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5

Minimum Variance Unbiased Estimates  
Generalization of Thompson's Distribution  
Random Orthonormal Bases

By  
Andre G. Laurent

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TECHNICAL REPORT NO. 5

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Minimum Variance Unbiased Estimates.  
Generalization of Thompson's Distribution.  
Random Orthonormal Bases.

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Department of Mathematics  
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1.0 Let  $X_1, \dots, X_N$  be a sample of  $N$  independent observations with a normal distribution  $N(\underline{m}, \underline{\sigma})$ . In 1935, W. R. Thompson has obtained the probability distribution of an observation  $X_i$  chosen at random in the sample, when the mean  $\underline{m}$  and the standard deviation  $\underline{\sigma}$  are replaced by the corresponding sample characteristics [1]; namely  $(X_i - \bar{X})^2 / (N-1)S^2$  follows an incomplete beta distribution with parameters  $1/2$  and  $(N/2)-1$ .

This result has been generalized in several directions, in [2]; the distribution of a sample is given after its being centered and studentized by means of the mean and standard deviation of an independent sample; the distribution of a subsample is derived when  $\underline{m}$  and  $\underline{\sigma}$  are replaced by the corresponding sample characteristics, and the conditional distribution of a subsample is obtained given  $\bar{X}$  and  $S$ .

Further generalizations have been presented in [3] where the results just mentioned are extended to the case of the multivariate normal distribution. Applications to Bombing Theory were proposed in [4].

1.1 Let  $(\xi) = (\xi_1, \dots, \xi_1, \dots, \xi_k)$  with  $\xi_i = (x_i^1, \dots, x_i^2, \dots, x_i^p)$  be a subset of observations from a set of  $N$  independent observations  $(X) = (X_1, \dots, X_1, \dots, X_N)$  with the same  $p$ -variate normal distribution  $N(m, \Sigma)$ ; let  $m'$  and  $S$  be the vector mean and covariance matrix of the subset  $(\xi)$ . The distribution of  $\xi$  given  $m'$  and  $S$  for  $k < N-p$  is

$$f(\xi | m'S) d\xi = c \cdot |I - (k/N)S^{-1}S_\xi - [k/(N-k)]S^{-1}S^*|^{(N-k-p-2)/2} d\xi / S^{k/2}$$

in the domain where the determinants are positive, with

$$S^* = (m'_\xi - m')'(m'_\xi - m')$$

and

$$c = \frac{\prod_{i=1}^k \Gamma((N-k-p-2)/2) \prod_{i=1}^{N-k} \Gamma((N-k-p)/2)}{\pi^{kp/2} N^{(k-1)p/2} (N-k)^{p/2}}$$

This result was given in [3].

2.1 In view of the completeness of  $m'S$ , it results from Blackwell's theorem that

$$\int \dots \int \varphi(\xi) f(\xi | m'S) d\xi$$

will provide a uniformly minimum variance unbiased estimate of  $E[\varphi(\xi)]$  if it exists.

As a special application, if

$$g(A, m, \Sigma) = \int_A \dots \int N(m, \Sigma) dX,$$

a minimum variance unbiased estimate of  $g(A, m, \Sigma)$  will be given by

$$\int_A \dots \int f(\xi | m'S) d\xi \quad \text{with } k = 1,$$

since an unbiased estimate of  $g(A, m, \Sigma)$  is provided by the characteristic function  $I_A(\xi)$  of the set  $A$  whose conditional expected value is given by the integral above. Further, in view of the fact that summation keeps unbiasedness, the minimum variance unbiased estimate of the density  $N(m, \Sigma)$  is  $f(\xi | m'S)$ .

The already established results for the univariate distribution can be obtained by considering it as a special case of the multivariate situation.

2.2 The conditional probability distribution of  $m_{\xi}^*$ ,  $S_{\xi}$  given  $m^*$ ,  $S$  is obtained by writing the joint distribution of the mean  $m_n^*$  and covariance matrix  $S_n$  of a sample of size  $n = N-k$  and of  $m_{\xi}^*$  and  $S_{\xi}$ , then performing the change of variable

$$m_n^* = Nm^*/n - km_{\xi}^*/n$$

$$S_n = NS/n - kS_{\xi}/n - S^*kN/n^2$$

subsequently dividing by the distribution of  $S$ .

$m_n^*$  is  $N(m, \mu/n)$  distributed,  $S_n$  is  $W(\mu, n-1)$  distributed, the Jacobian of the change of variables is

$$\frac{D(m_n^*, S_n)}{D(m^*, S)} = (N/n)^{p(p+3)/2}$$

$m^*$  is  $N(m, \mu/N)$ ,  $S$  is  $W(\mu, N-1)$  distributed.

After grouping terms, one obtains for  $p < k < N-p$

$$f(m_{\xi}^*, S_{\xi} | m^*, S) dm_{\xi}^* dS_{\xi} =$$

$$C \left| I - (k/N) S_{\xi} S^{-1} - [k/(N-k)] S^* S^{-1} \right|^{(N-k-p-2)/2} \left| S_{\xi} S^{-1} \right|^{(k-p-2)/2}$$

$$\left( \frac{dm_{\xi}^*}{|S|^{1/2}} \right) \left( \frac{dS_{\xi}}{|S|^{(p+1)/2}} \right)$$

where

$$C = \frac{k^k}{(N-k)N^{k-1}} \left[ \frac{1}{\pi} \right]^{p/2} \frac{[1/\pi]^{p(p+1)/4}}{\prod_{j=1}^p \frac{\Gamma[(N-j)/2]}{\Gamma[(N-k-j)/2] \Gamma[(k-j)/2]}}$$

in the domain where the determinants are positive.

2.3 In view of the completeness of  $m^*$ ,  $S$ , the kernel

$f(m_{\xi}^*, S_{\xi} | m^*, S)$  is of great interest

If  $\varphi_n(m_n^*, S_n)$  is an unbiased estimate of a function  $g(m, \mu)$  of the parameters, then, as a consequence of the uniqueness of

minimum variance unbiased linear estimates in case of completeness

$$\varphi_N(m^*, S) = \int \dots \int \varphi_k(m_{\xi}^*, S_{\xi}) f(m_{\xi}^*, S_{\xi} | m^*, S) dm_{\xi}^* dS_{\xi}$$

i.e. minimum variance unbiased linear estimates are a subclass of the solutions of this integral equation which defines a multivariate transformation.

3.0 In case  $X$  is normally distributed  $N(0, \sigma^2)$ , univariate situation, or  $N(0, \Sigma)$ , multivariate situation, other generalizations are of interest.

3.1 In the univariate case, let  $X$  be a set of  $N$  observations  $S_1 \dots X_N$  with probability distribution  $N(0, \sigma^2)$ , the conditional distribution of a subsample  $\xi$  of  $k$  items drawn from  $X$  at random, given the moment of order two  $m_2^*(X)$  of the set  $X$ , is obtained by writing the joint distribution of the moment of order two,  $m_2^*(X_n)$  of a sample  $X_n$  of size  $n = N-k$  and of an independent sample  $\xi$  of  $k$  items, performing the change of variables

$m_2^*(X_n) = (N/n)m_2^*(X) - (k/n)m_2^*(\xi)$  whose Jacobian is  $N/n$ , then dividing by the distribution of  $m_2^*(X)$ , one obtains with  $\xi =$

$(x_1 \dots x_k)$ ,  $k < N$

$$f[\xi | m_2^*(X)] d\xi = \frac{\pi^{-(N/2)}}{\pi^{k/2} \Gamma[(N-k)/2]} \left[ 1 - \frac{(k/N) m_2^*(\xi)}{m_2^*(X)} \right]^{(N-k-2)/2} \frac{d\xi}{[N m_2^*(X)]^{k/2}}$$

in the domain where the bracket is positive.

if  $\zeta = \xi / \sqrt{m_2^*(X)}$ , one obtains

$$\frac{\Gamma(N/2)}{\pi^{k/2} \Gamma[(N-k)/2]} (1 - \frac{\sum \xi_i^2}{N})^{(N-k-2)/2} (d\xi / N^{k/2}) \quad \sum \xi_i^2 \leq N$$

i.e. the distribution of a subsample normalized by  $\sqrt{m_2^*(X)}$  is dependent from  $m_2^*$ .

Also, in case  $k = N-1$ ,  $u = \xi / \sqrt{Nm_2^*(X)}$ ,  $u_1, \dots, u_{N-1}$  are  $N-1$  independent coordinates of a  $N$  dimensional unit vector, and one has  $[\Gamma(N/2) / \pi^{N/2}] (1 - uu')^{-1/2} du \quad uu' \leq 1$ .

The distribution of any coordinate  $u_i$  being

$$\frac{1}{\beta(\frac{N-1}{2}, \frac{1}{2})} (1 - u_i^2)^{(N-3)/2} du_i \quad u_i^2 \leq 1$$

(if  $N = 3$  one obtains a rectangular distribution).

Clearly, the quantity  $v = \sqrt{N-1} u_i / \sqrt{1-u_i^2}$  is Student distributed with  $N-1$  degrees of freedom.

$u_i$  is the cosine of the angle  $\varphi_i$  of the unit vector with any direction,  $u_i = \cos \varphi_i$  and  $v = \sqrt{n-1} \cot. \varphi_i$ .

3.2 This result can be slightly generalized to the case of a spherical distribution.

Let  $X = (X_1, \dots, X_n)$  have probability distribution  $f(X)dX$ ,

let  $R^2 = XX'$ , let  $f(X)dX$  be spherical, i.e.  $f(X) = h(XX')$ ,

then  $R^2$  is a sufficient statistic for  $f(X)dX$

$f(X)dX = h(R^2)dX = h(R^2)R^{n-1}dR d\sum_{n,1}$  ( $\sum_{n,1}$  = the area of the unit

sphere in the  $n$  dimensional space); integrating out  $d\sum_{n,1}$ , one

obtains  $\sum_{n,1} h(R^2)R^{n-1}dR$  as the probability distribution of  $R$ , a

well known result.

More explicitly, using polar coordinates

$$X_1 = R \cos \psi_1$$

$$X_{n-1} = R \sin \psi_1 \sin \psi_2 \dots \sin \psi_{n-2} \cos \psi_{n-1}$$

$$X_n = R \sin \psi_1 \dots \sin \psi_{n-2} \sin \psi_{n-1}$$

$\psi_i$  in  $(0, \pi)$  if  $i < n-1$ ,  $\psi_i$  in  $(0, 2\pi)$  if  $i = n-1$

$$dX = J dR d\psi_1 \dots d\psi_{n-1}, \quad \text{where}$$

$$J = [D(X)/D(R, \psi)] = R^{n-1} \sin^{n-2} \psi_1 \dots \sin \psi_{n-2}$$

$$\text{and } d\sum_{n,i} = \sin^{n-2} \psi_1 \dots \sin \psi_{n-2} d\psi_1 \dots d\psi_{n-1} = J d\psi / R^{n-1}$$

$$= dX_1 \dots dX_{n-1} / |X_n| R^{n-2}$$

hence, the conditional distribution of  $X$  given  $R^2$  is

$$\frac{d\sum_{n,i}}{\sum_{n,i}} = \frac{\Gamma(n/2)}{2\pi^{n/2}} \sin^{n-2} \psi_1 \dots \sin \psi_{n-2} d\psi_1 \dots d\psi_{n-1}$$

it is the distribution of the independent polar coordinates of a unit vector equidistributed on the unit sphere. In terms of the  $X_i$ 's and  $u_i$ 's, this gives (taking into account both signs of  $X_n$ ),

$$\frac{\Gamma(n/2)}{\pi^{n/2}} \frac{dX_1 \dots dX_{n-1}}{|X_n| R^{n-2}} = \frac{\Gamma(n/2)}{\pi^{n/2}} (I - uu^*)^{-1/2} du$$

Consequently the distribution of any subsample is the one given in paragraph 3.1.

This shows that "Thompson's distribution depends only on the sphericity of the universe. If we consider a random unit vector, with fixed origin, "chosen at random" in the  $n$  space i.e., with a probability distribution invariant under the rotation group, we see that we can construct such a vector by normalizing a vector whose distribution is spherical.

3.4 Let us consider a set (X) of independent N observations of a p-dimensional vector with distribution  $N(0, \Sigma)$ .  $(X) = (X_1, \dots, X_i, \dots, X_N)' = (X_i^j) = (X^1, \dots, X^j, \dots, X^p)$ ,  $X^j = (X_1^j, \dots, X_i^j, \dots, X_N^j)'$  let  $(\xi)$  be a subset of K observations. Let S and  $S_\xi$  be the maximum likelihood estimates of  $\Sigma$  obtained respectively with (X) and  $(\xi)$ . The distribution of  $(\xi)$  given the sufficient statistic S, is obtained by writing the joint distribution of  $(\xi)$  and the maximum likelihood estimate  $S_n$  of  $\Sigma$  obtained with a sample of  $n = N-k$  observations, making the change of variables  $S_n = (N/n) S - (k/n) S_\xi$

and dividing by the distribution of  $S_n$ . One obtains

$$f(\xi | S) d\xi = C \cdot \left| I - (k/N) S^{-1} S_\xi \right|^{(N-k-p-1)/2} d\xi / |S|^{k/2}$$

with

$$C = \frac{\prod_{j=1}^p \Gamma[(N+1-j)/2]}{\prod_{j=1}^p \Gamma[(N-k+1-j)/2]} \left[ 1/(N\pi) \right]^{pk/2}$$

valid for  $p \leq N-k$ , in the domain where the determinants are positive.

$$\text{Now } k S_\xi = \xi' \xi = \xi_1' \xi_1 + \dots + \xi_k' \xi_k$$

so that

$$\left| I - (k/N) S^{-1} S_\xi \right| = \left| I - (S^{-1}/N) \sum_{j=1}^k \xi_j' \xi_j \right| = |NS|^{-1} \left| S - \sum_{j=1}^k \xi_j' \xi_j \right|$$

there exists a unique triangular matrix T with positive diagonal such that  $NS = X' X = T T'$  then

$$\left| I - (k/N) S^{-1} S_\xi \right| = \left| T^{-1} \right| \left| S - \sum_{j=1}^k \xi_j' \xi_j \right| \left| T'^{-1} \right| = \left| I - \sum_{j=1}^k T^{-1} \xi_j' \xi_j T'^{-1} \right| = \left| I - \sum_{j=1}^k \eta_j' \eta_j \right| \text{ if } \eta_j' = T^{-1} \xi_j' \text{ and } \eta_j = \xi_j T'^{-1}$$

Now, it is known that

$$\left| I - \sum \eta_j' \eta_j \right| = \left| g_{il} - \eta_i \eta_l' \right|$$

so that

$$\left| I - (k/N)S^{-1}S_{\xi} \right| = \left| g_{il} - \xi_i T^{-1} T^{-1} \xi_l' \right| = \left| g_{il} - \xi_i (S^{-1}/N) \xi_l' \right|$$

$$f(\xi | s) d\xi = c \cdot \left| g_{il} - \xi_i (S^{-1}/N) \xi_l' \right|^{(N-k-p-1)/2} d\xi / |s|^{k/2}$$

One can also work with the variables  $\eta$  making the transformation

$\eta = \xi T^{-1}$ , one obtains for the distribution of  $\eta$ ,

$$c \cdot \left| I - \eta' \eta \right|^{(N-k-p-1)/2} d\eta$$

since the Jacobian  $D(\xi)/D(\eta) = |T|^k = |s|^{k/2}$ ,

$$\eta = \xi T^{-1}$$

is a procedure of studentization of the set  $\xi$  and generalizes the studentization by  $R = \sqrt{Nm_2^2(X)}$  in the univariate case to the multivariate case.

In some problems the following remark is of interest.

Consider the orthogonal matrix  $O$  that diagonalizes  $(S^{-1}/N)$  into  $\Lambda^{-1}$  say, then premultiplying and postmultiplying by  $O^{-1}$  and  $O$  we have

$$\left| I - (k/N)S^{-1}S_{\xi} \right| = \left| I - \Lambda^{-1} \sum_{j=1}^k O^{-1} \xi_j' \xi_j O \right| = \left| I - \Lambda^{-1} \sum_{j=1}^k u_j' u_j \right|$$

where  $u_j$  is the vector of the coordinate of  $\xi_j$  w.r. to the system of coordinates constituted by the eigen vectors of  $(S^{-1}/N)$  i.e. of  $NS$ , (which is almost surely of rank  $p$ ).

One can consider the normalization of these coordinates by the roots  $\sqrt{h_1} \dots \sqrt{h_p}$  of  $\Lambda^{1/2}$ , we get, in terms of the new variables  $V$

$$c \left| I - V'V \right|^{(N-k-p-1)/2} dV$$

and the corresponding normalization is

$$v = \xi S^{p-1/2}$$

according to the usual definition of  $S^{1/2}$  (see, for instance, [13]).

3.5 The conditional distribution of  $S_{\xi}$  given  $S$  is obtained by writing the joint distribution of  $S_{\xi}$  and  $S_n$ , making the change of variables indicated in 3.4 and dividing by the distribution of  $S$ , one obtains

$$f(S_{\xi} | S) dS_{\xi} = c \cdot \left| I - (k/N) S_{\xi} S^{-1} \right|^{(N-k-p-1)/2} \left| S_{\xi} S^{-1} \right|^{(k-p-1)/2} dS_{\xi} / |S|^{(p+1)/2}$$

with

$$c = (k/N)^{pk/2} \frac{1}{(\pi)^{p(p-1)/4}} \frac{\prod_{j=1}^p \Gamma[(N+1-j)/2]}{\prod_{i=1}^p \Gamma[(k+1-j)/2] \prod_{j=1}^p \Gamma[(N-k+1-j)/2]}$$

$p \leq k \leq N-p$ , in the domain where the determinants are positive.

In terms of  $\eta$  one obtains

$$c \left| I - \eta' \eta \right|^{(N-k-p-1)/2} \left| \eta' \eta \right|^{(k-p-1)/2} d\eta \eta$$

since the Jacobian of the transformations from  $\xi' \xi = k S_{\xi}$  to  $\eta' \eta$  is  $|T|^{p+1} = |S|^{(p+1)/2}$ . This is a generalization of the incomplete beta distribution.

3.6 The results in 3.4 and 3.5 can be generalized to the case of a probability distribution which depends on the set  $X$  through  $X'X = NS$ , i.e.

$$f(X) dX = h(X'X) dX$$

Then  $X'X$  is a sufficient statistic for the distribution. It has been shown by Hsu that the probability distribution of  $X'X = NS$ , under such circumstances, is

$$g(X'X)dX'X = \frac{\pi^{Np/2 - p(p-1)/4}}{\prod_{j=1}^p \Gamma[(N-j+1)/2]} |X'X|^{(N-p-1)/2} h(X'X)dX'X \quad [9]$$

It is well known that there exists a unique triangular matrix  $T$  with positive diagonal such that  $X'X = TT'$  and that the transformation  $X = YT'$  defines the so-called "rectangular coordinates"  $T$  [8]. To a considerable extent, what follows overlaps the theory of rectangular coordinates, though the approach is somewhat different.

3.6.1 The vectors  $(X^1, \dots, X^j, \dots, X^p)$ , span a space that is almost surely  $p$ -dimensional. We want to construct an orthonormal basis  $(Y^1, \dots, Y^j, \dots, Y^p)$  in that space. This can be done by use of the Schmidt's orthogonalization process, starting with  $X^1$ .

$$(X^1, \dots, X^j, \dots, X^p) = U(Y^1, \dots, Y^j, \dots, Y^p)$$

where  $U$  is an upper triangular operator (i.e. in the system  $(Y^1, \dots, Y^p)$ ,  $u$  is represented by an upper triangular matrix with positive diagonal.)

$$\text{now } X'X = NS = ((X^i, X^j)) = ((UY^i, UY^j)) = ((Y^i, U'UY^j))$$

i.e. the  $i, j$  element of  $U'U$  matrix of  $U'U$  with respect to

$(Y^1, \dots, Y^p)$  is the  $i, j$  element of  $NS$  that is  $U'U = TT'$  and

from the uniqueness of the factorization  $U = T'$ ; consequently

$$X = YT'$$

$Y$  has orthogonal columns and  $Y'Y = I_{pp}$ .

$$X_j = t_{j1} Y^1 + \dots + t_{jj} Y^j$$

X involves np free random variables, Y, np-p(p+1)/2 as orthogonal and T, p(p+1)/2 as triangular, Y = X T^{-1} is a random orthonormal basis.

Now

$$\int f(X) dX = h(X^*X) \prod_j dX^j \dots dX^j_N$$

let us study

$$\prod_j dX^j \dots dX^j_N = \prod_j dX^j$$

$$\text{One has } X^j = (t_{j,1} Y^1 + \dots + t_{j,j-1} Y^{j-1}) + t_{j,j} Y^j$$

Consider the j-1 dimensional space spanned by Y^1 .. Y^{j-1} and complete that basis by an orthonormal basis in the N-j+1 dimensional complementary space; t\_{j,j} Y^j is the projection of X^j in that space where its coordinates are X\_j^{j\*}, ..., X\_N^{j\*} say. In the new N dimensional basis X^j has coordinates t\_{j,1} .. t\_{j,j-1} X\_j^{j\*} .. X\_N^{j\*} and since one passes from the old basis to the new basis by a rotation one has

$$dX^j = dt_{j,1} \dots dt_{j,j-1} dX_j^{j*} \dots dX_N^{j*}$$

Using polar coordinates in the N-j+1 complementary space we have

$$dX_j^{j*} \dots dX_N^{j*} = t_{j,j}^{N-j} dt_{j,j} d\sum_{N-j+1}$$

where d\sum\_{N-j+1} denotes the elementary area of the unit sphere in the N-j+1 dimensional space.

Therefore,

$$dX = dT^t \prod_j t_{j,j}^{N-j} d\sum_{N-j+1}$$

and since dT^t = 2^{-p} \prod\_j t\_{j,j}^{p+j-1} dX^\*X and \prod\_j t\_{j,j} = |T|, one has

$$dX = 2^{-p} |X^*X|^{(N-p-1)/2} \prod_j d\sum_{N-j+1} dX^*X$$

$d\sum_{N-j+1}$  symbolizes a differential expression involving  $N-j$  angles functionally independent [i.e.  $N-p(p+1)$  variables for all  $j$  from 1 to  $p$ .]

$$f(X) dX = h(X^*X) 2^{-p} |X^*X|^{(N-p-1)/2} \prod_j d\sum_{N-j+1} dX^*X$$

integrating the angles out, one obtains  $g(X^*X) dX^*X$ , mentioned above, so that the conditional distribution of  $X$  given  $X^*X$  is

$$f(X|X^*X) dX = \prod_j \frac{d\sum_{N-j+1}}{\sum_{N-j+1}}$$

when expressed in function of the angular coordinates of the polar systems of coordinates.

$$\frac{d\sum_{N-j+1}}{\sum_{N-j+1}}$$

is the distribution of a unit vector  $Y^j$  equidistributed on the unit sphere of the  $N-j+1$  dimensional space.

The distribution of  $X$  given  $X^*X$  is independent from the nature of  $f$ , therefore, all the results of 3.4 and 3.5 are valid without the assumption of normality, it is enough that  $f(X) = h(X^*X)$ . Making the change of variables  $X = YT'$  and integrating out  $T'$ , we obtain the distribution above, (since  $T$  and  $X^*X$  are equivalent as sufficient statistics), i.e. the  $Y$  have the distribution

$$\prod_j \frac{d\sum_{N-j+1}}{\sum_{N-j+1}}; \text{ more specifically, the random basis } (Y^1, \dots, Y^p)$$

is constituted with unit vectors  $Y^j$  that are uniformly distributed, given  $Y^1 \dots Y^{j-1}$ , on the unit sphere of the  $N-j+1$  dimensional space and the distribution of  $Y$  is the same as that of  $X$  given  $X^*X = I$ .

$Y = XT^{-1}$  can be considered as a studentization or normalization of  $X$  generalizing the normalization by  $\sqrt{Nm_2^*(X)}$  in the univariate situation. One can write

$$\prod_j \frac{d \sum_{N-j+1}}{\sum_{N-j+1}} = \prod_j \frac{\Gamma[(N-j+1)/2] d \sum_{N-j+1}}{2^p \pi^{Np/2 - p(p-1)/4}}$$

$Y^j$  has coordinates  $Y_j^{j*} \dots Y_N^{j*}$  with respect to any orthonormal basis spanning the  $N-j+1$  space it belongs to, and the distribution of those is

$$\frac{\Gamma[(N-j+1)/2] dY_j^{j*} \dots dY_{N-1}^{j*}}{\pi^{(N-j+1)/2} [1 - (Y_j^{j*2} + \dots + Y_{N-1}^{j*2})]^{1/2}}$$

$$= \frac{2 \Gamma[(N-j+1)/2]}{\pi^{(N-j+1)/2}} \frac{dY_j^{j*} \dots dY_{N-1}^{j*}}{|Y_N^{j*}|}$$

( $Y_N^{j*}$  can take two values when the other dependent coordinates of  $Y^j$  are fixed.)

3.6.2 To obtain the distribution of  $X$  given  $X'X$  as an explicit function of the  $X_i^j$  or the  $Y_i^j$ , one ought to pass from the  $Y^*$  to the  $Y$ , choosing  $Np-p(p+1)$  functionally independent  $Y_i^j$ , then, if needed, come back to  $X$ , through  $X = YT^*$ ; only  $Np-p(p+1)/2$   $X_i^j$  are functionally independent, given  $X'X$ . As a matter of fact, only  $N-j$  coordinates of  $X^j$  are independent, given  $X'X$ ; this, in view of the fact that  $T$  is triangular and that only  $N-j$  coordinates of  $Y^j$  are independent. This shows that it is possible to get the

distribution of a subsample of  $k$  observations  $\xi$ , given  $X^*X$ , only if the minimum of  $N-j$ , namely  $N-p$ , is at most equal to  $k$ . As an explicit function of the independent  $Y_i^j$  (that we will denote as  $Y^*$ ) the probability distribution of  $Y$ , i.e. of  $Y^*$  is

$$\frac{2^p D(\sum_{N_1}, \dots, \sum_{N-p+1})}{D(Y^*)} \frac{\prod dY_i^j}{\prod \sum_{N-j+1}}$$

To obtain the Jacobian, one can refer to [10] where it is shown that the Jacobian  $D(X)/D(T, Y^*)$  of the change of variables  $X = YT^*$  is given by

$$\frac{D(X)}{D(T, Y^*)} = \frac{D(X, Y^*Y)}{D(T, Y)} \cdot \frac{D(Y^*Y)}{D(Y^{**})}$$

(where  $Y^{**}$  denotes the set of dependent  $Y_i^j$ )

$$\text{Now, } \frac{D(X, Y^*Y)}{D(T, Y)} = 2^p \prod_{i=1}^p f_{ii}^{n-1} \text{ see [10]}$$

$$\text{and } \frac{D(T)}{D(X^*X)} = 2^{-p} \prod_{i=1}^p f_{ii}^{-p+i-1}$$

so that

$$\int (X^*) dX = h(X^*X) \left| X^*X \right|^{(N-p-1)/2} dX^*X \frac{D(Y^{**})}{D(Y^*Y)} dY^*$$

$$\frac{D(\sum_{N_1}, \dots, \sum_{N-p+1})}{D(Y^*)} = \frac{D(Y^{**})}{D(Y^*Y)}$$

the distribution of  $Y^*$  is

$$(2^p / \prod_{N-j+1}^{\infty} \sum) dY^* / [D(Y^*Y) / D(Y^{**})]$$

where the general element of  $Y^*Y$  is  $\sum_1^N Y_h^i Y_h^j$  and of  $Y^{**}$  is  $Y_{N-k+1}^k$  for  $k \leq l$  and 0 for  $k > l$ .

3.6.3 In case  $N = p$ , one can also use Cayley's parametric representation of  $Y$ .

If  $|I+Y| \neq 0$ , i.e. if  $Y$  is non-exceptional, then there exists a skew-symmetric matrix  $S$  such that

$$Y = (I+S)^{-1}(I-S) = (I-S)(I+S)^{-1} = (I-S)/(I+S)$$

$$S = (I-Y)(I+Y)^{-1} = (I+Y)^{-1}(I-Y) = (I-Y)/(I+Y),$$

$S$  non-exceptional since skew symmetric.

if  $|I+Y| = 0$ , i.e. if  $Y$  is exceptional, then it is known that if one defines a sequence  $\{J_r\}$  of diagonal matrices

$$(\pm 1, \dots, \pm 1)$$

$$J_1 = (1, \dots, 1)$$

$$J_2 = (-1, 1, \dots, 1)$$

$$J_3 = (-1, -1, 1, \dots, 1)$$

in such a way that  $J_r$  differs from  $J_{r-1}$  by changing the sign of only one diagonal element and if  $M_r$  is that set of matrices  $Y$  such that

$$|Y+I| = 0$$

$$|J_2 Y + I| = 0$$

$$|J_{r-1} Y + I| = 0$$

$$|J_r Y + I| \neq 0$$

the set of exceptional matrices is  $\bigcup_{r>1} M_r$  and Cayley's represen-

tation for such a matrix is

$$Y = J_r (I-S)/(I+S), S = (I-J_r Y)/(I+J_r Y)$$

for some  $r > 1$

Let us, then, partition the space of  $X$  into the sets  $X^*$  and  $X^{**}$  such that, respectively,

$$X^* = Y^* T^*, Y^* \text{ non-exceptional}$$

$$\text{and } X^{**} = Y^{**} T^{**}, Y^{**} \text{ exceptional,}$$

one has

$$\int f(X^*) dX^* = h(X^{**}) dX^{**}$$

Let us make the change of variables

$$X^* = [(1-S^*)/(1+S^*)] T^{**}$$

whose Jacobian, after Hsu [11] is

$$D(X^*)/D(T^{**} S^*) = \prod_{i=1}^p \frac{p-1}{1} 2^{p(p-1)/2} |I+S^*|^{-(p-1)}$$

then

$$dX^* = 2^{-p} 2^{p(p-1)/2} |X^{**} X^*|^{-1/2} |I+S^*|^{-(p-1)} dX^{**} X^* dS^*$$

hence, the distribution of  $S^*$  given  $X^{**} X^*$  is independent from  $X^{**} X^*$  and given by

$$K dS^* / |I+S^*|^{p-1} \quad \text{with } K = \int_{M_1} dS^* / |I+S^*|^{p-1}$$

$K$  is given in [12] as  $2^{-(p-2)(p-1)/2} \prod_{i=1}^p \pi^{K/2} / \Gamma(K/2)$

In case  $Y$  is exceptional, one makes the transformation

$$X^{**} = J_r [(I-S^{**})/(I+S^{**})] T^{**} \text{ when } Y^{**} \in M_r$$

It results from [12] that  $M_2$  has same measure as  $M_1$  and  $M_r$  has measure zero if  $r > 2$ . Considering only the set  $M_2$  we will have

$$dX^{**} = 2^{-p} 2^{p(p-1)/2} |X^{***} X^{**}|^{-1/2} |I+S^{**}|^{-(p-1)} dX^{***} X^{**} dS^{**}$$

and the distribution of  $S^{**}$  given  $X^{***} X^{**}$  is the same as that of  $S^*$  given  $X^* X^*$ .

Given the proper definition of  $S=S^*$  on  $M_1$ ,  $S=S^{**}$  on  $M_2$  we will have.

$$f(x)dx = 2^{-p} 2^{p(p-1)/2} h(x^*x)(x^*x)^{-1/2} |I+S|^{-(p-1)} dX^*XdS$$

and given  $X^*X$ ,  $S$  has distribution

$$[(2^{p(p-1)/2}) / (\prod_{j=1}^p \sum_{p-j+1}^p)] ds / |I+S|^{p-1} \text{ i.e.}$$

$$\prod_{j=1}^p \frac{2^{-p(p+1)/4} \Gamma_j[(p-j+1)/2] 2^{(p-1)(p-2)/2-1}}{|I+S|^{-(p-1)}} ds$$

3.7 Consider the space defined by all  $X$  given  $X^*X$ , if  $G$  is a group operating transitively on  $X|X^*X$ , there exists at most one probability measure on that space that is invariant under  $G$ .

If we consider the group  $G$  of orthogonal matrices  $O_{n,n}$  operating on  $X$ ,  $X \rightarrow OX$ , it is geometrically obvious that  $G$  operates transitively on  $X|X^*X$ , on the other hand, it is straight forward that the conditional distribution of  $X$  given  $X^*X$  is invariant under  $G$  since  $(OX)^*(OX) = X^*X$ , therefore it is the unique distribution on  $X|X^*X$  invariant under  $G$ .

Suppose now we want to "pick at random" a  $p$  dimensional orthonormal basis in the  $N$  dimensional space; by this is meant choosing  $X$  such that  $X^*X = I$  and that the probability distribution of  $X$  be invariant under the orthogonal group; it suffices to take  $X$  with probability distribution  $f(X)dx = g(X^*X)dx$  and normalize by  $X=YT^*$ ,  $Y$  will be the basis desired. To construct such a basis, one can take  $X^1, \dots, X^p$  and use the Schmidt's orthogonalization process. Alternatively, one sees that one can choose at random a unit vector  $Y^1$  in the  $N$  dimensional space, then a random unit vector  $Y^2$  in the space complement to  $Y^1$ , and so on; at the  $j$ th step, one takes a random unit vector  $Y^j$  in the space complement to that spanned by  $Y^1 \dots Y^{j-1}$ . By random unit vector is meant a vector that is equidistributed on the unit sphere of the  $N-j+1$  dimensional space.

3.8 From 3.5 we see that, in case  $f(X)dX = h(X'X)dX$ , a minimum variance unbiased estimate of a function of the parameters, if it exists and except for a condition of completeness on  $S$ , will have to be found among the solutions of the integral equation

$$\psi_N(s) = \int \dots \int \varphi_k(s) f(s_s | s) ds$$

4.0 Further generalizations of Thompson's distributions can be tried in several directions.

4.1 One can try to obtain the conditional distribution of a sub-sample  $\xi$  of a sample  $X$  with probability distribution  $f(X)dX$ , given the mean  $\bar{X}$  and the variance  $S^2$  of the sample, in the general case, when  $f(X)$  is not spherical. This is possible when one has the probability distribution of the couple of statistics  $\bar{X}, S^2$  for a sample of any size; references [5], [6], [7] deal with the derivation of such a distribution.

One writes the joint distribution of  $\xi = (\xi_1 \dots \xi_k)'$  and of the variance and mean  $S_n^2, \bar{X}_n$  of an independent sample of size  $n = N-k$ , (where  $N$  is the size of  $X$ ), makes the change of variables, (as shown in [2]),

$$\bar{X}_n = \bar{X}N/(N-k) - \bar{\xi}k/(N-k)$$

$$S_n^2 = S^2N/(N-k) - (\bar{\xi} - \bar{X})^2/k^2(N-k)^2 - (\bar{\xi} - \bar{X}) \cdot (\bar{\xi} - \bar{X})/(N-k)$$

whose Jacobian is  $(N/N-k)^2$  and divides by the distribution of  $\bar{X}, S^2$ .

If  $\gamma_n(\bar{X}_n, S_n^2)$  is the probability density of  $\bar{X}_n, S_n^2$  one has

$$f(\xi | \bar{x}, s^2) = \frac{\gamma_n[\bar{x}_n(\bar{x}, \bar{\xi}), s_n^2(\bar{x}, s^2, \bar{\xi}, s_{\xi}^2)] (N/N-k)^2 f(\xi) d\xi}{\gamma_N(\bar{x}, s^2)}$$

over the proper domain (namely  $s_n^2 > 0$ ) and the recurrence formula

$$\gamma_N(\bar{x}, s^2) = \int \dots \int \gamma_n[\bar{x}_n(\bar{x}, \bar{\xi}), s_n^2(\bar{x}, s^2, \bar{\xi}, s_{\xi}^2)] f(\xi) (N/N-k)^2 d\xi$$

where the summation is performed over the proper domain.

This formula may be useful when  $\gamma_n$  can be obtained directly without too much difficulty for small  $n$ .

In case  $n=2$  one will perform the rotation

$$\begin{aligned} Y_1 &= (X_1 - X_2) / \sqrt{2} & X_1 &= (Y_1 + Y_2) / \sqrt{2} \\ Y_2 &= (X_1 + X_2) / \sqrt{2} = \sqrt{2}\bar{X} & X_2 &= (Y_2 - Y_1) / \sqrt{2} \end{aligned}$$

$$\bar{X} = Y_2 / \sqrt{2} \quad s^2 = Y_1^2 / 2.$$

4.2 One can also be interested in the distribution of the studentized  $\bar{X}$ , namely  $t = (\bar{X} - \bar{x}) / S$ .

One writes the joint distribution of  $\xi = (\xi_1, \dots, \xi_k)$ , and of  $s_n^2, \bar{x}_n$ , makes the change of variables  $t = (\bar{X} - \bar{x}_n) / s_n$  and integrates out  $\bar{x}_n, s_n$ ; the density of  $t$  is

$$h(t) = \iint \gamma_n(\bar{x}_n, s_n^2) f(\bar{x}_n + t s_n) s_n^k d\bar{x}_n ds_n$$

then one makes the change of variables

$$\begin{aligned} t &= (t + \bar{t}k/N-k) S / S_n \\ &= (t + \bar{t}k/N-k) / [N/(N-k) - \bar{t}^2 k^2 / (N-k)^2 - t^2 / (N-k)]^{1/2} \\ &= [t + \bar{t}k/(N-k)] / \left\{ (N/N-k) [1 - \bar{t}^2 k / (N-k) - s_{\xi}^2 k / N] \right\}^{1/2} \end{aligned}$$

The Jacobian of the transformation is

$$\begin{aligned} \frac{D(t)}{D(\bar{t})} &= [(N-k)/N]^{(k-2)/2} \left\{ 1 - [\bar{T}^2 k^2 / (N-k) + t^* t] / N \right\}^{-(k+2)/2} \\ &= [(N-k)/N]^{(k-2)/2} [1 - \bar{T}^2 k / (N-k) - S_{\bar{t}}^2 k / N]^{-(k+2)/2} \end{aligned}$$

This can be shown as follows, let

$$\ell = [N - t^* t - k^2 \bar{T}^2 / (N-k)] / (N-k)$$

$$\varphi = t + k \bar{T} / (N-k)$$

direct computations shows that

$$\frac{D(t)}{D(\bar{t})} = \ell^{-3k/2} (N-k)^{-k} \left| [(N-k) \delta_{ij} + 1] \ell + \varphi_i \varphi_j \right|$$

the determinant is that of the matrix

$$(N-k) \ell (I + [1/(N-k)] u^* u + [1/(N-k) \ell] \varphi^* \varphi)$$

where u is a row of one u = (1, ..., 1), it results from the identity

$$\left| I + X_1^* Y_1 + X_2^* Y_2 \right| = \left| \begin{array}{cc} 1 + Y_2 X_2^* & Y_2 X_1^* \\ Y_1 X_2^* & 1 + Y_1 X_1^* \end{array} \right|$$

that this determinant is

$$\begin{aligned} (N-k)^k \ell^k \left[ \left( 1 + \frac{\varphi \varphi^*}{(N-k) \ell} \right) \left( 1 + \frac{k}{N-k} \right) - \frac{k^2 \bar{T}^2}{(N-k)^2 \ell} \right] \\ = (N-k)^{k-2} \ell^{k-1} N^2 \end{aligned}$$

which gives the result above. Then, g(t)dt is easily obtained as

$$g(t) = h(t) \frac{D(t)}{D(\bar{t})}$$

In case k=1 this reduces to

$$t = t[N(N-1)]^{1/2} / (1-t^2/N-1)^{1/2}$$

$$\frac{D(t)}{D(\bar{t})} = [N(N-1)]^{1/2} / (1-t^2/N-1)^{3/2}$$

4.3 Another line of generalization arises from the consideration of distributions admitting a sufficient statistic (scalar or vector). The problem is to find the conditional parameter free distribution of a subsample  $\bar{X}$  of size  $k$  given the sufficient statistic obtained with a sample  $X$  of size  $N$ . Nothing very general can be said about this problem without making further assumptions.

In case the distribution of an individual observation belongs to Koopman's family, with one parameter) namely

$$f(x_i) dx_i = \exp[g(\theta) + t(x_i)h(\theta) + \psi(x_i)] dx_i, h'(\theta) \neq 0$$

a change of scale of the parameter in an interval where  $h(\theta)$  is monotone gives

$$f(x_i) dx_i = g(\theta) e^{t(x_i)\theta + \psi(x_i)} dx_i$$

Then, it is well known that the characteristic function of the sufficient statistic  $T(X) = \sum_1^N t(x_i)$  is

$$\varphi_{T(X)}(u) = [g(\theta)]^N / [g(\theta + iu)]^N$$

This is straight forward since

$$\int e^{iut(x)} f(x) dx = g(\theta) = \int e^{t(x)[i u + \theta] + \psi(x)} dx = g(\theta) / [g(\theta + iu)]$$

In any case, the probability distribution of  $T(X)$  can be obtained by recurrence, using the classical convolution formulas. Under proper conditions of continuity and boundedness, [14] and with obvious notations,

$$\begin{aligned} p_{n+k}(T_{n+k}) &= \int p_k(T_{n+k} - T_k) p_n(T_n) dT_n \\ &= \int p_n(T_{n+k} - T_k) p_k(T_k) dT_k \end{aligned}$$

To obtain the distribution of a subsample  $\xi$  of size  $k$  given the sufficient statistic  $T(X)$  of a sample  $X$  of size  $N$ , one gets the distribution of  $T(\xi_1) \dots T(\xi_k)$  given  $T(X)$  and comes back to the original variable  $\xi$ . The characteristic function of  $T(\xi_1) \dots T(\xi_k)$  is  $[g(\theta)]^k / \prod_j^k g(\theta + i u_j)$ , that of  $T_{n-k} =$

$T(X) - T(\xi)$  is  $[\psi(\theta) / \psi(\theta + i v)]^{N-k}$  that of  $T(\xi_1) \dots T(\xi_k), T_{n-k}$  is  $[g(\theta)]^N / [\psi(\theta + i v)]^{N-k} \prod_j^k g(\theta + i u_j)$ , hence the characteristic

function of  $T(\xi_1) \dots T(\xi_k), T(X)$  is

$$[g(\theta)]^N / [g(\theta + i v)]^{N-k} \prod_j^k g[\theta + i(u_j + v)]$$

since  $E[e^{u_1 T(\xi_1) + \dots + u_k T(\xi_k) + v T(X)}]$

$$= E[e^{(u_1 + v) T(\xi_1) + \dots + (u_k + v) T(\xi_k) + v T_{N-k}}]$$

from which one gets the characteristic function of  $T(\xi_1) \dots T(\xi_k)$  given  $T(X)$  whose characteristic function is

$$[g(\theta)/g(\theta+iv)]^N$$

The result is

$$\frac{\varphi(u_1, \dots, u_k)}{T(\xi_1) \dots T(\xi_k) | T(X)} = \frac{\int e^{-ivT(X)} [g(\theta+iv)]^{k-N} \prod_j^k (g[\theta+i(u_j+v)])^{-1} dv}{\int e^{-ivT(X)} [g(\theta+iv)]^{-N} dv}$$

It is always possible to obtain the probability distribution of  $\xi$  given  $T(X)$  by writing the joint probability distribution of  $T(\xi_1) \dots T(\xi_k)$  and  $T_{N-k} = T(X) - T(\xi)$ , making the change of variables  $T_{N-k} = T(X) - T(\xi)$  and dividing by the distribution of  $T(X)$ . To obtain the distribution of  $T(\xi)$  given  $T(X)$  one writes the joint distribution of  $T(\xi)$  and  $T_{N-k}$ , makes the change of variables  $T_{N-k} = T(X) - T(\xi)$  and divides by the distribution of  $T(X)$ . If  $T = T(X)$  is complete and if  $\{\varphi_n(T_n)\}$  is a sequence of unbiased estimates of a function  $g(\theta)$  of the parameter  $\theta$ , then

$$\varphi_N(T) = \int \varphi_k[T(\xi)] f(T(\xi) | T) dT(\xi)$$

i.e., minimum variance unbiased estimates are to be found among the solutions of the integral equation above. Also,

$$\int \dots \int \psi(\xi) f(\xi | T) d\xi$$

will be an unbiased minimum variance estimate of  $E[\psi(\xi)]$  if it exists.

Since the characteristic function  $I_A(X_i)$  of a set A is an unbiased estimate of

$$\int_A f(x_i) dx_i$$

the best estimate of this measure will be given by

$$\int_A f(\xi_i | T) d\xi_i$$

and consequently the best estimate of  $f(x_i)$  by  $f(\xi_i | T)$ . The discrete case can be treated in a similar way.

4.4 In this paragraph, the straightforward technique mentioned above is applied to several important distributions.

a) Poisson's distribution.

In case

$$P(X_i) = e^{-m} m^{X_i} / X_i!$$

$X_1 + \dots + X_N = N\bar{X}$  is a sufficient statistic, Poisson distributed and it is complete, one obtains

$$P(\xi | \bar{X}) = [1 - (k/N)]^{N\bar{X}} [1/(N-k)]^{k\bar{\xi}} \frac{(N\bar{X})!}{\xi_1! \dots \xi_k! [N\bar{X} - k\bar{\xi}]!} \quad k \neq N$$

on the domain  $N\bar{X} - k\bar{\xi} > 0$

In case  $k=N$ , one obtains the well known result

$$P(\xi | \bar{X}) = (N\bar{X})! / \left( \prod_{i=1}^k N^{\xi_i} \xi_i! \right)$$

the best estimate of  $P(X_i \leq X)$  is

$$P(\xi_i \leq x) = \sum_1^x [1 - (1/N)]^{N\bar{x}} [1/(N-1)]^{\xi_i} \binom{N\bar{x}}{\xi_i}, \quad x \leq N\bar{x}$$

also

$$P(\bar{\xi} | \bar{x}) = (k/N)^k \bar{\xi}^{-k} [1 - (k/N)] \binom{N\bar{x}}{k\bar{\xi}}, \quad N\bar{x} - k\bar{\xi} \geq 0$$

This is a Binomial distribution.

b) Gamma distribution  $\Gamma(\alpha, h)$

$$f(x_i) dx_i = [a^h / \Gamma(h)] x_i^{h-1} e^{-ax_i} dx_i,$$

if  $h$  is known  $\bar{x}$  is sufficient for  $\alpha$ , complete, and  $N\bar{x}$  is  $\Gamma(\alpha, Nh)$  distributed, one obtains

$$f(\bar{\xi} | \bar{x}) d\bar{\xi} = \frac{\Gamma[Nh]}{[\Gamma(h)]^k \Gamma[(N-k)h]} \prod_1^k (\bar{\xi}_i / N\bar{x})^{h-1}$$

$$[1 - (\bar{\xi} / \bar{x})(k/N)]^{(N-k)h-1} d\bar{\xi} / (N\bar{x})^k, \quad \bar{\xi} k \leq N\bar{x}$$

i.e. the  $\bar{\xi}_i / N\bar{x}$  are parameter free

the distribution of  $\bar{\xi} | \bar{x}$  is

$$f(\bar{\xi} | \bar{x}) d\bar{\xi} = \frac{1}{B[kh, (N-k)h]} \left(\frac{k}{N}\right)^{kh-1} \left(1 - \frac{k}{N}\right)^{(N-k)h-1} d(k\bar{\xi} / N\bar{x})$$

i.e.  $k\bar{\xi} / N\bar{x}$  is  $B[kh, (N-k)h]$  distributed.

The best estimate of  $P(X_i \leq x)$  is, for  $x \leq N\bar{x}$ ,

$$\int_0^x \frac{1}{B[h, (N-1)h]} \left(\frac{\xi_i}{N\bar{x}}\right)^{h-1} \left(1 - \frac{\xi_i}{N\bar{x}}\right)^{(N-1)h-1} d\xi_i / N\bar{x} \quad \xi_i \leq N\bar{x}$$

b\*) In case  $h=1$ ,  $\Gamma(\alpha, h)$  is the exponential distribution, one obtains,

$$f(\bar{\xi} | \bar{x}) d\bar{\xi} = [\Gamma(N) / \Gamma(N-k)] \cdot \left(1 - \frac{\bar{\xi}}{\bar{x}}\right)^{N-k-1} \cdot \frac{k}{N} d\bar{\xi} / (N\bar{x})^k$$

$$f(\bar{\xi} | \bar{x}) d\bar{\xi} = [1/B(k, N-k)] \left(\frac{k}{N}\right)^{k-1} \left(1 - \frac{k}{N}\right)^{N-k-1} d(k\bar{\xi}/N\bar{x})$$

$k\bar{\xi} \leq N\bar{x}$

The best estimate of  $P(X_i \leq x)$  is, for  $n \leq N\bar{x}$ ,

$$\int_0^x [1/B(1, N-1)] (1 - \xi_i/N\bar{x})^{N-2} d(\xi_i/N\bar{x}) \quad \xi_i \leq N\bar{x}$$

b") The case of the so-called Weibull's distribution

$$f(X_i) dX_i = \alpha P(PX_i)^{\alpha-1} e^{-(PX_i)^\alpha} dX_i$$

for known  $\alpha$  is not different from that of the exponential distribution except for a change of variable.

c) Binomial distribution

$X_i$  is 0 with probability  $1-p$  and 1 with probability  $p$ ,

$X^* = X_1 + \dots + X_N$  is sufficient, complete and Binomial distributed

$$P(\xi | X^*) = \binom{N-k}{X^* - \sum \xi_i} / \binom{N}{X^*}$$

with  $\xi^* = \xi_1 + \dots + \xi_k$

$$P(\xi^* | X^*) = \binom{k}{\xi^*} \binom{N-k}{X^* - \xi^*} / \binom{N}{X^*}$$

a hypergeometric distribution.

d) Multinomial distribution

$$P(X^1 \dots X^m) = (N! / X^1! \dots X^m!) p_1^{X^1} \dots p_m^{X^m}, \quad X^1 + \dots + X^m = N$$

$(X^1, \dots, X^m)$  is a complete sufficient statistic  $X$ ,  $\xi = (\xi^1, \dots, \xi^m)$

$$P(\xi | X) = \frac{X_1^{\xi_1} \dots X_m^{\xi_m}}{\xi^k} \quad \xi^i \leq X^i$$

e) Rectangular distribution,

$$f(X_i) dX_i = dX_i / \theta, \quad 0 \leq X_i \leq \theta, \theta > 0$$

The observations being ordered  $X_{(1)} \leq \dots \leq X_{(i)} \leq \dots \leq X_{(N)}$ ,  
 $X_{(N)}$  is a sufficient and complete statistic.

One picks a set of  $k$  observations  $\xi_{(1)} \dots \xi_{(k)}$ .

The probability that

$$\xi_{(k)} = X_{(N)} \text{ is } k/N \text{ and that } \xi_{(k)} \neq X_{(N)} \text{ is } [1 - (k/N)].$$

$$\text{If } \xi_{(k)} \neq X_{(N)} \quad \xi = (\xi_{(1)} \dots \xi_{(k)})$$

$$f(\xi | X_{(N)}) d\xi = k! [(N-k)/N] d\xi / X_{(N)}^k \quad 0 \leq \xi_{(k)} < X_{(N)}$$

$$f(\xi_{(k)} | X_{(N)}) d\xi_{(k)} = [(k(N-k))/N] (\xi_{(k)} / X_{(N)})^{k-1} d\xi_{(k)} / X_{(N)} \quad 0 \leq \xi_{(k)} < X_{(N)}$$

f) Epstein-Sobel's distribution

$$f(X_i) dX_i = \alpha e^{-\alpha(X_i - \theta)} dX_i \quad X_i \geq \theta$$

$$X_{(1)} \leq \dots \leq X_{(i)} \leq \dots \leq X_{(N)}$$

then  $(X_{(1)}, \bar{X} - X_{(1)})$  is a sufficient and complete statistic.

The probability that  $\xi_{(1)} = X_{(1)}$  is  $k/N$  and that  $\xi_{(1)} \neq X_{(1)}$  is  $1 - (k/N)$ .

The distribution of  $(X_{(1)}, \bar{X} - X_{(1)} = Y)$  is

$$\frac{(\alpha N)^N}{\Gamma(N-1)} \gamma^{N-2} e^{-\alpha(X_{(1)} + Y - \theta)} dX_{(1)} dY$$

Considering  $\xi_{(1)} \neq X_{(1)}$ , let  $X^c$  be the complementary sample of  $\xi$ , then

$$x_{(1)}^* = x_{(1)}$$

$$Y = \bar{X} - x_{(1)} = \frac{(N-k)\bar{X} + k\bar{\xi}}{N} - x_{(1)} = [(N-k)/N]Y' + (k/N)(\bar{\xi} - x_{(1)})$$

$$\text{i.e. } Y' = [NY/(N-k)] - (k/N)(\bar{\xi} - x_{(1)})$$

the usual technique gives

$$f(\bar{\xi} | x_{(1)}, \bar{X} - x_{(1)}) = \frac{k!(N-2)!(N-k)}{(N-k-2)!N} \left[ 1 - \frac{k}{N} \frac{\bar{\xi} - x_{(1)}}{\bar{X} - x_{(1)}} \right]^{N-k-2} d\bar{\xi} / N^k (\bar{X} - x_{(1)})^k$$

in the domain where the quantity between brackets is positive.

$$f(\bar{\xi}_{(1)}, \bar{\xi} - \bar{\xi}_{(1)} | x_{(1)}, \bar{X} - x_{(1)}) = k(k-1) \binom{N-2}{k} [(N-k)/N] \left[ \frac{k(\bar{\xi} - \bar{\xi}_{(1)})}{N(\bar{X} - x_{(1)})} \right]^{k-2} \left[ 1 - \frac{k(\bar{\xi} - x_{(1)})}{N(\bar{X} - x_{(1)})} \right]^{N-k-2} \frac{dk \bar{\xi}_{(1)}}{N(\bar{X} - x_{(1)})} \cdot \frac{dk(\bar{\xi} - \bar{\xi}_{(1)})}{N(\bar{X} - x_{(1)})}$$

in the same domain.

The first formula is in agreement with a result by R. F. Tate, [15], p. 361, formula (7.8).

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Log-Normal Distribution  
Estimation Problems

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The use of the Log-Normal distribution has been rarely advocated in the literature devoted to industrial life testing and failure theory, though considerable emphasis has been put on that tool in papers dealing with sensitivity data and it has been widely used as a model in the field of biological assays. (Refer, however, to "The Exponential Distribution and Its Role in Life Testing", Technical Report No. 2, May, 1958, by B. Epstein).

A somewhat extensive exposition of the history of the Log-Normal distribution is given in [1].

A broad variety of realistic and "reasonable" hypotheses about the mechanism of failure leads to a Log-Normal distribution for the age of death or the intensity of the stimulus under which failure occurs. Among them, that class of hypotheses which involve random effects that are multiplicative instead of additive and lead to the use of the central limit theorem. For example, if failure is caused by an accumulation of infinitesimal random shocks whose effects are not independent but are sequentially proportional to the already accumulated total effect. This is a well known set up and a plausible model for failure data when an ageing process takes place.

Let  $X$  be Log-normal distributed, i.e.  $Y = \text{Log } X$  is  $N(m, \sigma^2)$  distributed. The moments of  $X$  are

$$M_r(X) = e^{r^2 \sigma^2 / 2} e^{rm}$$

whence the expected value and variance of  $X$

$$\mu_1 = E[X] = e^{\sigma^2 / 2} e^m$$

$$\mu_2 = \text{var}[X] = e^{2m} (e^{\sigma^2} - 1)$$

while

$$E[Y] = m, \text{ logarithmic mean}$$

$$\text{var}[Y] = \sigma^2, \text{ logarithmic variance}$$

the minimum variance unbiased estimates of  $m$  and  $e^{-\sigma^2}$  being  $\bar{Y}$  and  $[n/(n-1)]S_Y^2$ .

$m$  and  $\sigma^2$  are measures of central tendency and dispersion in the logarithmic scale, useful corresponding characteristics in the original scale of  $X$  are  $\mu = e^m$  that is the median of the distribution and  $\sigma^2 = e^{\sigma^2}$  or  $\sigma = e^{\sigma^2 / 2}$ .

"Naive" estimates of  $\mu$  and  $\sigma^2$  are  $e^{\bar{Y}}$  and  $e^{S_Y^2}$ .

$$e^{\bar{Y}} = \exp\left[\frac{\sum \log X_i}{n}\right] = \left(\prod X_i\right)^{1/n}$$

is the geometric mean of the  $X_i$ 's

These two estimates are not unbiased;

$$\text{since } \bar{Y} \text{ is } N(m, \sigma^2/n) \quad E[e^{\bar{Y}}] = e^m \cdot e^{\sigma^2 / 2n}$$

and since the generating function of  $S^2$  is  $(1 - 2\sigma^2/n)^{-(n-1)/2}$

$$E[e^{S^2}] = (1 - 2\sigma^2/n)^{-(n-1)/2}$$

We want to obtain estimates without bias

$$1) e^{-\sigma^2/2n} = \sum_k [(-1)^k/k!] [\sigma^{2k}/(2n)^k]$$

also

$$E[S_Y^{2k}] = 2^k \frac{\Gamma[k+(n-1)/2]}{\Gamma[(n-1)/2]} (\sigma^{2k}/n^k)$$

$$\therefore (n^k/2^k) \frac{\Gamma[(n-1)/2]}{\Gamma[k+(n-1)/2]} S_Y^{2k} \text{ is an unbiased estimate of } \sigma^{2k}$$

therefore

$$\Gamma[(n-1)/2] \sum_k [(-1)^k/k!] \frac{S_Y^{2k}}{\Gamma[k+(n-1)/2] 2^{2k}} \text{ is an unbiased estimate of } e^{-\sigma^2/2n} \text{ but this is}$$

$$[2^{(n-3)/2}/S_Y^{(n-3)/2}] \Gamma[(n-1)/2] J_{(n-3)/2}(S_Y)$$

where  $J$  denotes the Bessel function of first kind of order  $(n-3)/2$ .

Since  $\bar{Y}$  and  $S_Y$  are independent variables

$$e^{\bar{Y}} [2^{(n-3)/2}/S_Y^{(n-3)/2}] \Gamma[(n-1)/2] J_{(n-3)/2}(S_Y)$$

is an unbiased estimate of  $e^{\mu} = \mu$ ; since  $\bar{Y}$  and  $S_Y$  are sufficient statistics, it is a minimum variance unbiased estimate of  $\mu$ .

2) One has

$$e^{h\sigma^2} = \sum_k [h^k/k!] \sigma^{2k}$$

therefore

$$\Gamma[(n-1)/2] \sum_k [h^k/k!] [n^k/2^k] \frac{S_Y^{2k}}{\Gamma[k+(n-1)/2]} \text{ is an un-}$$

biased estimate of  $e^{h\sigma^2}$ , now this is

$$\Gamma^{[(n-1)/2]} \frac{2^{(n-3)/2}}{(S_Y \sqrt{2hn})^{(n-3)/2}} I_{(n-3)/2}(S_Y \sqrt{2hn})$$

where I denotes the modified Bessel function of first kind of order  $(n-3)/2$ .

Since  $S_Y$  is a sufficient statistic, the minimum variance unbiased estimate of  $e^{\sigma^2}$  and  $e^{\sigma^2/2}$  are respectively

$$\Gamma^{[(n-1)/2]} \frac{2^{(n-3)/2}}{(S_Y \sqrt{2n})^{(n-3)/2}} I_{(n-3)/2}(S_Y \sqrt{2n}) \quad \text{and}$$

$$\Gamma^{[(n-1)/2]} \frac{2^{(n-3)/2}}{(S_Y \sqrt{n})^{(n-3)/2}} I_{(n-3)/2}(S_Y \sqrt{n})$$

As a by-product, one will obtain the minimum variance unbiased estimate of any moment  $M_r(x)$  of X.

Since

$$E[e^{r\bar{Y}}] = e^{rm} \cdot e^{r^2 \sigma^2 / 2n}$$

depends only <sup>on</sup>  $\bar{Y}$ , and we have an unbiased estimate of  $e^{h\sigma^2}$  that depends only on  $S_Y$ , making  $h = (n-1)r^2 / 2n$ , we get a minimum variance unbiased estimate of  $M_r$  as

$$e^{r\bar{Y}} \cdot \Gamma^{[(n-1)/2]} \frac{2^{(n-3)/2}}{(rS_Y \sqrt{n-1})^{(n-3)/2}} I_{(n-3)/2}(rS_Y \sqrt{n-1})$$

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