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# CORNELL UNIVERSITY

*Center for Radiophysics and Space Research*

RESEARCH REPORT RS 15



# An Analysis of the Effect of an Imposed Magnetic Field on the Spectrum of Incoherent Scattering

T. Laaspere



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ABSTRACT

Although the Thomson scattering cross section of a free electron is quite small, it leads to detectable scattering of radio waves even at frequencies thought to be so high, until recently, as to pass through the ionosphere unperturbed. The body of theoretical work on the subject of scattering of the incoherent type has increased rapidly, but the effect of the earth's magnetic field has not yet been analyzed. In radio wave scattering the characteristics of the frequency spectrum are related to time variations of electron density in the direction of the "scattering wave vector," which is directed perpendicular to the surfaces of constant path length, measured from the transmitter to the receiver via the scattering element. Since drift motions of charged particles across magnetic lines of force are restricted in the upper ionosphere, we should expect the presence of the magnetic field to have an effect on the spectrum of scattering at least when the scattering wave vector is directed perpendicular to the field. In fact, it is shown in our analysis that in this case the spectrum will in the first approximation consist of lines, if the electrons can be assumed to gyrate freely in the magnetic field. The separation of the lines is equal to the electron gyro-magnetic frequency.

The results of some recent theoretical work indicate that the assumption of free gyration of electrons probably cannot be made unless both the scale of scattering and the electron gyro radius are smaller than the Debye length. An ionized medium tends to remain neutral over scales larger than the Debye length, and we show that if at such scales of scattering the ions could be assumed to gyrate unperturbed, our electron line spectrum would be replaced

by one in which the separation of the lines is equal to the gyromagnetic frequency of the positive ions. At large scales, however, the effects of the internal electric fields on the behavior of the charged particles should be taken into account, and until this is done in the presence of an imposed magnetic field, it will not be clear to what extent the ion line spectrum is smeared by these fields. Even if the charged particles could be assumed to gyrate unperturbed, some smearing of the lines is shown to result if particles drift through the antenna beam, and also if the magnetic lines of force do not lie in the surfaces of constant path length. Curves given in the paper reveal that even if we started out with a line spectrum, it would get smeared out quite rapidly as the angle between the surfaces of constant path length and the magnetic field is increased from zero.

The envelope of the line spectrum is derived under the assumption that the spatial distribution of particles is random and that their velocity distribution is Maxwellian. The results show that if the radius of gyration of the particles is large compared to the scale of scattering, the envelope is given simply by the thermal Doppler-spectrum curve that would have existed in the absence of any magnetic field. In that case the presence of a magnetic field causes no change in the total width of the spectrum, although it should in general lead to its narrowing, if the gyro radii were small compared to the scale of scattering.

# I. INTRODUCTION

An entirely new branch of the field of radio physics was opened by W. E. Gordon of Cornell University when he made his prediction[1] that detectable scattering of radio waves in the ionosphere will occur even at frequencies so high that the characteristic scale of scattering ($\lambda/4\pi$ for backscatter) is much smaller than the electron mean free path.

If the process of scattering is completely incoherent, if all electrons can be thought to move undisturbed through spatial distances much larger than the scale of scattering, and if the effects of the magnetic field can be neglected, then the derivation of the spectrum of scattering can be accomplished simply by converting the thermal velocity spectrum of electrons into the power spectrum of scattering by the concept of Doppler shifts. The interesting problem of incoherent scattering in the presence of a constant external magnetic field, which has not been treated previously, will be analyzed in the chapters to follow. We will find that if the radar beam is directed normal to the magnetic field, its presence can have a profound effect on the spectrum of scattering. We will also find that the gyromagnetic frequency of the charged particles plays an important role in the whole problem.

The fundamental idea of the analysis is quite simple. It was first discussed by the author in a seminar at Cornell in January, 1959. We assume each electron to gyrate freely about a magnetic line of force at its thermal velocity. The gyration of an electron will have the effect of modulating the phase of the radiation scattered by that electron. The spectrum of the received scattered field of each electron will thus consist of lines with the separation equal to the electron gyromagnetic frequency. The resultant spectrum is a superposition of such line spectra. We have determined this by assuming the spatial distribution of electrons to be random (incoherent scattering) and the velocity distribution to be Maxwellian (thermal equilibrium). It now appears fairly certain from some recent theoretical and experimental results that the assumption of a random distribution and unperturbed electron motion cannot be made at scales of scattering comparable to, or larger than, the Debye length. For our analysis to be applicable, it is probably necessary for both the scale of scattering as well as for the average radius of gyration of the electrons to be much smaller than the Debye length. However, we will conclude in Section VIII-B that our results might be useful even at large scales of scattering, if the mass, and thus also the gyro frequency, of the positive ions is substituted for that of the electrons.

It is evident that a satisfactory understanding of the problem of incoherent scattering in the ionosphere requires a good understanding of the dynamics of thermal plasma irregularities in the presence of an imposed magnetic field. It is also clear that much work still needs to be done on this subject. We hope that a comparison of our theoretical results with experimental data will in some measure facilitate future progress in understanding the behavior of plasma irregularities.

The results of this study will be presented as follows. Chapter II gives a review and discussion of the general problem of the type of scattering predicted by W. E. Gordon. In Chapter III some expressions are derived for subsequent use for scattering by moving electrons. In this chapter we also investigate the effect of an external magnetic field on the forced vibration of electrons.

In Chapter IV we derive a line spectrum of backscatter with the radar beam directed perpendicular to the magnetic field. We also discuss here the various effects tending to smear the line spectrum. The envelope of this line spectrum is derived in Chapter V, and is shown to be very well approximated by the Gaussian thermal Doppler spectrum that would have existed in the absence of a magnetic field. Some sample spectra are given at the end of Chapter V.

Chapter VI shows that it is possible to modify the results of Chapters IV and V so as to include propagation at any angle to the magnetic field for both backscatter and forward scatter. It is shown that the spectrum of backscatter gets smeared at relatively small deviations from the condition of perpendicularity. Finally, in Chapter VII the application of our work to scattering in the ionosphere is discussed.

Except for the strength of the magnetic field, which is in gauss, all other parameters used in the analysis are measured in the mks system of units.

## II. REVIEW AND DISCUSSION OF THE GENERAL PROBLEM

### A. REVIEW

As has already been mentioned in the Introduction, Gordon recognized in 1958 that the radar-techniques and powerful equipment of today are sufficient to permit the detection of incoherent scattering by individual electrons in the ionosphere as well as in the adjacent space.[1] If scattering is incoherent, the scattering coefficient $\sigma$ (power scattered per unit power density incident at the scattering volume, per unit volume of scattering, per unit solid angle) is given simply by

$$\sigma = N \sigma_e \quad \text{per meter} , \qquad (2.1)$$

where $N$ is the electron density per cubic meter. The symbol $\sigma_e$ denotes the scattering cross section of a single electron, $\sigma_e = 7.95 \times 10^{-30} \sin^2 \chi$ square meters, $\chi$ being the angle between the incident electric field and the direction of scattering.

A consequence of the approach used by Gordon is the prediction that the spectrum of backscatter would have a considerable spread due to the Doppler shifts introduced by the random thermal motion of the electrons. The spacing $\delta_{\frac{1}{2}}$ of the points of this spectrum, where the power density has fallen to one-half of its maximum, can in the absence of an imposed magnetic field be easily shown to be[2]

$$\delta_{\frac{1}{2}} = 0.424 \frac{1}{\lambda} \sqrt{\frac{T}{M_e}} \quad \text{kilocycles/sec} ,$$

$$= 18.5 \frac{1}{\lambda} \sqrt{T} \quad \text{kilocycles/sec} , \qquad (2.2)$$

if $\lambda$ is in meters, and T is in degrees absolute; $M_e$ denotes the ratio of the mass of an electron to the mass of an atom of unit atomic weight.

We now know that Equation (2.1) holds whenever the distribution of the electrons is completely random (irregular) over distances large compared to the scale of scattering.* For Equation (2.2) to hold, it must in addition be true that all electrons can be thought to move in straight-line paths without collisions. We now also believe that Equations (2.1) and (2.2) should simply be regarded as special cases of more general results that apply to scattering of radio waves by an ionized gas in thermal equilibrium.

In the search for a more complete theory of such scattering, a paper written earlier by Pines and Bohm[3] has been of some importance. These authors analyze the behavior of electrons in the presence of a uniform smear of neutralizing positive charge. In Appendix I of their paper Pines and Bohm derive an expression for $<|\rho_{\vec{k}}|^2>$ for the isotropic case, where $\rho_{\vec{k}}$ is the spatial Fourier component with the vector wave number $\vec{k}$ of the electron density distribution. (Triangular brackets denote the probability average.) Since the electron density distribution is isotropic, the result, given in Equation (2.3), depends only on the magnitude, k, of the wave number:

$$<|\rho_k|^2> = \frac{Nk^2}{(1/l_D)^2 + k^2} \quad , \tag{2.3}$$

---

* A random distribution of particles exists, for instance, within a box containing a gas of neutral molecules in thermal equilibrium. The essential idea here is that if a box of volume V is divided into subvolumes of size $\Delta V_i$, a particle will have the probability $\Delta V_i/V$ of being found in the $i^{th}$ subvolume. If the gas consisted of electrons alone, it is intuitively obvious that the Coulomb forces of interaction would tend to oppose the creation of a random distribution, tending to smooth out the irregularities.

where $l_D$ is the so-called "Debye length."

It can be seen from Villars, Weisskopf[4] or our Equation (3.14) that in isotropic scattering, the scattering coefficient $\sigma$ is, for unit volume, given simply by

$$\sigma = \sigma_e < \left| \rho_K \right|^2 > \quad . \tag{2.4}$$

In Equation (2.4) K denotes the magnitude, $(4\pi/\lambda) \sin \theta/2$, of the "scattering wave vector" $\vec{K}$. The vector $\vec{K}$ is in the direction of the sum of two unit vectors: one drawn in the direction of incidence, i.e., from the transmitter toward the element of scattering; the other in the direction opposite to that of scattering, i.e., from the receiver toward the element of scattering. The "scattering angle" $\theta$ is measured between the direction of incidence and that of scattering (Figure 3-1). Use of Equation (2.4) then shows that the density-fluctuation formula (2.3) derived by Pines and Bohm[3] leads to the scattering coefficient

$$\sigma = \sigma_e N \frac{\left(\frac{4\pi}{\lambda} \sin \frac{\theta}{2}\right)^2}{\left(\frac{1}{l_D}\right)^2 + \left(\frac{4\pi}{\lambda} \sin \frac{\theta}{2}\right)^2} \quad . \tag{2.5}$$

For $l_D >> \frac{\lambda}{4\pi \sin \theta/2}$ the preceding equation reduces in fact to Equation (2.1), i.e., to $\sigma = \sigma_e N$. On the other hand, for $l_D << \frac{\lambda}{4\pi \sin \theta/2}$ Equation (2.1) would according to Equation (2.5) have in addition the reduction factor,

$$f_1 = \left(\frac{4\pi \, l_D \sin \frac{\theta}{2}}{\lambda}\right)^2 \quad . \tag{2.6}$$

At backscatter from the ionosphere, with $l_D \doteq 0.25 \times 10^{-2}$ meters, we have for $\lambda = 7$ meters

$$f_1 \doteq 2 \times 10^{-5} \quad \text{or} \quad -47 \text{ db} . \qquad (2.7)$$

Let us keep the preceding considerations in mind and proceed with the historical review.

Soon after Gordon's prediction of detectable incoherent backscatter from the ionosphere, Bowles was able to confirm the existence of such scattering experimentally.[5, 6] According to Bowles the results appeared to confirm the scattering coefficient [Equation (2.1)] of the incoherent scatter within $\pm$ 3 db. On the other hand, the width of the observed spectrum of backscatter apparently was much narrower than that of the electron thermal Doppler spectrum of Equation (2.2). Needless to say, if the reduction factor $f_1$ of Equation (2.6) had been applicable, Bowles would have been unable to detect any backscatter at all at his frequency of 41 Mc/s.

To explain the observed results, Bowles proposed "a modified theory of incoherent scatter." He pointed out that although it is true that at scales larger than the Debye length the plasma tends to be electrically neutral, in the ionosphere the positive charge does not consist of a uniform smear. Instead, thermal motions of the positive ions create irregularities in ion density which may also exist at scales much larger than the Debye length, since such large-scale fluctuations of ion density will be neutralized by corresponding irregularities induced in the electron density. It is the consequence of this idea that at large scales of scattering the width of the spectrum of scattering is determined by the thermal motion of the positive ions and not of the electrons (see Section VII-B). In his report[6] Bowles also derives an expression for the spectrum of backscatter that is narrower by about a factor of $\pi$ than the ion thermal Doppler spectrum that is obtained by evaluating expression (2.2) for the atomic weight $M_i$ of the positive ions. It should be pointed out, though, that in speaking of a "modified theory of incoherent scatter," Bowles apparently assumed that the scattering coefficient $\sigma = \sigma_e N$ should continue to apply without any modification.

A contribution to the problem at hand was made by Kahn,[7] although he was not directly concerned with the scattering of radio waves. Kahn derived formulas for both charge and electron density fluctuations in an ionized gas consisting of electrons and discrete positive ions which carry a multiple $q$ of the electronic charge. Kahn's results show that when the scale of scattering, $\lambda/(4\pi \sin\theta/2)$, is much smaller than the Debye length $l_D$, the scattering coefficient $\sigma = \sigma_e N$ of Equation (2.1) holds. On the other hand, if the scale of scattering is much larger than the Debye length,

$$\sigma = \frac{q}{1+q} \sigma_e N \quad . \tag{2.8}$$

According to Equation (2.8) the scattering coefficient of a gas consisting of electrons and singly charged positive ions would, at large scales of scattering, be reduced by a factor of two to $\frac{1}{2} \sigma_e N$. It is interesting that the scattering coefficient should depend on the multiplicity of charge of the positive ions. We should also note that Equation (2.8) does not depend on the mass of the positive ions.

In his analysis of the scattering of radio waves by an ionized gas in thermal equilibrium, Fejer[8] not only obtains a general formula for the scattering coefficient, but also derives expressions for the spectra of scattering. His analysis is for a gas consisting of neutral particles, electrons, and singly charged positive ions, and leads to the scattering coefficient,

$$\sigma = \sigma_e N \frac{1 + 2\left\{\dfrac{l_D}{\lambda/(4\pi \sin\theta/2)}\right\}^2}{2 + 2\left\{\dfrac{l_D}{\lambda/(4\pi \sin\theta/2)}\right\}^2} . \qquad (2.9)$$

Evidently

$$\sigma = \begin{cases} \sigma_e N & \text{for} \quad \dfrac{\lambda}{4\pi \sin\theta/2} \ll l_D , \\ \\ \frac{1}{2}\sigma_e N & \text{for} \quad \dfrac{\lambda}{4\pi \sin\theta/2} \gg l_D . \end{cases}$$

Fejer's derivation of the scattering coefficient is an extension of the analysis of Pines and Bohm discussed earlier. In the limit of short and long wavelengths Kahn and Fejer predict the same scattering coefficient, if the neutralizing ions are singly charged. The reduction in the intensity of both the ion and the electron density irregularities of scales comparable to or larger than the Debye length is brought about by electrostatic fields that exist in the plasma. These fields have the greatest effects at the largest scales. They make the electron density deviations follow the ion density deviations, but at the same time they also cause a slight smoothing out in the irregular distribution of both the ions and the electrons.

To derive the spectra, Fejer makes use of the idea that since the amplitude of the wave scattered in a given direction is determined by the spatial Fourier component with the scattering wave number $\vec{K}$ of the electron density deviation $\delta N$, the time variation of the scattered field may be determined from the time variation of that Fourier component. Fejer derives expressions for spectra for four special cases that cover the high and the low collision-frequency approximations for scales of scattering both long and short compared to the Debye length. Probably of greatest interest is his result that the 3-db loss of the scattered power, predicted for scattering at the large scales, i.e., when $\lambda >> 4\pi l_D \sin\frac{\theta}{2}$, is mainly at the expense of the power scattered near the frequency of the incident wave. This leads to the somewhat surprising result that the spectrum should assume the shape of a shallow saddle for

$$4\pi l_D \sin\frac{\theta}{2} << \lambda << \pi^2 l_{mi} \sin\frac{\theta}{2} \quad ,$$

where $l_{mi}$ is the mean free path of the positive ions. The dip of this spectrum is centered at the transmitted frequency, and two slightly raised shoulders are placed symmetrically about the dip. (See the solid curve of his Figure 1). As a result of the dip, the half-power width of this spectrum is larger by about a factor of two than that of the ion thermal Doppler spectrum.

At a colloquium, E. Salpeter[9] of Cornell presented some theoretical results that agree with those of Fejer. Recently a manuscript has come to our attention in which Dougherty and Farley[10] of the Cavendish Laboratory derive similar results by a different method. Also, the general expression of the scattering coefficient has been derived by Renau by a relatively simple analysis.[11] The existence of some spectral content at the frequencies

$f_T \pm nf_p$, where $f_p$ is the plasma frequency of the electrons, has now been predicted by several authors.[9, 10, 12]

We should finally like to point out that backscatter of the incoherent type has lately been observed at the Lincoln Laboratory[13] at 440 Mc/s. Their spectra have an approximate width appropriate to the thermal motion of the ions, and in addition even give an indication of the existence of a rather flat top such as that predicted by Fejer, Salpeter, and Dougherty and Farley.

## B. DISCUSSION

At least three approaches have been used to attack this problem. We hope in this section to add to the understanding of the problem by discussing the similarities as well as the differences of the various approaches.

First of all, there is general agreement that it is only the scattering by electrons that is of any importance, the positively charged particles being too heavy to scatter appreciably. Furthermore, everyone also seems to agree that the electrons can be assumed to scatter independently, i.e., the reradiation process of any one electron can be assumed to have no effect on any of the other electrons (single scattering by electrons). The assumption that electrons scatter independently does not mean that the process of scattering is incoherent, i.e., that the scattered wavelets of the individual electrons can be added directly in terms of power, nor does it mean that no mutual effects (such as Coulomb forces ) exist between the charged particles. We list the three approaches used in analyses of single scattering by electrons in what we think is their decreasing order of generality:

i) Scattering by individual electrons;

ii) Scattering by irregularities in electron density; and

iii) Incoherent scattering by individual electrons.

Approach (i) is the most general of the three. In this approach the total scattered electric field at time t is written as a sum (or integral) over the elementary wavelets contributed independently by all the electrons of the scattering volume. It will be shown in Chapter III that for linear polarization we have the expression

$$E(t) = \sigma_e^{\frac{1}{2}} \frac{E_o}{r^2} \sum_i \cos\{\omega_T t + \phi_i(t)\} , \tag{2.10}$$

where

$\sigma_e$ is the scattering cross section of a single electron,

$E_o$ is the amplitude of the electric field incident at the scattering volume;

$\omega_T$ is the radian transmitted frequency; and

$r$ is the distance of the scattering volume.

What makes the general method (i) very difficult is the fact that in order to determine the phase angle $\phi_i(t)$ of the radiation received at time t from the $i^{th}$ electron, we would need to know the trajectory of each electron in space-time.

Approach (ii) of scattering by irregularities in electron density is obtained from (i) by making the assumption that at each instant the total scattered field from the average electron density adds up to zero, and that therefore only the deviations of the electron density from the average have to be considered. Insofar as this assumption holds, (i) and (ii) should lead to the same results if the phase information is kept in the approach of scattering by irregularities as well.

Approach (iii) of incoherent scattering by individual electrons is obtained from (i) by making the assumption that each phase angle $\phi_i(t)$ of Equation (2.10) is distributed with equal probability between zero and $2\pi$ radians. This assumption about the distribution of phases is equivalent to the assumption that the spatial distribution of electrons is completely random (irregular) over distances large compared to a wavelength. If the phase angles of the wavelets are distributed with equal probability between zero and $2\pi$ radians, then the oscillations can be added directly in terms of power, because in that case the cross-product terms will vanish in the expression for the average value of $E^2(t)$. In connection with the approach of incoherent scattering the following should thus be noted.

a) Random distribution of electrons is not required over all the scattering volume. It will, therefore, not be objectionable if the electrons have a larger probability of being found either in the top, or bottom, part of the scattering volume. (We require a slowly varying, not a homogeneous, medium.)

b) Whether scattering is incoherent is determined solely by the spatial distribution of the electrons. If the spatial distribution of electrons is random, then scattering is incoherent, and $\sigma = \sigma_e N$. Electron distribution is apparently random over scales much smaller than the Debye length.

# III. SCATTERING BY MOVING FREE ELECTRONS

## A. DERIVATION OF EXPRESSIONS FOR ELECTRIC FIELD

We will now derive some expressions for the scattered electric field, which will be used in the succeeding chapters. In this derivation, let $\vec{r}_1 = \vec{r}_1(t)$ be the vector distance from the transmitter T to the free electron, and $\vec{r}_2 = \vec{r}_2(t)$ be the vector distance from the free electron to the receiver R. The vector wave number $\vec{k}_1$ is in the direction of the incident wave, i.e., $\vec{k}_1 = (2\pi/\lambda)(\vec{r}_1/r_1)$; and $\vec{k}_2$ is in the direction of the scattered wave, i.e., $\vec{k}_2 = (2\pi/\lambda)(\vec{r}_2/r_2)$ (see Figure 3-1). In all of our work $\omega_T$ denotes the radian transmitted frequency, and $\lambda$ (no subscript) the transmitted wavelength. Let us write the linearly polarized far-field electric field of the transmitted wave as

$$\vec{E}_1 = \frac{\vec{E}_o}{r_1} e^{i(\omega_T t - \vec{k}_1 \cdot \vec{r}_1 + \phi)} \quad . \tag{3.1}$$

The amplitude $\vec{E}_o$ could be a function of the direction of propagation, i.e., of $\vec{r}_1/r_1$.

The incident field giving rise to the scattered field that exists at the location of the receiver R at the time t was the incident field that existed at the time $t - [r_2]/c$ at the distance $[r_1]$ from the transmitter and $[r_2]$ from the receiver. The brackets indicate retarded quantities and refer to that position of the electron at which the radiation was emitted that gets to the receiver at precisely the time t. That is, the scattered field received at the receiver R at the time t from the electron is due to the excitation

$$[\vec{E}_1] = \frac{\vec{E}_o}{[r_1]} e^{i\{\omega_T (t - [r_2]/c) - [\vec{k}_1] \cdot [\vec{r}_1] + \phi\}} \quad .$$

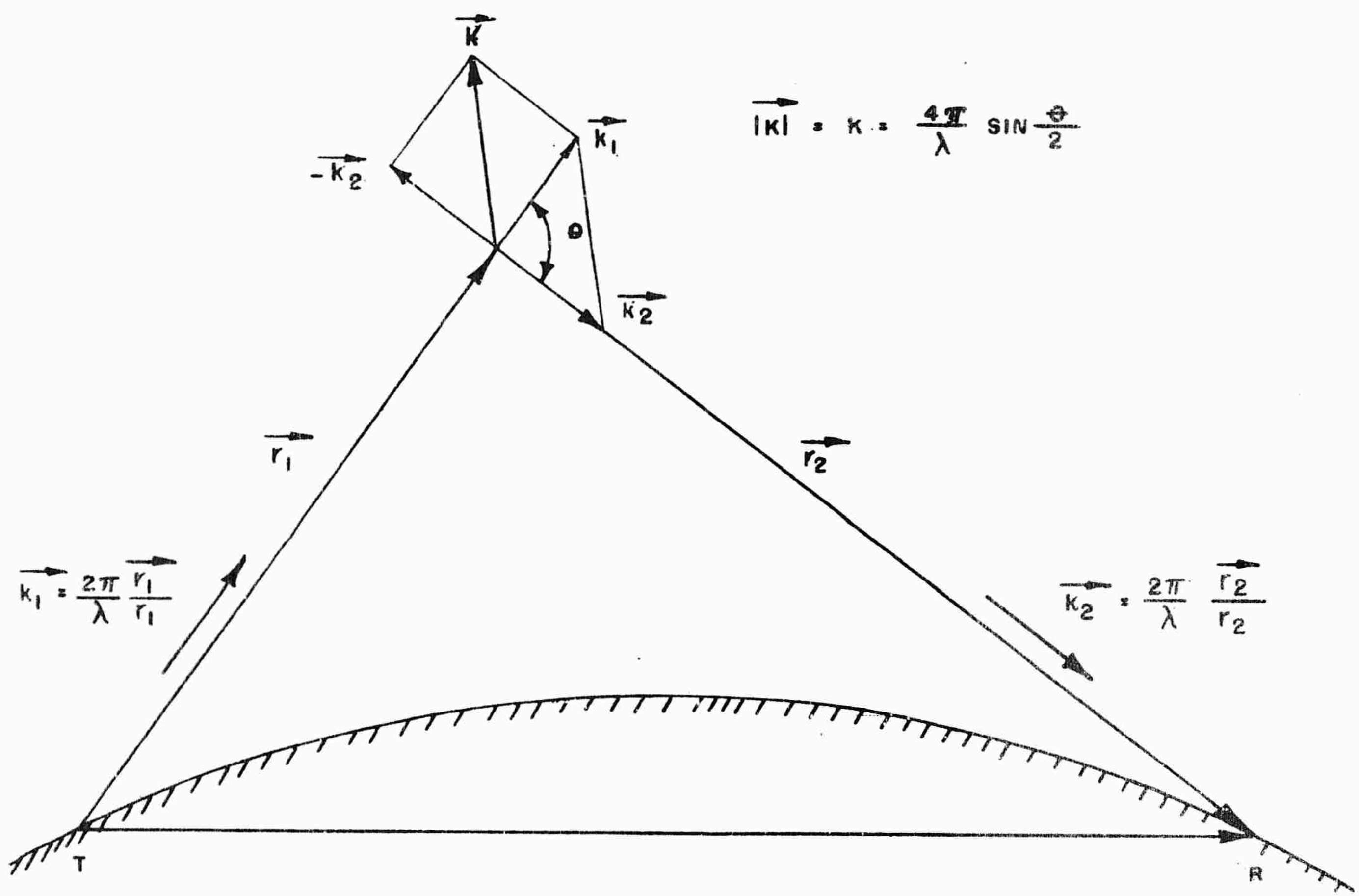


Figure 3-1. Geometry of the scattering problem.

This can be written in our vector notation as follows:

$$[\vec{E}_1] = \frac{\vec{E}_o}{[r_1]} e^{i\{\omega_T t - [\vec{k}_1]\cdot[\vec{r}_1] - [\vec{k}_2]\cdot[\vec{r}_2] + \phi\}} . \qquad (3.2)$$

The electric field of Equation (3.2), acting on the electron, gives rise to the acceleration of the electron

$$[\vec{a}] = -e\frac{[\vec{E}_1]}{m} = -\frac{e}{m}\frac{\vec{E}_o}{[r_1]} e^{i\{\omega_T t - [\vec{k}_1]\cdot[\vec{r}_1] - [\vec{k}_2]\cdot[\vec{r}_2] + \phi\}} . \qquad (3.3)$$

The far-field electric field of an accelerating particle of charge q is given by[14]

$$\vec{E}_2 = \frac{q}{4\pi\epsilon_o c^2}\frac{1}{[r]}[\vec{r}_o] \times ([\vec{r}_o] \times [\vec{a}]) , \qquad (3.4)$$

where r is the distance from the particle to the observer and $\vec{r}_o$ the unit vector in the direction of the observer. In our case, expression (3.4) becomes

$$\vec{E}_2 = \frac{-e}{4\pi\epsilon_o c^2}\frac{1}{[r_2]}\left[\frac{\vec{r}_2}{r_2}\right] \times \left\{\left[\frac{\vec{r}_2}{r_2}\right] \times \frac{-e}{m}[\vec{E}_1]\right\} . \qquad (3.5)$$

Now let $\chi$ denote the angle between the direction of the retarded incident electric field $[\vec{E}_1]$ and the retarded direction of scattering $[\vec{r}_2]$. Then

$$\left[\frac{\vec{r}_2}{r_2}\right] \times \left\{\left[\frac{\vec{r}_2}{r_2}\right] \times [\vec{E}_1]\right\} = [E_1] \sin\chi \, [\vec{\eta}] , \qquad (3.6)$$

where $\vec{\eta}$ is a unit vector in the direction of $\vec{r}_2 \times \{\vec{r}_2 \times \vec{E}_1\}$. (The unit vector $\vec{\eta}$ is perpendicular to $\vec{r}_2$ and lies in the plane defined by the two vectors $\vec{E}_1$ and $\vec{r}_2$.) Let us further observe that

$$\frac{e^2 \sin\chi}{4\pi \epsilon_o mc^2} = \frac{e^2 \mu_o}{4\pi m} \sin\chi = \sigma_e^{\frac{1}{2}} , \tag{3.7}$$

where $\sigma_e$ is the scattering cross section of a single electron that appears in Equation (2.1). Using Equations (3.3), (3.6), and (3.7) in Equation (3.5), we obtain the following expression for the scattered field at the receiving site at time t:

$$\vec{E}_2 = \sigma_e^{\frac{1}{2}} \frac{E_o}{[r_1][r_2]} e^{i\{\omega_T t - [\vec{k}_1]\cdot[\vec{r}_1] - [\vec{k}_2]\cdot[\vec{r}_2] + \phi\}} [\eta] . \tag{3.8}$$

Since by definition $[\vec{k}_1]$ is in the direction of $[\vec{r}_1]$, and $[\vec{k}_2]$ in the direction of $[\vec{r}_2]$, we have

$$[\vec{k}_1]\cdot[\vec{r}_1] + [\vec{k}_2]\cdot[\vec{r}_2] = \frac{2\pi}{\lambda} \{[r_1] + [r_2]\} , \tag{3.9}$$

so that Equation (3.8) could be written simply as

$$\vec{E}_2 = \sigma_e^{\frac{1}{2}} \frac{E_o}{[r_1][r_2]} e^{i\{\omega_T t - \frac{2\pi}{\lambda}([r_1] + [r_2]) + \phi\}} [\vec{\eta}] .$$

Note that $[r_1]+[r_2]$ is the total path length from the transmitter T to the receiver R via the moving electron at the time the radiation is emitted that gets to the receiving site at precisely the time t.

Equation (3.8) can be put in a form that is often used in the scatter theory. It is easily verified that

$$\begin{aligned} \vec{k}_1 \cdot \vec{r}_1 + \vec{k}_2 \cdot \vec{r}_2 &= \vec{k}_2 \cdot \overrightarrow{TR} + (\vec{k}_1 - \vec{k}_2) \cdot \vec{r}_1 \\ &= \vec{k}_2 \cdot \overrightarrow{TR} + \vec{K} \cdot \vec{r}_1 , \end{aligned} \tag{3.11}$$

where $\vec{K}_1 = \vec{k}_1 - \vec{k}_2$ is the so-called "scattering wave vector" whose magnitude is $(4\pi/\lambda) \sin\theta/2$.

Setting $-\vec{k}_2 \cdot T\vec{R} + \phi = \Phi$, one can write Equation (3.8) for the scattered field from the moving electron as

$$E_2 = \sigma_e^{\frac{1}{2}} \frac{E_o}{[r_1][r_2]} e^{i\{\omega_T t - [\vec{K}] \cdot [\vec{r}_1] + [\Phi]\}}_{[\vec{\eta}]} . \tag{3.12}$$

It should be emphasized that we have only taken into consideration the acceleration of an electron resulting from the incident electric field. Our results show that for such radiation the formulas for moving electrons differ from those derived for electrons fixed in their equilibrium positions only by being in terms of the retarded distances $[r_1]$ and $[r_2]$ instead of simply in terms of $r_1$ and $r_2$.

In order to write down the expression for the total scattered electric field we really do not have to track individual electrons, but can pay attention to fixed points of space instead, and work in terms of electron density. The argument is as follows.

A contribution to the signal is received at the time $t$ from a distance $r_2$ from all those electrons that happened to be at that distance within the beams of both antennas at the time $t - r_2/c$. Every electron that was at the distance $r_2$ at the time $t - r_2/c$ was given, at that instant, by the incident electric field, the acceleration,

$$\vec{a} = -\frac{e}{m} \frac{\vec{E}_o}{r_1} e^{i\{\omega_T t - \vec{k}_1 \cdot \vec{r}_1 - \vec{k}_2 \cdot \vec{r}_2 + \phi\}} .$$

The field at the receiving site* R at the time t resulting from such an electron can be written immediately on the basis of our previous work:

$$\vec{E}_2 = \sigma_e^{\frac{1}{2}} \frac{E_o}{r_1 r_2} e^{i\{\omega_T t - \vec{K} \cdot \vec{r}_1 + \Phi\}} \vec{\eta} \quad .$$

Now let $N(\vec{r}_1, t - r_2/c)$ denote the spatial electron density, evaluated for the distance $r_2$ at the retarded time $t - r_2/c$. The total scattered field at the receiver R is then obtained by integrating over the scattering volume V.

$$\vec{E}_R = \sigma_e^{\frac{1}{2}} \int_V \frac{N(\vec{r}_1, t - r_2/c)}{r_1 r_2} \vec{\eta} E_o e^{i(\omega_T t - \vec{K} \cdot \vec{r}_1 + \Phi)} dV \quad . \tag{3.13}$$

According to the theory of scattering, only the deviations $\delta N$ of the electron density from its average value have to be taken into account, so that in the expression (3.13), $N(\vec{r}_1, t - r_2/c)$ could be replaced by $\delta N(\vec{r}_1, t - r_2/c)$. Except for the appearance of the retarded time $t - r_2/c$, Equation (3.13) is then equivalent to expressions derived previously[4] on the basis of a macroscopic approach.

It is usually assumed that $r_1$, $r_2$, $\vec{K}$ and $\rho$ remain practically constant throughout a volume $V_i$, which could be a subvolume of the total scattering volume V. On the basis of the definition of the scattering coefficient $\sigma$ given in Section II-A, we would then obtain from Equation (3.13)

---

* We avoid the term "the received field," since no account has been taken of the pattern of the receiving antenna.

for the subvolume $V_i$ the scattering coefficient,

$$\sigma = \sigma_e \frac{1}{V_i} \left\langle \left| \rho_{\vec{K},\, t - r_2/c} \right|^2 \right\rangle ,$$

where $\rho_{\vec{K},\, t}$ is the Fourier component of the electron density fluctuations defined by

$$\rho_{\vec{K},\, t} = \int_{V_i} \delta N(\vec{r}_1, t)\, e^{-i \vec{K} \cdot \vec{r}_1}\, dV_i \quad .$$

If the process of scattering is time stationary, then the ensemble average of $\left| \rho_{\vec{K},\, t} \right|$ does not depend on the time $t$ for which it is evaluated. In that case

$$\sigma = \sigma_e \frac{1}{V_i} \left\langle \left| \rho_{\vec{K}} \right|^2 \right\rangle . \qquad (3.14)$$

If the distribution of $\delta N$ is isotropic, then the preceding expression depends only on the magnitude, $(4\pi/\lambda) \sin \theta/2$, of the scattering wave vector $\vec{K}$, and not on the direction of it. Equation (3.14), derived on the basis of a microscopic approach for moving electrons, is identical with the expression of the scattering coefficient that follows from a macroscopic approach.[4] It should be noted that although Equation (3.14) applies in the case of an anisotropic distribution of the electron density deviations $\delta N$, it is based on Equation (3.3), in which the acceleration of the electron was assumed to be independent of the direction of the incident electric field. This assumption will be discussed in the next section.

## B. EFFECTS OF APPLIED MAGNETIC FIELD

In writing Equation (3.3) we have neglected the effects of an applied magnetic field on the oscillatory motion performed by an electron under the influence of the incident electric field. The question thus arises: If a magnetic field is in fact present, under what circumstances are the formulas of the preceding section still applicable?

We should note that in the Maxwell equation for the curl of the magnetic field of the wave,

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \vec{J} + \epsilon_o \frac{\partial \vec{E}}{\partial t} \quad , \tag{3.15}$$

the oscillatory motion of the electrons enters directly through the current density $\vec{J}$. It is well know that in the absence of an external magnetic field the oscillatory motion of electrons in such that the real and the displacement current terms of the preceding equation can be combined, leading to

$$\nabla \times \vec{H} = \epsilon_o \left(1 - \frac{\omega_p^2}{\omega_T^2}\right) \frac{\partial \vec{E}}{\partial t} = \epsilon \frac{\partial \vec{E}}{\partial t} \quad , \tag{3.16}$$

where $\omega_p$ is the electron plasma frequency. In the derivation of Equation (3.16) it is assumed that $\nu^2 << \omega_T^2$, where $\nu$ is the electron collision frequency. The dielectric constant $\epsilon/\epsilon_o$ of such a plasma is thus given simply by the quantity $1 - \omega_p^2/\omega_T^2$.

If the imposed electric field is sinusoidal, $\nu^2 << \omega_T^2$, and an external magnetic field is present, then the real and the displacement current terms of Equation (3.15) can still be combined, but Equation (3.16) now has to be written in tensor form as follows:[15]

$$
\begin{pmatrix} (\nabla \times H)_x \\ (\nabla \times H)_y \\ (\nabla \times H)_z \end{pmatrix} = \epsilon_o \begin{pmatrix} 1 - \dfrac{\omega_p^2}{\omega_T^2 - \omega_b^2} & \dfrac{-j\omega_b\, \omega_p^2}{\omega_T\,(\omega_T^2 - \omega_b^2)} & 0 \\ \dfrac{+j\omega_b\, \omega_p^2}{\omega_T\,(\omega_T^2 - \omega_b^2)} & 1 - \dfrac{\omega_p^2}{\omega_T^2 - \omega_b^2} & 0 \\ 0 & 0 & 1 - \dfrac{\omega_p^2}{\omega_T^2} \end{pmatrix} \begin{pmatrix} \dfrac{\partial E_x}{\partial t} \\ \dfrac{\partial E_y}{\partial t} \\ \dfrac{\partial E_z}{\partial t} \end{pmatrix}
$$

In this case the dielectric constant thus assumes the form of a tensor, given by the expression in the parentheses of Equation (3.17). In the preceding expression $\omega_b$ is the radian gyromagnetic frequency $eB/m$ of electrons.

It is easily seen that if both $\omega_b^2 \ll \omega_T^2$ and $\omega_p^2 \ll \omega_T^2$, then the off-diagonal terms of the tensor can be neglected, in which case Equation (3.17) reduces to Equation (3.16). That is, as far as the forced oscillation of an electron is concerned, the presence of an external magnetic field can be neglected if

$$
\omega_b^2 \ll \omega_T^2 \quad , \qquad \omega_p^2 \ll \omega_T^2 \quad .
$$

These inequalities do, in fact, hold at frequencies $\omega_T$ used in incoherent scattering.

## C. AMPLITUDES OF ELECTRON OSCILLATION

We saw in the preceding section that the effects of an external magnetic field on the oscillatory motion of an electron can be neglected if $\omega_b^2 << \omega_T^2$, $\omega_p^2 << \omega_T^2$. We will now also show that at the frequencies of tens and hundreds of megacycles per second used in incoherent scattering, the oscillatory motion of an electron is of a very small amplitude. Such a conclusion will be useful, since it will allow us to picture electrons as gyrating freely about the magnetic field, with the incident electric field causing only a small perturbation in such motion.

The far-field power density S of a radar of gain G is given by

$$S = \frac{P_T}{4\pi h^2} G \quad , \tag{3.18}$$

where h is the distance to the point of observation, measured along the center-line of the antenna beam, and $P_T$ is the transmitted power. Using the relation

$$G = \frac{4\pi A_{eff}}{\lambda^2} \quad , \tag{3.19}$$

where $A_{eff}$ is the effective area of the antenna, one can put Equation (3.18) in the form

$$S = \frac{P_T A}{h^2 \lambda^2} \quad , \tag{3.20}$$

If the electric field of the wave at the point of interest is assumed to be of the form $E = E_o \cos\omega_T t$, then $S = E_o^2/2\zeta$, where $\zeta$ is the characteristic impedance of free space, $\zeta = 120\,\pi$ mks units. In that case, use of Equation (3.20) for S gives

$$E_o = (2\zeta)^{\frac{1}{2}} \frac{P_T^{\frac{1}{2}} A^{\frac{1}{2}}}{h\lambda} \quad .$$

From $\frac{d^2\vec{r}}{dt^2} = -\frac{e}{m}\vec{E}$, where $\vec{r}$ is the displacement of an electron, by integration we obtain immediately

$$r = r_o \cos\omega_T t = \frac{eE_o}{m\omega^2} \cos\omega_T t \quad .$$

The amplitude of the oscillatory motion of the electron is thus given by

$$r_o = \frac{e}{m\omega^2} (2\zeta)^{\frac{1}{2}} \frac{P_T^{\frac{1}{2}} A^{\frac{1}{2}}}{h\lambda}$$

$$= \frac{e}{mc^2} \frac{(2\zeta)^{\frac{1}{2}}}{4\pi^2} \frac{P_T^{\frac{1}{2}} A^{\frac{1}{2}}}{h} \lambda$$

$$= 1.36 \times 10^{-6} P_T^{\frac{1}{2}} \frac{A^{\frac{1}{2}}}{h} \lambda \quad . \qquad (3.21)$$

Using the very large antenna of $A^{\frac{1}{2}} = 300$ meters and setting $h = 300$ km $= 3 \times 10^5$ meters and $P_T = 10^6$ watts, gives the amplitude of oscillation $r_o$ as only

$$r_o = 1.36 \times 10^{-6} \lambda \quad .$$

For $\lambda = 0.7$ meters ($\doteq$ 430 Mc/s) we obtain $r_o = 0.95 \times 10^{-6}$ meters, which is only of the order of the wavelength of visible light.

# IV. DERIVATION OF LINE SPECTRUM OF BACKSCATTER WITH RADAR BEAM DIRECTED PERPENDICULAR TO MAGNETIC FIELD

## A. INTRODUCTION

In this chapter we will derive the spectrum of incoherent backscatter under the assumption that each electron is free to gyrate about the magnetic field. The beam of the radar is supposed to be directed perpendicular to the magnetic field.

We will find that under the assumptions made in the analysis, the spectrum of backscatter will consist of lines, the separation of which is equal to the gyromagnetic frequency of the electrons. The envelope of this line spectrum will be derived in the next chapter. In Chapter VI the results will be extended to propagation at an arbitrary angle to the magnetic field, including both backscatter and forward scatter.

The results of Chapters IV-VI are strictly valid only if both the scale of scattering and the gyro radius of the electrons are much smaller than the Debye length. According to the discussion given in Chapter VII the results might, however, constitute a useful first approximation even at scales of scattering much larger than the Debye length, if in that case the mass (and thus the gyromagnetic frequency) of the positive ions is substituted for the mass of the electrons.

It should be stated that in our work all of the received signal is assumed to be due to incoherent scattering only. We will also assume for the moment that the magnetic lines of force lie in the surfaces of constant phase so that any electron drift along the magnetic field will produce no change in the phase of the signal of backscatter. Finally, drift motions of the whole medium are assumed to be absent so that our spectra are centered at the transmitted frequency.

B. ANALYSIS

According to the results of the previous chapter, if an electron is located near the $z$ axis and the field incident upon it is taken as

$$E_i(t) = \frac{E_o}{z} \sin(\omega_T t - kz + \phi) \quad , \tag{4.1}$$

then the electric field of backscatter at the radar from the electron is given by

$$E_R(t) = -\sigma_e^{\frac{1}{2}} \frac{E_o}{z^2} \sin(\omega_T t - 2k[z] + \phi) \quad , \tag{4.2}$$

where $\sigma_e^{\frac{1}{2}}$ is the scattering cross section of a single electron. The quantity $[z]$ is the $z$ component of the retarded position of the electron, i.e., the $z$ component of that position in which the radiation was emitted that gets to the radar at precisely $t = t$ .

Now let the $j^{th}$ electron gyrate freely about the magnetic field at the distance $z_{oj}$ from the radar at its thermal velocity. It was shown in Section III-B that if the frequency of the incident electric field is much higher than both the electron gyromagnetic as well as its plasma frequency, then the magnetic field has little effect on that component of the electron's motion that is due to the incident electric field. Thus the forced vibration of an electron, which we saw to be of an extremely small amplitude, can be assumed to be perpendicular to the $z$ axis, and the $z$-component of the position of the $j^{th}$ electron is given simply by

$$z_j(t) = z_{oj} + R_j \sin(\omega_b t + \mu_j) \quad , \tag{4.3}$$

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$$z_j(t) = z_{oj} + R_j \sin(\omega_b t + \mu_j) \quad , \tag{4.3}$$

where

$\omega_b = \frac{eB}{m}$ is the gyromagnetic frequency of electrons;

$R_j$ is the radius of gyration of the $j^{th}$ electron; and

$\mu_j$ is its phase of rotation relative to some reference.

For the retarded position of the electron we have

$$[z_j(t)] = z\left(t - \frac{[z_j(t)]}{c}\right)$$

$$\doteq z\left(t - \frac{z_{oj}}{c}\right) \quad ,$$

where the last expression holds if the thermal velocity of the $j^{th}$ electron is much smaller than c . Thus, approximately,

$$[z_j(t)] = z_{oj} + R_j \sin\left\{\omega_b\left(t - \frac{z_{oj}}{c}\right) + \mu_j\right\}$$

$$= z_{oj} + R_j \sin(\omega_b t + \theta_j) \quad , \tag{4.4}$$

where $\theta_j \equiv \mu_j - \omega_b \frac{z_{oj}}{c}$ .

If Equation (4.4) is substituted in Equation (4.2) and the arbitrary phase angle $\phi$ is set equal to $\pi$, then we obtain for the field of backscatter at time t resulting from the $j^{th}$ electron, the expression,

$$E_{Rj}(t) = \sigma_e^{\frac{1}{2}} \frac{E_o}{z_j^2} \sin\left\{\omega_T t - 2kR_j \sin(\omega_b t + \theta_j) - 2kz_{oj}\right\} . \tag{4.5}$$

The total field of backscatter at the radar is given by the sum over all electrons that participate in the process of scattering, i.e., by

$$E(t) = \sigma_e^{\frac{1}{2}} E_o \sum_j \frac{1}{z_j^2} \sin\left\{\omega_T t - 2kR_j \sin(\omega_b t + \theta_j) + \psi_j\right\} , \tag{4.6}$$

where we have set $-2kz_{oj} \equiv \psi_j$.

It should be noted that we are dealing here simply with a case of phase modulation: the component of the electron's motion perpendicular to the surfaces of constant phase gives rise to the sinusoidal phase-modul term $-2kR_j \sin(\omega_b t + \theta_j)$ in the expression for the backscattered electri field. Expression (4.6) can be expanded by steps already known to those acquainted with the theory of frequency and phase modulation. If we assu $z_1 \doteq z_2 \doteq \ldots \doteq z_j \doteq z$, the result can be put in the form (see Appendix

$$E(t) = \sigma_e^{\frac{1}{2}} \frac{E_o}{z^2} \sum_j \Bigg\{ J_o(2kR_j) \sin(\omega_T t + \psi_j) + \sum_{n=1}^{\infty} (-1)^n J_n(2kR_j) \sin\Big((\omega_T + n\omega_b)t + n\theta_j + \psi_j\Big) + \sum_{n=1}^{\infty} J_n(2kR_j) \sin\Big((\omega_T - n\omega_b)t - n\theta_j + \psi_j\Big) \Bigg\}, \quad (4.7)$$

where $J_n(2kR_j)$ denotes the Bessel function of the first kind, order $n$, and argument $2kR_j$. According to the preceding expression, the spectral content of the scattered field is located at $\omega_T \pm n\omega_b$, $n = 0, 1, 2, \ldots$, where $\omega_b$ is the radian gyromagnetic frequency of the electrons. Note that in the preceding analysis it has not been necessary to assume the process of scattering to be incoherent: Equation (4.7) is based on the assumption of free gyration of electrons about the magnetic field.

In order to get an expression for the power spectrum of backscatter, we will now, in fact, make the assumption that the spatial distribution of electrons is random. From Equation (4.7) the ensemble average of the power at a frequency $\omega_T \pm n\omega_b$ is proportional to

$$\sum_j \sum_i \left\langle J_n(\xi_j)\, J_n(\xi_i)\, \sin\left((\omega_T \pm n\omega_b)t \pm n\theta_j + \psi_j\right) \sin\left((\omega_T \pm n\omega_b)t \pm n\theta_i + \psi_i\right)\right\rangle$$

$$= \sum_j \left\langle J_n^2(\xi_j)\, \sin^2\left((\omega_T \pm n\omega_b)t \pm n\theta_j + \psi_j\right)\right\rangle$$

$$+ \sum_{\substack{i,j \\ j \neq i}} \left\langle J_n(\xi_j)\, J_n(\xi_i)\, \sin\left((\omega_T \pm n\omega_b)t \pm n\theta_j + \psi_j\right) \sin\left((\omega_T \pm n\omega_b)t \pm n\theta_i + \psi_i\right)\right\rangle$$

(4.8)

where we have set $\xi_j \equiv 2kR_j$. In writing Equation (4.8) we have already assumed that the average of all terms involving products of different frequencies is zero.

In the preceding expression the phase angles $n\theta_j + \psi_j$ and $-n\theta_j + \psi_j$ are determined by both the location of the center of gyration of the $j^{th}$ particle and also by its phase of rotation. We could thus write

$$n\theta_j + \psi_j = \alpha_{pj} + \alpha_{rj} \quad ,$$

where the phase angle $\alpha_{pj}$ is determined by the location of the center of rotation, and $\alpha_{rj}$ by the phase of rotation relative to some reference. A similar expression can be written for $-n\theta_j + \psi_j$. We now assume $\alpha_{pj}$ and $\alpha_{rj}$ to be independent random variables with a uniform probability distribution between zero and $2\pi$ radians. We also assume the distribution of $\xi_j = 2kR_j$ to be independent of that of the phase angles $\alpha_{pj}$ and $\alpha_{rj}$. In that case the second term of Equation (4.8) vanishes, and the first term reduces to

$$\tfrac{1}{2} \sum_j \left\langle J_n^2(\xi_j)\right\rangle \quad . \qquad (4.9)$$

For random distribution the average power density at a frequency $\omega_T + n\omega_b$, or $\omega_T - n\omega_b$, is thus equal to

$$P_n = \sigma_e \frac{E_o^2}{2\zeta z^4} \sum_j \langle J_n^2(\xi_j) \rangle \tag{4.10}$$

watts per square meter. As before, $\zeta$ is the characteristic impedance of free space. If a total of NV electrons participate, then

$$P_n = NV \sigma_e \frac{E_o^2}{2\zeta z^4} \langle J_n^2(\xi) \rangle \quad . \tag{4.11}$$

Summation over all lines gives the total power density

$$W = P_o + 2 \sum_{n=1}^{\infty} P_n = NV \sigma_e \frac{E_o^2}{2\zeta z^4} \quad , \tag{4.12}$$

since

$$J_o + 2 \sum_{n=1}^{\infty} J_n^2 = 1 \quad . \tag{4.13}$$

According to the preceding, $\langle J_n^2(\xi) \rangle$ can also be interpreted as the fraction of the total power contained in both the line at $\omega_T + n\omega_b$ and the line at $\omega_T - n\omega_b$. It should also be noted that Equation (4.12) yields the scattering coefficient $\sigma = \sigma_e N$ of Equation (2.1). This is what should be expected, since Equation (4.12) is based on the assumption that the spatial distribution of electrons is completely random.

## C. DISCUSSION

### 1. Line Spectrum and the Irregularity Approach

We should first like to point out that if the electrons can be assumed to gyrate unperturbed about the magnetic field, a line spectrum of backscatter would also result in the first approximation, if we looked at the problem completely from the point of view of scattering by irregularities in electron density. Let r denote the radial distance from the radar, and let us assume that the magnetic field vector lies in the surfaces of constant phase. We can then say that the free gyration of electrons recreates periodically the electron density versus r; that is, we would have

$$N\left(r,\ t + \frac{n}{f_b}\right) = N(r,\ t) \quad ,$$

where $n = 0, 1, 2, \ldots$, and $f_b$ is the gyromagnetic frequency of the electrons. In that case we should expect a signal of backscatter of the form

$$E(t) = A(t)\ \cos\left\{\omega_T t + \Phi(t)\right\} \quad ,$$

where

$$A\left(t + \frac{n}{f_b}\right) \doteq A(t) \quad ,$$

and

$$\Phi\left(t + \frac{n}{f_b}\right) \doteq \Phi(t) \quad ,$$

for small values of n. As n increases, the preceding two approximations become progressively less accurate, if we permit the charged particles to drift along the magnetic lines of force, which they in fact do. (See the discussion at the end of the Section C-2 following.) If the expression for E(t) is expanded and Fourier series expansion used for A(t), cos Φ(t) and sin Φ(t), E(t) can be put in the form of the line spectrum derived in

Section IV-B,

$$E(t) \doteq A_1 \cos\omega_T t + B_1 \sin\omega_T t$$
$$+ A_2 \cos(\omega_T \pm \omega_b) t + B_2 \sin(\omega_T \pm \omega_b) t$$
$$+ A_3 \cos(\omega_T \pm 2\omega_b) t + B_3 \sin(\omega_T \pm 2\omega_b) t$$
$$+ \ldots,$$

where the A's and B's are undetermined constants.

2. Causes of Smearing of Spectral Lines

First of all, smearing of the spectral lines will be caused by any deviations from the assumed free gyration of the electrons. We know that in the upper ionosphere the collision frequency of electrons with other particles is much smaller than the electron gyromagnetic frequency. However, electron motion can be affected by relatively weak electric fields, both internal and external. Existence of any irregular electric fields of scales smaller than the scattering volume would certainly lead to some smearing of the spectral lines, but it is not well known how important such fields are in the upper ionosphere.

Even if the electrons could be assumed to gyrate unperturbed in the magnetic field , there would still be some smearing of each line. In ionospheric work it seems quite valid to assume that the lines of magnetic force have a negligible curvature relative to that of the spherical surfaces of constant phase of backscatter. Thus, even though the center of the radar beam is directed perpendicular to the magnetic field, except at the center of the scattering volume, a particle drifting along the magnetic field would have a component of motion either away or toward the radar. If $\Omega$ is the beamwidth of the radar in radians, then the resulting spreading of a line because of Doppler shifts is of the order of

$$\Delta f = 2 \frac{<|v|> \Omega/2}{\lambda} ,$$

where $<|v|>$ is the average speed of drift of an electron along a magnetic line of force. For a Maxwellian gas the relation between temperature and average speed of particle motion along a line can be shown to be given by

$$<|v|> = \sqrt{\frac{2kT}{\pi m}} ,$$

where $m$ is the mass of the particle in kilograms and $k$ is the Boltzmann constant, ($k = 1.38 \times 10^{-23}$ joule per degree Kelvin). For electrons

$$<|v|> = 3.1 \times 10^{3} \sqrt{T} \text{ meters/sec} ;$$

therefore

$$\Delta f = 3.1 \times 10^{3} \Omega \frac{\sqrt{T}}{\lambda} \text{ cycles/sec} , \qquad (4.14)$$

if $\lambda$ is in meters. On the other hand, the separation of the spectral lines, i.e., the gyromagnetic frequency of the electrons, is given by

$$f_b = 2.8 \times 10^{6} B_{gauss} \text{ cycles/sec} .$$

The spreading given by Equation (4.14) is thus a fraction,

$$1.11 \times 10^{-3} \frac{\Omega \sqrt{T}}{\lambda B} \qquad (4.15)$$

of the line separation. If we use $\Omega = \lambda/d$, where d is the effective diameter of the antenna aperture, then Equation (4.15) reduces to

$$1.11 \times 10^{-3} \frac{\sqrt{T}}{B\ d} , \qquad (4.16)$$

which does not depend on wavelength. For the values of temperature and magnetic field to be expected in the vicinity of the earth, this ratio is quite small for large diameter antennas used in incoherent scattering.

If we were dealing with incoherent scattering by electrons that had the thermal characteristics of ions of atomic weight M, then the corresponding fraction would be

$$47.7 \times 10^{-3} \frac{\sqrt{MT}}{B\ d} , \qquad (4.17)$$

Expression (4.17) is larger than (4.16) by the factor $\sqrt{M/M_{electron}} = 43\sqrt{M}$. If in Equation (4.17) we set $T = 1600^{\circ}K$, $B = 0.5$ gauss, and $\sqrt{M} = 4$ appropriate to the $O^{+}$ ions, then Equation (4.17) becomes $15.3/d$, where d is the effective antenna diameter in meters. Thus, unless the antenna diameter is considerably larger than 15 meters, the ion line spectrum would be smeared simply because in the ionosphere the curvature of the magnetic lines of force is quite small compared to that of the spherical surfaces of constant phase of backscatter.

Even if electrons gyrate unperturbed and the magnetic field lies in the surfaces of constant phase, some smearing of the spectral lines will occur, because electrons may drift along magnetic lines of force. An electron drifting along a magnetic line of force will cross the antenna beam. The amplitude of the radiation received from such a drifting electron will thus actually be a function of time, varying from zero through a maximum back to zero again. The spectrum of the signal received from each electron is for that reason really not a pure line spectrum as has been assumed previously, but is slightly smeared to begin with. This smearing is of the order of $1/\tau$ cps, where $\tau$ is the time taken by the electron to cross the beam.

# V. ENVELOPE OF THE LINE SPECTRUM

## A. INTRODUCTION

The result of the analysis of the previous chapter was a line spectrum, with the line separation equal to the gyromagnetic frequency of electrons. We will now derive the envelope of this spectrum. The main assumption is that the electron velocity distribution is that appropriate for a gas in thermal equilibrium.

## B. ANALYSIS

### 1. Distribution of $\xi = 4\pi R/\lambda$

It was shown in the previous chapter that the fraction of total power contained in both the line at $\omega_T + n\omega_b$ and also the line at $\omega_T - n\omega_b$ is equal to the average value of $J_n^2(\xi)$. We will now determine the distribution of $\xi$, which we need in an evaluation of $\langle J_n^2(\xi)\rangle$.

We should first note that

$$\xi = \frac{R}{\lambda/4\pi} = \frac{V/\omega_b}{\lambda/4\pi} = \frac{V}{V_o} \quad , \tag{5.1}$$

where $V$ is the linear velocity of a gyrating electron, measured in a plane perpendicular to the magnetic field, and $V_o = \lambda\omega_b/4\pi$. We are thus interested in the distribution of $V$, which can be interpreted as the magnitude of the projections of electron velocities on a plane.

According to the kinetic theory of gases, in a Maxwellian gas the probability that the $x$ component of a particle's velocity is between $v_x$ and $v_x + dv_x$, that the $y$ component is between $v_y$ and $v_y + dv_y$, and that the z component is between $v_z$ and $v_z + dv_z$, is given by

$$f(v_x, v_y, v_z)\, dv_x\, dv_y\, dv_z = \left(\frac{\beta}{\pi^{\frac{1}{2}}}\right)^3 e^{-\beta^2 (v_x^2 + v_y^2 + v_z^2)}\, dv_x\, dv_y\, dv_z \quad . \tag{5. 2}$$

If in the preceding expression the velocities are measured in meters per second, then in

$$\beta = \left(\frac{m}{2kT}\right)^{\frac{1}{2}} = \left(\frac{m_o M}{2kT}\right)^{\frac{1}{2}} , \tag{5. 3}$$

$m$ is the mass of the particle in kilograms,

$M$ its atomic weight,

$m_o = 1.66 \times 10^{-27}$ kg is the mass of a particle of unit atomic weight

$k = 1.38 \times 10^{-23}$ joule per degree absolute is the Boltzmann constant

$T$ is in degrees absolute.

It follows from Equation (5. 2) by integration over $v_z$ that the probability of having $v_x$ between $v_x$ and $v_x + dv_x$, and $v_y$ between $v_y$ and $v_y + dv_y$ is

$$f(v_x, v_y)\, dv_x\, dv_y = \frac{\beta^2}{\pi} e^{-\beta^2 (v_x^2 + v_y^2)}\, dv_x\, dv_y \quad .$$

Now let $v_x = V \cos\theta$, $v_y = V \sin\theta$. The elemental area in the new system is given by $V\, d\theta\, dV$. We obtain

$$f(V, \theta)\, dV\, d\theta = \frac{d\theta}{2\pi}\, 2\beta^2 V e^{-\beta^2 V^2}\, dV \quad .$$

Since $\theta$ is distributed uniformly and independently between zero and $2\pi$ radians, the distribution of $V$ can be obtained immediately from the preceding equation. It is

$$f(V)\,dV = 2\beta^2 V e^{-\beta^2 V^2} dV \quad , \tag{5.4}$$

with $\beta$ given in Equation (5.3). In a Maxwellian gas the magnitudes of projections of particle velocities on a plane thus obey the so-called "Rayleigh distribution."

We actually require the distribution of the variable $\xi = V/V_o$, with $\xi$ given in Equation (5.1). If $V = V_o \xi$ is inserted in Equation (5.4) we obtain

$$f(\xi)\,d\xi = 2\beta^2 V_o^2 \xi e^{-\beta^2 V_o^2 \xi^2} d\xi$$

$$= 2a^2 \xi e^{-a^2 \xi^2} d\xi \quad , \tag{5.5}$$

where

$$a^2 = \beta^2 V_o^2 = \frac{m_o M}{2kT} \left(\frac{\lambda \omega_b}{4\pi}\right)^2 \quad ,$$

$\omega_b$ being the radian gyromagnetic frequency of the particle. Insertion of the constants in the preceding expression gives

$$a^2 = 1.5 \times 10^{-5} \lambda^2 f_b^2 \frac{M}{T} \quad ,$$

where $\lambda$ is the transmitted wavelength in meters. But $f_b = 1.525 \times 10^3 \frac{B}{M}$ cps, if $B$ is in gauss; therefore

$$a^2 = 35 \frac{\lambda^2 B^2}{MT} = 35 \gamma^2 \quad , \tag{5.6}$$

where the parameter

$$\gamma = \frac{\lambda_{meters} B_{gauss}}{\sqrt{MT_{abs}}}$$

will appear often in the work to follow. The symbol M denotes the ratio of the mass of the particle involved (electron or positive ion) to the mass of an atom of unit atomic weight.

2. Evaluation of $< J_n^2(\xi) >$

The fraction of total power contained in both the line at $\omega_T + n\omega_b$ and $\omega_T - n\omega_b$, $n = 0, 1, 2, \ldots$, is given by the average value of $J_n^2(\xi)$, i.e., by

$$< J_n^2(\xi) > = \int_{\xi=0}^{\infty} J_n^2(\xi)\, f(\xi)\, d\xi \quad . \tag{5.7}$$

Insertion of $f(\xi)$ from Equation (5.5) gives

$$< J_n^2(\xi) > = \int_0^{\infty} 2a^2\, J_n^2(\xi)\, e^{-a^2\xi^2}\, d\xi \quad . \tag{5.8}$$

Integrals of the type of that appearing on the right-hand side of Equation (5.8) have been treated in the theory of Bessel functions.[16] Evaluation of Equation (5.8) gives

$$<J_n^2(\xi)> = e^{-x}\, I_n(x) \quad , \tag{5.9}$$

where $I_n(x)$ denotes the modified Bessel function of the first kind, order $n$, and argument $x$. In Equation (5.9)

$$x = \frac{1}{2a^2} = \frac{1}{70\,\gamma^2} = \frac{0.01429}{\gamma^2} \quad , \tag{5.10}$$

where

$$\gamma = \frac{\lambda_{meters}\, B_{gauss}}{\sqrt{M\, T_{abs}}} \quad .$$

3. Approximate Expressions for $<J_n^2(\xi)>$

In order to satisfy the requirements of incoherent scattering in the upper ionosphere, x turns out to be much larger than unity. We will now show that for such values of x the envelope of the line spectrum, if the latter is normalized to unit total power, is given approximately by

$$A(f) = \frac{1}{\sqrt{2\pi x}} \; e^{-\left\{\frac{f - f_T}{258\,(1/\lambda)\sqrt{T/M}}\right\}^2} , \qquad (5.11)$$

where $f_T$ denotes the transmitted frequency in cycles per second, and $\lambda$ the transmitted wavelength in meters. We should note that the exponential of the preceding expression represents precisely the envelope of the power density curve for incoherent scattering at long mean free paths in the absence of an external magnetic field.[2]

In a derivation of Equation (5.11), we first make use of an asymptotic series for $I_n(x)$, which enables us to write[16]

$$<J_n^2(x)> \doteq \frac{1}{\sqrt{2\pi x}} \sum_{m=0}^{\infty} \frac{(-1)^m \{4n^2 - 1\}\{4n^2 - 3^2\} \ldots \{4n^2 - (2m-1)^2\}}{(2x)^m\, 4^m\, m!}$$

After rearranging, the $m^{th}$ term of $\sqrt{2\pi x}\;<J_n^2(\xi)>$ can be put in the form

$$(-1)^m \frac{1}{m!} \left(\frac{n^2}{2x}\right)^m \left(1 - \frac{1^2}{4n^2}\right)\left(1 - \frac{3^2}{4n^2}\right) \ldots \left(1 - \frac{(2m-1)^2}{4n^2}\right) . \qquad (5.12)$$

Note that

$$e^{-n^2/2x} = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!} \left(\frac{n^2}{2x}\right)^m \quad . \tag{5.13}$$

Comparison of Equation (5.12) with the $m^{th}$ term of Equation (5.13) shows that they are approximately equal for $(2m-1)^2 \ll 4n^2$; i.e., for $m^2 \ll n^2$. We will thus have the result that if the series for $e^{-n^2/2x}$ converges sufficiently rapidly; i.e., if

$$e^{-n^2/2x} \doteq \sum_{m=0}^{\mathcal{M}} (-1)^m \frac{1}{m!} \left(\frac{n^2}{2x}\right)^m \quad ,$$

where $\mathcal{M}$ satisfies with sufficient stringency the requirement $\mathcal{M}^2 \ll n^2$, then

$$\langle J_n^2(\xi) \rangle \doteq \frac{1}{\sqrt{2\pi x}} \quad e^{-n^2/2x} \quad , \tag{5.14}$$

where

$$x = \frac{0.01429}{\gamma^2} \quad , \qquad \gamma = \frac{\lambda_{meters} B_{gauss}}{\sqrt{MT}} \quad ,$$

The requirements for the approximation (5.14) to hold are considered in more detail in Appendix II.

We next note that the gyromagnetic frequency of particles of atomic weight M in a magnetic field of B gauss is given by

$$f_b = 1.525 \times 10^3 \; B/M \; \text{cycles/sec} \quad . \tag{5.15}$$

In terms of the parameter $\gamma$,

$$f_b = 1.525 \times 10^3 \; \frac{\gamma}{\lambda} \sqrt{\frac{T}{M}} \quad \text{cycles/sec} \quad . \tag{5.16}$$

We can thus write

$$e^{-n^2/2x} = e^{-n^2\gamma^2/2(0.01429)} = e^{-\left(\frac{n\gamma}{0.169}\right)^2}$$

$$= e^{-\left\{\frac{n\gamma}{0.169}\,\frac{1.525\times 10^3(1/\lambda)\sqrt{T/M}}{1.525\times 10^3(1/\lambda)\sqrt{T/M}}\right\}^2}$$

$$= e^{-\left\{\frac{n f_b}{258\,(1/\lambda)\sqrt{T/M}}\right\}^2} \quad . \qquad (5.17)$$

A curve centered at the transmitted frequency, which for $(f - f_T)^2 = (nf_b)^2$ coincides with Equation (5. 17), is

$$e^{-\left\{\frac{f - f_T}{258\,(1/\lambda)\sqrt{T/M}}\right\}^2} \quad . \qquad (5.18)$$

Substitution of Equation (5. 18) in Equation (5. 14) then yields Equation (5. 11).

## C. DISCUSSION

It is important to note that in case the spectrum consists of many lines, its width is given simply by the thermal Doppler spectrum that would have existed in the absence of any magnetic field. This means that if the average radius of gyration of the particles is comparable to or larger than the wavelength, then the total width of the spectrum is not reduced, even though the radar beam is directed perpendicular to the magnetic field. However, as the parameter $\gamma = \lambda B/\sqrt{MT}$ approaches infinity, i.e., in the limit of very small radii of gyration, the spectrum reduces to a single line that is located at the transmitted frequency. In this case the particles, like beads on a wire, can only drift along the magnetic lines of force parallel to the surfaces of constant phase.

Three sample spectra are reproduced in Figures 5-1, 5-2, and 5-3 (see also Figure 6-3). Note that, as the parameter $\gamma$ is reduced, the Gaussian Doppler spectrum becomes an increasingly better fit to the envelope of the line spectrum. It can be shown that the average radius of gyration, $\langle R \rangle$, is equal to $0.012 \times \sqrt{MT}/B$ meters. Thus $\langle R \rangle/\lambda = 0.012/\gamma$. It follows that for Figures 5-1 and 5-2 the average radius of gyration is smaller than the wavelength, whereas for Figure 5-3 the opposite is true.

Figure 5-1 is drawn for $\gamma = \lambda B/\sqrt{MT} = 0.124$. For a wavelength of 0.7 meters (430 Mc/s), this value of $\gamma$ is satisfied in the ionosphere for electrons somewhere in the height range from 1,000 km to 3,000 km. For $\gamma = 0.069$ of Figure 5-2 the corresponding range is from about 2,000 to 4,500 km. However, since in the height ranges mentioned the Debye length $l_D$ is still smaller than 0.7 meters, the actual spectra of back-scatter from such heights may bear little resemblance to Figures 5-1 and 5-2. (See the discussion in Chapter VII.)

The relation $\gamma = \lambda B/\sqrt{MT} = 0.5975 \times 10^{-2}$, for which Figure 5-3 was computed, would be satisfied for $O^+$ ions at about 200 km for a wave-length of 1.5 meters (200 Mc/s). The same value of $\gamma$ would apply for electrons somewhere in the height range from about 1,000 km to 3,000 km, if a wavelength $\lambda = 0.03$ meters (10,000 Mc/s) is used.

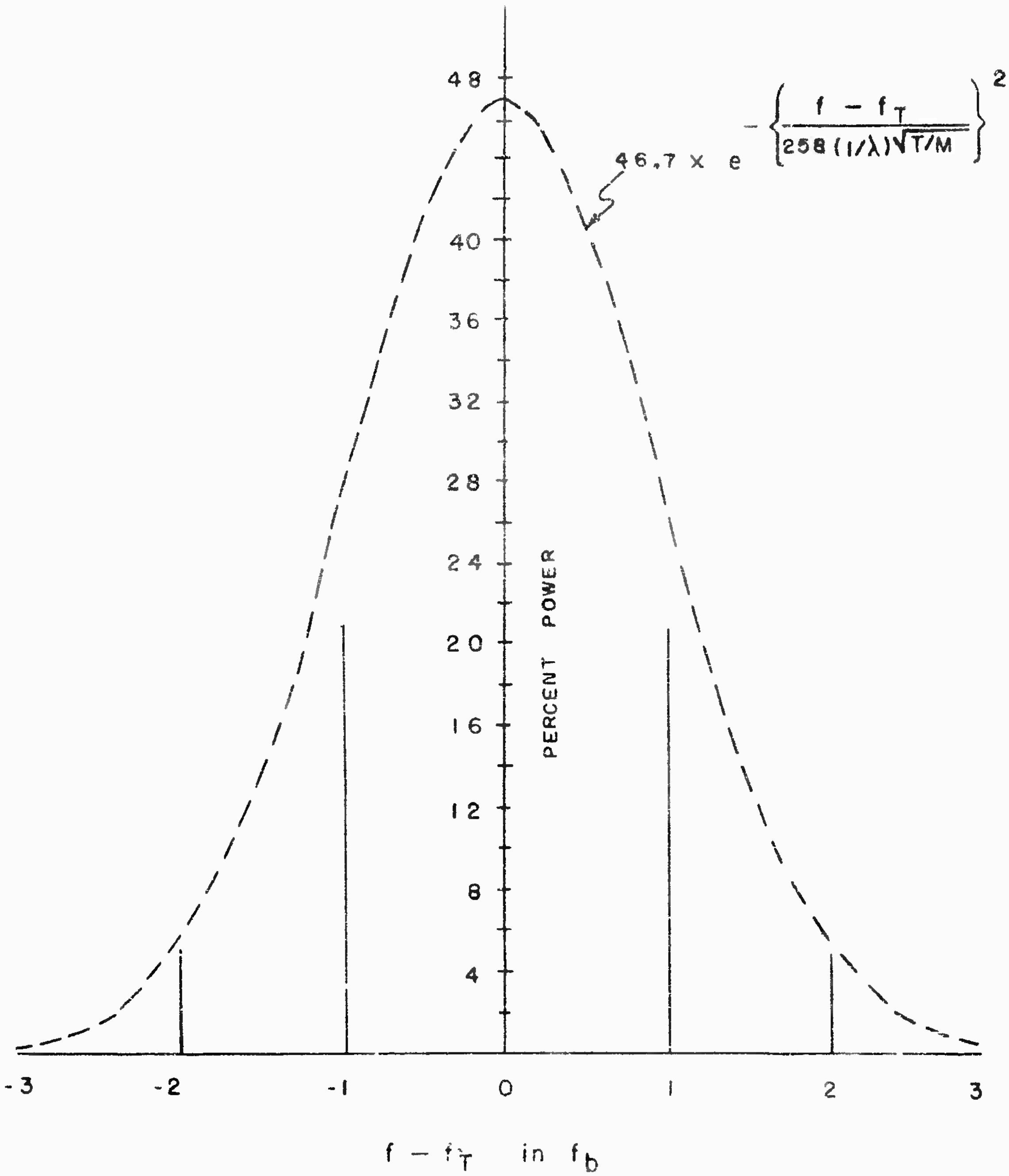


Figure 5-1. Line spectrum for $\gamma = \lambda B/\sqrt{MT} = 0.124$, and thus for $f_b = 189\ (1/\lambda)\ \sqrt{T/M}$. Gaussian Doppler-shift curve is given for comparison.

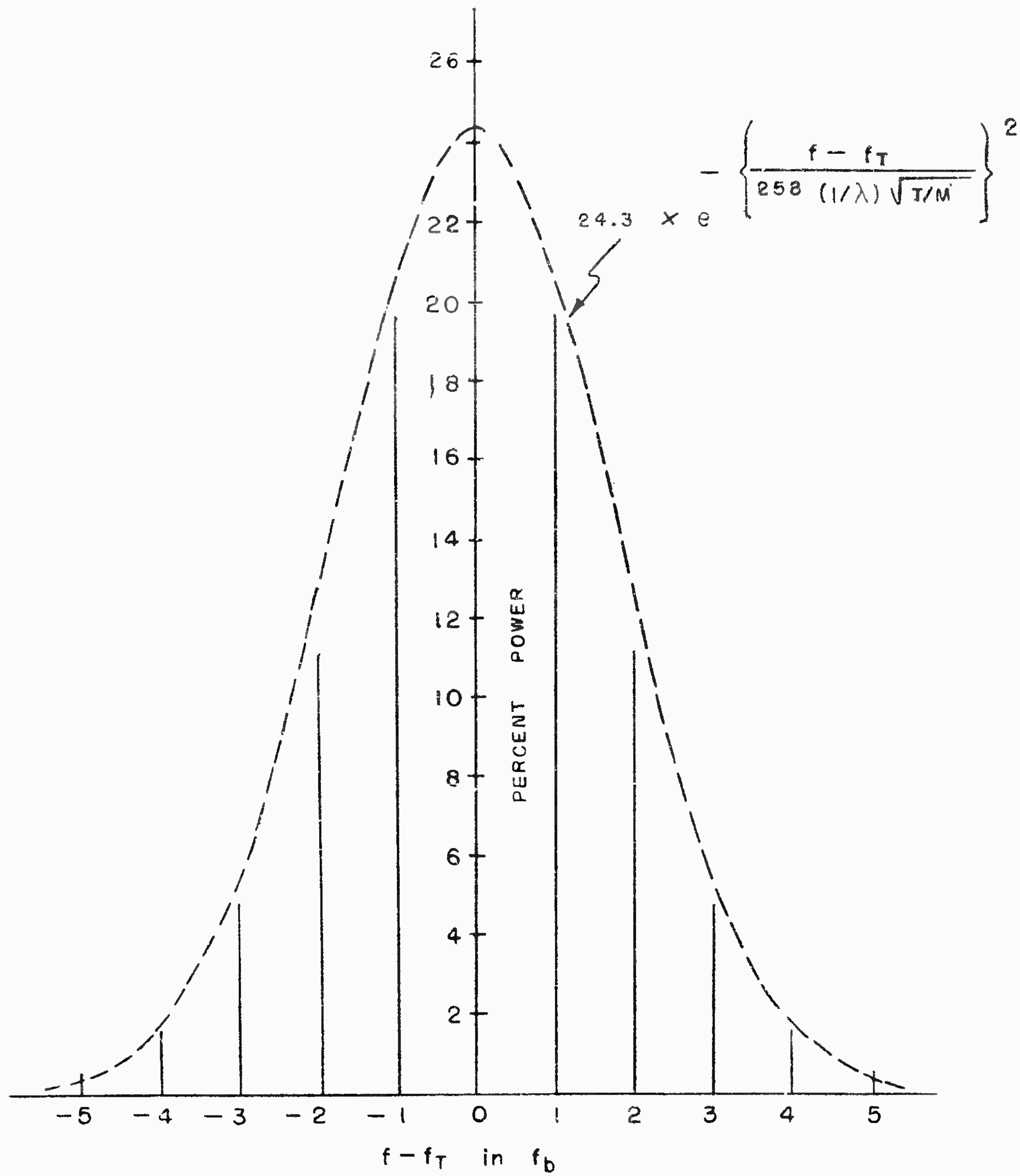


Figure 5-2. Line spectrum for $\gamma = 0.069$, and thus for $f_b = 105.2 \times \frac{1}{\lambda}\sqrt{\frac{T}{M}}$ cps .

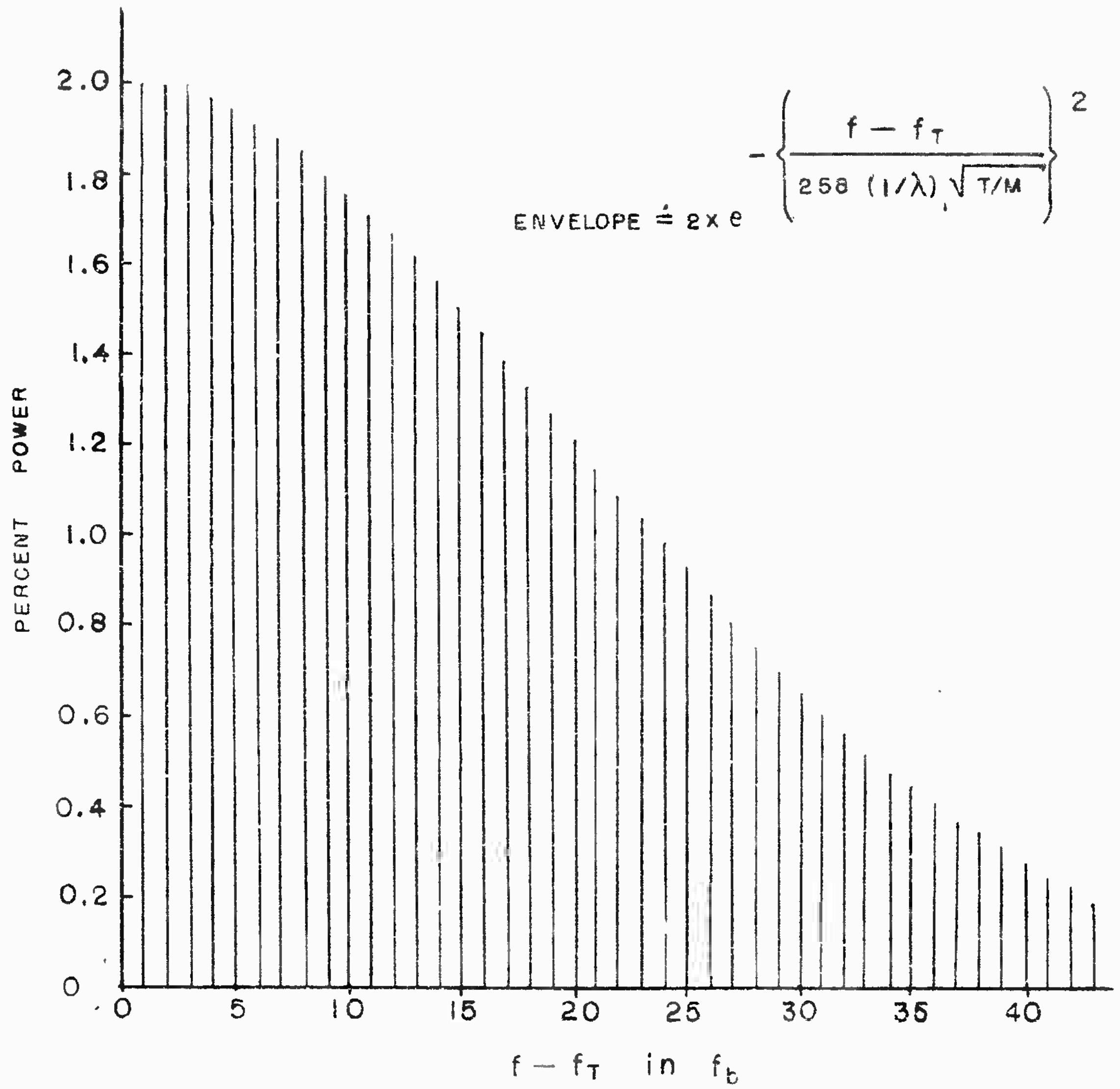


Figure 5-3. One wing of the symmetrical line spectrum for $\gamma = 0.5975 \times 10^{-2}$ and thus for $f_b = 9.11 \times (1/\lambda)\sqrt{T/M}$ cps. The envelope can be taken to be that of Gaussian Doppler spectrum with practically no error.

# VI. SPECTRUM WITH PROPAGATION AT ANGLE TO MAGNETIC FIELD (BACKSCATTER AND FORWARD SCATTER)

## A. INTRODUCTION

In the preceding two chapters, the spectrum of incoherent backscatter was analyzed under the assumption that the narrow beam of a radar is directed perpendicular to the magnetic field. We will show in this chapter that it is possible to modify the results derived previously to include the general cases of propagation at an angle to the magnetic field for both backscatter and forward scatter. In order to simplify the analysis the restriction is placed on forward scatter that the scattering volume, located at a great distance from the transmitter and the receiver, is at an equal distance from both.

## B. GEOMETRY

The phase-modulation spectrum of the previous chapters was the result of sinusoidal time variation in total path length in radians from the transmitter to the receiver via the moving electron. In the case already treated, this variation is given simply by

$$\frac{4\pi}{\lambda} R \sin(\omega_b t + \text{constant phase angle}) \quad . \tag{6.1}$$

We will now show that as far as the effects of the rotation of an electron are concerned, Equation (6.1) holds in the general case with the modification that instead of the actual transmitted wavelength $\lambda$, the formulas contain the modified wavelength,

$$\lambda_1 = \frac{\lambda}{\cos\alpha \, \sin\theta/2} \quad , \tag{6.2}$$

where $\alpha$ is the angle between the magnetic field and a plane that is parallel to the line TR and perpendicular to the plane of propagation, and $\theta$ is the angle between the direction of incidence and that of scattering (Figure 6-1). If the electrons are restricted to drift only along the magnetic field, then the Doppler shift results derived for backscatter and no magnetic field have to be modified by use of another modified wavelength,

$$\lambda_2 = \frac{\lambda}{\sin\alpha \, \sin\theta/2} \quad . \tag{6.3}$$

We begin the derivation of Equation (6.2) by noting from Figure 6-1 that for an electron that is rotating but not drifting,

$$\left|\vec{r}_1(t)\right| \doteq r_{1o} + R \cos\beta_1(t) \quad ,$$

and

$$\left|\vec{r}_2(t)\right| \doteq r_{2o} + R \cos\beta_2(t) \quad ,$$

where $\beta_1$ is the angle between $\vec{R}$ and $\vec{r}_1$, and $\beta_2$ the angle between $\vec{R}$ and $-\vec{r}_2$. Thus the total path length from the transmitter to the receiver by way of the rotating electron is given by

$$\left|\vec{r}_1(t)\right| + \left|\vec{r}_2(t)\right| \doteq r_{1o} + r_{2o} + R\left(\cos\beta_1(t) + \cos\beta_2(t)\right) \quad . \tag{6.4}$$

From Figure 6-1,

$$\vec{r}_{1o} = \frac{d}{2}\,\vec{a}_y + z_o\,\vec{a}_z \quad ,$$

$$-\vec{r}_{2o} = -\frac{d}{2}\,\vec{a}_y + z_o\,\vec{a}_z \quad ;$$

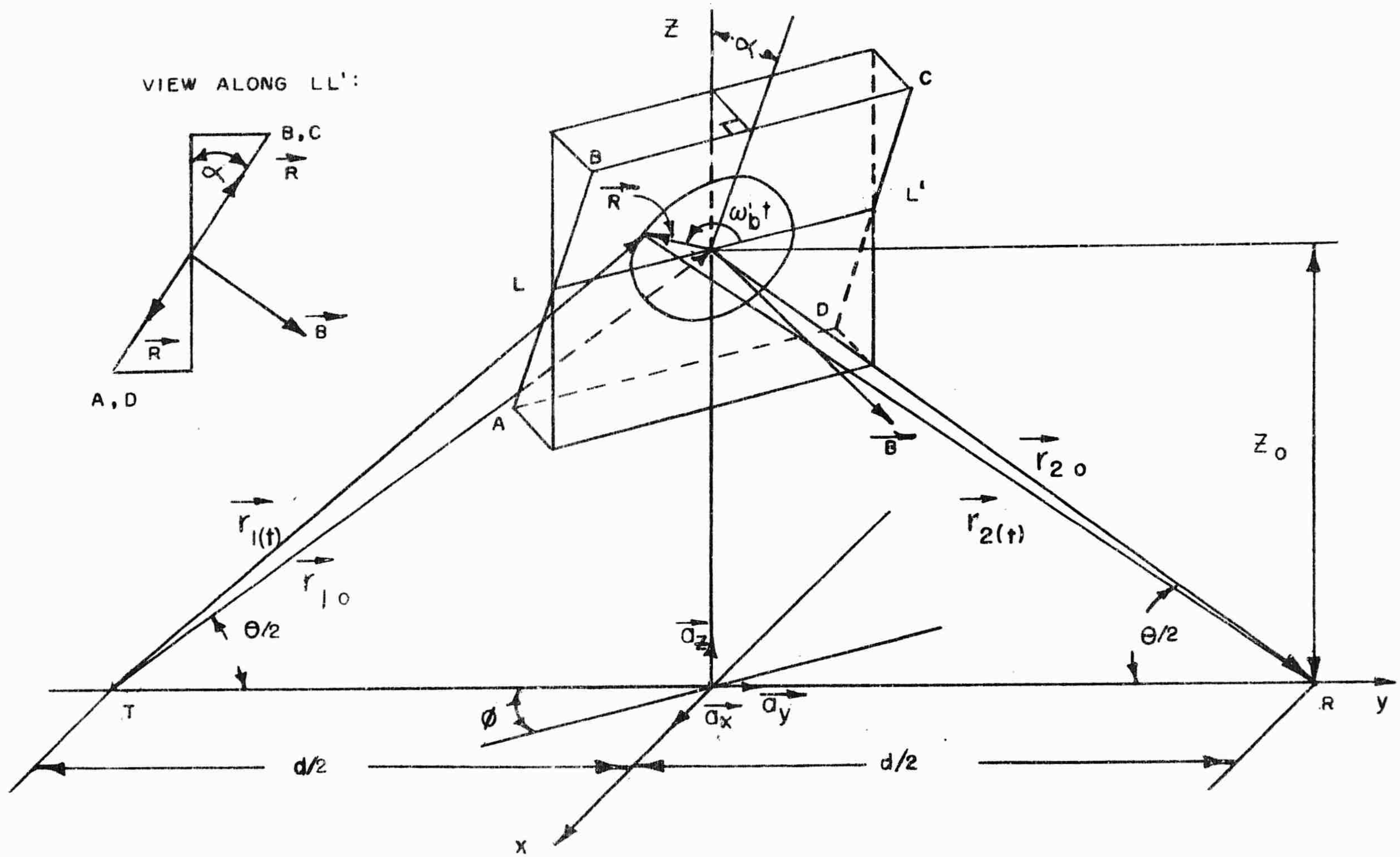


Figure 6-1. Geometry for the analysis in Chapter VI. Electron gyrates in the plane ABCD that is perpendicular to the magnetic field vector $\vec{B}$.

furthermore, in $\vec{R} = R_x \vec{a}_x + R_y \vec{a}_y + R_z \vec{a}_z$, we have

$$R_y = R \cos\omega_b t \cos\phi \quad ,$$

$$R_z = R \sin\omega_b t \cos\alpha \quad .$$

Thus

$$R \cos\beta_1 = \frac{\vec{R} \cdot \vec{r}_{1o}}{|\vec{r}_{1o}|} = \frac{R \frac{d}{2} \cos\omega_b t \cos\phi + R z_o \sin\omega_b t \cos\alpha}{r_{1o}} \quad ,$$

and, since $r_{1o} = r_{2o}$,

$$R \cos\beta_2 = \frac{-R \frac{d}{2} \cos\omega_b t \cos\phi + R z_o \sin\omega_b t \cos\alpha}{r_{1o}} \quad .$$

Therefore,

$$R(\cos\beta_1 + \cos\beta_2) = 2R \frac{z_o}{r_{1o}} \cos\alpha \sin\mu t \quad ,$$

$$= 2R \sin\frac{\theta}{2} \cos\alpha \sin\omega_b t \quad . \qquad (6.5)$$

The same distance in radians would be

$$\frac{4\pi}{\lambda} R \sin\frac{\theta}{2} \cos\alpha \sin\omega_b t \quad . \qquad (6.6)$$

As we wanted to show, this is of the same form as Equation (6.1) if in Equation (6.6) we use the modified wavelength

$$\lambda_1 = \frac{\lambda}{\cos\alpha \sin\theta/2} \quad . \qquad (6.2)$$

Turning our attention next to Doppler shifts, we note that if an electron is drifting along the magnetic field with a velocity $v$, this motion will give rise to a Doppler shift

$$f - f_T = \frac{v_R + v_T}{\lambda} \quad , \tag{6.7}$$

where $v_R$ is the component of electron's velocity toward the receiver and $v_T$ the component toward the transmitter. But

$$v_R = v_T = v_z \sin\theta/2 = v \sin\alpha \sin\theta/2 \quad ,$$

and therefore

$$f - f_T = 2 \frac{v}{\lambda} \sin\alpha \sin\theta/2 \quad . \tag{6.8}$$

At backscatter, in the absence of a magnetic field, the corresponding expression in the case of an isotropic velocity distribution would be

$$f - f_T = 2 \frac{v}{\lambda} \quad . \tag{6.9}$$

Expression (6.8) is of the same form as Equation (6.9) if in Equation (6.8) we use the modified wavelength

$$\lambda_2 = \frac{\lambda}{\sin\alpha \sin\theta/2} \quad . \tag{6.3}$$

This is what we wanted to prove.

## C. MODIFICATION OF LINE SPECTRUM

Let us dispose of the simplest extension to the results of the previous chapters first. This is the case of forward scatter with the magnetic field in a plane that is parallel to the line TR and perpendicular to the plane of propagation, i.e. $\alpha = 0$. At the magnetic equator, forward scatter in any direction satisfies this requirement. Here the <u>line spectrum</u>

still applies, but the fraction of power in both the line at $\omega_T + n\omega_b$ and $\omega_T - n\omega_b$ is, instead of Equation (5.10), now given by

$$\left\langle J_n^2 \left( \xi \sin \frac{\theta}{2} \right) \right\rangle = e^{-y} I_n(y) \quad , \tag{6.10}$$

where

$$y = \frac{0.01429}{\gamma_\theta^2} \quad ,$$

and

$$\gamma_\theta = \frac{\lambda_{meters}}{\sin \theta/2} \frac{B_{gauss}}{\sqrt{MT}} \quad . \tag{6.11}$$

Here, as before, $\xi = 4\pi R/\lambda$, and $\theta$ is the angle of scattering.

In order to see how Figures 5-1, 5-2, and 5-3 have to be modified to apply in this case, let us use Figure 5-1 as an example. The lines shown in that figure were computed for $\gamma = \lambda B/\sqrt{MT} = 0.124$, where $\lambda$ was the wavelength used in the backscatter experiment. The width of the line spectrum was approximately that of the spectrum given by

$$e^{-\left\{ \frac{f - f_T}{258\ (1/\lambda)\ \sqrt{T/M}} \right\}^2} \quad .$$

According to Equation (6.11), if $\lambda$ is the wavelength used at forward scatter at $\theta = \theta$, $\alpha = 0$, then the amplitudes of the lines of Figure 5-1 apply for $\gamma_\theta = (\lambda/\sin \frac{\theta}{2})\ B/\sqrt{MT} = 0.124$. The width of the line spectrum is given by that of the spectrum

$$e^{-\left\{ \frac{f - f_T}{258\ (1/\lambda)\ (\sin \theta/2)\ \sqrt{T/M}} \right\}^2} \quad .$$

With M assumed to be constant, we can thus say the following about the spectra of forward scatter for the special case $\alpha = 0$ :

i) If the strength of the magnetic field B and the temperature T are fixed, the spectrum of forward scatter at the wavelength $\lambda_{f.s.} = \lambda_{b.s.} \sin\frac{\theta}{2}$ is identical to the spectrum of backscatter at $\lambda_{b.s.}$.

ii) More generally, spectral lines are identical for

$$\gamma_\theta = \frac{\lambda}{\sin\theta/2} \quad \frac{B}{\sqrt{MT}} = \text{constant} \quad .$$

The width of the multiline spectrum is independent of the strength of the magnetic field. The spacing of the 3-db points of the envelope of the spectra is given according to Equation (2.2) by

$$0.424 \, \frac{\sin\theta/2}{\lambda} \quad \sqrt{\frac{T}{M}} \quad \text{kilocycles/sec} \quad .$$

In the general case of a scattering angle $\theta$ and an angle $\alpha$ between the magnetic field and a plane that is parallel to the line TR and perpendicular to the plane of propagation, the spectrum of scattering of one electron is as shown in Figure 6-2. According to the results of Section B of this chapter, the frequency $f_o$ of Figure 6-2 is not the transmitted frequency $f_T$ but the Doppler-shifted frequency,

$$f = f_T + 2\frac{v}{\lambda} \sin\alpha \, \sin\frac{\theta}{2}$$

$$= f_T + 2\frac{v}{\lambda_2} \quad . \qquad (6.12)$$

In Equation (6.12), v is the drift velocity of an electron along the magnetic field in that direction which yields a component along the negative direction of z .

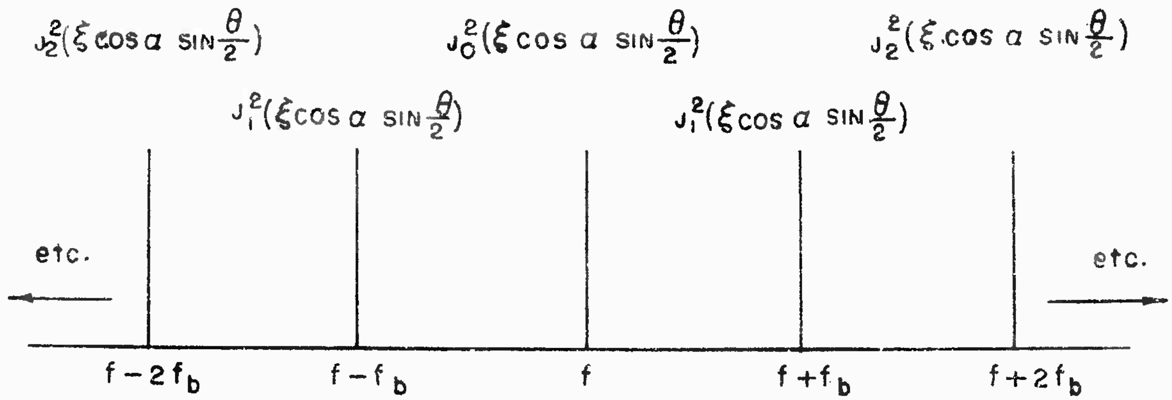


Figure 6-2. The spectrum of scattering resulting from one electron. $\xi = 4\pi R/\lambda$, where R is the electron gyromagnetic radius.

It is now important to note that although the resultant spectrum becomes smeared as a result of the Doppler shifts, each electron still has a line spectrum, as shown in Figure 6-2. We could thus imagine turning the smeared spectrum back into a line spectrum by shifting the spectrum of each electron along the frequency axis by the amount of its Doppler shift, but in the opposite direction. In such a corrected spectrum the fraction of total power at both the frequency $f_T + nf_b$ and $f_T - nf_b$ would according to Equations (5.10) and (6.2) be given by

$$\langle J_n^2(\xi \cos\alpha \sin\theta/2)\rangle = e^{-z} I_n(z) \quad , \tag{6.13}$$

where

$$z = \frac{0.01429}{\gamma_{\theta,\alpha}^2} \quad , \tag{6.14}$$

and

$$\gamma_{\theta,\alpha} = \frac{\lambda_{meters}}{\cos\alpha \sin\theta/2} \frac{B_{gauss}}{\sqrt{MT}}$$

$$= \frac{\lambda_1 B}{\sqrt{MT}} \tag{6.15}$$

The shape of the actual spectrum can be calculated by considering it to be a superposition of Gaussian spectra that are centered at each of the frequencies $f_T \pm n f_b$, $n = 0, 1, 2, \ldots$. The amplitudes of the component spectra are proportional to $e^{-z} I_n(z)$. Thus the component spectrum located at $f_T \pm n f_b$ is given by

$$p_n(f) = A \; e^{-z} I_n(z) \; e^{-\left\{\frac{f - f_T \mp n f_b}{258 \, (1/\lambda_2) \sqrt{T/M}}\right\}^2} , \qquad (6.16)$$

where

$$\lambda_2 = \frac{\lambda_{meters}}{\sin\alpha \, \sin\theta/2} ,$$

and $A$ is a constant of proportionality, which is the same for all component spectra.

The result, Equation (6.16), can be put in terms of the parameter $\gamma = \lambda B/\sqrt{MT}$ as follows: From Equation (5.16)

$$f_b = 1.525 \times 10^3 \, \frac{\gamma}{\lambda} \sqrt{\frac{T}{M}} \quad \text{cycles/sec} .$$

Thus

$$258 \, \frac{1}{\lambda_2} \sqrt{\frac{T}{M}} = 0.169 \, \frac{\sin\alpha \, \sin\frac{\theta}{2}}{\gamma} \, f_b ,$$

and

$$p_n(f) = A \; e^{-\frac{0.01429}{\gamma^2} \sin^2\frac{\theta}{2} \cos^2\alpha} \; I_n\left(\frac{0.01429}{\gamma^2} \sin^2\frac{\theta}{2} \cos^2\alpha\right)$$

$$e^{-\left\{\frac{f - f_T \mp n f_b}{0.169 \, \sin\alpha \, \sin\theta/2 \; f_b/\gamma}\right\}^2} . \qquad (6.17)$$

D. SAMPLE SPECTRA

Figures 6-3 to 6-5 show the effect on the spectrum of incoherent backscatter of increasing the angle $\alpha$ between the surfaces of constant phase and the magnetic field from zero to ten degrees. The results apply for

$$\gamma = \frac{\lambda_{meters}\ B_{gauss}}{\sqrt{MT}} = 0.0488 \quad . \qquad (6.18)$$

If we use $\lambda = 0.7$ meters (430 Mc/sec), and $\sqrt{M} = 1/43$ appropriate to the electrons, then $\gamma = 0.0488$ yields the ratio $B/\sqrt{T} = 1.62 \times 10^{-3}$, which might correspond to conditions existing at a height of about 3000 km above the surface of the earth. At that height the condition $4\pi l_D > \lambda$, where $l_D$ is the Debye length, might also be satisfied.

If, on the other hand, we set $\lambda = 12$ meters (25 Mc/s), and think in terms of incoherent scattering by electrons having the thermal characteristics of $O^+$ ions, then $\sqrt{M} = 4$, and $B/\sqrt{T} = 1.63 \times 10^{-2}$, which corresponds to low ionospheric heights. Of course, in this case the gyromagnetic frequency $f_b$, and thus the separation of the spectral lines, would be that of the $O^+$ ions (47.6 cps for $B = 0.5$ gauss). At the wavelength of 12 meters the parameter $\gamma = 0.0488$ would also apply at a height of about 2000 km, where the controling positive ions are protons.

The significant feature of Figures 6-3 to 6-5 is that they show a rather rapid smearing of the spectrum as $\alpha$ is increased. A curve was also calculated for $\alpha = 15^{\circ}$, but is not reproduced, since it would be indistinguishable from the Gaussian Doppler-shift curve that would apply in the absence of any magnetic field.

According to our assumptions, the smearing of the line spectrum is caused only by thermal drift motions of the particles along the magnetic field.

Since a reduction in temperature will reduce the velocity of such motions, we would expect the spectral smearing to be smaller if the temperature is reduced. This expectation is confirmed by a comparison of Figures 6-5 and 6-6. Figure 6-6 is computed for $\gamma = 0.069$ and thus, if $\lambda$, B, M are assumed to remain unchanged, the ratio of the temperature used in Figure 6-5 to that used in Figure 6-6 is $(0.069/0.0488)^2 = 2.0$. As the temperature is decreased there actually also exists an effect tending to oppose the resolution of the spectral lines: smaller thermal velocities imply smaller radii of gyration of the electrons; smaller radii of gyration, in turn, would give rise to weaker "sidebands."

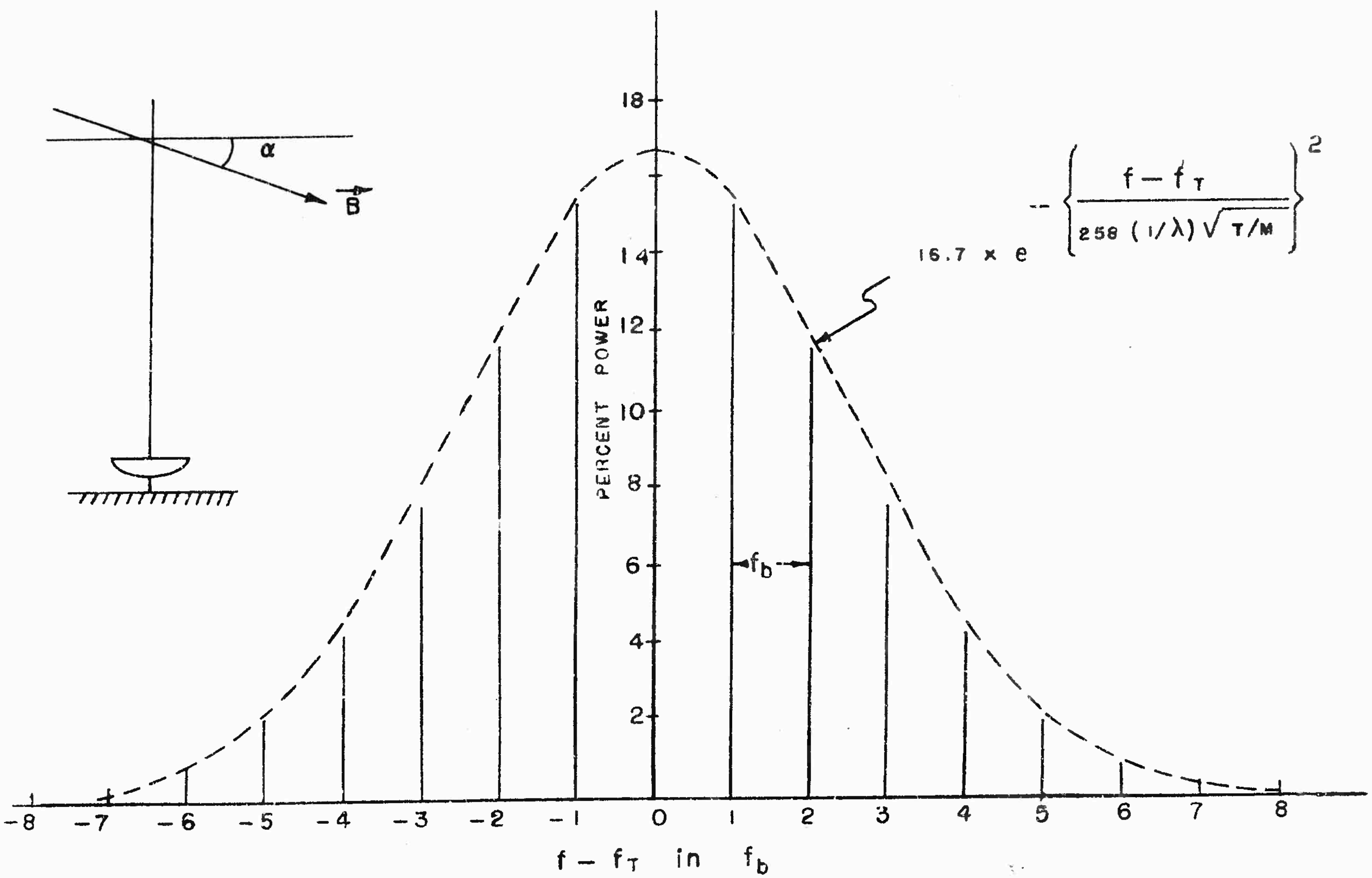


Figure 6-3. Line spectrum of backscatter with the radar beam normal to the magnetic field; i.e., with $\alpha = 0$. $\gamma = \lambda B/\sqrt{MT} = 0.0488$, and thus $f_b = 74.4\,(1/\lambda)\sqrt{T/M}$ cps.

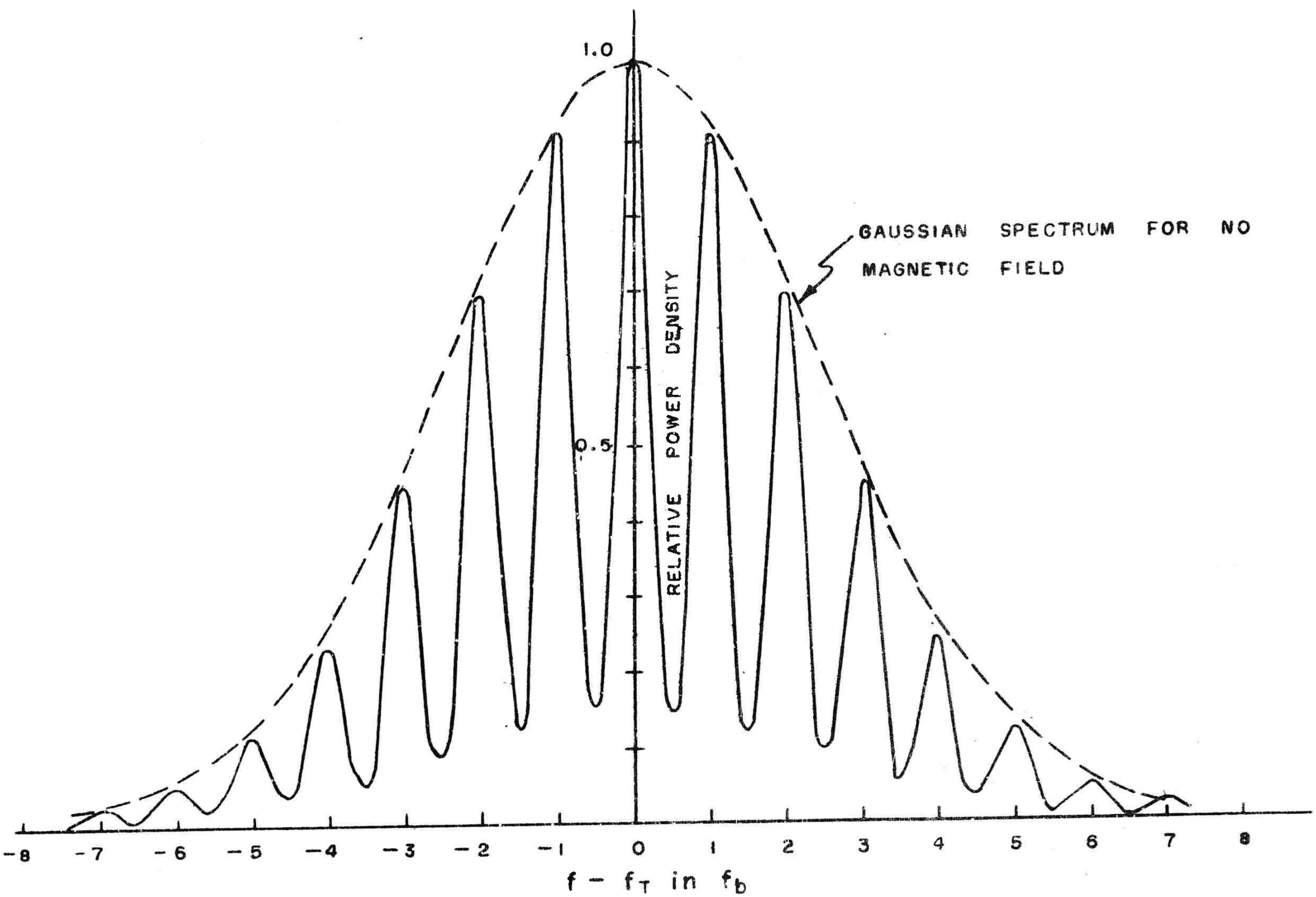


Figure 6-4. Spectrum of backscatter with $\alpha = 5^{\circ}$, $\gamma = 0.0488$.

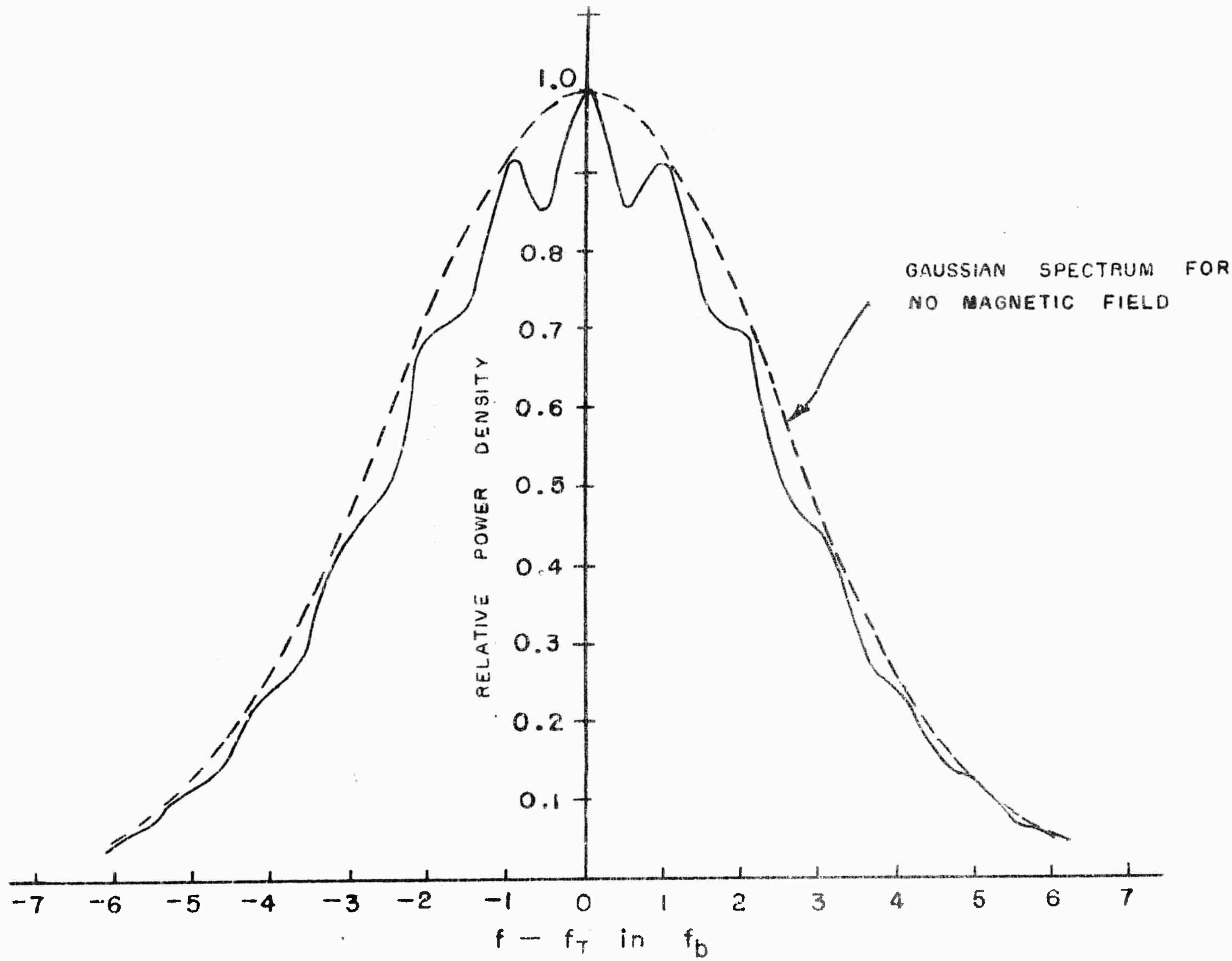


Figure 6-5. Spectrum of backscatter with $\alpha = 10^{\circ}$, $\gamma = 0.0488$ .

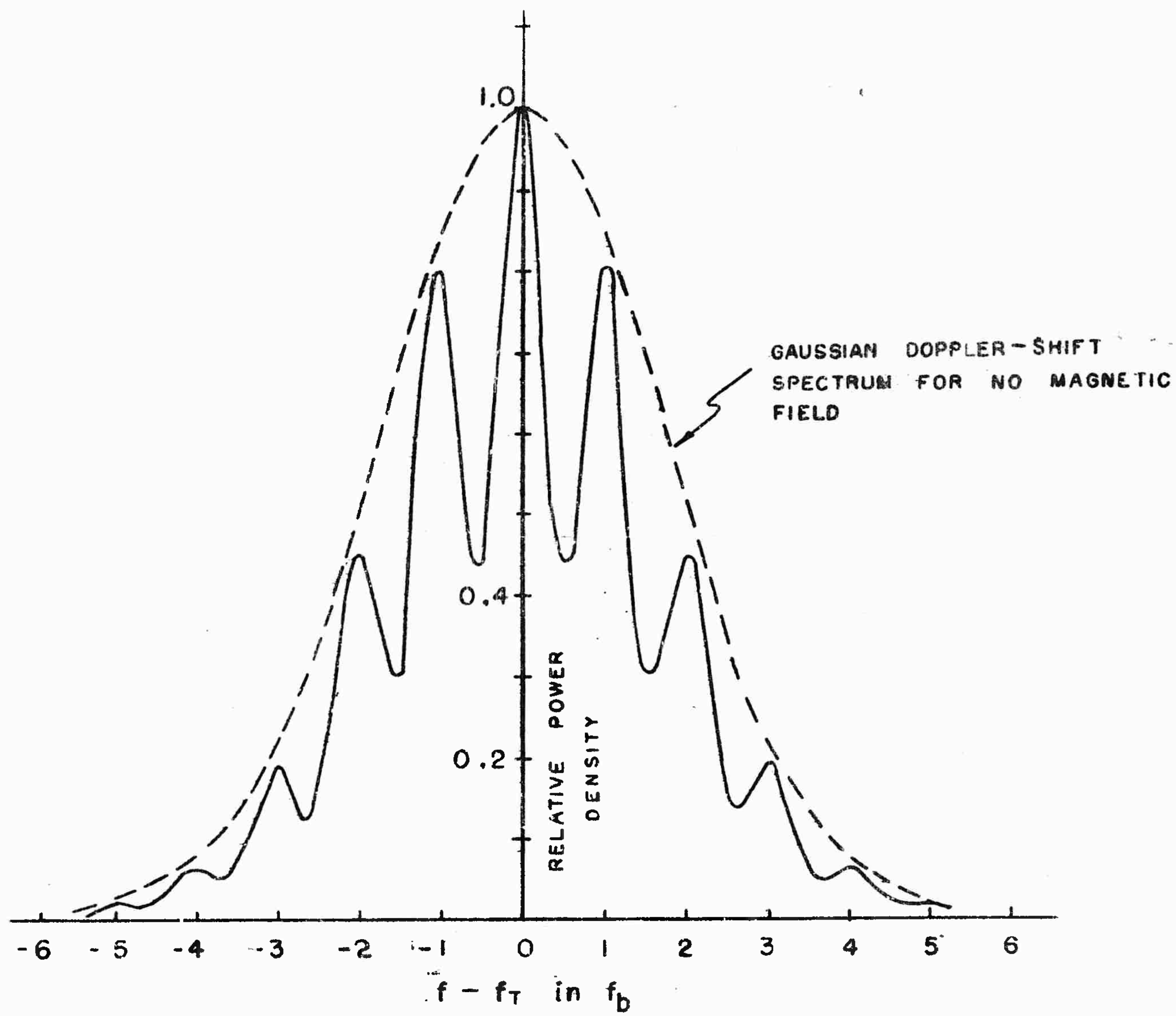


Figure 6-6. Spectrum of backscatter for $\alpha = 10^{\circ}$, $\gamma = 0.069$. $f_b = 105.2\ (1/\lambda)\sqrt{T/M}$ cps.

## VII. APPLICATION TO SCATTERING IN THE UPPER IONOSPHERE

### A. SCATTERING AT SMALL SCALES IN PRESENCE OF MAGNETIC FIELD

It is a consequence of recent theoretical work,[6, 8-10] thus far supported by experimental evidence,[5, 6, 13] that the spectra derived in Chapters IV-VI, if evaluated for the atomic weight of the electrons, should not be expected to apply unless the scale of scattering is smaller than both the Debye length and the electron mean free path.

It also appears that the magnitude of the electron gyro radius relative to the Debye scale is of some importance. For our analysis to apply, it is necessary for the electrons to gyrate unperturbed in the magnetic field. (By "unperturbed" we mean that any deviations from a helical path in directions perpendicular to the magnetic field should be small compared to the scale of scattering.) If the electron gyro radius is large compared to a Debye length, each gyration will carry an electron through many Debye volumes with the result that the chances for introducing perturbations (and thus of additional smearing of spectral lines) are increased. Electron drift sideways along the magnetic lines of force through many Debye distances may have the same effect. In the height range from about 1000 km to 10,000 km the electron gyro radius is of the same order of magnitude as the Debye length. Around the peak of the F layer the Debye length is considerably smaller than the electron gyro radius.

It is our opinion that much could be learned about the actual motion of electrons in the outer ionosphere by comparing our spectra with experimental results. According to Figure 7-1, such an experiment would have to be performed at a relatively small wavelength. An experiment at three centimeters with a steerable antenna that could be directed normal to the magnetic field would be extremely interesting.

With respect to Figure 7-1, it should be said that at the moment not very much is known about electron densities above the F layer. Even

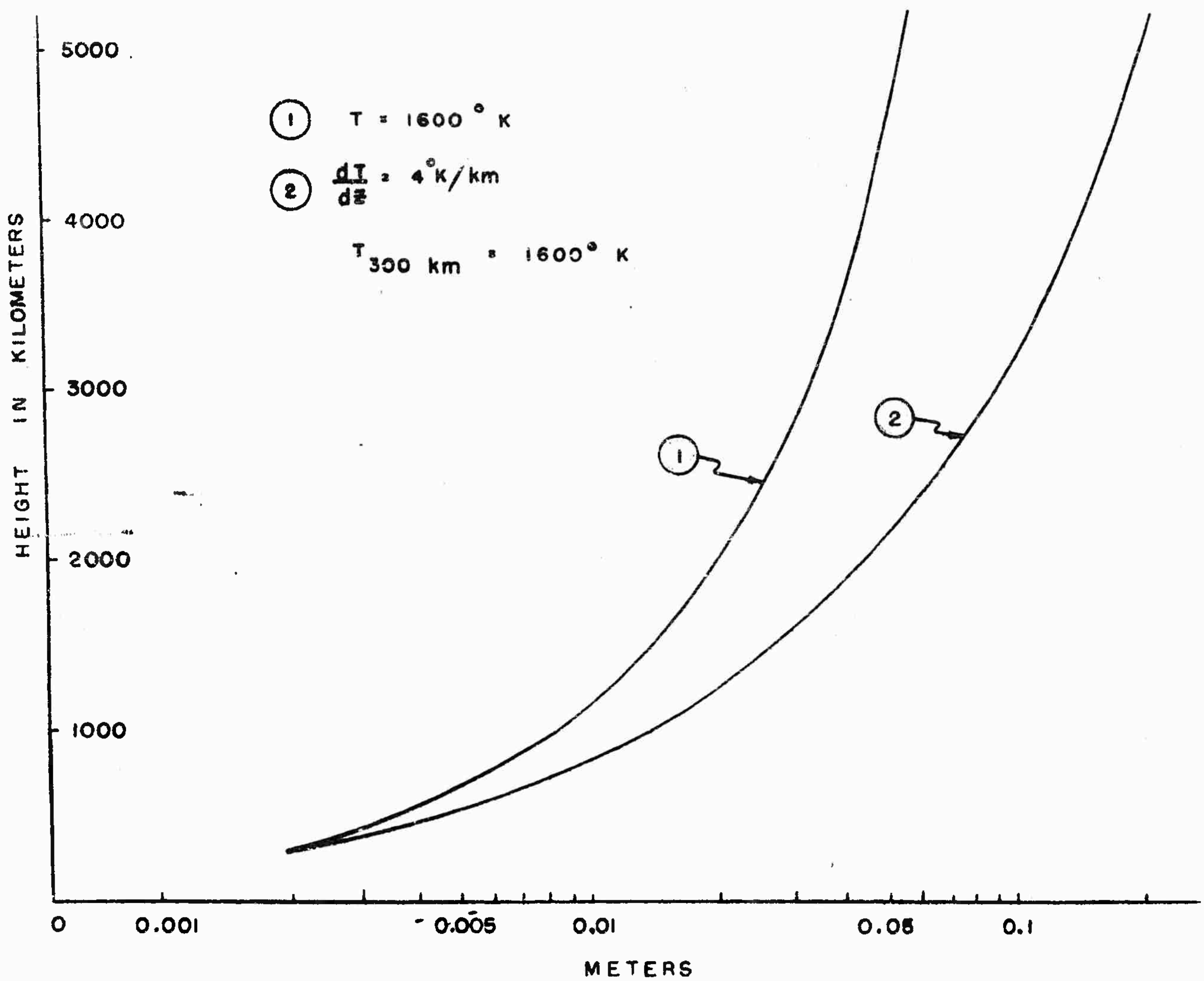


Figure 7-1. **Variation of the Debye length with height for two models of the outer ionosphere.**

less is known about temperatures at such heights. The electron densities used in our calculations were taken from a curve proposed for an ''average'' ionosphere at summer noon, middle latitudes, and sunspot maximum.[17] Smaller values of electron density would make the Debye length larger ($l_D = 69\sqrt{T/N}$ meters, if N is in per cubic meter).

As to the temperature, one current view is that an isothermal layer exists above the level where collisions become rare; the estimates for the temperature of that layer range from about $1000^\circ$ K to $2000^\circ$ K. The other idea, proposed by Chapman,[18, 19] is that the ionospheric temperature increases continually upwards until it reaches, at a few earth's radii, the temperature ($\doteq 200,000^\circ$ K) of a hypothetical solar interplanetary atmosphere. Figure 7-1 gives curves of Debye length for both temperature models.

## B. SPECTRA IN PRESENCE OF MAGNETIC FIELD AT LARGE SCALES OF SCATTERING

We will now discuss the statement, made in the Introduction, that our spectra might represent a useful first approximation even at scales of scattering much larger than the Debye length, if the gyromagnetic frequency of the positive ions is substituted for that of the electrons. One might now ask the question: Since only the electrons are supposed to do the scattering, why, then, should the gyromagnetic frequency of the ions have an effect on the received spectrum?

According to Equation (3.13), changes in the signal of backscatter are due to changes in electron density. It is thus of no consequence by what motions of individual electrons this change in distribution of electron density is brought about. Let us suppose the ions to gyrate freely, and let us also suppose the electron <u>density</u> to be equal to the ion density at each instant.

The resulting signal of scattering would be indistinguishable from that which would result if each gyrating ion were accompanied by one and the same free electron.

Now, since an ionized medium tends to remain neutral over scales much larger than the Debye length, the electron density will indeed follow the ion density over such scales. If the positive ions could be thought to gyrate freely, the signal of scattering at large wavelengths would be as one would expect if the electrons moved with the thermal characteristics of the positive ions.

For the ion gyromagnetic frequency to assume at large scales the role played by the electron gyromagnetic frequency at small scales, the crucial requirement is that the ions gyrate unperturbed. Such motion would recreate periodically the distribution of irregularities of ion density in directions perpendicular to that of the magnetic field. The large-scale irregularities of electron density would then assume the same periodicity. We should note in this connection that some electrostatic fields are set up in the process of maintaining neutrality. As has been shown by Fejer,[8] in the absence of a magnetic field these electrostatic forces have an effect, although small, on both ion distribution as well as on their motion. This is the reason why the spectrum deviates from the ion thermal Doppler spectrum. In the presence of a magnetic field even some additional internal electric fields might exist because the electrons, which have a relatively small gyro radius, have difficulty in following the ion density irregularities across the magnetic lines of force. However, as was first suggested by Bowles,[6] it is still possible that, at large scales of scattering, enhanced spectral content will be observed at frequencies determined by the gyromagnetic frequency of the ions, if the narrow beam of a radar is directed perpendicular, or at a small angle, to the magnetic field. On the other hand, it is certain that the smearing of the spectrum will be much more severe than that shown in Figures 6-4, 6-5, and 6-6. Also, it is probable that the envelope of such a spectrum will be approximated better by the shape of the ''saddle-spectrum'' derived by Fejer and others than by the shape of the Gaussian Doppler spectrum shown in all of our figures.

## APPENDIX I. DERIVATION OF PHASE-MODULATION SPECTRUM

Expression (4.6) can be expanded as follows. Let

$$2kR_j \equiv \xi_j \quad , \qquad \omega_T t + \psi_j \equiv x \quad , \qquad \omega_b t + \theta_j \equiv y \quad .$$

Then

$$\sin\{\omega_T t - 2kR_j \sin(\omega_b t + \theta_j) + \psi_j\}$$

$$= \sin\{x - \xi_j \sin y\}$$

$$= \sin x \cos(\xi_j \sin y) - \cos x \sin(\xi_j \sin y)$$

$$= \sin x \left\{ J_o(\xi_j) + 2 \sum_{n=1}^{\infty} J_{2n}(\xi_j) \cos 2ny \right\}$$

$$- \cos x \left\{ 2 \sum_{n=1}^{\infty} J_{2n-1}(\xi_j) \sin(2n-1)y \right\}$$

Here $J_n(\xi_j)$ denotes the Bessel function of the first kind, order $n$, and argument $\xi_j$. We can now make use of the relations

$$2 \sin x \cos 2ny = \sin(x + 2ny) + \sin(x - 2ny) \quad ,$$

and

$$2 \cos x \sin(2n-1)y = \sin\{x + (2n-1)y\} - \sin\{x - (2n-1)y\} \quad .$$

Thus

$$\sin \{x - \xi_j \sin y\}$$

$$= J_o(\xi_j) \sin x + \sum_{n=1}^{\infty} J_{2n}(\xi_j) \{\sin(x + 2ny) + \sin(x - 2ny)\}$$

$$- \sum_{n=1}^{\infty} J_{2n-1}(\xi_j) \{\sin\left(x + (2n-1)y\right) - \sin\left(x - (2n-1)y\right)\}$$

$$= J_o(\xi_j) \sin x + \sum_{n=1}^{\infty} (-1)^n J_n(\xi_j) \sin(x + ny) + \sum_{n=1}^{\infty} J_n(\xi_j) \sin(x - ny) .$$

That is,

$$\sin \{\omega_T t - 2kR_j \sin(\omega_b t + \theta_j) + \psi_j\}$$

$$= J_o(2kR_j) \sin(\omega_T t + \psi_j) + \sum_{n=1}^{\infty} (-1)^n J_n(2kR_j) \sin\{(\omega_T + n\omega_b)t + n\theta_j + \psi_j\}$$

$$+ \sum_{n=1}^{\infty} J_n(2kR_j) \sin\{(\omega_T - n\omega_b)t - n\theta_j + \psi_j\} .$$

## APPENDIX II. ANALYSIS OF VALIDITY OF THE GAUSSIAN APPROXIMATION

Let us now investigate in greater detail the conditions under which

$$\langle J_n^2(\xi) \rangle \doteq \frac{1}{\sqrt{2\pi x}} \quad e^{-n^2/2x} \quad . \tag{II. 1}$$

We have stated in Section V-B3 that Equation (II. 1) will hold if the series for $e^{-n^2/2x}$ converges so rapidly that

$$e^{-n^2/2x} \doteq \sum_{m=o}^{\mathcal{M}} (-1)^m \frac{1}{m!} \left( \frac{n^2}{2x} \right)^m \quad , \tag{II. 2}$$

where $\mathcal{M}$ satisfies with sufficient stringency the requirement $\mathcal{M}^2 << n^2$ to make

$$\left[ 1 - \frac{1^2}{4n^2} \right] \left[ 1 - \frac{3^2}{4n^2} \right] \quad . \; . \; . \quad \left[ 1 - \frac{(2\mathcal{M} - 1)^2}{4n^2} \right] \doteq 1 \quad .$$

Since we are dealing with an alternating series, Equation (II. 2) will hold if

$$\frac{1}{\mathcal{M}!} \left( \frac{n^2}{2x} \right)^{\mathcal{M}} << \eta\,(n, x) \quad , \tag{II. 3}$$

where $\eta\,(n, x) \equiv e^{-n^2/2x}$ is a number. In place of $\mathcal{M}^2 << n^2$ write

$$\mathcal{M}^2 = k^2 n^2 \quad ,$$

where $k^2 << 1$. Inequality (II. 3) then becomes

$$\eta\,(n, x)\, x^{kn} >> \frac{n^{2kn}}{2^{kn}\,(kn)\,!} \quad .$$

Write this as

$$p \eta x^{kn} \geqq \frac{n^{2kn}}{2^{kn} (kn)!} ,$$

where $0 < p \ll 1$. By Stirling's formula

$$(kn)! \doteq (2kn\pi)^{\frac{1}{2}} (kn)^{kn} e^{-kn} .$$

We thus need

$$x^{kn} \geqq p^{-1} \eta^{-1} \frac{n^{2kn}}{2^{kn} (2\pi kn)^{\frac{1}{2}} (kn)^{kn}} e^{kn} ,$$

or

$$x \geqq p^{-1/kn} \eta^{-1/kn} \frac{e}{2} \frac{n^2}{(2\pi kn)^{1/(2kn)} kn}$$

$$\doteq 1.4 \, p^{-1/kn} \frac{n \, e^{n/2kx}}{k(2\pi kn)^{1/(2kn)}} . \qquad \text{(II. 4)}$$

The preceding involves $x$ on both sides of the equation so that it has to be solved by trial and error. The results of such a solution are given in Figure II-1 for $k = 0.1$, $p = 0.01$. Actual calculations show that Equation (II. 1) is a good approximation in the range $20 \leqq n \leqq 40$ for $x$ as low as 400, which is about a third of $x_{min}$ of Figure II-1.

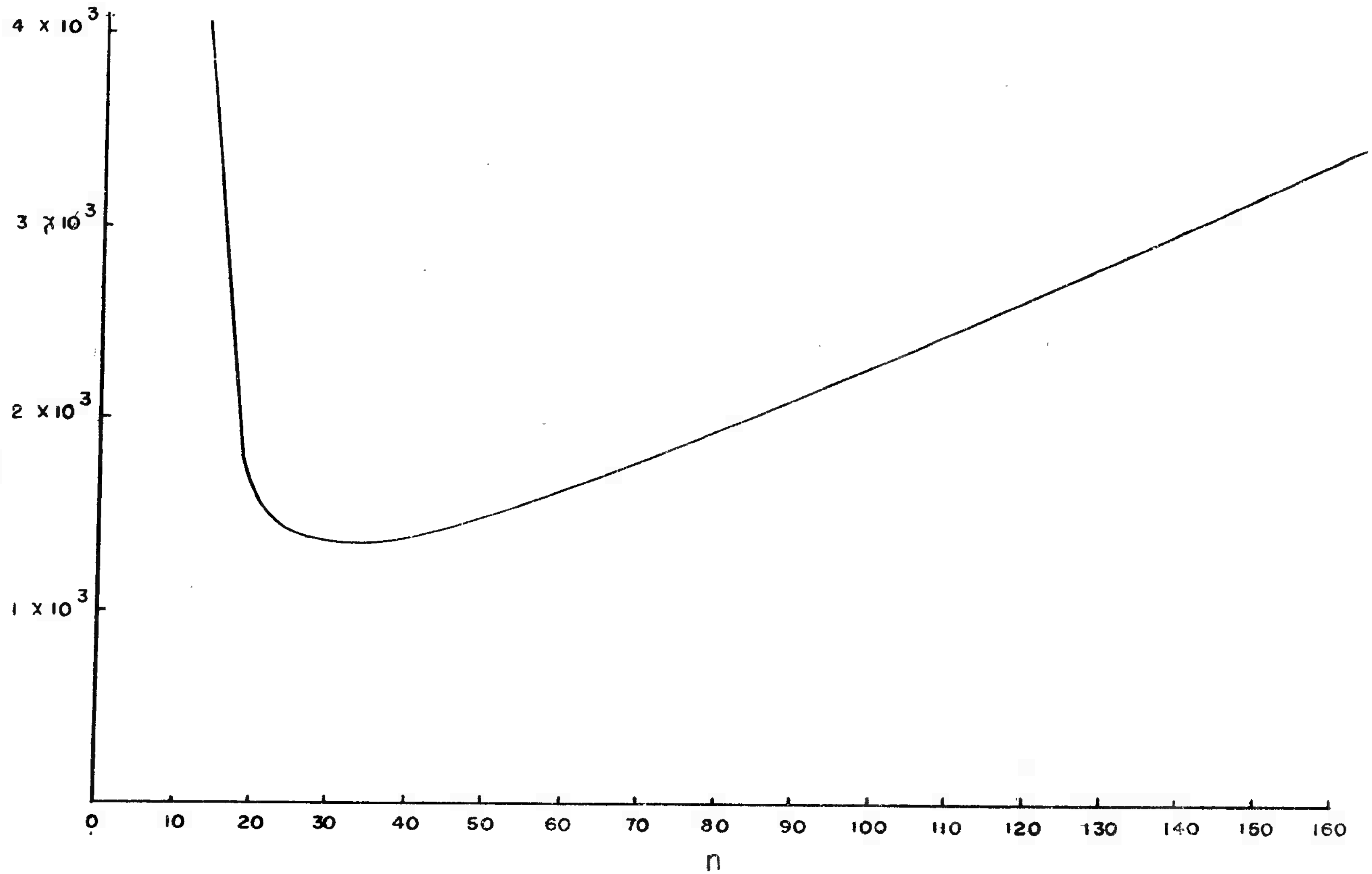


Figure II-1. Minimum value of x versus n for the Gaussian approximation to hold; $k = 0.1$, $p = 0.01$.

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