12}$ ,

$$(2.5) \quad P\{X_1 \geq \epsilon_1 \text{ or } \delta X_2 \geq \epsilon_2\} \leq \frac{\sigma_1^2(\sigma_1^2\sigma_2^2 - \sigma_{12}^2) + \sigma_1^2(x_2\sigma_{12}/\alpha_1)^2}{\sigma_1^2(\sigma_1^2\sigma_2^2 - \sigma_{12}^2) + (x_2^2\sigma_{12}^2/\alpha_1)}$$

where  $\alpha_i = \sigma_1^2 + \eta_1 \eta_i$ ,  $i = 1, 2$ , and  $\eta_2 = (\delta \epsilon_2 - \sigma_1^2 \mu_2) / \sigma_{12}$ ,  
 $\eta_1 = \min(\epsilon_1 - \mu_1, \eta_2)$ .

Proof: To obtain (2.5) from (2.3), make the change of variables

$$X'_1 = X_1 + \mu_1, \quad X'_2 = [\sigma'_{12}(X_1 + X_2) / \sigma_1'^2] + \mu_2, \quad \epsilon'_1 = \epsilon_1 + \mu_1 \quad \text{and} \quad \epsilon'_2 = \delta(\epsilon_2 \sigma'_{12} + \sigma_1'^2 \mu_2)$$

in (2.3), and then remove the primes.

3. An extension to continuous parameter martingales.

We begin by assuming that the underlying probability space is such that  $P$  is complete. Then we have the following

Theorem 3.1: Let  $\{Y_t, t \geq 0\}$  be a separable martingale such that for all  $t \geq 0$ ,  $E(Y_t) = 0$  and  $E(Y_t^2) = \sigma^2(t) < \infty$ . Then for every positive  $\epsilon$  and  $\tau$ ,

$$(3.1) \quad P \left\{ \sup_{t \in [0, \tau]} Y_t \geq \epsilon \right\} \leq \frac{\sigma^2(\tau)}{\epsilon^2 + \sigma^2(\tau)}.$$

Proof: Let  $0 = t_1 \leq t_2 \leq \dots \leq t_n = \tau$ . Then  $X_1 = Y_{t_1}$  and  $X_i = Y_{t_i} - Y_{t_{i-1}}$ ,

$i = 2, 3, \dots, n$  satisfy the conditions of Theorem 2.1, so that

$$(3.2) \quad P \left\{ \max_{1 \leq i \leq n} Y_{t_i} \geq \epsilon \right\} \leq \sigma^2(\tau) / (\epsilon^2 + \sigma^2(\tau)).$$

Let  $S$  be a countable set satisfying the definition of separability and containing the points  $0$  and  $\tau$ . Taking the supremum of the left side of (3.2) over all finite subsets of  $S \cap [0, \tau]$ , we obtain

$$P \left\{ \sup_{t \in S \cap [0, \tau]} Y_t \geq \epsilon \right\} \leq \sigma^2(\tau) / [\epsilon^2 + \sigma^2(\tau)].$$

But

$$P \left\{ \sup_{t \in S \cap [0, \tau]} Y_t \geq \epsilon \right\} = P \left\{ \sup_{t \in [0, \tau]} Y_t \geq \epsilon \right\},$$

and the proof is complete.

Theorem 3.2: Equality can be achieved in (3.1) if  $\sigma^2(\cdot)$  is right continuous.

Proof: In order to define a martingale that achieves equality in (3.1), let  $\Omega = \{-1\} \cup [0, \infty)$ ,  $\mathcal{B}$  be the Borel subsets of  $\Omega$ , and let  $\Gamma$  be the probability measure defined on  $\mathcal{B}$  by

$$P(B) = \left\{ \epsilon^2 / [\epsilon^2 + \lim_{x \rightarrow \infty} \sigma^2(x)] \right\} X_{B \cap \{-1\}} + \mu(B \cap [0, \infty)),$$

where  $X_E$  is the characteristic function of the set  $E$  and  $\mu$  is the measure induced on the Borel subsets of  $[0, \infty)$  by the right continuous distribution function  $\sigma^2(\cdot) / [\epsilon^2 + \sigma^2(\cdot)]$ . Let  $\{Z_t, t \geq 0\}$  be defined on  $(\Omega, \mathcal{B}, P)$  by

$$Z_t(\omega) = \begin{cases} -\sigma^2(t)/\epsilon & 0 \leq t < \omega \\ \epsilon & 0 \leq \omega \leq t \\ 0 & \omega = -1. \end{cases}$$

Then

$$P \left\{ \sup_{t \in [0, \tau]} Z_t \geq \epsilon \right\} = P \left\{ 0 \leq \omega \leq \tau \right\} = \sigma^2(\tau) / [\epsilon^2 + \sigma^2(\tau)] ,$$

and it remains only to verify that the process  $\{Z_t, t \geq 0\}$  satisfies the conditions of Theorem 3.1. We compute

$$E(Z_t) = [-\sigma^2(t)P(\omega > t)/\epsilon] + \epsilon P(0 \leq \omega \leq t) = 0 ,$$

and similarly obtain  $E(Z_t^2) = \sigma^2(t)$ ,  $t > 0$ . Clearly  $E(Z_t | Z_s = \epsilon) = Z_s$  where  $0 \leq s < t$  are fixed. Let  $\theta = E(Z_t | Z_s = -\sigma^2(s)/\epsilon)$ ; using the relation

$$0 = E(Z_t) = E[E(Z_t | Z_s)] = \epsilon P(Z_s = \epsilon) + \theta P(Z_s = -\sigma^2(s)/\epsilon) ,$$

we obtain  $\theta = -\sigma^2(s)/\epsilon$ . Hence the process  $\{Z_t, t \geq 0\}$  is a martingale satisfying the conditions of Theorem 3.1 and achieving equality in (3.1).

#### Reference

- [1] I.V. Uspensky, Introduction to Mathematical Probability, Mc Graw-Hill, New York, 1937.