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SPIN PRECESSION OF A CLASSICAL PARTICLE*

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1. INTRODUCTION

This paper is devoted to a discussion of the equation of motion for the intrinsic angular momentum of a classical particle placed in a homogeneous external electromagnetic field.[1] The interest in this problem has a number of facets. The first is practical in that the classical equation of motion is identical with the equation of motion for the expectation value of the intrinsic spin of a quantum particle.[2] Thus, for example, the results of this paper are applicable to the discussion of the motion of the spin of an electron in an external electromagnetic field. Another interest in these classical equations stems from Kramers, who in his book on quantum mechanics showed that for a particular choice of the equation of motion one is led to conclude that the gyromagnetic ratio for a classical particle is the same as that for the Dirac electron.[3] Although, as Kramers himself states, the equation of motion he discusses is a special case of the most general equation of motion which was first introduced much earlier by Frenkel,[4] the belief still seems to persist that the gyromagnetic ratio of a classical particle is uniquely implied by the

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equation of motion.[5] Another aspect of interest is in the choice of the covariant representation of the intrinsic angular momentum. Kramers and Frenkel, for example, choose to represent the intrinsic angular momentum by an antisymmetric second rank tensor, while more recently it has become fashionable to represent intrinsic angular momentum by a four-vector.[6] The two representations are, of course, equivalent.

In Section 2 we first adopt the antisymmetric tensor representation for the intrinsic angular momentum and then write down the most general equation of motion consistent with a set of reasonable restrictions. The resulting equation is essentially identical with that of Frenkel. We then repeat the discussion using the four-vector representation and show the equivalence of the two approaches. In Section 3 we use the equation of motion derived in Section 2 to obtain an explicit expression for the angular velocity of precession of the intrinsic spin.

In the following discussion we shall adopt the usual four-dimensional notation as given, for example, in Møller's book.[7] In addition we shall use cgs "natural" units for which c = 1. In the formulas which are of practical utility we put the c's back in.

2. THE EQUATION OF MOTION

The non-relativistic equation of motion for a particle with charge e, mass m, and intrinsic angular momentum $\vec{S} = (S_1, S_2, S_3)$ is

$$\frac{d\vec{S}}{dt} = \frac{ge}{2m}\vec{S} \times \vec{B} + \frac{fe}{2m}\vec{S} \times \vec{E} . \qquad (2.1)$$

Here we have assumed the particle has a magnetic dipole moment

$$\vec{m} = \frac{ge}{2m}\vec{S} , \tag{2.2}$$

and an electric dipole moment

$$\vec{d} = \frac{fe}{2m}\vec{S} . \tag{2.3}$$

Higher multipole couplings do not appear because the electric field $\vec{E}$ and the magnetic field $\vec{B}$ are assumed homogeneous.

Following Frenkel,[4] we introduce an antisymmetric tensor $(S_{\kappa\lambda})$, which in the particles rest frame has the form:[8]

$$(S'_{\kappa\lambda}) = \begin{pmatrix} 0 & S_3 & -S_2 & 0 \\ -S_3 & 0 & S_1 & 0 \\ S_2 & -S_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{2.4}$$

We want now to write down the most general equation of motion for $(S_{\kappa\lambda})$ which is form-invariant under Lorentz transformations and which is a generalization of the non-relativistic Eq. (2.1). In doing this we will be guided by the following requirements.

1. The differential equation should be of first degree in the time, or rather the proper time, τ, whose increment is related to the increment of laboratory time by

$$d\tau = \sqrt{1-v^2}\, dt, \tag{2.5}$$

where v is the velocity of the particle.

2. The equation should be first order in the electromagnetic field tensor,

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix} . \qquad (2.6)$$

In particular we assume the equation does not involve derivatives of the electromagnetic fields.

3. The equation should be homogeneous in $S_{\kappa\lambda}$.

4. The equation should be form-invariant under the improper Lorentz transformations of time reversal T and spatial inversion P. It is in regard to this requirement that we distinguish the magnetic dipole and electric dipole couplings. Thus, the non-relativistic equation is form-invariant under T and P only if we assume that the coupling constant g, which is just the Lande g-factor, does not change sign under either T or P and that the coupling constant f changes sign under T as well as under P. In other words, according to Watanabe's classification g is a regular scalar while f is a first kind pseudoscalar.[9]

5. Finally there are two dynamical requirements upon the equation of motion. The first is that the intrinsic angular momentum $\vec{S}$ must be unchanged in magnitude during the motion, i.e.,

$$S_{\kappa\lambda} S_{\kappa\lambda} = S_1^2 + S_2^2 + S_3^2 \qquad (2.7)$$

is a constant of the motion. The other requirement is that during the motion $S_{\kappa\lambda}$ must continue to have the form (2.4) in the rest frame of the particle. In other words, the requirement

$$S_{\kappa\lambda} U_\lambda \equiv 0 \qquad (2.8)$$

must be consistent with the equation of motion. Here U_λ is the proper four-velocity vector:

$$U_\kappa = \frac{dX_\kappa}{d\tau} = (\gamma\vec{v}, i\gamma), \quad \gamma = (1-v^2)^{-1/2}. \qquad (2.9)$$

These requirements are not all independent, e.g., requirement (2.7) is implied by the remaining requirements, but they do form a consistent set which uniquely determines the equation of motion.

The covariant quantities which can be used to form the equation of motion of consistent with the above requirements are the field tensor $F_{\mu\nu}$, the proper velocity of the particle U_κ, and the dual of the field tensor:

$$(F^*_{\mu\nu}) = \begin{pmatrix} 0 & -E_3 & +E_2 & -iB_1 \\ +E_3 & 0 & -E_1 & -iB_2 \\ -E_2 & +E_1 & 0 & -iB_3 \\ +iB_1 & +iB_2 & +iB_3 & 0 \end{pmatrix}. \qquad (2.10)$$

The reader can readily supply the arguments which show that the most general equation of motion for $S_{\kappa\lambda}$ fulfilling the requirements 1 to 4 has the form:[10]

$$\begin{aligned} \dot{S}_{\kappa\lambda} = & \frac{ge}{2m}(S_{\kappa\alpha}F_{\lambda\alpha} - S_{\lambda\alpha}F_{\kappa\alpha}) + \frac{g'e}{2m}(U_\kappa S_{\lambda\alpha}F_{\alpha\beta}U_\beta - U_\lambda S_{\kappa\alpha}F_{\alpha\beta}U_\beta) \\ & - \frac{fe}{2m}(S_{\kappa\alpha}F^*_{\lambda\alpha} - S_{\lambda\alpha}F^*_{\kappa\alpha}) - \frac{f'e}{2m}(U_\kappa S_{\lambda\alpha}F^*_{\alpha\beta}U_\beta - U_\lambda S_{\kappa\alpha}F^*_{\alpha\beta}U_\beta). \end{aligned} \qquad (2.11)$$

Here the fact that g and g' must be regular scalars while f and f' are first kind pseudoscalars follows from the fact that $F_{\mu\nu}$ is a second kind pseudotensor while $F^*_{\mu\nu}$ is a third kind pseudotensor.[9] The dynamical requirement (2.8) implies that for allowed solutions of (2.11),

$$\frac{d}{d\tau}(S_{\kappa\lambda}U_\lambda) = \dot{S}_{\kappa\lambda}U_\lambda + S_{\kappa\lambda}\dot{U}_\lambda \qquad (2.12)$$

must vanish identically. Using (2.11) together with the equation of motion of the particle itself:[11]

$$\dot{U}_\lambda = \frac{e}{m} F_{\lambda\alpha}U_\alpha \qquad (2.13)$$

we find that (2.12) becomes:

$$\frac{d}{d\tau}(S_{\kappa\lambda}U_\lambda) = (\frac{g'}{2} - \frac{g}{2} + 1)S_{\kappa\lambda}\dot{U}_\lambda - (f' - f)\frac{e}{2m} S_{\kappa\alpha}F^*_{\alpha\beta}U_\beta + O_{\kappa\alpha}S_{\alpha\lambda}U_\lambda, \qquad (2.14)$$

where

$$O_{\kappa\alpha} = \frac{ge}{2m} F_{\kappa\alpha} + \frac{g'e}{2m} U_\kappa U_\beta F_{\beta\alpha} - \frac{fe}{2m} F^*_{\kappa\alpha} - \frac{f'e}{2m} U_\kappa U_\beta F^*_{\beta\alpha} . \qquad (2.15)$$

Providing the first two terms on the right hand side of (2.14) vanish this equation is a homogeneous equation, first order in time for, the four vector $S_{\kappa\lambda}U_\lambda$. Such an equation clearly has the property that a solution which vanishes at any instant of time must vanish identically for all times. Hence the dynamical requirement (2.8) will be consistent with the equation of motion (2.11) providing:

$$g' = g - 2 , \quad f' = f. \qquad (2.16)$$

With these values for the coupling constants this equation becomes:

$$\dot{S}_{\kappa\lambda} = \frac{e}{2m}\Big\{ g(S_{\kappa\alpha}F_{\lambda\alpha} - S_{\lambda\alpha}F_{\kappa\alpha}) + (g - 2)(U_\kappa S_{\lambda\alpha}F_{\alpha\beta}U_\beta - U_\lambda S_{\kappa\alpha}F_{\alpha\beta}U_\beta) \\ - f(S_{\kappa\alpha}F^*_{\lambda\alpha} - S_{\lambda\alpha}F^*_{\kappa\alpha} + U_\kappa S_{\lambda\alpha}F^*_{\alpha\beta}U_\beta - U_\lambda S_{\kappa\alpha}F^*_{\alpha\beta}U_\beta)\Big\}, \qquad (2.17)$$

which is the desired equation of motion. It can readily be verified that (2.7) is a constant of the motion, at least for those solutions for which (2.8) holds. If one sets $f = 0$ (no electric dipole coupling), the resulting equation is identical with that of Frenkel.[4] If in addition one sets $g = 2$, the resulting equation is that discussed by Kramers.[3]

The choice of an antisymmetric tensor to represent intrinsic angular momentum is not unique, we could equally well have introduced a four-vector S_κ, which in the rest frame of the particle has the form:

$$(S'_\kappa) = (\vec{S}, 0). \qquad (2.18)$$

The requirements upon the equation of motion for S_κ are the same as those upon the equation for $S_{\kappa\lambda}$ except that the dynamical requirement (5) becomes

(5') The condition that the magnitude of $\vec{S}$ be constant is that

$$S_\kappa S_\kappa = S_1^2 + S_2^2 + S_3^2 \qquad (2.19)$$

be a constant of the motion. The condition that S_κ continue to have the form (2.18) in the rest frame is that

$$S_\kappa U_\kappa \equiv 0 \qquad (2.20)$$

must be consistent with the equation of motion.

The most general equation for S_κ consistent with the requirements 1-4 is:

$$\dot{S}_\kappa = \frac{ge}{2m} F_{\kappa\alpha}S_\alpha + \frac{g'e}{2m} U_\kappa F_{\alpha\beta}U_\alpha S_\beta - \frac{fe}{2m} F^*_{\kappa\alpha}S_\alpha - \frac{f'e}{2m} U_\kappa F^*_{\alpha\beta}U_\alpha S_\beta \tag{2.21}$$

If we now impose the requirement (2.20) we find

$$\begin{aligned}\frac{d}{d\tau} S_\kappa U_\kappa &= \dot{S}_\kappa U_\kappa + S_\kappa \dot{U}_\kappa \\ &= (g - 2 - g')\frac{e}{2m} F_{\kappa\alpha}U_\kappa S_\alpha \\ &+ (f' - f)\frac{e}{2m} F^*_{\kappa\alpha}S_\alpha U_\kappa .\end{aligned} \tag{2.22}$$

Hence (2.20) is consistent with the equation of motion providing

$$g' = g - 2 , \quad f' = f . \tag{2.23}$$

With this choice of coupling constants the equation for S_κ becomes

$$\dot{S}_\kappa = \frac{e}{2m} \left\{ g\, F_{\kappa\alpha}S_\alpha + (g - 2)\; U_\kappa F_{\alpha\beta}U_\alpha S_\beta - f\, F^*_{\kappa\alpha}S_\alpha - f\, U_\kappa F^*_{\alpha\beta}U_\alpha S_\beta \right\} . \tag{2.24}$$

This is the equation of motion obtained by Bargmann et al.[1] Again, one can verify that (2.19) is a constant of the motion providing (2.20) is satisfied.

The question still remains as to the relation between the four-vector S_κ and the antisymmetric tensor $S_{\kappa\lambda}$. It is easy to show that the following pair of relations are inverses of one another and reduce to the proper relations in the rest frame of the particle;[12]

$$S_\kappa = -\frac{i}{2} e_{\kappa\lambda\mu\nu}S_{\lambda\mu}U_\nu , \quad S_{\kappa\lambda} = -i\, e_{\kappa\lambda\mu\nu}S_\mu U_\nu \tag{2.25}$$

Again it is a straightforward matter using these relations to derive the equation of motion for $S_{\kappa\lambda}$ from that for S_κ or vice versa. Hence the two approaches are completely equivalent.

3. THE EQUATION OF MOTION FOR $\vec{S}$

The equations of motion obtained in the previous section, while quite general, are not in the most convenient form for the discussion of the motion of the intrinsic spin, $\vec{S}$. In order to obtain an explicit equation of motion for $\vec{S}$ we must introduce the Lorentz transformation, $L_{\kappa\lambda}$, from the rest frame to the laboratory frame. This must be a pure Lorentz transformation, without rotations, which has the property that

$$U_\kappa = L_{\kappa\lambda} W_\lambda , \qquad (3.1)$$

where U_κ is the proper velocity of the particle in the laboratory frame, given by (2.9), and

$$(W_\kappa) = (0, i) \qquad (3.2)$$

is the proper velocity in the rest frame. The transformation which fulfills these requirements is

$$L_{\kappa\lambda} = \delta_{\kappa\lambda} + \frac{1}{1+\gamma}\left[W_\kappa U_\lambda - (1+2\gamma)U_\kappa W_\lambda + W_\kappa W_\lambda + U_\kappa U_\lambda\right] , \qquad (3.3)$$

as can be verified by taking $\vec{v}$ along one of the coordinate axes and using the fact that

$$W_\kappa U_\kappa = - \gamma. \qquad (3.4)$$

The four-vector S_κ, whose equation of motion expressed in the laboratory frame is (2.24) is related to S'_κ, which has the form (2.18), by

$$S_\kappa = L_{\kappa\lambda} S'_\lambda \,. \tag{3.5}$$

Taking the derivative with respect to proper time of both sides of this equation and using (2.24) we get

$$\begin{aligned} L_{\kappa\lambda}\dot{S}'_\lambda = & - \dot{L}_{\kappa\lambda} S'_\lambda - \frac{e}{m} U_\kappa F_{\nu\lambda} U_\nu S_\lambda \\ & + \frac{ge}{2m}\left[F_{\kappa\lambda} S_\lambda + U_\kappa F_{\nu\lambda} U_\nu S_\lambda\right] \\ & - \frac{fe}{2m}\left[F^*_{\kappa\lambda} S_\lambda + U_\kappa F^*_{\nu\lambda} U_\nu S_\lambda\right] . \end{aligned} \tag{3.6}$$

Using (3.5) to express the right hand side of this equation in terms of S'_κ and then using the orthogonality property of $L_{\kappa\lambda}$:

$$L_{\kappa\mu} L_{\kappa\nu} = \delta_{\mu\nu} \,, \tag{3.7}$$

we find

$$\begin{aligned} \dot{S}'_\kappa = & \left\{ - L_{\nu\kappa}\dot{L}_{\nu\lambda} - \frac{e}{m} W_\kappa F'_{\nu\lambda} W_\nu + \frac{ge}{2m}\left[F'_{\kappa\lambda} + W_\kappa F'_{\nu\lambda} W_\nu\right] \right. \\ & \left. - \frac{fe}{2m}\left[F^{*\prime}_{\kappa\lambda} + W_\kappa F^{*\prime}_{\nu\lambda} W_\nu\right]\right\} S'_\lambda \,. \end{aligned} \tag{3.8}$$

Here

$$F'_{\kappa\lambda} = L_{\mu\kappa} L_{\nu\lambda} F_{\mu\nu} \,, \quad F^{*\prime}_{\kappa\lambda} = L_{\mu\kappa} L_{\nu\lambda} F^*_{\mu\nu} \,, \tag{3.9}$$

are the field tensor and its dual evaluated in the rest frame. These tensors have the form (2.6) and (2.10) resp. with[13]

$$\begin{aligned} \vec{E}' &= \gamma(\vec{E} + \vec{v} \times \vec{B}) - \frac{\gamma - 1}{v^2} \vec{v} \cdot \vec{E} \, \vec{v} \\ \vec{B}' &= \gamma(\vec{B} - \vec{v} \times \vec{E}) - \frac{\gamma - 1}{v^2} \vec{v} \cdot \vec{B} \, \vec{v} \,. \end{aligned} \tag{3.10}$$

We can now express $\dot{L}_{\nu\lambda}$ in terms of $F'_{\kappa\lambda}$ by taking the derivative of (3.3) with respect to proper time, using (2.13) to express $\dot{U}_\kappa$ in terms of $F_{\kappa\lambda}$, and finally using

$$F_{\kappa\lambda} = L_{\kappa\mu} L_{\lambda\nu} F'_{\mu\nu} \ , \tag{3.11}$$

which is the inverse of (3.9). Working all this out we find we can write

$$\begin{aligned} - L_{\nu\kappa}\dot{L}_{\nu\lambda} - \frac{e}{m} W_\kappa F'_{\nu\lambda} W_\nu = \frac{e}{m}\frac{1}{1+\gamma} \Big\{ & (U_\kappa F'_{\lambda\nu} W_\nu - U_\lambda F'_{\kappa\nu} W_\nu) \\ & - \gamma\, W_\kappa F'_{\lambda\nu} W_\nu + (1+2\gamma)\, W_\lambda F'_{\kappa\nu} W_\nu \Big\} \ . \end{aligned} \tag{3.12}$$

When this expression is put into (3.8) the last term vanishes since $W_\lambda S'_\lambda = 0$ and we get:

$$\begin{aligned} \dot{S}'_\kappa = \frac{e}{m} \Big\{ & \frac{1}{1+\gamma} \Big[U_\kappa F'_{\lambda\nu} W_\nu - U_\lambda F'_{\kappa\nu} W_\nu - \gamma\, W_\kappa F'_{\lambda\nu} W_\nu \Big] \\ & + \frac{g}{2} \Big[F'_{\kappa\lambda} + W_\kappa F'_{\nu\lambda} W_\nu \Big] - \frac{f}{2} \Big[F^{*\prime}_{\kappa\lambda} + W_\kappa F^{*\prime}_{\nu\lambda} W_\nu \Big] \Big\} S'_\lambda \end{aligned} \tag{3.13}$$

This is the desired equation for the intrinsic spin $\vec{S}$, whose components are the first three components of S'_κ. (One can readily check that $\dot{S}'_4 = -i\, W_\kappa \dot{S}'_\kappa = 0$.) When expressed in three-dimensional vector notation this equation takes the form:

$$\frac{d\vec{S}}{dt} = \vec{\Omega} \times \vec{S} \ , \tag{3.14}$$

where t is the laboratory time related to τ by (2.5) and $\vec{\Omega}$ is the angular velocity of precession given by

$$\vec{\Omega} = - \frac{e}{mc} \left[\frac{1}{1+\gamma} \frac{\vec{v}}{c} \times \vec{E}' + \frac{g}{2\gamma} \vec{B}' + \frac{f}{2\gamma} \vec{E}' \right] \ . \tag{3.15}$$

The first term within the bracket is the well-known Thomas precession term,[14] the second term is the magnetic dipole precession in the rest system magnetic field, modified by the time dilatation factor $\gamma^{-1} = \sqrt{1-v^2/c^2}$, and the third term is the corresponding electric dipole precession. Hence, this expression for the spin precession velocity is just what one could have written down from the beginning.

As the particle moves through the external electromagnetic field its velocity will change in both magnitude and direction as determined by the equation of motion (2.13). In particular from (2.13) one can show that:

$$\frac{d}{dt}\frac{\vec{v}}{v} = \vec{\Omega}_v \times \frac{\vec{v}}{v} \tag{3.16}$$

where $\vec{\Omega}_v$ is the angular velocity of precession of the particle's velocity, given by

$$\vec{\Omega}_v = \frac{e}{m(\gamma^2-1)} \frac{\vec{v}}{c^2} \times \vec{E}' \,. \tag{3.17}$$

The angular precession velocity of the intrinsic spin relative to the velocity of the particle is then:

$$\begin{aligned} \vec{\Omega}_{rel} &= \vec{\Omega} - \vec{\Omega}_v \\ &= -\frac{e}{mc}\left[\frac{\gamma}{\gamma^2-1}\frac{\vec{v}}{c} \times \vec{E}' + \frac{g}{2\gamma}\vec{B}' + \frac{f}{2\gamma}\vec{E}'\right] \end{aligned} \tag{3.18}$$

When $\vec{\Omega}$, $\vec{\Omega}_v$, and $\vec{\Omega}_{rel}$ are expressed in terms of the laboratory fields, using (3.10), they can be written:

$$\begin{aligned} \vec{\Omega} = -\frac{e}{mc}\Bigg[&\frac{1}{\gamma}\vec{B} - \frac{1}{\gamma+1}\frac{\vec{v}}{c} \times \vec{E} + \left(\frac{g}{2} - 1\right)\left(\vec{B} - \frac{1}{c^2}\frac{\gamma}{\gamma+1}\vec{v}\cdot\vec{B}\,\vec{v} - \frac{\vec{v}}{c} \times \vec{E}\right) \\ &+ \frac{f}{2}\left(\vec{E} - \frac{\gamma}{\gamma+1}\vec{v}\cdot\vec{E}\,\frac{\vec{v}}{c^2} + \frac{\vec{v}}{c} \times \vec{B}\right)\Bigg] \,, \end{aligned} \tag{3.19}$$

$$\vec{\Omega}_v = -\frac{e}{mc}\left[\frac{1}{\gamma}\vec{B} - \frac{\gamma}{\gamma^2-1}\frac{\vec{v}}{c}\times\vec{E} - \frac{\gamma}{\gamma^2-1}\vec{v}\cdot\vec{B}\frac{\vec{v}}{c^2}\right], \qquad (3.20)$$

$$\begin{aligned}\vec{\Omega}_{rel} = -\frac{e}{mc}\Bigg[&\frac{1}{\gamma^2-1}\frac{\vec{v}}{c}\times\vec{E} + \frac{\gamma}{\gamma^2-1}\vec{v}\cdot\vec{B}\frac{\vec{v}}{c^2}\\ &+ \left(\frac{g}{2}-1\right)\left(\vec{B} - \frac{\gamma}{\gamma+1}\vec{v}\cdot\vec{B}\frac{\vec{v}}{c^2} - \frac{\vec{v}}{c}\times\vec{E}\right) \qquad (3.21)\\ &+ \frac{f}{2}\left(\vec{E} - \frac{\gamma}{\gamma+1}\vec{v}\cdot\vec{E}\frac{\vec{v}}{c^2} + \frac{\vec{v}}{c}\times\vec{B}\right)\Bigg].\end{aligned}$$

The results (3.15-3.21) for the motion of a classical particle with spin are identical with those for the motion of the expectation value of intrinsic spin of a spin one-half quantum particle.

FOOTNOTES

1. The most recent article on this topic is that of Bargmann, Michel, and Telegdi, Phys. Rev. Lett. 2, 435 (1959).

2. This statement is simply a slight extension of Ehrenfest's theorem, Zeits. f. Physik., 45, 455 (1927). In this connection it seems to have been first remarked by F. Bloch, Phys. Rev. 70, 460 (1946).

3. H. A. Kramers, Quantum Mechanics (North Holland Publishing Company, Amsterdam, 1958) pp. 226-238.

4. J. Frenkel, Zeits. f. Physik, 37, 243 (1926).

5. See e.g., Corben and Stehle, Classical Mechanics, (John Wiley and Sons, Inc., New York, 1950), p. 358-359.

6. The four-vector representation of intrinsic angular momentum (or polarization) was first introduced by Michel and Wightman, Phys. Rev. 98, 1190 (1955). See also ref. 1.

7. C. Møller, The Theory of Relativity (Oxford Univ. Press, London, 1952), Ch. IV.

8. By the rest frame of the particle we mean a coordinate system in which the particle is instantaneously at rest and which is obtained by a pure Lorentz transformation, without rotation, from a fixed laboratory coordinate system. We shall denote quantities evaluated in the rest frame by a prime.

9. S. Watanabe, Rev. Mod. Phys. 27, 40 (1955). The classification of "kinds" of pseudo-tensors is according to the extra changes in sign under T or P and is indicated by the following table.

Kind	T	P
Reg.	+	+
1st	-	-
2nd	-	+
3rd	+	-

10. We use the notation in which the derivative with respect to proper time is indicated by a dot.

11. The most general particle equation of motion which includes spin orbit coupling is of the form,

$$\dot{U}_\lambda = \frac{e}{m} F_{\lambda\alpha} U_\alpha + A\, S_\beta \frac{\partial}{\partial X_\beta} F^*_{\kappa\alpha} U_\alpha .$$

12. Here $E_{\kappa\lambda\mu\nu}$ is the Levi-Civita symbol (see e.g., ref. 7, p. 113).

13. These expressions may be verified using (3.9). They are also given in ref. 7, p. 142.

14. See e.g., ref. 7, p. 56.

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