AD NUMBER: AD0227274
LIMITATION CHANGES
TO: Approved for public release; distribution is unlimited.
FROM: Distribution authorized to U.S. Gov't. agencies and their contractors; Administrative/Operational Use; 1 Sep 1958. Other requests shall be referred to Office of Naval Research, Washington, DC 20360.
AUTHORITY
ONR LTR 9 NOV 1977

UNIVERSITY OF CALIFORNIA
INSTITUTE OF ENGINEERING RESEARCH
BERKELEY, CALIFORNIA

WATER WAVES. II.

by

John V. Wehausen

Under Contract Number N-onr-222(30)

SERIES NO. 82
ISSUE NO. 7
DATE September, 1958

WATER WAVES. II.

by

John V. Wehausen

Under Contract Number N-onr-222(30)

University of California
Institute of Engineering Research
Berkeley, California

September, 1958

This report constitutes the second part of an article on Water Waves being prepared for the new edition of the Encyclopaedia of Physics (Handbuch der Physik) published by Springer Verlag.

INDEX

16. The solution of special boundary problems

In the next several sections we shall be considering a variety of problems, each associated with some special geometrical configuration.

In treating a particular boundary configuration one must first consider whether it is tractable at all by the theory of infinitesimal waves, i.e. whether it is possible to select a perturbation parameter ε satisfying the requirements mentioned in section 10. On this basis, for example, it would appear unreasonable to try to apply infinitesimal-wave theory to the waves generated by a vertical circular cylinder moving with constant velocity, for the slope of free surface may be expected to become very large near the front of the cylinder. On the other hand, in certain similar situations, notably the theory of planing surfaces, it is possible to strain the theory to accommodate such a situation. The choice of parameter will be discussed in each individual case. We call attention to the fact that in many cases it is a consequence of the linearization procedure that the boundary condition on a solid boundary is no longer to be satisfied on the physical boundary, but instead on some neighboring surface. The same situation occurred earlier in linearizing the free-surface condition. This should not be considered as a further approximation, but rather as one consistent with the infinitesimal-wave approximation.

The methods for finding a solution to a boundary-value problem, once it has been properly formulated, seem to fall into two or possibly three groups. One method is a combination of separation of variables and expansion of the factors in Fourier-type series or integrals. This requires, of course, a geometric configuration related in a suitable way to the coordinate surfaces of a set of variables which allows separation and a complete set of associated elementary solutions to be used in the expansion. If a Fourier-series expansion is to be used, orthogonality of the elementary solutions is desirable.

If the motion is harmonic in time with frequency σ and if the fluid is of finite depth h, then the functions

$$\{\cosh m_0(y+h),\ \cos m_i(y+h)\} \tag{16.1}$$

occurring as factors in (13.2) and (13.4), in (13.6), and in (13.8) may be shown easily by direct computation to be orthogonal on the interval $0 \geq y \geq -h$. Weinstein [1927] has shown that they form a complete set on this interval. The result may be used in the following way, for example. Suppose fluid occupies the region

$$x > 0,\quad 0 > y > -h,\quad 0 < z < \ell,$$

and that the boundary conditions on the walls and bottom are

$$\begin{aligned} \phi_x(0,y,z,t) &= F(y,z)\cos\sigma t, \\ \phi_z(0,x,y,t) &= \phi_z(\ell,x,y,t) = 0, \\ \phi_y(x,z,-h,t) &= 0. \end{aligned} \tag{16.2}$$

Then, by expressing $F(y,z)$ as a double series

$$\begin{aligned} F(y,z) = {} & \sum a_{0q}\cosh m_0(y+h)\cos\frac{\pi q}{\ell}z \\ & + \sum\sum a_{pq}\cos m_p(y+h)\cos\frac{\pi q}{\ell}z \end{aligned} \tag{16.3}$$

(with appropriate restrictions upon F), one may construct a solution from the elementary solutions in (13.6). Further conditions relating to boundedness and behavior as $x \to \infty$ are necessary in order to ensure a unique solution, but will not be discussed here. The elementary solutions (13.8) can be used in a similar way for the region exterior to a vertical cylindrical boundary. Still other configurations are possible corresponding to the various coordinate systems allowing separation of $\Delta_2 \varphi \pm m\varphi = 0$.

If the fluid is infinitely deep, it is possible to construct a Fourier-integral expansion using the functions

$$\left\{e^{\nu y},\ k\cos ky + \nu \sin ky\right\},\quad \nu = \sigma^2/g,\quad 0 < k < \infty. \tag{16.4}$$

In fact, Havelock [1929b] has remarked that the usual Fourier-integral representation of a function may be altered to give

$$f(y) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(\eta)\,\frac{(k\cos ky + \nu\sin ky)(k\cos k\eta + \nu \sin k\eta)}{k^2+\nu^2}\,d\eta\,dk$$
$$+ 2\nu e^{\nu y}\int_0^\infty f(\eta)\,e^{\nu\eta}\,d\eta. \tag{16.5}$$

If the problem is such that rectangular coordinates may be used conveniently, then (16.5) may be combined with a Fourier-series or Fourier-integral expansion in z and the elementary solutions (13.5) used to construct a solution analogous to (16.3). The necessary expressions in both rectangular and cylindrical coordinates can be found in the cited paper of Havelock.

If the fluid is of bounded horizontal extent and is bounded by vertical surfaces which are constant-coordinate surfaces in one of the coordinate systems allowing separation of $\Delta_2\varphi \pm m\varphi = 0$, the various possible modes of motion of the fluid may be obtained as the solution of an eigenvalue problem of a classical type. If the container is of more general shape, it is more difficult to obtain explicit solutions. The problem will be discussed in section 23.

The orthogonal functions (16.1) were associated with a single value of the frequency σ. It is possible to derive another result concerning orthogonality of solutions associated with different values of σ. Let $\varphi_1(x,y,z)\cos\sigma_1 t$ and $\varphi_2(x,y,z)\cos\sigma_2 t$, $\sigma_1 \neq \sigma_2$, be regular velocity potentials of harmonic oscillations of different frequencies. Furthermore, let any solid boundaries be fixed and, if the fluid is not bounded in extent, we suppose that $|\text{grad}\,\varphi| = O(R^{-1-\epsilon})$ as $R^2 = x^2+z^2 \to \infty$. Consider the fluid contained within a large cylinder Ω_R of radius R and above the plane $y = -R$. The fluid will be bounded partly by free surface F_R, partly by solid boundaries S_R, partly by the horizontal plane B_R and partly by the cylinder Ω_R. Applying Green's theorem to the two potential functions, one obtains

$$0 = \iint_{F_R+S_R+B_R+\Omega_R} (\varphi_1\varphi_{2n} - \varphi_{1n}\varphi_2)\,d\sigma =$$

$$\iint_{F_R} (\varphi_1\varphi_{2y} - \varphi_{1y}\varphi_2)\,d\sigma + \iint_{B_R+\Omega_R} (\varphi_1\varphi_{2n} - \varphi_{1n}\varphi_2)\,d\sigma. \tag{16.6}$$

As $R \to \infty$, the integral over $\Omega_R + B_R \to 0$, and one has

$$\iint_F (\varphi_1 \varphi_{2y} - \varphi_{1y} \varphi_2) \, d\sigma = 0. \tag{16.7}$$

From the free-surface condition

$$\varphi_{iy}(x,0,z) = -\frac{\sigma_i^2}{g} \varphi_i(x,0,z), \quad i = 1, 2, \tag{16.8}$$

and (16.7) becomes

$$\frac{\sigma_1^2 - \sigma_2^2}{g} \iint_F \varphi_1(x,0,z) \varphi_2(x,0,z) \, d\sigma = 0, \tag{16.9}$$

or simply

$$\iint_F \varphi_1(x,0,z) \varphi_2(x,0,z) \, d\sigma = 0. \tag{16.10}$$

Hence φ_1 and φ_2 are orthogonal over the free surface of the fluid. This theorem can be used for certain initial-value problems in a manner analogous to that in which the orthogonality of (16.1) can be used for boundary-value problems. This will be done in section 22.

A second method for solving special problems is the method of Green's functions or source functions (cf. Volterra [1934]). In this method one constructs first a potential function of the form

$$G(x,y,z;\xi,\eta,\zeta) = \frac{1}{r} + G_0(x,y,z;\xi,\eta,\zeta),$$
$$r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2, \tag{16.11}$$

such that G_0 is regular in $y<0$ and such that G satisfies the free-surface condition, conditions at infinity appropriate to the problem at hand, and, if the fluid is of finite depth, the boundary condition on the bottom. Such solutions are, of course, just the singular solutions derived in section 13 . Next, if there are surfaces S in (or on) the fluid upon which certain further boundary conditions must be satisfied, we attempt to satisfy them by a distribution of the modified sources (16.11) over the surface(s) S:

$$\phi(x,y,z,t) = \iint_S \gamma(\xi,\eta,\zeta,t)\, G(x,y,z;\xi,\eta,\zeta;t)\, d\sigma. \tag{16.12}$$

Here γ is an unknown function which is to be determined from the boundary condition on S. In most problems this boundary condition consists in specifying ϕ_n on S. Well known properties of surface distributions of sources then allow one to formulate an integral equation for γ:

$$\phi_n(x,y,z,t) = -2\pi\gamma(x,y,z,t) + \iint_S \gamma(\xi,\eta,\zeta,t)\, G_n(x,y,z;\xi,\eta,\zeta;t)\, d\sigma, \tag{16.13}$$
$$(x,y,z) \text{ on } S,$$

where n is the exterior normal to the surface S (taken here as a closed surface). When it is convenient, one may also use distributions of dipoles.

It is also possible, and sometimes advantageous, to construct solutions satisfying given boundary conditions on a closed surface S by distributing the singular solutions on surfaces, lines or points completely inside S. Examples will occur later.

A third method of approach is to seek first, instead of $\phi(x,y,z,t)$ or $f(z,t)$, the functions

$$\chi = \phi_{tt} + g\phi_y \quad \text{or} \quad g = f_{tt} + igf'.$$

These functions satisfy a simpler condition on the plane $y=0$:

$$\chi(x,0,z,t)=0 \quad \text{or} \quad \operatorname{Re} g(x-i0,t)=0.$$

If the other boundary conditions are such that they can be formulated simply in terms of χ or g, the new problem may be simpler to solve. After finding χ or g, one must then solve a differential equation in order to obtain the desired solution ϕ or f. This procedure is called the "reduction" method by Weinstein [1949]. It was apparently first introduced by Keldysh [1935] and has since been much exploited by him, Kochin, Sedov, Haskind, Lewy, Stoker and others. It has already been used in the derivation of (13.28) and will be applied in several other problems. (The method is used also by Muskhelishvili [Singular integral equations, Noordhoff, Groningen, 1953, § 74] to reduce a mixed boundary condition of more complicated type to a simple one.) The solution of the reduced problem may, of course, be carried out by one of the two methods already described above, or any other one which is convenient.

The methods outlined above do not exhaust the possible ones for finding analytic solutions. However, they will occur frequently in the next several sections. Several of the special problems treated in the following sections can be solved by each of the three approaches. The choice of a particular one has been made either to illustrate a method or because it happens to be convenient. Techniques for finding numerical solutions will not be discussed.

17. Two-dimensional progressive and standing waves in unbounded regions with fixed boundaries.

In this and the following section we shall consider situations in which the region occupied by fluid extends to infinity horizontally, the solid boundaries are fixed, but of more complicated shape than the simple flat bottom considered up to now, and the motion of the fluid at infinity is prescribed, or at least partly so. We shall assume that the velocity is bounded at all interior points of the fluid and also at the infinite limits of the fluid. The motion is taken to be periodic everywhere with period σ. Hence we shall assume (cf. § 11) that

$$\phi(x,y,t) = \varphi_1(x,y)\cos\sigma t + \varphi_2(x,y)\sin\sigma t = \operatorname{Re}\ \varphi e^{i\sigma t}.$$

The restriction to standing or progressive waves can be properly applied only at $x = \pm\infty$. Thus, we shall look for solutions which at $x = \infty$ behave like

$$(A \cos mx + B \sin mx) \cos \sigma t$$

or

$$A \cos (mx + \sigma t) + B \cos (mx - \sigma t),$$

and similarly at $x = -\infty$ if the fluid extends in that direction. As we shall see below, the coefficients cannot be chosen independently if φ remains bounded everywhere.

The parameter of linearization may be chosen as

$$\varepsilon = \max (Am, Bm).$$

If the solution φ is bounded everywhere, then as $\varepsilon \to 0$, $\varphi \to 0$ uniformly. However, if a singularity is allowed, then $\varphi \to 0$

uniformly only in a region excluding a neighborhood about the singularity. One may presume that the solution to the linearized problem loses physical significance within such a neighborhood. (It is assumed by Stoker [1947, p. 5] that singularities at the surface are associated with breaking of the waves.)

We shall discuss below two types of problems: obstacles in an infinite ocean and sloping beaches. For each type a special case will be discussed in some detail.

17α. Obstructions in an infinitely long canal.

Consider first the following situation. The fluid extends from $x = -\infty$ to $x = +\infty$; the bottom is given by $y = -h(x)$, where $h(x) = h_1 > 0$ for $x \geq x_1$, $h(x) = h_2 > 0$ for $x \leq x_2$; fixed obstacles may be present in the fluid or on the surface (see Figure 13). The

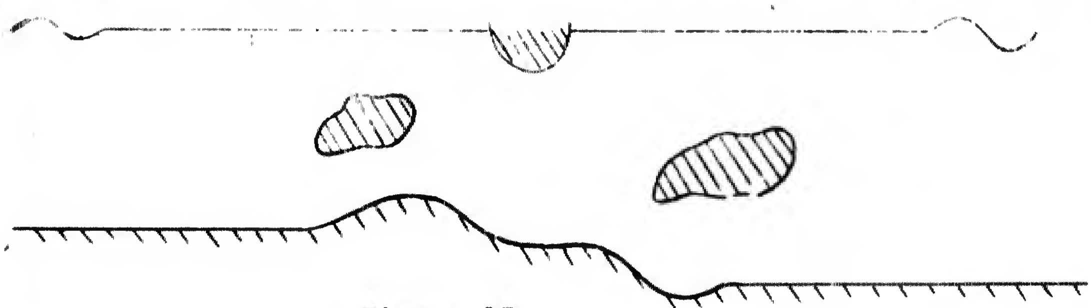

Figure 13.

surface at $x = +\infty$ will be assumed to behave like

$$\eta = A_1 \cos(m_1 x + \sigma t + \alpha_1) + B_1 \cos(m_1 x - \sigma t + \beta_1)$$

and at $x = -\infty$ like

$$\eta = A_2 \cos(m_2 x + \sigma t + \alpha_2) + B_2 \cos(m_2 x - \sigma t + \beta_2)$$

A proof of the existence of a solution to this problem does not seem to exist for the general case. One would not expect a uniqueness theorem since no statement has been made about singularities or circulation. For infinite depth and a submerged body Kochin [1939] has proved the existence for sufficiently large values of m (the situation is slightly different, but the proof carries over). Kreisel [1949] has established the existence of a solution and its uniqueness in two cases. In the first case $h_1 = h_2$, only obstacles on the bottom are allowed, ϕ is assumed bounded, and a certain constant, defined in terms of the conformal mapping of the fluid region onto the strip $o < y < h_1$, must be less than 1. Included are theorems comparing the values of this constant for different types of obstructions. The second result allows a shallow obstruction in the surface, but requires a flat bottom and sufficiently long waves and again bounded ϕ. Roseau [1952] has proved existence and uniqueness for no obstructions within the fluid, but for $h_1 \neq h_2$; the curve joining the two ends is of a special sort. John [1950, p. 78 ff.] has proved uniqueness for a flat bottom and for a body in the free surface with the property that every vertical line intersects either the free surface or the body just once; certain regularity properties of ϕ must also be assumed. If the body is convex and intersects the free surface perpendicularly, he is able to prove also existence of a solution.

Existence and uniqueness theorems have also been proved for several special configurations. In most of these cases explicit solutions are given. A vertical-line barrier extending from the free surface to a depth ℓ in an infinite fluid has been considered by Dean [1945], Ursell [1947], and Haskind [1948]. Both Dean and Ursell also consider a barrier extending from $-\infty$ to a distance ℓ below the surface. John [1948] has generalized both these problems to the case of a slanting barrier of slope $\pi/2n$, and obtained a more general solution even for the vertical barrier. Dean [1948] and Ursell [1950] have also considered submerged circular cylinders in an infinitely deep fluid, and Ursell has established a uniqueness theorem for this case. A horizontal obstruction of finite width on the water surface (the "finite-dock problem) has been treated by Rubin [1954], who proved existence of a solution by a variational method. Other references concerning the dock problem will be given below.

Reflection and transmission coefficients. If one assumes the existence of a solution to the general problem stated above, one may establish the form of the solution for $x > x_1$ and $x < x_2$, by using Weinstein's theorem [1927] asserting the completeness of the functions [cf. (13.6), reduced to two dimensions]

$$\{\cosh m_0(y+h),\ \cos m_n(y+h)\}$$

in the interval $-h \leq y \leq 0$ [cf. Kreisel, 1949, pp. 26-29; John, 1948, p. 152]. It is

$$\phi(x,y,t) = [A_i \cos(m_0^{(i)}x + \sigma t + \alpha_i) + B_i \cos(m_0^{(i)}x - \sigma t + \beta_i)] \cosh m_0^{(i)}(y+h_i) + \sum_{n=1}^{\infty} (a_{in}\cos\sigma t + b_{in}\sin\sigma t)\exp(-m_n^{(i)}|x|)\cos m_n^{(i)}(y+h_i), \tag{17.1}$$

where $i = 1,2$ and $\sigma^2 = gm_0^{(i)}\tanh m_0^{(i)}h_i = -gm_n^{(i)}\tan m_n^{(i)}h_i$.

Let us now apply the formula for dE/dt in equation (8.2) to the region of fluid bounded by the planes $x = c_2 < x_2$, $x = c_1 > x_1$, the bottom and any other obstructions, which we take to be between these two planes. Then, if $\varphi_x^2 + \varphi_y^2$ is bounded in the region considered,

$$\frac{dE}{dt} = \int_{-h_1}^{0} \rho\, \Phi_t\, \Phi_x(c_1, y, t)\, dy - \int_{-h_2}^{0} \rho\, \Phi_t\, \Phi_x(c_2, y, t)\, dy,$$

since on the "physical" boundaries [cf. (8.3)] either p = 0 or $\Phi_n = 0$. Anticipating that we are interested only in the asymptotic values for $c_1 \to \infty$ and $c_2 \to -\infty$, we compute the above expression using only the first term in (17.1) and average over a period $2\pi/\sigma$:

$$\left[\frac{dE}{dt}\right]_{av} = \pi m_0^{(1)} h_1 \left[1 + \frac{\sinh 2m_0^{(1)} h_1}{2m_0^{(1)} h_1}\right]\left[A_1^2 + B_1^2\right] -$$

$$-\pi m_0^{(2)} h_2 \left[1 + \frac{\sinh 2m_0^{(2)} h_2}{2m_0^{(2)} h_2}\right]\left[A_2^2 - B_2^2\right].$$

Since the average energy in the region is constant,

$$m_0^{(1)} h_1\left[1 + \frac{\sinh 2m_0^{(1)} h_1}{2m_0^{(1)} h_1}\right]\left[A_1^2 - B_1^2\right] = m_0^{(2)} h_2\left[1 + \frac{\sinh 2m_0^{(2)} h_2}{2m_0^{(2)} h_2}\right]\left[A_2^2 - B_2^2\right]. \quad (17.2)$$

This is, of course, a statement of the conservation of energy. If A_1 is given $\neq 0$ and $B_2 = 0$, then $A_2 = B_1$ are uniquely determined. For suppose two solutions ϕ and ϕ' are possible, both with the same A_1 and $B_2 = 0$, but one with A_2, B_1, the other with A_2', B_1'. Apply (17.2) to the difference $\phi - \phi'$:

$$-m_0^{(1)} h_1\left[1 + \frac{\sinh 2m_0^{(1)} h_1}{2m_0^{(1)} h_1}\right](B_1 - B_1')^2 = m_0^{(2)} h_2\left[1 + \frac{\sinh 2m_0^{(2)} h_2}{2m_0^{(2)} h_2}\right](A_2 - A_2')^2.$$

Each side must be zero since they differ in sign and are equal. Hence $A_2 = A_2'$ and $B_1 = B_1'$. This does not, of course, imply the uniqueness of ϕ itself.

If $h_1 = h_2$, then (17.2) simplifies in an obvious way:

$$A_1^2 - B_1^2 = A_2^2 - B_2^2 \quad . \tag{17.3}$$

Here h may also be infinite.

Setting $B_2 = 0$ and fixing A_1 as above corresponds physically to giving the amplitude of an incoming wave far to the right. B_1 is then the amplitude of the reflected wave and A_2 of the transmitted wave. The theorem of the preceding paragraph states that A_1 fixes them uniquely. We define $|B_1/A_1|$ as the reflection coefficient R and $|A_2/A_1|$ as the transmission coefficient T. They are uniquely determined and $R^2 + T^2 = 1$. Properly one should define both left and right coefficients since the channel is not symmetric. However, the uniqueness theorem implies that both have the same value. One can clearly arrange the phases so that A_1 and A_2 have the same sign. If this is done, $\alpha_2 - \alpha_1$ will be the phase shift caused by the obstacles.

Kreisel [1949] has proved several general theorems concerning the reflection coefficient if $h_1 = h_2$. In particular, if there are no obstacles within the fluid, he determines upper and lower bounds for the reflection coefficient in terms of the conformal mapping $z(\zeta)$ of the infinite strip $0 > \eta > -h$ onto the region occupied by fluid, with infinities corresponding. His bounds become closer as the wave length increases. He gives, for example, asymptotic expressions as $m_0 \rightarrow 0$ for the reflection coefficient from a horizontal reef of width a and height ε and from a flat plate in the surface of beam b, namely,

$$\frac{\varepsilon}{h} \quad \frac{2m_0 h \,|\sin 2m_0 a|}{\sinh 2m_0 h' \,(1 + 2m_0 h/\sinh 2m_0 h)}$$

and

$$\frac{m_0 \ell}{1 + 2m_0 h/\sinh 2m_0 h}$$

An interesting special result of Dean [1947] (see also Ursell [1950]) is that the reflection coefficient from a submerged circular cylinder in infinitely deep water vanishes. The proof may be briefly sketched. Let a be the radius and let the center be at (0,-b), b > a. Let the velocity potential be written as a sum of an incoming wave and a diverging wave:

$$\phi = A\nu e^{\nu y}\cos(\nu x + \sigma t) + \phi_0 ;$$

and suppose that ϕ_0 can be expressed as a sum of multipoles (13.31), starting with dipoles:

$$\phi_0 = \Sigma\, a_n\phi_n^{(s)}(x,y,t) + b_n\phi_n^{(a)}(x,y,t) + c_n\phi_n^{(s)}(x,y,t+\tfrac{\pi}{2\sigma}) + d_n\phi_n^{(a)}(x,y,t+\tfrac{\pi}{2\sigma}),$$

where $\phi_n^{(s)}$ is the potential for the symmetric potential of order n and strength Q = 1, and $\phi_n^{(a)}$ that for the antisymmetric one. The boundary condition on the cylinder (using the notation of (13.31)),

$$\left.\frac{\partial\phi_0}{\partial r}\right|_{r=a} = A\nu e^{-\nu b} e^{\nu a\cos\theta}\Big[\{\sin(\nu a\sin\theta)\sin\theta - \cos(\nu a\cos\theta)\cos\theta\}\cos\sigma t$$
$$+ \{\cos(\nu a\sin\theta)\sin\theta + \sin(\nu a\sin\theta)\cos\theta\}\sin\sigma t\Big],$$

gives the relation $a_n = -d_n$, $b_n = c_n$. The reflected wave at $+\infty$ from the antisymmetric functions then just cancels that from the symmetric functions. They reinforce each other at x = -∞. The phase change for b/a = 5/4 , $\sigma^2 a/g = 4/3$ was computed numerically by both Dean and Ursell and for this case was very close to 90°.

As mentioned above, straight-line barriers have been considered by Dean [1945], Ursell [1947], Haskind [1948], John [1948], the latter having treated also barriers inclined at an angle $\pi/2n$, and Levine [1958]. The last three authors use the reduction method, whereas the first two use a Fourier-integral method which leads to a singular integral equation. We shall treat this problem by the reduction method.

Vertical barrier. Let the barrier extend along the y-axis from y = 0 to $y = -\ell$ and suppose an incoming wave is given at $x = +\infty$ as

$$\eta = A\cos(\nu x + \sigma t + \alpha), \qquad \sigma^2 = g\nu.$$

We shall look for a velocity potential ϕ having the form

$$\phi = -A\frac{g}{\sigma}e^{\nu y}\sin(\nu x + \sigma t + \alpha) + \varphi_1\cos\sigma t + \varphi_2\sin\sigma t$$

and satisfying the following boundary conditions on the free surface and the barrier:

$$\phi_{tt} + g\phi_y(x,0,t), \quad |x|>0, \text{ and } \quad \phi_x(0,y,t)=0, \quad 0>y>-\ell.$$

As $x \to \pm\infty$, $\varphi_1\cos\sigma t + \varphi_2\sin\sigma t$ must represent outgoing waves. In the neighborhood of $(0,-\ell)$ it will be assumed that

$$\lim\,[x^2+(y+\ell)^2](\phi_x^2+\phi_y^2) = 0 \quad \text{as} \quad (x,y)\to(0,-\ell).$$

In the neighborhood of the intersection of the barrier and the surface (0,0) as well as in the region of fluid bounded away from the barrier, we shall assume $\phi_x^2 + \phi_y^2$ bounded. It should be noted, however, that this assumption excludes a large class of solutions of possible physical interest [cf. John, 1948].

If we introduce the stream functions Ψ, ψ_1, and ψ_2 corresponding to ϕ, φ_1 and φ_2 and the corresponding complex potentials F, f_1 and f_2, we have

$$F = \left(-\frac{Ag}{\sigma} i e^{-i(\nu z+\alpha)} + f_1\right)\cos\sigma t + \left(-\frac{Ag}{\sigma} e^{-i(\nu z+\alpha)} + f_2\right)\sin\sigma t = F_1 \cos\sigma t + F_2 \sin\sigma t$$

and the boundary conditions.

$$\operatorname{Re}\{-\nu F_n + i F_n'\} = 0, \quad y = 0, \ |x| > 0, \ n = 1, 2,$$

$$\operatorname{Re} F_n' = 0, \quad x = 0, \ 0 > y > \ell, \ n = 1, 2.$$

After finding F_1 and F_2 satisfying these conditions, constants occurring in the solutions must be adjusted so that f_1 and f_2 satisfy the radiation condition:

$$\lim_{x \to \pm\infty} (f_1' \pm \nu f_2) = 0, \quad \lim_{x \to \pm\infty} (f_2' \mp f_1) = 0.$$

Consider the function

$$G_1 = F_1' + i\nu F_1 = e^{-i\nu z}(e^{i\nu z} F_1)'.$$

Then the boundary conditions imply that G_1 satisfies

$$\operatorname{Im} G_1 = 0 \quad \text{for} \quad y = 0, |x| > 0,$$

$$\operatorname{Im} G_1' = 0 \quad \text{for} \quad y = 0, |x| > 0 \ \text{and} \ x = 0, 0 > y > -\ell.$$

The function G_1 may be extended into the upper half-plane by defining $G_1(x+iy) = \bar{G}_1(x-iy)$ for $y > 0$. Since we have assumed $|F'| \leq B$ for $|z| > b > \ell$, we may conclude that $|G_1| < B + C|z|$ for $|z| > b$ and expand G_1 in a Laurent series

$$G_1(z) = cz + d + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots, \quad |z| > b > \ell,$$

where all coefficients are real since $\mathrm{Im}\, G_1(x + i0) = 0$. The condition $|F_1'| \to 0$ as $y \to -\infty$ implies $|G_1'| \to 0$ as $y \to -\infty$ and hence $c = 0$. We may arbitrarily set $d = 0$ by redefinition of Ψ_1. Further, we may show as follows that $a_1 = 0$. Consider a contour containing the obstruction and lying in the region $|z| > b$. Then

$$\oint G_1(\zeta)\, d\zeta = 2\pi i\, a_1 .$$

Let this contour be contracted onto the barrier. Then, from the assumed behavior of F_1 on the barrier, the integral vanishes; hence $a_1 = 0$. Thus

$$G_1(z) = \frac{a_2}{z^2} + \frac{a_3}{z^3} + \cdots \quad .$$

Let us now exploit the boundary condition for G_1' by mapping the z-plane into a ζ-plane by the mapping

$$z = \sqrt{\zeta - \ell^2} ,$$

where the branch of the square root is chosen which makes $z \cong \zeta$ for large ζ. This maps the z-plane cut from $-i\ell$ to $i\ell$, i.e. along the barrier, onto the z-plane cut from $-\ell$ to $+\ell$, with infinities and upper and lower half-planes corresponding. Then $G_1'(z(\zeta)) = H_1(\zeta)$ is analytic in the whole lower half-plane with a singularity only at $\zeta = 0$, corresponding to $z = -i\ell$, and $\mathrm{Im}\, H_1(\xi + i0) = 0$. Since $H_1(\zeta)$ must agree with $G_1'(z)$ for large z, $H_1(\zeta)$ must have the form

$$H_1(\zeta) = \frac{b_3}{\zeta^3} + \frac{b_4}{\zeta^4} + \cdots , \quad b_n \text{ real.}$$

The condition on ϕ near the edge of the barrier, implies that $|z + i\ell| \cdot |F_1'| \to 0$ as $z \to -i\ell$, or $|\zeta^4 H_1(\zeta)| \to 0$ as $\zeta \to 0$ and hence that $b_n = 0$, $n \geq 4$. Thus

$$H_1(\zeta) = \frac{C_1}{\zeta^3}, \quad C_1 \text{ real},$$

or

$$G_1'(z) = \frac{C_1}{(\zeta^2+\ell^2)^{3/2}} .$$

Integrating, and writing $D_1 = C_1/\ell^2$,

$$G_1(\zeta) = D_1 \frac{\zeta - \sqrt{\zeta^2+\ell^2}}{\sqrt{\zeta^2+\ell^2}} = D_1 \frac{\zeta}{\sqrt{\zeta^2+\ell^2}} - D_1 ,$$

where the constant of integration has been chosen so as to make $G_1(z)$ behave like z^{-2} for large z. Then

$$F_1(\zeta) = E_1 e^{-i\nu\zeta} + D_1 e^{-i\nu\zeta} \int_{i\infty}^{\zeta} \frac{\zeta - \sqrt{\zeta^2+\ell^2}}{\sqrt{\zeta^2+\ell^2}} e^{i\nu\zeta} d\zeta ,$$

where the path of integration will be taken around the right-hand side of the barrier. The boundary condition Re $F_1'(0 + iy) = 0$, $0 < |y| < \ell$, relates E_1 and D_1 as follows. From $F_1(z)$:

$$F_1'(\zeta) = e^{-i\nu\zeta}\left[-i\nu E_1 + D_1 \frac{\zeta}{\sqrt{\zeta^2+\ell^2}} - i\nu D_1 \int_{i\infty}^{\zeta} \frac{\zeta e^{i\nu\zeta}}{\sqrt{\zeta^2+\ell^2}} d\zeta\right].$$

Take the path of integration along the y-axis, so that the integral becomes

$$\int_{i\infty}^{\zeta} = -i \int_{\ell}^{\infty} \frac{y e^{-\nu y}}{\sqrt{y^2-\ell^2}} dy \mp \int_{\ell}^{y} \frac{y e^{-\nu y}}{\sqrt{\ell^2-y^2}} dy$$

$$= -i\ell K_1(\nu\ell) \mp \int_{\ell}^{y} \frac{y e^{-\nu y}}{\sqrt{\ell^2-y^2}} dy , \quad x = \pm 0 .$$

Hence

$$Re\ F_1'(0+iy) = e^{\nu y}[+\nu\, Im\, E_1 - \nu \ell D_1 K_1(\nu\ell)] = 0$$

or

$$Im\ E_1 = +D_1 \ell K_1(\nu\ell)$$

Let $E_1 = e_1 + i\,\ell\, DK_1(\nu\ell)$. Then

$$F_1(z) = e^{-i\nu z}\left[e_1 + i\ell D_1 K_1(\nu\ell) + D_1 \int_{i\infty}^{z} \frac{z-\sqrt{z^2+\ell^2}}{\sqrt{z^2+\ell^2}}\, e^{i\nu z}\, dz\right].$$

We now compute the asymptotic expressions for $F_1(z)$ for $x \to \pm\infty$. If the path of integration is taken on a large arc of radius R in the first quadrant and then to z, and if R is allowed to become infinite, it follows from Jordan's lemma that the integral may also be written

$$\int_{\infty}^{z} \frac{z-\sqrt{z^2+\ell^2}}{\sqrt{z^2+\ell^2}}\, e^{i\nu z}\, dz.$$

Clearly,

$$F_1(z) \sim e^{-i\nu z}[e_1 + i\ell D_1 K_1(\nu\ell)] \quad \text{as } x \to +\infty.$$

As $x \to -\infty$,

$$F_1(z) \sim e^{-i\nu z}\left[e_1 + i\ell D_1 K_1(\nu\ell) + D_1 \int_{\infty}^{-\infty} \frac{z-\sqrt{z^2+\ell^2}}{\sqrt{z^2+\ell^2}}\, e^{i\nu z}\, dz\right],$$

where the path of integration passes below the barrier. By completing this path by a large semicircle in the upper half-plane, which gives a zero contribution in the limit, and then contracting the contour about the barrier, one sees that

$$\int_{\infty}^{-\infty} \frac{z-\sqrt{z^2+l^2}}{\sqrt{z^2+l^2}} e^{i\nu z}\,dz = +2\int_{-l}^{l} \frac{y\,e^{-\nu y}}{\sqrt{l^2-y^2}}\,dy = -2\nu\int_{-l}^{l} e^{-\nu y}\sqrt{l^2-y^2}\,dy$$

$$= -2\pi l I_1(\nu l).$$

Hence

$$F_1(z) \sim e^{-i\nu z}\left[e_1 + i l D_1 K_1(\nu l) - 2\pi l D_1 I_1(\nu l)\right] \quad \text{as } x \to -\infty.$$

Similar expressions hold for $F_2(z)$ with constants e_2 and D_2.

For f_1 and f_2 we have the asymptotic expressions:

$$\left.\begin{aligned} f_1(z) &\sim e^{-i\nu z}\left[\frac{Aq}{6} i e^{-i\alpha} + e_1 + i l K_1(\nu l) D_1\right] \\ f_2(z) &\sim e^{-i\nu z}\left[\frac{Aq}{6} e^{-i\alpha} + e_2 + i l K_1(\nu l) D_2\right] \end{aligned}\right\} \quad \text{as } x \to +\infty,$$

$$\left.\begin{aligned} f_1(z) &\sim e^{-i\nu z}\left[\frac{Aq}{6} i e^{-i\alpha} + e_1 + (iK_1 - 2\pi I_1) l D_1\right] \\ f_2(z) &\sim e^{-i\nu z}\left[\frac{Aq}{6} e^{-i\alpha} + e_2 + (iK_1 - 2\pi I_1) l D_2\right] \end{aligned}\right\} \quad \text{as } x \to -\infty.$$

The radiation condition gives simultaneous equations for the determination of e_1, e_2, D_1 and D_2. The solution may be written

$$l(D_1 + i D_2) = -\frac{Aq}{6} i e^{-i\alpha} \frac{1}{\pi I_1 + i K_1}, \quad e_1 + i e_2 = -\frac{Aq}{6} i e^{-i\alpha}\left(1 + \frac{\pi I_1}{\pi I_1 + i K_1}\right).$$

Substitution in the expressions for f_1 and f_2, and computation of $F_1 \cos \sigma t + F_2 \sin \sigma t$ gives, after a somewhat tedious calculation, the following asymptotic expressions for ϕ :

$$\Phi \sim \frac{Ag}{\sigma} e^{\nu y} \left\{ -\sin(\nu x + \sigma t + \alpha) + \frac{\pi I_1}{\sqrt{\pi I_1^2 + K_1^2}} \sin(\nu x - \sigma t - \alpha - \beta_R) \right\}, \quad x \to +\infty ,$$

$$\Phi \sim \frac{-Ag}{\sigma} e^{\nu y} \frac{K_1}{\sqrt{\pi^2 I_1^2 + K_1^2}} \sin(\nu x + \sigma t + \alpha + \beta_T), \quad x \to -\infty, \qquad (17.4)$$

where $\tan \beta_R = K_1/\pi I_1 = \cot \beta_T$, and $I_1 = I_1(\nu\ell)$, $K_1 = K_1(\nu\ell)$. Clearly the reflection and transmission coefficients are

$$R = \frac{\pi I_1}{\sqrt{\pi I_1^2 + K_1^2}} , \qquad T = \frac{K_1}{\sqrt{\pi I_1^2 + K_1^2}} . \qquad (17.5)$$

R, T and $\beta_R = \frac{1}{2}\pi - \beta_T$ are shown in Figure 14 as functions of $2\pi\ell/\lambda = \nu\ell$.

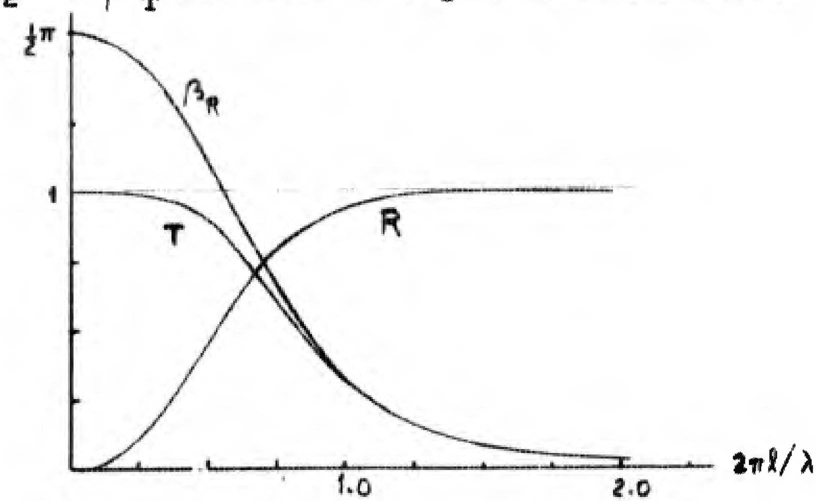

Figure 14

The reflection coefficient is practically one if $\ell/\lambda \geq \frac{1}{4}$.

One may now use the velocity potential to find the behavior of the fluid near the barrier, in particular, the water height and the pressure. The calculations will not be carried through, but may be found in Haskind [1948]. The elevation on either side of the barrier is given by

$$\eta(\pm 0, t) = A\left[\cos(\sigma t + \alpha) \mp \frac{1 + \nu\ell S(\nu\ell)}{\sqrt{\pi^2 I_1^2 + K_1^2}} \cos(\sigma t + \alpha + \beta_R)\right], \qquad (17.6)$$

where

$$S(\nu l) = \frac{\pi}{2\nu l}[I_1(\nu l) + L_1(\nu l)] = \int_0^1 e^{\nu l y}\sqrt{1-y^2}\, dy,$$

L_1 being a Struve function of imaginary argument [Watson, p. 329; L_1 is tabulated in J. Math. Phys. 25 (1946), 252-259]. Let the force and moment about the origin, per unit length of barrier, be denoted by X and M, the former being positive if directed along OX, the latter is counterclockwise. Then

$$X = +2\rho g A l\, X_0 \cos(kt + \alpha + \beta_R),$$

$$M = +2\rho g A l^2 M_0 \cos(kt + \alpha + \beta_R), \qquad (17.7)$$

where

$$X_0 = \frac{S}{\sqrt{\pi I_1^2 + K_1^2}}, \quad M_0 = \frac{1}{\nu l\sqrt{\pi I_1^2 + K_1^2}}\left(S - \frac{\pi}{4}\right).$$

Haskind also computes the average force and moment per unit length of the barrier. The results are:

$$X_{av.} = \frac{1}{2}\rho g A^2 \frac{\pi^2 I_1^2}{\pi^2 I_1^2 + K_1^2} = \frac{1}{2}\rho g A^2 R^2, \qquad (17.8)$$

$$M_{av.} = \frac{1}{2}\rho g A^2 l\left[S(-\nu l) - T(-\nu l) - \frac{\pi I_1(\nu l)}{2\nu l}\right]\frac{\pi I_1}{\pi^2 I_1^2 + K_1^2},$$

where

$$\nu l\, S(-\nu l) = \frac{1}{2}\pi\left[I_1(\nu l) - L_1(\nu l)\right],$$

$$T(-\nu l) = \frac{1}{2}\pi\left[I_0(\nu l) - L_0(\nu l)\right].$$

Figure 15 displays all four functions in dimensionless form.

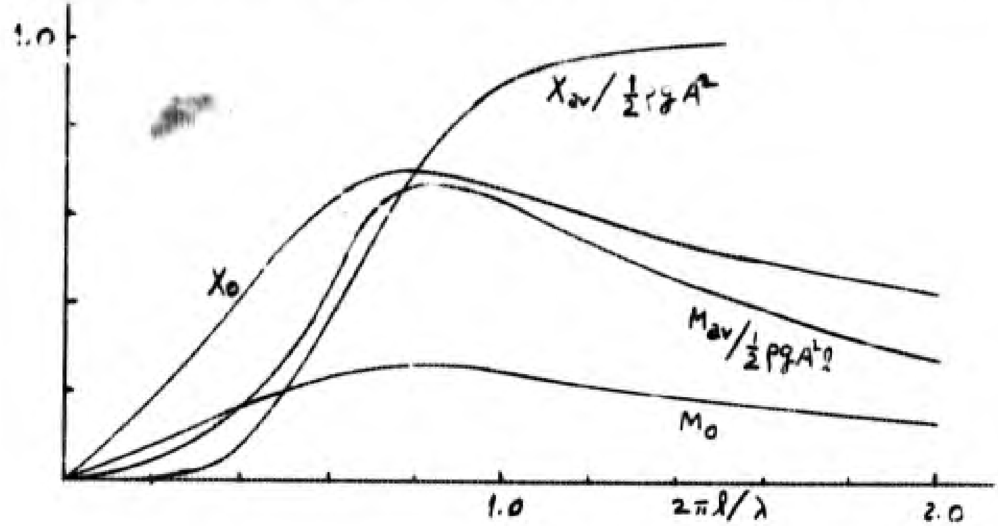

Figure 15.

The method of integral equations. This method for finding solutions has been frequently used, especially by Kochin [1937, 1939, 1940] and his colleagues. One of its advantages is that approximate solutions to the integral equation can frequently be found even when an explicit solution cannot be easily obtained. The following exposition follows approximately Kochin [1937] and Keldysh and Lavren-t'ev [1937].

Consider a submerged obstacle whose contour C is given parametrically by $z = z(s)$ and is oriented counterclockwise. Let $\beta(s)$ be the angle between the tangent vector and the positive x-direction (see Figure 16). We shall assume that as $x \to \infty$ the motion

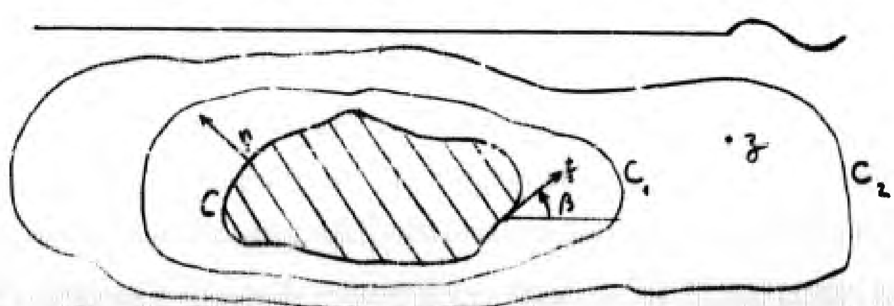

Figure 16

approximates to a standing wave:

$$\phi(x,y,t) \sim A\frac{g}{\sigma} e^{\nu y}\cos(\nu x+\alpha)\cos(\sigma t+\tau), \quad \nu = \frac{\sigma^2}{g}. \tag{17.9}$$

The other boundary conditions in terms of the complex potential $f(z) = \varphi(x,y) + i\,\Psi(x,y)$ are

$$\begin{aligned} &Im\{f'(x)+i\,f(x)\} = 0, \\ &Im\{f'(z(s))\,e^{i\beta(s)}\} = 0, \\ &\lim_{y\to-\infty} |f'| = 0. \end{aligned} \tag{17.10}$$

Write f(z) in the form

$$f(z) = f_1(z) + \frac{Ag}{\sigma} e^{-i(\nu z+\alpha)} = f_1(z) + a e^{-i\nu z}. \tag{17.11}$$

Then $f_1(z)$ must satisfy

$$\begin{aligned} &\lim_{x\to\infty} f_1(z) = 0 \\ &Im\left[f_1'(z(s)) - i a\nu e^{-i\nu z(s)}\right] e^{i\beta(s)} = 0, \end{aligned} \tag{17.12}$$

as well as the free surface condition and the condition as $y \to -\infty$.

We shall try to express $f_1(z)$ as a distribution of vortices over the contour C. However, the vortices are chosen so that the conditions on the free surface, at $x = \infty$ and at $y = -\infty$ are satisfied. As is apparent from the derivation of (13.28), the complex velocity potential for such vortices is given by

$$f_v(z;c) = \frac{\Gamma}{2\pi i}\left\{\log(z-c)(z-\bar{c}) - 2e^{-i\nu z}\int_{\infty}^{z} \frac{e^{i\nu u}}{u-\bar{c}}\,du\right\}. \tag{17.13}$$

We set $\Gamma = 1$ and try to express $f_1(z)$ as follows:

$$f_1(z) = \int_C \gamma(s) f_v(z, z(s)) \, ds \,, \tag{17.14}$$

where $\gamma(s)$ must be chosen so that the boundary condition on the body is satisfied.

In order to derive an integral equation for $\gamma(s)$, consider the following expression for $f_1'(z)$, a direct consequence of Cauchy's integral:

$$f_1'(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f'(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \int_{C_2} \frac{f'(\zeta)}{\zeta - z} \, d\zeta$$

$$= g_1(z) + g_2(z).$$

The function $g_1(z)$ is regular everywhere outside C_1 and $g_2(z)$ is regular everywhere inside C_2. One may contract C_1 onto C and extend $g_2(z)$ analytically into the whole lower half-plane (or fluid strip if the depth is finite).

Consider now (for infinite depth; the finite depth case is analogous) the following function:

$$g(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f_1'(\zeta)}{\zeta - z} \, d\zeta + \frac{1}{2\pi i} \int_{C_1} \overline{f_1'(\zeta)} \left[\frac{1}{z - \bar{\zeta}} - 2i\nu e^{-i\nu z} \int_{\infty}^{z} \frac{e^{i\nu u}}{u - \bar{\zeta}} \, du \right] d\bar{\zeta}.$$

The first summand is identical with $g_1(z)$ and the second is also regular in the whole half-plane. $g(z)$ satisfies the same boundary conditions as $f_1'(z)$. Hence $f_1'(z) - g(z)$ is regular in the whole lower half-plane, satisfies the free-surface condition and vanishes as $y \to -\infty$ and $x \to +\infty$. The uniqueness argument used in the derivation of (13.28) shows that $f_1'(z) \equiv g(z)$. Thus we have

$$f_1'(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f_1'(\zeta)}{z-\zeta}\,d\zeta - \frac{1}{2\pi i}\int_{C_1} f_1'(\zeta)\left[\frac{1}{z-\bar{\zeta}} - 2i\nu e^{-i\nu z}\int_{\infty}^{z}\frac{e^{i\nu u}}{u-\bar{\zeta}}\,du\right]d\bar{\zeta}. \tag{17.15}$$

Now contract C_1 to C. Then

$$[f_1'(\zeta) - ia\nu e^{-i\nu\zeta}]e^{i\beta} = v_t + i v_n = v_t$$

or

$$f_1'(\zeta) = v_t e^{-i\beta} + ia\nu e^{-i\nu\zeta}. \tag{17.16}$$

If one substitutes above, one finds that the contribution from the second summand in $f_1'(\zeta)$ vanishes and that, since $d\zeta/ds = e^{i\beta(s)}$,

$$f_1'(z) = \frac{1}{2\pi i}\int_C v_t(s)\left[\frac{1}{z-\zeta} - \frac{1}{z-\bar{\zeta}} \cdot 2i\nu e^{-i\nu z}\int_{-\infty}^{z}\frac{e^{i\nu u}}{u-\bar{\zeta}}\,du\right]ds \tag{17.17}$$

$$= \int_C v_t(s)\, f_v'(z;\zeta(s))\,ds.$$

This identifies $\gamma(s)$ as the tangential velocity $v_t(s)$ at a point of the contour.

Let us now consider the effect of letting $z \to z(s')$, a point of the contour C. Then, according to the Theorem of Plemelj-Sokhotskii,

$$\int_C \gamma(s) f_v'(\zeta;\zeta(s))\,ds = \int_C \gamma(s) f_v'(\zeta;\zeta(s))\,e^{-i\beta(s)}\,d\zeta(s) \to$$
$$\tfrac{1}{2}\gamma(s')\,e^{-i\beta(s')} + \quad PV\!\!\int_C \gamma(s) f_v'(\zeta(s');\zeta(s))\,ds, \tag{17.18}$$

whereas

$$f_1'(\zeta) \to v_t(s')\,e^{-i\beta(s')} + i a\nu e^{-i\nu\zeta(s')} = \gamma(s')\,e^{-i\beta(s')} + i a\nu e^{-i\nu\zeta(s')}.$$

Hence we have the integral equation for $\gamma(s)$:

$$-\tfrac{1}{2}\gamma(s') + PV\!\!\int_C \gamma(s) f_v'(\zeta(s');\zeta(s))\,e^{i\beta(s')}\,ds = iA\sigma e^{-i[\nu\zeta(s')-\beta(s')+\alpha]}. \tag{17.19}$$

This is really two integral equations. The imaginary part gives a singular integral equation of the first kind:

$$PV\!\!\int_C \gamma(s) K(s',s)\,ds = -2\pi A\sigma\, e^{\nu y(s')}\cos\left[\nu x(s') - \beta(s') + \alpha\right]. \tag{17.20}$$

The real part gives a Fredholm equation of the second kind with continuous kernel:

$$-\tfrac{1}{2}\gamma(s') + \frac{1}{2\pi}\int_C \gamma(s) L(s',s)\,ds = 2\pi A\sigma\, e^{\nu y(s')}\sin\left[\nu x(s') - \beta(s') + \alpha\right]. \tag{17.21}$$

Here

$$f_v'(\zeta(s');\zeta(s))\,e^{i\beta(s')} = \frac{1}{2\pi i}\left[K(s',s) + i\,L(s',s)\right]. \tag{17.22}$$

The kernel K(s',s) is of the form

$$K(s',s) = \frac{1}{s'-s} + C(s',s) \quad , \tag{17.23}$$

where $C(s',s)$ is continuous; the first term comes from $e^{i\beta(s')}/[z(s') - z(s)]$. If the curve C is sufficiently smooth,

$$\lim_{\sigma \to s} \operatorname{Im} \frac{e^{i\beta(s')}}{z(s') - z(s)} = \frac{1}{\rho(s)} , \tag{17.24}$$

where $\rho(s)$ is the radius of curvature of C at $z(s)$.

If the obstacle consists of a smooth arc, an analogous argument leads to only the singular integral equation above, but with $\gamma(s)$ now identified with the jump in $v_t(s)$ as one goes from the left to the right side of the arc.

There does not seem to be a published proof that a solution to either integral equation exists for all ν. However, Kochin [1936, pp. 119-126] shows the existence of a solution for both sufficiently large and sufficiently small values of ν for the equation of the second kind.

By adjusting the phases in (17.9) one may obtain two ϕ's which may be added to give an outgoing progressive wave. The behavior as $x \to -\infty$ will then be a superposition of an incoming and an outgoing wave. However, one may also modify the preceding arguments in order to treat the progressive wave problem directly. One specifies, say, an incoming wave from the right, writes

$$\phi(x,y,t) = \frac{Ag}{\sigma} e^{\nu y} \cos(\nu x + \sigma t) + \phi^*(x,y,t), \tag{17.25}$$

where ϕ^* must satisfy the radiation condition, and tries to express the corresponding complex potential as a distribution of the vortices (13.28) since they already satisfy the radiation condition. We shall not dwell on the details except to remark that the problem leads to a pair of coupled integral equations since one needs a distribution not only of (13.20) as it stands, but also of the vortices obtained by replacing t by t - $\pi/2\sigma$. This method could have been applied, for example, to the problem of the vertical barrier considered above.

Dock problems. This term is generally applied to water-wave problems in which the obstruction is a horizontal plane of finite or semi-infinite extent, either submerged or lying on the surface. The solution for the semi-infinite dock in infinitely deep water was given by Friedrichs and Lewy [1948], and at about the same time the same problem in water of finite depth was treated by A. Heins [1948] who also allowed a restricted type of three-dimensional motion. The methods were quite different. Subsequently Heins [1950] and Greene and Heins [1953] extended the treatment to submerged docks in water of finite and infinite depth. As was remarked earlier, Rubin [1954] has shown the existence of a solution for the finite dock in infinitely deep water. Sparenberg [1957] has deduced an integral equation of the second kind for this problem.

As an example, consider a submerged dock at depth b and extending from x = -a to x = a. The integral equation (17.20) then becomes

$$\mathrm{PV}\int_{-a}^{a} \gamma(\xi) K(x, \xi)\, d\xi = -2\pi A \sigma e^{-\nu b} \cos(\nu x + \alpha), \tag{17.26}$$

where $K(x, \xi) = K(x-\xi)$ with

$$K(x) = Re\left\{\frac{1}{x} - \frac{1}{x-2ib} + i\frac{2\nu}{\pi}e^{-i\nu x}\int_{\infty}^{x}\frac{e^{i\nu u}}{u-2ib}du\right\}$$

$$= \frac{1}{x} - \frac{x}{x^2+4b^2} - \frac{2\nu}{\pi}\int_{\infty}^{x}\frac{u\sin\nu(u-x)+2b\cos\nu(u-x)}{u^2+4b^2}du. \qquad (17.27)$$

Without actually establishing the existence of a solution to (17.26), Keldysh and Lavrent'ev [1937] in treating the flow about thin hydrofoils (see section 20 β) propose an approximate method of solution by expanding γ (x) and K(x) in a series in τ = a/2b:

$$\gamma(x) = \gamma_0(x) + \gamma_1(x)\tau + \cdots,$$

$$K(x) = \frac{1}{x} + a\sum K_n\left(\frac{x}{a}\right)^n\tau^{n+1}$$

and determining the $\gamma_n(x)$ recursively. In the problem treated by them the total vorticity was fixed by the Kutta-Joukowski condition, in the present problem the corresponding condition is still to be determined.

If the dock extends from $-\infty$ to 0, one may modify the earlier arguments so as to apply to an unbounded body and derive the integral equation

$$PV\int_{-\infty}^{0}\gamma(\xi)K(x-\xi)\,d\xi = -2\pi Ab e^{-\nu b}\cos(\nu x+\alpha). \qquad (17.28)$$

An integral equation of this form is known as a Wiener-Hopf integral equations and in many cases can be solved by use of Fourier transforms. It does not seem possible to expound the method briefly, so we refer to the paper of Greene and Heins [1953] where this problem is treated, but with the kernel expressed differently.

When the semi-infinite dock is on the surface, the dock may be considered as a limiting case of a beach in which the angle between the bottom and the free surface is 180°. Although waves on beaches are discussed in the next section, the methods which allow extension of the angle to 180° are also difficult and will not be considered there. They may be found in Stoker's Water waves [1957, §5.4].

17 β. Waves on beaches.

Let the fluid at rest be contained in the wedge defined by

$$\tan \gamma \leq \frac{-y}{x} \leq 1 \quad , x > 0, \quad \gamma > 0 \quad ,$$

i.e., the bottom is the plane $x \sin \alpha + y \cos \alpha = 0$. For such a body of fluid one may look for periodic waves which are either standing or progressive. The appropriate mathematical problem for standing waves is to find a velocity potential

$$\phi(x, y, t) = \varphi(x, y) \cos(\sigma t + \tau) \tag{17.29}$$

satisfying

1) $\Delta \varphi = 0$,

2) $\varphi_y(x, 0) - \frac{\sigma^2}{g} \varphi(x, 0) = 0$,

3) $\varphi_x \sin \gamma + \varphi_y \cos \gamma = 0$ for $x \sin \gamma + y \cos \gamma = 0$

4) $\lim\limits_{x^2+y^2 \to \infty} \varphi_x^2 + \varphi_y^2 = 0$ for $x \sin \gamma + y \cos \gamma = 0$

This problem, in both this form and the three-dimensional form to be considered in § 18, has received intensive study in recent years [e.g., Miche, 1944; Lewy, 1945; Stoker, 1947; Friedrichs, 1948; Isaacson, 1948, 1950; Weinstein, 1949; Peters, 1950, 1952; Roseau, 1952; Lehmann, 1954; Brillouët, 1957]. In particular, the cited work of Brillouët and chapter 5 of Stoker's <u>Water waves</u> [1957] contain a general exposition of the mathematical theory. We shall restrict the present treatment to simple cases.

Kirchhoff [1879] was apparently the first one to treat the two-dimensional case. The problem was taken up again by Hanson [1926] who considered both the two and three-dimensional cases. Both these authors restricted the solution to be bounded everywhere. This has the effect of excluding a physically important class of solutions with singularities at the origin. One may see this easily if $\gamma = 90^{\circ}$, i.e. when there is a vertical cliff. A bounded solution is obviously $\varphi(x,y) = Ae^{\nu y}\cos \nu x$, $\nu = \sigma^2/g$. This generates a standing wave behaving like $\cos \nu x$ at $x = \infty$. However, if we wish to construct a solution behaving, say, like an incoming wave at infinity we need also a standing-wave solution behaving like $\sin \nu x$ at infinity. No such solution exists which is bounded everywhere. However, as we shall see, it is possible to construct such a solution by allowing a singularity at the origin. If the two standing-wave solutions are used to construct an incoming progressive wave, the consequent loss of energy associated with the singularity is sometimes interpreted physically as representing loss of energy in breaking of the waves, at least when α is sufficiently small for this to happen. There is, of course, no a priori method of selecting the mathematical solution best representing the physical phenomena. The comparison between physical waves and mathematical solutions is discussed briefly in Stoker [1957, pp. 69-77].

Kirchhoff's approach to the solution is interesting historically because of its similarity to the method used later by Peters [1950] and Roseau [1951] in solving the problem. His reasoning runs as follows, with a slight change in notation. Let $f(z) = \varphi + i\psi$ be the complex potential. Then

$$2\varphi(x,y) = f(x+iy) + \bar{f}(x-iy),$$
$$2i\psi(x,y) = f(x+iy) - \bar{f}(x-iy).$$

The free-surface condition becomes

$$i[f'(x) - \bar{f}'(x)] = \nu[f(x) + \bar{f}(x)], \quad \nu = \sigma^2/g.$$

But then also

$$i[f'(z) - \bar{f}'(z)] = \nu[f(z) + f'(z)]. \tag{17.30}$$

The bottom must be a streamline. Hence

$$f(re^{-i\gamma}) - \bar{f}(re^{i\gamma}) = \text{const.};$$

we may take this constant as 0. From this

$$\bar{f}(z) = f(ze^{-i2\gamma}). \tag{17.31}$$

Hence

$$\frac{d}{dz}[f(z) - f(ze^{-i2\gamma})] = -i\nu[f(z) + f(ze^{-i2\gamma})]. \tag{17.32}$$

This differential difference equation must hold for all z for which f(z) and $f(ze^{-i2\gamma})$ are both defined, namely for

$$-\gamma < \arg z < \gamma.$$

Kirchhoff's formal arguments need to be supported in terms of analytic continuation by the reflection principle, but the essential idea is the same as that used more recently [cf., e.g., Lehmann, 1954, § 3, or Peters, 1950, § 3].

Kirchhoff proceeds to solve this equation in the special case $\gamma = m\pi/n$, m and n relatively prime integers, by assuming

$$f(z) = \sum_{k=0}^{n-1} A_k \exp i\lambda\nu z\beta^k, \quad \beta = \exp(-i2m\pi/n). \tag{17.33}$$

Substitution in (17.12) gives

$$A_k(\beta^k\lambda + 1) = A_{k-1}(\beta^k\lambda - 1), \quad k = 0, \cdots, n-1, \tag{17.34}$$

with $A_{-1} \equiv A_{n-1}$. Multiplying all equations together and remembering that $1, \beta, \ldots, \beta^{n-1}$ are all nth roots of unity, one finds

$$\lambda^n - (-1)^n = \lambda^n - 1,$$

which can hold only if n is even, say n = 2q (hence m is odd). With $\lambda = -1 = \beta^q$, the above equations determine successively $A_1, \ldots, A_{q-1}$ in terms of A_0, and $A_q = \ldots = A_{n-1} = 0$:

$$A_k = iA_{k-1}\cot k\gamma = i^k A_0 \cot\gamma \cot 2\gamma \ldots \cot k\gamma. \tag{17.35}$$

Then

$$f(z) = \sum_{k=0}^{q-1} A_k \exp(-i\nu\beta^k z). \tag{17.36}$$

A_0 is still an arbitrary complex constant. The differential-difference equation is a necessary condition for f(z), but not sufficient to ensure that all boundary conditions are satisfied. If one substitutes the above expression for f(z) in (17.31), one finds after some computation that one must take

$$A_0 = B_0 e^{-i\pi(q-1)/4}, \tag{17.37}$$

where B_o is pure imaginary (say iB_o') if both $\frac{1}{2}(m+1)$ and q are even and otherwise is real. With this choice of A_o one has

$$A_{q-k} = \bar{A}_{k-1} \, . \tag{17.38}$$

As Kirchhoff points out, the solution is physically acceptable for the problem at hand only if m = 1; otherwise, φ does not remain bounded as $x \to +\infty$. If m = 1, then for y = 0, the dominant term as $x \to \infty$ is given by

$$f(x) \sim B_o \exp\left(-i\nu x - i\pi \frac{q-1}{4}\right)$$

or

$$\varphi(x,0) \sim B_o \cos\left(\nu x + \frac{q-1}{4}\pi\right). \tag{17.39}$$

Here are several easily computable special cases of (17.36):

$\gamma = 90^\circ$ (m = 1, q = 1, β = -1):

$$f(z) = B_o e^{-i\nu z} = B_o e^{\nu y}(\cos \nu x - i \sin \nu x) \, ; \tag{17.40}$$

$\gamma = 45^\circ$(m = 1, q = 2, β = -1):

$$\begin{aligned} f(z) &= B_o e^{-i\frac{\pi}{4}}\left[e^{-i\nu z} + i e^{-\nu z}\right] \\ &= B_o\left[e^{\nu y}\cos\left(\nu x + \frac{\pi}{4}\right) + e^{-\nu x}\cos\left(\nu y - \frac{\pi}{4}\right)\right] - \\ &\quad - i B_o\left[e^{\nu y}\sin\left(\nu x + \frac{\pi}{4}\right) + e^{-\nu x}\sin\left(\nu y - \frac{\pi}{4}\right)\right] \, ; \end{aligned} \tag{17.41}$$

$\gamma = 30^\circ$(m = 1, q = 3, $\beta = \frac{1}{2}(\sqrt{3} - 1)$):

$$f(z) = B_o e^{-i\frac{\pi}{2}}\left[e^{-i\nu z} + i\sqrt{3}\, e^{-\frac{1}{2}(\sqrt{3}+i)\nu z} - e^{-\frac{1}{2}(\sqrt{3}-i)\nu z}\right]$$

$$= B_o\left\{-e^{\nu y}\sin \nu x - e^{-\frac{1}{2}\nu(x\sqrt{3}+y)}\sin\tfrac{1}{2}\nu(x - y\sqrt{3}) + \sqrt{3}\, e^{-\frac{1}{2}\nu(x\sqrt{3}-y)}\cos\tfrac{1}{2}\nu(x + y\sqrt{3})\right\} +$$

$$+ i B_o\left\{-e^{\nu y}\cos \nu x + e^{-\frac{1}{2}\nu(x\sqrt{3}+y)}\cos\tfrac{1}{2}\nu(x - y\sqrt{3}) - \sqrt{3}\, e^{-\frac{1}{2}\nu(x\sqrt{3}-y)}\sin\tfrac{1}{2}\nu(x + y\sqrt{3})\right\}.$$

Numerical computations for $\varphi(x,y)$ for $\gamma = 6^\circ$ ($q = 15$) as well as for the above cases were carried out by Stoker [1947] and are presented graphically in his paper.

Kirchhoff's solution is limited to the special choice of angle noted above and furthermore presents only solutions which are bounded at the origin. The solution of the differential-difference equation (17.32) for arbitrary γ, $0 < \gamma \leq \pi$, has been given by both Peters [1950], Isaacson [1950], and Roseau [1952, chap. V]. All use Laplace transforms. However, the method cannot be expounded briefly and we refer to either the original papers or Stoker's <u>Water waves</u> for the details.

The special case $\gamma = \pi/2q$ can be treated fairly simply by the reduction method used in the problem of the vertical barrier.

From (17.32) we have

$$f^{(k+1)}(z) + i\nu f^{(k)}(z) = \beta^{k+1} f^{(k+1)}(\beta z) - i\nu \beta^k f^{(k)}(\beta z), \quad k = 0, 1, \ldots \qquad (17.42)$$

The free surface condition [cf. (11.7)] implies

$$\operatorname{Im}\left\{f^{(k+1)}(x) + i\nu f^{(k)}(x)\right\} = 0, \quad x > 0. \qquad (17.43)$$

Hence also

$$\operatorname{Im}\left\{\beta^{k+1} f^{(k+1)}(x\beta) - i\nu \beta^k f^{(k)}(x\beta)\right\} = 0, \quad x > 0.$$

This last equation can also be written

$$\operatorname{Im}\left\{\beta^{k+1} f^{(k+1)}(z) - i\nu\beta^{k} f^{(k)}(z)\right\} = 0 \text{ for } z = re^{-2i\gamma}. \tag{17.44}$$

If the numbers a_k and a_k' are real, (17.43) and (17.44) imply

$$\begin{aligned} &\operatorname{Im}\left\{\sum_{k=0}^{s} a_k\left[f^{(k+1)}(x) + i\nu f^{(k)}(x)\right]\right\} = 0, \\ &\operatorname{Im}\left\{\sum_{k=0}^{s} a_k' \beta^{k}\left[\beta f^{(k+1)}(re^{-2i\gamma}) - i\nu f^{(k)}(re^{-2i\gamma})\right]\right\} = 0. \end{aligned} \tag{17.45}$$

We wish to find numbers $\{a_k\}$ and $\{a_k'\}$ such that

$$\sum_{k=0}^{s} a_k\left[f^{(k+1)}(z) + i\nu f^{(k)}(z)\right] \equiv \sum_{k=0}^{s} a_k'\beta^{k}\left[\beta f^{(k+1)}(z) - i\nu f^{(k)}(z)\right]. \tag{17.46}$$

Comparing coefficients of derivatives of the same order, one finds

$$\begin{aligned} a_0 &= -a_0', \\ a_{k-1} + i\nu a_k &= \beta^{k}(a_{k-1}' - i\nu a_k'), \quad k = 1, \ldots, s, \\ a_s &= \beta^{s+1} a_s' \end{aligned} \tag{17.47}$$

These relations will be satisfied if one takes $s = q - 1$ (for $\beta^q = -1$) and

$$\begin{aligned} a_k = -a_k' &= a_{k-1}\frac{1}{i\nu}\frac{\beta^{k}+1}{\beta^{k}-1} = a_{k-1}\frac{1}{\nu}\cot k\gamma, \\ &= \frac{a_0}{\nu^{k}}\cot\gamma\cot 2\gamma \ldots \cot k\gamma, \quad k = 1, \ldots, s. \end{aligned} \tag{17.48}$$

We note that $\nu^{q-k} a_{q-k} = \nu^{k-1} a_{k-1}$. With this choice of the coefficients $\{a_k\}$, define

$$g(z) = \sum_{k=0}^{q-1} a_k \{ f^{(k+1)}(z) + i\nu f^{(k)}(z) \} = P(\tfrac{d}{dz})(\tfrac{d}{dz} + i\nu) f(z),$$

$$= -\sum_{k=0}^{q-1} a_k \beta^k \{ \beta f^{(k+1)}(z) - i\nu f^{(k)}(z) \} = -P(\beta \tfrac{d}{dz})(\beta \tfrac{d}{dz} - i\nu) f(z)$$

$$= \sum_{k=0}^{q-1} a_k \{ f^{(k+1)}(\beta z) - i\nu f^{(k)}(\beta z) \} = P(\tfrac{d}{dz})(\tfrac{d}{dz} - i\nu) f(\beta z) \tag{17.49}$$

where the last equation follows from (17.42) and where

$$p(\lambda) = \sum_{k=0}^{q-1} a_k \lambda^k . \tag{17.50}$$

From the assumptions originally made concerning f(z) and from the method of selecting the $\{a_k\}$ it follows that g(z) is regular everywhere in the wedge

$$-2\gamma \leq \theta \leq 0$$

except possibly at the origin, that

$$Im\{g(z)\} = 0 \quad \text{for} \quad z = x > 0 \quad \text{and} \quad z = re^{-2i\gamma},$$

and finally, from the last of equations (17.24), that

$$g(\beta z) = -g(z).$$

Since f(z) is assumed bounded as $x \to \infty$, this is true also of g(z). These various conditions imply that g(z) must have the form

$$g(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^{(2n+1)q}}, \quad b_n \text{ real}. \tag{17.51}$$

We have thus shown that f(z) satisfies the differential equation

$$P\left(\frac{d}{d\zeta}\right)\left(\frac{d}{d\zeta} + i\nu\right) f(\zeta) = \sum_{n=0}^{\infty} \frac{b_n \nu}{\zeta^{(2n+1)q}} \quad , \quad b_n \text{ real.} \tag{17.52}$$

From the definition of P( ) it follows that

$$P(\lambda)(\lambda + i\nu) \equiv P(\beta\lambda)(-\beta\lambda + i\nu).$$

Since the coefficients in $P(\lambda)$ are real, $\bar{\lambda}$ is a root of $P(\lambda) = 0$ if λ is a root. Furthermore, from the identity above also $\beta\lambda$ is a root providing $\beta\lambda \neq i\nu$. Since $\lambda = -i\nu$ is an obvious root of the left hand member, $-i\beta\nu$ is also a root and hence $-i\beta^2\nu$, $-i\beta^3\nu, \ldots$. Since $\beta^q = -1$, no new roots are added by going further than $-i\beta^{q-1}\nu$, and since $i\beta^{-k}\nu = -i\beta^{q-k}\nu$, a complete set of roots of $P(\lambda)(\lambda + i\nu)$ is

$$-i\nu, -i\beta\nu, -i\beta^2\nu, \ldots, -i\beta^{q-1}\nu.$$

Thus the solution of the homogeneous equation can be expressed in the form

$$\sum_{k=0}^{q-1} A_k \exp(-i\nu\beta^k \zeta). \tag{17.53}$$

This is, of course, exactly the form of Kirchhoff's solution of (17.36). Since we have already determined the necessary form of the A_k in order to satisfy the boundary condition on the bottom, we need not pursue further the solution of the homogeneous equation.

The solution of the nonhomogeneous equation is straightforward. However, just as for the homogeneous equation, one must take care to satisfy the boundary condition on the bottom, i.e. $\mathrm{Im}\left\{e^{-i\delta} f'(re^{-i\delta})\right\} = 0$. The detailed considerations may be found in the several cited papers; Brillouët [1957] treats the matter thoroughly. If one considers (17.52) with the right-hand side replaced by only one of its summands, say $b_n z^{-(2n+1)q}$, then the complete solution can be put in the following form, as shown by Brillouët:

$$f(z) = \sum_{k=0}^{q-1} A_k \exp(-i\nu\beta^k z)\left[c_n + \tfrac{1}{2}(-1)^{nq+q-1}\frac{b_n}{\sqrt{q}}\int_{\Gamma_k}\frac{e^t\,dt}{t^{(2n+1)q}}\right], \qquad (17.54)$$

where c_n is an arbitrary real constant, B_o of (17.37) has been set equal to 1 , and where Γ_k indicates that the integral is to be carried out over each of the paths Γ_k^+ and Γ_k^- shown in Figure 17.

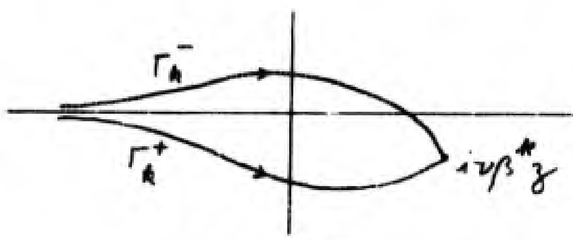

Figure 17.

However, one may obtain a variety of other forms for the solution.

An asymptotic expression as $x \to \infty$ and for $y = 0$ is given by

$$f(x) \sim \left[c_n + i\,b_n\frac{(-1)^{nq+q-1}\,\pi}{(2nq+q-1)!\sqrt{q}}\right]\exp\left(-i\nu x - i\pi\frac{q-1}{4}\right)$$

or (17.55)

$$\varphi(x,0) \sim c_n\cos\left(\nu x + \pi\frac{q-1}{4}\right) + b_n\frac{(-1)^{nq+q-1}\,\pi}{(2nq+q-1)!\sqrt{q}}\sin\left(\nu x + \pi\frac{q-1}{4}\right).$$

In the neighborhood of $z = 0$, $f(x)$ behaves like $\log z$ for $n = 0$ and like z^{-2nq} for $n > 0$.

It is not clear physically what type of singularity at $z = 0$ most nearly describes the behavior of real waves. However, most writers have restricted their treatment to the weakest possible singularity, i.e., the logarithmic one.

From the asymptotic expansion as $x \to \infty$ one sees that it is now possible to construct an incoming progressive wave by proper choice of the constants c_n and b_n. Thus, if we select

$$c_n = a\cos(\sigma t + \tau), \quad b_n = -(-1)^{nq+q}\pi^{-1}(2nq+q-1)!\sqrt{q}\, a\sin(\sigma t+\tau),$$

then the resulting solution will behave like

$$a\cos(\nu x + \sigma t + \tau)$$

as $x \to \infty$ for $y = 0$. In connection with (17.55) and the selection of b_n just made, it is apparent that the formulas (17.54) and (17.55) will be more directly connected with parameters with a simple physical interpretation if we replace b_n by

$$d_n = b_n \frac{(-1)^{nq+q-1}\pi}{(2nq+q-1)!\sqrt{q}}.$$

For $n = 0$ companion singular solutions to the regular solutions (17.40) and (17.41) are not difficult to write out:

$\gamma = 90^\circ (q = 1, n = 0)$:

$$\varphi(x,y) = d_0 e^{\nu y}\sin\nu x - \frac{d_0}{\pi}\int_0^\infty e^{-\sigma x}\frac{\sigma\cos\sigma y + \nu\sin\sigma y}{\nu^2+\sigma^2}\,d\sigma\,; \qquad (17.56)$$

$\gamma = 45^\circ (q = 2,\ n = 0)$:

$$\varphi(x,y) = \frac{d_0}{\pi} e^{\nu y}\left[\left(\frac{\pi}{2} + Si(\nu x)\right)\sin\left(\nu x + \frac{\pi}{4}\right) + Ci(\nu x)\cos\left(\nu x + \frac{\pi}{4}\right) + \frac{1}{2}\sqrt{\nu}\, e^{-\nu x} Ei(\nu x)\right] \tag{17.57}$$

Further formulas for $\gamma = 30^\circ$ and $\gamma = 6^\circ$ may be found in Brillouët [1957, p. 93ff.].

18. Three-dimensional progressive and standing waves in unbounded regions with fixed boundaries.

The general remarks at the beginning of section 17 apply here also. Although most of the solvable problems in the present category are such that they can be reduced to two-dimensional ones (however, see the end of section 19β), the methods of complex-function theory are no longer applicable to the same extent. The division of topics is the same as in the last section, namely, diffraction of waves by obstacles and waves on beaches.

18α. Diffraction of water waves.

In a horizontally unbounded ocean of uniform depth h assume that an incoming wave is specified by

$$\phi_I(x,y,z,t) = \frac{Ag}{\sigma}\cosh m(y+h)\cos(mx + \sigma t + \alpha) \tag{18.1}$$

and that it is scattered by one or more obstacles in the water. We wish to find the velocity potential for the motion of the water in the form

$$\phi(x,y,z,t) = \phi_I + \phi_S\ , \tag{18.2}$$

where ϕ_S is the scattered wave and satisfies the radiation condition if the body is of bounded extent.

As usual, we may write ϕ in the form

$$\phi(x,y,z,t) = Re\, \varphi(x,y,z)e^{-i\sigma t}, \quad \varphi = \varphi_1 + i\varphi_2, \tag{18.3}$$

where φ must be a potential function satisfying

$$\begin{aligned} &\varphi_y(x,0,z) - \nu\varphi(x,0,z) = 0, \qquad \nu = \sigma^2/g, \\ &\varphi_n = \varphi_{In} + \varphi_{Sn} = 0 \qquad \text{on the obstacles}, \\ &\lim_{R\to\infty} \sqrt{R}\left(\frac{\partial\varphi_S}{\partial R} - i\nu\varphi_S\right) = 0, \qquad \sqrt{R}\,\varphi_S = O(1) \quad \text{as } R\to\infty \end{aligned} \tag{18.4}$$

<u>General obstructions.</u> Consider a single submerged obstacle bounded by the surface S. We shall try to express the scattered wave $\phi_S = Re\, \varphi_S e^{-i\sigma t}$ by a distribution of sources over S. However, in order to satisfy the various boundary conditions, we take sources in the complex form (13.18) (or, in the case of infinite depth, in the form (13.17''):

$$\varphi_S(x,y,z) = \frac{1}{4\pi}\iint_S \gamma(\xi,\eta,\zeta)\, G(x,y,z;\xi,\eta,\zeta)\, dS, \tag{18.5}$$

where we have written $G = G_1 + iG_2$ for the complex form of (13.18). The boundary condition on the body now becomes

$$0 = \frac{\partial\varphi_I}{\partial n} + \frac{\partial\varphi_S}{\partial n} = \frac{\partial\varphi_I}{\partial n} - \frac{1}{2}\gamma(x,y,z) + \frac{1}{4\pi}\iint_S \gamma(\xi,\eta,\zeta)\frac{\partial}{\partial n}G(x,y,z;\xi,\eta,\zeta)\,ds$$

or

$$\gamma(x,y,z) = 2\frac{\partial\varphi_I}{\partial n} + \frac{1}{2\pi}\iint \gamma(\xi,\eta,\zeta)\frac{\partial G}{\partial n}\,dS. \tag{18.6}$$

Since $\partial\varphi_I/\partial n$ is a known function, this is a Fredholm

integral equation of the second kind for $\gamma(x,y,z)$.
(We note in passing that if the motion of the surface S had been prescribed to be $\partial\varphi_I/\partial n$, then the same integral equation for γ would have been obtained.)

This equation has been considered by Kochin [1940] in the case of infinite depth, and he proves that a solution exists if $\nu = \sigma^2/g$ is large enough. Iterative procedures for computing γ follow from the theory. Haskind [1946] has extended the argument to finite depth.

John [1950] has treated both the uniqueness and existence problem in great detail and has shown that a unique solution exists for a body whose surface intersects the free surface perpendicularly and which can be represented as a single-valued function over the area enclosed in the intersection. His result holds for all values of m (or ν if the depth is infinite). He also reduces the existence problem to solution of an integral equation.

<u>Vertical cylinders</u>. When the obstacle or obstacles are vertical cylinders extending from above the free surface to the bottom, it is possible to reduce the problem to one in diffraction of sound waves for which many special solutions are known (see, e.g., Havelock [1940]). In this case we may separate the y variable in the manner shown in section 13α:

$$\varphi(x,y,z) = \varphi(x,z)\,Y(y) \qquad (18.7)$$

where $Y(y) = \cosh m(y+h)\ \varphi(x,z)$

and

$$\varphi_{xx} + \varphi_{zz} + m^2 \varphi = 0. \tag{18.8}$$

Here m must be the same as in (18.1) since the frequency is fixed by the incoming wave. $\varphi(x, z)$ must now satisfy (18.8) and the second two conditions of (18.4). This is exactly the same mathematical problem encountered in the diffraction of sound waves by a cylindrical body (in that case the air pressure replaces φ). Thus, any solutions known for sound diffraction by cylinders may be taken over immediately for water-wave diffraction. For example, if the obstacle is a vertical circular post of radius a, the velocity potential of the scattered wave is given by [see Morse, Vibration and sound, 2d. ed., New York, 1948. pp. 347ff., 449]:

$$\varphi_s(R, \theta, y) = \frac{-Ag}{\sigma} \cosh m(y+h) \sum (-i)^n \varepsilon_n e^{-i\delta_n} \sin\delta_n \cos\theta \, H_n^{(1)}(mR), \tag{18.9}$$

where $\tan \delta_n = J_n'(ma)/Y_n'(ma)$ and

$$\varepsilon_0 = 1, \quad \varepsilon_n = 2 \quad \text{for} \quad n \geq 1.$$

Various approximations for large and small values of ma are known. The maximum wave amplitude at any point is given by $\frac{\sigma}{g}|\varphi|$.

The diffraction of water waves by a vertical half-plane may also be treated by transferring known solutions due to Sommerfeld for sound and electromagnetic waves to the present context. This has been done by Haskind [1948] for normal incidence and by Penney and Price [1952a] for both normal and oblique incidence. Peters and Stoker [1954] (see also Stoker [1956] and [1957, pp. 109-133]) have also solved this problem by a new and rather easy method, following an investigation of boundary conditions which will ensure uniqueness. The solution has an obvious application in predicting the effect of breakwaters. Let the breakwater be the half-plane $z = 0$, $x > 0$ and the incoming wave be given by

$$\eta = A \cos(m x \cos\alpha + m z \sin\alpha + \sigma t),$$
$$= A \cos(m R \cos(\theta - \alpha) + \sigma t),$$

where α is the angle between $-Ox$ and the direction of propagation, measured clockwise. Then the solution given by Peters and Stoker is

$$\varphi(R,\theta,y) = \frac{Ag}{\sigma} \cosh m(y+h) \Big[J_0(R) + 2 \sum_1^{\infty} e^{in\pi/4} J_{n/2}(R) \cos\frac{n\alpha}{2} \cos\frac{n\theta}{2} \Big]. \quad (18.10)$$

The result can also be expressed by means of integrals. In the case of normal incidence these reduce to Fresnel integrals, for which tables exist. Graphical representations of the behavior of the wave amplitudes may be found in Penney and Price [1952a].

Penney and Price also apply this analysis to an approximate treatment of diffraction by a breakwater of finite length and through a gap. The results are presumably applicable if the wave length is small compared to the length of the breakwater or the gap.

Periodic solutions for horizontal cylindrical obstacles.

In two physical situations the dependence upon z may be precipitated out, leaving a two-dimensional problem which in many cases can be solved by methods analogous to those used for the two-dimensional problems of section 17.

Let the obstruction be an infinitely long horizontal cylinder parallel to Oz. This might be, for example, a semi-infinite dock or submerged plane barrier, say $y = -b, x < 0$, a finite horizontal barrier, say $y = -b$, $|x| < a$, a vertical barrier, $x = 0, -b < Y < 0$, a beach, $y = -x \tan\gamma$, etc. Let an incoming plane wave at infinity propagate at an angle α to the x axis:

$$\eta_I(x,y,z,t) = A \cos[m(x \cos\alpha + z \sin\alpha) + \sigma t]. \qquad (18.11)$$

Although one will not expect the velocity potential ϕ to be periodic in x, it seems reasonable to assume that it will be periodic in z. In fact, we shall assume that

$$\phi(x,y,z,t) = \varphi(x,y)\, e^{-i(mz\sin\alpha + \sigma t)}, \qquad (18.12)$$

where $\varphi(x,y)$ must now satisfy, with $k = m \sin\alpha$,

$$\varphi_{xx} + \varphi_{yy} - k^2\varphi = 0 \qquad (18.13)$$

and the usual conditions on the free surface and rigid boundaries.

We should have come to the same conclusion if we had assumed an incoming wave at infinity of the form

$$\eta_I(x,y,z,t) = A\cos kz\cos(k_1 x + \sigma t)\ ,\quad k^2 + k_1^2 = m^2, \tag{18.14}$$

a so-called short-crested wave (note that we assume $k^2 < m^2$). That is, we shall now look for a solution in the form

$$\phi(x,y,z,t) = \varphi(x,y)\cos kz\, e^{-i\sigma t}, \tag{18.15}$$

satisfying equation (18.13) and the conditions on the free surface and rigid boundaries. Thus, a solution for one of these cases carries over easily to the other.

The problem is thus reduced to one almost identical with that of section 17, with the exception that the two-dimensional Laplacian is replaced by (18.13). Many of the same methods may be carried over, e.g., the reduction method and the integral-equation method. Haskind [1953] has considered some general aspects of the problem which will be outlined below, has derived the source solution of (18.13), and has treated the diffraction about a vertical barrier (an analogue of the problem treated in 17α) and a finite dock, all in infinitely deep water. MacCamy [1957] has derived a source solution of (18.13) and treated the finite dock problem in water of finite depth. Heins [1948, 1950, 1953] has given source solutions of (18.13) for finite depth and formulated and solved Wiener-Hopf integral equations for semi-infinite docks and submerged horizontal barriers. Greene and Heins [1953] treat the submerged barrier in water of infinite depth. The literature for beaches will be given in section 18β.

Suppose the fluid infinitely deep and let a cross-section of the obstacle have contour C. We wish then to find a solution $\varphi(x,y) = \varphi_1 + i\varphi_2$ of (18.13) such that

$$\varphi_n = 0 \quad \text{on } C,$$

$$\varphi_y(x,0) - \nu\varphi(x,0) = 0 \quad \text{on the free surface,} \tag{18.16}$$

$$\varphi \sim \frac{Ag}{\sigma} e^{\nu y} e^{-ik_1 x} + \frac{B^+ g}{\sigma} e^{\nu y} e^{ik_1 x} \quad \text{as } x \to +\infty,$$

$$\varphi \sim \frac{Ag}{\sigma} e^{\nu y} e^{-ik_1 x} + \frac{B^- g}{\sigma} e^{\nu y} e^{-ik_1 x} \quad \text{as } x \to -\infty,$$

where $k_1^2 < \nu^2$. Haskind [1953] applies the reduction method in the following manner (we follow his presentation closely). Introduce the function $f(x,y)$ by

$$\frac{\partial f}{\partial y} = \frac{\partial \varphi}{\partial y} - \nu\varphi. \tag{18.17}$$

Then f also satisfies (18.13) and

$$f_y(x,0) = 0 \quad \text{on the free surface.} \tag{18.18}$$

Consequently, f may be extended into the upper half-plane by defining $f(x,-y) = f(x,y)$ and f now satisfies (18.13) in the whole plane outside the contour C and its mirror image $\bar{C}$. Moreover, $|f| \to 0$ as $x^2 + y^2 \to \infty$. Assuming that f is known, one must now try to reconstruct φ from f in such a way that conditions (18.16) are satisfied. In order to do this, Haskind differentiates (18.17) with respect to y, subtracts from

$$f_{xx} + f_{yy} - k^2 f = 0,$$

and after some easy manipulation obtains

$$\frac{\partial^2}{\partial x^2}(\varphi - f) + k_1^2(\varphi - f) = -\nu\left(\frac{\partial f}{\partial y} + \nu f\right). \tag{18.19}$$

Treating this as a differential equation for $\varphi - f$, he finds the following solution for φ:

$$\varphi = f - \frac{\nu}{2ik_1}\left\{e^{ik_1x}\int_{\infty}^{x} e^{-ik_1\xi}(f_y + \nu f)\,d\xi - e^{-ik_1x}\int_{-\infty}^{x} e^{ik_1\xi}(f_y + \nu f)\,d\xi\right\} + \frac{Ag}{\sigma}e^{\nu y}e^{-ik_1x}, \tag{18.20}$$

the integrals being taken along half-lines parallel to the x-axis and below C. One may verify without great difficulty that φ satisfies (18.13). The asymptotic form of φ as $x \to \pm\infty$ may be written down immediately, and gives

$$\frac{\nu}{2ik_1}e^{\mp ik_1x}\int_{-\infty}^{\infty} e^{\pm ik_1\xi}(f_y + \nu f)\,d\xi + \frac{Ag}{\sigma}e^{\nu y}e^{-ik_1x}, \tag{18.21}$$

the path of integration being a line below the body. Consider now the region D bounded externally by this line and a large semicircle containing $C + \bar{C}$ and internally by $C + \bar{C}$. Application of Green's Theorem to f and $\chi = \exp(-\nu y + ik_1x)$ shows that

$$e^{-\nu y}\int_{-\infty}^{\infty} e^{ik_1\xi}(f_y + \nu f)\,d\xi = \int_{C+\bar{C}} (f\chi_n - \chi f_n)\,ds,$$

$$e^{-\nu y}\int_{-\infty}^{\infty} e^{-ik_1\xi}(f_y + \nu f)\,d\xi = \int_{C+\bar{C}} (f\bar{\chi}_n - \bar{\chi} f_n)\,ds.$$

Hence, the asymptotic conditions are satisfied and, moreover,

$$\frac{B^+q}{6} = \frac{\nu}{2ik_1}\int_{C+\bar{C}}(f\chi_n - \chi f_n)\,ds, \quad \frac{B^-q}{6} = \frac{\nu}{2ik_1}\int_{C+\bar{C}}(f\bar{\chi}_n - \bar{\chi} f_n)\,ds. \tag{18.22}$$

By a similar application of Green's Theorem Haskind shows that one may also write

$$\varphi = f + \nu e^{\nu y}\int_{\infty}^{y} f e^{-\nu\eta}\,d\eta + \frac{B^{\pm}q}{6}e^{\nu y \mp ik_1 x} + \frac{Aq}{6}e^{\nu y - ik_1 x}, \tag{18.23}$$

where the + sign is used for points to the right of C and the - sign for points to the left. It is easy to verify directly that φ satisfies (18.13) and (18.17); however, (18.20) allows one to investigate the asymptotic behavior more simply. If φ has no singularities, then (17.3) must also hold here, i.e., $(B^+)^2 + (B^-)^2 + 2AB^- = 0$.

This result may be used to find the source solutions giving outgoing waves at $\pm\infty$. For equation (18.13) the singular solutions for the whole plane are known to be the Bessel functions $K_n(kr)$, where $r^2 = (x-a)^2 + (y-b)^2$. To find the solution corresponding to (13.22), one assumes it may be expressed as

$$G(x, y; a, b) = \varphi_0 + K_0(kr) - (kr_1),$$

with $r_1^2 = (x-a)^2 + (y+b^2)$, where f has no singularities for $y < 0$. Then $f_{0y} = \varphi_{0y} - \nu\varphi_0$ may be extended as a regular solution of (18.13) to the whole plane. Also,

$$f_{0y}(x, 0) = 2\frac{\partial}{\partial y}K_0(kr_1)\Big|_{y=0}.$$

One may then show that this relation holds for all $y \leq 0$:

$$f_{0y}(x, y) = 2\frac{\partial}{\partial y}K_0(kr_1), \quad y \leq 0,$$

or

$$f_0(x,y) = 2K_0(kr_1).$$

Substitution in (18.23) with $A = 0$ and direct computation of $B^{\pm}$ from (18.22) by taking C as a small circle about the singularity gives

$$G = K_0(kr) + K_0(kr_1) + 2\nu e^{\nu y}\int_{\infty}^{-y} e^{-\nu y} K_0(kr_1)\,dy - \qquad (18.24)$$
$$-2\pi i \frac{\nu}{k_1} e^{\nu(y+b) \mp i k_1 (x-a)}.$$

For Haskind's application of this method to the diffraction about a vertical and a horizontal barrier we refer to the original paper. Force and moment are obtained in terms of Mathieu functions. For the horizontal barrier in water of finite depth we refer to MacCamy's paper [1957] where a formula analogous to (18.24) is derived.

18 β. Waves on beaches.

Much of the immediately preceding discussion of diffraction of plane waves approaching at an angle or of short-crested waves approaching normally applies also to this case. One is led to the following boundary-value problem for $\varphi(x,y) = \varphi_1 + i\varphi_2$:

1) $\varphi_{xx} + \varphi_{yy} - k^2\varphi = 0, \quad k^2 < \nu^2,$

2) $\varphi_y(x,0) - \nu\varphi(x,0) = 0,$ (18.25)

3) $\varphi_x \sin\gamma + \varphi_y \cos\gamma = 0 \quad \text{for } y + x\tan\gamma = 0,$

4) $\varphi \sim \frac{Ag}{\sigma} e^{\nu y} e^{-ik_1 x}$ as $x \to \infty$, $k_1^2 = \nu^2 - k^2$,

5) $\varphi_x^2 + \varphi_y^2 \to 0$ as $x^2 + y^2 \to \infty$ along $y + x \tan\gamma = 0$.

Many of the authors cited in section 17 β considered this problem along with the two-dimensional one. In particular, we refer to Hanson [1926], Miche [1944], Stoker [1947], Weinstein [1949], Roseau [1952], and Peters [1952]. Both Peters and Roseau solve the problem for arbitrary angle γ, $0 < \gamma \leq \pi$ (thus including the semi-infinite dock problem treated differently by Heins [1948]. The use of the reduction method limits one here, as in the two-dimensional case, to angles $\gamma = p\pi/2q$. We shall illustrate the procedure briefly for $\gamma = \pi/4$ and $\gamma = \pi/2$, following essentially Weinstein's [1949] treatment (see also Brillouët [1957, chaps. I, II]).

Since the boundary condition on the free surface and bottom is the same in the two- and three-dimensional cases, we may make use of the auxiliary function g constructed in (17.49) by using only the real part of the complex potential. Thus, for $\gamma = \pi/4$ one finds from (17.48) that $a_1 = a_0/\nu$. Hence, from (17.50)

$$P(\lambda) = a_0(1 + \lambda/\nu),$$

and

$$g(\zeta) = \frac{a_0}{\nu}\left(\frac{d}{d\zeta} + \nu\right)\left(\frac{d}{d\zeta} + i\nu\right)(\varphi + i\psi),$$

$$Im\ g(\zeta) = \frac{a_0}{\nu}\left(\frac{\partial}{\partial x} + \nu\right)\left(-\frac{\partial}{\partial y} + \nu\right)\varphi.$$

Thus, the boundary conditions 2) and 3) of (18.25) imply that

$$h(x,y) \equiv \left(\frac{\partial}{\partial x} + \nu\right)\left(\frac{\partial}{\partial y} - \nu\right)\varphi(x,y) = 0 \quad \text{on } y=0,\ x>0 \text{ and } x=0,\ y<0. \tag{18.26}$$

We recall that the definition of $\varphi(x,y)$ has been extended from the original wedge by reflection in the bottom. One must now find a function $h(x,y)$ satisfying equation 1) of (18.25) and the boundary conditions (18.26) and which is regular everywhere in the extended wedge, $0 \geq \theta \geq \frac{1}{2}\pi$, except possibly at the origin, bounded as $x^2+y^2 \to \infty$, and symmetric about the line $y=-x$. It is known that the general solution of this problem is given by

$$h(x,y) = \left(\frac{\partial}{\partial x} + \nu\right)\left(\frac{\partial}{\partial y} - \nu\right)\varphi(x,y) = \sum_{n=0}^{\infty} A_n K_{2(2n+1)}(kr)\sin 2(2n+1)\theta. \tag{18.27}$$

A similar analysis for waves approaching a vertical cliff ($\gamma = \frac{1}{2}\pi$) leads to

$$h(x,y) \equiv \left(\frac{\partial}{\partial y} - \nu\right)\varphi(x,y) = \sum_{n=0}^{\infty} A_n K_{2n+1}(kr)\sin(2n+1)\theta. \tag{18.28}$$

Let us take the weakest possible singularity in each case, i.e., K_1 for the 90° cliff and K_2 for the 45° beach. Consider first the vertical cliff. Taking account of the relation $K_0'(u) = -K_1(u)$, we have

$$\left(\frac{\partial}{\partial y} - \nu\right)\varphi(x,y) = -\frac{A_0}{k}\frac{\partial}{\partial y}K_0(kr).$$

We may then identify $-A_0 K_0/k$ with f and from (18.23), with $B^{\pm}=0$, we have

$$\varphi = -\frac{A_0}{k}K_0(kr) - A_0\frac{\nu}{k}e^{\nu y}\int_{\infty}^{y} e^{-\nu\eta}K_0(k\sqrt{x^2+\eta^2})\,d\eta + \frac{Ag}{\sigma}e^{\nu y - ik_1 x},$$

where A_0 must still be determined so that $\varphi_x(0,y)=0, y<0$. In computing φ_x as $x\to 0$, one must remember that $K_0(u)\sim \ln(2/u)$ as $u\to 0$. Hence, one finds

$$\varphi_x(0,y) = -\frac{A_0}{k}\nu e^{\nu y}\cdot\lim_{\epsilon\to 0}\lim_{x\to 0}\int_{-\epsilon}^{\epsilon}\frac{-x}{x^2+y^2}dy - i\frac{Agk_1}{\sigma}e^{\nu y}$$

$$= \frac{A_0}{k}\nu\pi e^{\nu y} - i\frac{Agk_1}{\sigma}e^{\nu y}.$$

Setting this equal to zero, one finds

$$\frac{A_0}{k} = \frac{Ag}{\sigma}\cdot\frac{ik_1}{\pi\nu}.$$

Substituting above and separating the real and imaginary parts of $\varphi=\varphi_1+i\varphi_2$, we obtain an everywhere regular solution φ_1 and a solution φ_2 with a singularity at the origin and 90° out of phase at $x=\infty$:

$$\varphi_1(x,y) = \frac{Ag}{\sigma}e^{\nu y}\cos k_1 x, \tag{18.29}$$

$$\varphi_2(x,y) = -\frac{Ag}{\sigma}\frac{k_1}{\pi\nu}\left[K_0(kr)+\nu e^{\nu y}\int_{\infty}^{y}e^{-\nu\eta}K_0(k\sqrt{x^2+\eta^2})\,d\eta\right] + \frac{Ag}{\sigma}e^{\nu y}\sin k_1 x.$$

The corresponding equation for (18.27) can be written in the form

$$\left(\frac{\partial}{\partial x}+\nu\right)\left(\frac{\partial}{\partial y}-\nu\right)f(x,y) = A_0K_2(kr)\sin 2\theta = \frac{2A_0}{k^2}\frac{\partial^2}{\partial x\partial y}K_0(kr). \tag{18.30}$$

One can find its integration discussed in Roseau [1952, ch. IV]. A solution for the next simplest case, $\gamma = 30°$, does not seem to have been published. For $\gamma = 45°$ the regular solution φ_1, and singular solution φ_2 as given by Roseau are:

$$\varphi_1 = A_1\left\{e^{\nu y}[k_1\cos k_1 x - \nu\sin k_1 x] + e^{-\nu x}[k_1\cos k_1 y + \nu \sin k_1 y]\right\},$$

$$\varphi_2 = A_2\left\{e^{\nu y}[\nu\cos k_1 x - k_1\sin k_1 x] + e^{-\nu x}[\nu\cos k_1 y + k_1\sin k_1 y]\right\} + \tag{18.31}$$

$$+A_2\frac{\pi k_1^2}{\nu}\Big\{-K_0(k\sqrt{x^2+y^2}) + \nu e^{-\nu x}\int_{-\infty}^{x} e^{\nu\xi}K_0(k\sqrt{\xi^2+y^2})\,d\xi +$$

$$+\nu e^{\nu y}\int_{y}^{\infty} e^{-\nu\eta}K_0(k\sqrt{x^2+\eta^2})\,d\eta - \nu^2 e^{-\nu y}\int_{y}^{\infty} d\eta\, e^{-\nu\eta}\Big(e^{-\nu x}\int_{-\infty}^{x} d\xi\, e^{\nu\xi}K_0(k\sqrt{\xi^2+\eta^2})\Big)\Big\}.$$

In order to satisfy condition 4) of (18.25) one must take

$$A_1 = \frac{-Ag}{\sigma}\cdot\frac{k_1 + i\nu}{k^2}, \quad A_2 = \frac{Ag}{\sigma}\cdot\frac{\nu + i k_1}{k^2}.$$

Edge Waves. In the investigation of diffraction of waves on horizontal cylindrical obstacles and of waves on beaches, it was specifically assumed that $k^2 < m^2$. This was automatically fulfilled for plane waves approaching at an angle, but needed to be assumed for short-crested waves. For the short-crested waves there also exist standing-wave solutions which can be exhibited in certain cases for $k^2 > m^2$. Such solutions were apparently first noticed by Stokes [1846, p. 7=1880, p. 167] in connection with the propagation of waves in a canal of non-rectangular cross-section. Certain peculiarities of these solutions have been pointed out by Ursell [1951, 1952].

Consider the first three conditions of (18.25) for waves on a sloping beach, but with $k^2 > \nu^2$. Then one may verify directly that

$$\varphi(x,y) = e^{k[y \sin\gamma - x \cos\gamma]}$$

is a solution. This gives a velocity potential for standing waves:

$$\phi(x,y,z,t) = e^{k[y \sin\gamma - x \cos\gamma]} \cos(kz+\epsilon)\cos(\sigma t + \tau), \qquad (18.32)$$

where

$$k \sin\gamma = \sigma^2/g.$$

The wave amplitude is bounded at the origin and drops off very quickly as x increases. Clearly, one must have $\gamma < \frac{1}{2}\pi$. Ursell has pointed out other interesting aspects. For a given γ and σ there is only one allowable k, i.e., it is a discrete point of the spectrum. In the case discussed earlier with $k^2 < \nu^2$ all values of k between 0 and ν were allowable. In addition, the total energy per unit length in the z direction is finite for the Stokes edge wave.

From (18.29) one may construct a progressive wave moving in the direction Oz with velocity.

$$c = \frac{g \sin\gamma}{\sigma}.$$

There is evidence that such waves have been observed in nature [cf. Munk, Snodgrass and Carrier, 1956; Dunn and Ewing, 1956].

Ursell [1952] has shown that (18.32) is only the first in a sequence of solutions of this nature for a sloping beach. He shows, in fact, that the following velocity potential also satisfies the condition:

$$\phi(x,y,z,t) = \Big\{ e^{-k[x\cos\gamma - y\sin\gamma]} + \sum_{m=1}^{n} A_{mn}\Big[e^{-k[x\cos(2m-1)\gamma + y\sin(2m-1)\gamma]} + e^{-k[x\cos(2m+1)\gamma - y\sin(2m+1)\gamma]} \Big]\Big\} \cos(kz+\varepsilon)\cos(\sigma t+\tau), \qquad (18.33)$$

where

$$A_{mn} = (-1)^m \prod_{r=1}^{m} \frac{\tan(n-r+1)\gamma}{\tan(n+r)\gamma}, \qquad \sigma^2 = gk\sin(2n+1)\gamma .$$

It follows from the last condition that one must have

$$(2n+1)\gamma \leq \frac{\pi}{2} \quad \text{or} \quad n < \frac{\pi}{4\gamma} + \frac{1}{2},$$

where $n = 0$ will be taken to indicate the Stokes edge wave. Thus, for fixed wave number k, the above formula gives one frequency σ if $\frac{1}{2}\pi > \gamma > \frac{1}{6}\pi$, two if $\frac{1}{6}\pi > \gamma > \frac{1}{10}\pi$, etc An experiment carried out by Ursell confirms the existence of these other modes of motion. The solution (18.33) for $\gamma = 30°$ has also been given by Lamb [Hydrodynamics, p. 450, eq. (30)]. At the critical angles $\pi/6$, $\pi/10$, etc., the solution (18.33) does not vanish as $x \to \infty$.

Keldysh [1938] has stated without proof that for $\gamma = 45°$ the Stokes edge wave and the function φ_1 from (18.31) constitute a complete set of bounded solutions in the sense that for any absolutely integrable function $f(x, z)$, $x = 0$, the following Fourier-integral-like theorem holds [cf. formula (16.5)]:

$$f(x,z) = \frac{2}{\pi^2}\int_0^\infty\int_0^\infty \frac{dk\,dk_1}{k^2+2k_1^2}\int_{-\infty}^{\infty} d\zeta\cos(z-\zeta)\int_0^\infty d\xi\, f(\xi,\zeta)\cdot$$

$$\cdot\Big\{\left[k_1 e^{-\nu x} + k_1\cos k_1 x - \nu\sin k_1 x\right]\left[k_1 e^{-\nu\xi} + k_1\cos k_1\xi - \nu\sin k_1\xi\right] +$$

$$+ 2k^2\exp\left(-k(x+\xi)/\sqrt{2}\right)\Big\}.$$

It is possible to construct other types of edge waves. First we rederive the Stokes wave from the third formula in (13.5) with $a = 0$. A surface satisfying $\phi_n = 0$ is defined by

$$\frac{dy}{dx} = \frac{\phi_y}{\phi_x} = -\frac{\nu}{\sqrt{k^2-\nu^2}},$$

or

$$y = -x\tan\gamma + C, \quad \tan\gamma = \nu/\sqrt{k^2-\nu^2},$$

where we may set $C = 0$ since it does not provide essentially different solutions for the bottom. This is just Stokes' solution.

One may expect to find a different type of solution by using the third equation of (13.6) with $a = 0$. Here the corresponding solution is

$$-\log\frac{\sinh m_0(y+h)}{\sinh m_0 h} = \frac{m_0^2}{\sqrt{k^2-m_0^2}}\,x, \qquad (18.34)$$

where again we have dropped an added constant. This describes a bottom which starts as a sloping beach and approaches, as $x \to \infty$, a flat bottom at depth h. The initial slope of the beach is $\sigma^2/g\sqrt{k^2-m_0^2}$. The velocity potential describes edge waves for such a configuration.

One may proceed in the same fashion with the last formulas of (13.5) and (13.6). They turn out to give identical bottoms:

$$\log \frac{\sin m_i(y+h)}{\sin m_i h} = \frac{m_i^2}{\sqrt{k^2+m_i^2}}\, x . \qquad (18.35)$$

This corresponds to edge waves along an overhanging cliff in water of finite depth. The initial backward slope of the cliff is $\sigma^2/g\sqrt{k^2+m_i^2}$.

A particularly interesting sort of edge wave, although the name is now a misnomer since there is no edge, has been discovered by Ursell [1951]. He has shown the existence of standing waves of the form

$$\varphi(x,y)\cos kz \cos \sigma t$$

in the neighborhood of a fixed submerged cylinder of radius a if ka is small enough. The waves are symmetric about the vertical plane through the axis of the cylinder and decay exponentially as $|x|$ increases. One can, of course, also construct waves progressing along the cylinder.

Ursell calls such modes of motion "trapping modes" since, if they occur in a canal with sides given by

$z=0$ and $z=n\pi/k$, no energy is radiated away, even though there is a path of escape. In fact, the motion is similar in this respect to standing waves in a basin of finite extent. The edge waves considered above also can be used to construct trapping modes.

18γ. Waves in canals.

The propagation of periodic waves along a canal leads to problems similar to those occurring in the propagation of waves parallel to a beach. Let the canal be parallel to Oz with cross-sectional contour C. We wish to find

$$\phi(x,y,z,t) = \varphi(x,y)\cos(kz-\sigma t)$$

where $\varphi(x,y)$ satisfies

$$\varphi_{xx} + \varphi_{yy} - k^2\varphi = 0, \qquad (18.36)$$

$$\varphi_y(x,0) - \nu\varphi(x,0) = 0, \quad \nu = \sigma^2/g \quad \text{, on the free surface,}$$

$$\varphi_n = 0 \quad \text{on } C.$$

It will also be assumed that $\varphi_x^2 + \varphi_y^2$ is bounded.

Clearly the same equations arise in searching for standing-wave solutions in a horizontally cylindrical basin with cross-sectional contour C bounded at either end by vertical walls at a distance ℓ apart. In this case k is restricted to the values $n\pi/\ell$. For progressive waves solutions with $k = 0$ are, of course, of no interest.

The special case when C is a rectangle has already been discussed in section 16γ. The configuration for C which seems to have attracted the next most attention is a triangular one in which the two sides are inclined at the same angle. Kelland [1844] was apparently the first to consider this problem for infinitesimal waves, limiting his treatment to angles of 45°. The matter was treated systematically by Macdonald [1894] who discovered that a solution with the properties of (18.36) exists only for angles $\gamma = 45°$ and $\gamma = 30°$. This does not exclude the possibility of the existence for other angles of a periodic progressive wave with a curved wave front, for these would not be described by the assumed form of ϕ.

The solutions for $\gamma = 45°$ can be obtained from the fundamental solutions of (13.6), but it is nearly as easy to find them directly. In the third formula of (13.6) let $a = b = \frac{1}{2}A$, $k^2 = 2m_0^2$. This gives the velocity potential, after forming a progressive wave,

$$\phi(x,y,z,t) = A\cosh\frac{k}{\sqrt{2}}(y+h)\cosh\frac{k}{\sqrt{2}}x\cos(kz-\sigma t). \tag{18.37}$$

Let the sides of the canal be given by $y = \pm x - h$. Then it is easy to verify that

$$\phi_n\big|_{y=x-h} = -\phi_x + \phi_y\big|_{y=x-h} = 0, \quad \phi_n\big|_{y=-x-h} = \phi_x + \phi_y\big|_{y=-x-h} = 0,$$

so that the boundary conditions are all satisfied. Since

$$\sigma^2 = g m_0 \tanh m_0 h = \frac{1}{\sqrt{2}} g k \tanh\frac{1}{\sqrt{2}} k h,$$

the wave velocity is given by

$$c^2 = \frac{g}{k\sqrt{2}}\tanh\frac{kh}{\sqrt{2}}. \tag{18.38}$$

If $m_0^2 > m_i^2$ (in the notation of eq.(13.6)), there will be i further symmetric modes. In (13.6), formula 4, set $a = b = \frac{1}{2}A$ and add this to formula 1 with $a = A, b = 0$

This gives

$$\phi(x,y,z,t) = A\left[\cos m_i(y+h)\cosh\sqrt{k^2+m_i^2}\,x + \cosh m_0(y+h)\cos\sqrt{m_0^2-k^2}\,x\right]\cos(kz-\sigma t)$$

One may again verify easily that $\phi_n = 0$ on the two sides of the canal if $k^2 = m_0^2 - m_i^2$. Hence, this mode of motion will exist only if $m_0^2 > m_i^2$. For given σ there will be no modes of this sort if h is small enough, for then $m_0^2 < m_1^2$. The number gradually increases as h increases. If h and k are fixed and σ allowed to increase, there will be an infinite sequence $\sigma_1, \sigma_2, \cdots$ for which $k^2 = m_0^2 - m_i^2$ will be satisfied; $\sigma_n^2 h/g \to (n+\frac{1}{4})\pi$ as $n \to \infty$. The situation is easily visualized by plotting on one graph $\tanh mh$, $-\tan mh$ and $(\sigma^2 h/g)/mh$. One may write the potential function in the form

$$\phi(x,y,z,t) = A\left[\cos m_i(y+h)\cosh m_0 x + \cosh m_0(y+h)\cos m_i x\right]\cos(kz-\sigma t),$$

where

$$m_0 \tanh m_0 h = \nu, \quad m_i \tan m_i h = -\nu; \quad k^2 = m_0^2 - m_i^2. \qquad (18.39)$$

The velocity is given by

$$c^2 = \frac{g m_0 \tanh m_0 h}{m_0^2 - m_i^2}. \qquad (18.40)$$

Asymmetric modes of motion also exist, having first been noticed by Greenhill [1886]. These cannot be deduced from (13.6) but must be found directly. The velocity potential corresponding to (18.37) is

$$\phi(x,y,z,t) = A \sinh \frac{k}{\sqrt{2}}(y+h) \sin \frac{k}{\sqrt{2}}x \cos(kz-\sigma t) \qquad (18.41)$$

The wave velocity is

$$c^2 = \frac{g}{k\sqrt{2}} \coth \frac{kh}{\sqrt{2}}, \tag{18.42}$$

which approaches infinity as $kh \to 0$. In addition to this mode, other asymmetric modes may exist under conditions similar to those required for (18.39). The velocity potential for these modes is

$$\phi(x, y, z, t) = A\left[\sin n_i(y+h)\sinh n_o x + \sinh n_o(y+h)\sin n_i x\right] \cdot \cos(kz - \sigma t), \tag{18.43}$$

where

$$n_o \coth n_o h = \nu, \quad n_i \cot n_i h = \nu, \quad k^2 = n_o^2 - n_i^2.$$

The velocity of propagation is given by

$$c^2 = \frac{n_o \coth n_o h}{n_o^2 - n_i^2}. \tag{18.44}$$

The solution for the angle $\gamma = 30°$ will not be discussed here. It can be found in Lamb's Hydrodynamics [1932, p. 449] as well as in Macdonald's paper cited above.

One may construct other possible contours for the canal cross-section by starting from one of the solutions (13.5) or (13.6) and finding surfaces for which $\phi_n = 0$. Thus, from the third equation of (13.5) for

$$\phi = A e^{\nu y} \sinh x\sqrt{k^2 - \nu^2} \cos(kz - \sigma t).$$

Solution of the differential equation $dy/dx = \phi_y/\phi_x$ leads easily to

$$y + h = \frac{\nu}{k^2 - \nu^2} \log \cosh x \sqrt{k^2 - \nu^2}$$

as an equation for the contour of a possible canal. The contour is reasonably shaped but varies with the choice of k . Also, the method is unsatisfactory in that it gives no information about other possible modes of motion.

19. Problems with steadily oscillating boundaries.

Such problems include waves resulting from forced oscillation of a submerged body and the waves associated with steady oscillations of a freely floating body in oncoming waves. In this section we shall assume the fluid of infinite extent. Waves in an oscillating bounded basin will be discussed later. The mathematical treatment has much in common with that of the last two sections, the scattered wave of those sections becoming the forced wave of this one.

19 α . Forced oscillations.

Suppose that the surface of the oscillator in its equilibrium position is represented by $F(x, y, z) = 0$. Let us take it, for example, to be oscillating vertically with amplitude ε . Then the equation of the oscillating surface S may be written $F(x, y, z, t) = F(x, y + \varepsilon a \sin \sigma t, z) = 0$ where a is some length dimension of the oscillator This ε will be taken as the expansion parameter in the perturbation procedure. In the perturbation theory of section 10, we were concerned only with the functions $\phi(x,y,z,t)$ and $\eta(x,y,t)$. However, we must similarly expand F before substituting it into the boundary condition satisfied on the surface of the oscillator, namely,

$$F_x \phi_x + F_y \phi_y + F_z \phi_z + F_t = 0 \text{ on } F(x,y,z,t) = 0. \qquad (19.1)$$

The expansion for this case is

$$F(x, y+\varepsilon \sin \sigma t, z) = F(x,y,z) + \varepsilon a \sin \sigma t\, F_y(x,y,z) + \tfrac{1}{2}\varepsilon^2 \sin^2 \sigma t\, F_{yy}(x,y,z) + \ldots \qquad (19.2)$$

We don't wish to restrict ourselves to this one mode of motion for the oscillator, but an examination of the form of this and similar expansions indicates that we may assume in general that the surface of the oscillator can be represented by the series

$$F(x,y,z,t) = F^{(0)}(x,y,z) + \varepsilon\left[F_1^{(1)}(x,y,z)\cos \sigma t + F_2^{(1)}(x,y,z)\sin \sigma t\right] \qquad (19.3)$$

+ time-periodic terms in higher powers of ε,

where $F^{(0)}(x,y,z) = 0$ is the equilibrium position of the oscillator. We may now assume either that ϕ is periodic, i.e.,

$$\phi(x,y,z,t) = \sum \varphi_{1n}(x,y,z)\cos n\sigma t + \varphi_{2n}(x,y,z)\sin n\sigma t \qquad (19.4)$$

or, more simply, that it is simple harmonic,

$$\phi(x,y,z,t) = \varphi_1(x,y,z)\cos \sigma t + \varphi_2(x,y,z)\sin \sigma t, \qquad (19.5)$$

where each function φ_{im} or φ_i is still to be expanded in a perturbation series. The two assumptions are not quite equivalent, even for the first-order theory, but since under certain conditions (18.4) leads to the same first-order equations as (18.5), we shall assume the latter form, together with

$$\eta(x,z,t) = \eta_1(x,z)\cos\sigma t + \eta_2(x,z)\sin\sigma t. \qquad (19.6)$$

Substitution of the perturbation series into the exact equations and boundary conditions, as in section 10, then leads to the following first-order equation and boundary conditions:

1) $\Delta\varphi_k^{(1)} = 0, \quad k = 1, 2,$

2) $\varphi_{ky}^{(1)}(x,0,z) - \frac{\sigma^2}{g}\varphi_k^{(1)}(x,0,z) = 0, \quad k = 1, 2,$

3) $\text{grad}\, F^{(0)} \cdot \text{grad}\, \varphi_1^{(1)} + \sigma F_2^{(1)} = 0 \quad \text{on} \quad F(x,y,z) = 0,$ (19.7)

4) $\text{grad}\, F^{(0)} \cdot \text{grad}\, \varphi_2^{(1)} - \sigma F_1^{(1)} = 0 \quad \text{on} \quad F(x,y,z) = 0.$

One should note that it is a natural consequence of the method that the boundary condition on the oscillator is to be satisfied at its equilibrium position. If we let

$$A_1(x,y,z) = \frac{-\sigma F_2^{(1)}}{|\text{grad}\, F^{(0)}|}, \quad A_2(x,y,z) = \frac{\sigma F_1^{(1)}}{|\text{grad}\, F^{(0)}|} \quad \text{for} \quad F^{(0)}(x,y,z) = 0, \qquad (19.8)$$

then conditions 3) and 4) of (18.7) may be written

$$\varphi^{(1)}_n = A(x, y, z) \quad \text{on} \quad F^{(0)} = 0, \tag{19.9}$$

where $\varphi^{(1)} = \varphi^{(1)}_1 + i\,\varphi^{(1)}_2$ and $A = A_1 + i A_2$.

We shall henceforth drop the superscripts and consider only the first-order equations. In addition to equations (19.7) the functions φ_i must also satisfy the usual conditions on fixed solid boundaries, $\varphi_{i n} = 0$, and, if the fluid is infinitely deep, $|\text{grad}\, \varphi| \to 0$ as $y \to -\infty$. Finally, one needs a boundary condition to ensure only outgoing waves at infinity. As has been pointed out by Ursell [1951], the foregoing conditions are not always sufficient to guarantee uniqueness of solution.

Kochin [1939, 1940] has considered the general mathematical problem in water of infinite depth for both two and three dimensions. Haskind [1942b, 1944, 1946] has extended Kochin's methods to water of finite depth. The frequently-cited paper by John [1950] treats the theoretical aspects of the problem in a thorough manner and includes many of the results of Kochin and Haskind. Special problems have been considered by numerous authors. Havelock [1929b] considers the waves generated by oscillation of a vertical plate extending to the bottom in water of infinite depth for both two and three dimensions, and in water of finite depth for two dimensions; he also considers waves generated by oscillations of a vertical cylinder. MacCamy [1957] has treated the three-dimensional problem in water of finite depth. Kennard [1949] has treated the two-dimensional problem as an initial-value problem. Ursell [1948] has considered waves generated by oscillation of a vertical strip with finite depth of immersion in water of infinite depth; the treatment is two dimensional. In a later paper Ursell [1949b] considered the waves generated by the rolling of long cylinders of ship-like cross-section. In addition, Ursell has treated the waves generated by a heaving half-submerged circular cylinder [1949a, 1953c, 1954] and by a pulsing submerged cylinder [1950]. Havelock [1955] has treated the wave motion generated by a half-submerged sphere. Certain mathematical aspects of this last problem have been examined in more detail by MacCamy [1954]. Because of its interest in connection with the heaving motion of a ship there exists many papers attempting to compute approximately the force and moment on a heaving shiplike body resulting from wave formation. We mention particularly one by Grim [1953]. In the cited papers by Kochin and Haskind certain special problems are solved approximately. In addition, Haskind [1942, 1943b] has considered the motion resulting from forced oscillation of a plate, or a system of plates, on the surface. In a more recent paper Haskind [1953a] has developed a method for finding solutions, and in particular the force and moment on the body, for a wide class of two-dimensional contours of ship-like cross-section. One should also consult a recent expository paper by Maruo [1957].

This brief summary of papers on forced water waves is by no means complete but lists many of the important papers and indicates the richness of the literature.

As was stated in the introductory remarks, the theory of forced water waves is mathematically almost identical with the diffraction theory. If one interprets the value of $-\partial\phi_I/\partial n$ on the body as the function describing the motion of the oscillator, then it is clear that the problems are the same. Hence, the general remarks about existence of solutions, uniqueness, and special methods carry over directly and will not be repeated. However, we wish to consider here one further topic in the general theory.

Kochin's H-function. The H-function was apparently first introduced by Kochin [1937] in connection with the theory of wave resistance. He later extended it [1939, 1940] to waves generated by oscillating bodies, and it has become a standard device among other Russian workers in this field, especially Haskind, who has extended its definition to other situations.

Each potential function φ satisfying the boundary conditions has associated with it an H-function which is related to it much in the same way that the Fourier transform of a function is related to the function. One of its chief virtues is that it allows one to give compact formulas for force and moment on an oscillating body (in the present context) as well as certain other quantities. It is also sometimes helpful in suggesting approximate solutions.

Let us suppose that the surface S of a body of bounded extent is oscillating in some manner in fluid of infinite depth and let $\varphi = \varphi_1 + i\varphi_2$ be the solution to the potential-theory problem formulated earlier. Let S_1 and S_2 be two closed surfaces lying below $y = 0$ with S_2 enclosing S_1 and S_1 enclosing S. Let us denote the source potential introduced in (13.17") by $G(x,y,z;\xi,\eta,\zeta)$, where (ξ,η,ζ) are the coordinates of the singularity, and let us write it as a contour integral:

$$G(x,y,z;\xi,\eta,\zeta) = \frac{1}{r} + \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \int_{0(L)}^{\infty} dk\, \frac{k+\nu}{k-\nu}\, e^{k(y+\eta - i(x-\xi)\cos\theta - i(z-\zeta)\sin\theta)}, \tag{19.10}$$

where the path L passes below the singularity at $k = \nu = \sigma^2/g$. (The residue at this point gives exactly the imaginary part of (13.17").)

Now apply Green's Theorem to the region between S_1 and S_2 (the following argument is very similar to a two-dimensional one used in section 17α in discussing the integral-equation method):

$$\varphi(x,y,z) = -\frac{1}{4\pi}\iint_{S_1}\left[\frac{1}{r}\frac{\partial\varphi}{\partial n}\left(\frac{1}{r}\right)\right]d\sigma + \frac{1}{4\pi}\iint_{S_2}\left[\frac{1}{r}\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]d\sigma \tag{19.11}$$
$$= \varphi_1 + \varphi_2 .$$

Then φ_1 may be extended to a function harmonic in the whole space exterior to S_1. φ_2 is harmonic in the whole interior of S_2, but since S_2 may be indefinitely enlarged as long as it remains below $y = 0$, φ_2 may be extended to be harmonic in the whole half-space, $y < 0$. Consider now the function

$$\Psi(x,y,z) = -\frac{1}{4\pi}\iint_{S_1}\left[G\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial G}{\partial n}\right]d\sigma \tag{19.12}$$

Ψ satisfies the free-surface condition and the condition at infinity. Moreover, since $G = r^{-1} +$ a function harmonic in the lower half-space, $\varphi - \Psi$ is harmonic in the lower half-space and satisfies the other boundary conditions. But then $\varphi - \Psi \equiv 0$, as follows from a uniqueness theorem proved by Kochin [1940, §1]. Hence, we may write

$$\varphi(x,y,z) = \frac{1}{4\pi}\iint_{S_1}[\varphi(\xi,\eta,\zeta)G_n(x,y,z;\xi,\eta,\zeta) - G\varphi_n]\,d\sigma \tag{19.13}$$

Now define

$$H(k,\theta) = \iint_{S_1} e^{k[\eta + i\xi\cos\theta + i\zeta\sin\theta]}\{\varphi_n(\xi,\eta,\zeta) - k\varphi[\cos(n,\eta) + i\cos\theta\cos(n,\xi) + i\sin\theta\cos(n,\zeta)]\}\,d\sigma \tag{19.14}$$

$$= \iint_{S_1} e^{k[\eta + i\xi\cos\theta + i\zeta\sin\theta]}\{\varphi_n(\xi,\eta,\zeta) + i\cos\theta[\varphi_\eta\cos(n,\xi) - \varphi_\xi\cos(n,\eta)] + i\sin\theta[\varphi_\eta\cos(n,\zeta) - \varphi_\zeta\cos(n,\eta)]\}\,d\sigma.$$

Then, after some manipulation with (19.13), one can show that

$$\varphi(x,y,z) = \frac{1}{4\pi}\iint_{S_1}\left[\varphi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial\varphi}{\partial n}\right]d\sigma - \frac{1}{8\pi^2}\int_{-\pi}^{\pi}d\theta\int_{0(L)}^{\infty}dk\,\frac{k+\nu}{k-\nu}\,e^{k(y - ix\cos\theta + iz\sin\theta)}H(k,\theta). \tag{19.15}$$

We give a few of Kochin's derived formulas. The asymptotic form of the free surface in a direction α is given by

$$\eta(R,\alpha,t) \cong Re\left[\frac{\sigma}{g}\sqrt{\frac{\nu}{2\pi R}}\,\bar{H}(\nu,\alpha)e^{i(\nu R-\sigma t-\frac{\pi}{4})}\right] \text{ as } R\to\infty. \tag{19.16}$$

The rate at which energy is being carried off by the waves (and hence also the power input) is given by

$$N=\frac{1}{8\pi}\frac{\rho\sigma^3}{g}\int_0^{2\pi}|H(\nu,\theta)|^2 d\theta. \tag{19.17}$$

The force components on the oscillating body, averaged over a period, are given by

$$X_{av}=\frac{\rho\nu^2}{8\pi}\int_{-\pi}^{\pi}|H(\nu,\theta)|^2\cos\theta\, d\theta,$$

$$Y_{av}=\rho g V+\frac{\rho}{16\pi^2}\int_{-\pi}^{\pi} PV\int_0^{\infty} k\,\frac{k+\nu}{k-\nu}|H(k,\theta)|^2 dk\, d\theta, \tag{19.18}$$

$$Z_{av}=\frac{\rho\nu^2}{8\pi}\int_{-\pi}^{\pi}|H(\nu,\theta)|^2\sin\theta\, d\theta.$$

The formulas can be derived from (8.4), (9.4), and asymptotic expressions for φ .

In formulas (19.14) and (19.15) the surface S_1 over which the integrals are taken may be contracted to S. This sometimes makes it possible to express H directly in terms of known boundary values. If φ can be expressed by means of a source distribution, say

$$\varphi(x,y,z)=\frac{1}{4\pi}\iint_S \gamma(\xi,\eta,\zeta)G(x,y,z;\xi,\eta,\zeta)\,d\sigma, \tag{19.19}$$

then one has

$$H(k,\theta) = -\iint_S \gamma(\xi,\eta,\zeta)\, e^{k[\eta + i\xi\cos\theta + i\zeta\sin\theta]}\, d\sigma . \qquad (19.20)$$

In order to find approximate answers, Kochin frequently uses the distribution γ which would be proper in an infinite fluid without free surface, substitutes this in (19.20) and then uses the resulting approximation to H in (19.17) and (19.18) above. The procedure may be looked upon as the first two steps in an alternating type of approximation in which one first satisfies the boundary condition on the body, neglecting the free surface, next corrects this so as to satisfy the free-surface condition, but now disturbing the condition on the body, then corrects again to satisfy the condition on the body, etc. This method of approximation has frequently been used by Havelock [e.g., 1929a].

Kochin [1939] has also defined the H-function for two-dimensional wave motion excited by an oscillating body. We simply reproduce the formulas. Let, as usual, $f(z,t) = f_1(z)\cos\sigma t + f_2(z)\sin\sigma t$ be the complex potential and let C_1 and C_2 be two contours in the lower half-plane containing C, C_1 inside C_2. Define

$$H_s(k) = \int_{C_1} e^{-ikz} f_s'(z)\, dz, \quad s = 1, 2, \qquad (19.21)$$

Then

$$f_s'(\zeta) = \frac{1}{2\pi i}\int_{C_1} \frac{f_s'(\zeta)}{\zeta - \zeta} d\zeta - \frac{1}{2\pi}\int_0^\infty \bar{H}_s(k) e^{-ik\zeta} dk$$
$$- \frac{\nu}{\pi} \; PV\int_0^\infty \frac{\bar{H}_s(k)}{k-\nu} e^{-ik\zeta} dk + (-1)^{s+1} H_{s+1}(\nu) e^{-i\nu\zeta}, \tag{19.22}$$

where $H_3 \equiv H_1$. This follows immediately from a formula similar to (17.15). For the asymptotic form of the waves one gets

$$\eta(x,t) \cong Re\, \frac{i\nu}{\sigma}\left[\bar{H}_1(\nu) - i\bar{H}_2(\nu)\right] e^{-i(\nu x - \sigma t)} \quad \text{as } x \to +\infty, \tag{19.23}$$
$$\eta(x,t) \cong Re\, \frac{-i\nu}{\sigma}\left[\bar{H}_1(\nu) + i\bar{H}_2(\nu)\right] e^{-i(\nu x + \sigma t)} \quad \text{as } x \to -\infty .$$

The rate of dissipation of energy is

$$N = \frac{1}{2}\rho\sigma\left[|H_1(\nu)|^2 + |H_2(\nu)|^2\right]. \tag{19.24}$$

The mean value of the force and moment, averaged over a period, is

$$X_{av} = \rho\nu\, Im\left\{\bar{H}_1(\nu) H_2(\nu)\right\},$$
$$Y_{av} = \frac{\rho}{4\pi}\; PV\int_0^\infty \frac{k+\nu}{k-\nu}\left\{|H_1(k)|^2 + |H_2(k)|^2\right\} dk, \tag{19.25}$$
$$M_{av} = -\frac{\rho}{4\pi}\, Im\left\{ PV\int_0^\infty \frac{k+\nu}{k-\nu}\left[H_1'\bar{H}_1 + H_2'\bar{H}_2\right] dk\right\} +$$
$$+ \frac{1}{2}\nu\rho\, Im\left\{H_1'(\nu)\bar{H}_2(\nu) - H_2'(\nu)\bar{H}_1(\nu)\right\}.$$

Roughly the same remarks apply to the use of the two-dimensional formulas as of the three-dimensional ones.

<u>Waves from an oscillator in a wall.</u> In order to illustrate the use of the H-function, we consider the following problem. Let the (y, ζ)-plane be a rigid wall

except for a certain bounded area S in which there is a membrane oscillating according to a given law

$$x = F(y,z) \sin \sigma t \quad , \quad (y,z) \text{ in } S. \tag{19.26}$$

The boundary condition which has to be satisfied on the plane $x = 0$ is then

$$\varphi_x(0,y,z) = \begin{cases} \sigma F(y,z) \; , & (y,z) \text{ in } S \\ 0 \; , & (y,z) \text{ not in } S \; , \end{cases} \tag{19.27}$$

where we still have $\varphi = \varphi_1 + i\,\varphi_2$.

This boundary condition, as well as the ones at infinity, can be satisfied by distributing "modified" sources (13.17") or (19.10) over S with density $-\sigma F(y,z)/2\pi$:

$$\varphi(x,y,z) = \frac{-\sigma}{2\pi} \iint_S F(\eta,\zeta) G(x,y,z;0,\eta,\zeta)\, d\eta\, d\zeta. \tag{19.28}$$

In order to compute the H-function, we shall interpret the source distribution as representing a thin body making symmetric pulsations in an infinite fluid. Hence, we may assume that the wall is removed and the membrane replaced by a doubled one. (That the requisite motion is physically impossible doesn't invalidate the considerations; a more realistic model can easily be devised.) In (19.14) we take S_1 to be both sides of the thin body. Then, remembering that

$$\varphi_n(+0,\eta,\zeta) = \varphi_x(0,\eta,\zeta) = \sigma F, \quad \varphi_n(-0,\eta,\zeta) = -\varphi_x(0,\eta,\zeta) = \sigma F,$$

$$\cos(n,\xi) = 1 \text{ for } x>0 \quad \text{and} \quad \cos(n,\xi) = -1 \text{ for } x<0,$$

one finds easily that

$$H(k,\theta) = 2\sigma \iint_S F(\eta,\zeta) e^{k(\eta + i\zeta\sin\theta)} d\eta\, d\zeta . \tag{19.29}$$

From (19.17) one then finds immediately, after carrying out the θ integration, that the rate of dissipation of energy to one side is given by

$$N = \frac{\rho\sigma^5}{4\pi g} \iint_S dy\, dz \iint_S d\eta\, d\zeta\, F(y,z) F(\eta,\zeta) e^{\nu(y+\eta)} J_0(\nu(z-\zeta)). \tag{19.30}$$

Expressions for Y_{av} and Z_{av} may also be written down. The result $X_{av} = 0$ is not really significant because the integral is over both sides of the thin pulsing body.

The theory for generation of two-dimensional waves in a semi-infinite channel by a vertical wave maker in the end-wall is easily derived in the same way. If the motion of the wave-maker is described by

$$x = F(y)\sin\sigma t, \quad a \le y \le b \le 0 \tag{19.31}$$

then

$$H_1(k) = \int_a^b e^{k\eta} F(\eta)\, d\eta, \quad H_2(k) = 0, \tag{19.32}$$

and, for example, the rate of dissipation of energy is given by

$$N = \rho \sigma^3 \left[\int_a^b e^{\nu y} F(y)\, dy \right]^2 \tag{19.33}$$

The generation of short-crested waves is subject to the limitations described in section 14 γ. Suppose, for example, that the water is of depth h, the channel of breadth , and that the motion of the wave-maker is described by

$$x = F(y) \cos kz \sin \sigma t, \quad k = n\pi/b, \quad -h \le y \le 0. \tag{19.34}$$

Then, since $\cos m_i (y+h)$, $\cosh m_0 (y+h)$ form a complete set of functions in $-h \le y \le 0$, there is no difficulty in representing $F(y)$ by a series of the fundamental solutions (13.6), but if $k^2 > m_0^2$, no progressive waves will move down the tank (within the limits of applicability of the linearized theory, of course). The analysis of the filtering effect of the tank on more complicated wave-maker motions can easily be carried through by Fourier analysis.

<u>Waves from an oscillator not in a wall.</u> Let us now suppose that we have a two-dimensional oscillator in infinitely deep water moving according to the law

$$x = F(y) \sin \sigma t, \quad a < y < b \le 0, \tag{19.35}$$

but with no wall present. This small change complicates the solution of the problem in a substantial way, the complication being associated with the now possible flow under (and over if $b < 0$) the oscillator. In addition, in order to ensure a unique solution some further condition analogous to the Kutta-Joukowski condition in airfoil theory is required; such a condition does not seem to have been formulated. Aside from this lack, the boundary conditions to be satisfied on the oscillator by the velocity potential $\phi(x, y, t) = \varphi_1 \cos \sigma t + \varphi_2 \sin \sigma t$ are

$$\begin{aligned} &\phi_x(0, y, t) = \sigma F(y) \cos \sigma t \qquad a < y < b \leq 0, \\ &\phi_y(0, a, t) = 0, \\ &\phi_y(0, b, t) = 0 \quad \text{if} \quad b < 0, \end{aligned} \tag{19.36}$$

The problem is clearly closely related to that of diffraction of plane waves by a vertical barrier and could be treated by a modification of the method used in section 17α for that problem. It may also be solved by the integral-equation method discussed in section 17 α . A modification of this method has been used by Ursell [1948].

Introduce the complex potential

$$\phi + i \Psi = Re_j \left\{ f(z) e^{-j \sigma t} \right\},$$

where (19.37)

$$f(z) = f_1(z) + j f_2(z) = (\varphi_1 + j \varphi_2) + i (\Psi_1 + j \Psi_2).$$

We try to construct a solution by means of a distribution of vortices of the form (13.28)

$$\begin{aligned} f_v(z; \zeta) &= \frac{1}{2\pi i} \log(z - \zeta)(z - \bar{\zeta}) + \frac{1}{\pi i} \, pv \int_0^\infty \frac{e^{-ik(z-\bar{\zeta})}}{k - \nu} dk - j i e^{-i\nu(z - \bar{\zeta})} \\ &= f_{v1} + j f_{v2}. \end{aligned} \tag{19.38}$$

with intensity

$$\gamma(\eta) = \gamma_1 + j\gamma_2 , \quad a < y < b, \tag{19.39}$$

along the oscillator:

$$f(z) = \int_a^b \gamma(\eta) f_v(z, i\eta) d\eta. \tag{19.40}$$

An analysis almost identical with that in 17α leads quickly to the integral equation

$$Re_i \int_a^b \gamma(\eta) f_v'(iy, i\eta) d\eta = 6F(y) + j\cdot 0 , \quad a < y < b. \tag{19.41}$$

Separating γ_1 and γ_2 and noting that $f_v'(iy; i\eta)$ is real with respect to i, one finds

$$\begin{aligned} &\int_a^b [\gamma_1(\eta) f_{v1}'(iy, i\eta) - \gamma_2(\eta) f_{v2}'(iy, i\eta)] d\eta = 6F(y), \\ &\int_a^b [\gamma_1(\eta) f_{v2}'(iy; i\eta) + \gamma_2(\eta) f_{v1}'(iy, i\eta)] d\eta = 0 . \end{aligned} \tag{19.42}$$

The equations can be uncoupled by applying the operator $[\partial/\partial y - \nu]$ to each (so that the reduction method enters in after all!). Introducing

$$\mu_k = \gamma_k' - \nu\gamma_k , \quad G(y) = F' - \nu F, \tag{19.43}$$

one finally obtains the pair of equations

$$\begin{aligned} &\int_a^b \mu_1(\eta) \frac{d\eta}{y^2 - \eta^2} = \frac{\gamma_1(b)}{y^2 - b^2} - \pi 6 \frac{G(y)}{y} \\ &\int_a^b \mu_2(\eta) \frac{d\eta}{y^2 - \eta^2} = \frac{\gamma_2(b)}{y^2 - b^2} , \end{aligned} \tag{19.44}$$

where we have taken advantage of the fact that

$\varphi_y(\pm 0, y) = \mp \gamma(y)$ and hence $\gamma(a) = 0$; if $b < 0$ also $\gamma(b) = 0$. The integral equations are easily reduced to a known type occurring in airfoil theory [see, e.g., W. Schmeidler, Integralgleichungen ..., Akademische Verlagsgesellschaft, Leipzig, 1950, pp. 55-56, or Mikhlin, Integral'nye uravneniya ..., Gostekhizdat, Moscow, 1949, pp. 149-154] by the transformation

$$r = y^2 - \tfrac{1}{2}(a^2 + b^2), \quad \rho = \eta^2 - \tfrac{1}{2}(a^2 + b^2).$$

The solution may be written in the form

$$\mu_1(\eta) = \frac{2\eta}{\pi\sqrt{(a^2-\eta^2)(\eta^2-b^2)}}\left[\nu\int_a^b \gamma_1(\eta)\,d\eta + 2\varepsilon b \; P\!\int_a^b G(y)\sqrt{(a^2-y^2)(y^2-b^2)}\,\frac{dy}{\eta^2-y^2}\right],$$

$$\mu_2(\eta) = \frac{2\eta\nu}{\pi\sqrt{(a^2-\eta^2)(\eta^2-b^2)}}\int_a^b \gamma_2(\eta)\,d\eta. \tag{19.45}$$

It is evident that the solution is not uniquely determined without some statement about the total circulation. Fixing the total circulation is equivalent to fixing $\gamma(b)$, as follows easily from the form of $\mu_1(\eta)$ and the relation

$$\gamma(\eta) = e^{\nu\eta}\int_a^\eta e^{-\nu s}\mu(s)\,ds. \tag{19.46}$$

It is possible to compute the H-function directly in terms of $\mu(s)$. First, we note that

$$H(\lambda) = \oint_{C_1} e^{-i\lambda\zeta} f'(\zeta)\,d\zeta = \oint_{C_1} e^{i\lambda\zeta}\,d\zeta \int_a^b \gamma(y) f_v'(\zeta; iy)\,dy$$

$$= \int_a^b \gamma(y)\,dy \oint_{C_1} e^{-i\lambda\zeta} f_v'(\zeta; iy)\,d\zeta = \int_a^b \gamma(y) e^{\lambda y}\,dy.$$

It then follows from (19.46) that

$$H(\lambda) = \frac{e^{\lambda b}}{\lambda+\nu}\gamma(b) - \frac{1}{\lambda+\nu}\int_a^b \mu(y)e^{\lambda y}\,dy. \tag{19.47}$$

One may now apply formulas (19.23) to (19.25) to find the quantities indicated there (note that $\bar{H} = H$).

One notes again that the function $H(\lambda)$ is determined only after $\gamma(b)$ is fixed. Taking $\gamma(b) \neq 0$ is equivalent to having a singularity at the end. If the oscillator is totally submerged, it seems reasonable to set $\gamma(b) = 0$, as we assumed in (19.36), for then the vertical velocity is continuous at the end, i.e., $\varphi_y(+0, b) = \varphi_y(-0, b)$, as has already been assumed for the lower end at $y = a$. It is not clear what is the proper assumption if $b = 0$, i.e., if the oscillator extends through the surface. In the similar problem of diffraction by a vertical plate, treated by the reduction method in section 17α, the assumption of no singularity at the surface is equivalent to assuming $\gamma(0) = 0$. We note that if $\gamma(b) = 0$, then it follows from (19.46) and the form of μ_2 in (19.45) that $\mu_2 \equiv 0$, and hence that $\gamma_2 \equiv 0$. This is not true, of course, for γ_1 .

<u>Waves generated by a heaving hemisphere</u>. We describe briefly a procedure used by Havelock [1955] and MacCamy [1954], and before them also by Ursell [1949a] for an analogous two-dimensional problem. Let a hemisphere of radius a have its center on the free surface in its undisturbed position and let it undergo forced vertical oscillations

described by

$$x^2+(y-b_0 \sin \sigma t)^2+z^2=a^2. \qquad (19.48)$$

Then the boundary condition to be satisfied by $\varphi(x,y,z)=\varphi_1+i\,\varphi_2$ on the hemisphere is

$$\frac{\partial \varphi_1}{\partial r}=b_0\sigma\,\frac{y}{a}=b_0\sigma\cos\theta,\quad \frac{\partial \varphi_2}{\partial r}=0 \quad \text{on } x^2+y^2+z^2=a^2,\ y\leq 0. \qquad (19.49)$$

φ must, of course, also satisfy the free-surface condition and the radiation condition, as stated in (19.7).

The method of the above-named authors is to represent φ as a series in which the first term is a source at the center, say (13.17), and the remaining terms represent only local disturbances of the sort shown in (13.21), with $m=0$ since we have radial symmetry. The source term is actually taken in the form (13.17'''). Since the source is at $(0,0,0)$, $r=r_1$ in the formulas and certain terms cancel and others double. Let

$$\varphi_0=\frac{1}{r}-\frac{2\nu}{\pi}\int_0^\infty[\nu\cos ky-k\sin ky]\frac{K_0(kR)}{k^2+\nu^2}\,dk$$
$$-\pi\nu e^{\nu y}\,Y_0(\nu R)+i\pi\nu e^{\nu y}\,J_0(\nu R), \qquad (19.50)$$

$$\varphi^{(n)}=\frac{-\nu}{2n}\,\frac{P_{2n-1}(\cos\theta)}{r^{2n}}+\frac{P_{2n}(\cos\theta)}{r^{2n+1}}.$$

Then the assumed form for φ is

$$\varphi(x,y,z)=\sum_{n=0}^{\infty}a^{n+2}(A_n+iB_n)\,\varphi^{(n)}(x,y,z). \qquad (19.51)$$

Substitution in the boundary condition (19.49) leads to an infinite set of linear equations for the coefficients A_n, B_n. Numerical methods may then be used to find any desired number of terms. MacCamy shows that the solution (19.51) is convergent if $\nu a < 12/7$.

Having found φ approximately, one may proceed to compute the vertical hydrodynamic force on the sphere by integrating the pressure $p = -\rho\,\partial\Phi/\partial t$ over the hemisphere. Havelock carried through an approximate calculation, expressing the result in the form

$$\begin{aligned} Y &= \tfrac{2}{3}\pi\rho a^3 b_0 \sigma^2 \left[k \sin\sigma t - 2h\cos\sigma t\right] \\ &= -Mk\,\frac{d^2 y_0}{dt^2} - M\,2h\sigma\,\frac{dy_0}{dt}\,, \end{aligned} \qquad (19.52)$$

where M is the mass of displaced fluid and y_0 the coordinate of the center. The parameter k is usually called the added-mass coefficient or the virtual-inertia coefficient; h is called the damping parameter. Figure 18 from Havelock's paper shows k and $2h$ as functions of νa.

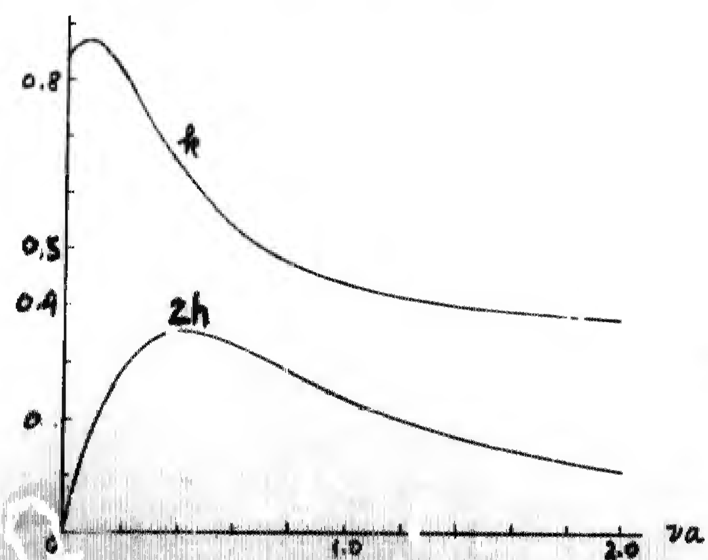

Figure 18

As $\nu a \to \infty$, $2h \to 0$ and $k \to \frac{1}{2}$; $k(0) = .828\cdots$. The average rate at which work is being done by the sphere is $\frac{2}{3} \pi \rho a^3 b_0^2 \sigma^3 h$ and does not involve k .

It is of interest to compare the same parameters as computed by Ursell [1949a] for a circular cylinder (per unit length). They are shown in Figure 19.

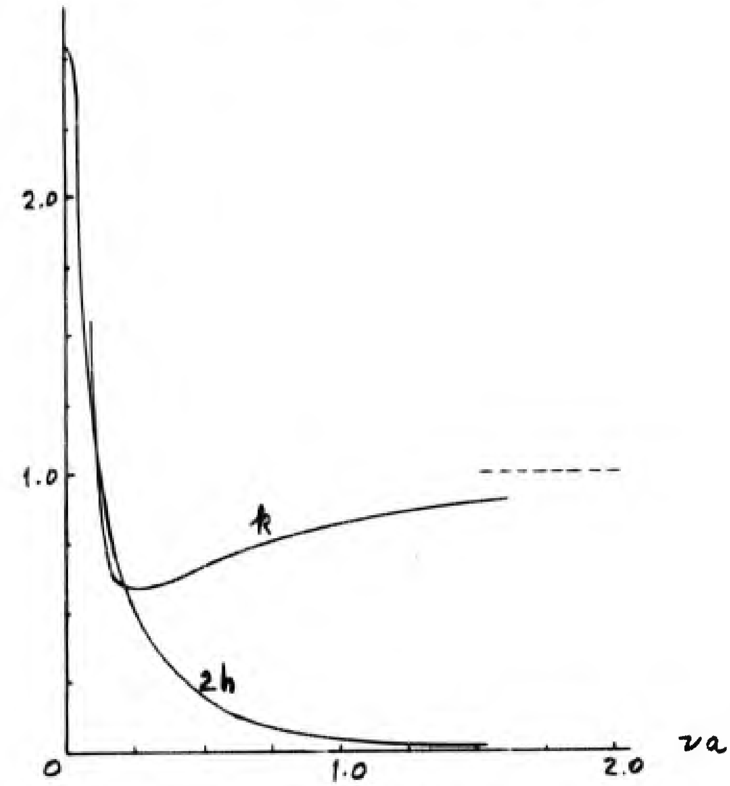

Figure 19

The asymptotic behavior of k is given by

(19.53)

$$k(\nu a) = \frac{8}{\pi^2}\left[\log\frac{1}{\nu a} + \frac{3}{2} - 2\log 2 - \gamma\right] + o(\nu a) = \frac{8}{\pi^2}\left[\log\frac{1}{\nu a} - .046\right] + o(\nu a) \quad \text{as } \nu a \to 0,$$

$$k(\nu a) = 1 - \frac{4}{3\pi\nu a} + O\left(\frac{1}{\nu a}\right) \quad \text{as } \nu a \to \infty .$$

19 β. Steady oscillations of a freely floating body in waves.

Let us suppose that a rigid body is floating in an infinite ocean with prescribed plane waves approaching from a fixed direction, say from $x = +\infty$. If the motion has persisted for some time, we may suppose that the body is moving with a simple periodic motion of the same frequency as the waves. With this assumption the proper formulation of the linearized equations and boundary conditions has been derived by John [1949].

Suppose the body is at rest in still water and let (x_0, y_0, z_0) be the coordinates of its center of gravity. Let $\bar{O}\bar{x}\bar{y}\bar{z}$ be a coordinate system fixed in the body with $\bar{O}$ at the center of gravity and the axes parallel to the space axes $Oxyz$. When the body is displaced, one may describe its position by giving the new position of the center of gravity $(\xi, \eta, \zeta) = (x_0 + \varepsilon x_1, y_0 + \varepsilon y_1, z_0 + \varepsilon z_1)$ and the Eulerian angles $\varepsilon\alpha, \varepsilon\beta, \varepsilon\gamma$ (we change notation from the customary ϕ, θ, ψ to avoid confusion with our other use of these letters). Thus the choice of ε implies that the amplitude of motion is small compared to some typical body length. The assumption of section 10 α that $\phi = \varepsilon\phi^{(1)} + \cdots$ implies that the amplitude of the prescribed incoming waves is also small compared to this length. The relationship between the two sets of coordinates may be easily found from the usual formulas concerning Eulerian angles to be of the form

$$\bar{x} = x - x_0 - \varepsilon[x_1 - \gamma(y - y_0) + \beta(z - z_0)] + \varepsilon^2[\ldots] + \cdots, \quad (19.54)$$

$$\bar{y} = y - y_0 - \varepsilon[\gamma(x - x_0) + y_1 - \alpha(z - z_0)] + \cdots,$$

$$\bar{z} = z - z_0 - \varepsilon[-\beta(x - x_0) + \alpha(z - z_0) + z_1] + \cdots.$$

Let the surface of the body be described by

$$F(\bar{x}, \bar{y}, \bar{z}) = 0 \quad (19.55)$$

in body coordinates. To find the position in space coordinates one must substitute from (19.54) in (19.55). The kinematic boundary condition (see (19.1)) then becomes

$$\varepsilon\{\operatorname{grad} F(x - x_0, y - y_0, z - z_0) \cdot \operatorname{grad} \phi^{(1)} + F_x(x - x_0, y - y_0, z - z_0)[-\dot{x}_1 + \dot{\gamma}(y - y_0) - \dot{\beta}(z - z_0)] +$$
$$+ F_y[-\dot{y}_1 + \dot{\alpha}(z - z_0) - \dot{\gamma}(x - x_0)] + F_z[-\dot{z}_1 + \dot{\beta}(x - x_0) - \dot{\alpha}(y - y_0)]\} +$$
$$+ \varepsilon^2\{\cdots\} + \cdots = 0. \quad (19.56)$$

Letting n_x, n_y, n_z be the components of the unit normal vector to the surface at rest, i.e.,

$$F(x - x_0, y - y_0, z - z_0) = 0 \quad (19.57)$$

(we shall call this surface S_0), and $\mathbf{q} = (\mathbf{r} - \mathbf{r}_0) \times \mathbf{n}$, i.e.

$$q_x = (y - y_0)n_z - (z - z_0)n_y, \; q_y = (z - z_0)n_x - (x - x_0)n_z, \; q_z = (x - x_0)n_y - (y - y_0)n_x, \quad (19.58)$$

we may express the first-order term in (19.56), after dropping the superscript, in the form

$$\phi_n = \dot{x}_1 n_x + \dot{y}_1 n_y + \dot{z}_1 n_z + \dot{\alpha} q_x + \dot{\beta} q_y + \dot{\gamma} q_z \quad \text{for } (x, y, z) \text{ on } S_0. \quad (19.59)$$

We call attention to the fact that it follows as a natural consequence of the linearization that the boundary condition is to be satisfied on the surface in its undisturbed position.

In order to state the dynamical conditions on the body we introduce the following notations. Let M be its mass and $I_x, I_y, I_z, I_{xx}, I_{yy}, \ldots$ its moments and moments and products of inertia about the body axes selected above. Let V be the volume bounded by the plane $y=0$ and the submerged part of the surface in its rest position, and let $I^V, I^V_x, I^V_y, I^V_z, I^V_{xx}, I^V_{xy}, \ldots$ be the volume, the moments, and the moments and products of inertia of this volume about the body axes in their rest position. Let A be the intersection of the body in its rest position with the surface $y=0$, and let $I^A, I^A_x, I^A_z, I^A_{xx}, I^A_{xz}, I^A_{zz}$ denote the area, the moments and the moments and products of inertia of A with respect to axes through $(x_0, 0, z_0)$ and parallel to the body axes; e.g.,

$$I^A_{xz} = \iint_A (x-x_0)(z-z_0)\,dx\,dz .$$

The exact dynamical equations are

$$\begin{aligned} M\ddot{\xi} &= \iint_S p\cos(n,x)\,d\sigma, \\ M\ddot{\eta} &= \iint_S p\cos(n,y)\,d\sigma - Mg, \\ M\ddot{\zeta} &= \iint_S p\cos(n,z)\,d\sigma, \end{aligned} \tag{19.60}$$

where S is the wetted surface of the body in its (to-be-determined) position at time t and

$$p = -\rho g y - \rho \phi_t - \tfrac{1}{2}\rho |\mathrm{grad}\ \phi|^2 ;$$

and three similar equations for $\ddot{\alpha}$, $\ddot{\beta}$, $\ddot{\gamma}$. Substitution of the perturbation series gives for the zero-order terms

$$M = \rho I_v \quad , \quad I_x^V = I_z^V = 0, \tag{19.61}$$

i.e., Archimedes' law and the statement that the center of buoyancy and center of gravity are on the same vertical line. The first-order equations, after dropping superscripts, are

$$\begin{aligned}
M\ddot{x}_1 &= -\rho \iint_{S_0} \phi_t n_x \, d\sigma, \\
M\ddot{y}_1 &= -\rho \iint_{S_0} \phi_t n_y \, d\sigma + \rho g \left(-I^A y_1 - I_x^A \gamma + I_z^A \alpha\right), \\
M\ddot{z}_1 &= -\rho \iint_{S_0} \phi_t n_z \, d\sigma,
\end{aligned} \tag{19.62}$$

$$-(I_{yy} + I_{zz})\ddot{\alpha} + I_{xy}\ddot{\beta} + I_{xz}\ddot{\gamma} = \rho \iint_{S_0} \phi_t \Omega_x \, d\sigma - \rho g \left[I_z^A y_1 + I_{xz}^A \gamma - I_{zz}^A \alpha - I_y^V \alpha\right],$$

$$I_{xy}\ddot{\alpha} - (I_{xx} + I_{zz})\ddot{\beta} + I_{yz}\ddot{\gamma} = \rho \iint_{S_0} \phi_t \Omega_y \, d\sigma,$$

$$I_{xz}\ddot{\alpha} + I_{yz}\ddot{\beta} - (I_{xx} + I_{yy})\ddot{\gamma} = \rho \iint_{S_0} \phi_t \Omega_z \, d\sigma + \rho g \left[I_x^A y_1 + I_{xx}\gamma - I_{xz}^A \alpha + I_y^V \gamma\right].$$

We note that the boundary conditions have been derived for general motions of the body and fluid, not just for the simply periodic ones for which they will be used below.

John [1949] has used the equations to investigate the stability of a floating body. We shall not reproduce the results but remark that he shows that the usual condition for stability, namely that the metacenter lie above the center of gravity, derived from purely hydrostatic considerations, is in fact still a sufficient condition for stability when the hydrodynamic equations are considered (within the limitations of the linearized theory).

It is also shown by John that the above equations have a unique solution for an initial-value problem, i.e., if at some instant the position and velocity of body and fluid are prescribed. However, for the problem with which we are concerned in this section, steady simple harmonic motion with a prescribed incoming wave, he proves uniqueness only for sufficiently large values of σ and for bodies such that a vertical line intersects the immersed surface only once (e.g., a floating sphere with its center at or above the free surface).

Knowledge of the motion of a floating body in surface waves is obviously of great importance to ship designers, and, as might be expected, there is a large amount of specialized literature. However, most of this literature may be considered irrelevant to this article for it is based upon the assumption that one may neglect the kinematic boundary condition (19.59) completely and, in the dynamic boundary condition (19.62), that one may take for ϕ simply the velocity potential for the oncoming wave, thus neglecting the effect of the diffracted waves and the waves generated by the ship's own motion. This assumption is usually called the Froude-Krylov Hypothesis. W. Froude [1861] introduced it in connection with an investigation of ship rolling in waves and A. N. Krylov [1896, 1898] investigated its implications rather thoroughly for general motions. In spite of its apparent crudeness this assumption has been useful in elucidating many aspects of ship motions.

In recent years there have appeared a number of papers in which an attempt has been made to take account of the proper boundary conditions, but no attempt will be made to summarize this literature. The most systematic investigation of the matter has been made by John [1949, 1950], Haskind [1946a], and Peters and Stoker [1957]. The papers by John consider the proper formulation of the linearized problem for a body with no average forward speed and the uniqueness and existence of solutions. Both Haskind and Peters and Stoker are primarily concerned with ships having a constant average forward speed. Peters and Stoker treat carefully the proper formulation of the linearized problem and conclude that Haskind's fundamental equations are not properly formulated in that some of his terms really belong with the second-order terms and should have been discarded. The objection applies also to part of his results for a stationary ship. The other part will be summarized below.

The motion of a ship in waves when it has a nonzero translational velocity will not be considered in this article. For this theory one should refer to the cited papers, to Stoker's Water waves [1957, ch.9], or to a recent survey by Maruo [1957]. The transient oscillatory motion of a floating body in calm water will be considered later.

Let us return to the problem of steady oscillation of a floating body in oncoming waves. Since we assume steady oscillation, we shall write

$$\phi = Re\{\varphi e^{-i\sigma t}\}, (x_1, y_1, z_1) = Re\{(a_0, b_0, c_0) e^{-i\sigma t}\}, (\alpha, \beta, \gamma) Re\{(\alpha_0, \beta_0, \gamma_0) e^{-i\sigma t}\}, \tag{19.63}$$

where $\varphi = \varphi_1 + i\varphi_2$, $a_0 = a_0' + i a_0''$, etc. The unknown function φ and the constants $a_0, \ldots, \gamma_0$ are to be determined from the equations and boundary conditions.

We shall assume that ϕ can be expressed as the sum of the velocity potentials of the incoming wave, say

$$\phi^e = \frac{Ag}{\sigma} e^{\nu y} \cos(\nu x + \sigma t) \tag{19.64}$$

if the fluid is infinitely deep, a diffracted wave $\phi^0 = \varphi^0 e^{-i\sigma t}$ and a forced wave $\phi_f = \varphi_f e^{-i\sigma t}$ resulting from the body's own motion:

$$\phi = \phi^e + \phi^0 + \phi_f \tag{19.65}$$

We shall express ϕ_f in the following form (following Haskind):

$$\phi_f = \varphi^1 \dot{x}_1 + \varphi^2 \dot{y}_1 + \varphi^3 \dot{z}_1 + \varphi^4 \dot{\alpha} + \varphi^5 \dot{\beta} + \varphi^6 \dot{\gamma}. \tag{19.66}$$

Then the kinematic boundary condition (19.59) implies:

$$\varphi^0_n = -\varphi^e_n , \tag{19.67}$$

$$\varphi^1_n = n_x, \quad \varphi^2_n = n_y, \quad \varphi^3_n = n_z,$$

$$\varphi^4_n = \Omega_x, \quad \varphi^5_n = \Omega_y, \quad \varphi^6_n = \Omega_z,$$

all to be satisfied on S_0 , the rest position of the body. The functions φ^k, $k = 0, 1, \ldots, 6$, are to satisfy also the radiation condition and the condition at $y = -\infty$ (or at $y = -h$ for a flat bottom). The dynamical condition (19.62) will be used to determine the amplitudes and phases (i.e., the complex amplitudes), but first we introduce some notation. Let

$$\mu_{jk} + \frac{i}{\sigma} \lambda_{jk} = \rho \iint_{S_0} \varphi^j \frac{\partial \varphi^k}{\partial n} d\sigma. \tag{19.68}$$

The constants μ_{jk} and λ_{jk} depend only upon the geometry of the body. It may be shown by an application of Green's Theorem that $\mu_{kj} = \mu_{jk}$ and $\lambda_{kj} = \lambda_{jk}$.

Let us now substitute the expanded expression for ϕ into, say, the first of equations (19.62) (the others may be treated similarly), remembering that $n_x = \varphi^1_n$ on S_0 :

$$M\ddot{x}_1 = -\rho \iint_{S_0} (\phi^e + \phi^o)_t n_x d\sigma - \rho \iint_{S_0} (\varphi^1 \ddot{x}_1 + \cdots + \varphi^6 \ddot{\gamma}) \varphi^1_n d\sigma. \tag{19.69}$$

Consider, for example, the second term of the second integral:

$$\rho \iint_{S_0} \varphi^2 \varphi^1_n \ddot{y}_1 d\sigma = (\mu_{21} + \frac{i}{\sigma} \lambda_{21}) \ddot{y}_1 = \mu_{21} \ddot{y}_1 + \lambda_{21} \dot{y}_1 , \tag{19.70}$$

where we have used the special form of $y_1 = b_0 e^{-i\sigma t}$.

Thus, (19.69) may be written

$$(M + \mu_{11}) \ddot{x}_1 + \mu_{21} \ddot{y}_1 + \cdots + \mu_{61} \ddot{\gamma} + \lambda_{11} \dot{x}_1 + \lambda_{21} \dot{y}_1 + \cdots + \lambda_{61} \dot{\gamma} = F_{ex} + F_{ox} , \tag{19.71}$$

where $F_{ex} = f_{ex} e^{-i\sigma t}$ and $F_{ox} = f_{ox} e^{-i\sigma t}$ represent the x-components of the forces resulting from the incoming and diffracted waves and are to be computed from the first integral in (19.69). The form of (19.71) explains the names given to the μ_{ij} and λ_{ij} : the μ_{ij} are called <u>added masses</u>, the λ_{ij} , <u>damping forces</u>. If one now writes $x_1, \ldots, \gamma$ in their assumed forms in (19.63) and substitutes in (19.71), one obtains

$$-\sigma^2(M+\mu_{11})a_0 - \sigma^2\mu_{21}b_0 - \cdots - \sigma^2\mu_{61}\gamma_0 - i\sigma\lambda_{11}a_0 - \cdots - i\sigma\lambda_{61}\gamma_0 = f_{ex} + f_{ox} \tag{19.72}$$

and five similar equations. Since the amplitudes $a_0, \ldots, \gamma_0$ are complex, this gives twelve equations to determine the twelve unknown quantities. It is thus clear that, providing these equations can be uniquely solved, the problem of finding the steady oscillatory motion of a freely floating body can be reduced to the solution of several problems of the type studied in sections 18 and 19 α . From the form of (19.72) and the similar equations, it is clear that the complex amplitudes are all proportional to the amplitude A of the incoming wave as would be expected.

Haskind has applied the method outlined above to a body symmetric with respect to the (x, y)-plane, e.g., a ship heading into waves. The only possible motions are heaving, pitching and surging. In carrying out some numerical computations he makes a further approximation that the kinematic boundary condition on the body may be satisfied on its plane of symmetry rather than on the surface. Although this approximation is perfectly consistent with the linearized theory in certain contexts, as will be seen in section 21, it is not consistent with the theory as formulated here and must be considered to be a further approximation of some sort.

Freely floating sphere. Computation of the motion of a freely floating sphere with its center at the undisturbed water level can be carried through without an unreasonable amount of numerical work. The procedure for the heaving motion has been carried up to the point of numerical computation by MacCamy [1954]. Part of the problem has already been solved in section 19α, i.e., the waves resulting from the forced motion.

Since the phase at infinity must be kept arbitrary, one must replace (19.48) by

$$x^2 + (y - h_0' \cos\sigma t - h_0'' \sin\sigma t)^2 + z^2 = a^2 \qquad (19.72)$$

However, the solution of that problem may be taken over with practically no change, for the velocity potential φ^2 in the notation of (19.66) must satisfy

$$\frac{\partial}{\partial r}\varphi^2 = \frac{y}{a} = \cos\theta \quad \text{for } x^2+y^2+z^2=a^2,\ y \le 0. \qquad (19.73)$$

Thus we need only set $h_0 \sigma = 1$ in (19.49) and later. In fact, from formula (19.52)

$$\mu_{22} = \frac{1}{3}\pi\rho a^3 \cdot k, \quad \lambda_{22} = \frac{2}{3}\pi\rho a^3 \cdot 2h. \qquad (19.74)$$

Finding the diffracted wave requires finding an outgoing wave satisfying

$$\left.\frac{\partial \varphi^0}{\partial r}\right|_{r=a} = -\frac{Ag}{\sigma}\,\nu[\cos\theta - i\sin\theta\cos\alpha]\,e^{\nu a\cos\theta}\,e^{-i\nu a\sin\theta\cos\alpha}, \quad (19.75)$$

where $x = r\sin\theta\cos\alpha,\ y = r\cos\theta,\ z = r\sin\theta\sin\alpha$. MacCamy shows that φ^0 can be found as a series in functions of the form (13.21) with $b = 0$ and account taken of certain symmetries. Let

$$\varphi_{2k}^{2m} = \left[\frac{P_{2k}^{2m}(\cos\theta)}{r^{2k+1}} - \frac{\nu}{2k-2m}\,\frac{P_{2k-1}^{2m}(\cos\theta)}{r^{2k}}\right]\cos 2m\alpha, \quad k = 1,2,\ldots;\ m = 0,\ldots k-1, \quad (19.76)$$

$$\varphi_{2k}^{2m-1} = \left[\frac{P_{2k+1}^{2m-1}(\cos\theta)}{r^{2k+2}} - \frac{\nu}{2k-2m+2}\,\frac{P_{2k}^{2m-1}(\cos\theta)}{r^{2k+1}}\right]\cos(2m-1)\alpha, \quad k = 1,2,\ldots;\ m = 1,\ldots k.$$

Then φ^0 may be expressed as follows:

$$\varphi^0 = \sum_{k=1}^{\infty}\sum_{m=0}^{k-1} A_{2k}^{2m}\,a^{2k+2}\,\varphi_{2k}^{2m} + i\sum_{k=1}^{\infty}\sum_{m=1}^{k} B_{2k}^{2m-1}\,a^{2k+3}\,\varphi_{2k}^{2m-1} \quad (19.77)$$

with the coefficients A_{2k}^{2m}, B_{2k}^{2m-1} to be determined by (19.75). MacCamy proves that the series converge if $\nu a < 12/7$ and analyzes the coefficients further. Unfortunately, no numerical data seems to be available.

20. Motions which may be treated as steady flows.

In this section we shall consider several problems which are time-independent, either by their formulation or by introduction of moving coordinates. The flow associated with a constant discharge rate through a canal is of the first type; the waves associated with a ship which has moved with constant velocity C over a long period is typical of the second.

The boundary conditions at the free surface have been derived in sections 10 and 11. For three-dimensional motion the velocity potential must satisfy (see eq. 11.3)

$$\varphi_y(x,0,z) + \frac{c^2}{g}\varphi_{xx}(x,0,z) = 0 \; ; \tag{20.1}$$

the equation of the free surface is

$$y = \eta(x,z) = \frac{c}{g}\varphi_x(x,0,z). \tag{20.2}$$

In two-dimensional motion, if the complex potential $f = \varphi + i\psi$ is used, then the boundary condition may be written

$$R_e\left\{f''(x+i0) + i\frac{g}{c^2}f'(x+i0)\right\} = 0 . \tag{20.3}$$

If the potential has been taken in the form $F(z) = -cz + f(z)$ with $\psi = 0$ as the free-surface streamline, then

$$Re\left\{f'(x+i0) + i\frac{g}{c^2}f(x+i0)\right\} = 0; \tag{20.3}$$

the surface is given by

$$y = \eta(x) = \frac{1}{c}\psi(x,0). \tag{20.4}$$

On obstructions, which are now all fixed, one has as usual

$$\varphi_n = 0 \quad \text{or} \quad \psi = \text{const}. \tag{20.5}$$

Far ahead of, or far upstream of, the obstruction the motion must approach either rest , or a uniform flow, respectively.

The general theory of steady free-surface flow about a submerged obstacle in infinitely deep fluid has been considered by Kochin [1937] for both two and three dimensions. Haskind [1945a, b] has extended Kochin's treatment to fluid of constant finite depth. The methods used for waves generated by oscillating bodies carry over with only slight change, so that we shall not consider here the general aspects of the theory but consider instead several special problems.

20 α . Flow over an uneven bottom.

Let us first derive the proper boundary condition on the bottom. We shall assume that the bottom may be represented in the form

$$y = -h + \varepsilon b^{(1)}(x) \tag{20.6}$$

and that the fluid flows from the right with discharge rate $q = ch$. As in the derivation of (10.19) we take

$$F(z) = -cz + \varepsilon f^{(1)}(z) + \varepsilon^2 f^{(2)}(z) + \cdots \tag{20.7}$$

Then the condition that the bottom be a streamline is

$$-c(-h + \varepsilon b^{(1)} + \cdots) + \varepsilon \psi^{(1)}(x, -h + \varepsilon b^{(1)} + \cdots) + \cdots = ch . \tag{20.8}$$

Expansion in the manner of section 10 and grouping of coefficients leads to the boundary condition for $\psi^{(1)}$:

$$\psi^{(1)}(x,-h) = c\, b^{(1)}. \tag{20.9}$$

We may hereafter write ψ for $\varepsilon\psi^{(1)}$ and b for $\varepsilon b^{(1)}$. We note that the choice of ε indicates that the amplitude of unevenness of the bottom must be small compared with h for the linearized theory to be applicable.

Consider now a bottom of the form (see Lamb [1932, p. 409], Wien [1900 p.200])

$$y = -h + b_0 \cos kx. \tag{20.10}$$

We look for a solution in the form

$$f(z) = A\cos kz + B\sin kz, \tag{20.11}$$

where A and B are complex. Substitution in (20.9), with $b^{(1)} = b_0 \cos kx$, shows that A must be pure imaginary, say iA', and B real, and further that

$$A'\cosh kh - B\sinh kh = c\, b_0. \tag{20.12}$$

Substitution in (20.3), i.e., $\psi_y(x,0) - g c^{-2}\psi(x,0) = 0$ yields

$$kB = \frac{g}{c^2}A'. \tag{20.13}$$

One then finds easily that

$$f(z) = \frac{\nu\sin kz + i k\cos kz}{k\cosh kh - \nu\sinh kh}\, c\, b_0, \quad \nu = \frac{g}{c^2}, \tag{20.14}$$

$$\eta(x) = \frac{k b_0}{k\cosh kh - \nu\sinh kh}\cos kx.$$

An interesting consequence is that when $c^2h/g < 1$, i.e., when the flow is subcritical, the crests and troughs just oppose those of the bottom, whereas, if $c^2h/g > 0$, they occur together. If $c^2h/g = 1$, there is no steady flow. Also, when c^2h/g is close to 1, it is clear that the assumption of small perturbations is no longer satisfied.

By use of the Fourier Integral Theorem one may now construct solutions for an arbitrarily shaped bottom, within the limitations of the theorem. For from

$$b(x) = \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^{\infty} b(\xi)\cos k(x-\xi)\, d\xi \tag{20.15}$$

one may derive

$$f(\zeta) = \frac{c}{\pi}\, PV\!\int_0^\infty dk \int_{-\infty}^{\infty} b(\xi)\, \frac{\nu \sin k(\zeta-\xi) + ik\cos k(\zeta-\xi)}{k\cosh kh - \nu \sinh kh}\, d\xi, \tag{20.16}$$

$$\eta(x) = \frac{1}{\pi}\, PV\!\int_0^\infty \frac{k}{k\cosh kh - \nu\sinh kh}\, dk \int_{-\infty}^{\infty} b(\xi)\cos k(x-\xi)\, d\xi.$$

An examination of the asymptotic properties of these integrals as $x \to +\infty$ shows that they do not vanish if $\nu h = gh/c^2 > 1$. Conditions for the validity of the Fourier Integral Theorem, say that $b(x)$ is of bounded variation and absolutely integrable, indicate that it applies to situations in which the bottom unevenness is somewhat localized. Hence, it is reasonable to require the additional boundary condition

$$\lim_{x \to \infty} \eta(x) = 0 .$$

Thus we must amend the solutions (20.10) if $gh/c^2 > 1$ by adding, respectively,

$$\frac{-\nu c}{\cosh^2 k_0 h - \nu h} \int_{-\infty}^{\infty} b(\xi) \cos k_0 (z - \xi + ih)\, d\xi, \qquad (20.17)$$

$$\frac{+\nu \sinh k_0 h}{\cosh^2 k_0 h - \nu h} \int_{-\infty}^{\infty} b(\xi) \sin k_0 (x - \xi)\, d\xi,$$

where k_0 is the real solution of $k \cosh kh - \nu \sinh kh = 0$. We note that the other boundary conditions are not spoiled, for the first expression in (20.17) satisfies (20.3) and its imaginary (stream-function) part vanishes for $y = -h$ so that (20.3) is still satisfied.

Thus, if $c^2 > gh$ there is a local disturbance of the fluid in the region of unevenness which eventually smooths out. If $c^2 < gh$ there is also a local disturbance given by (20.16), but as $x \to -\infty$ there remains a disturbance given by twice the expressions in (20.17)

We remark in passing that we might have obtained this solution by distributing along the bottom dipoles of the form (13.48) with $\alpha = 0$ and with moment density $c\,b(x)$.

Various special cases of $b(x)$ may be considered. Lamb [1932, p. 410] replaces the unevenness by a single dipole. Wien [1900, p. 202] takes $b(x) = \arctan ex$ and in the limit lets $e \to \infty$ in order to find the flow over a small step. However, Kochin [1938] has treated this problem by a different method and finds that Wien has made an error by a factor of two in the downstream waves (he had not satisfied the upstream condition). The flow about a vertical plate in the bottom may be treated by distributing vortices (13.47) along the plate with the intensity to be determined by solving an integral equation.

One will find an attractive discussion of the subject in four papers of W. Thomson (Lord Kelvin) [1886, 1887]. Ekman [1906] has applied the same method to three-dimensional flow. First he finds the form of the free surface over a doubly periodic bottom, then applies the double Fourier integral theorem to construct flows over irregular bottoms. He analyzes the asymptotic behavior of the surface for the case of an isolated dipole on the bottom and presents graphs showing the change in wave amplitude for different radial sections. The method of analysis may also be extended to superposed fluids of different densities (see Long [1953, §4]).

20β. Flow about submerged obstacles.

Linearization. The procedure for linearizing may be carried through in at least two ways, leading to somewhat different boundary conditions for the body. Consider a body moving in a fluid. For the time being in order to achieve somewhat greater generality, we shall not restrict the velocity to be constant. If the dimensions of the body are sufficiently small compared with the depth of submersion, it will not disturb the surface appreciably, and one will expect to be able to use the infinitesimal-wave approximation. However, the same end is obtained if the body approximates to a flat disc moving in its plane, a line segment moving along its line, a piece of a cylindrical surface moving along the cylinder, etc., various combinations being easily visualized. We consider the two situations separately.

Let $F(x,y,z,t)=0$ describe the surface of a bounded body at time t, and let λ be some typical dimension of the body, say its maximum diameter, and let h be the depth of submersion measured to some point of the body. Now, consider the family of flows associated with the family of surfaces

$$F^{(\varepsilon)}(x,y,z,t) = F\left(\frac{x}{\varepsilon}, \frac{y+h}{\varepsilon} - h, \frac{z}{\varepsilon}, t\right) = 0. \tag{20.18}$$

where $\varepsilon = a/h$. As $\varepsilon \to 0$ the surface $F^{(\varepsilon)} = 0$ contracts to a point and the fluid approaches a state of rest. Hence, as in section 10 α, it is allowable to expand ϕ and η in a perturbation series

$$\phi = \varepsilon\phi^{(1)} + \varepsilon^2\psi^{(2)} + \cdots, \quad \eta = \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)} + \cdots \tag{20.19}$$

The boundary condition to be satisfied on the body, namely,

$$\operatorname{grad} F^{(\varepsilon)} \cdot \operatorname{grad}\phi + F_t^{(\varepsilon)} = 0, \tag{20.20}$$

becomes

$$\operatorname{grad} F \cdot \operatorname{grad}\phi^{(1)} + F_t + \varepsilon \operatorname{grad} F \cdot \operatorname{grad}\phi^{(2)} + \cdots = 0,$$

and $\phi^{(1)}$ must satisfy

$$\operatorname{grad} F \cdot \operatorname{grad}\phi^{(1)} + F_t = 0 \quad \text{on} \quad F^{(\varepsilon)} = 0.$$

Thus, one finds that, in this method of linearizing, the boundary condition to be satisfied on the body is the exact one

$$\operatorname{grad} F \cdot \operatorname{grad} \phi + F_t = 0 \quad \text{on} \quad F(x,y,z,t) = 0 . \tag{20.21}$$

The boundary condition satisfied by ϕ on the free surface is, of course, the linearized one. The approximation to the exact solution is better, the deeper the relative submergence.

The second method of linearization will be illustrated with the so-called thin-ship approximation. Let the equation of a ship hull be given in the form

$$\bar{z} = \pm F(\bar{x}, \bar{y}) \tag{20.22}$$

in coordinates fixed in the ship. Let us write this in the form

$$\bar{z} = \pm \varepsilon F^{(1)}(\bar{x}, \bar{y}) \tag{20.23}$$

where ε is, say, the beam/length. Suppose the ship moves in direction Ox with velocity $c(t)$ and consider the family of flows generated by the motion of such bodies for different ε. Let the velocity potential be $\phi(x,y,z,t,\varepsilon)$. Then, since as $\varepsilon \to 0$ the hull approaches a flat disc S_0, the ship's centerplane section, the motion of the fluid will also approach a state of rest and we may expand

$$\phi = \varepsilon \phi^{(1)} + \varepsilon^2 \phi^{(2)} + \dots \tag{20.24}$$

and similarly for η. The assumed forms for ϕ and η lead immediately as in section 10 α to the linearized free-surface boundary condition for $\phi^{(1)}$. The exact condition on the hull is

$$F_x\left(x-\int^t c(\tau)\,d\tau,\, y\right)\phi_x\left(x,\, y,\, \epsilon F\left(x-\int^t c\,d\tau, y\right), t\right)+F_y\,\phi_y-\phi_z - c(t)F_x = 0. \tag{20.25}$$

After substituting (20.23) and (20.24) in (20.25), one finds that $\phi^{(1)}$ must satisfy

$$\phi_z^{(1)}(x, y, \pm 0, t) = \mp c(t)F_x^{(1)}\left(x-\int^t c(\tau)\,d\tau,\, y\right), \tag{20.26}$$

$\phi^{(2)}$ must satisfy

$$\phi_z^{(2)}(x,y,\pm 0,t) = \pm\left[F_x^{(1)}(x,y)\,\phi_x^{(1)}(x,y\pm 0,t)+F_y^{(1)}\phi_y^{(1)}-F^{(1)}\phi_{zz}^{(1)}\right], \tag{20.27}$$

and $\phi^{(i)}$ a relation of the form

$$\phi_z^{(i)}(x,y,\pm 0,t) = \pm\, C_i\left\{F^{(1)}, \phi^{(1)}, \dots, \phi^{(i-1)}\right\}, \tag{20.28}$$

where C_i is a functional of the functions in braces. We note especially that it is a consequence of the linearization that the boundary condition imposed by the presence of the body is to be satisfied on the centerplane section and not on the actual surface. One will expect this linearized theory to be more accurate the smaller ϵ is, i.e., the smaller the beam-to-length ratio.

It is clear that one may proceed similarly in the situations mentioned earlier. We record the results in several cases for reference.

First consider the **thin-wire approximation** for two-dimensional hydrofoils. In a coordinate system $\bar{O}\bar{x}\bar{y}$ fixed in the hydrofoil let the trailing edge of the hydrofoil be at $(-a, -h)$, and let the upper and lower surfaces be given by

$$\bar{y} = -h + u(\bar{x}) \text{ and } \bar{y} = -h + b(\bar{x}), \quad -a \le \bar{x} \le a, \tag{20.29}$$

respectively. Define

$$r(\bar{x}) = \tfrac{1}{2}[u(\bar{x}) + b(\bar{x})], \quad s(\bar{x}) = \tfrac{1}{2}[u(\bar{x}) - b(\bar{x})], \tag{20.30}$$

so that now the top and bottom are given by

$$\bar{y} = -h + r(\bar{x}) + s(\bar{x}) \text{ and } \bar{y} = -h + r(\bar{x}) - s(\bar{x}), \quad -a \le \bar{x} \le a, \tag{20.31}$$

respectively. The class of profiles in a form analogous to (20.23) is now given by

$$\bar{y} = -h + \varepsilon[r^{(1)}(\bar{x}) \pm s^{(1)}(\bar{x})], \qquad -a \le \bar{x} \le a. \tag{20.32}$$

It is clear that as $\varepsilon \to 0$, the profiles approach the line segment $\bar{y} = -h$, $0 \le \bar{x} \le a$, so that the perturbation procedure is allowable. The analysis leads to the linearized boundary condition

$$\varphi_y(x, -h \pm 0, t) = -c(t)r'\left(x - \int^t c(\tau)d\tau\right) \mp c(t)s'\left(x - \int^t c(t)d\tau\right), \tag{20.33}$$
$$-a \le x - \int^t c\,d\tau \le a.$$

The **slender-body approximation** is also consistent with the linearized free-surface condition. Let the body be defined by

$$(\bar{y}+h)^2+\bar{z}^2-r^2(\bar{x})=0 \qquad |\bar{x}|<a, \quad h>0, \tag{20.34}$$

in a coordinate system fixed in the body. If one considers the class of bodies defined by $\varepsilon r^{(1)}(\bar{x})$, then the appropriate condition to be satisfied by $\phi^{(1)}$ is

$$\lim_{\varepsilon\to 0} \phi_r^{(1)}\Big(x-\int^t c(\tau)\,d\tau, -h+\varepsilon r^{(1)}\cos\theta, \varepsilon r^{(1)}\sin\theta\Big)r^{(1)}\Big(x-\int^t c\,d\tau\Big) = -c\,r^{(1)}r^{(1)\prime}. \tag{20.35}$$

We note that the same problem may be approached by two linearized theories. For example, in approximating the flow about a hydrofoil, one may either consider it as a relatively deeply submerged body and satisfy the exact conditions on the surface, or consider it as a thin hydrofoil and use the conditions (20.33). Each method will have its own domain of excellence, but it is not proper in the present context to say that the thin-hydrofoil approximation is less exact than the other one, even though this is true in an unbounded fluid.

The H-functions. Kochin's H-function, introduced in section 19α, may also be used effectively for the flows considered in the present section. The definition for three dimensions is identical with (19.14). For two dimensions (19.21) is replaced by

$$H(k)=\int_{C_1} e^{-ik\zeta} f'(\zeta)\,d\zeta . \tag{20.36}$$

However, the formulas for the force on the body are somewhat different. For three dimensions they are

$$X = -\frac{\nu^2\rho}{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} |H(\nu\sec^2\theta, \theta)|^2 \sec^3\theta\, d\theta,$$

$$Y = \rho g V - \frac{\rho}{8\pi^2}\int_0^\infty\int_{-\pi}^{\pi} |H(k,\theta)|^2 k\, d\theta\, dk$$
$$+ \frac{\rho\nu^2}{4\pi^2}\int_{-\pi}^{\pi} \mathrm{PV}\!\int_{-\infty}^{1} \left|H\left(\frac{\nu(1-\lambda)}{\cos^2\theta}, \theta\right)\right|^2 \frac{1-\lambda}{\lambda} \sec^4\theta\, d\lambda\, d\theta, \qquad (20.37)$$

$$Z = -\frac{\nu^2\rho}{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} |H(\nu\sec^2\theta, \theta)|^2 \sec^4\theta \sin\theta\, d\theta,$$

where V is the displaced volume of fluid and $\nu = g/c^2$. The two-dimensional formulas are

$$X = -\nu\rho|H(\nu)|^2,$$

$$Y = \rho g A + \rho c\Gamma - \frac{\rho}{2\pi}\int_0^\infty |H(k)|^2 dk + \frac{\rho\nu}{\pi}\,\mathrm{PV}\!\int_{-\infty}^{1} |H(\nu - k\nu)|^2 \frac{dk}{k}, \qquad (20.38)$$

$$M = -g\rho A x_c - \rho c\,\mathrm{Re}\{iH'(0)\} - \rho\,\mathrm{Re}\left\{\frac{1}{2\pi i}\int_0^\infty H'(k)\overline{H(k)}\, dk,\right.$$
$$\left. + \nu H'(\nu)\overline{H(\nu)} + \frac{\nu i}{\pi}\,\mathrm{PV}\!\int_{-\infty}^{1} H'(\nu - k\nu)\overline{H(\nu - k\nu)}\frac{dk}{k},\right.$$

where A is the area of the profile, (x_c , y_c) are the coordinates of its centroid, and Γ is the circulation. The remarks made in section 19 α concerning the use of the H-function apply also here.

<u>Submerged circular cylinder</u>. The appropriate linearization for the circular cylinder is the one associated with deep submergence. Hence, one must try to satisfy the exact boundary condition on the cylinder.

This problem has been treated by Lamb [1913; see also 1932, § 247], Havelock [1927, 1929a, 1936], Sretenskii [1938], who considers also finite depth, Kochin [1937] and Haskind [1945a], who applies Kochin's methods for finite depth. Coombs [1950] considers the flow about a pair of submerged cylinders, and, as a preliminary, also about a single cylinder; numerical computations are carried though for two cases, one with the centers on a horizontal line and one with them on a vertical line. Coombs' method has wider applicability than just to circular cylinders. In all the cited papers, with the exception of Havelock's and Coombs', the problem is solved by placing at the center of the circle a dipole modified to satisfy the free-surface condition, i.e., (13.45) with $\alpha = 0$ and $M = 2\pi c a^2$, where a is the radius and c the velocity (the c of (13.45) is taken as $-ih$, h the depth of the center). This provides, of course, only an approximate solution, for in the presence of a free surface a dipole in a stream no longer generates a circle, as is testified to by the fact that the contour actually generated is subject to a moment. Havelock [1927, 1929a] gave second approximations for drag and lift and later [1936b] a complete solution.

The problem may be treated by a combination of Milne-Thomson's Circle Theorem [1956, p. 151] and a formula of Kochin's. The former states that if f(z) is the complex potential for a flow with its singularities all at a distance greater than a from the origin and with no solid boundaries, then

$$f(\zeta) + \bar{f}\left(\frac{a^2}{\zeta}\right) \tag{20.39}$$

is the complex potential for a flow with the same singularization but now with a circle of radius a and center at the origin situated in the fluid.

Kochin [1937] has proved that if f(z) is the complex potential for a bounded contour C under a free surface, then

$$f(\zeta) = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - \zeta}\, d\zeta + \frac{1}{2\pi i}\int_{C_1} \overline{f(\zeta)}\left[\frac{1}{\zeta - \bar{\zeta}} - 2i\nu e^{-i\nu\zeta}\int_{-\infty}^{\zeta} \frac{e^{i\nu u}}{u - \bar{\zeta}}\, du\right] d\bar{\zeta} \tag{20.40}$$

where C_1 is any contour in the lower half-plane containing C. The formula and its proof are almost identical with that given in (17.15). The first integral in (20.40) represents a function regular everywhere outside C, the second integral a function regular everywhere in the lower half-plane. If one starts with a function f(z) whose only singularities are contained inside C, then the operation

$$f + K\{f\}, \tag{20.41}$$

where

$$K\{f\} = \frac{1}{2\pi i}\int_{C_1} \overline{f(\bar{z})}\left[\frac{1}{z-\bar{z}} - 2i\nu e^{-i\nu z}\int_{\infty}^{z}\frac{e^{i\nu u}}{u-\bar{z}}\,du\right]d\bar{z}, \tag{20.42}$$

$$= \frac{1}{2\pi i}\int_{C_1} \overline{f(\bar{z})}\left[\frac{1}{z-\bar{z}} - 2\pi\nu e^{-i\nu(z-\bar{z})} + 2i\nu\, PV\!\int_{0}^{\infty}\frac{e^{-iu(z-\bar{z})}}{u-\nu}\,du\right]d\bar{z}$$

yields a complex potential function satisfying the free-surface condition and having the same singularities as f in the lower half-plane.

On the other hand, if one starts with a complex potential f(z) whose only singularities are in the upper half-plane, then

$$f + M\{f\} \tag{20.43}$$

where

$$M\{f\} = \bar{f}\left(\frac{a^2}{z+ih} + ih\right) + ich, \qquad a < h; \tag{20.44}$$

will be a complex potential for a flow about a circle of radius a and center $-ih$ and with the same singularities as f in the upper half-plane, the singularities of $M\{f\}$ being all inside the circle.

We start with the flow $f_0(z) = -cz$ representing a uniform flow from the right; the free-surface condition is satisfied for $y = 0$ in a trivial manner. Now form the sequence

$$f_0, f_1 = M\{f_0\}, f_2 = K\{f_1\}, \cdots, f_{2n+1} = M\{f_{2n}\}, f_{2n+2} = K\{f_{2n+1}\}, \cdots \tag{20.45}$$

Then $f_0 + f_1$, $f_2 + f_3$, $f_4 + f_5$, $\cdots$ each represent flows satisfying the boundary condition on the circle; hence, also their sum if the series converges. On the other hand, $f_0, f_1+f_2, f_3+f_4, \cdots$ each represent flows satisfying the free surface condition, and, hence, also the sum if it exists.

Let us now consider the two operators $\mathcal{M}$ and $\mathcal{K}$. $\mathcal{M}$ is always being applied to a function regular and bounded in the lower half-plane. Since $a^2(z+ih)^{-1}+ih$ maps the exterior of the circle onto the the interior, the maximum of $|\mathcal{M}\{f\}|$ for z in the lower half-plane does not exceed that for $|f|$ within or on the circle. We write this as

$$|\mathcal{M}\{f\}| \leq \|f\| \equiv \max_C |f| . \tag{20.46}$$

In particular,

$$\|f_{2m+1}\| \leq \|f_{2m}\| . \tag{20.47}$$

The operator $\mathcal{K}$ is always applied here to functions regular everywhere outside and on the boundary of the circle. Hence C_1 may be contracted to C and one can establish the following estimate for z in the lower half-plane:

$$\begin{aligned} |\mathcal{K}\{f\}| &\leq a \max_C |f| \left[\frac{1}{|z-\bar{\zeta}|} + 2\pi\nu e^{\nu(y+\eta)} + 2\nu e^{\nu(y+\eta)}|Ei(\nu|y+\eta|)|\right] \\ &\leq a\left[\frac{1}{h-a} + 2\pi\nu e^{-\nu(h-a)} + 2\nu e^{-\nu(h-a)} Ei(\nu(h-a))\right]\max_C |f| , \\ &\leq K\|f\| \end{aligned} \tag{20.48}$$

where in the second inequality h must be large enough that $\nu(h-a) > 0.4$. For fixed values of νa one may select

h/a large enough so that K is as small as one wishes, in any case, less than 1 . Thus, in particular,

$$\begin{aligned} \| f_{2n} \| &\leq K \| f_{2n-1} \| < \| f_{2n-1} \| \\ &\leq K^n \| f_0 \|, \\ \| f_{2n+1} \| &\leq \| f_{2n} \| \leq K^n \| f_0 \| . \end{aligned} \tag{20.50}$$

From this it follows easily that the series

$$f_0 + f_1 + f_2 + \cdots + f_{2n} + f_{2n+1} + \cdots \tag{20.51}$$

with terms defined by (20.45) converges uniformly in the part of the lower half-plane exterior to $|z + ih| < a$.

One may extend the method to flows about more general cylinders by combining the operator $\mathcal{M}$ with another defined in terms of conformal mapping of the given profile into a circle. The procedure carried through above is a natural generalization of the procedure used by Havelock in his first two papers [1926, 1929] to find the second approximation. However, in his later paper [1936b] he used a different procedure, one which has also been used by Ursell in analogous problems. This consists in expressing the potential as a sum of multipoles situated at the center and, of course, already modified so as to satisfy the condition on the free surface and as $x \to \infty$. This leads to an infinite set of equations for the coefficients. The method is quite suitable for approximate computation.

After computation of $H(k)$, the formulas (20.38) can be used to find the force and moment. In the computation of H only the terms with odd indices contribute. This leads to a considerable saving in effort. For example, if one had approximated the flow by the first three terms of (20.51) and computed the force by integrating the pressure over the cylinder, the result would be the same as that obtained by using the H-function evaluated for f_1 alone, and without the need of finding f_2. Havelock has frequently made use of this device without specifically introducing the H-function. Figure 20 from Havelock [1936b] shows $R = -X$ and Y plotted in units of $\pi g \rho a^2$ with abscissa $1/\sqrt{\nu h}$ for $a/h = \frac{1}{2}$. The curves labelled R_1 and Y_1 give the result when only the first approximation is used, i.e.,

$$
\begin{aligned}
&H(k) = 2\pi c a^2 k e^{-kh} \\
&R = \pi \rho g a^2 \cdot \pi \left(\frac{a}{h}\right)^2 \left(\frac{2gh}{c^2}\right)^2 e^{-2gh/c^2}, \qquad (20.52) \\
&Y = -\pi \rho g a^2 \cdot \left(\frac{a}{h}\right)^2 \frac{c^2}{2gh} \left[1 + \frac{2gh}{c^2} - \left(\frac{2gh}{c^2}\right)^2 - \left(\frac{2gh}{c^2}\right)^3 e^{-2gh/c^2} Ei\left(\frac{2gh}{c^2}\right)\right].
\end{aligned}
$$

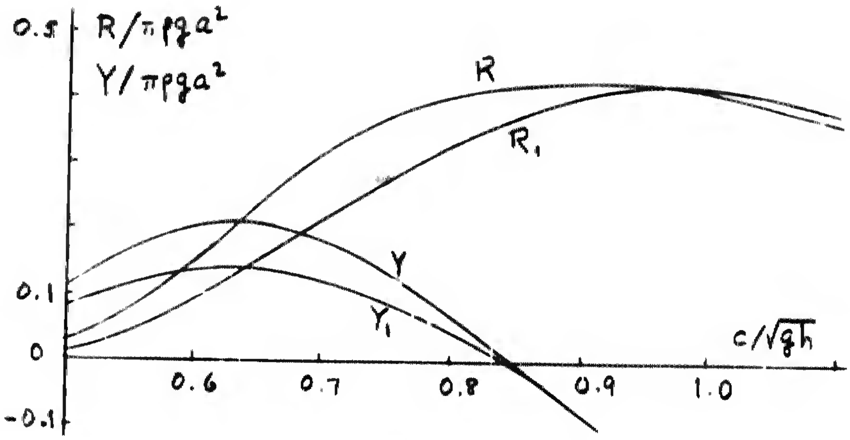

Figure 20

Computation of M gives, on this approximation, the anomolous result

$$M = h\left[1 - \frac{c^2}{gh}\right] \cdot R .$$

In the formula for Y the terms resulting from buoyancy and circulation are omitted. Havelock [1928b] has investigated the form of the surface over a moving dipole, i.e., over a sphere to the same degree of approximation.

<u>Some submerged three-dimensional bodies.</u> The flow about submerged ellipsoids and bodies of revolution in general has obvious interest in connection with the wave resistance of submarines. As a result there is a considerable amount of both theoretical and experimental work available, and even some tables for computation of wave resistance. Most of the theoretical work does not go beyond the approximation in which one represents the

body by the singularity distribution appropriate to an unbounded fluid, but with the potential function for the singularity modified to satisfy the conditions on the free surface and at $x = +\infty$. Thus, to find the flow about a submerged sphere one will in this approximation use a modified dipole with axis in the direction Ox and moment $\frac{1}{2} c a^3$. One should realize, however, that the boundary condition on the body appropriate to deep submergence has not been satisfied. The necessary refinements could be carried through for the sphere in a manner similar to that used for the circular cylinder. Since the sphere (and even more the circular cylinder) is a poor shape to which to apply perfect-fluid theory, such a computation is of only moderate interest. Both Pond [1951, 1952] and Havelock [1952] have considered methods for improving the accuracy with which the boundary condition on bodies of revolution is satisfied. This is particularly important in estimating the moment about the transverse horizontal axis, but, as Pond shows, of less importance for the wave resistance.

Havelock [1931a] treated by the approximate method the wave resistance of prolate and oblate spheroids moving both along and at right angles to their axes. Later [1931b] he extended the results to general ellipsoids moving in the direction of the longest axis. Weinblum [1936] has considered bodies of revolution using the slender-body approximation, but satisfying it only in the approximate sense described above; his aim was to find forms of minimum wave resistance. Weinblum [1951] returned to the problem, taking up in particular numerical computation of the wave resistance for a given shape. Tables and graphs are given to facilitate the computation for certain classes of bodies. Experiments were made by Weinblum, Amtsberg and Bock [1936] on three forms at several depths. Presumably, more recent experiments exist whose results are not publicly available. A general survey of the theory may be found in Bessho [1957].

If one has once computed the function $H(k,\theta)$ for a source and a dipole, it is usually straightforward to compute it for bodies generated by distributions of sources and dipoles, and hence to compute the force. Let S_1 be a surface containing a single submerged source of strength m at the point (a, b, c), $b > 0$ (i.e., (13.36) multiplied by $-m$); one finds

$$H(k,\theta) = 4\pi m e^{kb} e^{ik(a\cos\theta + c\sin\theta)} . \tag{20.53}$$

For a dipole of moment M in the direction Ox one finds

$$H(k,\theta) = 4\pi i M k e^{kb} e^{ik(a\cos\theta + c\sin\theta)} \cos\theta . \tag{20.54}$$

These may now be superposed as necessary for either discrete or continuous distributions. Thus, if we write $G(x,y,z;\xi,\eta,\zeta)$ for the function (13.36) with (a, b, c) replaced by (ξ,η,ζ), and if we can express φ for some problem by

$$\varphi = \iint_S \gamma(\xi,\eta,\zeta)\, G(x,y,z,\xi,\eta,\zeta)\, d\sigma\, , \tag{20.55}$$

then (cf. (19.20))

$$H(k,\theta) = -4\pi \iint_S e^{k\eta} e^{ik(\xi\cos\theta + \zeta\sin\theta)} \gamma(\xi,\eta,\zeta)\, d\sigma . \tag{20.56}$$

<u>Prolate spheroid.</u> We give an example of the preceding remarks. A prolate spheroid of major semi-axis a and minor semi-axes b moving with velocity c in the direction of its major axis can be represented in an unbounded fluid by a distribution of dipoles of moment density

$$\mu(\xi) = Ac\,(a^2e^2 - \xi^2)\, , \quad |\xi| < ae\, , \tag{20.57}$$

where

$$A^{-1} = \frac{4e}{1-e^2} - 2\log\frac{1+e}{1-e}\, , \quad e^2 = 1 - \frac{b^2}{a^2}\, ,$$

placed along the major axis between the two foci. Hence, with the center at depth h one has in this approximation

$$\begin{aligned} H(k,\theta) &= 4\pi i A c k e^{-kh} \cos\theta \int_{-ae}^{ae} (a^2e^2 - \xi^2) e^{ik\xi\cos\theta}\, d\xi \\ &= 8\sqrt{2}\,\pi i\, Ac(ae)^{3/2} \frac{e^{-kh}}{k^{1/2}\cos^{1/2}\theta} J_{3/2}(aek\cos\theta) \end{aligned} \tag{20.58}$$

Substituting in the first formula of (20.37), one finds

$$R = -X = +128\pi\rho\nu c^2 a^3 e^3 A^2 \int_0^{\frac{1}{2}\pi} e^{-2\nu h\sec^2\theta} \left[J_{3/2}(ae\nu\sec\theta)\right]^2 \sec^2\theta\, d\theta . \tag{20.59}$$

Figure 21 from Havelock [1931a] shows a graphical presentation of $R/\pi\rho g b^3$ for spheroids with various ratios of a/b and for $h = 2b$.

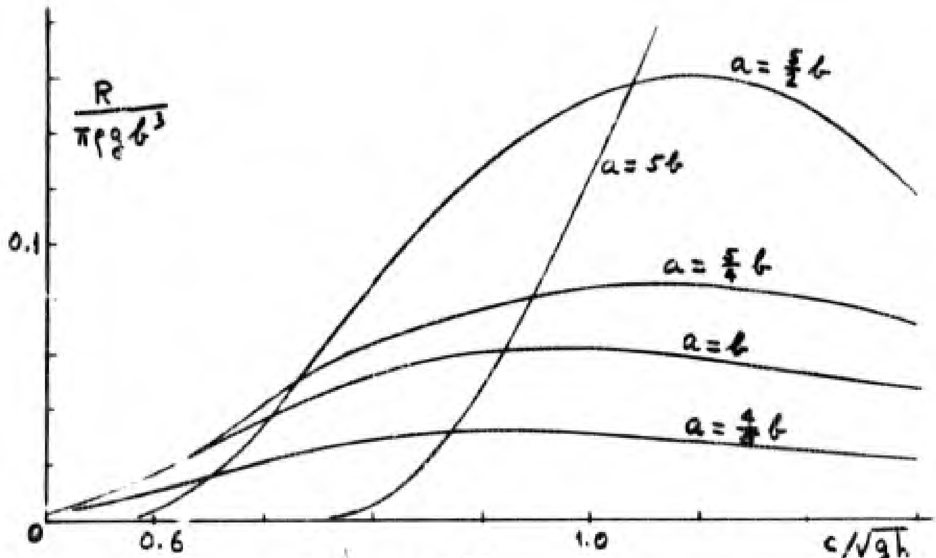

Figure 21

In comparing the different curves one should keep in mind the selected vertical scale; one based on displaced fluid, i.e. $R/\frac{4}{3}\pi\rho g a b^2$, would give the comparison a different aspect.

As mentioned earlier, it has been shown by both Pond and Havelock that this approximate treatment of the boundary condition on the body is inadequate for computation of the moment. Figure 22 is from Pond [1951] and shows the computed moment about the center for a Rankine ovoid, i.e., for the body generated in an unbounded fluid by a source and sink of equal strengths moving in the direction of their axis. The dashed curves show the result with the approximate computation; the solid curves were computed by a method in which the boundary condition on the body is more closely satisfied. The length ℓ of the body is 10.5 times the maximum diameter $d = 2b$. A positive moment is nose-up.

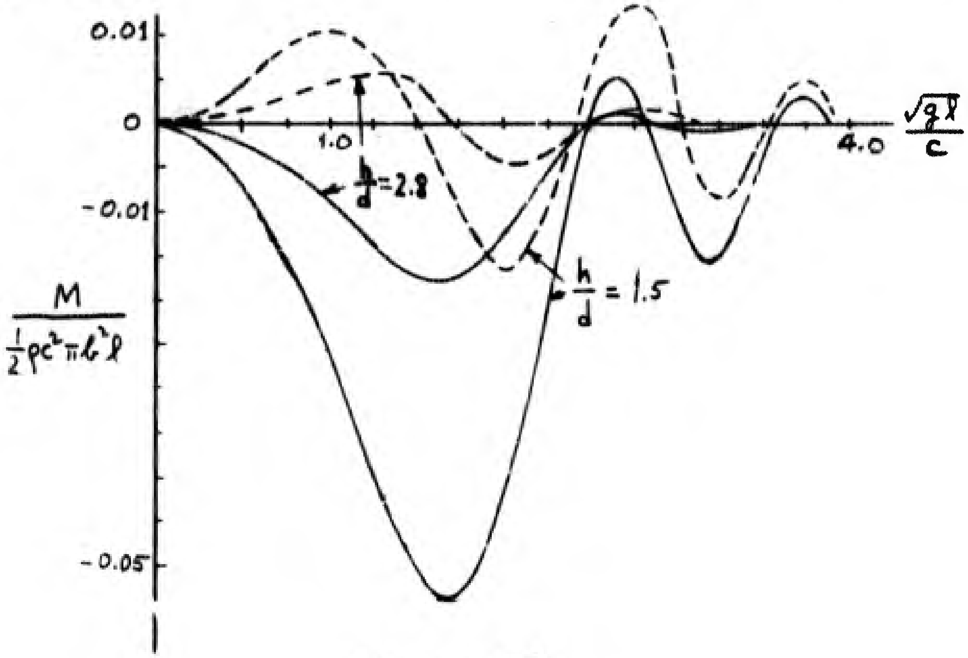

Figure 22

Slender bodies. It is known that, for bodies of revolution given in the form (20.34), the slender-body boundary condition (20.35) can be satisfied in an infinite fluid by a distribution of sources along the axis of strength density

$$\gamma(x) = \frac{1}{4} c \frac{d}{dx} r^2(x). \tag{20.60}$$

If one assumes that this same distribution of the modified sources (13.36) will satisfy approximately (20.35), then

$$\varphi(x,y,z) = \int_{-a}^{a} \gamma(\xi)\, G(x,y,z;\xi,-h,0)\, d\xi \tag{20.61}$$

and

$$H(k,\theta) = -4\pi e^{-kh} \int_{-a}^{a} e^{ik\xi\cos\theta}\, \gamma(\xi)\, d\xi . \tag{20.62}$$

From this one finds easily from (20.37)

$$\begin{aligned} R = -X &= 16\pi\rho\nu^2 \int_0^{\pi/2} d\theta \sec^3\theta\, e^{-2\nu h\sec^2\theta} \int_{-a}^{a}\int_{-a}^{a} d\xi\, d\xi' \cos[k(\xi-\xi')\sec\theta]\,\gamma(\xi)\gamma(\xi') \\ &= 16\pi\rho\nu^2 \int_0^{\frac{\pi}{2}} \sec^3\theta\, e^{-2\nu h\sec^2\theta} [\,P^2 + Q^2\,]\, d\theta , \end{aligned} \tag{20.63}$$

where

$$P(\theta) = \int_{-a}^{a} \gamma(\xi)\cos(\nu\xi\sec\theta)\, d\xi, \quad Q(\theta) = \int_{-a}^{a} \gamma(\xi)\sin(\nu\xi\sec\theta)\, d\xi .$$

As mentioned earlier, Weinblum [1951] has published tables allowing one to compute R for γ's representable as certain polynomials. An earlier paper [1936] considers the minimization of R among certain classes of polynomial γ's. Pond [1952] treats the necessary refinements to (20.61) in order to compute the moment.

<u>Thin ships.</u> Let the equation of the hull be given as in (20.22) by $z = \pm F(x,y)$ in a coordinate system moving with the ship, which we take to move with constant velocity c in the direction Ox. If we assume that a steady state has been reached, the boundary condition for

the hull appropriate to the thin ship approximation is, from (20.26) with $\phi(x,y,z,t) = \varphi(x-ct, y, z)$ and a change to a moving coordinate system,

$$\varphi_z(x, y, \pm 0) = \mp c\, F_x(x, y) \qquad (20.64)$$

for $(x, y, 0)$ in S_o, the centerplane section of the ship at rest. For $(x, y, 0)$ not in S_o one has $\varphi_z(x, y, \pm 0) = 0$ from symmetry considerations. φ must, of course, also satisfy (20.1) and the condition of vanishing motion as $x \to \infty$.

The boundary conditions may be satisfied immediately by distributing sources (13.26) over S_o. If we again denote the potential function in (13.36) by $G(x, y, z; \xi, \eta, \zeta)$, then, for infinitely deep fluid, the solution is

$$\varphi(x, y, z) = \frac{c}{2\pi} \iint_{S_o} G(x, y, z; \xi, \eta, 0)\, F_x(\xi, \eta)\, d\sigma. \qquad (20.65)$$

This follows easily from known theorems in potential theory [see, e.g., Kellogg, Foundations of potential theory, Springer, Berlin, 1929, pp. 160-166]. (The part of G regular in $y \leq 0$ does not interfere with the satisfying of (20.64) since the z-derivative of these terms vanishes for z = 0.)

The quantity of chief interest is the resistance resulting from the waves. This may be computed by using again (20.56) and (20.37) (and remembering to take account of both halves of the hull), or by direct integration of the pressure over the hull after taking account of linearization, i.e.

$$R = 2\rho c \iint_{S_0} \varphi_x(x,y,0)\, F_x(x,y)\, dx\, dy \qquad (20.66)$$

If the latter form is used, only the single-integral term in G gives a non-vanishing contribution. In either case one finds immediately, again for infinitely deep fluid,

$$R = \frac{4q^2\rho}{\pi c^2}\int_0^{\pi/2} \sec^3\theta\,[P^2(\theta) + Q^2(\theta)]\,d\theta,$$

$$P = \iint_{S_0} F_x(x,y)\, e^{\nu y \sec^2\theta} \cos(\nu x \sec\theta)\, dx\, dy, \qquad (20.67)$$

$$Q = \iint_{S_0} F_x(x,y)\, e^{\nu y \sec^2\theta} \sin(\nu x \sec\theta)\, dx\, dy .$$

The result may be, and has been, put into a variety of different forms by change of variable and order of integration. We give one of them. Let $\lambda = \sec\theta$. Then one may verify that

$$R = \frac{4q^2\rho}{\pi c^2}\iint_{S_0} dx\, dy \iint_{S_0} d\xi\, d\eta\, F_x(x,y) F_\xi(\xi,\eta)\, M(\nu(x-\xi), \nu(y+\eta)), \qquad (20.68)$$

$$M(x,y) = \int_1^\infty \frac{\lambda^2}{\sqrt{\lambda^2-1}} e^{\lambda^2 y} \cos\lambda x\, d\lambda .$$

This expression for R in terms of the hull form and velocity was first given by Michell [1898], but derivations by different methods have since been given by many other, e.g. Havelock [1932, 1951], Sretenskii [1937], Kochin [1937], Lunde [1951], and Timman and Vossers [1955]. It is usually called "Michell's integral."

Because Michell's integral gives R directly in terms of the hull geometry it has been intensively investigated by several persons in order to throw light upon the influence of variations of hull form upon wave resistance. Foremost among these investigators has been Havelock, who in a series of notable papers [1923, 1925 a,b, 1926 a, 1932 a, b, 1935] studied the effects of various systematic variations described by the titles of the papers. Much of this work is summarized in Havelock [1926]. In addition, there are numerous papers by G. P. Weinblum and W. C. S. Wigley devoted to comparison of experiment and theory. One can find surveys of much of this and related work, as well as further bibliography, in Wigley [1930, 1935, 1949], Weinblum [1950], Havelock [1951], Lunde [1957], and Wehausen [1957]. Lunde's 1951 paper contains derivations of practically all the general theoretical results, including the effect of finite depth, walls, and acceleration. Takao Inui [1954] has given an extensive survey of Japanese investigations on wave resistance and related topics, and in a later paper [1957] a complete survey.

In order to allow better exploitation of Michell's integral much attention has been give in recent years to its numerical computation. One can find a general discussion in Birkhoff and Kotik [1954], and various special proposals in Kabachinskii [1947], Reinov [1951], Guilloton [1951] and Weinblum [1955]. The last two papers both contain sets of tables to be used in evaluating Michell's integral.

In making a comparison of the theoretically predicted wave resistance with measured wave resistance one must examine critically the experimental method for estimating the wave resistance. The standard method consists in measuring the total resistance, estimating the part of the total resulting from the effects of visosity, and attributing the difference to wave making. Thus the accuracy of the experimentally estimated wave resistance depends upon the accuracy of the estimated viscous resistance and upon the validity of the assumption that the two may be added. In the case of a very thin body this estimate can be made accurately and, in addition, the physical assumption in the thin-ship linearization is well realized. Figure 23 from a report by Weinblum, Kendrick and Todd [1952] shows a comparison between estimated and computed values of $R_w/\frac{1}{2}\rho c^2 S$ for a towed "friction plane" 21 feet long with parabolic ends and 3 foot draft. These experimental data present Michell's integral in a most favorable light. For more ship-like forms the separation of viscous from wave-making resistance is more difficult and the compared

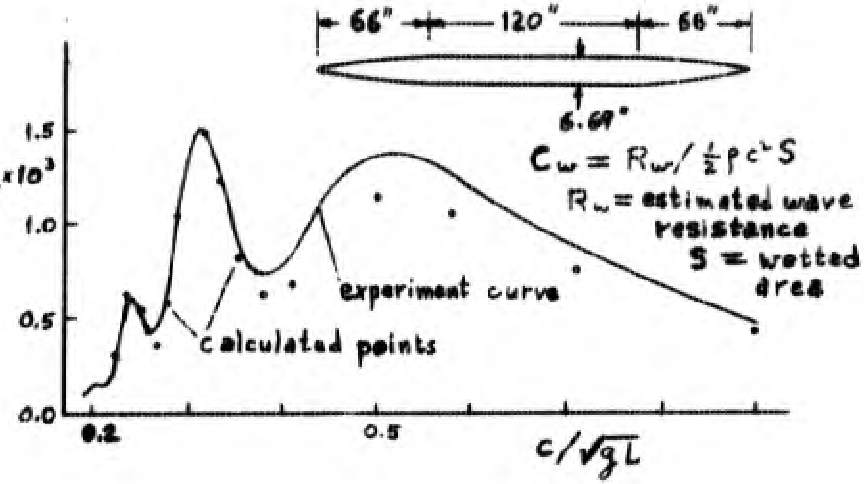

Figure 23

values seldom show such striking quantitative agreement, although it is still fai.̄ in many cases. We call attention to the fact that M'chell's integral predicts the same wave-making resistance no matter in which direction the ship moves.

So far we have discussed vessels moving in an infinitely deep fluid. However, if for our function G we had taken (13.37) instead of (13.36), the same analysis would have lead us to the following expression, first given by Sretenskii [1937]:

$$R = \frac{2\rho g c}{\pi}\int_{\mu_1}^{\infty}\left[P^2(\mu) + Q^2(\mu)\right]\sqrt{\frac{\mu}{\mu - \nu \tanh \mu h}}\, d\mu,$$

$$P(\mu) = \iint_{S_0} F_x(x,y)\,\frac{\cosh \mu(y+h)}{\cosh \mu h}\cos\left(x\sqrt{\nu\mu \tanh \mu h}\right) dx\, dy, \tag{20.69}$$

$$Q(\mu) = \iint_{S_0} F_x(x,y)\,\frac{\cosh \mu(y+h)}{\cosh \mu h}\sin\left(x\sqrt{\nu\mu \tanh \mu h}\right) dx\, dy .$$

Here μ_h is the nonzero solution of $\mu = \nu \tanh \mu h$ if such exists, i.e. if $c^2/gh > 1$; otherwise $\mu_h = 0$. As $h \to \infty$, $\mu_h \to \nu$ and one obtains one of the forms of Michell's integral.

An expression for the wave resistance of a thin ship moving down the center of a rectangular canal was derived independently by Sretenskii [1936, 1937] and Keldysh and Sedov [1937]. The result may be found in Lunde [1951].

One may naturally ask how the wave pattern illustrated in Figure 1 for a moving source is related to that for a ship. In the thin-ship approximation, the ship is replaced by a distribution of sources on the centerplane, so that each infinitesimal area of the centerplane contributes to such a pattern according to its strength. However, in many large vessels the middle part of the ship is cylindrical, so that $F_x = 0$ in this region and only the bow and stern regions contribute a nonvanishing source density. Thus, if one replaces the ship by a single source in the bow region and a single sink in the stern region, the resulting wave pattern will approximate to that of a ship, the approximation being better at higher values of the Fronde number $c/\sqrt{gL}$. Depending upon the value of $c/\sqrt{gL}$, the transverse wave systems from the two singularities may either reinforce or partially cancel one another. When they are in phase, a larger amount of

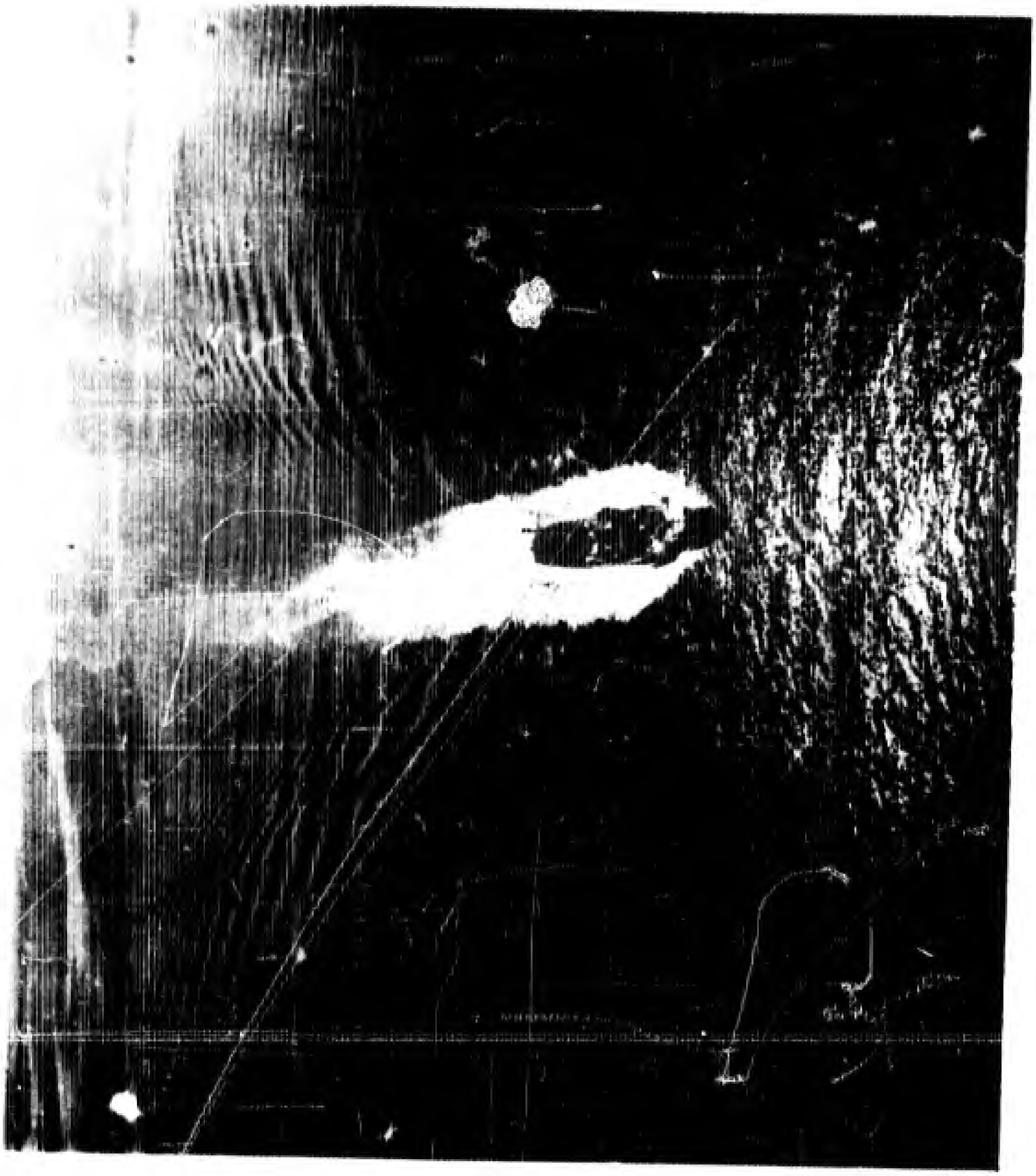

Figure 24

energy is being left behind in the wave system and the resistance curve shows a maximum, when they are out of phase a minimum, the so-called "humps and hollows" of the resistance curve; these show clearly in Figure 23. Replacing the ship by a source and sink is, of course, a gross simplification. However, it serves to explain qualitatively certain aspects of a ship's wave pattern and resistance curve, and, in fact, can be given a certain amount of validity as an approximate computation of Michell's integral for sufficiently large $c/\sqrt{gL}$. For very large values of $c/\sqrt{gL}$, the wave length of the transverse waves along the path, $2\pi c^2/g$, becomes much larger than L and one may approximate the ship by a dipole. Many photographs of the wave pattern made by a fast motor boat fall into this class. The photograph reproduced in Figure 24 shows clearly the diverging waves from the bow and stern, a third set possibly originating at the forward shoulder, and also the transverse waves, which presumably are here nearly in phase.

The angular opening of the wedge containing the wave pattern should be, according to (13.42) and Figure 1, 38° 56' in deep water. This is usually confirmed approximately in photographs, although difficulty in determining boundaries makes precise confirmation difficult. For ships moving in water of finite depth h the angular opening changes as shown in Figure 2, and for supercritical velocities, i.e. $c^2 > gh$, there are no transverse waves.

Jinnaka [1957] has recently published a brief survey of the theory of ship waves.

Thin hydrofoils. We take the hydrofoil as described in (20.29) and treat the problem two-dimensionally. Assuming constant velocity c and steady motion and taking our coordinate system moving with the hydrofoil, the boundary condition (20.33) becomes

$$\varphi_y(x, -h \pm 0) = -c\,r'(x) \mp c\,s'(x), \quad -a \leq x < a. \tag{20.70}$$

This problem has been treated very thoroughly by Keldysh and Lavrent'ev [1936]. They follow a procedure quite analogous to that used in section 19α to find the waves generated by a vertical oscillator not in a wall. Distribute vortices of intensity $\gamma(x)$ and sources of strength $\sigma(x)$ along the line $a \leq x \leq a$, $y = -h$, but taking them, of course, modified as in (13.43) in order to satisfy the free-surface condition and the conditions at infinity. To start with, we take $\sigma(x) = -2c\,s'(x)$. It then follows from the theorem of Plemelj-Sokhotskii (cf. 17.18) that

$$\varphi_y(x, -h+0) - \varphi_y(x, -h-0) = -2c\,s'(x), \tag{20.71}$$

a step toward satisfying (20.70). We now look for a complex potential in the form

$$f(z) = \int_{-a}^{a} \left[-2c\,s'(\xi) f_s(z; \xi - ih) + \gamma(\xi) f_v(z; \xi - ih)\right] d\xi, \tag{20.72}$$

where we have separated the source and vortex potentials in (13.43). The boundary condition (20.70) now yields an integral equation for $\gamma(x)$ in much the same manner that (17.18) was derived:

$$\operatorname{Im} \int_{-a}^{a} \left[-2c\,s'(\xi) f_s'(x - ih; \xi - ih) + \gamma(\xi) f_v'(x - ih; \xi - ih)\right] d\xi = c\,r'(x), \quad -a \leq x \leq a. \tag{20.73}$$

Noting from the third expression in (13.43) that f_s and f_v are functions of the difference $x - \xi$, we define

$$2\pi i f_s'(x-ih_1-ih) = i\left[\frac{1}{x} - \frac{1}{x-2ih}\right] - 2\nu e^{-i\nu x}\int_{\infty}^{x}\frac{e^{i\nu t}}{t-2ih}\,dt =$$

$$= H(x) + i\,J(x),$$

$$2\pi i f_v'(x-ih_1-ih) = \frac{1}{x} + \frac{1}{x-2ih} - 2i\nu e^{-i\nu x}\int_{\infty}^{x}\frac{e^{i\nu t}}{t-2ih}\,dt$$

$$= K(x) + i\,I(x).$$

Here, for example,

$$K(x) = \frac{1}{x} + \frac{x}{x^2+4h^2} + 2\nu\int_{\infty}^{x}\frac{2h\cos\nu(t-x) + t\sin\nu(t-x)}{t^2+4h^2}\,dt\,.$$

The integral equation (20.73) can now be written in the following form:

$$\int_{-a}^{a}\gamma(\xi)K(x-\xi)\,d\xi = -2\pi c\,r'(x) + \int_{-a}^{a}2c\,s'(\xi)H(x-\xi)\,d\xi, \tag{20.74}$$

where the right-hand side is a known function.

This integral equation is the hydrofoil analogue of the thin-wing integral equation of airfoil theory:

$$\int_{-a}^{a}\gamma(\xi)\frac{1}{x-\xi}\,d\xi = -2\pi c\,r'(x)\,. \tag{20.75}$$

In the latter equation the kernel is simpler and in addition only the function r' describing the camber and the angle of attack occurs on the right side. Since the wing thickness does not enter into the determination of γ in (20.75) it may be neglected, for only γ is needed to find the lift.

The situation is clearly different for hydrofoils. Even a symmetric wing with zero angle of attack may have a circulation, and hence lift. This is a consequence, of course, of the presence of the free surface and the associated wave motion.

Kochin [1936] has also considered hydrofoils, but from a somewhat different viewpoint. He has essentially used the "deep-submersion" linearization described first in this section. Thus he must satisfy the exact boundary conditions on the wing as well as the Kutta-Joukowski condition. However, one cannot say here, as one could for an infinite fluid, that his method is more exact than that of Keldysh and Lavrent'ev. Their approximation is more accurate the thinner the wing, for a given submersion. Kochin's is more accurate the deeper the submersion, for a given wing.

Equation (20.74) is not sufficient to determine $\gamma'(x)$ uniquely. One must still add some further condition. We shall assume a finite velocity at the trailing edge, i.e.

$$\varphi_y(-a-0,-h)=\int_{-a}^{a}[\gamma(\xi)K(-a-\xi)-2cs'(\xi)H(-a-\xi)]d\xi \quad \text{finite,} \qquad (20.76)$$

Keldysh and Lavrent'ev propose solving the integral equation (without actually proving that a solution exists) by expanding K, H and γ in a power series in $\tau=a/\nu h$ and then determining recursively the coefficients. Let

$$K(x)=\frac{1}{x}+\frac{\tau}{a}\sum_{n=0}^{\infty}K_n(2\nu h)\tau^n\left(\frac{x}{a}\right)^n, \qquad (20.77)$$

$$H(x)=\frac{\tau}{a}\sum_{n=0}^{\infty}H_n(2\nu h)\tau^n\left(\frac{x}{a}\right)^n,$$

$$\gamma(x)=\sum_{n=0}^{\infty}\gamma_n(x)\tau^n.$$

Then (20.74) gives the following sequence of integral equations.

$$\int_{-a}^{a} \gamma_0(\xi)\frac{d\xi}{x-\xi} = -2\pi c\, r'(x),$$

$$\int_{-a}^{a} \gamma_1(\xi)\frac{d\xi}{x-\xi} = \frac{1}{a} H_0 \int_{-a}^{a} 2c\, s'(\xi)\, d\xi, \qquad (20.78)$$

$$\cdots$$

$$\int_{-a}^{a} \gamma_{n+1}(\xi)\frac{d\xi}{x-\xi} = \frac{1}{a^{n+1}} H_n \int_{-a}^{a} 2a\, s'(\xi)(x-\xi)^n d\xi - \sum_{k+l=n} \frac{1}{a^{l+1}} K_l \int_{-a}^{a} \gamma(\xi)(x-\xi)^l d\xi,$$

$$\cdots$$

This procedure has the obvious advantage of reducing the solution to the airfoil integral equation for which an explicit solution satisfying the trailing-edge condition is known. If we denote temporarily the right-hand sides of the equations (20.78) by $F_n(x)$, respectively, then the general solution is [see, e.g., W. Schmeidler, Integralgleichungen . . . , Akademische Verlagsgesellschaft, Leipzig, 1950, pp. 55-56]

$$\gamma_n(x) = \frac{1}{\pi^2\sqrt{a^2-x^2}}\left[\int_{-a}^{a} \frac{F_n(\xi)\sqrt{a^2-\xi^2}}{\xi - x}\, d\xi + \pi\int_{-a}^{a} \gamma_n(\xi)\, d\xi\right], \qquad (20.79)$$

where the value of $\int_{-a}^{a}\gamma_n\, d\xi$ is undetermined. In terms of the series expansion, condition (20.76) states that

$$\sum_{n=0}^{\infty} \tau^n \int_{-a}^{a} \gamma_n(\xi)\frac{d\xi}{x-\xi} + \sum_{n=1}^{\infty} \tau^n \sum_{k+l=n-1} a^{-l-1} K_l \int_{-a}^{a} \gamma_k(\xi)(x-\xi)^l d\xi$$

$$-\sum_{n=1}^{\infty} a^{-n-1} H_{n-1} \int_{-a}^{a} 2c\, s'(\xi)(x-\xi)^{n-1} d\xi, \qquad (20.80)$$

must remain finite for $x \to -a$. We assume $s'(-a)$ finite. γ_k may possibly have a singularity of the form $1/\sqrt{a+x}$ near $x=-a$. However, the integral

$$\int_{-a}^{a} \frac{(x-\xi)^l}{\sqrt{a^2-\xi^2}}\, d\xi = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (x - a\sin\alpha)^l \, d\alpha,$$

is a polynomial in x for $l \geq 0$. Thus the last two summands of (20.80) remain finite at $x=-a$. However, the first summand potentially contributes terms like

$$\int_{-a}^{a} \frac{d\xi}{(x-\xi)\sqrt{a^2-\xi^2}} = \frac{\pi}{\sqrt{x^2-a^2}} .$$

In order to avoid this singularity at $x=-a$, we select the total circulation $\int_{-a}^{a}\gamma_n \, d\xi$ so that

$$\int_{-a}^{a} \gamma_n(\xi)\, d\xi = -\frac{1}{\pi}\int_{-a}^{a} \frac{F_n(\xi)\sqrt{a^2-\xi^2}}{\xi+a}\, d\xi . \qquad (20.81)$$

Substituting into (20.79), one finds finally

$$\gamma_n(x) = \frac{1}{\pi^2}\sqrt{\frac{a+x}{a-x}} \int_{-a}^{a} F_n(\xi)\sqrt{\frac{a-\xi}{a+\xi}}\, \frac{d\xi}{\xi - x} . \qquad (20.82)$$

$\gamma(x)$ itself is given by the sum displayed in (20.77). Although the singularity at the trailing edge has been removed, there is still one at the leading edge; this occurs also in thin-airfoil theory and corresponds roughly to the fact that the conditions of linearzation (i.e. of small disturbance) are not satisfied near the leading edge stagnation point.

Keldysh and Lavrent'ev compute the integrals which will be necessary if $r(x)$ and $s(x)$ are given as polynomials and apply their computational method to a flat-plate and circular-arc airfoil at a small angle of attack.

In order to find the force and moment on the wing it is convenient to fall back on the H-function. One finds easily

$$H(k) = e^{-hk}\int_{-a}^{a}[\gamma(\xi) - 2ics'(\xi)]e^{-i\lambda s}\,ds\,,$$

$$|H(k)|^2 = e^{-2hk}\int_{-a}^{a}\int_{-a}^{a}\{[\gamma(\xi)\gamma(x) + 4c^2 s'(\xi)s'(x)]\cos k(\xi - x) + 2c[\gamma(\xi)s'(x) - \gamma(x)s'(\xi)]\sin k(\xi - x)\}\,d\xi\,dx. \quad (20.83)$$

Formulas (20.38) allow one to complete the calculation for special cases.

The theory analogous that that described above for fluid of finite depth h_0 has been carried through by Tikhonov [1940]. He has applied the method to calculate the lift and drag coefficients for a flat plate at a small angle of attack.

Rather than reproduce the graphical presentations of Keldysh and Lavrent'ev and Tikhonov for the flat plate, we shall give instead the lift and drag coefficients for a submerged vortex. Here one may give relatively simple analytic expressions, and the qualitative behavior of the curves is similar to that of a flat plate. The formulas for lift L and drag D are as follows:

$$D_\infty = \rho \nu \Gamma^2 e^{-2\nu h},$$

$$L_\infty = \rho \Gamma c - \frac{\rho \Gamma^2}{4\pi h} + \frac{\rho \Gamma^2}{\pi} \nu e^{-2\nu h} Ei(2\nu h), \tag{20.84}$$

$$D_{h_0} = \rho \nu \Gamma^2 \frac{\sinh^2 m_0(h_0 - h)}{\nu h_0 - \cosh^2 m_0 h_0} \quad \text{if } \nu h_0 > 1, \; = 0 \text{ if } \nu h_0 < 1,$$

$$L_{h_0} = \rho \Gamma c - \frac{\rho \Gamma^2}{4\pi} \frac{1}{h_0 - h} \cdot \frac{\rho \Gamma^2}{2\pi} \int_0^\infty (\nu + k) e^{-k h_0} \frac{\sinh 2k(h_0 - h)}{\nu \sinh k h_0 - k \cosh k h_0} \, dk,$$

where m_0 is the real root of $m = \nu \tanh m h_0$. For finite depth the expression for D stems from the last term in (13.47). The dimensionless coefficients $C_D = Dh/\rho\Gamma^2$ and $C_L = (L - \rho c \Gamma) h/\rho \Gamma^2$ are shown in Figure 25a for infinite depth as functions of c^2/gh and in Figures 25b and 25c as functions of c^2/gh_0 for various values of $\beta = h/h_0$. For infinite depth C_L starts with a value $1/4\pi$ and tends asymptotically to $-1/4\pi$, crossing the axis at $c^2/gh = 2.47$. For finite depth the coefficients have a discontinuity at $c^2/gh_0 = 1$. As $c^2/gh \to 0$, $C_D \to 0$, and as $c^2/h_0 \to 1$, $C_D \to \frac{3}{2}\beta(1-\beta)^2$. For $c^2/gh_0 > 1$, C_L is always negative and increasing with a vertical asymptote at $c^2/gh_0 = 1$ and a horizontal one as $c^2/gh_0 \to \infty$ at $-\beta/4 \sin \beta\pi$; these curves start at $\frac{1}{4}\beta \cot \beta\pi$.

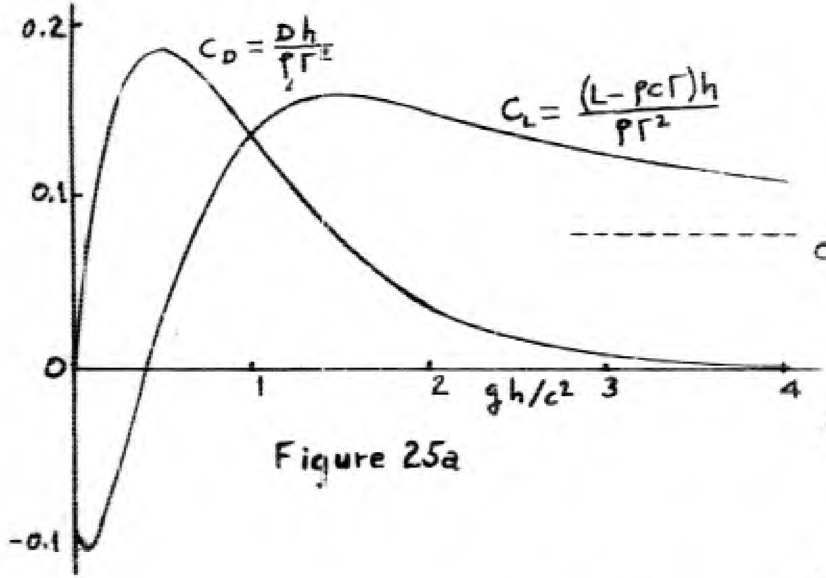

Figure 25a

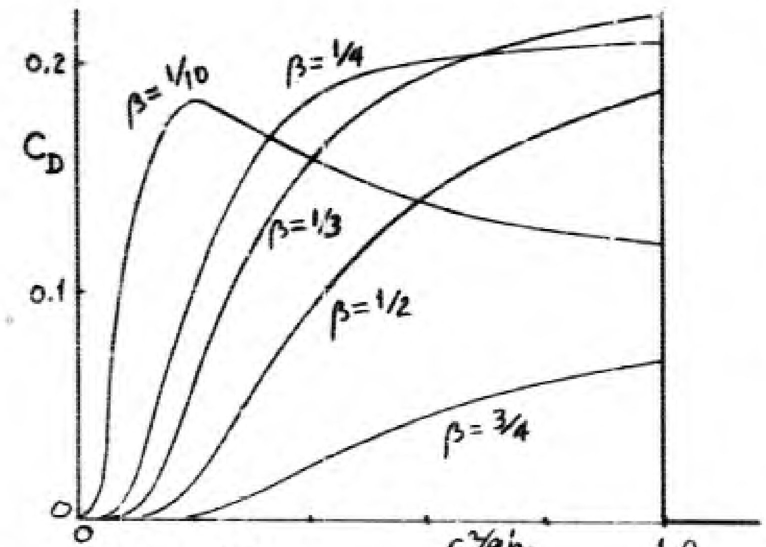

Figure 25b

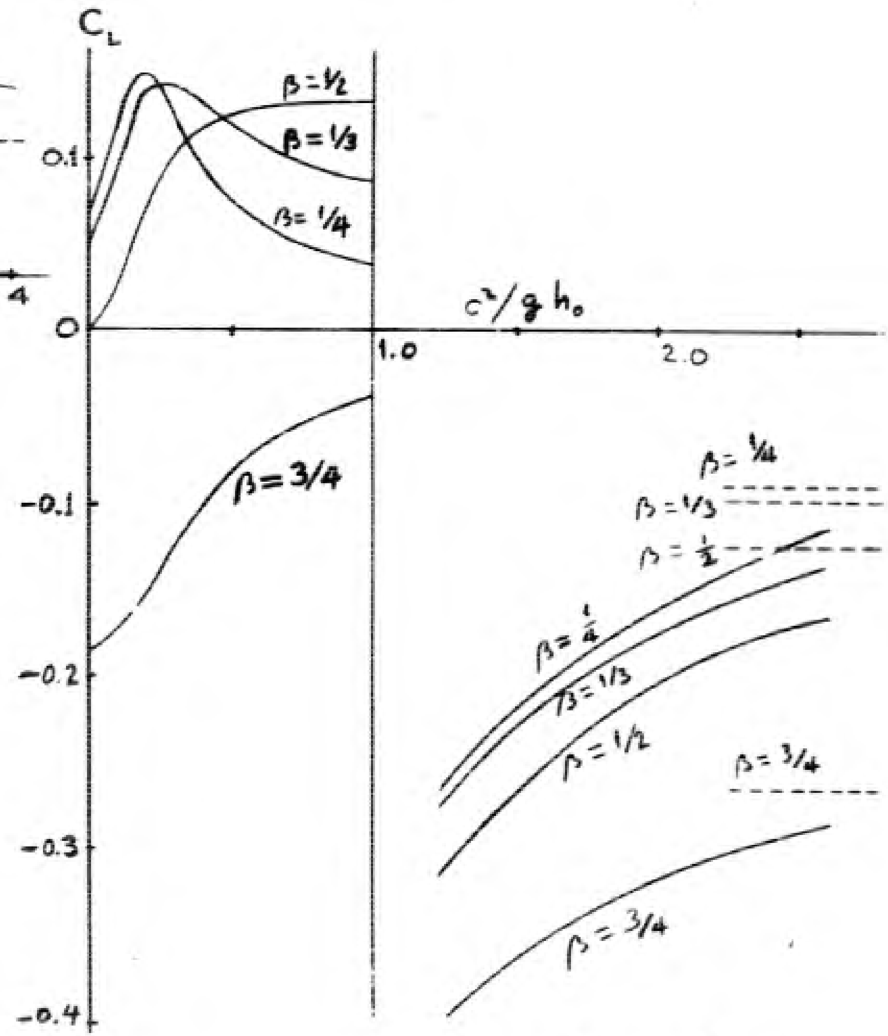

Figure 25c

Further development of hydrofoil theory has taken place in several directions. Haskind [1945a] has extended Kochin's "deep-submersion" theory to water of finite depth. However, he does not discuss the steps necessary for fulfilment of the Kutta-Joukowski condition, as does Kochin. The lifting-line theory for airfoils of finite span has been extended to hydrofoils by Wu [1954], Breslin [1957], and Haskind [1956]. Parkin, Perry and Wu [1956] and Laitone [1954,1955] have investigated both theoretically and experimentally the effect of bringing a given hydrofoil so close to the surface that the infinitesimal-wave approximation breaks down completely. There exists also a considerable amount of work on flow about cavitating hydrofoils. However, since the effect of gravity is neglected, this work is not considered in the present article. Experimental data relevant to the theoretical development outlined above is scanty. Reports by Benson and Land [1942] and by Land [1943] give results of an experimental investigation of the effect of depth of submersion. However, the investigations were not designed to test the validity of the theory and do not, for example, include the region of maximum C_D . Ausman [1953], in connection with an experimental investigation of the pressure distribution on the upper surface of a hydrofoil, measured the lift coefficient and compared it with that predicted by the thin-hydrofoil theory. The theory failed when gh/c^2 became too small because the associated free surface over the hydrofoil no longer approximated infinitesimal waves, or, in other words, the thin hydrofoil was not thin enough for these values of gh/c^2 for the theory to be applicable. It should also be emphasized that for small values of gh/c^2 the occurrence of cavitation on the upper surface must be taken into account for a complete theory.

A comprehensive survey of hydrofoil theory is given in a recent paper by Nishiyama [1957].

20γ . Planing surfaces.

The following discussion is limited to two-dimensional motion, for the theory of three-dimensional planing surfaces for flows with gravity does not appear to have been developed.

For the linearized problem it is natural to consider the planing surface or glider as an approximation to a flat plate moving along the surface of the undisturbed fluid, i.e. the curvature, angle of attack and vertical displacement are all assumed small. In order to formalize the perturbation procedure, let the planing surface be represented by

$$y = h + F(x), \quad |x| \leq a, \quad F(-a) = 0, \qquad (20.85)$$

in coordinates fixed in space, and let the fluid have velocity $-c$ at $x = +\infty$. Thus, we are going to consider the flow to be a perturbation of a uniform flow.

First let us consider briefly in a qualitative fashion the exact solution. There will be a stagnation point A somewhere behind the leading edge and a jet will be thrown out ahead of the glider. We take it to be of thickness b and to make an angle β with OX. If $\Phi = -cx + \varphi(x,y)$ and $\Psi = -cy + \psi(x, y)$ are potential and stream function, respectively, we shall take the free surface ahead of the glider to be given by Ψ = -bc and behind the glider by Ψ = 0 (see Figure 26).

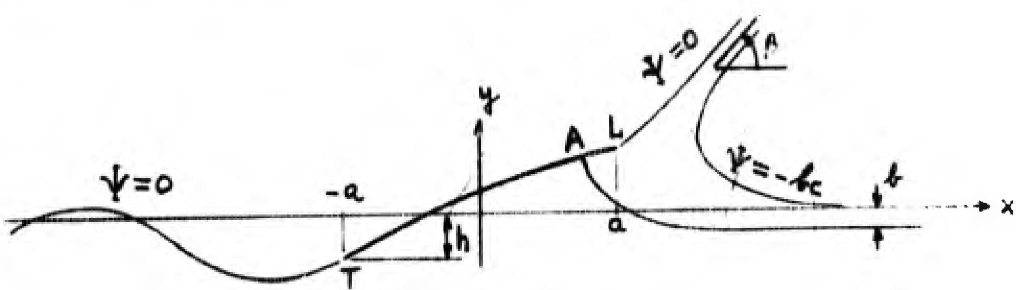

Figure 26

Then b/a, and AL/a will all be functions of ga/c^2 and gh/c^2. It will be assumed as one of the boundary conditions of the problem, in analogy with the Kutta-Joukowski condition, that the velocity is continuous at the trailing edge. It is obvious that the flow near the leading edge cannot conform to the requirement that it be a small perturbation of a uniform flow. However, we shall give arguments below to indicate that, except in the neighborhood of the leading edge, this effect is of the second order.

In order to get some idea of the relative size of the jet, consider the simpler problem of a flat plate of length ℓ and angle of attack α gliding on a weightless fluid. This problem can be solved exactly, (see, e.g., Milne-Thomson [1956, § 12.26]; A.E. Green [1935, 1936]). The asymptotic expression, for small α, of both the ratios b/ℓ and AL/ℓ s $\frac{\pi}{2}\frac{\alpha^2}{1+\cos\beta}$, i.e. they are both of the second order. We shall suppose that this relation continues to hold when gravity is acting.

We now carry through the perturbation procedure of section 10 α (see especially (10.16)), writing

$$\phi = -cx + \varepsilon\varphi^{(1)} + \cdots, \quad \Psi = -cy + \varepsilon\psi^{(1)} + \cdots, \quad \eta = \varepsilon\eta^{(1)} + \cdots,$$

$$h + F(x) = \varepsilon F^{(1)}(x) + \varepsilon h^{(1)} + \varepsilon^2 h^{(2)} + \cdots, \qquad (20.86)$$

$$b = \varepsilon^2 b^{(2)} + \cdots .$$

Substitution in the exact boundary conditions then yields the following linearized conditions:

$$\psi^{(1)}(x,0) - \frac{c^2}{g}\psi_y^{(1)}(x,0) = 0, \quad |x| > a,$$

$$\psi^{(1)}(x,0) = c\left(h^{(1)} + F^{(1)}(x)\right), \quad |x| < a. \qquad (20.87)$$

The free surface is given by

$$\eta^{(1)}(x) = \frac{1}{c}\Psi^{(1)}(x,0) = \frac{c}{g}\varphi_x^{(1)}(x,0), \quad |x| > a. \qquad (20.88)$$

We require as usual that the disturbance vanish as $x \to \infty$. One will expect that the behavior near the leading edge will reflect in some manner the inconsistency of the exact solution with the notion of a small perturbation. It will turn out that it will be necessary to allow a singularity at the leading edge. (Almost the same situation exists in the thin-hydrofoil or thin-wing theory since the stagnation point near the leading edge also prevents the flow in that region from being a small perturbation of a uniform flow.)

A singularity at the trailing edge, although mathmatically possible, has been specifically proscribed. The strength of the singularity and the elevation h of the trailing edge will be determined as functions of ga/c^2 in the course of solving the problem,

The problem as formulated above has been considered, for infinite depth, by Sretenskii [1933, 1940], Sedov [1937], Kochin [1938], and Maruo [1951]. Haskind has extended Sedov's analysis to finite depth [1943a], and later [1955] has treated a glider moving on a wavy surface. Yu. S. Chaplygin [1940] has apparently carried through a fairly comprehensive numerical analysis for a flat plate making use of Sedov's method of analysis (see Sretenskii [1951, p. 83]). Sretenskii's papers are expounded in terms of a flat plate, but the method clearly has wider applicability, as he remarks in his first paper. Sretenskii's 1940 paper gives the results of rather extensive calculations for flat plates. Marno's paper is conceptually very similar to those of Sretenskii, but his method is not quite as efficient for computation. However, he also gives computational results and includes a correction to take account of the failure of the linearized theory near the leading edge.

Both Sedov and Kochin introduce the complex potential $f(z) = \varphi + i\psi$ and thereafter the function $f' + i\nu f$, $\nu = g/c^2$. Although the two methods are not by any means the same, they have much in common with the treatment of hydrofoils given above. Consequently, we shall outline below the method followed by Sretenskii

As a preliminary we need a result from section 21 below. Suppose that a pressure distribution $p(x)$, which we take to be absolutely integrable, is given on the free surface. Then the complex velocity potential must satisfy

$$\mathrm{Re}\left\{f'(x+i0) + i\nu f(x+i0)\right\} = \frac{1}{\rho c}\, p(x), \qquad (20.89)$$

and the free surface is given by

$$y = \eta(x) = \frac{1}{c}\,\psi(x,0) = \cdots\, \varphi_x(x,0) - \frac{1}{\rho g}\, p(x). \qquad (20.90)$$

The function $f(z)$ which satisfies (20.89) and which vanishes as $x \to \infty$ can be written in several forms, of which we select the following (see eq.(21.37)):

$$f(z) = \frac{1}{\pi i \rho c}\int_{-\infty}^{\infty} d\xi\, p(\xi)\; PV\!\int_0^{\infty} \frac{e^{-i\lambda(z-\xi)}}{\lambda-\nu}\, d\lambda - \frac{1}{\rho c}\int_{-\infty}^{\infty} p(\xi)\, e^{-i\nu(z-\xi)}\, d\xi \qquad (20.91)$$

The free surface is given by

$$\eta(x) = \lim_{y\to 0} \frac{1}{\pi \rho c^2}\int_{-\infty}^{\infty} d\xi\, p(\xi)\; PV\!\int_0^{\infty} \frac{\cos\lambda(x-\xi)}{\lambda-\nu}\, e^{\lambda y}\, d\lambda + \frac{e^{\nu y}}{\rho c}\int_{-\infty}^{\infty} p(\xi)\sin\nu(x-\xi)\, d\xi\,; \qquad (20.92)$$

the reason for leaving y explicitly in the formulas will appear below.

When a glider is moving on a free surface, the streamline $y = c^{-1}\psi(x,o)$ will consist partly of free surface, where $p(x) = 0$, and partly of the wetted surface of the glider, where $p(x)$ is some unknown function.

Equation (20.92) may then be written as the following integral equation for this unknown function $p(x)$:

$$h + F(x) = \lim_{y\to -0} \frac{1}{\pi \rho c^2}\int_{-a}^{a} d\xi\, p(\xi)\; PV\!\int_0^{\infty} \frac{\cos\lambda(x-\xi)}{\lambda-\nu}\, e^{\lambda y}\, d\lambda + \frac{e^{\nu y}}{\rho c}\int_{-\infty}^{\infty} p(\xi)\sin\nu(x-\xi)\, d\xi\,, \quad |x| < a\,. \qquad (20.93)$$

Once $p(x)$ has been determined, one may substitute back into (20.92) in order to find the form of the free surface for $|x| > a$.

It is possible to work directly with (20.93), and this is the procedure followed by Maruo. However, Sretenskii differentiates twice with respect to x and adds ν^2 times (20.93). This yields

$$F''(x)+\nu^2F(x)+\nu^2h = \lim_{y\to -0}\frac{-1}{\pi\rho c^2}\int_{-a}^{a}d\xi\, p(\xi)\int_0^\infty(\lambda+\nu)\cos\lambda(x-\xi)e^{\lambda y}d\lambda$$

$$= \lim_{y\to 0}\frac{1}{\pi\rho c^2}\int_{-a}^{a}p(\xi)\left(\nu+\frac{\partial}{\partial y}\right)\frac{y}{(x-\xi)^2+y^2}\,d\xi \qquad (20.94)$$

$$= -\frac{\nu}{\rho c^2}p(x) - \lim_{y\to 0}\frac{1}{\pi\rho c^2}\frac{\partial}{\partial x}\int_{-a}^{a}p(\xi)\frac{x-\xi}{(x-\xi)^2+y^2}\,d\xi$$

$$= \frac{-\nu}{\rho c^2}p(x) - \frac{1}{\pi\rho c^2}\frac{\partial}{\partial x}\,\mathrm{pv}\int_{-a}^{a}\frac{p(\xi)}{x-\xi}\,d\xi .$$

Although this last equation is a necessary condition for $p(x)$, it obviously cannot determine it uniquely, for the last term of (20.93), assuring vanishing of the disturbance far ahead of the glider, was lost in the formation of (20.94). Thus one still has need for (20.93). Equation (20.94) is essentially the equation derived by Sretenskii.

Let us now integrate (20.94) with respect to x from $x = -a$ to x, and denote

$$P(x) = \frac{1}{\rho c^2}\int_{-a}^{x}p(\xi)\,d\xi .$$

Then equation (20.94) becomes

$$F'(x) - F'(-a) + \nu^2\int_{-a}^{x}F(\xi)d\xi + \nu^2h(x+a) = -\nu P(x) - \frac{1}{\pi}\int_{-a}^{a}\frac{P'(\xi)}{x-\xi}\,d\xi , \qquad (20.95)$$

where an additive constant has been discarded since h itself is an undetermined constant. Equation (20.95) is just Prandtl's integro-differential equation for the circulation about an airfoil of finite span [see, e.g., N.I. Muskhelishvili , Singular integral equations, Noordhoff, Groningen, 1953, ch. 17]. Thus known methods for solving the airfoil equation can be carried over to the study of this equation. However, the solutions themselves cannot be taken over directly, for different boundary conditions are imposed: in the airfoil equation the unknown function is the circulation $\Gamma(x)$ and it is usually assumed that $\Gamma(-x) = \Gamma(x)$ and $\Gamma(-a) = \Gamma(a) = 0$; in the present problem $P(x)$ is not necessarily symmetric and P (-a) = P' (-a) = 0, but P (a) is not restricted except to be finite. The theory of the Prandtl integro-differential equation without the customary additional requirements associated with airplane wings has been developed by L.G. Magnaradze [Soobshch. Akad. Nauk Gruzin. SSR 3(1942), 503-508].

The equation can be solved by an extension of Glauert's method [e.g., H. Glauert, The elements of airfoil and airscrew theory, Cambridge, 1943, ch XI]. This is the method which has been used by both Maruo and Sretenskii.

However, each expands $P' = p$ rather than P in a Fourier series in order to obtain the correct behavior at the two end points. Introduce the new variables θ and γ by the equations

$$x = -a\cos\theta, \qquad \xi = -a\cos\gamma$$

and assume the following expansion for $p(x)$:

$$\frac{1}{\rho c^2}p(x) = \frac{1}{\rho c^2}p(-a\cos\theta) = a_0\tan\tfrac{1}{2}\theta + a_1\sin\theta + \cdots + a_n\sin n\theta + \ldots \quad (20.96)$$
$$= a_0\sqrt{\frac{a-x}{a+x}} + a_1\sqrt{a^2-x^2} + \cdots .$$

Maruo substitutes (20.96) into (20.93), Sretenskii into (20.94). The latter, which seems less laborious, leads to

$$a\left[F''(-a\cos\theta) + \nu^2 F(-a\cos\theta) + \nu^2 h\right]\sin\theta =$$
$$= -\nu a a_0(1-\cos\theta) - \nu a\sum_{n=1}^{\infty} a_n \sin\theta\sin n\theta - \sum_{n=1}^{\infty} n a_n \sin n\theta. \quad (20.97)$$

We shall not discuss Sretenskii's further steps to determine the coefficients a_n. However, they lead to expressions for the coefficients of the following kinds:

$$a_{2m-1} = A_{2m-1}a\nu^2 h + B_{2m-1}a\nu a_0 + C_{2m-1},$$
$$a_{2m} = \qquad\qquad\qquad B_{2m}a\nu a_0 + C_{2m}, \quad m = 1, 2, \ldots, \quad (20.98)$$

where A_n, B_n, C_n are functions of νa. Substitution of the coefficients into (20.93) and into (20.93) differentiated once with respect to x and evaluated at $x = -a$

results in equations of the form

$$\nu h = Q_1 a_0 + R_1 \nu h + S_1 \,, \qquad (20.99)$$

$$F'(-a) = Q_2 a_0 + R_2 \nu h + S_2 \,,$$

where Q_i , R_i , S_i are functions of νa ; these equations may be used to determine νh and a_0 as functions of νa . As long as $a_0 \neq 0$ there will be a singularity at the leading edge.

Once p(x) has been determined approximately, one can compute the lift, drag and moment about, say, the center. To the order of approximation appropriate to the linear theory they are

$$L = \int_{-a}^{a} p(x)\, dx \,,$$

$$R = \int_{-a}^{a} p(x) f'(x)\, dx \,, \qquad (20.100)$$

$$M = \int_{-a}^{a} p(x)\, x\, dx \,,$$

For the flat-plate glider it is possible to give the following asymptotic expressions for these quantities when $\nu a \to 0$.

(20.101)

$$L = \pi a \varrho c^2 \alpha \left[1 - \nu a\left(\pi + \tfrac{4}{\pi}\right)\right] + \Theta(\nu^2 a^2 \log \nu a), \quad R = \alpha L,$$

$$M = \tfrac{1}{2} \pi a^2 \varrho c^2 \alpha \left[1 - \nu a\left(\pi + \tfrac{8}{3\pi}\right)\right] + \Theta(\nu^4 a^4 \log \nu a).$$

These were first given by Sedov, but are also derived in the papers by Kochin and Sretenskii.

Figure 27 reproduces several of Sretenskii's computed pressure distributions for a flat plate. The predictions of the linearized theory cannot, of course, be expected to be accurate near the leading edge. Maruo has corrected his computed points in this region by using the exact theory for a weightless fluid. Both Maruo and Sretenskii give further computational results which we do not reproduce.

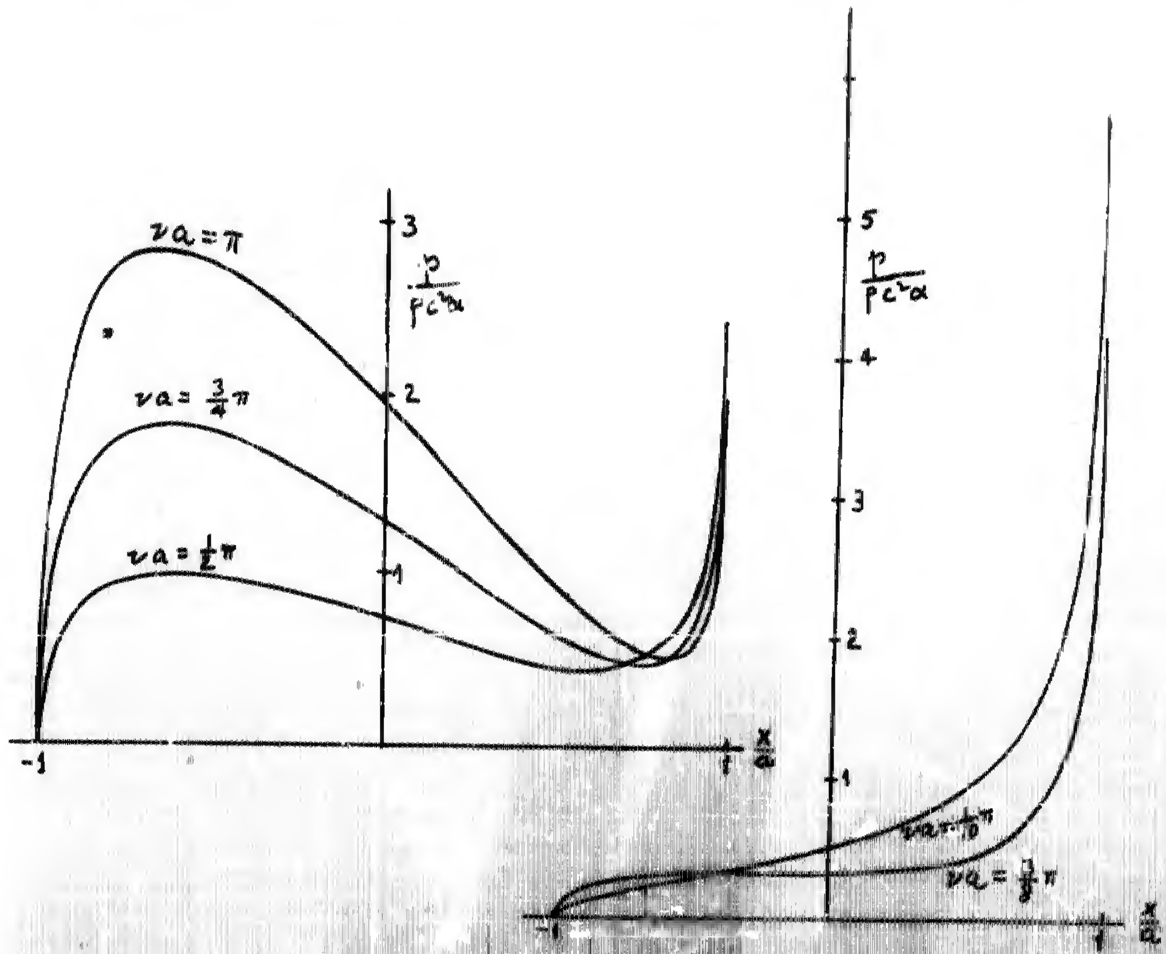

Figure 27

21. Waves resulting from variable pressure distributions

In the situations considered up to now in this chapter the pressure at the free surface has been taken as constant. We now consider the result of allowing the pressure over the free surface to be a given function of both position and time. Otherwise the fluid is taken to be infinite in horizontal extent and to be either infinitely deep or of uniform depth h. The time variation in pressure will be limited to two cases. In section 21 α a periodically varying pressure is considered; in section 21 β the pressure is taken to move with uniform velocity; section 21 γ gives some references to a combination of these two. In section 22 waves from pressure distributions will be considered again in connection with initial-value problems. Since the methods for finding the velocity potential are similar in most respects to those used in finding the velocity potential for a source, we shall, with one exception, give the results without proof.

Just as in the cases of the stationary source of periodic strength and the moving source of constant strength treated in section 13 γ, we must in the present situation impose boundary conditions at infinity in order to ensure a unique solution. The imposed conditions, namely the radiation condition and the vanishing of the fluid motion far ahead, respectively, are selected as being physically reasonable. However, one may proceed differently, derive formulas analogous to (13.50) and (13.51) and then find the limit as $t \to \infty$. The resulting velocity potentials automatically satisfy the correct boundary conditions at infinity. This method has been used, for example, by G. Green [1948] in the two-dimensional problems considered in the following two sections, and also by Stoker [1953, 1954].

The theory of wave generation by pressure distributions has an obvious application in oceanographic problems. However, the theory was apparently first developed in an attempt to explain the wave pattern produced by a ship. We shall not attempt to disentangle the history of the subject. For the material covered in section 21 β we call attention to a survey by J.K. Lunde [1951 b] which also contains a useful bibliography.

21 α. Pressure distributions periodic in time

Three dimensions. The boundary conditions have already been given in section 11. If ϕ and p are represented by

$$\phi(x,y,z,t) = Re\,\varphi(x,y,z)e^{-i\sigma t}, \quad p(x,z,t) = p(x,z)e^{-i\sigma t},$$
$$\varphi = \varphi_1 + i\varphi_2, \quad p = p_1 + ip_2, \tag{21.1}$$

then the condition on the free surface may be written

$$\varphi_y(x,0,z) - \frac{\sigma^2}{g}\varphi(x,0,z) = \frac{i\sigma}{\rho g}p(x,z), \tag{21.2}$$

and the form of the surface is given by

$$\eta(x,z,t) = Re\left\{\frac{i\sigma}{g}\varphi(x,0,z) - \frac{1}{\rho g}p(x,z)\right\}e^{-i\sigma t}. \tag{21.3}$$

In addition, a radiation condition is assumed at infinity (see (13.9), eq. 5) and a condition appropriate to the depth of fluid. We shall also assume $p(x,z)$ to be absolutely integrable.

The velocity potential can be expressed as follows:

$$\varphi(x,y,z) = \frac{-i\sigma}{2\pi\rho g}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\xi,\zeta)\,d\xi\,d\zeta \;\; PV\int_{0}^{\infty}\frac{k e^{ky}}{k-\nu}\,J_0\left(k\sqrt{(x-\xi)^2+(z-\zeta)^2}\right)dk$$

$$+\frac{\sigma\nu e^{\nu y}}{2\rho g}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\xi,\zeta)\,J_0\left(\nu\sqrt{(x-\xi)^2+(z-\zeta)^2}\right)d\xi\,d\zeta, \quad \nu=\frac{\sigma^2}{g}, \tag{21.3}$$

and in cylindrical coordinates $x = R\cos\alpha$, $z = R\sin\alpha$ in the form

$$\varphi(R,\alpha,y) = \frac{-i\sigma}{2\pi\rho g}\int_0^{2\pi}\int_0^{\infty} p(R',\alpha')R'\,dR'\,d\alpha' \;\; PV\int_{-\infty}^{\infty}\frac{k e^{ky}}{k-\nu}\,J_0\left(k\sqrt{R^2+R'^2-2RR'\cos(\alpha'-\alpha)}\right)dk$$

$$+\frac{\sigma\nu e^{\nu y}}{2\rho g}\int_0^{2\pi}\int_0^{\infty} p(R',\alpha')\,J_0\left(\nu\sqrt{R^2+R'^2-2RR'\cos(\alpha'-\alpha)}\right)R'\,dR'\,d\alpha'. \tag{21.4}$$

The addition theorem for J_0 [see Watson, A treatise on the theory of Bessel functions, Cambridge, 1944, p. 353] allows one to write

$$J_0\left(k\sqrt{R^2+R'^2-2RR'\cos(\alpha'-\alpha)}\right) = \sum_{n=0}^{\infty}\varepsilon_n J_n(kR)\,J_n(kR')\cos n(\alpha'-\alpha),$$

$$\varepsilon_0 = 1, \quad \varepsilon_n = 2, \quad n \geq 1. \tag{12.5}$$

If p is independent of α, one may derive easily

$$\varphi(R,y) = \frac{-i\sigma}{\rho g}\int_0^\infty p(R')R'\,dR' \quad PV\int \frac{k e^{ky}}{k-\nu} J_0(kR)J_0(kR')\,dk$$
$$+ \frac{\pi\sigma\nu e^{\nu y}}{\rho g} J_0(\nu R) \int_0^\infty p(R')J_0(\nu R')R'\,dR' . \tag{21.6}$$

The asymptotic form for large R of (21.6) is a relatively simple expression:

$$\varphi(R,y) \sim \frac{\pi\sigma\nu e^{\nu y}}{\rho g}\sqrt{\frac{2}{\pi\nu R}}\, e^{i(\nu R - \frac{\pi}{4})} \int_0^\infty p(R')J_0(\nu R')R'\,dR' . \tag{21.7}$$

We note in passing that the potential function (21.3) or (21.4) can also be obtained as a distribution of sources on the surface (see Hudimac [1953, p. 78]). This may be easily verified as follows. In (13.17") let $b = 0$. Then, using (13.12), one obtains (substituting ξ, ζ for a, c)

$$2\, PV\int^\infty \frac{k}{k-\nu} e^{ky} J_0\left(k\sqrt{(x-\xi)^2+(z-\zeta)^2}\right) dk + i\, 2\pi\nu e^{\nu y} J_0\left(\nu\sqrt{(x-\xi)^2+(z-\zeta)^2}\right).$$

A distribution of these sources over the plane $y = 0$ of strength $+i\sigma p(\xi,\zeta)/4\pi\rho g$ yields (21.3) (we recall that a source of strength m behaves like $-m/r$ near the singularity).

The rate at which the pressure distribution does work upon the fluid can be calculated directly or by using equation (8.2). Consider the volume of fluid contained in a large cylinder of radius R_0. Then, from (8.2), after appropriate linearization, the rate of increase of energy of the fluid is given by

$$\frac{dE}{dt} = \mathrm{Re}\left\{\int_0^{2\pi}\int_0^{R_0} p(R,\alpha,t)\,\phi_y(R,\alpha,0,t)\,R\,dR\,d\alpha + \rho\int_0^{2\pi}\int_{-\infty}^{0}\phi_t(R_0,\alpha,y,t)\,\phi_R(R_0,\alpha,y,t)\,R_0\,dy\,d\alpha\right\}.$$

Now substitute (21.1) and take the average over a period, which will clearly be zero. The result may be written:

$$0 = \left[\frac{dE}{dt}\right]_{av} = \mathrm{Re}\left\{-\frac{1}{2}\iint p(R,\alpha)\,\bar{\varphi}_y(R,\alpha,0)\,R\,dR\,d\alpha + \frac{1}{2}\rho\sigma\iint i\,\varphi_R(R_0,\alpha,y)\,\bar{\varphi}(R_0,\alpha,y)\,R_0\,dy\,d\alpha\right\}. \quad (21.8)$$

The first integral gives the average rate W_{av} at which the pressure distribution is working upon the fluid. It must equal minus the second integral. If $p(R,\alpha) = p(R)$, then we may apply (21.7) to obtain a relatively simple expression for the average rate over the whole fluid:

$$W_{av} = \frac{\pi^2\sigma\nu^2}{\rho g}\left|\int_0^\infty p(R')\,J_0(\nu R')\,R'\,dR'\right|^2. \quad (21.9)$$

To carry through the computation when p is not circularly symmetric is more complicated arithmetically, but can be carried through by use of (21.5).

One can find an investigation of the waves resulting from a doubly modulated pressure distribution over a rectangular domain,

$$p = Ae^{-i\sigma t}\cos mx\cos nz,\ |x| \le a,\ |z| \le b,$$

in a paper of Sretenskii [1956].

If the fluid is of uniform depth h, the expression for the velocity potential is

$$\varphi(x,y,z) = \frac{-i\sigma}{2\pi\rho g}\iint_{-\infty}^{\infty} p(\xi,\zeta)\,d\xi\,d\zeta\; PV\!\!\int_0^{\infty} \frac{k\cosh k(y+h)}{k\sinh kh - \nu\cosh kh}\, J_0\!\left(k\sqrt{(x-\xi)^2+(z-\zeta)^2}\right) dk$$

$$+\frac{\sigma}{2\rho g}\,\frac{m_0\cosh m_0(y+h)\sinh m_0h}{\nu h + \sinh^2 m_0 h}\iint_{-\infty}^{\infty} p(\xi,\zeta)\,J_0\!\left(m_0\sqrt{(x-\xi)^2+(z-\zeta)^2}\right)d\xi\,d\zeta, \qquad (21.10)$$

where, as usual, m_0 is the real solution of

$$m_0\tanh m_0 h - \nu = 0.$$

Other forms of this expression similar to (21.4), (21.6) and (21.7) can be found with no difficulty. We give only the analogue of (21.9):

$$W_{av} = \frac{1}{2}\pi^2\frac{\sigma\nu m_0}{\rho g}\,\frac{\sinh 2m_0h}{\nu h + \sinh^2 m_0 h}\left|\int_0^{\infty} p(R')\,J_0(\nu R')\,R'\,dR'\right|^2. \qquad (21.11)$$

The identities following (13.18) may be used to put both (21.10) and (21.11) into other forms.

<u>Two dimensions</u>. The derivation of the velocity potential will be carried through, at least in part, since it illustrates a nice application of the Plemelj-Sokhotskii formulas. Two complex units will be introduced, as described at the end of section 11. That is, we shall write

$$\phi(x,y,t) = \varphi_1(x,y)\cos\sigma t + \varphi_2(x,y)\sin\sigma t = Re_j\, \varphi e^{-j\sigma t},$$

$$p(x,t) = p_1(x)\cos\sigma t + p_2(x)\sin\sigma t = Re_j\, p e^{-j\sigma t},$$

$$\varphi = \varphi_1 + j\varphi_2\,, \tag{21.12}$$

and also introduce a stream function $\psi = \psi_1 + j\psi_2$ and a complex potential

$$f(\zeta) = f_1(\zeta) + j f_2(\zeta), \quad f_k = \varphi_k + i\psi_k\,, \quad k = 1,2. \tag{21.13}$$

Then the boundary condition on the free surface may be formulated as follows

$$Im_i\left\{f'(x-i0) + i\nu f(x-i0)\right\} = -\frac{\sigma}{\rho g}\, j\, p(x). \tag{21.14}$$

The definition of $g = f' + i\nu f$ may be extended to the whole complex ζ-plane by reflection, i.e.

$$g(x+iy) = \overline{g(x-iy)}\,, \quad y > 0.$$

Then the condition (21.14) may be written in the form

$$Im_i\left\{f'(x\pm i0) + i\nu f(x\pm i0)\right\} = \pm\frac{\sigma}{\rho g}\, j\, p(x). \tag{21.15}$$

We shall suppose $p(x)$ to be absolutely integrable on the infinite interval. In addition, we shall suppose that either $p(x)$ satisfies a Hölder condition (and is hence continuous) on the whole infinite interval, or else that there are a finite number of segments $(-\infty, b_1)$ (a_2, b_2), ... (a_r, ∞), $b_n < a_{n+1}$, such that $p(x)$ satisfies a Hölder condition on any closed segment interior to one of the above segments, and at an end-point may be expressed in the form

$$p(x) = \frac{q(x)}{(x-c)^{\alpha}}, \quad 0 \leq \alpha < 1, \quad c = a_i \text{ or } b_i,$$

where $q(x)$ satisfies a Hölder condition at the end. Here a Hölder condition means that for any pair of points x_1, x_2, $p(x)$ satisfies

$$|p(x_1) - p(x_2)| < A\,|x_1 - x_2|^{\mu}, \quad \mu > 0$$

In the first case f' will be assumed to have no singularities in the whole lower half-plane. In the second case the behavior of $f'(z)$ near an end-point c will be restricted so that it must satisfy

$$|f'(z)| < \frac{C}{|z-c|^{\alpha}}, \quad 0 < \alpha < 1.$$

As usual it will be assumed that $|f'|$ is bounded as $z \to \infty$ and that only outgoing waves are generated.

The solution of this boundary-value problem for the function

$g(z) = f' + i\nu f$ is determined, up to are additive real constant which may be discarded here, by [see, e.g., Muskhelishvili, Singular integral equations, Noordhoff, Groningen, 1953, §§ 43, 78]

$$f'(z) + i\nu f(z) = j\frac{\sigma}{\pi\rho g}\int_{-\infty}^{\infty}\frac{p(s)}{z-s}\,d\ , \quad y<0. \tag{21.16}$$

After integrating the differential equation and selecting the solution so as to represent outgoing waves at $x = \pm\infty$, one obtains finally [the derivation is similar to that of (13.28)]

$$f(z) = j\frac{\sigma}{\pi\rho g}e^{-i\nu z}\int_{-\infty}^{\infty}p(s)\,ds\int_{\infty}^{z}\frac{e^{i\nu\zeta}}{\zeta-s}\,d\zeta - \frac{\sigma}{\rho g}(1+ij)e^{-i\nu z}\int_{-\infty}^{\infty}p(s)e^{i\nu s}\,ds, \tag{21.17}$$

where the path of integration for ζ is taken in the lower half-plane. The asymptotic form of the time-dependent velocity potential is given by

$$Re_j\, f(z)e^{-j\sigma t} \sim \frac{\sigma}{\rho g}e^{-i(\nu z \mp \sigma t)}\int_{-\infty}^{\infty}e^{i\nu s}\left[p_1(s) \mp i p_2(s)\right]ds \quad \text{as } x\to\pm\infty, \tag{21.18}$$

and the asymptotic form of the free surface by

$$\eta(x,t) \sim Im_i\left\{\frac{\sigma^2}{\rho g^2}e^{-i(\nu x \mp \sigma t)}\int_{-\infty}^{\infty}e^{i\nu s}\left[p_1 \mp i p_2\right]ds\right\} \quad \text{as } x\to\pm\infty. \tag{21.19}$$

From this last expression one can easily derive the average rate at which the pressure system is transferring energy to the fluid:

$$W_{av} = \frac{1}{4}\frac{\sigma\nu}{\rho g}\left\{\left|\int_{-\infty}^{\infty}(p_1 - ip_2)e^{i\nu s}ds\right|^2 + \left|\int_{-\infty}^{\infty}(p_1 + ip_2)e^{i\nu s}ds\right|^2\right\} \quad (21.20)$$

$$= \frac{1}{2}\frac{\sigma\nu}{\rho g}\left\{\left[\int_{-\infty}^{\infty}p_1\cos\nu s\,ds\right]^2 + \left[\int_{-\infty}^{\infty}p_1\sin\nu s\,ds\right]^2 + \left[\int_{-\infty}^{\infty}p_2\cos\nu s\,ds\right]^2 + \left[\int_{-\infty}^{\infty}p_2\sin\nu s\,ds\right]^2\right\}.$$

The expression for $f(z)$ can be put into several different forms by changing variables and deforming the path of integration appropriately. Thus, if one introduces a new variable λ by $\nu(\zeta - z) = -\lambda(z - s)$ and deforms the resulting path to the x-axis, one obtains

$$f(z) = -j\frac{6}{\pi\rho g}\int_{-\infty}^{\infty}p(s)\,ds\ \ \mathrm{PV}\int_{0}^{\infty}\frac{e^{-ik(z-s)}}{k-\nu}dk + \frac{6}{\rho g}e^{-i\nu z}\int_{-\infty}^{\infty}p(s)e^{i\nu s}ds.$$

(21.21)

A different deformation of the path leads to

$$f(z) = -j\frac{6}{\pi\rho g}\int_{-\infty}^{x}p(s)\,ds\int_{0}^{\infty}\frac{e^{-\mu(z-s)}}{\mu - i\nu}d\mu - j\frac{6}{\pi\rho g}\int_{x}^{\infty}p(s)ds\int_{0}^{\infty}\frac{e^{\mu(z-s)}}{\mu + i\nu}d\mu$$

$$+\frac{6}{\rho g}(1+ij)\int_{-\infty}^{x}p(s)e^{-i\nu(z-s)}ds + \frac{6}{\rho g}(1-ij)\int_{x}^{\infty}p(s)e^{-i\nu(z-s)}ds. \quad (21.22)$$

For fluid of depth h an expression for the complex velocity potential analogous to (21.10) and (21.21) is

$$f(z) = -i\frac{6}{\pi \rho g}\int_{-\infty}^{\infty} ds\, p(s)\; PV\!\int_0^{\infty} \frac{\cos k(z-s+ih)}{k \sinh kh - \nu \cosh kh}\, dk$$

$$+ \frac{6}{\rho g}\frac{\sinh m_0 h}{\nu h + \sinh^2 m_0 h}\int_{-\infty}^{\infty} p(s) \cos m_0 (z-s+ih)\, ds \qquad (21.23)$$

One will find both the two- and the three-dimensional case of a periodic pressure distribution over infinitely deep water discussed in Lamb [1904, pp. 387-393]. Stoker [1957, ch. 4] discusses in considerable detail the two-dimensional problem of waves generated by a periodic uniform pressure applied over a finite interval.

21β. Moving pressure distributions.

In this section we shall suppose that a fixed pressure distribution is moving with a constant velocity c. Thus the motion may be treated as time-independent in a coordinate system moving with the pressure distribution. The boundary condition at the free surface is given by [see (11.3)]

$$\varphi_y(x,0,z) + \frac{1}{\nu}\varphi_x(x,0,z) = \frac{c}{\rho g} p_x(x,z), \quad \nu = \frac{g}{c^2}, \qquad (21.24)$$

and the form of the free surface by

$$\eta(x,z) = \frac{c}{g}\varphi_x(x,0,z) - \frac{1}{\rho g} p(x,z). \qquad (21.25)$$

"NOTICE: When Government or other drawings, specifications c other data are used for any purpose other than in connection a definitely related Government procurement operation, the U.S Government thereby incurs no responsibility, nor any obligatio whatsoever; and the fact that the Government may have formu furnished, or in any way supplied the said drawings, specificatic or other data is not to be regarded by implication or otherwis in any manner licensing the holder or any other person or cor tion, or conveying any rights or permission to manufacture, use sell any patented invention that may in any way be related th

In addition, we shall assume vanishing of the fluid motion far ahead, i.e. as $x \to +\infty$, and the usual conditions appropriate to infinite or finite depth. p will be assumed to be absolutely integrable and to vanish for sufficiently large values of $x^2 + z^2$; however, the latter condition can be weakened.

Results will be given without proof since their derivations are similar to those in 13 γ and 21 α. The results for two and three dimensions will be separated.

Three dimensions. The expression for the velocity potential for infinite depth of fluid can be given as follows:

$$\varphi(x,y,z) =$$

$$= \frac{1}{\pi^2 \rho c} \iint_{-\infty}^{\infty} d\xi\, d\zeta\, p(\xi,\zeta) \int_0^{\frac{1}{2}\pi} d\vartheta \sec\vartheta \; \mathrm{P}\!\int_0^{\infty} dk \frac{k e^{ky}}{k - \nu\sec^2\vartheta} \sin[k(x-\xi)\cos\vartheta] \cos[k(z-\zeta)\sin\vartheta] \tag{21.26}$$

$$- \frac{\nu}{\pi \rho c} \iint_{-\infty}^{\infty} d\xi\, d\zeta\, p(\xi,\zeta) \int_0^{\frac{1}{2}\pi} d\vartheta \sec^3\vartheta\, e^{y\nu\sec^2\vartheta} \cos[\nu(x-\xi)\sec\vartheta] \cos[\nu(z-\zeta)\sec^2\vartheta \sin\vartheta].$$

The rate at which the pressure distribution is transferring energy to the fluid is given by

$$W = -\iint_{-\infty}^{\infty} p(x,z)\, \varphi_y(x,-0,z)\, dx\, dz. \tag{21.27}$$

This may be computed directly from (21.26). The first term gives no contribution since it is an odd function of $x-\xi$ (cf. the evaluation of 20.66). The final result may be expressed as follows:

$$W = \frac{\nu^2}{\pi \rho c} \int_0^{\frac{1}{2}\pi} d\vartheta \sec^5\vartheta \iint_{-\infty}^{\infty} dx\, dz \iint_{-\infty}^{\infty} d\xi\, d\zeta\, p(x,z)\, p(\xi,\zeta) \cos\left(\nu \sec^2\vartheta\left[(x-\xi)\cos\vartheta + (z-\zeta)\sin\vartheta\right]\right)$$

$$= \frac{\nu^2}{\pi \rho c} \int_0^{\frac{1}{2}\pi} d\vartheta \sec^5\vartheta \left[P^2(\vartheta) + Q^2(\vartheta)\right], \tag{21.28}$$

$$P(\vartheta) = \iint_{-\infty}^{\infty} dx\, dz\; p(x,z) \cos\left[\nu \sec^2\vartheta (x \cos\vartheta + z \sin\vartheta)\right],$$

$$Q(\vartheta) = \iint_{-\infty}^{\infty} dx\, dz\; p(x,z) \sin\left[\nu \sec^2\vartheta (x \cos\vartheta + z \sin\vartheta)\right].$$

If the pressure distribution is given in cylindrical coordinates, $p = p(R,\alpha)$, $x = R\cos\alpha$, $z = R\sin\alpha$, then one may express, say, $P(\vartheta)$ in the form

$$P(\vartheta) = \int_0^{\infty} dR \int_0^{2\pi} d\alpha\; R\, p(R,\alpha) \cos\left[\nu R \sec^2\vartheta \cos(\alpha - \vartheta)\right] \tag{21.29}$$

and a similar formula for $Q(\vartheta)$. If p depends only upon R, then $Q(\vartheta) \equiv 0$ and

$$P(\vartheta) = 2\pi \int_0^{\infty} R\, p(R)\, J_0(\nu R \sec^2\vartheta)\, dR. \tag{21.30}$$

If the fluid is of depth h, the velocity potential is given by

$$\varphi(x,y,z) = \frac{1}{\pi^2 \rho c} \iint_{-\infty}^{\infty} d\xi\, d\zeta\, p(\xi,\zeta) \int_0^{\frac{1}{2}\pi} d\vartheta \sec^{\cdot 4} \times$$

$$\text{P.V.}\int_0^{\infty} dk \frac{k \cosh k(y+h)\,\operatorname{sech} kh}{k - \nu \sec^2\vartheta \tanh kh} \sin[k(x-\xi)\cos\vartheta] \cos[k(z-\zeta)\sin\vartheta]$$

$$-\frac{1}{\pi \rho c} \iint_{-\infty}^{\infty} d\xi\, d\zeta\, p(\xi,\zeta) \int_{\vartheta_0}^{\frac{1}{2}\pi} d\vartheta \sec\vartheta \frac{k_0 \cosh k_0(y+h)\,\operatorname{sech} k_0 h}{1 - \nu h \sec^2\vartheta \operatorname{sech}^2 k_0 h} \times \tag{21.31}$$

$$\cos[k_0(x-\xi)\cos\vartheta] \cos[k_0(z-\zeta)\sin\vartheta],$$

where $k_0 = k_0(\vartheta)$ is the positive real root of

$$k - \nu \sec^2\vartheta \tanh kh = 0, \quad \vartheta_0 < \vartheta < \tfrac{1}{2}\pi,$$

and

$$\vartheta_0 = \begin{cases} \operatorname{Arc}\cos\sqrt{\nu h} & \text{if } \nu h = \frac{gh}{c^2} < 1, \\ 0 & \text{if } \nu h > 1. \end{cases}$$

The rate of transfer of energy may again be computed from (21.27) and again only the second integral gives a nonvanishing contribution. The result may be expressed in several forms analogous to (21.28)-(21.30):

$$W = \frac{c}{\pi \rho g} \int_{\vartheta_0}^{\frac{1}{2}\pi} d\vartheta \frac{k_0^3 \cos\vartheta}{1 - \nu h \sec^2\vartheta \operatorname{sech}^2 k_0 h} \iint_{-\infty}^{\infty} dx\, dz \iint_{-\infty}^{\infty} d\xi\, d\zeta\, p(x,z)\, p(\xi,\zeta) \times$$

$$\cos\left(k_0[(x-\xi)\cos\vartheta + (z-\zeta)\sin\vartheta]\right) \tag{21.32}$$

$$= \frac{c}{\pi \rho g} \int_{\vartheta_0}^{\frac{1}{2}\pi} d\vartheta \frac{k_0^2 \sec\vartheta}{1 - \nu h \sec^2\vartheta \operatorname{sech}^2 k_0 h} \left[P^2(\vartheta) + Q^2(\vartheta)\right],$$

$$P(\vartheta) = \iint_{-\infty}^{\infty} p(x,z) \cos[k_0(x\cos\vartheta + z\sin\vartheta)]\,dx\,dy,$$

$$Q(\vartheta) = \iint_{-\infty}^{\infty} p(x,z) \sin[k_0(x\cos\vartheta + z\sin\vartheta)]\,dx\,dy.$$

In cylindrical coordinates formulas (21.29) and (21.30) carry over to the present situation with ν replaced by k_0.

The asymptotic form of the free surface for either infinite or finite depth is much more complicated to analyze than for the stationary periodically oscillating pressure distribution of the preceding section. Although it is not strictly necessary to do so, it has been customary in this type of analysis to consider the special case of a "concentrated pressure point". To derive the velocity potential for the pressure point consider the pressure distribution defined by

$$p(R) = \begin{cases} \dfrac{p_0}{\pi R_0^2}, & R \le R_0, \\ 0, & R > R_0. \end{cases} \tag{21.33}$$

Substitute in (21.26) or (21.31) and take the limit as $R_0 \to 0$.
Then (21.26) becomes

$$\varphi(x,y,z) = \frac{p_0}{\pi^2 \rho c}\int_0^{\frac{1}{2}\pi} d\vartheta \sec\vartheta \;\; \mathrm{PV}\!\!\int_0^{\infty} dk \frac{k e^{ky}}{k - \nu\sec^2\vartheta} \sin(kx\cos\vartheta)\cos(kz\sin\vartheta)$$
$$- \frac{\nu p_0}{\pi \rho c}\int_0^{\frac{1}{2}\pi} d\vartheta \sec^3\vartheta\, e^{y\nu\sec^2\vartheta} \cos(\nu x\sec\vartheta)\cos(\nu z\sec^2\vartheta\sin\vartheta). \tag{21.34}$$

The velocity potential for the pressure point in fluid of finite depth is derived similarly. The equation representing the free surface may now be obtained immediately from (21.25):

$$\eta(x,z) = \frac{c}{g}\,\varphi_x(x,0,z)\,, \quad x^2+z^2>0. \tag{21.35}$$

The velocity potential (21.34) is very similar to that of a submerged source in steady motion (see 13.36) and the method sketched in section 13 for the derivation of the asymptotic expression (13.42) can be carried over directly to the moving pressure point.

The result, expressed in cylindrical coordinates, is as follows:

for $\quad 0 \leq \alpha < \pi - \text{Arc}\sin\frac{1}{3} = \alpha_c$:

$$\eta(R,\alpha) = O\left((\nu R)^{-2}\right);$$

for $\quad \alpha = \alpha_c$: (21.36)

$$\eta(R,\alpha_c) = -\frac{p_0\nu^2}{\pi\rho g}\, 2^{-3/2}\, 3^{4/3}\, \Gamma\!\left(\tfrac{1}{3}\right)(\nu R)^{-1/3} \sin\left(\frac{\sqrt{3}}{2}\nu R\right) + O\left((\nu R)^{-2/3}\right);$$

for $\quad \alpha_c < \alpha < \pi$:

$$\eta(R,\alpha) = \frac{p_0\nu^2}{\pi\rho g}\sqrt{\frac{2\pi}{\nu R}}\,\frac{1}{[1-9\sin^2\alpha]^{1/4}}\left\{\sec^{5/2}\vartheta_1 \sin\left(\nu R\mu_1 - \frac{\pi}{4}\right) + \sec^{5/2}\vartheta_2 \sin\left(\nu R\mu_2 + \frac{\pi}{4}\right)\right\} + O\left((\nu R)^{-1}\right);$$

for $\alpha = \pi$:

$$\eta(R,\pi) = -\frac{p_0 \nu^2}{\pi \rho g}\sqrt{\frac{2\pi}{\nu R}}\ \sin\left(\nu R + \frac{\pi}{4}\right)$$

The variables ϑ_1 , ϑ_2 , μ_1 , μ_2 are the same ones defined following (13.42), where certain properties are also given. Figure 1 is not quite accurate as a description of the wave crests in the region $|\pi - \alpha| < \alpha_c$ for the pressure point since the phase in (21.36) has been shifted by $\frac{1}{2}\pi$; the wave crests in Figure 1 should be moved back a distance $\frac{1}{2}\pi$.

The wave pattern resulting from a moving pressure distribution has been the subject of many investigations, starting apparently with Kelvin [1906]. His aim was to explain the typical wave pattern found behind a ship. The procedure is quite reasonable as a method for obtaining a qualitative prediction of a ship's wave pattern, since a moving ship has associated with it a pressure distribution around the wetted hull. The obvious disadvantage of the method is that it gives no connection between the geometry of the hull and the wave pattern. For this the "thin ship" approximation of section 20 β is better within its range of applicability. The single pressure point can be taken to represent approximately a ship moving at high speed (more accurately, at high Froude number $c/\sqrt{Lg}$. where L is the length), say a fast motor boat.

For detailed investigation of the asymptotic expression one should refer to Hogner [1923], Peters [1949], Bartels and Downing [1955] (who do not restrict themselves to a pressure point) and Stoker [1957, ch. 8]. The necessary modifications for finite depth have been made by Havelock [1908] and Tetsurô Inui [1936] and are described qualitatively in the discussion following (13.42). One can find an exposition of the theory of waves generated by moving pressure distributions in a report by Lunde [1951b]. Several papers by Havelock [1909, 1914b, 1919, 1922] take up the wave resistance ($-W/c$) of a pressure distribution. Hogner [1928] has also considered the wave resistance and gives essentially (21.28).

In the preceding considerations we have assumed that, as $x^2+z^2 \to \infty$, $p(x,z)$ approached zero sufficiently quickly so that it might be represented as a Fourier integral. It is also possible to proceed somewhat differently, assume $p(x,y)$ periodic in one or both variables and use a Fourier series representation. This has been done, for example, by Voit [1957a], who has considered for both infinite and finite depth a moving pressure distribution of the following form:

$$p(x,z) = \begin{cases} P(z)\sum_{n=1}^{\infty} a_n \cos nkx, & |z| < h \\ 0, & |z| > h. \end{cases}$$

The waves resulting from a pressure point moving parallel to beaches forming angles of 30° and 45° have been treated by Hanson [1926]; in the same paper he also treats the waves formed by a pressure point moving over a two-layered fluid. A detailed investigation of this last topic is given in a paper of Sretenskii's [1934].

Two dimensions. By introducing a stream function $\Psi(x,y)$ and a complex potential $f = \varphi + i\Psi$, the free surface boundary condition can be put into a form analogous to (21.14), namely,

$$\mathrm{Re}\left\{f'(x-i0)+i\nu f(x-i0)\right\} = \frac{1}{\rho c}\,p(x), \quad \nu = \frac{g}{c^2}. \tag{21.37}$$

In addition, we assume $|f'|$ bounded as $z \to \infty$ and also $\lim_{x\to\infty}|f'| = 0$. We shall assume $p(x)$ subject to the same limitations as in section 21α.

One may apply the same method of analysis to derive the following forms for the complex velocity potential:

$$\begin{aligned} f(z) &= \frac{e^{-i\nu z}}{\pi i \rho c}\int_{-\infty}^{\infty} ds\, p(s)\int_{\infty}^{z}\frac{e^{i\nu\zeta}}{\zeta - s}\,d\zeta \\ &= -\frac{1}{\pi i\rho c}\int_{-\infty}^{\infty} ds\, p(s)\ \mathrm{PV}\int_0^{\infty}\frac{e^{-i\lambda(z-s)}}{\lambda-\nu}\,d\lambda - \frac{e^{-i\nu z}}{\rho c}\int_{-\infty}^{\infty} p(s)\,e^{i\nu s}\,ds \\ &= -\frac{1}{\pi i\rho c}\int_{-\infty}^{x} ds\, p(s)\int_0^{\infty}\frac{e^{-\mu(z-s)}}{\mu - i\nu}\,d\mu - \frac{1}{\pi i\rho c}\int_{x}^{\infty} ds\, p(s)\int_0^{\infty}\frac{e^{\mu(z-s)}}{\mu+i\nu}\,d\mu \\ &\quad - \frac{2}{\rho c}\,e^{-i\nu z}\int_x^{\infty} p(s)\,e^{i\nu s}\,ds, \end{aligned} \tag{21.38}$$

where the path of integration for ζ in the first expression is taken in the lower half-plane. The rate at which the pressure distribution transfers energy to the fluid is easily found from formula (21.27) and the second expression for $f(z)$ to be

$$W = \frac{\nu}{\rho c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x)\,p(\xi)\cos\nu(x-\xi)\,dx\,d\xi. \tag{21.39}$$

If the fluid is of depth h, the complex velocity potential may be given in a form analogous to the second formula of (21.37):

$$f(\zeta) = \frac{\nu c}{\pi \rho g} \int_{-\infty}^{\infty} ds\, p(s)\, PV\!\!\int_{-\infty}^{\infty} dk \frac{\sin k(\zeta - s + ih) \operatorname{sech} kh}{k - \nu \tanh kh}$$

$$- \frac{\nu c}{\rho g} \int_{-\infty}^{\infty} p(s) \frac{\cos k_0(\zeta - s + ih) \operatorname{sech} k_0 h}{1 - \nu h \operatorname{sech}^2 k_0 h}\, ds, \qquad (21.40)$$

where k_0 is the real positive root of

$$k_0 - \nu \tanh k_0 h = 0$$

and exists only if $\nu h = gh/c^2 > 1$; if $\nu h \leq 1$, the last term in (21.40) must be deleted. The rate at which the pressure is doing work upon the fluid is given by

$$W = \begin{cases} \dfrac{k_0^2 c}{\rho g} \displaystyle\iint_{-\infty}^{\infty} p(x)\, p(\xi) \frac{\cos k_0(x - \xi)}{1 - \nu h \operatorname{sech}^2 k_0 h}\, dx\, d\xi & \text{if } \nu h > 1, \\ 0 \quad \text{if } \nu h < 1. \end{cases} \qquad (21.41)$$

The absence of the second term in (21.40) and the vanishing of W when $\nu h < 1$ correspond to the absence of an infinite train of trailing waves. A similar phenomenon occurs behind a moving singularity in two dimensions [cf. (13.46)-(13.48) and the following remarks].

For either (21.37) or (21.39) the form of the free surface can be written down immediately from the formula

$$\eta(x) = \frac{1}{c} \Psi(x, 0). \qquad (21.41)$$

We shall not carry out the details. The asymptotic form of the surface behind a two-dimensional "pressure point", or also a distributed pressure, is much easier in two than in three dimensions and we again omit a detailed statement. However, the problem has been treated by Kelvin [1905] and is discussed in Lamb [1932, § 242-5], both for infinite and finite depth. It is also discussed in the paper of Peters [1949] already cited in connection with the three-dimensional problem.

Derivations of the complex velocity potential may be found in the papers of Sretenskii [1934, 1940], Sedov [1936], Kochin [1939] and Haskind [1943a] already cited in connection with planing surfaces. We refer also to papers of Dean [1947] and Timman and Vossers [1955].

20 γ. Moving periodic pressure distributions.

It is clearly possible to combine the cases considered in sections 20 α and 20 β and consider the waves resulting from a pressure distribution expressible in the form

$$p(x,z,t) = p_1(x-ct,z)\cos\sigma t + p_2(x-ct,z)\sin\sigma t,$$

where the coordinates are fixed in space. The resulting velocity potential will be analogous to (13.52) for the three-dimensional case, if one is dealing with a "pressure point".

We shall not reproduce the formulas here. However the analogue of (13.49) for pressure distributions may be found in the cited report of Lunde [1951b], and from this the required velocity potential may be found. For two-dimensional motion the details, carried out by this procedure, may be found in papers by Kaplan [1957] and Wu [1957].