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PROPAGATION OF WAVES OVER AN OBSTACLE
IN
WATER OF FINITE DEPTH

BY

T. FRANCIS OGILVIE

Under Contract Number N-onr-222(30)

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I. INTRODUCTION

In the first half of the nineteenth century, the problem of progressive gravity waves in a fluid of constant finite depth was solved within the framework of linearized perfect fluid theory. However, it has been only in the last two decades that any success has been attained in treating progressive waves where the fluid depth is non-constant. Stoker[1] discusses in considerable detail water waves on a beach of constant slope. Other problems and methods are summarized by Wehausen[2].

The work of Kreisel[3] is of particular relevance to the present paper. He showed that a large class of progressive wave problems have a solution and that these solutions satisfy a certain integral equation. He further showed that this integral equation can actually be solved by iteration and that the solution so obtained is unique.

Although Kreisel's method of solution is complete for the problems to which it is applicable, it is of interest to investigate certain approximation techniques for these problems. One approach is that of considering the case of very short wavelengths. For example, in a recent paper Keller[4] has shown how this leads to a "geometrical optics" type of solution.

At the other end of the spectrum is the case of very long waves, and it is with this case that the present paper

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[1] Superscript numbers refer to references listed at the conclusion of the paper.

is concerned. Consider the following situation:

We have a channel of infinite horizontal extent, with a uniform fluid depth except in a finite region, where the contour of the bottom is specified by some function, $y = b(x)$. (If the coordinate system is taken in the undisturbed free surface, $b(x) = -h$ for $|x|$ greater than some fixed constant.) If sinusoidal waves are incident from the left, than as $t \longrightarrow \infty$, there will be a progressive (reflected) wave moving toward the left and a progressive (transmitted) wave moving toward the right.

Under appropriate conditions on the function $b(x)$, several results will be proved:

(1) The velocity potential can be expanded in an asymptotic series in powers of the wave number, k_o, valid as $k_o \longrightarrow 0$. This expansion is valid for every x, and it is uniform in x in any bounded domain of the x-axis, but it is not uniform in the infinite interval, $-\infty < x < +\infty$.

(2) As $|x| \longrightarrow \infty$, the potential varies sinusoidally with x, and this asymptotic (in x) solution also has an asymptotic expansion in powers of k_o. The magnitude of the coefficient of k_o^n becomes unbounded like x^n as $|x| \longrightarrow \infty$.

(3) If the complete coefficients of k_o^n are required to satisfy the usual boundary conditions and in addition are required to have the behavior at $|x| \longrightarrow \infty$ specified by the expansion described in (2) above, then they are uniquely described.

(4) The reflection and transmission coefficients can be expressed in terms of asymptotic series in powers of k_o.

In other words, we have potentials which are bounded in the fluid domain, even as $|x| \longrightarrow \infty$. These can be expanded into asymptotic power series, the individual terms of which become unbounded as $|x| \longrightarrow \infty$. In fact, each successive term is unbounded to a higher degree in x than the previous terms. These individual terms are uniquely specified when they are required to become infinite in a certain way as $|x| \longrightarrow \infty$.

The reflection coefficient, R, depends explicitly only on the solution as $|x| \longrightarrow \infty$. In spite of the fact that each term of the asymptotic expansion becomes overwhelmingly larger than the previous terms as $|x| \longrightarrow \infty$, it will be shown that if we have the potential expansion to N terms, we can find an expansion giving R with a comparable number of terms. The latter expansion of course does not depend on x, and since it is valid as $k_o \longrightarrow 0$, it can be used to calculate R for small k_o.

Finally, a specific problem is worked out by the procedure developed. We consider the case of a bottom which is completely flat for all x, interrupted only by a vertical barrier of vanishing thickness. Curves are given for the reflection coefficient as a function of wave number, with (barrier height)/(undisturbed water depth) as a parameter.

II. FORMULATION OF THE PROBLEM

We consider a channel of infinite horizontal extent, containing fluid of depth h for $x \leq x_A$ and $x \geq x_B$. For the time being, we take the origin in the free surface, as shown in Figure 1. Let the curve of the bottom be given by:

$$y = b(x). \tag{1}$$

($b(x) = -h$, for $x \leq x_A$ and $x \geq x_B$.) The fluid is assumed to be non-viscous, incompressible, and unable to sustain surface tension, so that the fluid motion can be described in terms of a velocity potential, $\Phi(x,y,t)$, which satisfies the following conditions:

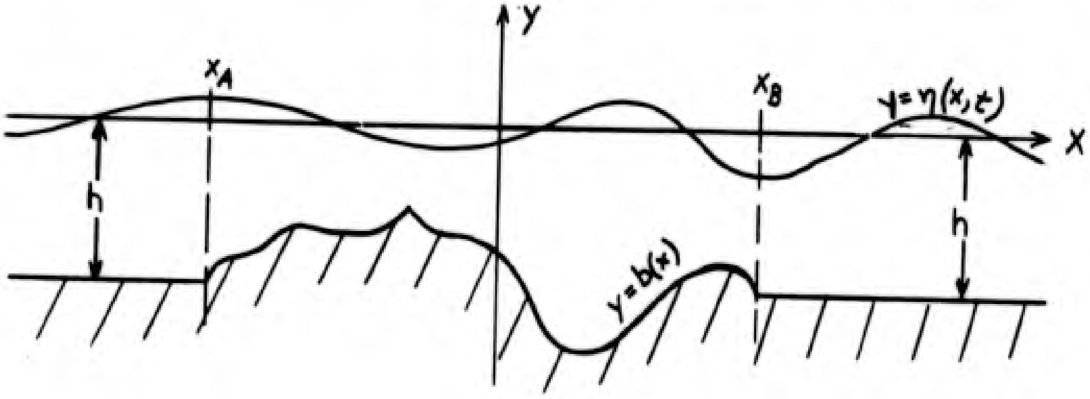

FIGURE 1.

$$\Phi_{xx} + \Phi_{yy} = 0, \quad \text{for } (x,y) \text{ in the fluid domain;}$$

$$\left.\frac{\partial\Phi}{\partial n}\right|_{y=b(x)} = 0,$$

where $\partial/\partial n$ indicates the rate of change along the vector normal to the boundary;

$$\left.\begin{aligned} \Phi_x \eta_x - \Phi_y + \eta_t &= 0 \\ g\eta + \Phi_t + \tfrac{1}{2}[\Phi_x^2 + \Phi_y^2] &= 0 \end{aligned}\right\} \quad \text{on} \quad y = \eta(x,t),$$

where $y = \eta(x,t)$ is the curve of the free surface.

In addition, we must have:

$|\Phi(x,y,t)|$ bounded in the fluid domain and on the boundaries.

The magnitude of the fluid velocity, $|\text{grad}\ \Phi|$, will be bounded everywhere as well, except possibly at a finite number of points on the bottom, where the contour has corners which project into the fluid.

We linearize the boundary conditions in the usual way[1,2]. Now let there be a train of waves of given frequency incident from the left. If we let $t \longrightarrow \infty$, the motion everywhere will vary sinusoidally in time with the same circular frequency, say σ. We can represent the potential by:

$$\Phi(x,y,t) = \varphi_1(x,y) \cos\sigma t + \varphi_2(x,y) \sin\sigma t, \qquad (2)$$

or, if we set $\varphi(x,y) = \varphi_1(x,y) + i\,\varphi_2(x,y)$, (3)

by:

$$\Phi(x,y,t) = \text{Re}\{\varphi(x,y)\, e^{-i\sigma t}\}\ .^{(a)}$$

We find that $\varphi(x,y)$ satisfies:

$\varphi(x,y)$ is bounded and harmonic in the fluid domain; (4a)

$$\frac{\partial\varphi(x,b(x))}{\partial n} = 0\ ; \qquad (4b)$$

$$\sigma^2\varphi(x,0) - g\,\varphi_y(x,0) = 0. \qquad (4c)$$

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[a] Although $\varphi_1(x,y)$ and $\varphi_2(x,y)$ are each harmonic functions, the sets of curves: $\varphi_1 = c$, $\varphi_2 = d$ are not generally orthogonal, and $\varphi_1 + i\varphi_2$ is not generally an analytic function of $z = x + iy$.

The shape of the free surface is given by:

$$\eta(x,t) = -\frac{\sigma}{g}\,\mathrm{Im}\left\{\varphi(x,0)\,e^{-i\sigma t}\right\},$$

to the accuracy of this linearized model.

If the bottom were completely flat for all x, i.e., $b(x) \equiv -h$, then the solution would be the classical result:

$$\varphi(x,y) = a\,\frac{\cosh k_0(y+h)}{\cosh k_0 h}\,e^{ik_0x}, \qquad (5)$$

where k_0 is the real positive root of

$$\sigma^2 = gk_0 \tanh k_0 h\,. \qquad (6)$$

In the statement of the problem above, it was implied that σ would be considered as a given quantity. From (6) it is apparent that k_0^2 can be expanded into a series in powers of σ^2, so that this approach is entirely legitimate[(a)]. However, it will be more convenient to use k_0 as independent variable, with σ^2 always given by (6), even when the bottom is not completely flat. Physically, this means only that we shall specify the wavelength at infinity as the given variable, rather than the frequency. From now on, this will be indicated explicitly by writing k_0 as one of the arguments of the functions considered or sought.

[(a)] If we write out (6) with the right side expanded in a series,

$$\sigma^2/g = h\,k_0^2 - 1/3\,h^3(k_0^2)^2 + \ldots,$$

there is no constant term in the series, so that it can be inverted to give a series expression for

$$k_0^2 = k_0^2(\sigma^2)\,.$$

The general problem to be solved involves showing that:

(a) under certain conditions,

$$\varphi(x,y;k_o) = \sum_{n=0}^{N} \varphi_n(x,y)\, k_o^n + o(k_o^N), \quad \text{as } k_o \longrightarrow 0,$$

valid for all (x,y) in the fluid domain; and

(b) the expansion is unique under these conditions.

The specific problem to be solved involves a reformulation of the problem as stated above (in a more convenient, but basically the same, form) and the determination of the first several terms in the expansion.

III. EXISTENCE OF THE ASYMPTOTIC EXPANSION OF THE VELOCITY POTENTIAL

This section depends very heavily on several facts proved by Kreisel[3]. The first subsection is devoted entirely to a restatement of these facts. The second subsection contains the proof of the existence of the expansion for all (x,y) in the fluid domain. The third subsection presents the proof that the potential has an expansion even when $x \longrightarrow \pm\infty$.

A. Kreisel's results

For convenience, we redefine the scales of length and time, respectively, so that h = 1 and g = 1. Then k_o is the positive real root of:

$$\sigma^2 = k_o \tanh k_o \ . \tag{6'}$$

Let k_n, n = 1,2,3,..., be the positive real root of:

$$\sigma^2 = - k_n \tan k_n \ , \tag{7}$$

for which

$$(n - 1/2)\pi < k_n < n\pi.^{(a)} \tag{7'}$$

Kreisel proves:

LEMMA I. A potential function, $\varphi(x,y;k_o)$, satisfying (4), is of the form:

(a) (ik_n) is a pure imaginary root of: $\sigma^2 = k \tanh k$.

$$\varphi(x,y;k_0) = \cosh k_0(y+1)\,[a(\operatorname{sgn} x;k_0)\,e^{ik_0x} + b(\operatorname{sgn} x;k_0)\,e^{-ik_0x}] + \sum_{n=1}^{\infty} a_n(\operatorname{sgn} x;k_0)\,e^{-k_n(k_0)|x|}\cos k_n(k_0)(y+1) \tag{8}$$

over the flat portions of the bottom extending to either infinity. Also,

$$\sum_{n=1}^{\infty} |a_n(\operatorname{sgn} x;k_0)|\,e^{-k_n(k_0)|x|} = O(e^{-\pi|x|/2}) \quad \text{as } |x| \to \infty. \tag{9}$$

In our problem of incident waves from the left, there will be a transmitted wave going toward $x = +\infty$ and a reflected wave returning toward $x = -\infty$. In accordance with (8), let[(a)]

$$\varphi(x,0;k_0) \longrightarrow [a'(k_0)e^{ik_0x} + b'(k_0)e^{-ik_0x}], \text{ as } x \to -\infty, \tag{10a}$$

$$\longrightarrow a(k_0)e^{ik_0x}, \text{ as } x \to +\infty. \tag{10b}$$

(As $x \to +\infty$, the solution represents a pure progressive wave.) Define:

$$R = \text{reflection coefficient} = |b'|/|a'|; \tag{11a}$$

$$T = \text{transmission coefficient} = |a|/|a'|. \tag{11b}$$

Then Kreisel proves:

[(a)] The notation: $f(x) \longrightarrow g(x)$, as $x \longrightarrow \pm\infty$, will be used consistently to mean:

$$f(x) - g(x) \longrightarrow 0, \text{ as } x \longrightarrow \pm\infty.$$

LEMMA II. The coefficients R and T are uniquely defined by the above statement of the problem. In the general case that $\varphi(x,y;k_0)$ has the asymptotic forms, as $x \longrightarrow \mp\infty$,

$$\varphi(x,0;k_0) \xrightarrow[x\to-\infty]{} a'(k_0)e^{ik_0x} + b'(k_0)e^{-ik_0x},$$

$$\xrightarrow[x\to+\infty]{} a(k_0)e^{ik_0x} + b(k_0)e^{-ik_0x},$$

then:

$$|a|^2 - |b|^2 = |a'|^2 - |b'|^2.$$

(This expresses the constancy of transmission of energy.) For the special case of (10a) and (10b), this reduces to:

$$R^2 + T^2 = 1.$$

Now map the fluid domain onto the strip:

$$-\infty < \xi < +\infty, \quad -1 < \eta < 0,$$

by the function $\zeta(z)$, where $\zeta = \xi + i\eta$, $z = x + iy$. The mapping is conformal and unique, except for a possible translation parallel to the ξ-axis. The free surface, $y = 0$, goes into the real axis, $\eta = 0$, in the ζ-plane, and the bottom goes into the line $\eta = -1$. (At certain points of the bottom, where corners may occur in the z-plane, the mapping is not generally conformal.) Sometimes we shall refer to the image of the fluid domain in the ζ-plane also as the "fluid domain". No confusion should result.

Again, call the potential in the ζ-plane $\varphi(\xi,\eta;k_0)$, although, of course, it is a different function of its arguments. It satisfies:

φ is bounded and harmonic in $-1 < \eta < 0$; (12a)

$$\varphi_\eta = 0, \quad \text{on} \quad \eta = -1; \tag{12b}$$

$$\sigma^2\varphi - \varphi_\eta + \sigma^2\left(\frac{dz}{d\zeta} - 1\right)\varphi = 0, \quad \text{on } \eta = 0. \tag{12c}$$

From Kreisel again, we have:

<u>LEMMA III.</u> As $|x| \longrightarrow \infty$ (or as $|\xi| \longrightarrow \infty$),

$$\left|\frac{dz}{d\zeta} - 1\right|_{\eta=0} = O(e^{-\pi|\xi|}).$$

That is, $\zeta \sim z$.

Thus $\varphi(\xi,\eta;k_0)$ has the same asymptotic form for $|\xi| \longrightarrow \infty$ that $\varphi(x,y;k_0)$ has for $|x| \longrightarrow \infty$. In particular, for our problem of waves incident from the left, redefine $a(k_0)$ slightly so that, as $\xi \longrightarrow +\infty$,

$$\varphi(\xi,\eta;k_0) \longrightarrow a(k_0)\, e^{ik_0\xi} \cosh k_0(\eta+1)/\cosh k_0.$$

Let:

$$\varphi(\xi,\eta;k_0) = \varphi_1(\xi,\eta;k_0) + a\, e^{ik_0\xi} \cosh k_0(\eta+1)/\cosh k_0. \tag{13}$$

(We now take the amplitude of the outgoing wave, a, as the known quantity and consider the amplitude of the incoming wave, $a'(k_0)$, as one of the unknown quantities.) Then $\varphi_1(\xi,\eta;k_0)$ satisfies (12a) and (12b). In place of (12c), we have:

$$\begin{aligned} \sigma^2(k_0)\,\varphi_1(\xi,0;k_0) &- \varphi_{1\eta}(\xi,0;k_0) \\ &= g(\xi;k_0)\,\varphi_1(\xi,0;k_0) + h(\xi;k_0), \end{aligned} \tag{14}$$

where

$$g(\xi;k_o) = -\sigma^2(k_o)\left[\frac{dz}{d\zeta} - 1\right]_{\eta=0}, \tag{15a}$$

$$h(\xi;k_o) = -\sigma^2(k_o)\left[\frac{dz}{d\zeta} - 1\right]_{\eta=0} a e^{ik_o\xi} . \tag{15b}$$

In place of (10a) and (10b), we now have, as $\xi \to \mp\infty$:

$$\varphi_1(\xi,0;k_o) \xrightarrow[\xi\to-\infty]{} [a'(k_o) - a]e^{ik_o\xi} + b'(k_o)e^{-ik_o\xi}, \tag{16a}$$

$$\xrightarrow[\xi\to+\infty]{} 0 . \tag{16b}$$

Kreisel proves the following lemmas:

<u>LEMMA IV.</u> If $\alpha < 1$, where

$$\alpha = [1 + 2k_o/\sinh 2k_o]^{-1}\left\{2k_o\int_{-\infty}^{+\infty}\left|\frac{dz}{d\zeta} - 1\right|_{\eta=0} d\xi + [1 - 2k_o/\sinh 2k_o]\left[\max_{\eta=0}\left|\frac{dz}{d\zeta} - 1\right|\right]\right\}, \tag{17}$$

then $\varphi_1(\xi,0;k_o)$ exists and is uniquely given as the solution of the integral equation:

$$\varphi_1(\xi,0;k_o) = \int_{-\infty}^{+\infty} g(\xi';k_o)\,\varphi_1(\xi',0;k_o)\,f(\xi-\xi',0;k_o)\,d\xi' + \int_{-\infty}^{+\infty} h(\xi';k_o)\,f(\xi-\xi',0;k_o)\,d\xi' , \tag{18}$$

where

$$f(x,y;k_o) \equiv \frac{1}{2\pi}\int_{-\infty+i\rho}^{+\infty+i\rho} \frac{\cosh k(y+1)}{\cosh k}\,\frac{e^{ikx}\,dk}{\sigma^2(k_o)-k\tanh k} , \tag{19}$$

and $0 < \rho < \pi/2$.

The condition, $\alpha < 1$, is sufficient but certainly not necessary. We also note that for sufficiently small k_o it is always true that $\alpha < 1$.

<u>LEMMA V.</u> If $\alpha < 1$, then

$$\varphi_1(\xi,\eta;k_o) = \int_{-\infty}^{+\infty} f(\xi-\xi',\eta;k_o)\,[\sigma^2(k_o)\,\varphi_1(\xi',0;k_o) - \varphi_{1_{\eta'}}(\xi',0;k_o)]\,d\xi'$$

$$= \int_{-\infty}^{+\infty} f(\xi-\xi',\eta;k_o)\,[g(\xi';k_o)\,\varphi_1(\xi',0;k_o) + h(\xi';k_o)]\,d\xi'.$$

<u>LEMMA VI.</u> As $\xi \longrightarrow -\infty$,

$$\varphi_1(\xi,0;k_o) \longrightarrow i[\tanh k_o\ (1 + 2k_o/\sinh 2k_o)]^{-1} \cdot \int_{-\infty}^{+\infty} [g(\xi';k_o)\varphi_1(\xi',0;k_o) + h(\xi';k_o)] \cdot [e^{ik_o(\xi-\xi')} - e^{-ik_o(\xi-\xi')}]\,d\xi'.$$

<u>LEMMA VII.</u> Let $f(x,0;k_o) = f(x;k_o)$. Then $f(x;k_o)$ has the following properties:

(a) For $x > 0$,

$$f(x;k_o) = -\sum_{n=1}^{\infty} \frac{e^{-k_n(k_o)x}}{\tan k_n(k_o)\,[1 + 2k_n(k_o)/\sin 2k_n(k_o)]}$$

(b) For $x < 0$,

$$f(x;k_o) = f(-x;k_o) + i\,\frac{e^{ik_ox} - e^{-ik_ox}}{\tanh k_o\,[1 + 2k_o/\sinh 2k_o]}.$$

<u>B. Existence of the asymptotic expansion for all ξ</u>

In this section we shall work almost exclusively with the function $\varphi_1(\xi,\eta;k_o)$. For simplicity in writing, we drop the subscript $_1$. Thus the function $\varphi(\xi,\eta;k_o)$ in this section satisfies (12a), (12b), (14), and (16).

Before proving the existence of the asymptotic expansion, we need several more lemmas, besides those proved by Kreisel.

<u>LEMMA VIII.</u> Consider σ^2, k_0, and k_n as complex variables. Then $k_n(k_0)$ and $\sigma^2(k_0)$ are analytic functions uniformly in some neighborhood of $k_0 = 0$, i.e., a finite neighborhood of $k_0 = 0$ can be found in which these are analytic functions for all n.

<u>Proof.</u> For σ^2, this follows immediately from:

$$\sigma^2 = k_0 \tanh k_0. \tag{6'}$$

To prove the lemma for k_n, let

$$k_n = n\pi - \varepsilon_n. \tag{20}$$

Then, from (7) and (7'), we have:

$$\sigma^2 = (n\pi - \varepsilon_n) \tan \varepsilon_n = n\pi \varepsilon_n - \varepsilon_n^2 + \ldots, \tag{21}$$

which is convergent for $|\varepsilon_n| < \pi/2$. Since $n\pi \neq 0$, this series can be inverted, giving a series for ε_n in terms of σ^2, with a positive radius of convergence. From (20), k_n can then be represented by a Taylor series expansion about $\sigma^2 = 0$. But from (6'), σ^2 can also be represented in a Taylor series expansion about $k_0 = 0$, in powers of k_0. (In fact, only even powers of k_0 appear.) Thus k_n is an analytic function of σ^2, which is an analytic function of k_0 near $k_0 = 0$.

We now need to prove only that the circles of convergence of $k_n(k_o)$ have radii which remain larger than R (some finite positive number), as $n \longrightarrow \infty$. It will do just as well to prove it for the series giving $\varepsilon_n(\sigma^2)$. Let $|\sigma^2| < \pi/8$. Since in any case we restrict $|\varepsilon_n| < \pi/2$, we have:

$$|\tan \varepsilon_n| = \left|\frac{\sigma^2}{n\pi - \varepsilon_n}\right| < \frac{\pi/8}{(n-1/2)\pi} = \frac{1}{8(n-1/2)} .$$

On any circle $|\varepsilon_n|$ = constant $< \pi/2$, $|\tan \varepsilon_n|$ has a minimum where $\mathrm{Re}\{\varepsilon_n\} = 0$, so that $\tanh|\varepsilon_n| \leq |\tan \varepsilon_n|$.(a) Thus $\tanh|\varepsilon_n| < 1/8(n-1/2)$, which requires that $|\varepsilon_n| < \pi/4$, for all n, since $\tanh \pi/4 \cong 0.66$.

Equation (21), of course, has other roots besides the one near $\varepsilon_n = 0$. However, for $n > 1$, it is easily seen that the next nearest root is $(\varepsilon_{n-1}+\pi)$ or $(\varepsilon_{n+1}-\pi)$. The same proof as above, applied to $\varepsilon_{n\pm1}$ shows that $|\varepsilon_{n\pm1}| < \pi/4$ if $|\sigma^2| < \pi/8$, and thus the minimum distance between ε_n $(n>1)$ and the nearest other root of (21) is greater than $\pi/2$.

Now consider the integral:

(a)Let $y = \tan(Ae^{i\gamma})$, A = real constant $< \pi/2$, $-\pi < \gamma \leq \pi$. Then:

$$|y|^2 = y\bar{y} = \tan(Ae^{i\gamma})\tan(Ae^{-i\gamma}),$$

and

$$\frac{d|y|^2}{d\gamma} = 0 \quad \text{when} \quad \gamma = 0, \pm\pi/2, \pi.$$

But when $\gamma = 0$ or π, $|y|^2 = \tan^2 A$ has a maximum, because the coefficients in the series for tan u are all positive and thus $|\tan u| \leq \tan|u|$. Since $|y|^2$ is a continuous function of γ, it has a minimum when $\gamma = \pm\pi/2$, i.e., when $|y| = |\tanh A|$.

$$I(\sigma^2) = \frac{1}{2\pi i} \int_C \frac{u\,[\tan u - (n\pi - u)\sec^2 u]}{\sigma^2 - (n\pi - u)\tan u}\, du ,$$

where C is the circle $|u| = \pi/4$. The integrand has a simple pole at $u = u_0$, where u_0 is the zero of the denominator. Since this is the same as equation (21), we have $u_0 = \varepsilon_n$, and there is only one such pole inside C. It is easily shown from residue theory then that $I(\sigma^2) = u_0(\sigma^2) = \varepsilon_n(\sigma^2)$. But $I(\sigma^2)$ is an analytic function of σ^2 for all σ^2 such that $|u_0| < \pi/4$. We have shown that if $|\sigma^2| < \pi/8$, then $|\varepsilon_n| < \pi/4$, for all n, i.e., $|u_0| < \pi/4$. So $I(\sigma^2) = \varepsilon_n(\sigma^2)$ is analytic in σ^2 at least for $|\sigma^2| < \pi/8$, for all $n > 1$.

We note specifically that the series for ε_n and k_n can be written:

$$\varepsilon_n = \sum_{\nu=1}^{\infty} \varepsilon_{n,2\nu}\, k_0^{2\nu},$$

$$k_n = n\pi - \sum_{\nu=1}^{\infty} \varepsilon_{n,2\nu}\, k_0^{2\nu},$$

valid for $|k_0| < \mu$ = some positive constant.

<u>LEMMA IX.</u> Let

$$f_n(|x|;k_0) = -\frac{e^{-k_n|x|}}{\tan k_n + k_n \sec^2 k_n} + \frac{e^{-n\pi|x|}}{n\pi} . \qquad (22)$$

Then $f_n(|x|;k_0)$ is analytic in k_n near $k_n = n\pi$, and thus in k_0 near $k_0 = 0$. This is true for all real x.

<u>Proof.</u> The first term is a quotient of analytic functions (in k_n), with the denominator non-zero for $k_n = n\pi$, which is all that is necessary, because of Lemma VIII.

We note that the second term of $f_n(x;k_o)$ just cancels the constant term of the expansion of the first term. Thus we can write:

$$f_n(|x|;k_o) = \sum_{\nu=1}^{\infty} f_{n,2\nu}(|x|)\, k_o^{2\nu}.$$

From Lemma VII, it is easily shown that:

$$f(x;k_o) = \sum_{n=1}^{\infty} f_n(|x|;k_o) - \frac{1}{\pi}\log\left[1 - e^{-\pi|x|}\right] \qquad (23)$$

$$+ \frac{i}{2}(1 - \operatorname{sgn} x)\,\frac{e^{ik_o x} - e^{-ik_o x}}{\tanh k_o[1 + 2k_o/\sinh 2k_o]}$$

<u>LEMMA X.</u> (a) $\sum_1^{\infty} f_n(|x|;k_o)$ is an analytic function of k_o near $k_o = 0$, for all real x. (b) The coefficient of k_o^{ν} in its power series is obtained by summing the coefficients of k_o^{ν} in the expansions of $f_n(|x|;k_o)$. (c) For large $|x|$, the expansion has the form:

$$\sum_{n=1}^{\infty} f_n(|x|;k_o) = \sum_{\nu=1}^{\infty} k_o^{2\nu}\left[e^{-\pi|x|}P_\nu(|x|) + o(e^{-\pi|x|})\right], \qquad (24)$$

converging uniformly in x for all real x, where $P_\nu(|x|)$ is a polynomial of degree ν in x.

<u>Proof.</u> Parts (a) and (b) will follow immediately if we show that Σf_n converges uniformly in k_o for all real x. This we do first:

From (22) and (20), we obtain:

$$f_n(|x|;k_o) = -\frac{e^{-(n\pi-\varepsilon_n)|x|}}{-\tan\varepsilon_n + (n\pi-\varepsilon_n)\sec^2\varepsilon_n} + \frac{e^{-n\pi|x|}}{n\pi} \qquad (25)$$

$$= e^{-n\pi|x|/2}\left\{\frac{\pi n[e^{-n\pi|x|/2}\sec^2\varepsilon_n - e^{-(n\pi/2-\varepsilon_n)|x|}] - [\tan\varepsilon_n + \varepsilon_n\sec^2\varepsilon_n]e^{-n\pi|x|/2}}{\pi n[-\tan\varepsilon_n + (n\pi-\varepsilon_n)\sec^2\varepsilon_n]}\right\}, \qquad (25')$$

by a simple reorganization of terms. Restrict ε_n and σ^2 so that

$$|\varepsilon_n| < \pi/4 \qquad \text{and} \qquad |\sigma^2| < \pi/8.$$

Then, also, $|\tan \varepsilon_n| < 1/8(n-1/2)$, from the proof of Lemma VIII. To estimate the denominator, we observe first that:

$$\begin{aligned}
&|-\tan \varepsilon_n + (n\pi - \varepsilon_n)\sec^2\varepsilon_n| \\
&\qquad \geq n\pi - |\tan \varepsilon_n| - |\varepsilon_n \sec^2 \varepsilon_n| - n\pi|\tan^2\varepsilon_n| \\
&\qquad \geq n\pi - \frac{1}{8(n-1/2)} - \frac{\pi}{4}\left[1 + \frac{1}{64(n-1/2)^2}\right] - n\pi\,\frac{1}{64(n-1/2)^2} \\
&\qquad > n\pi/2 \text{ , for all n.}
\end{aligned}$$

Thus the denominator in (25') is $> \frac{1}{2}\pi^2 n^2$ in magnitude.

For $|x| = 0$,

$$\begin{aligned}
|f_n(0;k_o)| &\leq \frac{|\pi n \tan^2\varepsilon_n - \tan \varepsilon_n - \varepsilon_n \sec^2\varepsilon_n|}{\pi^2 n^2/2} \\
&\leq \frac{\pi n|\tan^2\varepsilon_n| + |\tan \varepsilon_n| + |\varepsilon_n \sec^2\varepsilon_n|}{\pi^2 n^2/2}.
\end{aligned}$$

From the estimate of $|\tan \varepsilon_n|$, we see that each of the terms in the numerator is uniformly bounded for all n. So there exists a positive constant, K_o, independent of n, such that

$$|f_n(0;k_o)| \leq \frac{K_o}{n^2} \qquad \text{for all n.}$$

Thus $\Sigma f_n(0;k_o)$ converges uniformly in k_o.

For $|x| > 0$, we have for the numerator:

$$\Big| \pi m \, [e^{-n\pi|x|/2}\sec^2\varepsilon_n - e^{-(n\pi/2 - \varepsilon_n)|x|}]$$

$$- [\tan\varepsilon_n + \varepsilon_n\sec^2\varepsilon_n]\, e^{-n\pi|x|/2} \Big|$$

$$= e^{n\pi|x|/2}\Big| \pi m(1+\tan^2\varepsilon_n) - \pi m e^{\varepsilon_n|x|} - \tan\varepsilon_n - \varepsilon_n\sec^2\varepsilon_n \Big|$$

$$\leq e^{-n\pi|x|/2}[\pi m|\tan^2\varepsilon_n| + \pi m|e^{\varepsilon_n|x|} - 1| + |\tan\varepsilon_n| + |\varepsilon_n\sec^2\varepsilon_n|].$$

Since

$$|e^{\varepsilon_n|x|} - 1| < |e^{|\varepsilon_n||x|} - 1|,$$

and

$$e^{-n\pi|x|/2}\,|e^{|\varepsilon_n||x|} - 1| < \frac{|\varepsilon_n|}{n\pi/2 - |\varepsilon_n|} < \frac{1}{2(n-1/2)},$$

we find again that each of the terms in the numerator is uniformly bounded for all n. So finally,

$$|f_n(|x|;k_o)| \leq e^{-n\pi|x|/2}\,[\frac{K_o + \pi}{n^2}] \leq \frac{K}{n^2}\, e^{-n\pi|x|/2}, \qquad (26)$$

where K is a positive constant independent of n.

So we have proved more than was needed: The series

$$\sum_{n=1}^{\infty} e^{+n\pi|x|/2}\, f_n(|x|;k_o)$$

converges uniformly with respect both to k_o (near $k_o = 0$) and to $|x|$ (for all $|x| \geq 0$).

To prove (c), we note that equation (25) gives:

$$f_n(|x|;k_o) = e^{-n\pi|x|} \sum_{\nu=1}^{\infty} g_{n,\nu}(|x|)\, k_o^{2\nu},$$

where $g_{n,\nu}(|x|)$ is a polynomial of degree ν in x. Thus:

$$\sum_{n=1}^{\infty} f_n(|x|;k_o) = \sum_{n=1}^{\infty} \sum_{\nu=1}^{\infty} e^{-n\pi|x|}\, g_{n,\nu}(|x|)\, k_o^{2\nu}$$

$$= \sum_{\nu=1}^{\infty} k_o^{2\nu} \sum_{n=1}^{\infty} e^{-n\pi|x|} g_{n,\nu}(|x|)$$

$$= \sum_{\nu=1}^{\infty} k_o^{2\nu} \left[e^{-\pi|x|} g_{1,\nu}(|x|) + o(e^{-\pi|x|})\right].$$

Note that in general it would be wrong to remove the factor $e^{-\pi|x|}$ from the summation, because there is no guarantee of the uniform convergence at large $|x|$ of $\Sigma g_{1,\nu}(x)$. However, as proved in (b), we can remove a factor $e^{-\pi|x|/2}$ without difficulty.

<u>LEMMA XI.</u> The integral

$$\int_{-\infty}^{+\infty} h(\xi';k_o)\, f(\xi-\xi';k_o)\, d\xi'$$

can be expanded in an asymptotic power series in k_o, valid as $k_o \longrightarrow 0$, uniformly in ξ for $-\infty < -M < \xi$, where M is any real constant. If the expansion is given by

$$\sum_{n=0}^{N} H_n(\xi)\, k_o^n,$$

then $H_o(\xi) = H_1(\xi) = 0$, and

$$H_n(\xi) = \begin{cases} O(\xi^{n-1} e^{-\pi\xi}), & \text{as } \xi \longrightarrow +\infty, \\ O(\xi^{n-1}), & \text{as } \xi \longrightarrow -\infty, \end{cases}$$

for $n \geq 2$.

<u>Proof.</u> From (15b), Lemma III, and (23), we have for the integral:

$$\int_{-\infty}^{+\infty} h(\xi';k_0)\, f(\xi-\xi';k_0)\, d\xi'$$

$$= \frac{a\sigma^2}{\pi}\int_{-\infty}^{+\infty} e^{ik_0\xi'}\left[\frac{dz}{d\zeta'}-1\right]_0 \log\left[1-e^{-\pi|\xi-\xi'|}\right] d\xi' \qquad \text{(a)}$$

$$+\, a\sigma^2\int_{-\infty}^{+\infty} e^{ik_0\xi'}\left[\frac{dz}{d\zeta'}-1\right]_0 \left\{\sum_{n=1}^{\infty} f_n(\xi-\xi';k_0)\right\} d\xi'$$

$$-\, \frac{i\, a\, \sigma^2}{\tanh k_0\,[1+2k_0/\sinh 2k_0]}\int_{\xi}^{+\infty} e^{ik_0\xi'}\left[\frac{dz}{d\zeta'}-1\right]_0 \cdot\left[e^{ik_0(\xi-\xi')}-e^{-ik_0(\xi-\xi')}\right] d\xi'.$$

Consider the first integral. Let

$$I_N = \int_{-\infty}^{+\infty}\left[e^{ik_0\xi'}-\sum_{n=0}^{N}\frac{(i\xi')^n}{n!}k_0^n\right]\left[\frac{dz}{d\zeta'}-1\right]_0 \log\left[1-e^{-\pi|\xi-\xi'|}\right] d\xi' .$$

For $|\xi'| < M$, where M is any large positive number,

$$\left[e^{ik_0\xi'}-\sum_{n=0}^{N}\frac{(i\xi')^n}{n!}k_0^n\right] = o(k_0^N) \text{ uniformly in } \xi', \text{ as}$$

$k_0 \longrightarrow 0$. By Lemma III, the same is true if we multiply this quantity by $[\frac{dz}{d\zeta'}-1]_0$. But the multiplication by $[\frac{dz}{d\zeta'}-1]_0$ guarantees that the product will have a finite absolute bound at some finite value of ξ' and will approach zero as $|\xi'| \longrightarrow \infty$, provided only that $|k_0| < \pi$. Thus

$$\left[\frac{dz}{d\zeta'}-1\right]_0\left[e^{ik_0\xi'}-\sum_{n=0}^{N}\frac{(i\xi')^n}{n!}k_0^n\right] = o(k_0^N) \quad \text{as } k_0 \longrightarrow 0,$$

uniformly in ξ' for all ξ' (even as $\xi' \longrightarrow \pm\infty$).

Since $\log[1-e^{-\pi|\xi-\xi'|}]$ is integrable over the

(a) $[\frac{dz}{d\zeta'}-1]_0$ means: $[\frac{dz}{d\zeta'}-1]$ evaluated at $\eta' = 0$.

infinite interval, we can multiply the asymptotic expansion by it and integrate term-by-term, obtaining another asymptotic expansion. (See reference 6.)

$$I \equiv \int_{-\infty}^{+\infty} e^{ik_0\xi'}\left[\frac{dz}{d\zeta'}-1\right]_0 \log\left[1-e^{-\pi|\xi-\xi'|}\right] d\xi'$$

$$= \sum_{n=0}^{N} k_o^n \int_{-\infty}^{+\infty} \frac{(i\xi')^n}{n!}\left[\frac{dz}{d\zeta'}-1\right]_0 \log\left[1-e^{-\pi|\xi-\xi'|}\right] d\xi' \quad + o(k_o^N)$$

$$\equiv \sum_{n=0}^{N} I_n(\xi)k_o^n \quad + \quad o(k_o^N) \; , \qquad \text{as } k_o \longrightarrow 0.$$

We note that

$$I_n(\xi) \;=\; O(e^{-\pi|\xi|}\xi^n) \qquad \text{as } \xi \longrightarrow \pm\infty.$$

All of the above steps can be carried out on the integral:

$$J(\xi;k_o) \;=\int_{-\infty}^{+\infty} e^{ik_0\xi'}\left[\frac{dz}{d\zeta'}-1\right]_0 \sum_{n=1}^{\infty} f_n(|\xi-\xi'|;k_o)\; d\xi' \;,$$

although it is first necessary to rearrange terms in the product of the respective expansions of $e^{ik_0\xi'}$ and Σf_n. The exponential bounds on the coefficients of the latter series guarantee existence of all of the integrals. It is found that $J_0(\xi) \equiv J_1(\xi) \equiv 0$ and:

$$J_n(\xi) \;=\; O(\xi^{n-1}\, e^{-\pi|\xi|}) \qquad \text{as } \xi \longrightarrow \pm\infty.$$

The third integral can be written:

$$K(\xi;k_o) \;=\int_{\xi}^{+\infty} e^{ik_0\xi'}\left[\frac{dz}{d\zeta'}-1\right]_0 \left[e^{ik_0(\xi-\xi')}-e^{-ik_0(\xi-\xi')}\right] d\xi'$$

$$= \; e^{ik_0\xi}\int_{\xi}^{\infty}\left[\frac{dz}{d\zeta'}-1\right]_0 d\xi' \;-\; e^{-ik_0\xi}\int_{\xi}^{\infty} e^{2ik_0\xi'}\left[\frac{dz}{d\zeta'}-1\right]_0 d\xi'$$

$$\equiv \sum_{n=1}^{N} K_n(\xi)\, k_o^n + o(k_o^N) \qquad \text{as } k_o \longrightarrow 0,$$

with

$$K_n(\xi) = \begin{cases} O(\xi^n e^{-\pi|\xi|}) & \text{as } \xi \longrightarrow +\infty, \\ O(\xi^n) & \text{as } \xi \longrightarrow -\infty. \end{cases}$$

We previously had the fact that $\sigma^2(k_o)$ can be expressed as a power series in k_o, containing only even powers and starting with the k_o^2 term. Also, the denominator of the quantity which multiplies $K(\xi;k_o)$ can be expanded into a series starting with a term linear in k_o. Since asymptotic power series have the multiplicative property, we can combine and rearrange these series formally, as needed, and we obtain for the desired integral:

$$\int_{-\infty}^{+\infty} h(\xi';k_o)\, f(\xi-\xi';k_o)\, d\xi'$$

$$= \frac{a\sigma^2(k_o)}{\pi} \left[\sum_{n=0}^{N} I_n(\xi)\, k_o^n + o(k_o^N) \right]$$

$$- a\sigma^2(k_o) \left[\sum_{n=2}^{N} J_n(\xi)\, k_o^n + o(k_o^N) \right]$$

$$- \frac{ia\sigma^2(k_o)}{\tanh k_o\, [1 + 2k_o/\sinh 2k_o]} \left[\sum_{n=1}^{N} K_n(\xi)\, k_o^n + o(k_o^N) \right]$$

$$\equiv \sum_{n=2}^{N} H_n(\xi)\, k_o^n + o(k_o^N) \qquad \text{as } k_o \longrightarrow 0. \qquad (a)$$

where

$$H_n(\xi) = \underline{\begin{cases} O(\xi^{n-1} e^{-\pi\xi}) & \text{as } \xi \longrightarrow +\infty, \\ O(\xi^{n-1}) & \text{as } \xi \longrightarrow -\infty. \end{cases}}$$

(a) This will be called "an asymptotic expansion to N terms" although strictly there are at most N-1 non-zero terms.

LEMMA XII. If $\alpha < 1$ (See Lemma IV), then there exists a sequence of functions, $\{\varphi_n(\xi)\}$, such that

$$\varphi(\xi,0;k_o) = \sum_{n=2}^{N} \varphi_n(\xi)\, k_o^n + o(k_o^N) \quad \text{as } k_o \longrightarrow 0;$$

$$\varphi_n(\xi) = \begin{cases} O(\xi^{n-1}) & \text{as } \xi \longrightarrow -\infty, \\ O(\xi^{n-1}e^{-\pi\xi}) & \text{as } \xi \longrightarrow +\infty. \end{cases}$$

That is, $\varphi(\xi,0;k_o)$ has an asymptotic expansion in powers of k_o for all ξ, uniform in ξ for $-\infty < -M < \xi$, where M is any real constant.

Proof. Kreisel showed that $\varphi(\xi,0;k_o)$ is the solution of the integral equation, (18), and for $\alpha < 1$ the solution is given by:

$$\varphi(\xi,0;k_o) = \sum_{r=0}^{\infty} \psi_r(\xi,0;k_o),$$

where

$$\psi_0(\xi,0;k_o) = \int_{-\infty}^{+\infty} h(\xi';k_o)\, f(\xi-\xi';k_o)\, d\xi',$$

$$\psi_{r+1}(\xi,0;k_o) = \int_{-\infty}^{+\infty} g(\xi';k_o)\, f(\xi-\xi';k_o)\, \psi_r(\xi',0;k_o)\, d\xi'.$$

It was shown in Lemma XI that $\psi_0(\xi,0;k_o)$ can be expanded in an asymptotic power series to N terms, the first non-zero term containing k_o^2. Because of the properties of $g(\xi;k_o)$ (See (15a), (6'), and Lemma III.), and by the same arguments as used in Lemma XI, $\psi_1(\xi,0;k_o)$ can be expanded into an asymptotic series with N+2 terms, the first non-zero term containing k_o^4. Similarly, $\psi_r(\xi,0;k_o)$ can be expanded into a series with N+2r terms, starting with the k_o^{2+2r} term. Thus, existence of the expansion of $\psi_0(\xi,0;k_o)$ to N terms guarantees the existence of the expansions of $\psi_r(\xi,0;k_o)$, $r > 0$, to at least N terms. Furthermore, only a finite number of the $\psi_r(\xi,0;k_o)$ will have terms in their expansions containing k_o^n, for $n < N$. Thus $\varphi_n(\xi)$ can be found by summing the coefficients of k_o^n in the expansions of $\psi_r(\xi,0;k_o)$. Such sums will have only a finite number of terms.

The highest power of ξ appearing in $\varphi_n(\xi)$ (as $\xi \longrightarrow -\infty$) arises in the contribution from $\psi_o(\xi,0;k_o)$. Thus

$$\varphi_n(\xi) = O(\xi^{n-1}) \qquad \text{as} \quad \xi \longrightarrow -\infty.$$

Analysis similar to that of Lemma XI shows that the predominant behavior as $\xi \longrightarrow +\infty$ is also determined by $\psi_o(\xi,0;k_o)$.

We are now ready to prove the theorem which is the main purpose of this section:

<u>THEOREM I.</u> If $\alpha < 1$ (See Lemma IV.), there exists a sequence of functions, $\{\varphi_n(\xi,\eta)\}$, such that:

$$\varphi(\xi,\eta;k_o) = \sum_{n=2}^{N} \varphi_n(\xi,\eta)\, k_o^n + o(k_o^N) \qquad \text{as } k_o \longrightarrow 0,$$

for every (ξ,η) in the fluid domain, with

$$\varphi_n(\xi,\eta) = \begin{cases} O(\xi^{n-1}) & \text{as} \quad \xi \longrightarrow -\infty, \\ O(\xi^{n-1}e^{-\pi\xi}) & \text{as} \quad \xi \longrightarrow +\infty. \end{cases}$$

<u>Proof.</u> Lemma V provides the methods for calculating $\varphi(\xi,\eta;k_o)$ from $\varphi(\xi,0;k_o)$. We must prove that the integrand appearing in the statement of that lemma has an asymptotic expansion uniformly in ξ, for all ξ. Then the integral will have an asymptotic expansion.

From (14) and Lemma XII,

$$\sigma^2(k_0)\,\varphi(\xi,0;k_0) \;-\; \varphi_{\eta'}(\xi,0;k_0)$$

has an asymptotic power series expansion, valid as $k_0 \longrightarrow 0$. Because the factor $[\frac{dz}{d\zeta}-1]_0$ appears in both terms (See (14) and (15).), this expansion is valid uniformly in ξ, for all ξ. In addition, the coefficient of each power of k_0 is bounded at least by the product of a polynomial times a decaying exponential, as $|\xi| \longrightarrow \infty$. So it is necessary to prove only that $f(\xi-\xi',\eta;k_0)$ has an asymptotic expansion in which the coefficients of k_0^n are of finite degree as $|\xi| \longrightarrow \infty$, for any n.

$f(x,y;k_0)$ was defined in Lemma IV. It can be evaluated by residue theory in the same way that Kreisel evaluated it for the case that $y = 0$. It is easily seen that the only poles of the integrand are at the zeros of $\sigma^2 - k \tanh k$, i.e., at $\pm k_0$ and $\pm ik_n$, $n = 1,2,3,\ldots$. (See (6'), (7), and the footnote to (7').) For $x > 0$,

$$f(x,y;k_0) = -\sum_{n=1}^{\infty} \frac{\cos k_n(y+1)}{\cos k_n}\;\frac{e^{-k_n x}}{\tan k_n\,[1+2k_n/\sin 2k_n]}\,.$$

Now, $|\cos k_n(y+1)| = |\cos(n\pi-\varepsilon_n)(y+1)|$

$$= |\cos n\pi(y+1)\cos\varepsilon_n(y+1) + \sin n\pi(y+1)\sin\varepsilon_n(y+1)|$$

$$< |\cos\varepsilon_n(y+1)| + |\sin\varepsilon_n(y+1)|\,.$$

Since $|\cos k_n| \neq 0$,

$$\left|\frac{\cos k_n(y+1)}{\cos k_n}\right| < \left|\frac{\cos \varepsilon_n(y+1)}{\cos \varepsilon_n}\right| + \left|\frac{\sin \varepsilon_n(y+1)}{\cos \varepsilon_n}\right|$$

$$< \quad 2 \qquad \text{for large } n,$$

since $-1 \leq y \leq 0$, and since $|\varepsilon_n| \longrightarrow 0$ as $n \longrightarrow \infty$. Thus the convergence properties of the sum representing $f(x,0;k_0)$ (See Lemma X.) apply to the above sum for $f(x,y;k_0)$, $y \neq 0$.

Let:

$$f_n(|x|,y;k_0) = -\frac{\cos k_n(y+1)}{\cos k_n}\,\frac{e^{-k_n|x|}}{\tan k_n\,[1+2k_n/\sin 2k_n]} + \frac{(-1)^n \cos n\pi(y+1)\, e^{-n\pi|x|}}{\pi n}\,.$$

Then:

$$f(x,y;k_0) = \sum_{n=1}^{\infty} f_n(|x|,y;k_0) \tag{27}$$

$$-\frac{1}{2\pi}\log[1+2e^{-\pi|x|}\cos \pi(y+1) + e^{-2\pi|x|}]$$

$$+\frac{i}{2}(1-\operatorname{sgn} x)\,\frac{\cosh k_0(y+1)}{\cosh k_0}\,\frac{e^{ik_0x}-e^{-ik_0x}}{\tanh k_0\,[1+2k_0/\sinh 2k_0]}\,.$$

In the same way as for $f(x,0;k_0)$, it can be shown that $f(x,y;k_0)$, $y \neq 0$, has the desired properties: It can be expanded into a Taylor series (and thus an asymptotic series) in powers of k_0, for all x, (except the logarithm term, which does not depend on k_0, may become infinite at $x = 0$) with the coefficient of k_0^n of finite degree as $x \longrightarrow -\infty$ and approaching zero as x ⟶ $+\infty$.

As in Lemma XI, the predominant behavior of $\varphi_n(\xi,\eta)$ as $\xi \longrightarrow \pm\infty$ is found from the last term in $f(x,y;k_o)$ above. Since

$$\frac{\cosh k_o(y+1)}{\cosh k_o} = 1 + o(k_o),$$

this predominant behavior is unaffected by y, so that the asymptotic behavior of $\varphi_n(\xi,\eta)$, as $\xi \longrightarrow \pm\infty$, is identical with that of $\varphi_n(\xi,0) = \varphi_n(\xi)$.

In the above theorem and in several of the lemmas, estimates have been provided concerning the order of magnitude of the terms in the expansions as $\xi \longrightarrow \pm\infty$. It has not always been essential to make these estimates, and, in fact, the next section provides an explicit statement concerning the behavior of the terms as $\xi \longrightarrow -\infty$. (All terms approach zero as $\xi \longrightarrow +\infty$.) However, the estimates have been included for the purpose of emphasizing the non-uniformity of the asymptotic expansion. Lemma I showed that the potential remains bounded, even infinitely far away. But it has been shown now that the individual terms in the expansion become unbounded at $-\infty$. This is not particularly remarkable in itself; the potential acts like a sine function at $-\infty$, and if, say, sin $k_o\xi$ were expanded into an asymptotic series in powers of k_o, the same behavior would be found. However, we propose presently to use these expansions in a way which depends explicitly on their behavior at infinity, viz., to calculate the reflection and transmission coefficients. Before doing so, it is necessary to show that this unboundedness of the terms at infinity must take a particular form, and that this form enables us to determine the terms completely and uniquely. This will be done in the next sections.

To complete this section, it is necessary to state one more fact, which is now rather obvious:

COROLLARY. The complete potential in the ζ-plane, that is,

$$\varphi_1(\xi,\eta;k_o) \quad + \quad a e^{ik_o\xi}\,\frac{\cosh k_o(\eta+1)}{\cosh k_o} \quad ,$$

(See (13) and the first paragraph of this section.) has an asymptotic expansion in powers of k_o, valid as $k_o \longrightarrow 0$, holding uniformly for $|\xi| < M < \infty$. The same is true of the potential in the z-plane.

C. Existence of the asymptotic expansion as $\xi \longrightarrow \pm\infty$

Lemma I showed that, as $\xi \longrightarrow \pm\infty$, the potential is a sum of a sinusoidally-varying function plus a function which diminishes exponentially. Theorem I showed that this sum can be represented by an asymptotic expansion, for all ξ. We now show that the sinusoidal part of the solution can be represented separately by an asymptotic expansion.

THEOREM II. As $\xi \longrightarrow -\infty$,

$$\varphi_1(\xi,\eta;k_o) \longrightarrow \frac{\cosh k_o(\eta+1)}{\cosh k_o}\left[\alpha_1(k_o)e^{ik_o\xi} - \alpha_2(k_o)e^{-ik_o\xi}\right],$$

where $\alpha_1(k_o)$ and $\alpha_2(k_o)$ can each be represented by asymptotic expansions in powers of k_o, each starting with terms linear in k_o. As $\xi \longrightarrow +\infty$, $\varphi_1(\xi,\eta;k_o) \longrightarrow 0$.

Proof. The result for $+\infty$ is already given in equation (16b). Lemma VI provides the method for calculating $\varphi_1(\xi,0;k_o)$ as $\xi \longrightarrow -\infty$. The integrands appearing in Lemma VI can each be expanded in asymptotic series in powers of k_o, starting with terms containing k_o^2. This proves the lemma for $\eta = 0$.

Kreisel proved Lemma VI by showing that only the unsymmetrical part of $f(\xi-\xi';k_o)$ (See (23).) contributes to the asymptotic potential for $\xi \longrightarrow -\infty$. By substituting (27) into the equation of Lemma V, one finds that this is still true when $\eta \neq 0$, i.e.,

$$\varphi_1(\xi,\eta;k_o) \longrightarrow \frac{\cosh k_o(\eta+1)}{\cosh k_o}\,\varphi_1(\xi,0;k_o) \quad \text{as } \xi \longrightarrow -\infty.$$

The second part of this proof could also be implied directly from Lemma I.

<u>COROLLARY.</u> The errors in the estimates of Theorem II diminish exponentially as $\xi \longrightarrow \pm\infty$.

<u>Proof.</u> This follows directly from Lemma I.

Theorem II is equivalent to the statement that the amplitude and phase of the sinusoidal solutions at infinity can be represented by asymptotic expansions in powers of k_o. Clearly, the same is true for the complete potential in the ζ-plane and for the potential in the z-plane.

IV. UNIQUENESS OF THE ASYMPTOTIC EXPANSION OF THE VELOCITY POTENTIAL

We consider again in this section the function $\varphi_1(\xi,\eta;k_0)$ and again we drop the subscript. So the function $\varphi(\xi,\eta;k_0)$ here satisfies:

φ bounded and harmonic in the fluid domain; (28)

$$\varphi_\eta = 0 \quad \text{on} \quad \eta = -1 ; \tag{29}$$

$$\varphi_\eta = \sigma^{-2}(k_0)\frac{dz}{d\zeta}\varphi - h(\xi;k_0) \quad \text{on} \quad \eta = 0; \tag{30}$$

$$\varphi(\xi,0;k_0) \xrightarrow[\xi\to-\infty]{} i[\tanh k_0\,(1+2k_0/\sinh 2k_0)]^{-1} \cdot \int_{-\infty}^{+\infty} [g(\xi';k_0)\varphi(\xi',0;k_0) + h(\xi';k_0)] \cdot [e^{ik_0(\xi-\xi')} - e^{-ik_0(\xi-\xi')}]\,d\xi'. \tag{31}$$

(30) comes directly from (14) and (15a); (31) is just Lemma VI. $g(\xi;k_0)$ and $h(\xi;k_0)$ are defined in (15a) and (15b). As $\xi \longrightarrow +\infty$, $\varphi(\xi,0;k_0) \longrightarrow 0$.

In the expansion of $\varphi(\xi,\eta;k_0)$, $\varphi_n(\xi,\eta)$ satisfies:

φ_n bounded and harmonic in the fluid domain; (28')

$$\varphi_{n_\eta} = 0 \quad \text{on} \quad \eta = -1; \tag{29'}$$

$$\varphi_{n_\eta} = \text{a function of } (\varphi_2, \varphi_3, \dots, \varphi_{n-1}, \xi) \quad \text{on} \quad \eta = 0; \tag{30'}$$

$$\begin{aligned} \varphi_n &\xrightarrow[\xi\to-\infty]{} \text{a polynomial in } \xi \text{ which depends only on } \varphi_2, \varphi_3, \dots, \varphi_{n-1}, \text{ on } \eta = 0; \\ &\xrightarrow[\xi\to+\infty]{} 0 \quad \text{on} \quad \eta = 0. \end{aligned} \tag{31'}$$

(30') and (31') follow from (30) and (31) on noting that any particular $\varphi_n(\xi,0)$ is multiplied by a higher power of

k_0 on the right side than on the left. When coefficients of given powers of k_0 are separated, the primed conditions result. We also note that $\varphi_0(\xi,\eta) = \varphi_1(\xi,\eta) = 0$, from Theorem I.

<u>THEOREM III.</u> If $\alpha < 1$ (See Lemma IV.), then $\varphi_n(\xi,\eta)$ is uniquely given by the above statement of the problem.

<u>Proof.</u> We prove this by induction. For $n = 2$ (the smallest n for which $\varphi_n(\xi,\eta)$ is not identically zero), we have the conditions:

$$\varphi_{2_\eta} = 0 \quad \text{on} \quad \eta = -1 \, ;$$

$$\varphi_{2_\eta} = a\left[\frac{dz}{d\zeta} - 1\right] \quad \text{on} \quad \eta = 0 \, ;$$

$$\varphi_2 \longrightarrow \begin{cases} c_1 \xi + c_2 & \text{as } \xi \longrightarrow -\infty, \text{ on } \eta = 0 \, ; \\ 0 & \text{as } \xi \longrightarrow +\infty, \text{ on } \eta = 0 \, ; \end{cases}$$

where

$$c_1 = a \int_{-\infty}^{+\infty} \left[\frac{dz}{d\zeta} - 1\right]_0 d\xi \, ,$$

$$c_2 = a \int_{-\infty}^{+\infty} \xi \left[\frac{dz}{d\zeta} - 1\right]_0 d\xi \, .$$

A solution for this problem exists; in fact, it was already shown that

$$\varphi_2(\xi,0) = \lim_{k_0 \to 0} \frac{1}{k_0^2} \int_{-\infty}^{+\infty} h(\xi';k_0)\, f(\xi-\xi';k_0)\, d\xi' \, ,$$

and $\varphi_2(\xi,\eta)$ is obtained by the methods of the previous section.

Now suppose that there is some other solution for $\varphi_2(\xi,\eta)$. Let $\psi(\xi,\eta)$ be their difference. Then ψ satisfies:

$\psi(\xi,\eta)$ bounded and harmonic in the fluid domain; (28")

$$\psi_\eta = 0 \quad \text{on} \quad \eta = -1 ; \qquad (29'')$$

$$\psi_\eta = 0 \quad \text{on} \quad \eta = 0 ; \qquad (30'')$$

$$\psi \longrightarrow 0 \quad \text{as} \quad \xi \to \pm\infty, \text{ on } \eta = 0. \qquad (31'')$$

By (29") and (30"), $\psi(\xi,\eta)$ can be extended into the whole plane as a bounded, harmonic function, periodic in η :

$$\psi_\eta(\xi,\eta) = \sum_{n=1}^{\infty} f_n(\xi) \sin n\pi\eta \quad + \quad f_o(\xi).$$

But $f_o(\xi) = 0$, by (29") or (30"). Thus $\psi(\xi,\eta)$ can also be represented in the whole plane as a periodic function which is bounded and harmonic everywhere. By (31") it is bounded even as $\xi \to \pm\infty$. Thus it is a constant, and again by (31") that constant must be zero.

For any other n, $n > 2$, a solution is given for $\varphi_n(\xi,\eta)$ by the methods of the previous section. Suppose a second distinct solution were possible. We assume that, for $m < n$, $\varphi_m(\xi,\eta)$ is unambiguously known. Then the difference between the solutions must again be a function satisfying (28") to (31"), and the same arguments show it to be identically zero.

V. THE SOLUTION AS A FUNCTION OF A COMPLEX VARIABLE

A. The differential-difference equation

We seek a potential function which can be written:

$$\Phi(x,y,t;k_0) = \varphi_1(x,y;k_0)\cos\sigma t + \varphi_2(x,y;k_0)\sin\sigma t \tag{2}$$

φ_1 and φ_2 each satisfy:

$\varphi_i(x,y;k_0)$ is bounded and harmonic in the fluid domain; (4a)

$$\frac{\partial\varphi_i}{\partial n} = 0 \qquad \text{on} \quad y = b(x); \tag{4b}$$

$$\sigma^2\varphi_i - \frac{\partial\varphi_i}{\partial y} = 0 \qquad \text{on} \quad y = 0. \tag{4c}$$

(We still take the units of time such that $g = 1$.) φ_1 and φ_2 are real-valued functions.

Let us now map the fluid domain into a strip in a new ζ-plane:

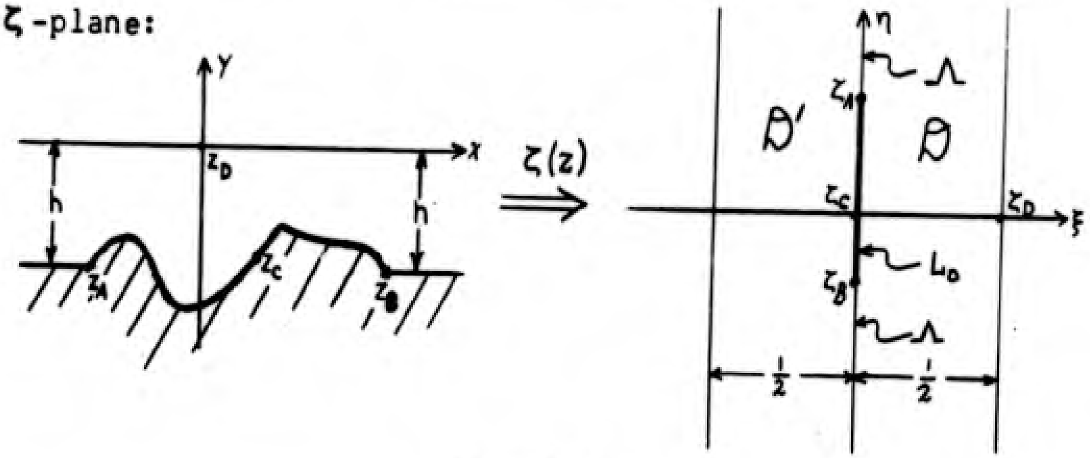

FIGURE 2

The mapping is conformal except perhaps at a finite number of points on the bottom. Also it is unique except for the possibility of a translation of the strip parallel to the η-axis in the ζ-plane. The fluid domain corresponds to the strip: $0 < \xi < 1/2$, $-\infty < \eta < +\infty$, in the ζ-plane.

ζ_A and ζ_B are the images of the ends of the flat portions of the bottom. The part of the η-axis between ζ_A and ζ_B is called "L_o"; the rest of the η-axis is called "Λ".

Let

$$f(\zeta) = \frac{1}{2ih} \frac{dz(\zeta)}{d\zeta}. \tag{32}$$

Since $\text{Im}[f(\zeta)] = 0$ on Λ and on $\xi = 1/2$, $f(\zeta)$ can be extended into the whole plane as a periodic function, with period unity:

$$f(\zeta+1) = f(\zeta). \tag{33a}$$

Also,

$$f(\zeta) = \overline{f(-\overline{\zeta})}. \tag{33b}$$

There will generally be branch cuts at the periodic repetitions of L_o. If we define:

$$L_n = \left\{\zeta : (\zeta-n) \in L_o\right\}, \tag{34}$$

then each L_n, $n = 0, \pm 1, \pm 2, \ldots$, is a branch cut for the function $f(\zeta)$.

Let $\psi(\xi,\eta;\delta)$ be the harmonic function in the ζ-plane corresponding to either φ_1 or φ_2, i.e.,

$$\varphi_i(x,y;k_o) = \psi(\xi(x,y),\eta(x,y);\delta(k_o)), \quad i = 1 \text{ or } 2,$$

where

$$\delta = 2hk_o.$$

k_o is now to be considered strictly as a real variable. From (4b) and (4c), $\psi(\xi,\eta;\delta)$ satisfies:

$$\psi_\xi - (2h\sigma^2)\, f(\zeta)\, \psi = 0 \quad \text{on } \xi = 1/2; \tag{35a}$$

$$\psi_\xi = 0 \quad \text{on } \xi = +0. \tag{35b}$$

We now change the problem to one of finding a function of a complex variable, $\zeta = \xi + i\eta$, which satisfies conditions to be determined. We state the new problem in the following:

THEOREM IV. Let $G(\zeta;\delta)$ be a function defined in $\mathcal{D}$ (the image of the fluid domain), such that

$$\psi(\xi,\eta;\delta) = \mathrm{Re}[iG(\zeta;\delta)] = -\,\mathrm{Im}[G(\zeta;\delta)]$$

satisfies (35a) and (35b). Then $G(\zeta;\delta)$ can be continued analytically into the whole plane in such a way that it satisfies:

$$[G'(\zeta+1;\delta) - G'(\zeta;\delta)] - 2h\sigma^{2}f(\zeta)[G(\zeta+1;\delta)+G(\zeta;\delta)] = 0, \tag{36}$$

for all ζ, and it also satisfies:

$$\mathrm{Re}[G(\zeta;\delta)] = 0 \quad \text{on} \quad L_0 \cup \Lambda . \tag{37}$$

The prime denotes differentiation with respect to ζ. (Generally it will be necessary for branch cuts to be located on every L_n; in addition, to make $G(\zeta;\delta)$ single-valued, it may be necessary to cut the plane further, viz., along the real axis from $\xi = 1$ to $\xi = \infty$ and from $\xi = -1$ to $\xi = -\infty$.)

Proof. By (35b), $\mathrm{Re}[G(\zeta;\delta)]$ = constant on $L_0 \cup \Lambda$. Since there are no restrictions which prevent the addition of an arbitrary real constant to $G(\zeta;\delta)$ (ψ determines directly only the imaginary part of G), we set:

$$\mathrm{Re}[G(\zeta;\delta)] = 0 \quad \text{on} \quad L_0 \cup \Lambda .$$

We then extend the definition of $G(\zeta;\delta)$ into $\mathcal{D}'$ (See Figure 2) by reflection about the imaginary axis. Let the function so defined on $\mathcal{D} \cup \mathcal{D}'$ again be called $G(\zeta;\delta)$; then (a)

(a) $\mathcal{D}$ is strictly an open set. We define $G(\zeta;\delta)$ on the boundaries as the limiting value when the boundary is approached from the interior, if such a limiting value exists. Where such limit does not exist, the function remains undefined.

$$G(\zeta;\delta)\Big|_{\zeta=\zeta_1} = -\overline{G(\zeta;\delta)\Big|_{\zeta=-\bar{\zeta}_1}}, \quad \zeta_1 \in \mathcal{D} \cup \mathcal{D}', \qquad (38a)$$

$$G'(\zeta;\delta)\Big|_{\zeta=\zeta_1} = +\overline{G'(\zeta;\delta)\Big|_{\zeta=-\bar{\zeta}_1}}, \quad \zeta_1 \in \mathcal{D} \cup \mathcal{D}'. \qquad (38b)$$

Since $\psi(\xi,\eta;\delta) = -\operatorname{Im}[G(\zeta;\delta)]$, $\psi_\xi(\xi,\eta;\delta) = -\operatorname{Im}[G'(\zeta;\delta)]$, we can write:

$$\psi(\xi,\eta;\delta) = -\frac{1}{2i}\left[G(\xi+i\eta;\delta) + G(-\xi+i\eta;\delta)\right];$$

$$\psi_\xi(\xi,\eta;\delta) = -\frac{1}{2i}\left[G'(\xi+i\eta;\delta) - G'(-\xi+i\eta;\delta)\right].$$

Then equation (35a) becomes, for $\zeta = 1/2 + i\eta$:

$$[G'(\zeta;\delta) - G'(\zeta-1;\delta)] - 2h\sigma^2 f(\zeta)[G(\zeta;\delta) + G(\zeta-1;\delta)] = 0.$$

So far, $G(\zeta;\delta)$ has been defined only for $\zeta \in \mathcal{D} \cup \mathcal{D}'$, i.e., for $-1/2 \leq \xi \leq +1/2$. We can extend $G(\zeta;\delta)$ into the whole plane now by this equation. Let us again call the extended function $G(\zeta;\delta)$; then, for example, if $1/2 < \xi \leq 3/2$, $G(\zeta;\delta)$ is given by the solution of the differential equation:

$$G'(\zeta;\delta) - 2h\sigma^2 f(\zeta)G(\zeta;\delta) = G'(\zeta-1;\delta) + 2h\sigma^2 f(\zeta)G(\zeta-1;\delta).$$

The right hand side is completely defined, since $f(\zeta)$ was previously extended into the whole (cut) plane. We note that the differential equation has singular points on the branch cuts, L_n. So it is expected that $G(\zeta;\delta)$ will likewise have singularities on L_n, for all n. Otherwise, the differential equation has no singular points, and so it defines $G(\zeta;\delta)$ as an analytic function in the neighborhood of any point $\zeta \notin L_n$.

For $-1 < \xi < +1$, the differential equation defines $G(\zeta;\delta)$

as a single-valued function. But for $|\xi| > 1$, it may be necessary to introduce other branch cuts connecting the L_n's, in order to make $G(\zeta;\delta)$ single-valued. If such be necessary, let us make the cuts along the real axis, so that $G(\zeta;\delta)$ may be defined uniquely as a single-valued analytic function everywhere in the cut plane:

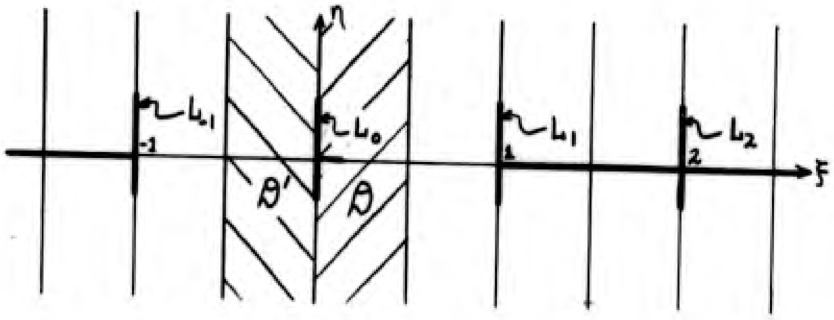

FIGURE 3

Clearly the extension of $G(\zeta;\delta)$ into $\mathcal{D}'$ also satisfies (37).

The converse of Theorem IV is also true:

THEOREM V. Let $G(\zeta;\delta)$ be a single-valued analytic function in the cut plane of Figure 3, satisfying (36) and (37). Then

$$\psi(\xi,\eta;\delta) = \mathrm{Re}[iG(\zeta;\delta)] \tag{38}$$

satisfies (35a) and (35b).

Proof. Equation (37) requires that

$$\begin{aligned} G(\tfrac{1}{2}+i\eta;\delta) + G(-\tfrac{1}{2}+i\eta;\delta) &= 2i\,\mathrm{Im}[G(\tfrac{1}{2}+i\eta;\delta)] \\ &= -2i\,\psi(\tfrac{1}{2},\eta;\delta); \\ G'(\tfrac{1}{2}+i\eta;\delta) - G'(-\tfrac{1}{2}+i\eta;\delta) &= -2i\,\psi_\xi(\tfrac{1}{2},\eta;\delta). \end{aligned}$$

When these are substituted into (36), we obtain (35a). Also, (37) requires that $\mathrm{Im}[G'(\zeta;\delta)] = -\psi_\xi = 0$, on $\xi = 0$.

B. Boundedness of the complex potential

In the z-plane it was permitted that the velocity might be infinite at a finite number of points on the bottom. We now show that the analogous situation cannot occur in the ζ -plane.

THEOREM VI. $G(\zeta;\delta)$ is analytic, thus bounded, on and near L_0; $G'(\zeta;\delta)$ is thus also bounded on and near L_0.

Proof. Let z_0 be one of the points of the bottom at which the velocity is infinite, and let its image in the ζ-plane be $\zeta_0 \in L_0$. Since there are only a finite number of such points, there is a neighborhood of ζ_0 which contains no other such points. In this neighborhood, $G(\zeta;\delta)$ is single-valued (except that it may be undefined at ζ_0 itself), since $\mathrm{Re}[G(\zeta;\delta)] = 0$ on $L_0 U \Lambda$, and this implies that $\mathrm{Im}[G(\zeta;\delta)]$ has the same value on both sides of $L_0 U \Lambda$. (For points near but not on $L_0 U \Lambda$, $G(\zeta;\delta)$ is obviously single-valued.) Also, $G(\zeta;\delta)$ is analytic in this neighborhood, except possibly at ζ_0 itself. Thus either ζ_0 is a regular point or else $G(\zeta;\delta)$ has an isolated singularity at ζ_0.

Now $\psi(\xi,\eta;\delta) = - \mathrm{Im}[G(\zeta;\delta)]$ is bounded for $\zeta \in \mathcal{D}$ or $\zeta \in L_0 U \Lambda$. The extension of $G(\zeta;\delta)$ into $\mathcal{D}'$ shows that $\mathrm{Im}[G(\zeta;\delta)]$ is bounded there also. Thus $\mathrm{Im}[G(\zeta;\delta)]$ is bounded in the complete neighborhood of ζ_0. At an isolated singularity, it is impossible for the imaginary part of the function to be bounded in a complete neighborhood and the real part unbounded. So $G(\zeta;\delta)$ is bounded in the neighborhood of ζ_0 and ζ_0 is a regular point of $G(\zeta;\delta)$.

We thus have the fact that L_0 is actually not needed as a branch cut for $G(\zeta;\delta)$; $G(\zeta;\delta)$ is analytic for all $\zeta \in \mathcal{D}' \cup \mathcal{D}' \cup L_0 \cup \Lambda$. And, in fact, from the differential equation which extends $G(\zeta;\delta)$ into the whole plane, $G(\zeta;\delta)$ is analytic for all ζ such that $-1 < \xi < +1$, $-\infty < \eta < +\infty$. But it should be noted that L_0 is still a branch cut for $f(\zeta)$, and the cuts on L_n, $n \neq 0$, are still needed for $G(\zeta;\delta)$.

C. The asymptotic expansion of the complex potential

We now assume that $G(\zeta;\delta)$ can be expressed asymptotically as $\delta \longrightarrow 0$ by a power series in δ:

$$G(\zeta;\delta) = \sum_{n=0}^{N} G_n(\zeta)\,\delta^n + o(\delta^N), \text{ as } \delta \longrightarrow 0. \tag{40}$$

Let:

$$2h\sigma^2 = \sum_{n=1}^{\infty} \gamma_{2n}\,\delta^{2n}. \tag{41}$$

Since $2h\sigma^2 = 2k_0h \tanh k_0h = \delta \tanh \delta/2$, it is easily seen from the well-known expansion for the hyperbolic tangent that

$$\gamma_{2n} = \frac{2(2^{2n}-1)}{(2n)!}\,B_{2n}\,, \tag{41'}$$

where B_{2n} = $(2n)^{th}$ Bernoulli number.[a]

Also, define the two operators:

---

[a] The definition of B_{2n} is not standard. We are following reference 7, where:

$$\sum_0^{\infty} \frac{t^n}{n!} B_n = t/(e^t-1).$$

Then, for example, $B_0 = 1$; $B_1 = -1/2$; $B_2 = 1/6$; $B_3 = 0$; $B_4 = -1/30$; etc. Also, $\gamma_2 = 1/2$; $\gamma_4 = -1/24$; $\gamma_6 = 1/240$; etc.

$$\Delta F(\zeta) = F(\zeta+1) - F(\zeta),$$

$$\Sigma F(\zeta) = F(\zeta+1) + F(\zeta).$$

Then we have the following:

THEOREM VII. $G_n(\zeta)$ satisfies the following conditions, for all n:

$$\Delta G_n'(\zeta) = f(\zeta) \sum_{m=1}^{q} \gamma_{2m} \Sigma G_{n-2m}(\zeta) \ , \tag{42}$$

where

$$q = \begin{cases} n/2, \text{ for n even,} \\ (n-1)/2, \text{ for n odd.} \end{cases}$$

$$\mathrm{Re}[G_n(\zeta)] = 0 \quad \text{for} \quad \xi = 0. \tag{43}$$

$$|G_n(\zeta)| , \ |G_n'(\zeta)| \quad \text{bounded in } \mathcal{D} \cup \mathcal{D}' \cup L_0 \cup \Lambda \ , \tag{44}$$

except as $\eta \longrightarrow \pm\infty$, where each is of finite degree in ζ.

Proof. (42) is obtained by substituting (40) and (41) into (36) and then identifying corresponding coefficients of δ. When the expansion, (40), is substituted into (37), the latter must hold for the coefficient of each power of δ, which gives (43). Likewise, Theorem VI must apply to each coefficient of δ^n in (40).

From Theorem II, $\mathrm{Im}[G(\zeta;\delta)]$ behaves sinusoidally in η as $\eta \longrightarrow \pm\infty$. From Lemma I, the error in approximating $\mathrm{Im}[G(\zeta;\delta)]$ by a sinusoidal function, as $\eta \longrightarrow \pm\infty$, decreases exponentially with increasing $|\eta|$. That is,

$$\mathrm{Re}[iG(\zeta;\delta)] = \mathrm{Re}[A_1 \cosh(\delta\zeta - i\varepsilon_1) + iA_2 \sinh(\delta\zeta - i\varepsilon_2)] + O(e^{-c|\eta|}), \quad \text{as } \eta \longrightarrow \pm\infty,$$

where A_1, A_2, ε_1, ε_2 are real constants depending only on sgn η and on δ, and c is a real positive constant depending only on the geometry. Since $\mathrm{Re}[G(\zeta;\delta)] = 0$ on $\xi = 0$, this requires that

$$iG(\zeta;\delta) \longrightarrow A_1\cosh(\delta\zeta - i\epsilon_1) + iA_2\sinh(\delta\zeta - i\epsilon_2)$$

as $\eta \rightarrow \pm\infty$, again with exponentially diminishing error. By Theorem II, the right hand side here can be expanded into an asymptotic power series in δ; clearly then $G_n(\zeta)$ will behave like a polynomial in ζ, as $\eta \longrightarrow \pm\infty$.

If we write:

$$G_n(\zeta) = \sum_{m=0}^{N} a_{nm}(\operatorname{sgn}\eta)\,\zeta^m + O(e^{-c|\eta|}), \text{ as } \eta \longrightarrow \pm\infty,$$

then:

$$G'_n(\ ) = \sum_{m=1}^{N} ma_{nm}(\operatorname{sgn}\eta)\,\zeta^{m-1} + O(e^{-c|\eta|}).$$

This is possible only because of the rapid decrease in the error term. We thus have the last part of the theorem.

<u>COROLLARY.</u> The polynomial expression for $G_n(\zeta)$ as $\eta \longrightarrow \pm\infty$ can be differentiated to give the expression for the behavior of $G'_n(\zeta)$ in the same limits.

The above statement of the asymptotic behavior of $G_n(\zeta)$ and $G'_n(\zeta)$ is not sufficient to determine the function $G_n(\zeta)$. We postpone to a later section the formulation of a more explicit condition at infinity.

VI. SOLUTION WHEN THE BOTTOM IS SYMMETRICAL

The determination of the asymptotic expansions can be simplified in cases where the bottom is symmetrical about a vertical axis. In this and the following sections, we assume that the function which specifies the shape of the bottom has the property:

$$b(x) = b(-x). \qquad (45)$$

We also require that the mapping into the ζ-plane be performed so that the symmetry axis in the z-plane goes into the real axis in the ζ-plane. (See Figure 2.) Then it is apparent from the symmetry that $\mathrm{Im}[f(\zeta)] = 0$ on $\eta = 0$, $0 < \xi < 1/2$. Thus:

$$f(\zeta) = \overline{f(\bar{\zeta})} ; \qquad (46a)$$

which, with (33b), implies that

$$f(\zeta) = f(-\zeta). \qquad (46b)$$

For reference, we note that:

$$f(\zeta) = 1 + O(e^{-2\pi|\eta|}) \quad \text{as} \quad \eta \longrightarrow \pm\infty. \qquad (47)$$

A. Separation into even and odd solutions

THEOREM VIII. If (45) holds, then any solution can be represented as the sum of an odd and an even function, each of which is separately a solution.

Proof. Let $G(\zeta;\delta)$ be a solution as defined in Section V, i.e.,

$$\psi(\xi,\eta;\delta) = \mathrm{Re}[iG(\zeta;\delta)] \qquad (48)$$

satisfies (35a) and (35b). Let G be continued into $\mathcal{D}'$ as given by (38a) and (38b). Then (35a) is equivalent to:

$$\mathrm{Im}\left[\frac{dG(\zeta_1;\delta)}{d\zeta_1} - 2h\sigma^2 f(\zeta_1)G(\zeta_1;\delta)\right] = 0, \qquad (49)$$

for $\zeta_1 = 1/2 + i\eta$. By (38a) and (38b),

$$\mathrm{Im}\left[-\frac{dG(\zeta_2;\delta)}{d\zeta_2} - 2h\sigma^2 f(\zeta_2)G(\zeta_2;\delta)\right] = 0\ , \qquad (49')$$

for $\zeta_2 = -1/2 + i\eta$. If we take $\zeta_2 = -\zeta_1$, with $\zeta_1 = 1/2 + i\eta$, then:

$$G(\zeta_2;\delta) = G(-\zeta_1;\delta), \quad \text{and} \quad \frac{dG(\zeta_2;\delta)}{d\zeta_2} = -\frac{dG(-\zeta_1;\delta)}{d\zeta_1}.$$

Thus, for $\zeta_1 = 1/2 + i\eta$, (49') gives:

$$\mathrm{Im}\left[\frac{dG(-\zeta_1;\delta)}{d\zeta_1} - 2h\sigma^2 f(\zeta_1)G(-\zeta_1;\delta)\right] = 0.$$

(46b) has been used here. This is the same as the condition on $G(\zeta_1;\delta)$. That is, if $G(\zeta;\delta)$ satisfies (49), then so does $G(-\zeta;\delta)$.

Equation (35b) requires that $\mathrm{Re}[G(\zeta;\delta)] = 0$ on $\xi = 0$. Obviously, if $G(\zeta;\delta)$ satisfies this condition, then so does $G(-\zeta;\delta)$.

Now we need only note that

$$G(\zeta;\delta) = \tfrac{1}{2}[G(\zeta;\delta) + G(-\zeta;\delta)] + \tfrac{1}{2}[G(\zeta;\delta) - G(-\zeta;\delta)],$$

the first bracketed quantity being even in ζ, the second odd in ζ, and each a solution separately satisfying (35a) and (35b) (when (48) is used).

If a solution satisfies (37) (as it must), and if it is odd or even in ζ, then it is easily seen that the real and imaginary parts are each odd or even in η. The mapping between the z-plane and the ζ-plane retains this symmetry, so that the real potential in the z-plane can also be represented as the sum of odd and even solutions in x.

B. Asymptotic form of the even and odd solutions

The original form of the solution in the z-plane was:

$$\Phi(x,y,t;k_0) = \varphi_1(x,y;k_0)\cos\sigma t + \varphi_2(x,y;k_0)\sin\sigma t. \tag{50}$$

(See equation (2).) We can now rewrite this:

$$\begin{aligned}\Phi(x,y,t;k_0) = {} & \tfrac{1}{2}\Big\{[\varphi_1(x,y;k_0)+\varphi_1(-x,y;k_0)]\cos\sigma t \\ & \qquad + [\varphi_2(x,y;k_0)+\varphi_2(-x,y;k_0)]\sin\sigma t\Big\} \\ & + \tfrac{1}{2}\Big\{[\varphi_1(x,y;k_0)-\varphi_1(-x,y;k_0)]\cos\sigma t \\ & \qquad + [\varphi_2(x,y;k_0)-\varphi_2(-x,y;k_0)]\sin\sigma t\Big\}.\end{aligned} \tag{50a}$$

Each quantity in square brackets is a solution of the time-independent problem:

$$(A)\begin{cases}\chi(x,y;k_0) \text{ bounded and harmonic in the fluid domain;} \\ \dfrac{\partial\chi}{\partial n} = 0 \quad \text{on } y = b(x); \\ \dfrac{\partial\chi}{\partial y} - \sigma^2\chi = 0 \quad \text{on } y = 0.\end{cases}$$

LEMMA. Let

$$\chi_1(x,y;k_0) = \tfrac{1}{2}[\varphi_1(x,y;k_0) + \varphi_1(-x,y;k_0)],$$
$$\chi_2(x,y;k_0) = \tfrac{1}{2}[\varphi_2(x,y;k_0) + \varphi_2(-x,y;k_0)].$$

Then:

$$\chi_1(x,y;k_0) = A\,\chi_2(x,y;k_0),$$

where A = a real constant; that is, the two even solutions are linearly dependent. Similarly, the two odd solutions are linearly dependent.

Proof. Suppose the two even solutions to be independent. As $x \longrightarrow \mp\infty$, each must vary sinusoidally in x, by Lemma I. Since each is even in x, let their asymptotic behavior be given by:

$$\chi_1(x,0;k_0) \longrightarrow A_1 \cos\,(k_0x \pm \gamma_1),$$

$$\chi_2(x,0;k_0) \longrightarrow A_2 \cos\,(k_0x \pm \gamma_2).$$

Now $[\chi_1(x,y;k_0)\cos\sigma t]$ and $[\chi_2(x,y;k_0)\sin\sigma t]$ are legitimate solutions for $\Phi(x,y,t;k_0)$; so also must be the linear combination of these:

$$\Phi_1(x,y,t;k_0) = \chi_1(x,y;k_0)\cos\sigma t + \frac{A_1}{A_2}\chi_2(x,y;k_0)\sin\sigma t.$$

As $x \longrightarrow \mp\infty$,

$$\begin{aligned}\Phi_1(x,0,t;k_0) \longrightarrow\ & \cos\,(k_0x+\sigma t)\,[\,\cos\gamma_1 \pm \sin\gamma_2]\\ &+ \sin\,(k_0x+\sigma t)\,[\mp\sin\gamma_1 + \cos\gamma_2]\\ &+ \cos\,(k_0x-\sigma t)\,[\,\cos\gamma_1 \mp \sin\gamma_2]\\ &+ \sin\,(k_0x-\sigma t)\,[\mp\sin\gamma_1 - \cos\gamma_2].\end{aligned}$$

The transmission of energy principle (Lemma II) then requires, after some reduction, that

$$\gamma_2 = \gamma_1.$$

That is, χ_1 and $(A_1/A_2)\chi_2$ have the same asymptotic behavior. Now let:

$$\chi(x,y;k_0) = \chi_1(x,y;k_0) - \frac{A_1}{A_2}\chi_2(x,y;k_0)\ .$$

χ satisfies (A) and in addition:

$$\chi(x,y;k_0) \longrightarrow 0 \quad \text{as } x \longrightarrow \mp\infty.$$

In the ζ-plane used in Section III, $\chi(x(\xi,\eta),y(\xi,\eta);k_0)$ satisfies:

$$\sigma^2\chi - \chi_\eta = -\sigma^2\left[\frac{dz}{d\zeta}-1\right]\chi \qquad \text{on } \eta = 0.$$

This is exactly of the form for application of Lemma IV, but with $h(\xi;k_0) = 0$. The solution by iteration of the integral equation[a] in Lemma IV shows that χ is identically zero for $\eta = 0$. Equation (14) and Lemma V then show that χ is identically zero for all (ξ,η), hence for all (x,y). This proves that χ_1 and χ_2 are linearly related. The same proof applies to the odd solutions as well.

THEOREM IX. For the symmetrical geometry, the potential can be written:

$$\Phi(x,y,t;k_0) = \psi_1(x,y;k_0)\cos(\sigma t+\alpha) + \psi_2(x,y;k_0)\sin(\sigma t+\beta), \tag{51}$$

where, in the notation of (50),

$$\psi_1(x,y;k_0)\cos\alpha = \tfrac{1}{2}[\varphi_1(x,y;k_0) + \varphi_1(-x,y;k_0)],$$

$$\psi_1(x,y;k_0)\sin\alpha = \tfrac{1}{2}[\varphi_2(x,y;k_0) + \varphi_2(-x,y;k_0)],$$

$$\psi_2(x,y;k_0)\cos\beta = \tfrac{1}{2}[\varphi_2(x,y;k_0) - \varphi_2(-x,y;k_0)],$$

$$\psi_2(x,y;k_0)\sin\beta = \tfrac{1}{2}[\varphi_1(x,y;k_0) - \varphi_1(-x,y;k_0)],$$

---

[a] The application of the iteration procedure is discussed in detail by Kreisel in reference 3.

and α and β are functions of k_o only. Furthermore, if

$$\begin{aligned}\Phi(x,0,t;k_o) \xrightarrow[x\to-\infty]{} & \cos(\sigma t - k_o x) + a_1(k_o)\cos(\sigma t + k_o x) \\ & + a_2(k_o)\sin(\sigma t + k_o x), \\ \xrightarrow[x\to+\infty]{} & b_1(k_o)\cos(\sigma t - k_o x) + b_2(k_o)\sin(\sigma t - k_o x),\end{aligned} \tag{50b}$$

then:

$$\psi_1(x,0;k_o) \longrightarrow \cos(k_o x \pm \alpha) \qquad \text{as } x \longrightarrow \mp\infty, \tag{52a}$$

$$\psi_2(x,0;k_o) \longrightarrow \sin(k_o x \pm \beta) \qquad \text{as } x \longrightarrow \mp\infty. \tag{52b}$$

Proof. The first result, (51), comes immediately from application of the Lemma to (50a). The second result, (52a) and (52b), is obtained by writing the asymptotic form of ψ_1 and ψ_2, as $x \longrightarrow \mp\infty$:

$$\psi_1(x,0;k_o) \longrightarrow A_1\cos(k_o x \pm \gamma_1),$$

$$\psi_2(x,0;k_o) \longrightarrow A_2\sin(k_o x \pm \gamma_2).$$

These forms must exist, since ψ_1 and ψ_2 are even and odd, respectively, and they behave sinusoidally at infinity, by Lemma I. Detailed comparison with a reduction of (50b) shows however that these forms can exist only if:

$$A_1 = A_2 = 1; \qquad \gamma_1 = \alpha \; ; \quad \gamma_2 = \beta \; .$$

Thus the second part of the theorem.

Further comparison of (50b) with (52a) and (52b) shows that:

$$\begin{aligned} a_1 &= \sin(\alpha+\beta)\sin(\beta-\alpha); \\ b_1 &= \cos(\alpha+\beta)\cos(\beta-\alpha); \\ a_2 &= \cos(\alpha+\beta)\sin(\beta-\alpha); \\ b_2 &= -\sin(\alpha+\beta)\cos(\beta-\alpha). \end{aligned} \tag{53}$$

We again map (conformally) the fluid domain onto the strip: $0 < \xi < 1/2$, $-\infty < \eta < +\infty$, in the ζ-plane, as indicated in Figure 2. The symmetry of the z-plane is to be preserved in the image in the ζ-plane. Let ψ_1 and ψ_2 be the real parts of two functions of the complex variable, ζ, as follows:

$$\psi_1(x,y;k_0) = \operatorname{Re}[iG(\zeta;\delta)]; \tag{54a}$$

$$\psi_2(x,y;k_0) = \operatorname{Re}[iH(\zeta;\delta)]. \tag{54b}$$

The function G and H will have all of the properties described for the G of Section V. In addition,

$$G(\zeta;\delta) = G(-\zeta;\delta); \tag{55a}$$

$$H(\zeta;\delta) = -H(-\zeta;\delta). \tag{55b}$$

Also, as $\eta \longrightarrow \pm\infty$, it is easily seen that (52a) and (52b) require that:

$$G(\zeta;\delta) \longrightarrow -i\cosh[\delta(\zeta \pm i\varepsilon) \mp i\alpha(\delta)]; \tag{56a}$$

$$H(\zeta;\delta) \longrightarrow \sinh[\delta(\zeta \pm i\varepsilon) \mp i\beta(\delta)]; \tag{56b}$$

where ε is the amount by which the fluid domain is "stretched" in the mapping. Specifically, we have:

$$z(\zeta) = \int_{1/2+i0}^{\zeta} \frac{dz(\zeta')}{d\zeta'}\, d\zeta'$$

$$= 2ih(\zeta - \tfrac{1}{2}) + 2ih \int_{1/2+i0}^{\zeta} [f(\zeta') - 1]\, d\zeta'$$

$$\xrightarrow[x \to \mp\infty]{} 2ih(\zeta - \tfrac{1}{2}) \mp ih \int_{-i\infty}^{+i\infty} [f(\zeta) - 1]\, d\zeta .$$

If we write:

$$z \longrightarrow 2ih(\zeta \pm i\varepsilon) - ih, \qquad \text{as } x \longrightarrow \mp\infty,$$

(See Figure 2.) then:

$$\varepsilon = -\frac{1}{2}\int_{-\infty}^{+\infty}[f(\tfrac{1}{2}+i\eta)-1]\,d\eta. \tag{57}$$

C. The reflection and transmission coefficients

In accordance with the definitions in Equations (11a) and (11b), the reflection and transmission coefficients are now:

$$R = \sqrt{a_1^2 + a_2^2}\ ;$$

$$T = \sqrt{b_1^2 + b_2^2}\ .$$

We can express R and T in terms of the asymptotic "phases" of the even and odd solutions.

THEOREM X. If $\Phi(x,y,t;k_0)$ has the asymptotic form of (50b), then:

$$R = |\sin(\alpha - \beta)|, \tag{58a}$$

$$T = |\cos(\alpha - \beta)|, \tag{58b}$$

where α and β both are functions of δ.

Proof. From (53),

$$\begin{aligned} R^2 &= \sin^2(\alpha+\beta)\sin^2(\beta-\alpha) + \cos^2(\alpha+\beta)\sin^2(\beta-\alpha) \\ &= \sin^2(\alpha-\beta)\ . \\ T^2 &= \cos^2(\alpha+\beta)\cos^2(\beta-\alpha) + \sin^2(\alpha+\beta)\cos^2(\beta-\alpha) \\ &= \cos^2(\alpha-\beta)\ . \end{aligned}$$

D. Expansions of the even and odd solutions

Let:

$$G(\zeta;\delta) = \sum_{n=o}^{N} G_n(\zeta)\delta^n + o(\delta^N) \quad \text{as } \delta \longrightarrow 0; \qquad (59a)$$

$$H(\zeta;\delta) = \sum_{n=o}^{N} H_n(\zeta)\delta^n + o(\delta^N) \quad \text{as } \delta \longrightarrow 0. \qquad (59b)$$

The functions $G_n(\zeta)$, $H_n(\zeta)$ satisfy all of the conditions found in Section V-C, which we collect here for reference.

(A) $$\Delta G_n'(\zeta) = f(\zeta) \sum_{m=1}^{q} \gamma_{2m} \Sigma G_{n-2m}(\zeta) \,, \qquad (60)$$

where

$$q = \begin{cases} n/2 \,, & n \text{ even}, \\ (n-1)/2, & n \text{ odd}. \end{cases} \qquad (61)$$

(B) $\mathrm{Re}[G_n(\zeta)] = 0$ on $L_o \cup \Lambda$,

which implies more generally that

$$\mathrm{Re}[G_n^{(m)}(\zeta)] = 0 \,, \quad \text{for } m \text{ even, on } L_o \cup \Lambda; \qquad (62a)$$

$$\mathrm{Im}[G_n^{(m)}(\zeta)] = 0 \,, \quad \text{for } m \text{ odd, on } L_o \cup \Lambda; \qquad (62b)$$

where

$$G_n^{(m)}(\zeta) = \frac{d^m}{d\zeta^m} G_n(\zeta) \,.$$

(C) $G_n(\zeta)$ is analytic for all $\zeta \in \mathcal{D} \cup \mathcal{D}' \cup L_o \cup \Lambda$,

which implies in particular that:

$G_n^{(m)}(\zeta)$ is bounded and harmonic near L_o (all m). (63)

(D) The above all apply to $H_n(\zeta)$ as well. Since G is even in ζ and H is odd, we have in addition that:

$$G_n^{(m)}(\zeta) = (-1)^m G_n^{(m)}(-\zeta); \quad (\text{all } m) \qquad (64)$$

$$H_n^{(m)}(\zeta) = -(-1)^m H_n^{(m)}(-\zeta). \quad (\text{all } m) \qquad (65)$$

(E) Let:

$$\alpha(\delta) = \sum_{n=o}^{N} \alpha_n \delta^n + o(\delta^N) \quad \text{as } \delta \longrightarrow 0; \qquad (66a)$$

$$\beta(\delta) = \sum_{n=0}^{N} \beta_n \delta^n + o(\delta^N) \quad \text{as } \delta \longrightarrow 0. \qquad (66b)$$

We now expand (56a) and (56b) into asymptotic power series in δ, and then we identify coefficients with the respective coefficients in (59a) and (59b), as $\eta \longrightarrow \pm\infty$. (This is allowed by Theorem II.) The results are as follows, as $\eta \longrightarrow \pm\infty$:

$$G_0(\zeta) \longrightarrow -i\cos\alpha_0; \qquad (67a)$$

$$G_1(\zeta) \longrightarrow \mp(\zeta \pm i\varepsilon \mp i\alpha_1)\sin\alpha_0; \qquad (67b)$$

$$G_2(\zeta) \longrightarrow -\frac{1}{2!}i(\zeta \pm i\varepsilon \mp i\alpha_1)^2\cos\alpha_0 + i\alpha_2\sin\alpha_0; \qquad (67c)$$

$$\begin{aligned} G_3(\zeta) \longrightarrow {} & \mp\alpha_2(\zeta \pm i\varepsilon \mp i\alpha_1)\cos\alpha_0 \\ & + [\mp\frac{1}{3!}(\zeta \pm i\varepsilon \mp i\alpha_1)^3 + i\alpha_3]\sin\alpha_0; \end{aligned} \qquad (67d)$$

$$\begin{aligned} G_4(\zeta) \longrightarrow {} & [-\frac{1}{4!}i(\zeta \pm i\varepsilon \mp i\alpha_1)^4 \mp \alpha_3(\zeta \pm i\varepsilon \mp i\alpha_1) + \frac{1}{2}i\alpha_2^2]\cos\alpha_0 \\ & + [\frac{1}{2}i\alpha_2(\zeta \pm i\varepsilon \mp i\alpha_1)^2 + i\alpha_4]\sin\alpha_0; \end{aligned} \qquad (67e)$$

etc.;

$$H_0(\zeta) \longrightarrow \mp i\sin\beta_0; \qquad (67f)$$

$$H_1(\zeta) \longrightarrow (\zeta \pm i\varepsilon \mp i\beta_1)\cos\beta_0; \qquad (67g)$$

$$H_2(\zeta) \longrightarrow \mp i\beta_2\cos\beta_0 \mp \frac{1}{2!}i(\zeta \pm i\varepsilon \mp i\beta_1)^2\sin\beta_0; \qquad (67h)$$

$$\begin{aligned} H_3(\zeta) \longrightarrow {} & [\frac{1}{3!}(\zeta \pm i\varepsilon \mp i\beta_1)^3 \mp i\beta_3]\cos\beta_0 \\ & - \beta_2(\zeta \pm i\varepsilon \mp i\beta_1)\sin\beta_0; \end{aligned} \qquad (67i)$$

$$\begin{aligned} H_4(\zeta) \longrightarrow {} & [\mp\frac{1}{2}i\beta_2(\zeta \pm i\varepsilon \mp i\beta_1)^2 \mp i\beta_4]\cos\beta_0 \qquad (67j) \\ & + [\mp\frac{1}{4!}i(\zeta \pm i\varepsilon \mp i\beta_1)^4 - \beta_3(\zeta \pm i\varepsilon \mp i\beta_1) \pm \frac{1}{2}i\beta_2^2]\sin\beta_0; \end{aligned}$$

etc.

These are the conditions which were omitted in Section

V-C, now stated only for the case of the symmetrical bottom. Although the coefficients in the series for $\alpha(\delta)$ and $\beta(\delta)$ are still unknown, it will be shown presently that sufficient information is now available to find both solutions, G and H, as well as α and β, uniquely.

We note specifically that the asymptotic behavior as $\eta \longrightarrow \pm\infty$ of $G_n^{(m)}(\zeta)$ and $H_n^{(m)}(\zeta)$ can be found by differentiation of the appropriate expressions in (67a)-(67j). See the corollary to Theorem VII.

E. The even solutions

We now show that the conditions (60)-(67) determine $G_n(\zeta)$ completely. In fact, the α_n's will be determined as well.

First consider $G_n(\zeta)$ with n odd. For n = 1, from (60),

$$\Delta G_1'(\zeta) = 0.$$

Thus, $G_1'(\zeta) = P_1(\zeta) =$ a periodic function of unit period. $G_1'(\zeta)$ has no singularities, from this result and (63). The only periodic function with this property, which remains of finite degree as $\eta \longrightarrow \pm\infty$, is $P_1(\zeta) =$ constant. From (67b), $G_1'(\zeta) \longrightarrow \mp \sin \alpha_0$ as $\eta \longrightarrow \pm\infty$. The only constant which can have this behavior is zero. Thus α_0 is an integer multiple of π. But addition of π to α does not change the value of

$$\cos(k_0 x \pm \alpha)\cos(\sigma t + \alpha).$$

(See (51) and (52a).) So we can set

$$\alpha_0 = 0. \qquad (68a)$$

It then follows from (67b) that

$$G_1(\zeta) = \text{constant} = 0. \qquad (69a)$$

Similarly,

$$\Delta G_3'(\zeta) = 0.$$

Again, $G_3'(\zeta)$ = constant for all ζ. By (67d) and (68a), $G_3'(\zeta) \longrightarrow \mp \alpha_2$. Thus,

$$\alpha_2 = 0, \qquad (68b)$$

and:

$$G_3(\zeta) = 0. \qquad (69b)$$

The same result follows for all α_{2n} and all $G_{2n+1}(\zeta)$, $n = 0, 1, 2, \ldots$, by induction. For suppose it has been shown that

$$G_{2m+1}(\zeta) = 0, \quad m \leq n,$$

$$\alpha_{2m} = 0, \quad m \leq n.$$

Since $\alpha_0 = 0$, the expansion of $G(\zeta;\delta)$, as $\eta \rightarrow \pm\infty$, is:

$$G(\zeta;\delta) \longrightarrow -i \sum_{n=0}^{\infty} \frac{1}{(2n)!}[(\zeta \pm i\varepsilon \mp i\alpha_1)\delta \mp i\alpha_3\delta^3 \mp \ldots \mp i\alpha_{2n+1}\delta^{2n+1} \mp i\alpha_{2n+2}\delta^{2n+2} \mp \ldots]^{2n} \qquad (70)$$

The coefficient of δ^{2n+3} is $\mp\alpha_{2n+2}(\zeta \pm i\varepsilon \mp i\alpha_1)$. But:

$$\Delta G_{2n+3}'(\zeta) = 0,$$

which means again that $G_{2n+3}'(\zeta)$ = constant, and the constant must be zero to give the asymptotic behavior corresponding to the coefficient of δ^{2n+3}. Thus:

$$\alpha_{2n+2} = 0 \,, \quad \text{for all } n, \tag{68c}$$

$$G_{2n+3}(\zeta) = 0 \,, \quad \text{for all } n. \tag{68d}$$

We now find $G_n(\zeta)$ for even n. As in the cases above,

$$G_0'(\zeta) = \text{constant}.$$

By (67a), the constant is zero and

$$G_0(\zeta) = \text{constant} = -i.^{(a)} \tag{69d}$$

Next we have:

$$\Delta G_2'(\zeta) = f(\zeta)\gamma_2\Sigma G_0(\zeta) = -if(\zeta).$$

Since $f(\zeta)$ is periodic, we can write down the general solution of this equation immediately:

$$G_2'(\zeta) = -i\zeta\, f(\zeta) + P_2(\zeta),$$

where $P_2(\zeta)$ is again a periodic function of unit period. (We shall occasionally refer to the first term as a "particular solution" of the non-homogeneous difference equation and to the second term as the solution of the associated homogeneous equation.) We now find that $P_2(\zeta)$ does have singularities, so that the solution is non-trivial.

Since $\mathrm{Im}[f(\zeta)] = 0$ on Λ, $P_2(\zeta)$ satisfies:

$$\mathrm{Im}[P_2(\zeta)] = 0 \quad \text{on } \Lambda; \tag{71a}$$

$$\mathrm{Im}[P_2(\zeta)] = \mathrm{Im}[i\zeta f(\zeta)] \quad \text{on } L_0. \tag{71b}$$

(a) Thus $G_0(\zeta)$ adds nothing to the flow, since its derivative is zero. However, this term may not be dropped, because equation (4c) removed the possibility of adding an arbitrary constant to the real potential, and the selection of the bottom as the zero-streamline (See proof of Theorem IV.) fixed the additive constant in the harmonic function conjugate to the real potential. Thus the complex potential may not be altered by the addition of a (real or imaginary) constant.

(See (62b).) In general, $f(\zeta)$ is singular at least at the ends of L_o. By (63) we must have:

$$|-i\zeta f(\zeta) + P_2(\zeta)| \text{ bounded near } L_o. \tag{71c}$$

Since $f(\zeta) \longrightarrow +1$ as $\eta \longrightarrow \pm\infty$,

$$P_2(\zeta) \longrightarrow \pm(\varepsilon - a_1). \tag{71d}$$

(See (67c).) And, of course,

$$P_2(\zeta+1) = P_2(\zeta). \tag{71e}$$

Let

$$P_2(\zeta) = \frac{1}{2\pi i} \int_{L_o} \pi \cot \pi(\zeta'-\zeta)\ i\zeta'[f^+(\zeta') - f^-(\zeta')]\ d\zeta' + Q_2(\zeta), \tag{72}$$

where the direction of the integration is taken from ζ_B to ζ_A. $f^+(\zeta)$ is the value of $f(\zeta)$ as the point ζ approaches L_o from the left, and $f^-(\zeta)$ is the value from the right.

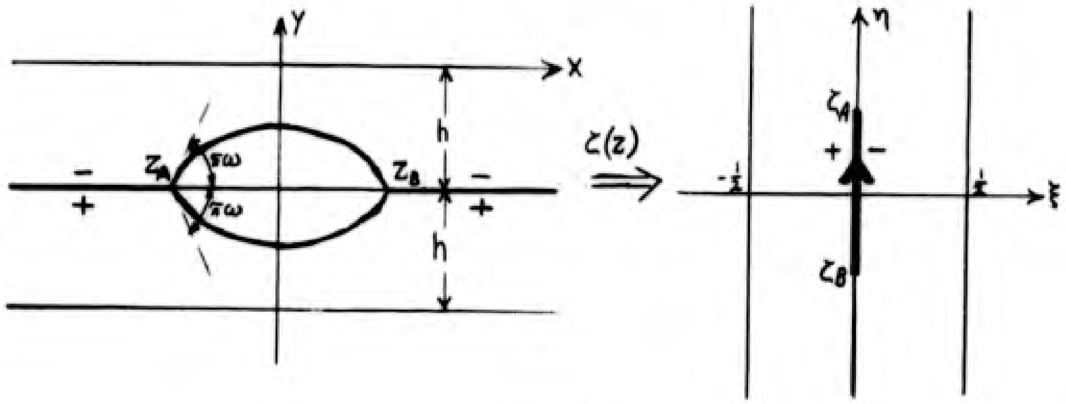

FIGURE 4

(The naming of + and - sides follows from the convention that the + side is always the left side as the contour is traversed. See reference (8).) The function $Q_2(\zeta)$ is analytic near L_o.

Since $\cot \pi(\zeta'-\zeta-1) = \cot \pi(\zeta'-\zeta)$, the integral is periodic. Thus $Q_2(\zeta)$ must also be periodic.

For ζ near L_0, write $P_2(\zeta)$ as follows:

$$P_2(\zeta) = \frac{1}{2\pi i} \int_{L_0} \frac{i\zeta'[f^+(\zeta') - f^-(\zeta')]}{\zeta' - \zeta}\, d\zeta' + Q_2(\zeta) \qquad (72')$$

$$+ \frac{1}{2\pi i} \int_{L_0} [\pi \cot \pi(\zeta' - \zeta) - \frac{1}{\zeta' - \zeta}] i\zeta'[f^+(\zeta') - f^-(\zeta')]\, d\zeta'.$$

Let ζ_0 be a point on L_0, but not at either end or at any ζ for which $f(\zeta)$ is unbounded. The Plemelj formulae[8] give for the first integral in (72'):

$$\frac{1}{2\pi i} \int_{L_0} \frac{i\zeta'[f^+(\zeta') - f^-(\zeta')]}{\zeta' - \zeta}\, d\zeta' \quad \xrightarrow[\zeta \to \zeta_0 \mp 0]{}$$

$$\pm \frac{1}{2} i\, \zeta_0 [f^+(\zeta_0) - f^-(\zeta_0)] + \frac{1}{2\pi i}\, \mathrm{PV}\!\!\int_{L_0} \frac{i\zeta'[f^+(\zeta') - f^-(\zeta')]}{\zeta' - \zeta_0}\, d\zeta'$$

$$= i\, \mathrm{Im}[i\zeta_0 f^{\pm}(\zeta_0)] + \frac{1}{2\pi i}\, \mathrm{PV}\!\!\int_{L_0} \frac{i\zeta'[f^+(\zeta') - f^-(\zeta')]}{\zeta' - \zeta_0}\, d\zeta' ,$$

where we have used the fact that

$$f^+(\zeta_0) - f^-(\zeta_0) = 2i\, \mathrm{Im}[f^+(\zeta_0)] = -2i\, \mathrm{Im}[f^-(\zeta_0)].$$

This last fact also shows that the term with the principal value integral is pure real.

The second term in $P_2(\zeta)$ as expressed in (72') can be written:

$$\frac{1}{2\pi i} \int_{L_0} [\pi \cot \pi(\zeta' - \zeta) - \frac{1}{\zeta' - \zeta}] i\zeta'[f^+(\zeta') - f^-(\zeta')]\, d\zeta'$$

$$= \frac{1}{2\pi i} \int_{L_0} \sum_{1}^{\infty} [\frac{1}{\zeta' - \zeta - n} + \frac{1}{\zeta' - \zeta + n}]\, i\zeta'[f^+(\zeta') - f^-(\zeta')]\, d\zeta'$$

$$= \frac{1}{2\pi i} \int_{L_0} \sum_{1}^{\infty} \frac{2(\zeta' - \zeta)}{(\zeta' - \zeta)^2 - n^2}\, i\zeta'[f^+(\zeta') - f^-(\zeta')]\, d\zeta' .$$

This is pure real for $\zeta \in L_o$ or $\zeta \in \Lambda$. Also, it is single-valued and bounded on and near L_o (including in the neighborhoods of the singular points of $f(\zeta)$). Thus we see that the integral in (72) has the properties required of $P_2(\zeta)$ by (71a), (71b), and (71e). We next investigate the behavior of $P_2(\zeta)$ near a point at which $f(\zeta)$ becomes unbounded, to show that (71c) is satisfied. A typical such point is one end of L_o.[a]

Suppose that the bottom turns up through an angle $\pi\omega$ at $z = z_A$. (See Figure 4.) Then near $z = z_A$ or $\zeta = \zeta_A$,

$$z - z_A = [i(\zeta - \zeta_A)]^{1-\omega} e^{-i\omega\pi} + \dots,$$

the omitted terms being of order: $o(\zeta - \zeta_A)$ near $\zeta = \zeta_A$. (Similarly in subsequent formulae) The branch of the root is taken so that this relation corresponds to the figure. In particular, on the + side in the ζ-plane,

$$[i(\zeta - \zeta_A)]^{1-\omega} = |\zeta - \zeta_A|^{1-\omega}.$$

Then:

$$\frac{dz}{d\zeta} = i(1-\omega)e^{-i\pi\omega}[i(\zeta - \zeta_A)]^{-\omega} + \dots,$$

and:

$$i\zeta f(\zeta) = \frac{i\zeta}{2h}(1-\omega)e^{-i\pi\omega}[i(\zeta - \zeta_A)]^{-\omega} + \dots.$$

On the cut near ζ_A,

---

[a] If $f(\zeta)$ is unbounded at some ζ in the interior of L_o, we reduce it to this case by considering L_o as being composed of two segments which abut upon each other.

$$i\zeta'[f^+(\zeta') - f^-(\zeta')] \;=\; \frac{\zeta'(1-\omega)\,\sin\pi\omega}{h[i(\zeta-\zeta_A)]^{\omega}} \;+\; \ldots,$$

with the root in the denominator selected for the + side of the cut. Near ζ_A, the integral then has the behavior:

$$-\frac{e^{-i\pi\omega}}{2i\,\sin\pi\omega}\left[\frac{\zeta_A(1-\omega)\,\sin\pi\omega}{h[i(\zeta-\zeta_A)]^{\omega}}\right] \;+\; R(\zeta)$$

$$= \frac{i\zeta_A}{2h}\,(1-\omega)e^{-i\pi\omega}[i(\zeta-\zeta_A)]^{-\omega} \;+\; R(\zeta),$$

where $R(\zeta)$ is bounded near ζ_A and tends to a definite limit as $\zeta \longrightarrow \zeta_A$. (See page 74, reference 8.) This is the same as the behavior of $i\zeta f(\zeta)$ near ζ_A. Thus $P_2(\zeta)$ has the desired behavior near ζ_A. (See (71c).)

Finally, consider $P_2(\zeta)$ as $\eta \longrightarrow \pm\infty$.

$$P_2(\zeta) - Q_2(\zeta) \;\longrightarrow\; \pm\,\frac{1}{2}\int_{L_0} i\zeta'[f^+(\zeta') - f^-(\zeta')]\,d\zeta' \;.$$

It is now apparent that $Q_2(\zeta)$ satisfies:

$$\mathrm{Im}[Q_2(\zeta)] \;=\; 0 \quad \text{on} \quad L_0 \cup \Lambda;$$

$$Q_2(\zeta+1) \;=\; Q_2(\zeta);$$

$$|Q_2(\zeta)| \text{ bounded near } L_0;$$

$$Q_2(\zeta) \longrightarrow \text{constant, as } \eta \longrightarrow \pm\infty.$$

Obviously, $Q_2(\zeta)$ = constant. Then:

$$P_2(\zeta) \xrightarrow[\eta\to\pm\infty]{} Q_2 \pm \frac{1}{2}\int_{L_0} i\zeta'[f^+(\zeta') - f^-(\zeta')]\,d\zeta' \;.$$

The only value of Q_2 consistent with this relation and (71d) is $Q_2 = 0$. So now $P_2(\zeta)$ is completely determined, thus providing $G_2^*(\zeta)$. And, in fact, α_1 has been found, as is seen by comparing the last result with (71d):

$$\alpha_1 = \varepsilon - \frac{1}{2}\int_{L_0} i\zeta'[f^+(\zeta') - f^-(\zeta')]\, d\zeta'. \qquad (68d)$$

We note that α_1 is pure real, as desired. Also, $[f^+ - f^-]$ is odd in ζ' on L_0. So this integral is not generally zero.

Now that $G_2'(\zeta)$ is known, (67c) provides for the unambiguous determination of $G_2(\zeta)$. In fact, if ζ_1 is any point in the fluid domain,

$$\begin{aligned} G_2(\zeta) &= \int_{\zeta_1}^{\zeta} G_2'(\zeta')\, d\zeta' + G_2(\zeta_1) \\ &= -\frac{1}{2!} i\zeta^2 + (\varepsilon - \alpha_1)\zeta + \Big\{\frac{1}{2!} i\zeta_1^2 - (\varepsilon - \alpha_1)\zeta_1 \\ &\qquad + G_2(\zeta_1) + \int_{\zeta_1}^{\zeta}\Big[-i\zeta'[f(\zeta') - 1] + [P_2(\zeta') - \varepsilon + \alpha_1]\Big] d\zeta'\Big\}. \end{aligned}$$

As $\zeta \longrightarrow i\infty$, the integrals here exist. Comparison with (67c) shows that:

$$\frac{1}{2!} i(\varepsilon - \alpha_1)^2 = \frac{1}{2!} i\zeta_1^2 - (\varepsilon - \alpha_1)\zeta_1 + G_2(\zeta_1) + \int_{\zeta_1}^{i\infty}\Big[-i\zeta'[f(\zeta') - 1] + P_2(\zeta') - \varepsilon + \alpha_1\Big] d\zeta',$$

which gives $G_2(\zeta_1)$.

In the same way, we can find $G_4'(\zeta)$ uniquely by satisfying (60)-(67). We obtain a special solution of the non-homogeneous difference equation:

$$\Delta G_4'(\zeta) = f(\zeta)\,[\gamma_2 \Sigma G_2(\zeta) + \gamma_4 \Sigma G_0(\zeta)],$$

and add to it a solution of the homogeneous equation, that is, a periodic function. Again, the periodic function will be expressed as a Cauchy integral which approaches ± a constant as $\eta \longrightarrow \pm\infty$. Comparison with (67e) then determines α_3. It is easily seen also that the special solution of the non-homogeneous equation supplies the parts of $G_4'(\zeta)$ that act like ζ^3, ζ^2, and ζ, as $\eta \longrightarrow \pm\infty$. This comes most directly from consideration of the last form given above for $G_2(\zeta)$.

The same procedure is possible for each $G_{2n}(\zeta)$. At each stage of the solution, the asymptotic form of $G_{2n}'(\zeta)$ introduces a new one of the unknown constants, viz., α_{2n-1}. These arise from the $n = 1$ term in (70).

F. The odd solutions

Consider first the $H_n(\zeta)$ with n even.

$$\Delta H_0'(\zeta) = 0.$$

So $H_0'(\zeta) = R_0(\zeta) =$ a periodic function of unit period, with no singularities. Thus $H_0'(\zeta) =$ constant. By (67f), this constant is zero, and hence $\beta_0 = 0$ or $m\pi$, m an integer. But addition of $m\pi$ to β makes no difference in

$$\sin(k_0 x \pm \beta)\sin(\sigma t + \beta).$$

(See (51) and (52b).) So we set:

$$\beta_0 = 0\ . \tag{73a}$$

Also, then,

$$H_0(\zeta) = 0\ . \tag{74a}$$

Next,

$$\Delta H_2'(\zeta) = 0.$$

So $H_2'(\zeta) = R_2(\zeta) =$ constant $\longrightarrow \mp i\beta_2$ as $\eta \longrightarrow \pm\infty$. Thus:

$$\beta_2 = 0, \tag{73b}$$

$$H_2(\zeta) = 0. \tag{74b}$$

Similarly, for all n,

$$\beta_{2n} = 0, \tag{73c}$$

$$H_{2n}(\zeta) = 0. \tag{74c}$$

The general proof is identical with that for the even solutions.

Now consider $H_n(\zeta)$ for n odd. First,

$$\Delta H_1'(\zeta) = 0.$$

Then, by (67g) and (73a),

$$H_1'(\zeta) = R_1(\zeta) = 1.$$

We can integrate this uniquely because $H_1(\zeta) = - H_1(-\zeta)$:

$$\begin{aligned} H_1(\zeta) &= \int_{\zeta_1}^{\zeta} H_1'(\zeta')\,d\zeta' + H_1(\zeta_1) \\ &= \zeta + [H_1(\zeta_1) - \zeta_1] . \end{aligned}$$

The term is brackets is, of course, a constant. Since $H_1(\zeta)$ is odd and the first term on the right is odd, then the second term must be odd also. But this can be so only if the constant is zero:

$$H_1(\zeta) = \zeta . \tag{74d}$$

By (67g), $H_1(\zeta) \longrightarrow \zeta \pm i(\varepsilon - \beta_1)$ as $\eta \longrightarrow \pm\infty$. Obviously,

$$\beta_1 = \varepsilon . \tag{73d}$$

See (57) for a formula for ε .

Next,

$$\Delta H_3'(\zeta) = f(\zeta)\gamma_2\Sigma H_1(\zeta) = \tfrac{1}{2} f(\zeta)(2\zeta + 1).$$

The general solution of this difference equation is:

$$H_3'(\zeta) = \tfrac{1}{2}\zeta^2 f(\zeta) + R_3(\zeta).$$

$R_3(\zeta)$ satisfies:

$$\mathrm{Im}[R_3(\zeta)] = 0 \quad \text{on } \Lambda\ ;$$
$$\mathrm{Im}[R_3(\zeta)] = -\,\mathrm{Im}[\tfrac{1}{2}\zeta^2 f(\zeta)] \quad \text{on } L_o;$$
$$\left|\tfrac{1}{2}\zeta^2 f(\zeta) + R_3(\zeta)\right| \quad \text{bounded near } L_o;$$
$$R_3(\zeta+1) = R_3(\zeta);$$
$$R_3(\zeta) \longrightarrow 0 \text{ as } \eta \longrightarrow \pm\infty.$$

(The last condition comes from (67i). As before, we represent R_3 by a Cauchy integral along L_o:

$$R_3(\zeta) = -\frac{1}{2\pi i}\int_{L_o} \pi\cot\pi(\zeta'-\zeta)\,\tfrac{1}{2}\zeta'^2[f^+(\zeta')-f^-(\zeta')]\,d\zeta'.$$

An extra additive function (Cf. Q_2 on p. 57.) can be shown to be identically zero. This function has the same kind of properties as $P_3(\zeta)$, with one exception. As $\eta \longrightarrow \pm\infty$,

$$R_3(\zeta) \longrightarrow \mp\frac{1}{4}\int_{L_o} \zeta'^2[f^+(\zeta')-f^-(\zeta')]\,d\zeta'.$$

Since $[f^+(\zeta')-f^-(\zeta')] = 2i\,\mathrm{Im}[f^+(\zeta')]$ is odd with respect to the real axis, this integral is zero, i.e.,

$R_3(\zeta) \longrightarrow 0$ as $\eta \longrightarrow \pm\infty$, as required above.

It is readily seen that the other conditions on $R_3(\zeta)$ are also met.

We note that $R_3(\zeta) = R_3(-\zeta)$. We use this fact now in integrating $H_3'(\zeta)$:

$$H_3(\zeta) = \int_{+0+i0}^{\zeta} H_3'(\zeta')\,d\zeta' + H_3(+0+i0)$$
$$= \int_{-0+i0}^{\zeta} H_3'(\zeta')\,d\zeta' + H_3(-0+i0)\ .$$

Since H_3 is odd in ζ , we must have $H_3(+0+i0) = - H_3(-0+i0)$. Thus,

$$H_3(+0+i0) = \frac{1}{2}\int_{-0+i0}^{+0+i0} H_3'(\zeta)\, d\zeta .$$

By deforming the contours for these integrations in an appropriate manner and using the even behavior of $H_3'(\zeta)$, we find:

$$\begin{aligned} H_3(\zeta) &= \frac{1}{2}\int_{-\zeta}^{\zeta} H_3'(\zeta')\, d\zeta' \\ &= \frac{1}{2}\int_{-\zeta}^{\zeta} \frac{1}{2}\zeta'^2[f(\zeta') - 1]\, d\zeta' + \frac{1}{2}\int_{-\zeta}^{\zeta} R_3(\zeta')\, d\zeta' + \frac{1}{3!}\zeta^3 . \end{aligned}$$

Since $\beta_1 = \varepsilon$, by (67i) we have:

$$\beta_3 = - \frac{1}{2}\int_{-i\infty}^{+i\infty} \frac{1}{2}\zeta^2[f(\zeta) - 1]\, d\zeta - \frac{1}{2}\int_{-i\infty}^{+i\infty} R_3(\zeta)\, d\zeta . \qquad (73e)$$

Similarly, the successive $H_{2n+1}(\zeta)$ can be found. In each case, the function must be odd in ζ , thus supplying the means to determine a new β_{2n+1} with each $H_{2n+1}(\zeta)$.

VII. SOLUTION FOR A SUBMERGED BARRIER

A. Formulation of the problem

Now we consider a specific problem, the case of sinusoidal waves passing over (and being reflected from) a submerged wall of vanishing thickness and height ℓ . See Figure 5:

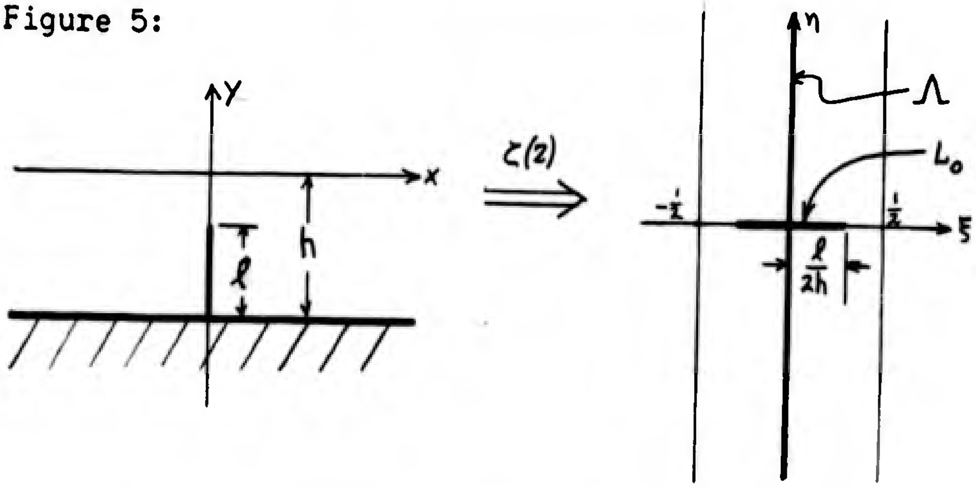

FIGURE 5.

Since the bottom is symmetrical with respect to x = 0, the results of Section VI all hold. We make repeated use of the fact that the solutions can be separated into even and odd parts, which are separately solutions. The procedure is essentially that of the last section, although here we take advantage of the particular simplicity of the boundary to alter the procedure somewhat.

Instead of mapping the fluid domain into a strip, we simply rotate and compress it. In particular, let:

$$\zeta = \frac{1}{2ih} z .$$

Thus the function $f(\zeta)$ used in Section VI becomes identically equal to unity. The purpose of using the ζ-plane at all is just to simplify the form of the difference equations which must be solved.

As before, we need an even and an odd function, which behave, respectively, like $\cos(k_0 x \pm \alpha)$ and $\sin(k_0 x \pm \beta)$ as $x \longrightarrow \mp\infty$, $y = 0$. But we do not even have to look for the even solution; it is the classical standing wave solution:

$$\cos k_0 z \ .$$

That is, $\alpha = 0$ and $\alpha_n = 0$, for all n. The real potential is thus:

$$\Phi(x,y,t;k_0) = \mathrm{Re}[\cos k_0 z \cos \sigma t + F(z;k_0) \sin(\sigma t + \beta)], \tag{75}$$

where

$$F(z;k_0) \xrightarrow[x \longrightarrow \mp\infty]{} \sin (k_0 z \pm \beta) \ , \tag{75'}$$

and β is a function only of k_0. (See Theorem IX.)

From Theorem X, the reflection and transmission coefficients will be given by:

$$R = |\sin \beta| \ , \tag{76a}$$

$$T = |\cos \beta| \ . \tag{76b}$$

As before, we define:

$$\delta = 2hk_0,$$

and we consider the complex potential in the ζ-plane:

$$iH(\zeta;\delta) = F(z;k_0).$$

$H(\zeta;\delta)$ satisfies, by Theorem IV:

$$\Delta H'(\zeta;\delta) - 2h\sigma^2(\delta)\ \Sigma H(\zeta;\delta) = 0 \ ; \tag{77}$$

$$\mathrm{Re}[H(\zeta;\delta)] = 0 \quad \text{on} \quad L_0 \cup \Lambda \ .$$

Since L_0 is part of the real axis and Λ is part of the imaginary axis (See Figure 5.), this implies that:

$$\mathrm{Re}[H^{(m)}(\zeta;\delta)] = 0 \text{ on } L_0, \text{ for all } m; \tag{78}$$

$$\mathrm{Re}[H^{(m)}(\zeta;\delta)] = 0 \text{ on } \Lambda, \text{ even } m; \tag{79a}$$

$$\mathrm{Im}[H^{(m)}(\zeta;\delta)] = 0 \text{ on } \Lambda, \text{ odd } m. \tag{79b}$$

Note that L_0 includes the reflection of the barrier, as shown in Figure 5.

We still require that $H(\zeta;\delta)$ be bounded in the fluid domain and on its boundaries. However, we now must allow that $H'(\zeta;\delta)$ be infinite at $(\ell/2h + i0)$. It is easy to see, in fact, that $H'(\zeta;\delta)$ has an inverse square root behavior near $\zeta = (\ell/2h + i0)$. This can be deduced from the fact that if the fluid domain be mapped into a simple strip then the complex potential is analytic on the image of the bottom, by Theorem VI. The inverse mapping into the present ζ-plane gives the described behavior, as is found by application of the Schwarz-Christoffel type of transformation. (We could have assumed that the potential has this behavior by referring to the known solution for the passage of waves over a submerged barrier in infinitely deep water.[9,10])

For convenience, define the complex number $\rho = \ell/2h + i0$. That is, $\zeta = \rho$ is the point at the top of the barrier. We shall also occasionally use ρ to mean the real number, $\ell/2h$, when no confusion can arise therefrom.

We now assume the expansion:

$$H(\zeta;\delta) = \sum_{n=o}^{N} H_n(\zeta)\delta^n + o(\delta^N) \quad \text{as } \delta \to 0. \tag{80}$$

From Theorem VII, the individual terms satisfy:

$$\Delta H_n'(\zeta) = \sum_{m=1}^{q} \Upsilon_{2m}\, \Sigma H_{n-2m}(\zeta) , \tag{81}$$

where

$$q = \begin{cases} n/2 , \text{ for n even,} \\ (n-1)/2, \text{ for n odd.} \end{cases}$$

$$\mathrm{Re}[H_n^{(m)}(\zeta)] = 0 \quad \text{on } L_o, \text{ for all m, n.} \tag{82}$$

$$\mathrm{Re}[H_n^{(m)}(\zeta)] = 0 \quad \text{on } \Lambda, \text{ for all n, even m;} \tag{83a}$$

$$\mathrm{Im}[H_n^{(m)}(\zeta)] = 0 \quad \text{on } \Lambda, \text{ for all n, odd m.} \tag{83b}$$

$$|\zeta \mp \rho|^{1/2} |H_n'(\zeta)| \text{ bounded near } \zeta = \pm\rho,$$

which implies that:

$$|\zeta \mp \rho|^{(2m-1)/2} |H_n^{(m)}(\zeta)| \text{ bounded near } \zeta = \pm\rho. \tag{84}$$

As shown in Section VI-F, we can expand β in an asymptotic series in odd powers of δ:

$$\beta = \beta_1\delta + \beta_3\delta^3 + \beta_5\delta^5 + \dots + \beta_{2n+1}\delta^{(2n+1)} + o(\delta^{2n+1}) , \tag{85}$$

as $\delta \to 0$. From (75'),

$$H(\zeta;\delta) \longrightarrow [\zeta \mp i\beta_1]\delta + [\tfrac{1}{3!}(\zeta \mp i\beta_1)^3 \mp i\beta_3]\delta^3 + [\tfrac{1}{5!}(\zeta \mp i\beta_1)^5 \mp \tfrac{1}{2} i\beta_3(\zeta \mp i\beta_1)^2 \mp i\beta_5]\delta^5 + \dots ;$$

Identification of coefficients with (80) yields, as $\eta \longrightarrow \pm\infty$:

$$H_1(\zeta) \longrightarrow (\zeta \mp i\beta_1); \tag{86a}$$

$$H_3(\zeta) \longrightarrow \frac{1}{3!}(\zeta \mp i\beta_1)^3 \mp i\beta_3; \tag{86b}$$

$$H_5(\zeta) \longrightarrow \frac{1}{5!}(\zeta \mp i\beta_1)^5 \mp \frac{1}{2}i\beta_3(\zeta \mp i\beta_1)^2 \mp i\beta_5; \tag{86c}$$

etc.

The behavior of $H_n^{(m)}(\zeta)$ as $\eta \longrightarrow \pm\infty$ is obtaining by differentiating these expressions m times. We also recall from the last section that $H_n(\zeta) = 0$ for all even n.

There are also some derived conditions which will be used. Since $H(\zeta)$ is odd in ζ, we have:

$$H_n^{(m)}(\zeta) = -(-1)^m H_n^{(m)}(-\zeta). \tag{87}$$

Combination of (83a) and (83b) with (87) yields:

$$\mathrm{Im}[H_n^{(m)}(\zeta)] = 0 \quad \text{on R for all m, n,} \tag{88}$$

where

$$R = \left\{ \zeta : \eta = 0 \text{ and } \rho < |\xi| < 1/2 \right\}. \tag{89}$$

But (88) is equivalent to:

$$H_n^{(m)}(\zeta) = \overline{H_n^{(m)}(\bar{\zeta})} \quad \text{for all m, n.} \tag{88'}$$

Finally, for future reference, we define some new functions:

$$p_n(\zeta) = \frac{\sin \pi\zeta}{[\cos^2\pi\rho - \cos^2\pi\zeta]^{(n+1/2)}} ; \tag{90a}$$

$$q_n(\zeta) = \frac{\cos \pi\zeta}{[\cos^2\pi\rho - \cos^2\pi\zeta]^{(n+1/2)}} . \tag{90b}$$

The branch of the denominator is taken as follows:

$$[\cos^2\pi\rho - \cos^2\pi\zeta]^{(n+1/2)} \longrightarrow \pm\,(-1)^n\,\frac{i}{2^{2n+1}}\,e^{\mp(2n+1)i\pi\zeta}$$

as $\eta \longrightarrow \pm\infty$. Then:

$$p_n(\zeta) \longrightarrow \frac{\mp\frac{1}{2i}\,e^{\mp i\pi\zeta}}{\pm(-1)^n\frac{i}{2^{2n+1}}\,e^{\mp(2n+1)i\pi\zeta}} \longrightarrow \begin{cases} 1, & n=0, \\ 0, & n>0; \end{cases} \tag{91a}$$

$$q_n(\zeta) \longrightarrow \frac{\frac{1}{2}\,e^{\mp i\pi\zeta}}{\pm(-1)^n\frac{i}{2^{2n+1}}\,e^{\mp(2n+1)i\pi\zeta}} \longrightarrow \begin{cases} \mp i, & n=0, \\ 0, & n>0. \end{cases} \tag{91b}$$

We note that the numerators and the denominators of $p_n(\zeta)$ and $q_n(\zeta)$ are periodic, with period 2, and that $p_n(\zeta)$ and $q_n(\zeta)$ are periodic, with period 1, i.e.,

$$p_n(\zeta+1) = p_n(\zeta), \tag{92a}$$

$$q_n(\zeta+1) = q_n(\zeta). \tag{92b}$$

B. Solution for $H_1(\zeta)$

From (81), we have

$$\Delta H_1'(\zeta) = 0.$$

Thus $H_1'(\zeta)$ is a periodic function, with unit period. By (84), we must allow $H_1'(\zeta)$ to have a square root singularity at $\zeta = \pm\rho$. Since it is periodic, it will then have similar singularities at every point given by $\zeta = \pm\rho + n$, $n = 0, \pm1, \pm2, \ldots$.

Now consider the function $[H_1'(\zeta)]^2$. From (82) and (88'), $H_1'(\zeta)$ is pure imaginary on L_0, with opposite values on the two sides. Thus $[H_1'(\zeta)]^2$ is single-valued and real on L_o. On R, $H_1'(\zeta)$ is single-valued, and thus so is $[H_1'(\zeta)]^2$. From its periodicity, $[H_1'(\zeta)]^2$ is thus single-valued everywhere on the real axis, except that it remains undefined at the ends of L_n. (The point $\zeta \in L_n$ if the point $(\zeta - n) \in L_o$.) It is also single-valued at all points near the real axis. Thus the ends of the L_n must be isolated singularities, and, by (84), the singularities must be simple poles. By the Mittag-Leffler theorem, the most general form for $[H_1'(\zeta)]^2$ is:

$$[H_1'(\zeta)]^2 = \frac{g(\zeta)}{\cos^2 \pi\rho - \cos^2 \pi\zeta} + h(\zeta),$$

where $g(\zeta)$ and $h(\zeta)$ are both entire and periodic. Because they are entire as well as periodic, they can each be expanded into Fourier series valid in the whole plane. Let:

$$g(\zeta) = \sum_{-\infty}^{+\infty} a_n e^{2ni\pi\zeta},$$

$$h(\zeta) = \sum_{-\infty}^{+\infty} b_n e^{2ni\pi\zeta}.$$

By (86a), $[H_1'(\zeta)]^2$ is bounded as $\eta \to \pm\infty$. Thus:

$$g(\zeta) = a_{-1} e^{-2i\pi\zeta} + a_o + a_{+1} e^{2i\pi\zeta};$$

$$h(\zeta) = b_o;$$

$$[H_1'(\zeta)]^2 \longrightarrow -4a_{\mp 1} + b_o, \text{ as } \eta \longrightarrow \pm\infty.$$

The other coefficients in the expansions of $g(\zeta)$ and $h(\zeta)$ must all be zero. Again from (86a),

$$-4a_{+1} + b_o = -4a_{-1} + b_o = 1,$$

which requires that:

$$a_{-1} = a_{+1};$$
$$b_o = 1 + 4a_{+1}.$$

Thus:

$$[H_1'(\zeta)]^2 = \frac{A - \cos^2\pi\zeta}{\cos^2\pi\rho - \cos^2\pi\zeta},$$

where

$$A = -2a_1 + a_o + (1 + 4a_1)\cos^2\pi\rho.$$

For $H_1'(\zeta)$ we have now:

$$H_1'(\zeta) = \frac{\sqrt{A - \cos^2\pi\zeta}}{\sqrt{\cos^2\pi\rho - \cos^2\pi\zeta}}.$$

From (82) and (83b), this is zero for $\zeta = 0 \pm i0$. That is, $\sqrt{A - 1} = 0$, and $A = 1$. Therefore, finally,

$$H_1'(\zeta) = \frac{\sin\pi\zeta}{\sqrt{\cos^2\pi\rho - \cos^2\pi\zeta}} = p_o(\zeta).$$

This can be integrated explicitly in any of several forms. Using a result of Section VI-F, we obtain:

$$H_1(\zeta) = \frac{1}{2}\int_{-\zeta}^{\zeta} p_o(\zeta')\,d\zeta'$$

$$= -\frac{1}{\pi}\cos^{-1}\left[\frac{\cos\pi\zeta}{\cos\pi\rho}\right] \qquad (93a)$$

$$= \frac{i}{\pi}\log\left[\frac{i\cos\pi\zeta + \sqrt{\cos^2\pi\rho - \cos^2\pi\zeta}}{i\cos\pi\rho}\right] \qquad (93b)$$

$$= -\frac{i}{\pi}\log\left[\frac{i\cos\pi\zeta - \sqrt{\cos^2\pi\rho - \cos^2\pi\zeta}}{i\cos\pi\rho}\right]. \qquad (93c)$$

We decided previously to take the branch of the root so that:

$$[\cos^2\pi\rho - \cos^2\pi\zeta]^{1/2} \longrightarrow \pm i\cos\pi\zeta, \text{ as } \eta \longrightarrow \pm\infty.$$

From (93b), as $\eta \longrightarrow +\infty$,

$$H_1(\zeta) \longrightarrow \frac{i}{\pi}\log\left[\frac{2i\cos\pi\zeta}{i\cos\pi\rho}\right] \longrightarrow \zeta - \frac{i}{\pi}\log[\cos\pi\rho] + 2k,$$

where k is any integer. From (93c), which is entirely equivalent to (93b), but is more convenient for estimates as $\eta \rightarrow -\infty$, we find that, as $\eta \rightarrow -\infty$:

$$H_1(\zeta) \longrightarrow -\frac{i}{\pi}\log\left[\frac{2i\cos\pi\zeta}{i\cos\pi\rho}\right] \longrightarrow \zeta + \frac{i}{\pi}\log[\cos\pi\rho] + 2k.$$

Comparison with (86a) shows that the branch of the logarithm is to be taken such that $k = 0$, and that then:

$$\beta_1 = \frac{1}{\pi}\log[\cos\pi\rho]. \tag{94a}$$

Thus we have found the first non-zero coefficient in the expansion for β. We note that it can be expressed by integral representations:

$$\beta_1 = i\left(\int_{0+i0}^{i\infty} + \frac{1}{2}\int_{0-i0}^{0+i0}\right)[p_0(\zeta) - 1]\,d\zeta \tag{94b}$$

$$= \frac{1}{2}i\int_{-i\infty}^{+i\infty}[p_0(\zeta) - 1]\,d\zeta\ . \tag{94c}$$

It is easily seen that this is the same as the result given in Section VI-F.

It is of some interest to note that $H_1(\zeta)$ is the complex potential for a simple physical problem, viz., the flow between two walls with a protruding barrier from one or both walls. It can be shown rather easily that the problem as stated for $H_1(\zeta)$ is equivalent to replacing the free surface by a rigid wall. As $\eta \longrightarrow \pm\infty$, $H_1(\zeta) \propto \zeta$, which means that $H_1(\zeta)$ is the potential for a uniform flow at infinity. These two facts together show that $H_1(\zeta)$ is just the complex potential for the flow between two rigid walls, with a barrier perpendicular to one wall partially blocking the flow. Since $H_1(\zeta)$ is symmetrical about the image of the undisturbed free surface, we could consider the fluid domain and its reflection about the free surface as forming a new region filled with fluid, and $H_1(\zeta)$ would be the complex potential for the flow in this region which becomes uniform at infinity. Obviously in this case, there will be a barrier protruding from each wall. By considering the fluid domain and its reflection about the bottom as a new fluid-filled region, we could utilize $H_1(\zeta)$ as the complex potential for a flow between two walls with a barrier perpendicular to the walls—but in midstream. These three situations are indicated in the figure.

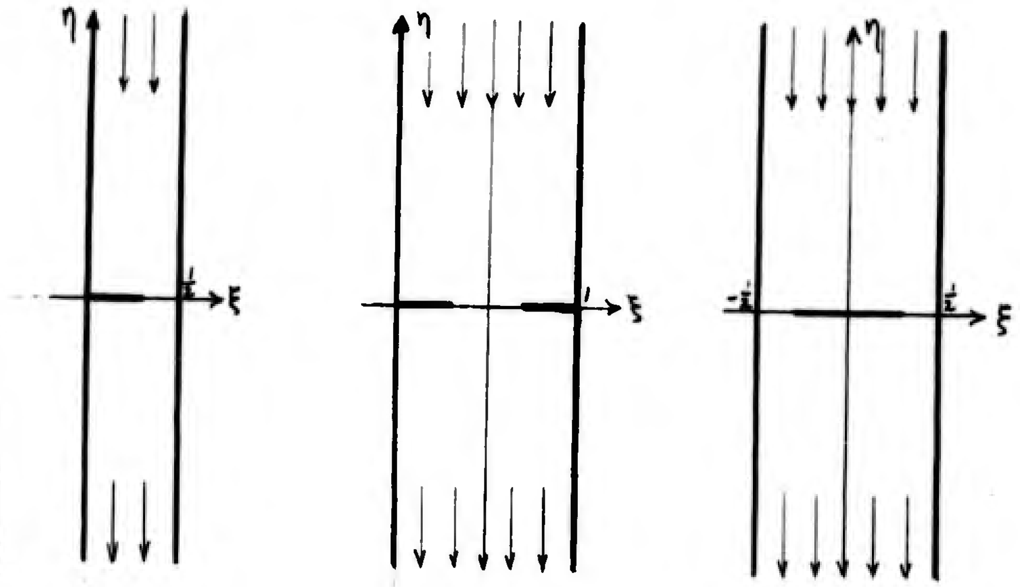

FIGURE 6

Each $H_n(\zeta)$ which is found can be interpreted as the complex potential for some complete problem. Of course, none of these problems will be a free-surface problem, and in fact $H_1(\zeta)$ seems to be the only one with real physical interest.

C. Solution for $H_3(\zeta)$

Rather than find $H_3'(\zeta)$ directly from (81), we differentiate (81) once with respect to ζ (for n = 3). Then we solve:

$$\Delta H_3''(\zeta) \;=\; \gamma_2\, \Sigma H_1'(\zeta) \;=\; \tfrac{1}{2}\cdot 2p_0(\zeta) \;=\; p_0(\zeta).$$

Since $p_0(\zeta)$ is periodic, the general solution is:

$$H_3''(\zeta) \;=\; \zeta\, p_0(\zeta) \;+\; P_3(\zeta), \tag{95a}$$

where $P_3(\zeta)$ is a periodic function of unit period. We note that $p_0(\zeta)$ is pure real on Λ and pure imaginary on L_0. Thus $\zeta\, p_0(\zeta)$ is pure imaginary on Λ and on L_0. From (78) and (79a) we see then that:

$$\mathrm{Re}[P_3(\zeta)] \;=\; 0 \quad \text{on } \Lambda \text{ and on } L_0. \tag{95b}$$

Also, $\zeta\, p_0(\zeta) \longrightarrow \zeta$, as $\eta \longrightarrow \pm\infty$. So by (86b),

$$P_3(\zeta) \longrightarrow \mp i\beta_1, \text{ as } \eta \longrightarrow \pm\infty. \tag{95c}$$

From (84) we have:

$$|\zeta \mp \rho|^{3/2}|P_3(\zeta)| \text{ is bounded near } \zeta = \pm\rho. \tag{95d}$$

Also, of course,

$$P_3(\zeta) \;=\; -\,P_3(-\zeta). \tag{95e}$$

This, combined with (95b), yields:

$$\mathrm{Im}[P_3(\zeta)] \;=\; 0 \quad \text{on } R, \tag{95f}$$

or, what is the same thing,

$$P_3(\zeta) \;=\; \overline{P_3(\bar{\zeta})}. \tag{95g}$$

Now consider $[P_3(\zeta)]^2$, which is meromorphic. It may have third order poles at the branch points of $f(\zeta)$, that is, at the ends of each L_n. So it can be written:

$$[P_3(\zeta)]^2 = \frac{k_1(\zeta)}{[\cos^2\pi\rho - \cos^2\pi\zeta]} + \frac{k_2(\zeta)}{[\cos^2\pi\rho - \cos^2\pi\zeta]^2} + \frac{k_3(\zeta)}{[\cos^2\pi\rho - \cos^2\pi\zeta]^3} ,$$

where the $k_i(\zeta)$ are entire periodic functions such that the whole expression remains bounded as $\eta \longrightarrow \pm\infty$. (An additive entire function is eliminated as before.) Each $k_i(\zeta)$ must have a Fourier expansion valid in the whole plane.

We must also be able to write:

$$P_3(\zeta) = \frac{\ell_1(\zeta)}{[\cos^2\pi\rho - \cos^2\pi\zeta]^{1/2}} + \frac{\ell_2(\zeta)}{[\cos^2\pi\rho - \cos^2\pi\zeta]^{3/2}} , \qquad (96)$$

since the presence of a term containing $[\cos^2\pi\rho - \cos^2\pi\zeta]^{-1}$ would be inconsistent with (95b) and (95f). If this expression is squared and compared with the previous expression for $[P_3(\zeta)]^2$, we see that the $\ell_i(\zeta)$ each have Fourier expansions valid in the whole plane. Also, from the behavior of the denominators in (96) and the fact that $P_3(\zeta)$ is periodic, we must have:

$$\ell_i(\zeta+1) = -\ell_i(\zeta).$$

Thus:

$$\ell_i(\zeta) = \sum_{n=0}^{\infty} a_{in}\cos(2n+1)\pi\zeta + \sum_{n=0}^{\infty} b_{in}\sin(2n+1)\pi\zeta .$$

Boundedness of $P_3(\zeta)$ as $\eta \longrightarrow \pm\infty$ requires that the coefficients all be zero except a_{10}, b_{10}, a_{20}, b_{20}, a_{21}, and b_{21}.

Thus:

$$P_3(\zeta) = \frac{a_{10}\cos\pi\zeta + b_{10}\sin\pi\zeta}{[\cos^2\pi\rho - \cos^2\pi\zeta]^{1/2}}$$

$$+ \frac{a_{20}\cos\pi\zeta + a_{21}\cos 3\pi\zeta + b_{20}\sin\pi\zeta + b_{21}\sin 3\pi\zeta}{[\cos^2\pi\rho - \cos^2\pi\zeta]^{3/2}} .$$

The sine terms cannot satisfy (95b). So the b's are all zero. The other constants must be pure real. Thus:

$$P_3(\zeta) = \frac{a_{10}\cos\pi\zeta}{[\cos^2\pi\rho - \cos^2\pi\zeta]^{1/2}} + \frac{a_{20}\cos\pi\zeta + a_{21}\cos 3\pi\zeta}{[\cos^2\pi\rho - \cos^2\pi\zeta]^{3/2}}$$

$$= \frac{[a_{10} - 4a_{21}]\cos\pi\zeta}{[\cos^2\pi\rho - \cos^2\pi\zeta]^{1/2}}$$

$$+ \frac{[a_{20} + 4a_{21}(\cos 3\pi\rho/\cos\pi\rho)]\cos\pi\zeta}{[\cos^2\pi\rho - \cos^2\pi\zeta]^{3/2}} .$$

If we let: $a_{30} = a_{10} - 4a_{21}$; $a_{31} = a_{20} + 4a_{21}(\cos 3\pi\rho/\cos\pi\rho)$;

then:

$$P_3(\zeta) = \frac{a_{30}\cos\pi\zeta}{[\cos^2\pi\rho - \cos^2\pi\zeta]^{1/2}} + \frac{a_{31}\cos\pi\zeta}{[\cos^2\pi\rho - \cos^2\pi\zeta]^{3/2}}$$

$$= a_{30}\, q_0(\zeta) + a_{31}\, q_1(\zeta).$$

From (95c) and (91b), we have:

$$P_3(\zeta) \longrightarrow \mp i a_{30} = \mp i\beta_1, \quad \text{as } \eta \longrightarrow \pm\infty.$$

So:

$$a_{30} = \beta_1 .$$

We must still determine a_{31}.

Our solution now has the form:

$$H_3''(\zeta) = \zeta p_0(\zeta) + \beta_1 q_0(\zeta) + a_{31} q_1(\zeta).$$

From (82) and (83b), $H_3'(0 \pm i0) = 0$. Thus:

$$H_3'(\zeta) = \int_{0+i0}^{\zeta} [\zeta' p_0(\zeta') + \beta_1 q_0(\zeta') + a_{31} q_1(\zeta')]\, d\zeta' .$$

Clearly we could just as well have taken the lower limit as $(0-i0)$, and since the integrand is odd with respect to ζ, this shows that:

$$H_3'(\zeta) \;=\; \int_{0-i0}^{-\zeta} H_3''(\zeta')\,d\zeta' \;=\; H_3'(-\zeta).$$

That is, $H_3'(\zeta)$ is even, as required by (87).

We can rewrite $H_3'(\zeta)$ in a more convenient form as follows:

$$H_3'(\zeta) = \tfrac{1}{2}\zeta^2 - i\beta_1\zeta + \int_{0+i0}^{\zeta}\Big[\zeta'[p_0(\zeta')-1] + \beta_1[q_0(\zeta')+i] + a_{31}q_1(\zeta')\Big]\,d\zeta'.$$

As $\eta \longrightarrow +\infty$, by (86b),

$$H_3'(\zeta) \longrightarrow \tfrac{1}{2}(\zeta - i\beta_1)^2 = \tfrac{1}{2}\zeta^2 - i\beta_1\zeta - \tfrac{1}{2}\beta_1^2 .$$

Thus:

$$-\tfrac{1}{2}\beta_1^2 = \int_{0+i0}^{i\infty}\Big[\zeta'[p_0(\zeta')-1] + \beta_1[q_0(\zeta')+i] + a_{31}q_1(\zeta')\Big]\,d\zeta',$$

which supplies the means of determining a_{31}. In fact,

$$a_{31} = \frac{-\tfrac{1}{2}\beta_1^2 - \displaystyle\int_{0+i0}^{i\infty}\Big[\zeta[p_0(\zeta)-1] + \beta_1[q_0(\zeta)+i]\Big]\,d\zeta}{\displaystyle\int_{0+i0}^{i\infty} q_1(\zeta)\,d\zeta}. \qquad (97)$$

Each of these integrals exists, because $[p_0(\zeta)-1]$, $[q_0(\zeta)+i]$, and $q_1(\zeta)$ all approach zero exponentially as $\eta \longrightarrow +\infty$. So we now have determined $H_3'(\zeta)$ uniquely.

As in Section VI-F, we can integrate $H_3'(\zeta)$ explicitly and uniquely, obtaining the result in any of several forms:

$$H_3(\zeta) = \left(\int_{0+i0}^{\zeta} + \frac{1}{2}\int_{0-i0}^{0+i0}\right) H_3'(\zeta')\, d\zeta'$$

$$= \frac{1}{2}\int_{-\zeta}^{\zeta} H_3'(\zeta')\, d\zeta'$$

$$= \frac{1}{3!}\zeta^3 - \frac{1}{2}i\beta_1\zeta^2 - \frac{1}{2}\beta_1^2\zeta$$

$$- \left(\int_{0+i0}^{\zeta} + \frac{1}{2}\int_{0-i0}^{0+i0}\right) d\zeta' \int_{\zeta'}^{i\infty} d\zeta'' \Big[\zeta''[p_0(\zeta'') - 1]$$

$$+ \beta_1[q_0(\zeta'') + i] + a_{31}q_1(\zeta'')\Big]$$

Comparison with (84b) shows that:

$$i\beta_3 = \frac{1}{3!}i\beta_1^3 + \left(\int_{0+i0}^{i\infty} + \frac{1}{2}\int_{0-i0}^{0+i0}\right) d\zeta \int_{\zeta}^{i\infty} d\zeta' \Big[\zeta'[p_0(\zeta') - 1]$$

$$+ \beta_1[q_0(\zeta') + i] + a_{31}q_1(\zeta')\Big] \qquad (98)$$

This provides the second non-zero coefficient in the expansion of β.

D. Solution for $H_5(\zeta)$

With n = 5, we differentiate (81) twice, obtaining:

$$\Delta H_5'''(\zeta) = \gamma_2\, \Sigma H_3''(\zeta) + \gamma_4\, \Sigma H_1''(\zeta).$$

The solution is:

$$H_5'''(\zeta) = \frac{1}{2}\zeta^2 p_0(\zeta) + \zeta\, [\beta_1 q_0(\zeta) + a_{31}q_1(\zeta) - \frac{1}{12}p_0'(\zeta))]$$

$$- \frac{1}{2}\beta_1^2 p_0(\zeta) + a_{51}p_1(\zeta) + a_{52}p_2(\zeta)\ .$$

The constants a_{51} and a_{52} are to be found.

As before, this result can be integrated:

$$
\begin{aligned}
H_5''(\zeta) = {} & \frac{1}{3!}\zeta^3 - \frac{1}{2}i\beta_1\zeta^2 - \frac{1}{2}\beta_1^2\zeta \\
& + \left(\int_{0+i0}^{\zeta} + \frac{1}{2}\int_{0-i0}^{0+i0}\right)\Big\{\frac{1}{2}\zeta'^2[p_0(\zeta') - 1] \\
& + \zeta'[\beta_1[q_0(\zeta') + i] + a_{31}q_1(\zeta') - \frac{1}{12}p_0'(\zeta')] \\
& - \frac{1}{2}\beta_1^2[p_0(\zeta') - 1] + a_{51}p_1(\zeta') + a_{52}p_2(\zeta')\Big\}\, d\zeta' \\
\longrightarrow {} & \frac{1}{3!}\zeta^3 - \frac{1}{2}i\beta_1\zeta^2 - \frac{1}{2}\beta_1^2\zeta + \frac{1}{3!}i\beta_1^3 - i\beta_3 ,
\end{aligned}
$$

as $\eta \longrightarrow +\infty$, by (86c). This provides a linear algebraic equation connecting a_{51} and a_{52}:

$$
a_{51}\left(\int_{0+i0}^{i\infty} + \frac{1}{2}\int_{0-i0}^{0+i0}\right)p_1(\zeta)d\zeta + a_{52}\left(\int_{0+i0}^{i\infty} + \frac{1}{2}\int_{0-i0}^{0+i0}\right)p_2(\zeta)d\zeta = g(\beta_1,\beta_3) , \qquad (99)
$$

where

$$
\begin{aligned}
g(\beta_1,\beta_3) = {} & \frac{1}{3!}i\beta_1^3 - i\beta_3 \qquad (99') \\
& - \left(\int_{0+i0}^{i\infty} + \frac{1}{2}\int_{0-i0}^{0+i0}\right)\Big\{\frac{1}{2}[\zeta^2 - \beta_1^2][p_0(\zeta) - 1] \\
& + \zeta[\beta_1[q_0(\zeta) + i] + a_{31}q_1(\zeta) - \frac{1}{12}p_0'(\zeta)]\Big\}d\zeta.
\end{aligned}
$$

Another integration yields:

$$H_5'(\zeta) = \frac{1}{4!}\zeta^4 - \frac{1}{3!}i\beta_1\zeta^3 - \frac{1}{4}\beta_1^2\zeta^2 + \frac{1}{3!}i\beta_1^3\zeta - i\beta_3\zeta$$

$$- \int_{0+i0}^{\zeta} d\zeta' \int_{\zeta'}^{i\infty} d\zeta'' \Big\{ \frac{1}{2}[\zeta''^2 - \beta_1^2][p_o(\zeta'') - 1]$$

$$+ \zeta''[\beta_1[q_o(\zeta'') + i] + a_{31}q_1(\zeta'') - \frac{1}{12}p_o'(\zeta'')]$$

$$+ a_{51}p_1(\zeta'') + a_{52}p_2(\zeta'')\Big\}$$

$$\longrightarrow \frac{1}{4!}\zeta^4 - \frac{1}{3!}i\beta_1\zeta^3 - \frac{1}{4!}\beta_1^2\zeta^2 + \frac{1}{3!}i\beta_1^3\zeta + \frac{1}{4}\beta_1^4 - i\beta_3\zeta - \beta_1\beta_3 ,$$

as $\eta \longrightarrow +\infty$, by (86c). By taking the integrals to $i\infty$, we obtain another linear algebraic equation for a_{51} and a_{52}:

$$a_{51} \int_{0+i0}^{i\infty} d\zeta \int_{\zeta}^{i\infty} p_1(\zeta')d\zeta' + a_{52} \int_{0+i0}^{i\infty} d\zeta \int_{\zeta}^{i\infty} p_2(\zeta')d\zeta'$$

$$= h(\beta_1,\beta_3) , \qquad (100)$$

where

$$h(\beta_1,\beta_3) = -\frac{1}{4!}\beta_1^4 + \beta_1\beta_3$$

$$- \int_{0+i0}^{i\infty} d\zeta \int_{\zeta}^{i\infty} d\zeta' \Big\{ \frac{1}{2}[\zeta'^2 - \beta_1^2][p_o(\zeta') - 1]$$

$$+ \zeta'[\beta_1[q_o(\zeta') + i] + a_{31}q_1(\zeta') - \frac{1}{12}p_o'(\zeta')]\Big\}. \qquad (100')$$

Equations (99) and (100) determine a_{51} and a_{52} if the determinant of the coefficients is non-zero. In fact,

$$D = \begin{vmatrix} \left(\int_{0+i0}^{i\infty} + \frac{1}{2}\int_{0-i0}^{0+i0}\right) p_1(\zeta)d\zeta & \left(\int_{0+i0}^{i\infty} + \frac{1}{2}\int_{0-i0}^{0+i0}\right) p_2(\zeta)d\zeta \\ \int_{0+i0}^{i\infty} d\zeta \int_{\zeta}^{i\infty} p_1(\zeta')d\zeta' & \int_{0+i0}^{i\infty} d\zeta \int_{\zeta}^{i\infty} p_2(\zeta')d\zeta' \end{vmatrix}$$

$$= - \frac{i}{3\pi^3 \cos^4 \pi\rho \ \sin^2 \pi\rho} \neq 0.$$

Thus solutions exist for all $\rho < \pi/2$. (β_1 and β_3 are undefined for $\rho = \pi/2$.) In particular,

$$Da_{51} = \begin{vmatrix} g(\beta_1, \beta_3) & \left(\int_{0+i0}^{i\infty} + \frac{1}{2}\int_{0-i0}^{0+i0}\right) p_2(\zeta)d\zeta \\ h(\beta_1, \beta_3) & \int_{0+i0}^{i\infty} d\zeta \int_{\zeta}^{i\infty} p_2(\zeta')d\zeta' \end{vmatrix}$$

$$Da_{52} = \begin{vmatrix} \left(\int_{0+i0}^{i\infty} + \frac{1}{2}\int_{0-i0}^{0+i0}\right) p_1(\zeta)d\zeta & g(\beta_1, \beta_3) \\ \int_{0+i0}^{i\infty} d\zeta \int_{\zeta}^{i\infty} p_1(\zeta')d\zeta' & h(\beta_1, \beta_3) \end{vmatrix}$$

One more integration can be performed. The expression for $H_5(\zeta)$ will not be used subsequently, since we shall terminate the approximation with this term; so we do not give the result. However, as $\eta \longrightarrow +\infty$, it provides the following result:

$$
\begin{aligned}
-i\beta_5 = {} & \frac{1}{5!} i\beta_1^5 - \frac{1}{2} i\beta_1^2\beta_3 \\
& + \left(\int_{0+i0}^{i\infty} + \frac{1}{2}\int_{0-i0}^{0+i0} \right) d\zeta \int_{\zeta}^{i\infty} d\zeta' \int_{\zeta'}^{i\infty} d\zeta'' \Big\{ \frac{1}{2}[\zeta''^2 - \beta_1^2][p_0(\zeta'')-1] \\
& \qquad + \zeta''[\beta_1[q_0(\zeta'') + i] + a_{31}q_1(\zeta'') - \frac{1}{12}p_0'(\zeta'')] \\
& \qquad + a_{51}p_1(\zeta'') + a_{52}p_2(\zeta'') \Big\} \; .
\end{aligned}
$$

Thus we have a third term in the series for β .

E. The reflection coefficient

From equation (76),

$$
\begin{aligned}
R = {} & |\sin\beta(\delta)| = |\sin[\beta_1\delta + \beta_3\delta^3 + \beta_5\delta^5 + \ldots]| \\
= {} & \Big| [\beta_1]\delta + [\beta_3 - \frac{1}{3!}\beta_1^3]\delta^3 + [\beta_5 - \frac{1}{2}\beta_1^2\beta_3 + \frac{1}{5!}\beta_1^5]\delta^5 \\
& + o(\delta^5) \Big| \; .
\end{aligned}
$$

Numerical results have been obtained and are presented in Figure 7, with

$$\delta = 2hk_0 = 4\pi h/\lambda$$

as abscissa. The parameter $2\rho = \ell/h$ is indicated on the different curves. The solid curves give the complete results to three terms. To show the contributions of the δ^3 and δ^5 terms, curves are also presented for $|\beta_1\delta|$ and $|\beta_1\delta + [\beta_3 - \frac{1}{3!}\beta_1^3]\delta^3|$.

We note that the first term in the expansion for R is exactly the same as Kreisel's approximation for this problem. We also note that R ⟶ 0 as $\lambda \longrightarrow \infty$, in contrast to the situation in infinitely deep water, where R ⟶ 1 as $\lambda \longrightarrow \infty$. (See references 9, 10.)

The method of performing the calculations will be briefly indicated. Because the integrands are $O(\eta^m e^{-n\pi\eta})$ as $\eta \longrightarrow +\infty$, the following transformations are valid:

$$\int_{0+i0}^{i\infty} d\zeta \int_{\zeta}^{i\infty} d\zeta' f(\zeta') = \int_{0+i0}^{i\infty} \zeta f(\zeta)\, d\zeta;$$

$$\left(\int_{0+i0}^{i\infty} + \frac{1}{2}\int_{0-i0}^{0+i0}\right) d\zeta \int_{\zeta}^{i\infty} d\zeta' f(\zeta') = \left(\int_{0+i0}^{i\infty} + \frac{1}{2}\int_{0-i0}^{0+i0}\right)\zeta f(\zeta) d\zeta;$$

$$\left(\int_{0+i0}^{i\infty} + \frac{1}{2}\int_{0-i0}^{0+i0}\right) d\zeta \int_{\zeta}^{i\infty} d\zeta' \int_{\zeta'}^{i\infty} d\zeta'' f(\zeta'')$$

$$= \left(\int_{0+i0}^{i\infty} + \frac{1}{2}\int_{0-i0}^{0+i0}\right) \frac{1}{2}\zeta^2 f(\zeta) d\zeta .$$

All of the integrals except those containing $[q_0(\zeta) + i]$ can then be transformed:

$$\left(\int_{0+i0}^{i\infty} + \frac{1}{2}\int_{0-i0}^{0+i0}\right) d\zeta\ f(\zeta) = \frac{1}{2}\int_{1/2-i\infty}^{1/2+i\infty} f(\zeta)\, d\zeta .$$

Also,

$$\left(\int_{0+i0}^{i\infty} + \frac{1}{2}\int_{0-i0}^{0+i0}\right) \zeta^{2n+1}[q_0(\zeta) + i] d\zeta = \frac{i}{2n+2}(1/2)^{2n+2}$$

$$+ \frac{1}{2}\int_{1/2-i\infty}^{1/2+i0} \zeta^{2n+1}[q_0(\zeta) - i] d\zeta + \frac{1}{2}\int_{i/2+i0}^{1/2+i\infty} \zeta^{2n+1}[q_0(\zeta) + i] d\zeta .$$

With these relations, all of the integrals of sub-sections B, C, and D can be reduced to single integrals along straight contours parallel to the imaginary axis.

When the expressions for $p_n(\zeta)$ and $q_n(\zeta)$ are inserted into the integrands, a typical integral obtained is:

$$I_u = \int_0^\infty \eta^u [\frac{\cosh \pi\eta}{\sqrt{\sinh^2 \pi\eta + \cos^2 \pi\rho}} - 1]\, d\eta .$$

We note that:

$$\frac{1}{\sqrt{\sinh^2 \pi\eta + \cos^2 \pi\rho}} = 2\, e^{-\pi\eta} \sum_{k=0}^{\infty} (-1)^k e^{-2\pi k\eta}\, P_k(\cos 2\pi\rho),$$

where $P_k(\cos 2\pi\rho)$ is the Legendre polynomial of order k and argument $\cos 2\pi\rho$. We substitute into I_u and integrate term-by-term. It is easy to see that convergence is no problem, and neither is the term-wise integration. The final form of I_u is:

$$I_u = \frac{u!}{(2\pi)^{u+1}} \left\{ P_0(\cos 2\pi\rho) + \sum_{k=1}^{\infty} (-1)^k P_k(\cos 2\pi\rho)[\frac{1}{k^{u+1}} + \frac{1}{(k+1)^{u+1}}]. \right.$$

In this way, all of the integrals were evaluated. The values of the Legendre polynomials were obtained from Reference 5.

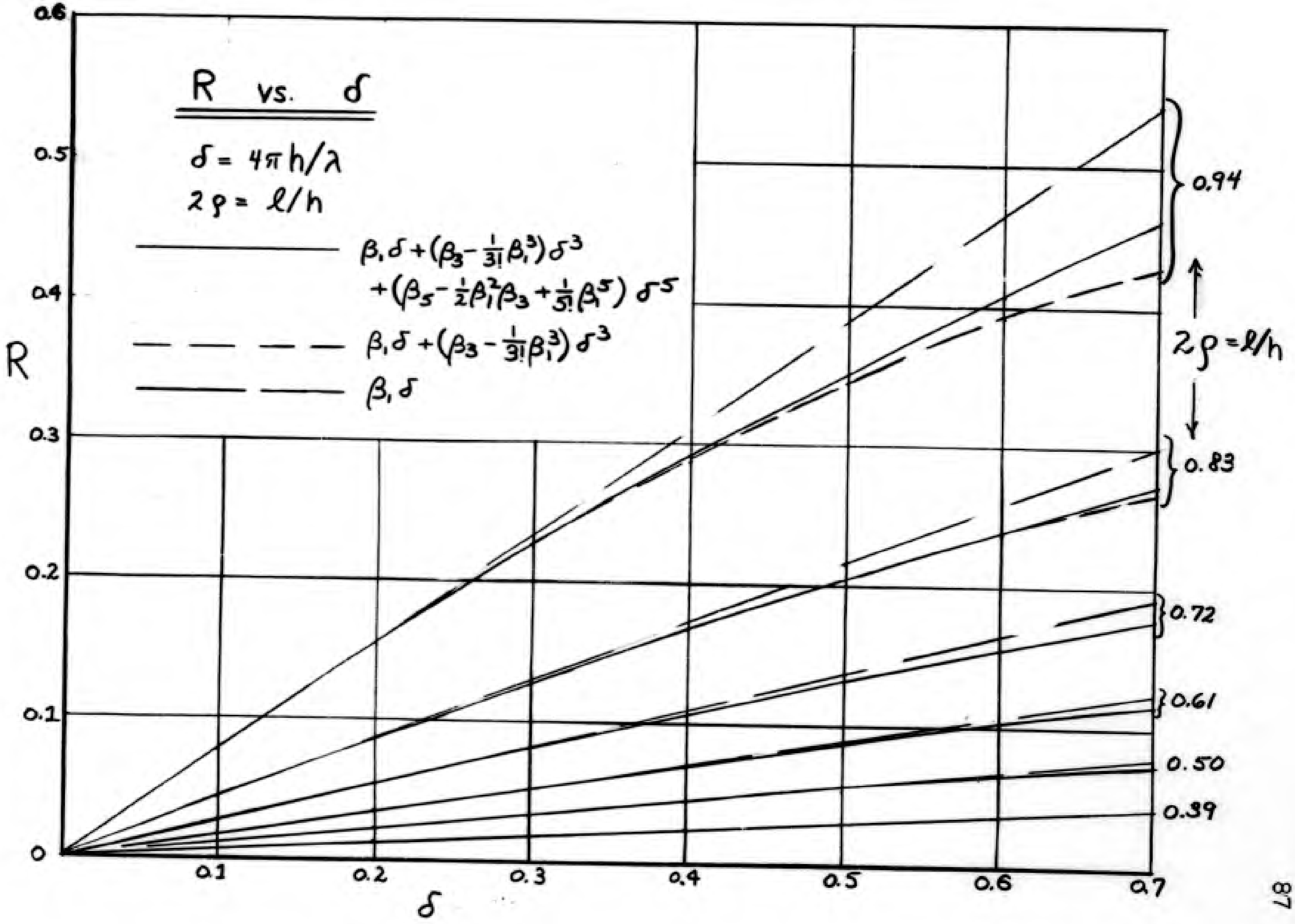
R vs. δ
$\delta = 4\pi h/\lambda$
$2\rho = \ell/h$
$\beta_1\delta + (\beta_3 - \frac{1}{3!}\beta_1^3)\delta^3 + (\beta_5 - \frac{1}{2}\beta_1^2\beta_3 + \frac{1}{5!}\beta_1^5)\delta^5$
$\beta_1\delta + (\beta_3 - \frac{1}{3!}\beta_1^3)\delta^3$
$\beta_1\delta$
R
0
0.1
0.2
0.3
0.4
0.5
0.6
δ
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.94
$2\rho = \ell/h$
0.83
0.72
0.61
0.50
0.39

FIGURE 7

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