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WADC TECHNICAL REPORT 59-509 96-500

DAMPING OF FLEXURAL VIBRATIONS BY ALTERNATE VISCO-ELASTIC AND ELASTIC LAYERS

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WADC TECHNICAL REPORT 59-509

**DAMPING OF FLEXURAL VIBRATIONS BY ALTERNATE
VISCO-ELASTIC AND ELASTIC LAYERS**

Eric E. Ungar, Donald Ross, and Edward M. Kerwin, Jr.

*Bolt Beranek and Newman Inc.
Cambridge, Massachusetts*

NOVEMBER 1959

Materials Laboratory
Contract No. AF 33(616)-5426 (453 700)
Project No. 7360

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AIR RESEARCH AND DEVELOPMENT COMMAND
UNITED STATES AIR FORCE
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500 — January 1960 — 18-506

FOREWORD

This report was prepared by Bolt Beranek and Newman Inc., Cambridge, Massachusetts, under subcontract of the basic USAF Contract No. AF 33(616)-5426 with the University of Minnesota, Department of Aeronautical Engineering. The contract was initiated under Project No. 7360, "Materials Analyses and Evaluation Techniques", Task No. 73604, "Fatigue and Creep of Materials". The work was monitored by the Materials Laboratory, Directorate of Laboratories, Wright Air Development Center, under the direction of Mr. W. J. Trapp.

This report covers work during the period of December 1958 to July 1959.

WADC TR 59-509

ABSTRACT

Previous work dealing with the damping of flexural vibrations by application of single "damping tapes" consisting of metal foils and dissipative adhesives is summarized and extended to multiple tapes. A general analysis of damping due to N equal tapes is presented; the effect of using non-equal tapes is investigated for double tape applications. Suitable dimensionless parameters are used where possible in order to maintain generality.

It is shown that additional tapes provide a considerable increase in damping at low frequencies, but only a very small increase at high frequencies. It is found that multiple tapes and single tapes incorporating an equivalent amount of metal provide nearly the same damping, a fact which results in great design flexibility.

Experimental and theoretically predicted results are shown to be in reasonably good agreement.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:



WALTER J. TRAPP
Chief, Strength and Dynamics Branch
Metals and Ceramics Division
Materials Laboratory

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DAMPING OF FLEXURAL VIBRATIONS BY ALTERNATE VISCO-ELASTIC AND ELASTIC LAYERS

SECTION I

INTRODUCTION

The recent increase in the occurrence of intense noise and vibration environments has led to a search for improved methods of damping of flexural vibrations. Damping treatments such as automobile undercoat use the extensional motion of single homogeneous visco-elastic layers to obtain dissipation of vibrational energy. Extensive analyses of such single-layer treatments appear in the literature¹⁻⁶/*

Other damping treatments, such as the damping tapes which are being used in aircraft, ships, and appliances employ a visco-elastic layer in conjunction with a relatively stiff constraining layer. Kerwin^{7,8} has shown that in such damping tapes the dissipation of energy may be ascribed essentially to the shearing motion of the constrained visco-elastic layer. In a recent report, Röss, Kerwin, and Dyer⁹ dealt with the design and optimization of damping treatments consisting of single tapes.

As a step in the direction of optimization of damping treatments one is naturally led to consider the use of a number of tapes, one applied on top of the other. How does the damping increase due to the addition of a second, third, etc. tape vary with frequency? How does a multiple tape treatment compare to a single tape treatment of equal weight? We attempt to answer these questions in the present report.

* Superscript numbers refer to the Bibliography at the end of this report.

Manuscript released by the authors in September 1959 for publication as a WADC Technical Report.

SECTION II

SUMMARY OF RESULTS FOR SINGLE TAPE TREATMENT

It is convenient to review some of the results obtained for single tapes, so that multiple tape analyses may be understood more easily. A basic plate to which a single damping tape has been applied comprises a three-layer structure composed of

- 1) the basic plate to be damped
- 2) a visco-elastic (dissipative adhesive) layer, and
- 3) an outer "constraining" layer, generally of metal foil.

Figures 8 and 9 appearing on page 20 of the Appendix may be used to visualize this construction and define the thicknesses H_1 , H_2 , H_3 of the three aforementioned layers and the distances H_{31} and D .

The damping effectiveness η of such a three-layer arrangement* is given by^{9/}

$$\frac{\eta}{B} = \frac{12k_3 h_3^2 g}{[1 + k_3 h_3^2][1 + 2g(1 + k_3 + g^2 + \beta^2)(1 + k_3)^2] + 12k_3 h_{31}^2 g[1 + g(1 + \beta^2)(1 + k_3)]} \quad (1)$$

where

β = loss factor of visco-elastic adhesive**

$$h_{31} = H_{31}/H_1, \quad h_3 = H_3/H_1, \quad k_3 = K_3/K_1 \quad (2a)$$

and where g is a convenient "shear parameter" defined by

$$g \equiv G_2/K_3 H_2 b^2 \quad (2b)$$

in terms of the shear modulus G_2 of the adhesive, the stiffness K_3 of the constraining layer, the thickness H_2 of the adhesive, and

* Damping effectiveness η is defined as the imaginary portion of the complex combined flexural rigidity B^* , that is, $B^* = B(1 + j\eta)$. The real part B is associated with elastic behavior, whereas the imaginary part $B\eta$ is associated with hysteretic, or dissipative, behavior.

** β denotes the imaginary portion of the complex shear modulus G_2^* of the visco-elastic adhesive; $G_2^* = G_2(1 + j\beta)$. The real part of G_2^* accounts for the elastic, the imaginary part for the hysteretic action of the adhesive.

the wave number p of the vibration. The wave number associated with the n^{th} mode may be computed from

$$p^4 = \omega_n^2 \left(\frac{M}{B} \right) \quad (3)$$

where m denotes the plate mass per unit length in the direction of propagation of a flexural wave and ω_n is the circular natural frequency corresponding to the n^{th} mode.

The previously defined shear parameter g has been found to be a convenient dimensionless quantity which characterizes the tape. Kerwin^{7,8/} has shown that $\sqrt{g}$ is directly proportional to the ratio of the length of a flexural wave to the distance within which a localized shear disturbance decays to $1/e$ times its initial value. Alternately, since p is inversely proportional to the bending wavelength, one finds that g is also proportional to the ratio K_s/K_t , where K_s is the shear stiffness of unit width of an adhesive layer one wavelength long and where K_t is the tension stiffness of unit width of a constraining layer one wavelength long. Thus, g is inversely proportional to frequency and directly proportional to K_{s2}/K_3 , where the K 's refer to the same stiffness as before, except that now unit length is implied.

Equation (1) is quite cumbersome. However, if the constraining layer is considerably less stiff than the basic plate, as fortunately is the case in many practical applications, then $K_3/K_1 = k_3 \ll 1$, and Eq (1) reduces to

$$\eta(1) = \frac{12k_3 h_{31}^2 \beta g}{1 + 2g + g^2(1 + \beta^2)} \quad (4)$$

where the (1) appended to the η is intended to signify that the relation applies for a single tape application. From the foregoing one may readily verify that for fixed β , η increases nearly linearly with g for small values of g , reaches a maximum at $g = (1 + \beta^2)^{-1/2}$, and decreases nearly linearly for very large g . The curve labeled "single tape" in Fig 3 is a plot of Eq (4) for $\beta = 1.0$. Equation (4) may be used as a basis for the design of optimum single damping tapes; however, in practice the frequency and temperature-dependence of the loss factor complicates the problem^{7,8/}.

SECTION III

TREATMENTS CONSISTING OF TWO TAPES

Since the damping performance of a single tape treatment is definitely limited in accordance with the previous discussion, one might well be interested in investigating the possible benefit one may derive from the use of the two tapes, one applied over the other. As derived in the Appendix, the loss-factor obtained with a double-tape treatment may be expressed as

$$\eta = 12\kappa_3 \beta h_{31}^2 P_N / P_D,$$

$$P_N = g[r^2 + 2rg + sg^2(1 + \beta^2)] + \alpha_{53} \frac{h_{51}^2}{h_{31}^2} g^2 [2r + (r+s)g(1 + \beta^2)] \quad (5)$$

$$P_D = r^2 + 2r(r+s)g + 2rg^2(1 - \beta^2) + (r+s)^2 g^2(1 + \beta^2) \\ + 2(r+s)g^3(1 + \beta^2) + g^4(1 + \beta^2)^2$$

if the thicknesses and stiffnesses of the layers comprising the tapes are considerably smaller than the corresponding properties of the basic plate. The parameters r , s and α_{53} appearing in the foregoing equation are related to the thicknesses according to

$$r = H_4 H_5 / H_2 H_3, \quad s = 1 + \alpha_{53}, \quad \alpha_{53} = H_5 / H_3 \quad (6)$$

Also, h_{51} is defined as

$$h_{51} = H_{51} / H_1 \quad (7)$$

where the dimension H_{51} is the distance from the middle of layer 5 (the outer constraining layer) to the mid-plane of the basic plate, as shown in Fig (9).

Inspection of Eq (5) reveals that for two tapes, as for a single one, η varies essentially as g for very small values of g , and as $1/g$ for very large values of g . Hence, η must reach a maximum at intermediate g and the damping curve for a double tape application possesses the same general characteristics as that for a single tape. The results of computations based on Eq (5) further show that the damping obtained with two tapes is very nearly independent of the ratios of the dimensions of the layers comprising the two tapes. [These ratios enter the relation in terms of the parameters r , s , α_{53} .] The damping curve is found to depend essentially only on the total thickness of the constraining layers, i.e., the total weight of the tape.

Figure 1 illustrates this independence of tape thickness ratios. It shows the "reduced" damping factor per total tape extensional stiffness

$$\frac{\eta}{12(k_3+k_5)h_{31}^2\beta} = \frac{\eta}{12sk_3h_{31}^2\beta}$$

as a function of a modified shear parameter

$$g = \frac{G}{(K_3+K_5)H_2p^2} = \frac{g}{s}$$

which is also defined in terms of the total tape stiffness, for several double-tape applications having the same total foil thicknesses (i.e., essentially same total weight) but different thickness ratios. For this illustration both tapes were assumed to be made of the same material, and β was assumed constant and equal to unity. The curves for all cases considered, except one, coincide so very nearly with each other and with that for a single tape of equal weight, that they were not labeled individually in the Figure. Even the case for which the damping curve deviates most strongly from the others of the group results in the same damping curve maximum and shape. (This case corresponds to the condition where each of the layers of the outer tape is twice as thick as the corresponding layer of the tape directly in contact with the basic plate.)

For two identical tapes $r = 1$ and $s = 2$, and Eq (5) reduces to

$$\eta(2) = \frac{12k_3h_{31}^2\beta g[1+4g+5(1+\beta^2)g^2]}{1+6g+(11+7\beta^2)g^2+6(1+\beta^2)g^3+(1+\beta^2)^2g^4} \quad (8)$$

where the (2) appended to the η indicates applicability of the relation to two tapes. In view of the foregoing discussion this equation also gives a good approximation to the damping behavior of non-identical tapes. It is instructive to compute the ratio

$$\frac{\eta(2)}{\eta(1)} = \frac{1+6g+(14+6\beta^2)g^2+14(1+\beta^2)g^3+5(1+\beta^2)^2g^4}{1+6g+(11+7\beta^2)g^2+6(1+\beta^2)g^3+(1+\beta^2)^2g^4} \quad (9)$$

from Eqs (4) and (8) in order to compare the damping effectiveness of a single tape to that obtained by use of two tapes of the same construction as the single tape. Figure 2 shows the variation of this ratio with the shear parameter g for three typical values of the loss factor β . One may observe that for low values of g (corresponding to high frequencies) addition of a second tape contributes little additional damping, but that for large g (low frequencies) the double tape configuration tends to be 5 times as effective as a single tape. The damping curves for single and double tapes having $\beta = 1$ are compared in Fig 3. It shows that the aforementioned damping improvement for large values of g may be visualized as due

to an upward shifting of the single tape curve plus a translation of this curve toward higher g .

This shifting of the damping curve occurs at least in part as a result of increasing the total constraining layer stiffness by addition of the second tape. Thus, one may obtain a better evaluation of the effectiveness of dual tapes by comparing a double tape application to one consisting of a single tape whose foil thickness is equal to the total foil thickness of the double tape (or to twice the thickness of the foil of a single tape). The damping effectiveness of such a single tape with double foil thickness varies with the shear parameter g (for the original single foil thickness) as

$$\eta'(1) = \frac{12k_3 \beta h_{31}^2 g}{1 + g + \frac{1}{4}(1 + \beta^2)g^2} \quad (10)$$

which may be obtained by setting $s = 2$ and $r = 0$ in Eq (5). By use of Eq (8) one may then compute the ratio

$$\frac{\eta(2)}{\eta'(1)} = \frac{1 + 5g + \frac{1}{4}(37 + 21\beta^2)g^2 + 6(1 + \beta^2)g^3 + \frac{5}{4}(1 + \beta^2)^2 g^4}{1 + 6g + (11 + 7\beta^2)g^2 + 6(1 + \beta^2)g^3 + (1 + \beta^2)^2 g^4} \quad (11)$$

which compares the damping effectiveness of a double tape to a single tape of equal foil thickness (or weight). As is apparent from Fig 4, which is a plot of Eq (11) for three typical values of β , this ratio does not differ markedly from unity. Its minimum value is about 0.85 and occurs for g of the order of 0.5, and it approaches its maximum value of $5/4$ asymptotically as g becomes very large. Therefore, one may conclude that two layers of damping tape are not inherently more effective than a single tape whose constraining layer is of the same thickness as the total of the two corresponding layers of the double tape application.

SECTION IV

MULTIPLE TAPE TREATMENTS

The theory applicable to N identical thin tapes is applied in the Appendix to the computation of the damping effectiveness $\eta(3)$ and $\eta(4)$ of triple and quadruple tape applications. The resulting expressions appearing in Eqs (I-52) and (I-55). One finds that such multiple tape treatments possess characteristics similar to those of double tapes; for small g (high frequencies) additional tapes contribute very little added damping, but for large g (low frequencies) additional tapes lead to considerably improved damping effectiveness. For large g the triple tape tends to be 14 times as effective as a single tape; for quadruple tapes the corresponding factor is 30. However, as with double tapes, this advantage of multiple tapes is considerably reduced when these are compared to single tapes of equal total foil thickness. One finds that $\eta(3)/\eta'(1)$ and $\eta(4)/\eta'(1)$ are somewhat less than unity for values of g of the order of 0.1 and approach 1.55 and 1.87, respectively, for large g.

The low frequency advantage of multiple tapes vs single tapes seems to obey the recursion relation

$$\left[\frac{\eta(N)}{\eta(1)} \right]_{g \rightarrow \infty} = \left[\frac{\eta(N-1)}{\eta(1)} \right]_{g \rightarrow \infty} + N^2 \quad (12)$$

where $\eta(N)$ denotes the damping effectiveness of N tapes. From the equations presented in this report it may easily be verified that this relation holds for N up to 4. We have verified its validity for N through 6, but have omitted the details here for the sake of brevity. Although we have not proven the general validity of the foregoing expression, the recurrence of the manipulations inherent in the multiple-tape theory outlined in the Appendix makes the validity of such a recursion relation extremely likely. The results of applying Eq (12) for N up to 10 are plotted in Fig 6.

SECTION V

EXPERIMENTAL VERIFICATION

In order to obtain some check on the validity of the foregoing theoretical results, we undertook to compare these to experimental data incidentally obtained by Bolt Beranek and Newman Inc. in connection with a study sponsored by the Convair Division, General Dynamics Corporation. The results of carrying out such a comparison for a typical tape appear in graphic form in Figures 6 and 7.

The particular basic single tape to which these figures pertain is No. 425 Sound Damping Tape manufactured by Minnesota Mining and Manufacturing Co. It consists of a 2.5 mil layer of adhesive attached to a 3 mil aluminum foil. The experiments were performed with this tape applied to 0.125 inch thick aluminum test bars, using the apparatus and methods described in reference 9.

Figure 6 shows how the damping effectivenesses obtained with one, two and four identical tapes vary with frequency. For convenience, a scale of shear parameter g is also given. The solid lines of the graph, representing theoretically calculated results are seen to be in reasonably good agreement as to magnitudes and trends with the dashed curves, which represent the experimental data.

Figure 7 shows the ratios of damping effectivenesses $\eta(2)/\eta(1)$ and $\eta(4)/\eta(2)$ as functions of the shear parameter g , which is an inverse function of frequency. Figure 7 was obtained directly from the data of the previous figure in order to demonstrate more effectively the relatively good agreement of the theoretically predicted with the experimentally obtained effectiveness ratios.

There are, unfortunately, some uncertainties involved in the experimental results, and also in some of the data concerning the adhesive properties which were used in the theoretical calculations. Because of the size of the experimental specimens used, the low frequency results (below about 100 cps) are probably somewhat in error. Also, since it is difficult to apply tapes uniformly and to measure the resulting adhesive thicknesses, the values of these are somewhat uncertain. The values of the dynamic shear modulus G and loss factor β of the adhesive, which were used in the computation, were obtained by smoothing and extrapolation of a relatively small amount of experimental data (supplied by the manufacturer*). These values thus are also likely to contribute to discrepancies.

* Data and extrapolations appear in Fig 16 of Reference 9.

In view of these uncertainties, the agreement between theory and experiment indicated in the two aforementioned figures is rather gratifying. It may be noted that the corrections of adhesive thickness H_2 and shear modulus G_2 would result in corrections of the calculated shear parameter g . Such corrections would have the effect of translating the theoretical curves with respect to the experimental ones, and might result in even better agreement.

SECTION VI

CONCLUSIONS

The analyses presented in the present report point out that the use of additional tapes results in additional damping at all frequencies. The increase may be negligible at "high" frequencies, however, even though it may be quite considerable at "low" frequencies - especially for a large number of tapes.

On the other hand, it was demonstrated that it is essentially only the sum of all constraining layer thicknesses that determines the damping characteristics of a damping tape treatment with a given adhesive material; the number of tapes used and their relative thicknesses have only a small effect. Thus, a single tape with a thicker foil is equivalent to several tapes of essentially the same weight insofar as damping is concerned.

Besides the small theoretical damping advantage multiple tapes have at low frequencies over a single tape of equal weight, multiple tapes may be considerably more desirable from a practical viewpoint. They may be applied in confined spaces or to strongly curved areas, where handling of a single stiff tape may be difficult. In addition, there exists the economic advantage that one or two standard tapes may be superposed to obtain the damping effect of single tapes that otherwise would have to be specially manufactured for each application.

SECTION VII

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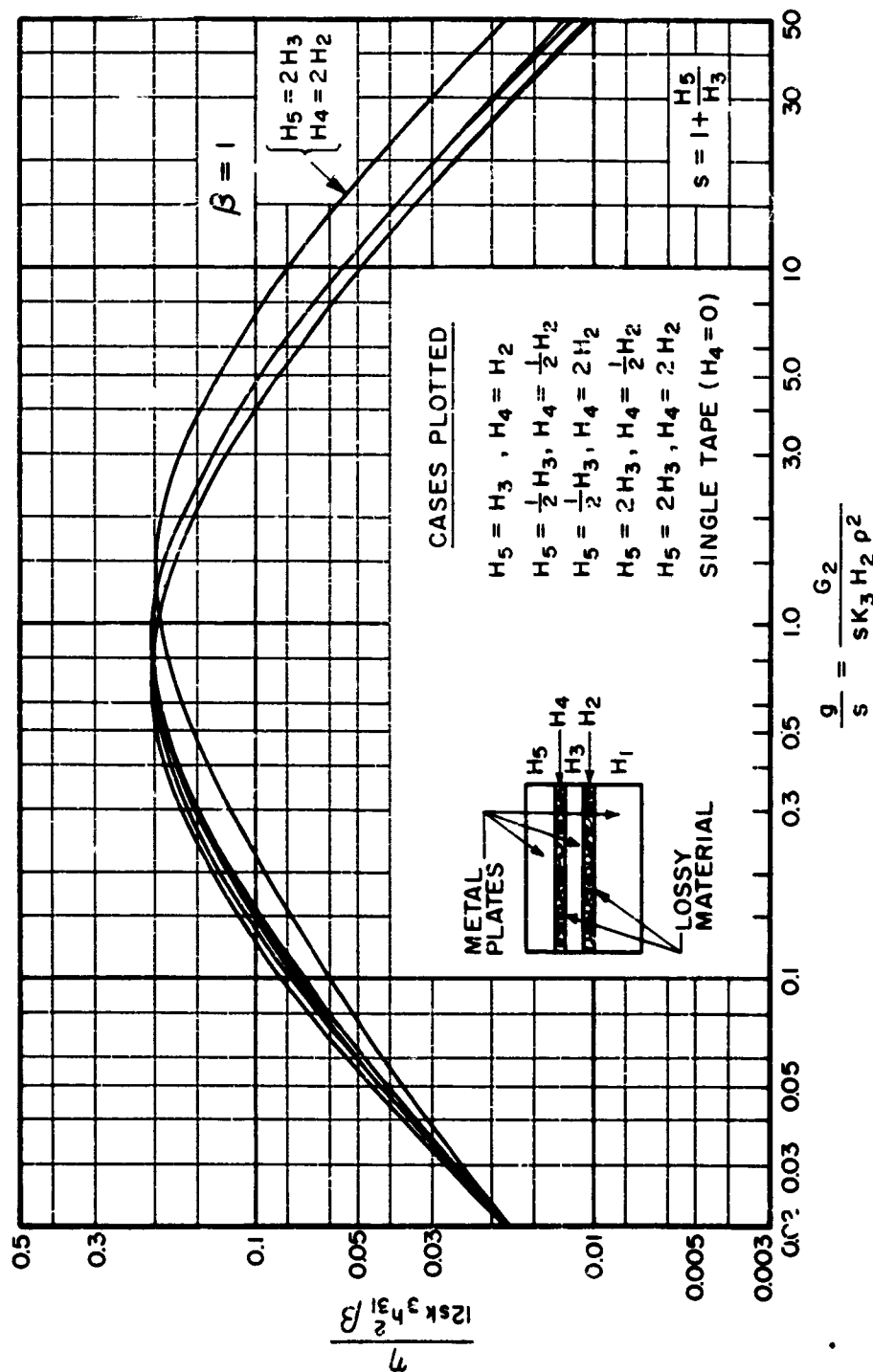


FIG. 1 DAMPING CURVES FOR 6 TAPES HAVING SAME TOTAL FOIL THICKNESS.

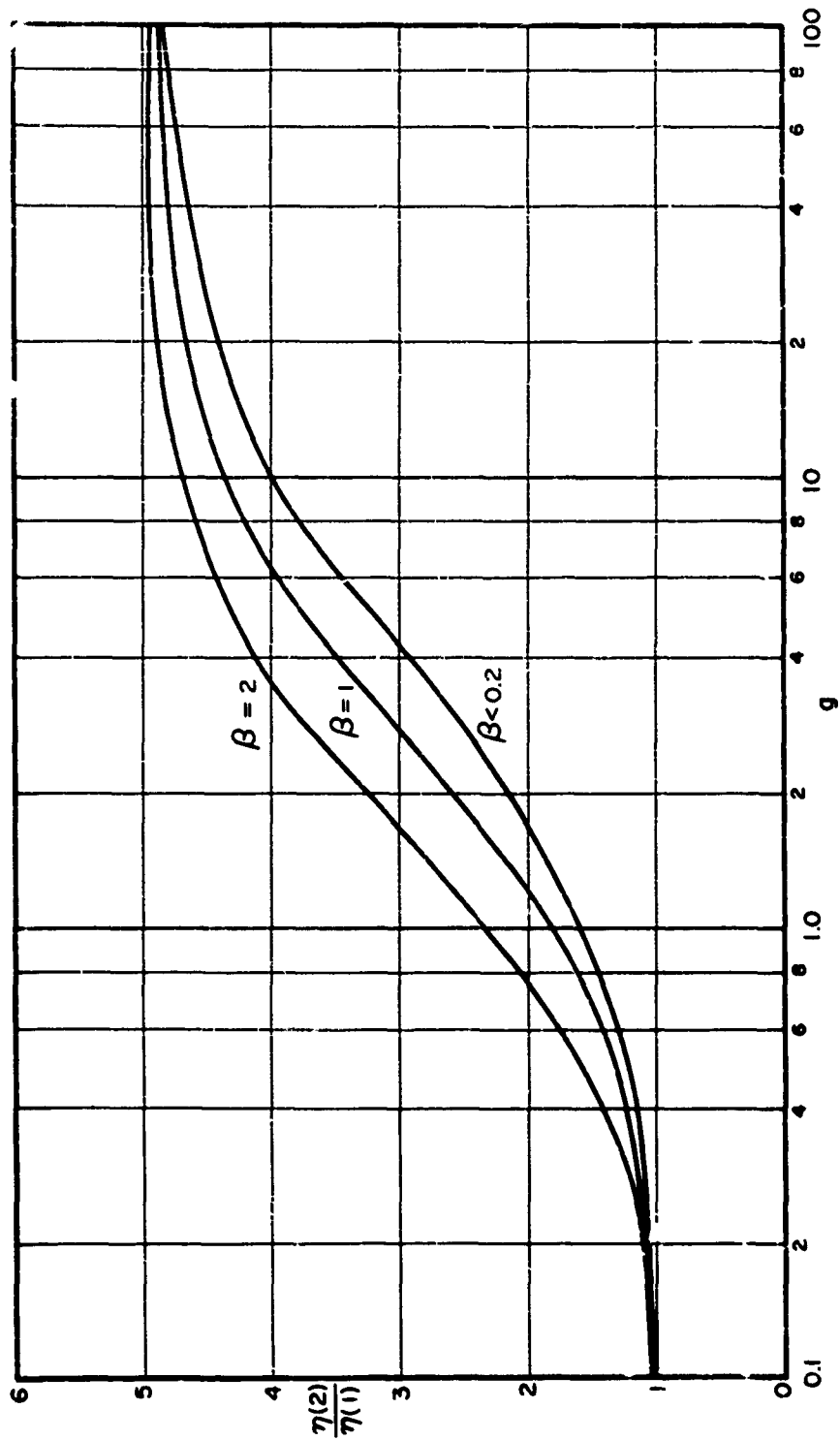


FIG. 2 VARIATION OF RATIO OF DOUBLE TAPE TO SINGLE TAPE DAMPING WITH SHEAR PARAMETER FOR THREE VALUES OF ADHESIVE LOSS FACTOR.

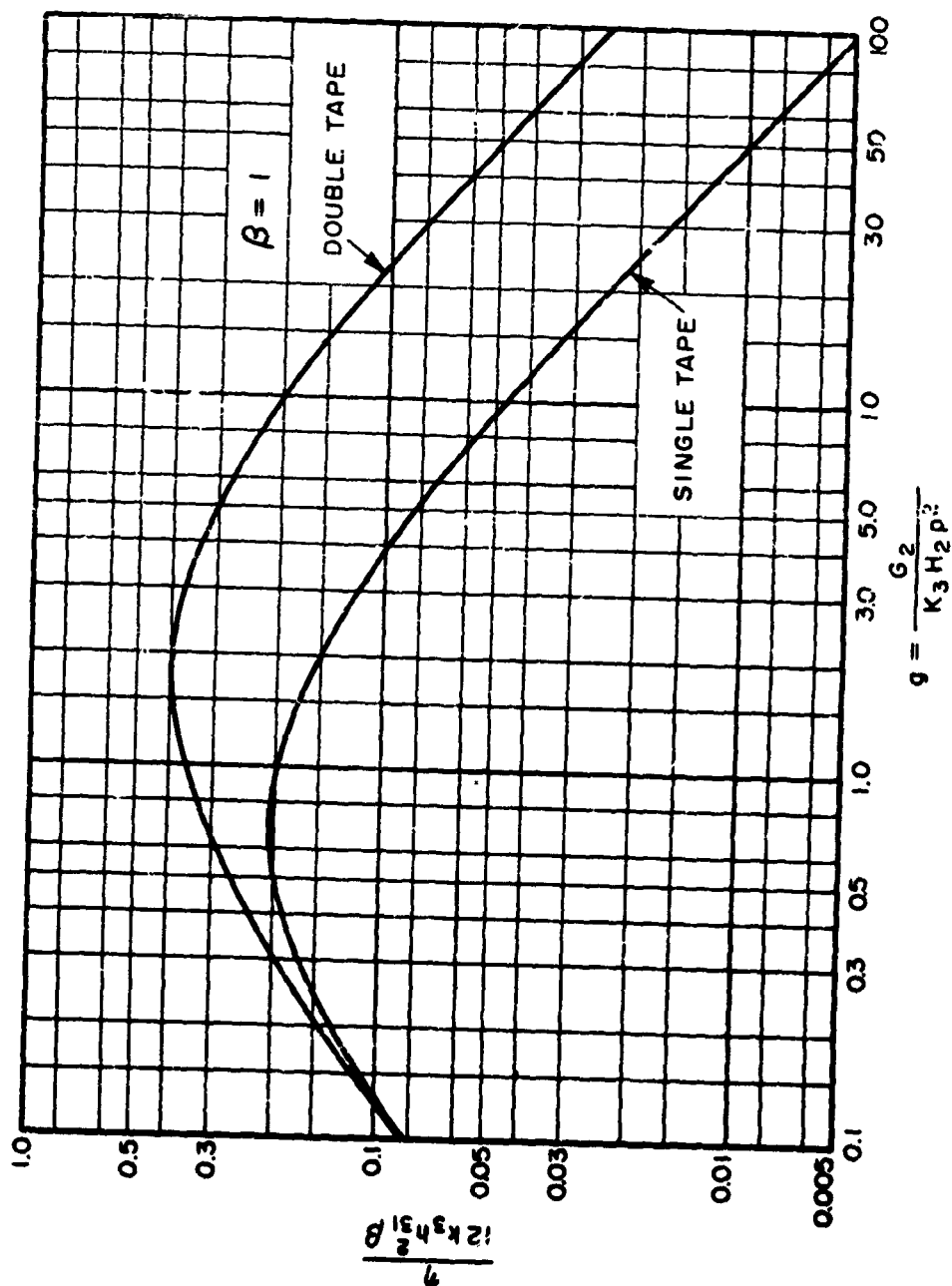


FIG. 3 COMPARISON OF DOUBLE AND SINGLE TAPE DAMPING WITH LOSS FACTOR UNITY

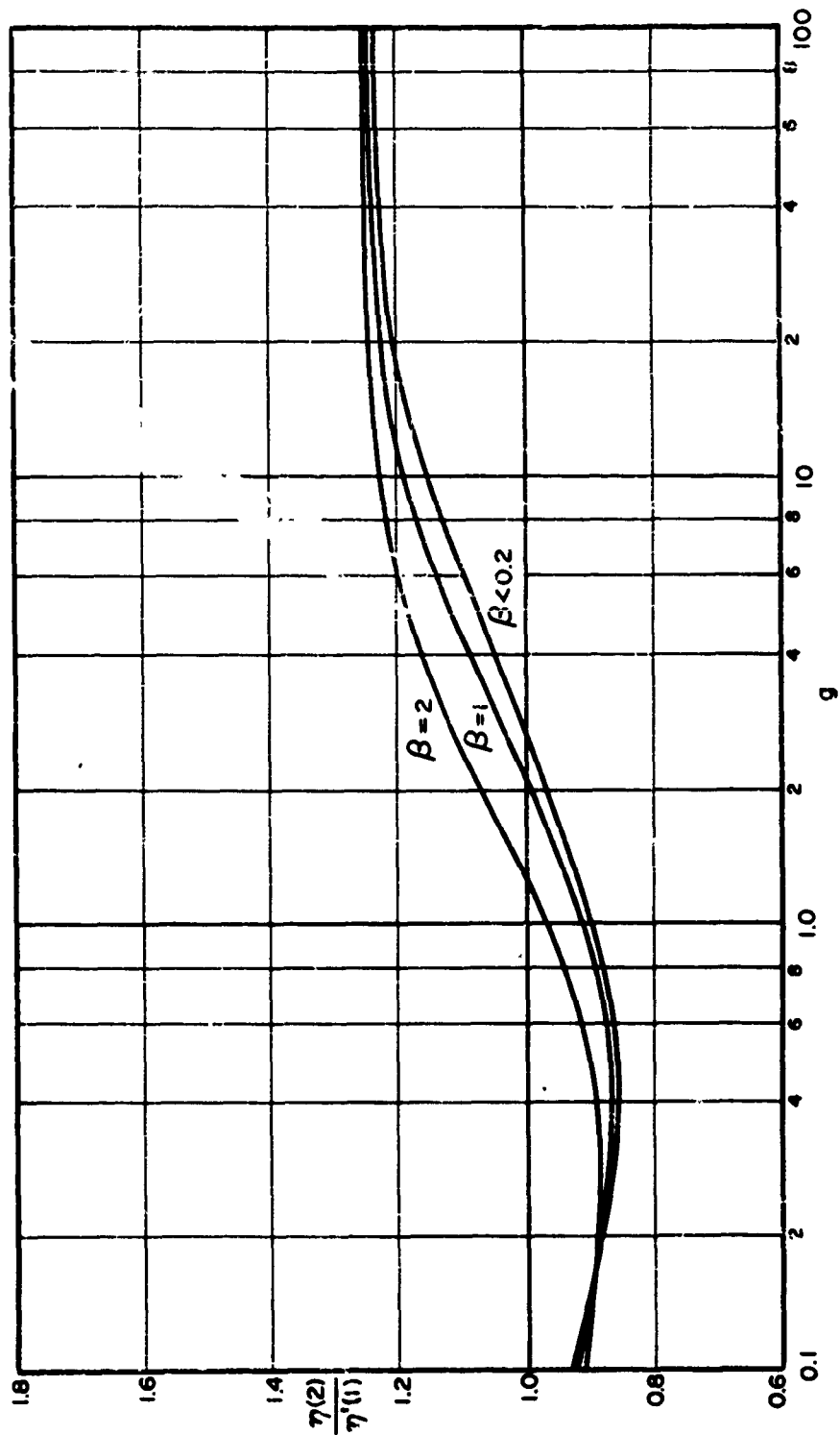


FIG. 4 COMPARISON OF DOUBLE TAPE TO SINGLE TAPE
OF EQUAL TOTAL FOIL THICKNESS (WEIGHT)

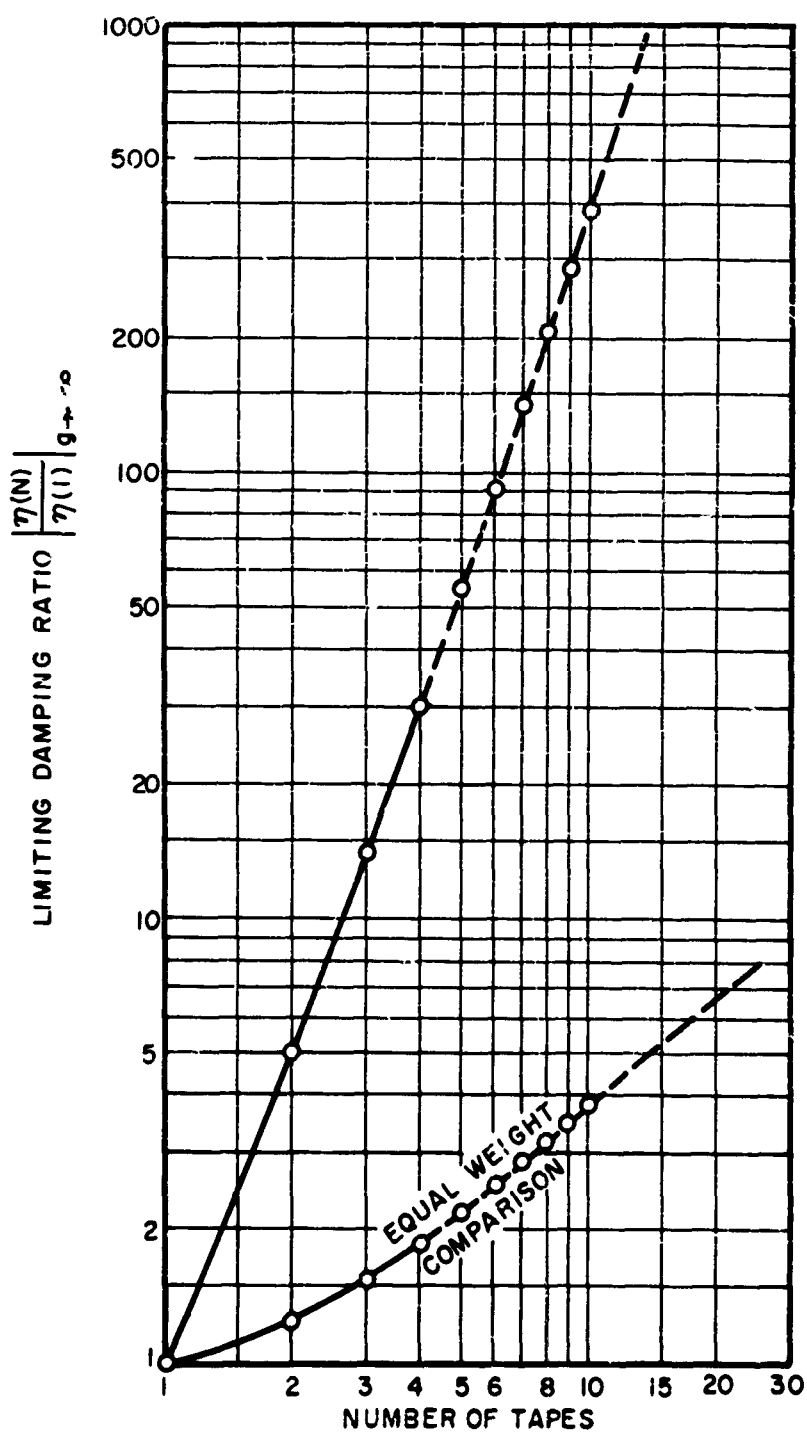


FIG. 5 LOW-FREQUENCY ADVANTAGE OF MULTIPLE TAPES

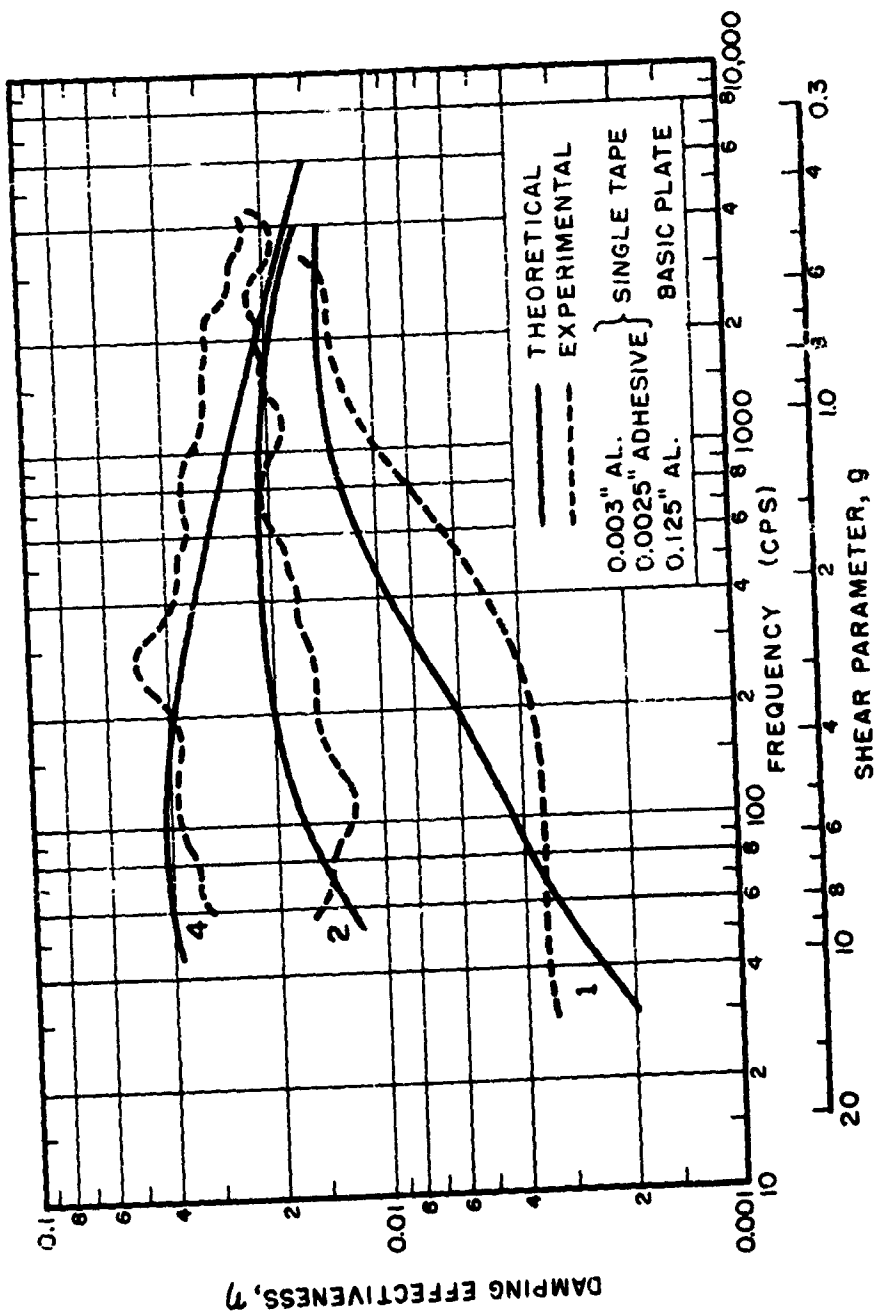


FIG.6 COMPARISON OF THEORETICAL AND EXPERIMENTAL DAMPING FOR 1, 2 AND 4 MM NO. 425 TAPES.

SECTION VIII

APPENDIX I

THEORY OF MULTIPLE DAMPING TAPES

General Analysis

As a basis for the analysis of specific configurations of multiple damping tapes, we consider the formal analysis of the flexural vibrations of a plate with an applied damping treatment consisting of N damping tapes. Each damping tape consists of a layer of visco-elastic (adhesive) damping material and of an elastic (metal) foil. The composite structure is thus composed of $2N+1$ alternate layers of elastic and visco-elastic material. The energy losses are assumed to be entirely attributable to the shear motion of the visco-elastic damping layers. The thickness of the entire composite plate is assumed to be small compared to a wavelength.

The method of analysis employed here consists of developing a general expression for the bending rigidity, B , of the composite plate in terms of the dimensions and elastic moduli of the individual layers, and of including the damping effect by representing the elastic moduli of the dissipative layers by complex quantities.

Figure 8 represents an element of a plate to which two layers of damping tape have been applied; layer 1 represents the original undamped metal plate, the even numbered regions represent the damping layers, the odd numbered regions represent the elastic layers. Bending of the element is measured by the flexural angle ϕ , and shearing of the adhesive layers is measured by the shear strains ψ_2 and ψ_4 , with the conventions for positive angles as indicated in the figure. The present analysis deals with $n = 2N+1$ layers; the extension of the figure to n layers may readily be visualized. The x -direction is chosen as the direction of propagation of a straight-crested flexural wave. It is assumed that

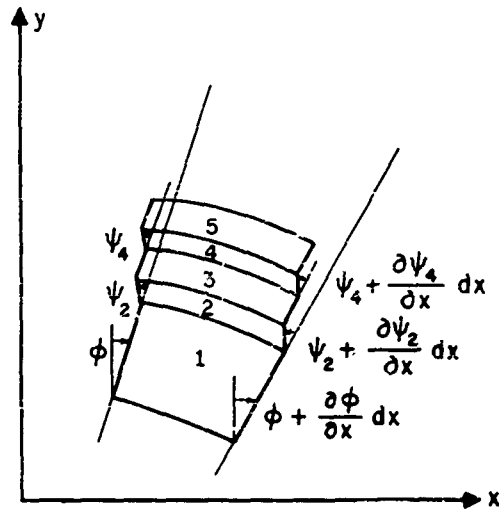


FIG. 8 ELEMENT OF 5-LAYER PLATE; CONVENTIONS FOR BENDING AND SHEAR ANGLES.

THE EVEN-NUMBERED (DAMPING) LAYERS POSSESS NEGLIGIBLE EXTENSIONAL STIFFNES AND THAT THE SHEAR STRAINS OF THE ODD-NUMBERED (ELASTIC) LAYERS ARE NEGLIGIBLE.

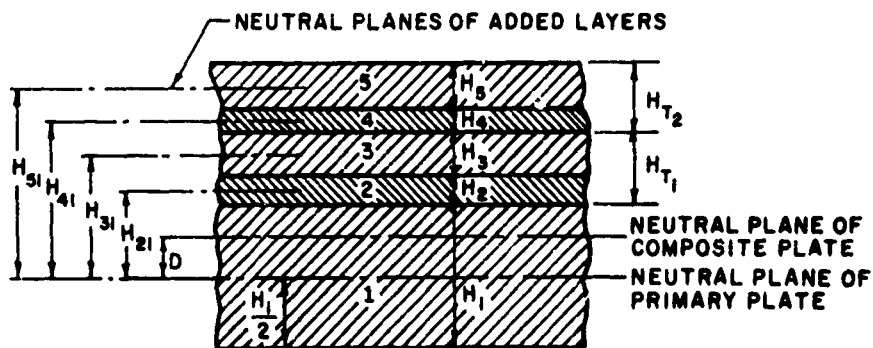


FIG. 9 ELEMENT OF 5-LAYER PLATE, DEFINING IMPORTANT DIMENSIONS.

The distances used in the analysis are shown in Fig 9. The thicknesses of the various layers are H_i , and the distances from the neutral planes of the added layers to that of the primary plate are H_{i1} . The thickness of each complete tape is represented by H_{T1} , where

$$H_{T1} = H_{2i} + H_{2i+1}, \quad i = 1, 2, 3, \dots, N \quad (I-1)$$

The displacement of the neutral plane of the composite plate from that of the primary plate is denoted by D .

By applying the definition of flexural rigidity one may express that of a general n -layered composite plate as

$$B = \sum_{i=1}^n [m_{i1} + f_i(H_{i1}-D)] \quad (I-2)$$

If one assumes that the shear strains of the odd-numbered (metal) layers and the extensional stresses of the even-numbered (adhesive) layers are negligible, the terms in the foregoing equation are given by:

$$m_{i1} = \frac{M_{i1}}{\frac{\partial \phi}{\partial x}} = \frac{1}{12} K_1 H_1^2, \quad i \text{ odd} \quad (I-3a)$$

$$f_i = \frac{F_i}{\frac{\partial \phi}{\partial x}} = K_1 \left[(H_{i1}-D) - \sum_{j=2}^{i-1} \mu_j \right]; \quad i \text{ odd}, j \text{ even} \quad (I-3b)$$

where

$$\mu_j = H_j \frac{\partial \psi_j}{\partial x} / \frac{\partial \phi}{\partial x}$$

M_{11} = bending moment of i^{th} layer about its own neutral plane

F_1 = extensional force of i^{th} layer

K_1 = extensional stiffness of unit length (in x-direction) of i^{th} layer

Expressions relating the various extensional forces to each other can be derived from considerations of equilibrium of the various layers. Figure 10 shows the forces acting on the layers of an element of an n-layer plate, and Fig 11 shows these specifically for a two-tape treatment. In both figures, dF_1 represents

$$\frac{\partial F_1}{\partial x} dx$$

The shearing forces acting between layers 1 and 2 may be evaluated from consideration of equilibrium of layer 1. Then the forces between layers 2 and 3 may be determined from equilibrium of layer 2, and so on. In general, the shear forces between layers q and $q+1$ may be directly evaluated from equilibrium of layer q , if all shear forces acting on the layers bearing numbers less than q have been evaluated.

Equilibrium of the n^{th} or outermost layer (or equilibrium of the entire element) demands that

$$\sum_{i=1}^n dF_1 = \sum_{i=1}^n \frac{\partial F_1}{\partial x} dx = 0, \quad i \text{ odd} \quad (\text{I-4})$$

The relation between the shear stresses and strains of the even-numbered layers may be expressed in terms of the shear moduli G_j . With the stresses as determined from Fig 10, one obtains the set of relations

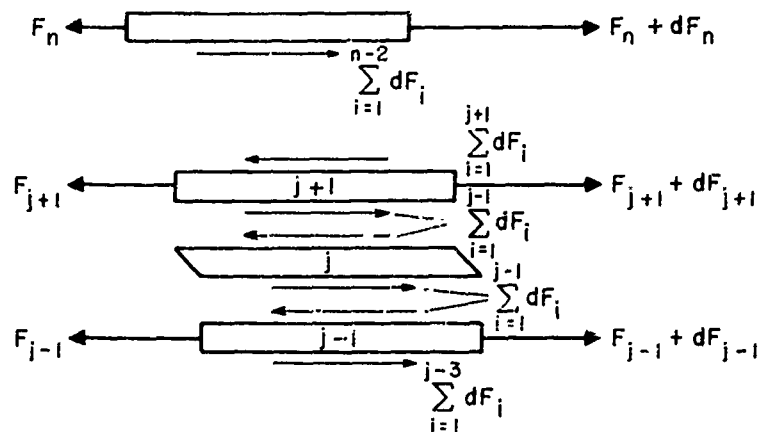


FIG. 10 FORCES ACTING ON LAYERS OF n -LAYER PLATE ELEMENT.

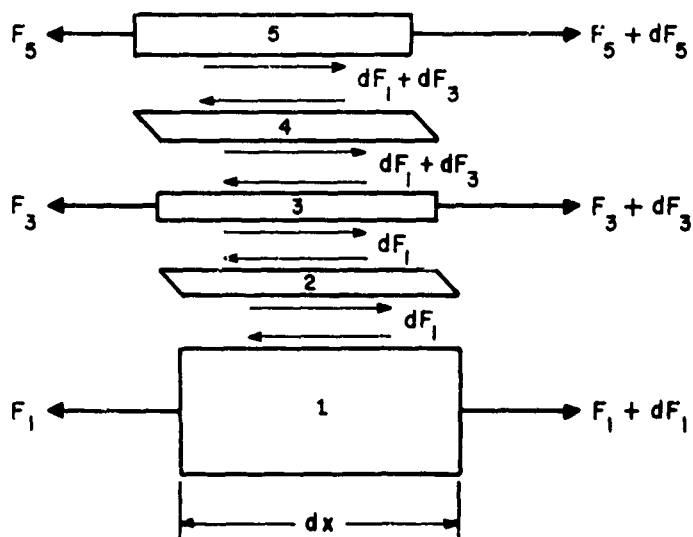


FIG. 11 FORCES ACTING ON THE FIVE LAYERS OF THE ELEMENT OF FIG. A-1.

$$G_j \psi_j = \sum_{i=1}^{j-1} \frac{\partial F_1}{\partial x} ; \quad i \text{ odd}, j \text{ even} \quad (\text{I-5})$$

Since the added layers are assumed to experience the same wave motion as the primary layer, the shear strains are related to their second derivatives by

$$\psi_j = -\frac{1}{p^2} \frac{\partial^2 \psi_j}{\partial x^2}, \quad j \text{ even} \quad (\text{I-6})$$

where p is the wave number of the vibration.

It follows from the assumption of simple flexural wave motion that the space derivatives of the shear angles ψ_j are proportional to the corresponding derivatives of the flexural angle ϕ . The expressions for f_1 given by Eq (I-3b) are therefore independent of position, and

$$\frac{\partial f_j}{\partial x} = \frac{1}{\frac{\partial \phi}{\partial x}} \left[\frac{\partial F_1}{\partial x} - f_1 \frac{\partial^2 \phi}{\partial x^2} \right] = 0$$

It follows that

$$\frac{\partial F_1}{\partial x} = f_1 \frac{\partial^2 \phi}{\partial x^2} \quad (\text{I-7})$$

Combining this result with Eqs (I-5) and (I-6) and using the definition of μ_j , one can derive the following relations between the shear strains and the flexural angle:

$$\mu_j = H_j \frac{\partial \psi_j / \partial x}{\partial \phi / \partial x} = H_j \frac{\partial^2 \psi_j / \partial x^2}{\partial^2 \phi / \partial x^2} = -\frac{H_j p^2}{G_j} \sum_{i=1}^{j-1} f_i \quad (\text{I-8})$$

In view of Eq (I-7), Eq (I-4) can now be rewritten as:

$$\sum_{i=1}^n f_i = 0 \quad (I-9)$$

Finally, then, one may at least in principle solve this equation plus the $N+1$ relationships of Eq (I-3b) and the N Eqs (I-8) for the $N+1$ reduced forces f_i , the N shear terms μ_j , and the displacement of the neutral plane D . In practice, however, the general solution is algebraically unwieldy - and substitution of the resultant expressions in Eq (I-2) for the bending rigidity is absurd. In the following sections we consider several special cases of practical importance.

Two Tapes (N=2)

If $N=2$, then there are a total of five layers and Eqs (I-9), (I-8) and (I-3b) reduce to six equations for six unknowns:

$$\begin{aligned} f_1 + f_3 + f_5 &= 0 \\ f_1 &= -K_1 D \\ f_3 &= K_3(H_{31}-D) - K_3 \mu_2 \\ f_5 &= K_5(H_{51}-D) - K_5(\mu_2 + \mu_4) \\ \mu_2 &= -\frac{H_2 p^2}{G_2} f_1 \\ \mu_4 &= -\frac{H_4 p^2}{G_4} (f_1 + f_3) = \frac{H_4 p^2}{G_4} f_5 \end{aligned} \quad (I-10)$$

The relative shear stiffness of the two damping layers are described by the dimensionless shear parameters

$$g_2 = \frac{G_2}{K_3 H_2 p^2} \quad \text{and} \quad g_4 = \frac{G_4}{K_5 H_4 p^2} \quad (\text{I-11})$$

The bending rigidity of the composite plate may be reduced to the dimensionless form

$$b = 1 + k_3 h_3^2 + k_5 h_5^2 + 12dh_{51} + \frac{12d}{g_2}(h_{51} - h_{31}) - 12k_3(h_{51} - h_{31})(h_{31} - d) \quad (\text{I-12})$$

by introduction of dimensionless properties represented by lower case letters and defined as

$$b = B/\frac{1}{12}K_1 H_1^2, \quad h_3 = H_3/H_1, \quad d = D/H_1, \quad \text{etc.}$$

i.e., by comparing the various dimensions and stiffnesses to the corresponding properties of the primary plate. Solving the six equations of Eq (I-10) for the displacement of the neutral plane, we find

$$d = \frac{g_2 k_3 h_{31}(1 + g_4 + k_{53} g_4) + g_2 g_4 k_5 (h_{51} - h_{31})}{(1 + g_2 + g_2 k_3)(1 + g_4) + k_{53} g_4 (1 + g_2 k_3)} \quad (\text{I-13})$$

where $k_{53} = K_5/K_3 = k_5/k_3$ is the ratio of the stiffnesses of the two metal layers. When Eq (I-13) is substituted into Eq (I-12) there results a final expression for the bending rigidity of the composite plate. If the tapes are somewhat thinner than the primary plate, we can neglect the square of the ratio of tape thickness to base plate thickness, and we may neglect $(h_{51} - h_{31})^2$ relative to h_{31}^2 or h_{51}^2 . With these simplifications,

$$b \approx 1 + \frac{12g_2(1 + g_4)k_3 h_{31}^2 + 12g_2 g_4 k_5 h_{51}^2}{(1 + g_2 + g_2 k_3)(1 + g_4) + k_{53} g_4 (1 + g_2 k_3)} \quad (\text{I-14})$$

If we assume that the two tapes are made of the same materials and only differ in their thicknesses then the properties of the second tape can be expressed in terms of those of the first tape and their relative dimensions:

$$k_5 = k_3 \frac{h_5}{h_3} = k_3 \alpha_{53}$$

$$k_4 = \frac{G_4}{K_5 H_4 p^2} = \frac{H_3}{H_5} \frac{H_2}{H_4} \frac{G_2}{K_3 H_2 p^2} = \frac{G_2}{\alpha_{53} \alpha_{42}}$$

Dropping the subscript on the shear parameter, then

$$b \doteq 1 + \frac{12gk_3 [(g + \alpha_{53} \alpha_{42}) n_{31}^2 + g \alpha_{53} h_{51}^2]}{(1 + g + gk_3)(g + \alpha_{53} \alpha_{42}) + \alpha_{53} g (1 + gk_3)} \quad (I-15)$$

To find an expression for the loss factor η , the shear parameter is made complex, $g^* = g(1 + j\beta)$. The ratio of the imaginary part of b to the real part is then the loss factor for the composite plate. The relationship obtained is so cumbersome that it cannot readily be inspected with a view toward revealing significant trends. A somewhat simplified expression having most of the characteristics of the more complete solution can be obtained by neglecting the stiffness of the foil in the denominator of Eq (I-15); i.e., by assuming $k_3 \ll 1$. With this restriction

$$\eta \doteq \frac{12k_3 \beta h_{31}^2 [r^2 g + 2rg^2 + sg^3(1 + \beta^2)] + 12k_3 \beta \alpha_{53} h_{51}^2 [2rg^2 + (r+s)g^3(1 + \beta^2)]}{r^2 + 2r(r+s)g + 2rg^2(1 + \beta^2) + (r+s)^2 g^2(1 + \beta^2) + 2(r+s)g^3(1 + \beta^2) + g^4(1 + \beta^2)^2} \quad (I-16)$$

where

$$r \equiv \alpha_{53}\alpha_{42} = \frac{h_5}{h_3} \cdot \frac{h_4}{h_2}$$

$$s \equiv 1 + \alpha_{53} = 1 + \frac{h_5}{h_3}$$

This expression for η can be used to estimate the damping characteristics of a two-tape treatment, provided only that the two tapes are composed of the same materials.

N Identical Tapes

Although the general solution of the N-tape problem is hopelessly unwieldy, solutions of practical importance can be obtained for N identical thin tapes. In this case one may define a shear parameter applicable to each tape by

$$g \equiv \frac{G_j}{K_{j+1}H_jp^2} = \frac{G_2}{K_3H_2p^2} \quad (I-17)$$

so that Eq (I-8) for the relative shear strain becomes

$$K_{j+1}\mu_j = K_3\mu_j = -\frac{1}{g} \sum_{i=1}^{j-1} f_i \quad (I-18)$$

and so that the expressions Eq (I-3b) for the force per unit flexural angle for each tape metal layer may be rewritten as

$$\phi_i = \frac{f_i}{K_3} = H_{31} + H_T \left(\frac{1-3}{2} \right) - D + \frac{1}{g} \sum_{j=2}^{i-1} \left(\sum_{s=1}^{j-1} \phi_s \right) \quad (I-19a)$$

(i > 1 and odd, j even, s odd)

where H_T denotes the tape thickness, and the distances H_{i1} have been expressed in terms of H_{31} and the tape thickness H_T ; and

$$\phi_1 = \frac{f_1}{K_1} = - \frac{K_1 D}{K_3} = - \frac{D}{k_3} \quad (I-19b)$$

in view of Eq (I-3b) with $i = 1$. Eq (I-9) for the sum of the forces can now be written

$$\sum_{i=1}^n \phi_i = 0 \quad (i \text{ odd}) ; \quad (I-20)$$

this, in conjunction with Eqs (I-19), suffices to establish values of the $N+1$ ϕ 's and D , there being $N+2$ equations to be solved for $N+2$ unknowns.

A useful relation may be obtained by subtracting ϕ_1 from ϕ_{i+2} :

$$\phi_{i+2} - \phi_1 = H_T + \frac{1}{g} \sum_{s=1}^i \phi_s, \quad (i > 1, \text{ odd}, s \text{ odd}) \quad (I-21)$$

Combining this last result and Eq (I-20) with Eq (I-19) one finds:

$$\phi_1 = \left(1 + \frac{1}{g}\right) \phi_{i+2} + \frac{1}{g} \sum_{s=i+4}^n \phi_s - H_T, \quad (i > 1, \text{ odd}, s \text{ odd}) \quad (I-22)$$

which expresses each ϕ_1 in terms of only ϕ 's with higher subscripts. One may thus use this relation to express ϕ_{n-2} in terms of ϕ_n ; ϕ_{n-4} in terms of ϕ_{n-2} and ϕ_n ; ϕ_{n-6} in terms of ϕ_{n-4} , ϕ_{n-2} , and ϕ_n ; etc. Clearly, one may then express each of the ϕ_i 's for $i=3, 5, \dots, n-2$, in terms of ϕ_n by performing the proper substitutions in turn. The last of these equations is that for $i=3$. By eliminating D between this equation and the equation of Eqs (I-19) for $i=3$

$$\phi_3 = H_{31} + \left(k_3 + \frac{1}{g} \right) \phi_1, \quad (I-23)$$

one can obtain an expression for ϕ_1 in terms of ϕ_n .

Finally, by summing all the ϕ_i 's, which are now functions of ϕ_n , and by using Eq (A-20) one can solve for ϕ_n itself. The expression for ϕ_n can then be used to evaluate the other ϕ_i 's.

An expression for the bending stiffness can be obtained by substituting the various ϕ_i 's into Eq (I-2), which can be rewritten as

$$B = \sum_{i=1}^n m_{ii} + K_3 \sum_{i=3}^n \phi_i \left(H_{31} + \frac{1-3}{2} H_T \right) \quad (I-24)$$

by use of Eqs (I-19) and (I-20), and by expressing the distances H_{11} in terms of H_{31} and multiples of H_T .

In view of Eq (I-20), this may also be written as

$$B = \sum_{i=1}^n \left(\frac{K_1 H_1^2}{12} + \frac{H_T}{2} K_3 i \cdot \phi_i \right) - (H_{31} - H_T) K_3 \phi_1 \quad (I-25)$$

Because of the term involving all the ϕ_i 's this is a rather complicated expression, especially when we let g be complex in order to find the loss factor. It is apparent that the algebra could be greatly simplified if the terms involving H_T were negligibly small.

N Thin Identical Tapes - If the individual tapes are thin compared to the primary plate, we may assume that for $i > 1$, $H_1^2 \ll H_1^2$, and we may simplify the expression Eq (I-24) for the bending rigidity to:

$$B \doteq \frac{1}{12} K_1 H_1^2 + H_{31} K_3 \sum_{i=3}^n \left(1 + \frac{1-3}{2} \epsilon \right) \phi_i \quad (I-26)$$

where $\epsilon = H_T/H_{31}$ is small relative to unity. If one uses Eq (I-20) and neglects k_3 compared to unity, Eq (I-23) for ϕ_3 may be written as*

$$(1+g) \phi_3 + \sum_{s=5}^n \phi_s \doteq g H_{31} \quad (I-27)$$

Substitution of this equation into Eq (I-26) results in an expression for the flexural rigidity that involves only ϕ_1 's with indices greater than 3:

$$\frac{B - I}{K_3 H_{31}^2} \doteq \frac{g}{1+g} + \frac{1}{H_{31}} \sum_{s=5}^n \left(\frac{g}{1+g} + \frac{1-3}{2} \epsilon \right) \phi_s \quad (I-28)$$

Here,

$$I = \sum_{i=1}^n \frac{1}{12} K_1 H_1^2 \doteq \frac{1}{12} K_1 H_1^2$$

When the shear parameter is made complex to account for dissipation, the right-hand side of Eq (I-28) must be a rational algebraic function of $g^* = g(1+j\beta)$ and one may write:

$$\frac{B^* - I}{K_3 H_{31}^2} = \frac{C_1 + j\beta C_2}{C_3 + j\beta C_4} \quad (I-29)$$

*Since ϕ_3 and $\sum_{s=5}^n \phi_s$ are of the same order of magnitude, one may neglect $gk_3 \sum_{s=5}^n \phi_s$ compared to $g\phi_3$ if $k_3 \ll 1$. Note that $k_3 \ll 1$ does not imply $k_3 \ll 1/g$ for large g .

where the C's are real functions of g and e. Rationalization of the right-hand side of Eq (I-29) results in:

$$\frac{B(1+j\eta)}{K_3 h_{31}^2} = \frac{\frac{1}{12} K_1 h_1^2}{K_3 h_{31}^2} + \frac{C_1 C_3 + \beta^2 C_2 C_4 + j\beta(C_2 C_3 - C_1 C_4)}{C_3^2 + \beta^2 C_4^2} \quad (I-30)$$

whence

$$\begin{aligned} \frac{\eta}{12k_3 h_{31}^2 \beta} &= \frac{C_2 C_3 - C_1 C_4}{C_3^2 + \beta^2 C_4^2 + 12k_3 h_{31}^2 (C_1 C_3 + \beta^2 C_2 C_4)} \\ &\doteq \frac{C_2 C_3 - C_1 C_4}{C_3^2 + \beta^2 C_4^2 + 3k_3 (C_1 C_3 + \beta^2 C_2 C_4)} \end{aligned} \quad (I-31)$$

When the relative stiffness of the individual tapes is strictly negligible, we can drop the term in the denominator involving k_3 , since all C's are of the same order in g, and obtain the simplified expression

$$\frac{\eta}{12k_3 h_{31}^2 \beta} \doteq \frac{C_2 C_3 - C_1 C_4}{C_3^2 + \beta^2 C_4^2} \quad (I-32)$$

Single Tape - It is natural to evaluate multiple tapes in terms of results applicable to a single tape. For a single tape, Eq (I-12) for the bending rigidity of a two-tape treatment reduces to

$$b(1) = 1 + k_3 h_3^2 + 12d h_{31} \quad (I-33)$$

where d is now given by Eq (I-13) with $k_5 = 0$:

$$d(1) = \frac{g k_3 h_{31}}{1 + g + g k_3} \quad (I-34)$$

Substituting Eq (I-34) into Eq (I-33) and replacing g by $g^* = g(1+\beta)$, we readily find**

$$\eta(1) = \frac{12gk_3h_{31}^2\beta}{(1+k_3^2)(1+2g(1+k_3)+g^2(1+\beta^2)(1+k_3)^2)+12k_3gh_{31}^2[1+g(1+\beta^2)(1+k_3)]} \quad (I-35)$$

which agrees with a previous derivation.*** For thin tapes, Eq (I-16) leads immediately to

$$\eta(1) \doteq \frac{12gk_3h_{31}^2\beta}{1+2g+g^2(1+\beta^2)} \quad (I-36)$$

As a check on the multiple tape analysis, we may solve for η when $n=3$. For $n=3$, Eq (I-28) becomes:

$$\frac{B^* - \frac{1}{12} K_1 H_1^2}{K_3 H_{31}^2} = \frac{g^*}{1+g^*} = \frac{g + j\beta g}{1 + g + j\beta g} \quad (I-37)$$

whence:

$$\begin{aligned} C_1 &= C_2 = C_4 = g \\ C_3 &= 1 + g \end{aligned} \quad (I-38)$$

** The numbers in parentheses following η indicate the number of damping tapes applied; $\eta(N)$ denotes the loss factor obtained with N tapes.

***BBN Report No. 564, "Flexural Vibration Damping of Multiple-Layer Plates", 26 June 1958, ONR Contract Nonr 2321(00).

and Eq (I-31) for the loss factor gives

$$\frac{\eta(1)}{12k_3^2 h_{31}^2} = \frac{g(1+g) - g^2}{(1+g)^2 + \beta^2 g^2 + 3k_3[(1+g)g + \beta^2 g^2]}$$

$$= \frac{g}{1 + 2g + g^2(1+\beta^2) + 3k_3g[1 + g(1+\beta^2)]} \quad (I-39)$$

This is seen to be an approximation which falls between those of Eqs (I-35) and (I-36). The assumption of $k_3 \ll 1$ or use of Eq (I-32) for η leads directly to the result of Eq (I-36). In comparing multiple-tape treatments with single tapes it is important that one use expressions that involve approximations of the same order.

Two Equal Tapes - For a double tape, $n=5$ and Eq (I-28) becomes

$$\frac{B-I}{K_3 H_{31}^2} \pm \frac{g}{1+g} \left(1 + \frac{\phi_5}{H_{31}} \right) + \frac{\epsilon}{H_{31}} \phi_5 \quad (I-40)$$

Eqs (I-22) and (I-27) relate ϕ_3 to ϕ_5 and here become

$$(1+g) \phi_3 + \phi_5 \pm g H_{31}$$

$$g \phi_3 = (1+g) \phi_5 - \epsilon H_{31} g \quad (I-41)$$

whence we find that

$$\phi_5 = \frac{g^2 H_{31} \left[1 + \epsilon \left(1 + \frac{1}{g} \right) \right]}{1 + 3g + g^2} \quad (I-42)$$

Substituting this result into Eq (I-40) and neglecting ϵ results in the following equation for the flexural rigidity of a plate with a thin double tape:

$$\frac{B-I}{K_3 h_{31}^2} = \frac{g + 2g^2}{1 + 3g + g^2} \quad (I-43)$$

Making the shear parameter complex, one finds

$$\begin{aligned} C_1 &= g + 2g^2(1-\beta^2) \\ C_2 &= g + 4g^2 \\ C_3 &= 1 + 3g + g^2(1+\beta^2) \\ C_4 &= 3g + 2g^2 \end{aligned} \quad (I-44)$$

and finally, from Eq (I-32)

$$\frac{\eta(2)}{12k_3 h_{31}^2 \beta} = \frac{g[1+rg+5(1+\beta^2)g^2]}{1+6g+(11+7\beta^2)g^2+6(1+\beta^2)g^3+(1+\beta^2)^2 g^4} \quad (I-45)$$

This same result can be derived from Eq (I-16) for the loss factor of a general non-equal, two-tape treatment. Assuming two equal thin tapes, we have

$$\begin{aligned} \alpha_{53} &= \alpha_{42} = 1 \\ r &= \alpha_{53} \alpha_{42} = 1 \\ s &= 1 + \alpha_{53} = 2 \\ h_{51}^2 &= h_{31}^2 \end{aligned}$$

whence

$$\frac{\eta(2)}{12k_3 h_{31}^2 \beta} = \frac{g+4g^2+5g^3(1+\beta^2)}{1+6g+2(1-\beta^2)g^2+9(1+\beta^2)g^2+6(1+\beta^2)g^3+g^4(1+\beta^2)^2} \quad (I-46)$$

This is equivalent to Eq (I-45), which was derived by application of the general equal-tape method.

Three Equal Tapes - For a triple tape, $n=7$ and Eq (I-22) yields the two relationships

$$\begin{aligned}\phi_5 &= \left(1 + \frac{1}{g}\right) \phi_7 - \epsilon H_{31} \\ \phi_3 &= \left(1 + \frac{1}{g}\right) \phi_5 + \frac{1}{g} \phi_7 - \epsilon H_{31}\end{aligned}\tag{I-47}$$

and Eq (I-27) gives

$$\phi_3 = \frac{g}{1+g} H_{31} - \frac{1}{1+g} (\phi_5 + \phi_7)\tag{I-48}$$

By solving these three equations simultaneously for ϕ_5 and ϕ_7 , we obtain

$$\begin{aligned}\frac{\phi_5}{H_{31}} &= \frac{g^2(1+g)}{1 + 5g + 6g^2 + g^3} \\ \frac{\phi_7}{H_{31}} &= \frac{g^3}{1 + 5g + 6g^2 + g^3}\end{aligned}\tag{I-49}$$

if we neglect the terms involving ϵ .

The flexural rigidity, found by substituting these two values into Eq (I-28), then is given by

$$\begin{aligned}\frac{B-I}{K_3 H_{31}^2} &= \frac{g}{1+g} \left(1 + \frac{g^2(1+2g)}{1 + 5g + 6g^2 + g^3}\right) \\ &= \frac{g(1 + 4g + 3g^2)}{1 + 5g + 6g^2 + g^3}\end{aligned}\tag{I-50}$$

In accordance with Eq (I-29), the C's here turn out to be

$$\begin{aligned}
C_1 &= g[1 + 4g(1-\beta^2) + 3g^2(1-3\beta^2)] \\
C_2 &= g[1 + 8g + 3g^2(3-\beta^2)] \\
C_3 &= 1 + 5g + 6g^2(1-\beta^2) + g^3(1-3\beta^2) \\
C_4 &= g[5 + 12g + g^2(3-\beta^2)]
\end{aligned} \tag{I-51}$$

and, from Eq (I-32), the loss factor is then found to be

$$\frac{\eta(3)}{12k_3 h_{31}^2 \beta} = \frac{g + 8g^2 + (23 + 11\beta^2)g^3 + 28(1 + \beta^2)g^4 + 14(1 + \beta^2)^2 g^5}{1 + 10g + (37 + 13\beta^2)g^2 + 2(31 + 27\beta^2)g^3 + 2(23 + 36\beta^2 + 13\beta^4)g^4 + 12(1 + \beta^2)^2 g^5 + (1 + \beta^2)^3 g^6} \tag{I-52}$$

Four Equal Tapes - In an entirely similar manner, one can use Eqs (I-22) and (I-27) to find expressions for the various Φ 's for a quadruple tape where $n=9$. One obtains

$$\begin{aligned}
\frac{\Phi_9}{H_{31}} &= \frac{g^4}{1 + 7g + 15g^2 + 10g^3 + g^4} \\
\frac{\Phi_7}{H_{31}} &= \frac{g^3(1+g)}{1 + 7g + 15g^2 + 10g^3 + g^4} \\
\frac{\Phi_5}{H_{31}} &= \frac{g^2(1+3g+g^2)}{1 + 7g + 15g^2 + 10g^3 + g^4}
\end{aligned} \tag{I-53}$$

By substituting these into Eq (I-28) one finds

$$\frac{B-I}{K_3 H_{31}^2} = \frac{g(1 + 6g + 10g^2 + 4g^3)}{1 + 7g + 15g^2 + 10g^3 + g^4} \tag{I-54}$$

whence the loss factor is given by

$$\frac{\eta(4)}{12k_3 h_{31}^2 \beta} = \frac{g W_N(\varepsilon, \beta)}{W_D(\varepsilon, \beta)} \quad (I-55)$$

where

$$W_N(\varepsilon, \beta) = 1 + 12g + g^2(57 + 17\beta^2) + 8g^3(17 + 13\beta^2) + 9g^4(19 + 26\beta^2 + 7\beta^4) \\ + 108g^5(1 + \beta^2)^2 + 30g^6(1 + \beta^2)^3$$

$$W_D(\varepsilon, \beta) = 1 + 14g + g^2(79 + 19\beta^2) + 10g^3(23 + 15\beta^2) + g^4(367 + 438\beta^2 + 87\beta^4) \\ + 2g^5(157 + 286\beta^2 + 129\beta^4) + 5g^6(26 + 87\beta^2 + 54\beta^4 + 14\beta^6) \\ + 20g^7(1 + \beta^2)^3 + g^8(1 + \beta^2)^4$$

In a like manner, the theory for multiple thin tapes could be applied to five or more layers. However, the algebra becomes increasingly involved as the number of layers increases. The solutions for three and four tapes serve to demonstrate the method.