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CONSTRAINT QUALIFICATIONS IN MAXIMIZATION PROBLEMS, II

BY
KENNETH J. ARROW AND HIROFUMI UZAWA

TECHNICAL REPORT NO. 84
MAY 23, 1960

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Applied Mathematics and Statistics Laboratories
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CONSTRAINT QUALIFICATIONS IN MAXIMIZATION PROBLEMS, II

Kenneth J. Arrow and Hirofumi Uzawa

1. Introduction

In the previous article [4], an investigation was made of the interrelationship between various conditions under which the classical Lagrange method remains valid for maximization problems subject to inequality constraints. In the present paper, further results on the subject will be discussed and simplified proofs given. First, the Kuhn-Tucker Constraint Qualification ([5], p. 483) will be slightly weakened so that the meaning of the qualification becomes more straightforward. It will be shown in Theorem 1 below that the Lagrangian method can be applied to those constraint maxima for which the present version of the Constraint Qualification is satisfied. Then we show that the Constraint Qualification in the present formulation is the weakest requirement for the Lagrange method to be applicable; namely, in Theorem 2 below, it is proved that if the Lagrange method is justified for any differentiable maximand, then the constraint function satisfies the Constraint Qualification provided the constraint set is convex. Finally, in Section 4 it will be shown that the Constraint Qualification is implied by the condition that the constraint functions corresponding to a certain set of indices are convex. The latter condition includes all those given earlier in [3] and [2].

2. Definitions and Preliminary Remarks.

Let $g(x) = [g^1(x), \dots, g^m(x)]$ be an m -vector valued function defined for n -vectors $x = (x_1, \dots, x_n)$. The set of all n -vectors x for which $g(x) \geq 0$ will be called the constraint set defined by $g(x)$, denoted by C ; i.e.,

$$(1) \quad C = \{x : g(x) \geq 0\} .$$

Given an n -vector \bar{x} in C and an arbitrary n -vector $\xi = (\xi_1, \dots, \xi_n)$, the contained path with origin \bar{x} and direction ξ is defined as an n -vector valued function $\psi(\theta)$ satisfying:

$$(2) \quad \psi(\theta) \text{ is defined and is continuous for all } 0 \leq \theta \leq \bar{\theta} \\ \text{with some positive } \bar{\theta};$$

$$(3) \quad \psi(0) = \bar{x}; \quad \psi(\theta) \in C \quad \text{for all } 0 \leq \theta \leq \bar{\theta};$$

$$(4) \quad \psi(\theta) \text{ has a right-hand derivative at } \theta = 0 \text{ such that} \\ \psi'(0) = \xi .$$

An n -vector ξ for which there is a contained path with origin \bar{x} and direction ξ will be referred to as an attainable direction; and the set of all attainable directions will be denoted by A . The closure of the convex cone spanned by A will be denoted by K_1 ; the elements of K_1 will be referred to as weakly attainable directions.

The set of indices $\{1, \dots, m\}$ is divided into two parts E and F . E is the set of all indices effective indices at \bar{x} ; namely,

$$(5) \quad E = \{j: g^j(\bar{x}) = 0\},$$

and F is the set of all ineffective indices at \bar{x} ; namely,

$$(6) \quad F = \{j: g^j(\bar{x}) > 0\}.$$

Let K_2 be the set of n -vectors defined by

$$(7) \quad K_2 = \{\xi: \bar{g}_x^E \xi \geq 0\};$$

the elements of K_2 may be termed locally constrained directions.

Lemma 1. Every weakly attainable direction is locally constrained.

Proof: Let $\psi(\theta)$ be a contained path with origin \bar{x} and direction ξ . Then, for any $j \in E$,

$$g^j[\psi(0)] = 0 \quad \text{and} \quad g^j[\psi(\theta)] \geq 0.$$

Hence

$$\bar{g}_x^j \xi \geq 0.$$

Q.E.D.

A function $g(x)$ will be called to satisfy the (Kuhn-Tucker)
Constraint Qualification at \bar{x} if

$$(CQ) \quad K_2 \subset K_1 ,$$

i.e., every locally constrained direction is weakly attainable. Kuhn and Tucker ([5], p. 483) required that every locally constrained direction be attainable.

Let us define the set K_3 by

$$(8) \quad K_3 = \text{the closure of the set } \{\lambda(x-\bar{x}) : \lambda \geq 0, x \in C\} .$$

Lemma 2. If the constraint set C is convex, then

$$K_3 \subset K_1 .$$

Proof: If $x \in C$, then by the convexity of the set C ,

$$\bar{x} + \theta(x - \bar{x}) \in C \quad \text{for all} \quad 0 \leq \theta \leq 1 .$$

Hence, $x - \bar{x}$ is attainable and therefore weakly attainable. Since K_1 is a cone, the conclusion follows.

Let B be any set of vectors. The negative polar cone, to be denoted by B^- , is defined by

$$B^- = \{u: ux \leq 0 \quad \text{for all} \quad x \in B\} .$$

We have (see, e.g., Fenchel [4], pp. 8-10),

(9) B^- is a closed convex cone;

(10) $B_1 \subset B_2$ implies that $B_1^- \supset B_2^-$;

(11) if B is a closed convex cone, $B^{--} = B$.

3. Lagrange Regularity and the Constraint Qualification.

The classical Lagrange method for constrained maxima (see, e.g. [1], p. 153) has been adapted by Kuhn and Tucker ([5], Theorem 1, p. 484).

(L) If \bar{x} maximizes a differentiable function $f(x)$ subject to $x \in C$, then there exists a nonnegative m -vector \bar{y} such that

$$(12) \quad \bar{f}_x + \bar{y} \bar{g}_x = 0,$$

$$(13) \quad \bar{y} g(\bar{x}) = 0 .$$

An m -vector valued function $g(x)$ will be termed Lagrange regular if, for any differentiable function $f(x)$, the condition (L) holds.

Lemma 3. If \bar{x} maximizes $f(x)$ subject to $x \in C$, then

$$\bar{f}_x \in K_1^- ,$$

where K_1^- is the negative polar cone of K_1 .

Proof: Let $\psi(\theta)$ be a contained path with origin \bar{x} and direction ξ , then

$$f[\psi(\theta)] \leq f[\psi(0)] = f(\bar{x}) \quad \text{for all} \quad 0 \leq \theta \leq \bar{\theta} .$$

Then

$$\bar{f}_x \xi = \bar{f}_x \psi'(0) \leq 0 ,$$

for any ξ in A and, by continuity and convexity, for any $\xi \in K_1$.

Q.E.D.

Theorem 1. If $g(x)$ satisfies the Constraint Qualification (CQ), then $g(x)$ is Lagrange regular.

Proof: Let $f(x)$ be a differentiable function and \bar{x} maximize $f(x)$ subject to $x \in C$. Then, by Lemma 3, $\bar{f}_x \in K_1^-$. On the other hand, the condition (CQ) implies, from (10), that

$$K_1^- \subset K_2^-$$

Hence, we have

$$(14) \quad \bar{f}_x \in K_2^- .$$

By applying the Minkowski-Farkas Lemma we have from (7) and (14),

$$-\bar{f}_x = \bar{y}^E \bar{g}_x^E \quad \text{for some} \quad \bar{y}^E \geq 0 .$$

Define

$$\bar{y} = (\bar{y}^E, \bar{y}^F) \quad \text{with} \quad \bar{y}^F = 0 .$$

Then, \bar{y} satisfies conditions (12) and (13).

Q.E.D.

Theorem 1 is the basic necessity theorem for non-linear programming ([5], Theorem 1) extended to the weaker constraint qualification of this paper.

Theorem 2. If $g(x)$ is Lagrange regular and if the constraint set C defined by it is a convex set, then $g(x)$ satisfies the Constraint Qualification (CQ).

Proof: It will be first shown that

$$(15) \quad K_3^- \subset K_2^- .$$

Let $a \in K_3^-$; then for $\lambda = 1$,

$$(16) \quad a(x - \bar{x}) \leq 0 \quad \text{for all} \quad x \in C .$$

Then \bar{x} maximizes the function $f(x) = ax$ subject to $x \in C$. By the Lagrange regularity of $g(x)$, there is an m -vector \bar{y} such that

$$(17) \quad a + \bar{y} \bar{g}_x = 0 ; \quad \bar{y} \geq 0 ,$$

and

$$(18) \quad \bar{y} g(\bar{x}) = 0 .$$

The conditions (17) and (18) imply that

$$(19) \quad a + \bar{y}^E \bar{g}_x^E = 0, \quad \bar{y}^E \geq 0 .$$

The condition (19) implies that

$$a\xi \leq 0 \quad \text{for all } \xi \text{ such that } \bar{g}_x^E \xi \geq 0 ;$$

namely,

$$a \in K_2^- .$$

Hence, we have the relation (15).

Then, by (10) and (11),

$$(20) \quad K_3 \supset K_2 .$$

Applying Lemma 2,

$$K_1 \supset K_3 \supset K_2 .$$

Q.E.D.

4. A Sufficient Condition for the Constraint Qualification.

Let E' be a subset of the effective indices defined by

$$(21) \quad E' = \{j : j \in E, \text{ u } \bar{g}_x^E = 0, \text{ for some } u \geq 0 \text{ with } u_j > 0\} .$$

Theorem 3. If $g^j(x)$ is convex for every $j \in E'$, then $g(x)$ satisfies the Constraint Qualification (CQ) at \bar{x} .

Proof: By the Minkowski-Farkas Lemma, an index j belongs to E' if and only if

$$(22) \quad \bar{g}_x^j \xi \leq 0 \quad \text{for all } \xi \text{ such that } \bar{g}_x^k \xi \geq 0, \quad k \in E, k \neq j.$$

Hence, for any $j \in E'' = E - E'$, there exists ξ^j such that

$$(23) \quad \bar{g}_x^j \xi^j > 0, \quad \bar{g}_x^k \xi^j \geq 0 \quad \text{for all } k \in E, k \neq j.$$

Let

$$(24) \quad \xi^* = \sum_{j \in E''} \xi^j.$$

Then, by (23) we have

$$(25) \quad \bar{g}_x^{E''} \xi^* > 0.$$

By (22) and the definition (7) of K_2 , we have

$$\bar{g}_x^{E'} \xi = 0 \quad \text{for all } \xi \in K_2.$$

Hence,

$$(26) \quad \bar{g}_x^{E'} \xi = 0 \quad \text{for all } \xi \in L,$$

where L is the linear space spanned by K_2 .

Since $g^j(x)$ is convex for $j \in E'$, the condition (26) implies that $g^j(\bar{x} + \xi)$ has its minimum in L at $\xi = 0$; hence,

$$(27) \quad g^{E'}(\bar{x} + \xi) \geq 0 \quad \text{for all } \xi \in L,$$

since $E' \subset E$, so that $g^{E'}(\bar{x}) = 0$.

We shall now prove that K_2 is contained in K_1 . Let ξ be any element of K_2 . For any positive $\alpha > 0$, define

$$(28) \quad \psi(\theta) = \bar{x} + (\xi + \alpha \xi^*) \theta.$$

From (27),

$$(29) \quad g^{E'}[\psi(\theta)] \geq 0 \quad \text{for all } \theta \geq 0.$$

From (25) and the definition (7) of K_2 ,

$$\left. \frac{dg^{E''}[\psi(\theta)]}{d\theta} \right|_{\theta=0} = \bar{g}_x^{E''}(\xi + \alpha \xi^*) = \bar{g}_x^{E''} \xi + \alpha \bar{g}_x^{E''} \xi^* > 0.$$

Since $E' \subset E$, we have $g^{E'}[\psi(0)] = 0$; hence,

$$(30) \quad g^{E'}[\psi(\theta)] > 0 \quad \text{for } \theta > 0 \text{ and sufficiently small.}$$

Finally, $g^F[\psi(0)] > 0$, so that

$$(31) \quad g_x^F[\psi(\theta)] > 0 \quad \text{for} \quad \theta > 0 \quad \text{and} \quad \text{sufficiently small.}$$

The relations (29-31) imply that

$$g[\psi(\theta)] \underset{=}{\geq} 0 \quad \text{for} \quad \theta \quad \text{sufficiently small.}$$

Hence $\psi(\theta)$ is a contained path, and $\xi + \alpha\xi^*$ is an attainable direction for all $\alpha > 0$. Therefore, ξ being a limit of $\xi + \alpha\xi^*$ as α tends to zero, is weakly attainable. Q.E.D.

Obviously, the conclusion of Theorem 3 will hold if either $g^j(x)$ is convex for all j or E' is the null set.

Corollary 1: If $g^j(x)$ is convex (in particular, linear) for every j , then $g(x)$ satisfies the Constraint Qualification.

Corollary 1 extends Theorem 2 of [3].

Corollary 2: ([3], Theorem 3). Suppose $g(x)$ is concave, and $g^E(x^*) > 0$ for some x^* . Then $g(x)$ satisfies the Constraint Qualification.

Proof: Since $g^E(x)$ is concave,

$$g^E(x^*) - g^E(\bar{x}) \leq \bar{g}_x^E(x^* - \bar{x}) .$$

But $g^E(x^*) > 0$ and $g^E(\bar{x}) = 0$; hence,

$$\bar{g}_x^E(x^* - \bar{x}) > 0 .$$

If $u \bar{g}_x^E = 0$, then $u \bar{g}_x^E(x^* - \bar{x}) = 0$, which, for $u \geq 0$, is only possible for $u = 0$. Then E' is null. Q.E.D.

This condition was used by Slater [6] for $f(x)$ concave.

Corollary 3: ([2], Theorem 2, p. 18). Suppose

(32) $g(x)$ is quasi-concave;

(33) $g^E(x^*) > 0$, for some x^* ;

and

(34) for each $j \in E$, either $g^j(x)$ concave or $\bar{g}_x^j \neq 0$.

Then the Constraint Qualification is satisfied.

Proof: If $g^j(x)$ is concave, for $j \in E$, then by (33)

(35) $\bar{g}_x^j \neq 0$.

Hence, by assumption (34), the relation (35) holds for all $j \in E$. Since the set C is convex, $x - \bar{x}$ is attainable for all $x \in C$ (see Lemma 2). Since x^* is an interior point of C , $x^* - \bar{x}$ is an interior point of A ; hence, the set L , the space spanned by K_2 , is the entire space.

If, for some $j \in E$,

$$\bar{g}_x^j \xi = 0 \quad \text{for all } \xi \in K_2 .$$

Then

$$\bar{g}_x^j \xi = 0 \quad \text{for all } \xi ;$$

hence

$$\bar{g}_x^j = 0,$$

contradicting (35). Therefore, the set E' is null.

Q.E.D.

Corollary 4: (non-degeneracy). If \bar{g}_x^E has a rank equal to the number of effective constraints, then $g(x)$ satisfies the Constraint Qualification.

Proof: By the assumption of the maximum rank

$$u \bar{g}_x^E = 0 \quad \text{implies} \quad u = 0 ;$$

hence, the set E' is null.

Q.E.D.

Remark: The case of nonlinear equality constraints is not handled by Theorem 3 or Corollaries 1-3. Indeed, it could easily happen in that case there is no linear contained path. Corollary 4 does remain valid if some or all constraints are equalities; see [3], Appendix 1.

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