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A NOTE ON WAVEGUIDE IMAGE TECHNIQUES

BY

R.E. COLLIN

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ABSTRACT

Floquet's theorem and the reflection symmetry properties of Maxwell's equations are used to establish the relationship between scattering by two dimensional periodic structures in free space and equivalent waveguide problems. For structures, such as strip gratings, having periodicity in one direction only it is shown that the free space problem is equivalent to the sum of two waveguide problems and vice-versa. For structures with periodicity along two axis it is shown that four waveguide problems must be solved in order to determine the complete solution of the free space problem. Only when more than one basic cell of the periodic structure is contained within the waveguide walls will a single waveguide problem be equivalent to the free space problem as far as dominant mode fields are concerned.

1. Introduction

It is well known that under certain conditions the problem of scattering from a periodic array of obstacles in free space can be reduced to an equivalent waveguide problem. For arrays, in free-space, with periods less than one half-wavelength all the higher order modes that are excited are evanescent. For those angles of incidence which permit reducing the problem to an equivalent waveguide problem this results in at least two periods of the structure being located within the waveguide walls in order that a waveguide mode will propagate. When only one period of the free-space array is contained within the waveguide walls the waveguide problem and free-space problem are not fully equivalent. The reason is that under these conditions one or more higher order diffraction modes can propagate in free-space. This particular aspect of the waveguide image technique does not appear to have been considered in detail before.

In this note Floquet's theorem is used to establish the equivalence between the problem of scattering from periodic arrays in free space and the corresponding waveguide problems. In particular, it is shown that by solving two (or more) waveguide problems the solution for the reflection coefficients for the two (or more) propagating reflected waves in the free-space problem may be found and vice versa.

2.1 Scattering from One-Dimensional Periodic Gratings

Consider a one-dimensional periodic grating (infinite and uniform in the y direction) with spacing a along the x axis as in

Fig. 1.

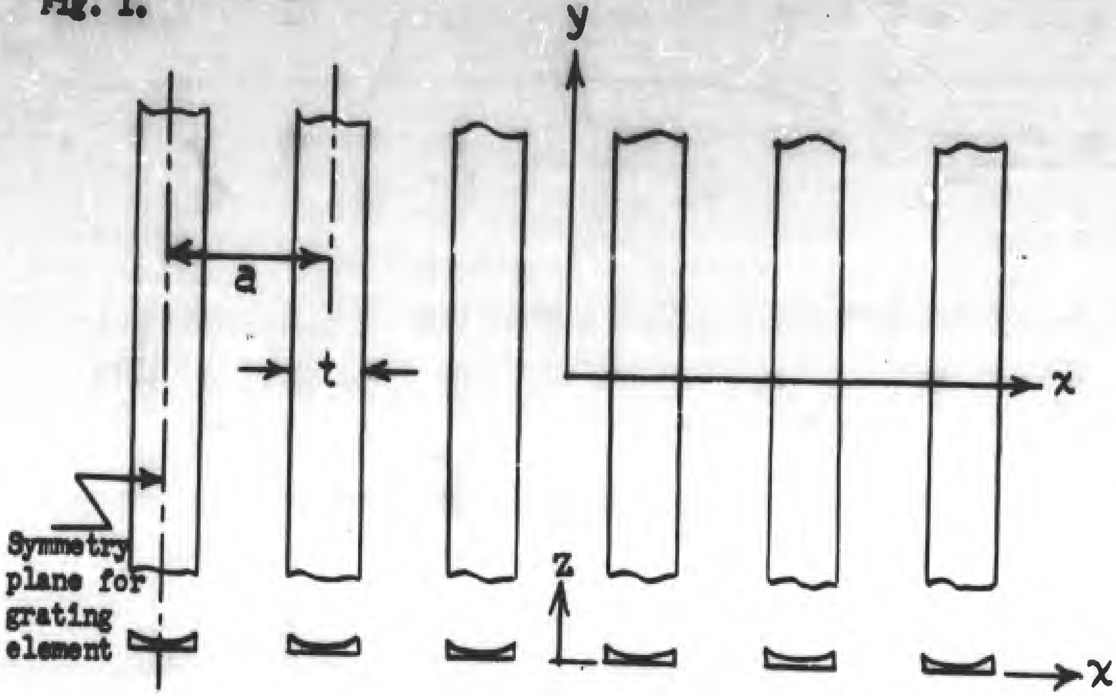

Fig. 1, A one-dimensional periodic grating

Each grating element is of width t and is restricted to have a cross-section that is symmetrical about its mid-plane as indicated in Fig. 1.

Let a perpendicular-polarized TEM wave be incident at an angle θ_i, relative to the z axis, in the xz plane as in Fig. 2. In view of the nature of the incident field and the uniformity of the structure in the y direction only TE modes are excited. The field components H_x and H_z for these may be derived from the single component E_y of electric field that is present.

Let the incident electric field be

$$E_{yi} = a_o e^{-jhx-\Gamma_o z} \qquad (1)$$

where $h = k_o \sin \theta_i$, $\Gamma_o = jk_o \cos \theta_i$. Since the structure is periodic

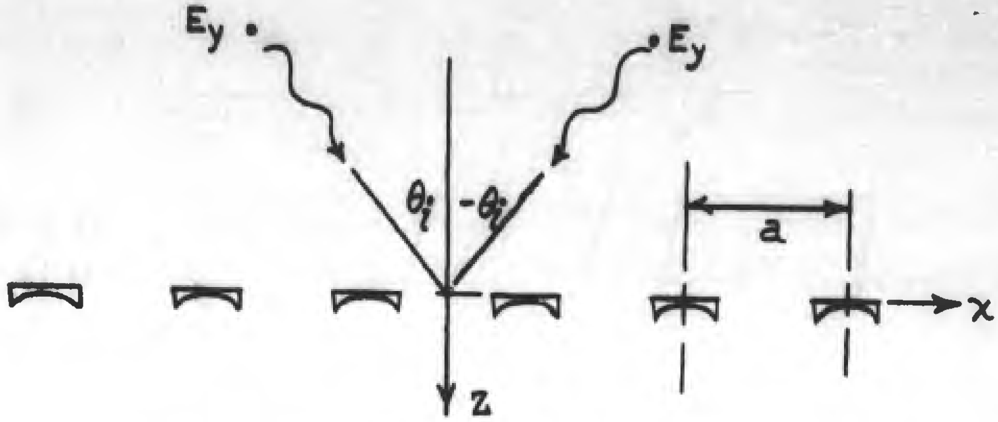

Fig. 2, Incidence of TEM wave on grating

in the x direction the scattered field for $z < 0$ will, according to Floquet's theorem, have the form

$$E_{yr} = \sum_{n=-\infty}^{\infty} b_n \, e^{-j(h + 2n\pi/a)x} \, e^{\Gamma_n z} \qquad (2)$$

where

$$\Gamma_n^2 = (h + 2n\pi/a)^2 - k_o^2$$

For $z > 0$ the transmitted field will be

$$E_{yt} = \sum_{n=-\infty}^{\infty} c_n \, e^{-j(h + 2n\pi/a)x} \, e^{-\Gamma_n z} \qquad (3)$$

If we begin with an incident field

$$E_{yi} = a_o \, e^{jhx - \Gamma_o z} \qquad (4)$$

incident at an angle $-\theta_i$ the symmetry of the grating elements and array shows that the reflected and transmitted field must be

$$E_{yr} = \sum_{n=-\infty}^{\infty} b_n e^{j(h + 2n\pi/a)x} e^{\Gamma_n z} \qquad (5)$$

$$E_{yt} = \sum_{n=-\infty}^{\infty} c_n e^{j(h + 2n\pi/a)} e^{-\Gamma_n z} \qquad (6)$$

It is only necessary to replace x by -x in (1), (2) and (5). If we superimpose the above two solutions we obtain

$$E_{yi} = 2a_o \cos hx\, e^{-\Gamma_o z} \qquad (7)$$

$$E_{yr} = 2 \sum_{n=-\infty}^{\infty} b_n \cos(h + 2n\pi/a)x\, e^{\Gamma_n z} \qquad (8)$$

$$E_{yt} = 2 \sum_{n=-\infty}^{\infty} c_n \cos(h + 2n\pi/a)\; e^{-\Gamma_n z} \qquad (9)$$

If the angle of incidence is chosen so that $h = k_o \sin \theta_i = \pi/a$, i.e.

$$\sin \theta_i = \lambda_o/2a \qquad (10)$$

then (7), (8) and (9) become

$$E_{yi} = 2a_o \cos \frac{\pi x}{a} e^{-\Gamma_o z} \qquad (11)$$

$$E_{yr} = 2b_o \cos \frac{\pi x}{a} e^{\Gamma_o z} + 2b_{-1} \cos \frac{\pi x}{a} e^{\Gamma_o z}$$

$$+ 2 \sum_{\substack{n=-\infty \\ n\neq 0,-1,}}^{\infty} b_n \cos(1 + 2n) \frac{\pi x}{a} e^{\Gamma_n z} \qquad (12)$$

$$E_{yt} = (2c_0 + 2c_{-1}) \cos \frac{\pi x}{a} e^{-\Gamma_0 z}$$

$$+ 2 \sum_{\substack{n=-\infty \\ n \neq 0,-1}}^{\infty} c_n \cos(1 + 2n) \frac{\pi x}{a} e^{-\Gamma_n z} \qquad (13)$$

since $\Gamma_{-1} = \Gamma_0$. It is assumed that all Γ_n except Γ_0 and Γ_{-1} are real and hence only the n=0,-1, modes propagate. The necessary condition is $\frac{3\pi}{a} > k_0$. An examination of this solution shows that at $x = \pm \frac{a}{2} + ma$ the electric field vanishes. Conducting planes may, therefore, be inserted at $x = \pm a/2$. We may place conducting planes at y = o,b as well since the electric field is everywhere perpendicular to the xz plane. Hence we obtain the waveguide structure illustrated in Fig. 3a.

Equations (11)-(13) give the solution to the waveguide problem of Fig. 3a. The reflection coefficient R_1, from (12), is seen

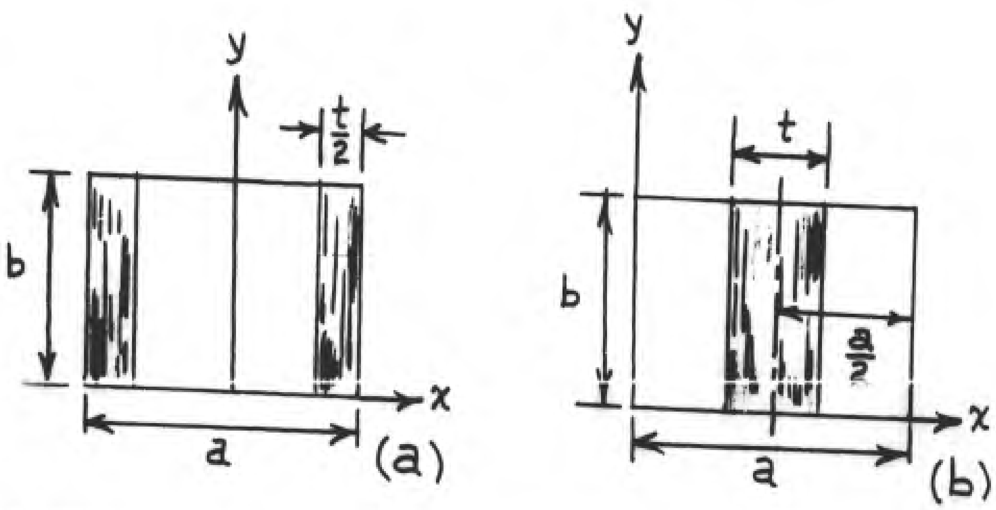

Fig. 3, Waveguide structures derived from the grating structure.

to be given by

$$R_1 = \frac{b_o + b_{-1}}{a_o} \qquad (14)$$

If B_1 is the shunt inductive susceptance of the diaphragm in Fig. (5a) for infinitely thin flat strips then

$$R_1 = \frac{jB_1}{2 - jB_1} \qquad (15)$$

It should be noted that a solution for R_1 determines the sum of the amplitudes $b_o + b_{-1}$ for the n = 0 and n = -1 propagating modes for the free-space problem. Conversely, both b_o and b_{-1} must be found in order to determine the reflection coefficient R_1 for the waveguide problem.

By choosing a different incident field we can also arrive at the waveguide structure of Fig. 5b. A combination of the two problems provides a solution for both b_o and b_{-1}. In the waveguide it is assumed that only the H_{10} mode propagates. Because of the symmetry the H_{20}, H_{40}, etc., modes are not excited.

Let the incident fields be those given by (1), and (4) multiplied by -1, thus

$$E_{yi} = a_o e^{-\Gamma_o z} (e^{-jhx} - e^{jhx})$$

$$= -2ja_o \sin hx \, e^{-\Gamma_o z} \qquad (16)$$

The reflected field is now given by (2) and (5) multiplied by (-1) and is

$$E_{yr} = -2j \sum_{n=-\infty}^{\infty} b_n \sin(h + 2n\pi/a)x \, e^{\Gamma_n z} \qquad (17)$$

When θ_1 is again given by (10) we obtain

$$E_{yi} = -2j\ a_o \sin \frac{\pi x}{a} e^{-\Gamma_o z} \tag{18}$$

$$E_{yr} = (-2j\ b_o + 2jb_{-1}) \sin \frac{\pi x}{a} e^{\Gamma_o z}$$

$$-2j \sum_{\substack{n=-\infty \\ n \neq 0,-1}}^{\infty} b_n \sin (1 + 2n) \frac{\pi x}{a} e^{\Gamma_n z} \tag{19}$$

Note the change in sign of b_{-1} since $\sin (1 + 2n) \frac{\pi x}{a} = - \sin \pi x/a$ for $n = -1$.

The above field vanishes at $x = o, a, \ldots ma$ and conducting walls may therefore be placed at $x = o, a$. This leads to the waveguide structure of Fig. 5b. The waveguide reflection coefficient for the H_{10} mode in this case is R_2 where

$$R_2 = \frac{b_o - b_{-1}}{a_o} \tag{20}$$

For an infinitely thin flat strip the shunt inductive susceptance is B_2 and

$$R_2 = \frac{jB_2}{2 - jB_2} \tag{21}$$

If the two waveguide problems have been solved so that R_1 and R_2 are known, then

$$b_o = (R_1 + R_2) \frac{a_o}{2} \tag{22a}$$

$$b_{-1} = (R_1 - R_2) \frac{a_o}{2} \tag{22b}$$

On the other hand if the coefficients b_o and b_{-1} for the two propagating reflected modes in the free-space problem are known the solutions for the two waveguide problems illustrated in Fig. 3 are determined by (14) and (20). Relations similar to the above hold for the coefficients of the higher order evanescent modes as well.

2.2 Asymmetrical Case

Let the incident TE field be chosen as

$$E_{yi} = a_o e^{-jhx - \Gamma_o z} \tag{23a}$$

The corresponding reflected field will be

$$E_{yr} = \sum_{n=-\infty}^{\infty} b_n e^{-j(h + 2n\pi/a)x + \Gamma_n z} \tag{23b}$$

Choose a second incident field

$$E_{yi} = -a_o e^{jhx - \Gamma_o z} \tag{24a}$$

Symmetry considerations show that the reflected field will now be

$$E_{yr} = - \sum_{n=-\infty}^{\infty} b_n e^{j(h + 2n\pi/a)x + \Gamma_n z} \tag{24b}$$

Superposition of the two above solutions gives

$$E_{yi} = - 2ja_o \sin hx\, e^{- \Gamma_o z} \tag{25a}$$

$$E_{yr} = - 2j \sum_{n=-\infty}^{\infty} b_n \sin (h + 2n\pi/a)x\, e^{\Gamma_n z} \tag{25b}$$

Let the angle of incidence Θ_i be chosen so that

$$h = k_o \sin \Theta_i = 2\pi/a \qquad (26)$$

Furthermore let the spacing a be chosen so that

$$a/2 < \lambda_o < a \qquad (27)$$

Under these conditions the modes n=0,-1, and -2 propagate. The mode n=-1 corresponds to a TEM wave since $h + 2(-1)\ \pi/a = 0$. This TEM wave mode is present in the solutions (25b) and (24b) but is absent in (25b) since $\sin (h + 2n\pi/a)x = 0$ for n=-1.

For the specified value of h the function $\sin (h + 2n\pi/a)x$ vanishes at $x = 0, \pm ma/2$. Conducting planes may be inserted at x = 0 and x = a/2 in Fig. 1 and the waveguide problem of Fig. 4 is obtained. In this waveguide only the H_{10} mode propagates. Examination of (25b) shows that the reflected H_{10} mode in the waveguide is (the n=0,-2, modes)

$$-2j\left[b_o \sin hx + b_{-2} \sin (h - 4\pi/a)x \right] e^{+\Gamma_o z}$$

$$= -2j \left(b_o - b_{-2} \right) \sin 2\pi x/a \ e^{\Gamma_o z} \qquad (28)$$

A solution of the waveguide problem of Fig. 4 thus provides a solution for the difference $b_o - b_{-2}$ between the amplitudes of the n=0 and n=-2 reflected modes for the free space problem. On the other hand the waveguide problem provides no information on the amplitude b_{-1} of the

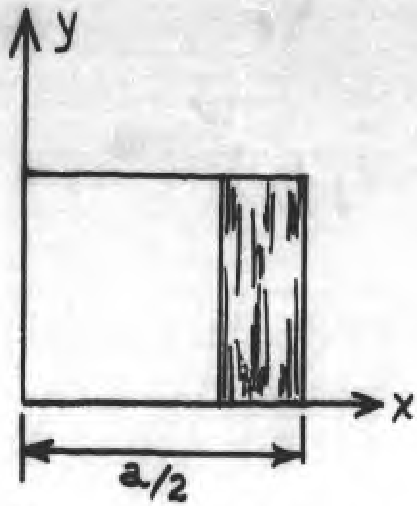

Fig. 4, Waveguide problem for asymmetrical case

reflected TEM mode in the free space problem. To solve the waveguide problem both b_0 and b_{-2} must be determined in the free space problem.

2.3 Equivalent Problems

If two or more periods of the structure is confined between the waveguide walls and only the H_{10} mode propagates in the waveguide then only the n=0 mode propagates in the free space problem. In this case the free space problem and waveguide problem are fully equivalent as far as the dominant mode fields are concerned. The reflection coefficient is the same for both. For example, with $h = \pi/2a$ and $2a < \lambda_0 < 4a$ only the n=0 mode in (8) and (17) propagates. Corresponding to the solutions (8) and (17) are the waveguide problems illustrated in Fig. 5a and b. The reflection coefficient in the two waveguide problems and the free space problem are all equal to b_0/a_0.

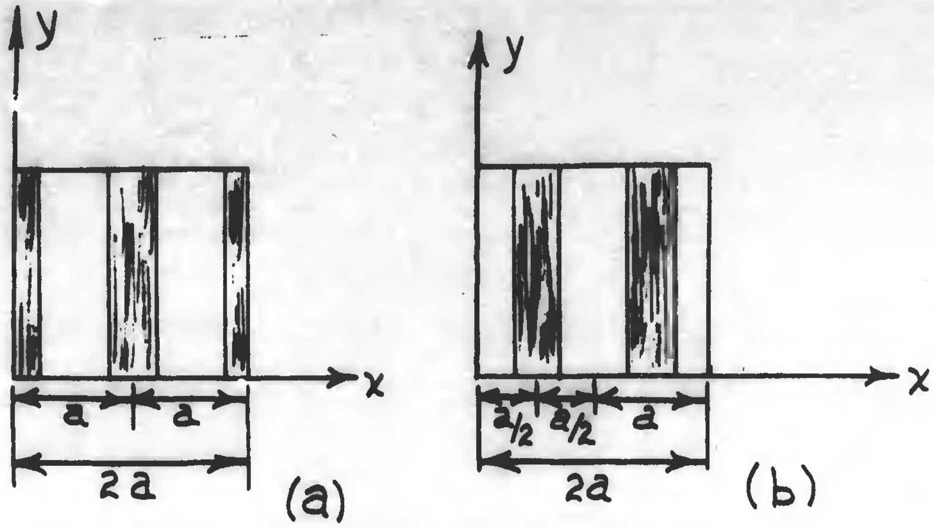

Fig. 5, Waveguide problems equivalent to a free space problem

3. Parallel-Plate Media

Figure 6 illustrates an infinite array of equi-spaced, infinitely thin, and perfectly conducting parallel plates. The incident wave is a TE wave of the type considered in Sec. 2.1. If the angle of incidence is chosen so that $\sin \theta_i = \lambda_o/2a$ then the free space problem is equivalent to the two waveguide problems illustrated in Fig. 7a and b. In Fig. 7a the parallel plates coincide with the waveguide walls and hence no reflection occurs. Thus we must have

Fig. 6, Parallel-plate array

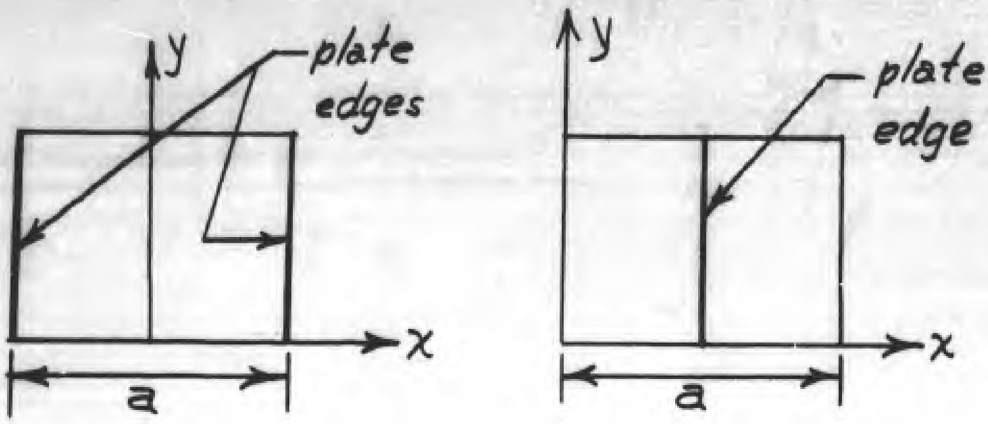

Fig. 7, Waveguide problems equivalent to free space parallel-plate problem for $\sin \theta_i = \lambda_o/2a$

$$b_o + b_{-1} = 0$$

or

$$b_o = - b_{-1} \tag{29}$$

For the problem of Fig. 7b the reflection coefficient is

$$R = \frac{b_o - b_{-1}}{a_o} = \frac{2b_o}{a_o} \tag{30}$$

The result expressed by (29), obtained here from rather elementary considerations, shows that in the free space problem, with the angle of incidence given by $\theta_i = \sin^{-1} \lambda_o/2a$, just as much power is reflected in the mode n=-1 as in the mode n=0. This result checks with that obtained from the formal solution of the parallel-plate problem.†

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†F. Berz, "Reflection and Refraction of Microwaves at a Set of Parallel Metallic Plates" Proc. IEE (London) vol. 98, part III, pp 47-55, Jan. 1951.

4. Structures With Periodicity in Two Dimensions

Figure 8a illustrates a two dimensional periodic structure consisting of thin circular disks with uniform spacing a in the x direction and b in the y direction. Any other shape of disk with cymmetry about the x and y axis would serve equally well for the discussion presented below (see Fig. 8b).

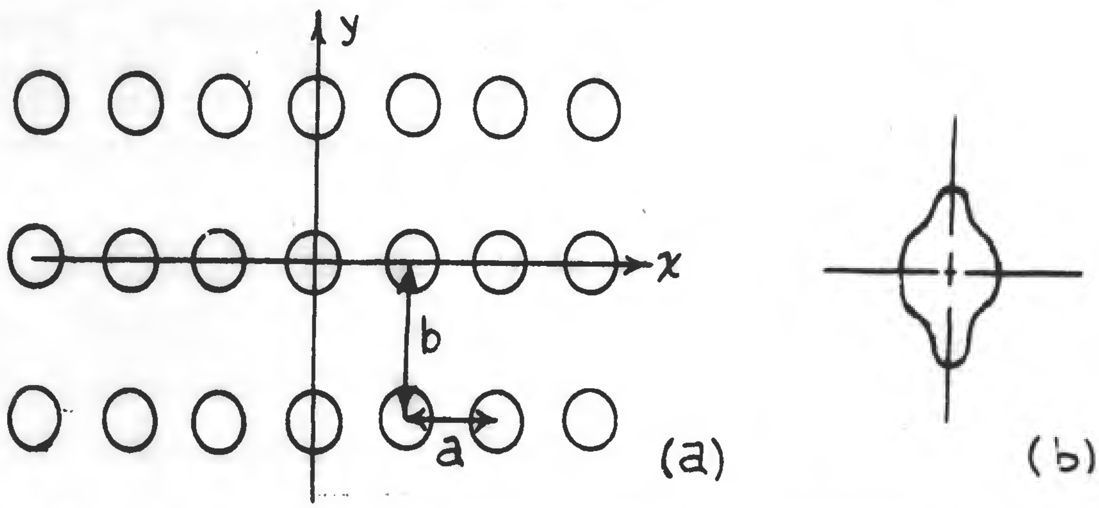

Fig. 8 (a), Two-dimensional array of disks, (b) a general shaped disk with two axis of symmetry.

Let a plane TEM wave be incident, on the array of disks, along a direction defined by the unit wave normal $\underset{\sim}{n}$. The electric field may be written as

$$\underset{\sim}{E}_{i1} = \underset{\sim}{E}_o \, e^{-jhx-j\ell y-\Gamma_{oo} z} \qquad (31)$$

where $h = k_o n_x$, $\ell = k_o n_y$ and $\Gamma_{oo} = jk_o n_z = (h^2 + \ell^2 - k_o^2)^{1/2}$ and $\underset{\sim}{E}_o$ is a constant vector perpendicular to $\underset{\sim}{n}$. The scattered field will in general consist of both TE and TM modes with respect to the z axis. Each component of the reflected electric and magnetic field can be represented in the following form

$$\psi = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} A_{nm} e^{-j(h + 2n\pi/a)x} e^{-j(\ell + 2m\pi/b)y} e^{\Gamma_{nm} z} \tag{32}$$

where $\Gamma_{nm} = \left[(h + 2n\pi/a)^2 + (\ell + 2m\pi/b)^2 - k_o^2\right]^{1/2}$ and ψ can be any field component provided the amplitude constants A_{nm} are appropriately chosen.

In order to carry out the required symmetry arguments it will be convenient to introduce z directed electric and magnetic Hertzian potentials π_e and π_m from which the fields are expressed as

$$\underset{\sim}{E} = j\omega\mu \, \underset{\sim}{a}_z \times \nabla \pi_m - \nabla \times (\underset{\sim}{a}_z \times \nabla \pi_e) \tag{33a}$$

$$\underset{\sim}{H} = -j\omega\epsilon \, \underset{\sim}{a}_z \times \nabla \pi_e - \nabla \times (\underset{\sim}{a}_z \times \nabla \pi_m) \tag{33b}$$

The potentials π_e and π_m generate the E and H modes respectively. A general incident field can be expressed in terms of the potentials

$$\pi_{el} = C_{el} e^{-jhx - j\ell y - \Gamma_{oo} z} \tag{34a}$$

$$\pi_{ml} = C_{ml} e^{-jhx - j\ell y - \Gamma_{oo} z} \tag{34b}$$

where C_{el} and C_{ml} are constants. The corresponding potentials for the reflected fields may be expressed as

$$\pi_{el,r} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} A_{nm} e^{-j(h + 2n\pi/a)x} e^{-j(\ell + 2m\pi/b)y} e^{\Gamma_{nm} z} \tag{34c}$$

$$\pi_{m1,r} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} B_{nm}\, e^{-j(h + 2n\pi/a)x}$$

$$e^{-j(\ell + 2m\pi/b)y}\, e^{\Gamma_{nm} z} \qquad (34d)$$

Similar expansions may also be written for the potentials that give rise to the transmitted fields in the region $z > 0$ (the array of disks is assumed located in the $z=0$ plane)

In addition to the incident fields specified by the potentials (34a) and (34b) three additional incident fields derivable from the following potentials may be specified,

$$\pi_{e2} = C_{e2}\, e^{jhx - j\ell y - \Gamma_{00} z} \qquad (35a)$$

$$\pi_{m2} = C_{m2}\, e^{jhx - j\ell y - \Gamma_{00} z} \qquad (35b)$$

$$\pi_{e3} = C_{e3}\, e^{-jhx + j\ell y - \Gamma_{00} z} \qquad (36a)$$

$$\pi_{m3} = C_{m3}\, e^{-jhx + j\ell y - \Gamma_{00} z} \qquad (36b)$$

$$\pi_{e4} = C_{e4}\, e^{jhx + j\ell y - \Gamma_{00} z} \qquad (37a)$$

$$\pi_{m4} = C_{m4}\, e^{jhx + j\ell y - \Gamma_{00} z} \qquad (37b)$$

The corresponding potentials for the reflected fields are of the same form as given by (34)

The array of disks exhibits reflection symmetry in the x axis and the y axis. Upon reflection in the x axis the solutions of Maxwell's equations transform as follows[†]

$$E_x(x,y,z) \longrightarrow \pm E_x(-x,y,z)$$

$$E_y(x,y,z) \longrightarrow \mp E_y(-x,y,z)$$

$$E_z(x,y,z) \longrightarrow \mp E_z(-x,y,z) \qquad (38)$$

$$H_x(x,y,z) \longrightarrow \mp H_x(-x,y,z)$$

$$H_y(x,y,z) \longrightarrow \pm H_y(-x,y,z)$$

$$H_z(x,y,z) \longrightarrow \pm H_z(-x,y,z)$$

The corresponding transformations for the z directed Hertzian potentials must be

$$\pi_e(x,y,z) \longrightarrow \mp \pi_e(-x,y,z) \qquad (39a)$$

$$\pi_m(x,y,z) \longrightarrow \pm \pi_m(-x,y,z) \qquad (39b)$$

in order that the above transformation of the field components will result. For a reflection in the y axis a similar transformation of the potentials holds. Finally, for a reflection in both the x axis and the y axis it follows that

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[†]C.G. Montgomery, et. al. "Principles of Microwave Circuits," Rad. Lab. Series vol. 8, McGraw-Hill Book Company, Inc., New York, 1948.

$$\pi_e(x,y,z) \longrightarrow \pm \pi_e(-x,-y,z) \qquad (40a)$$

$$\pi_m(x,y,z) \longrightarrow \pm \pi_m(-x,-y,z) \qquad (40b)$$

With the aid of the transformations (39) and (40) it is clear that the incident fields specified in (35), (36) and (37) correspond to solutions obtained from (34) by reflection in the x axis, y axis, and both x and y axis provided the coefficients are chosen so that $C_{e2} = \mp C_{e1}$, $C_{m2} = \pm C_{m1}$; $C_{e3} = \mp C_{e1}$, $C_{m3} = \pm C_{m1}$; and $C_{e3} = \pm C_{e1}$, $C_{m3} = \pm C_{m1}$. The solutions for the reflected fields follow by similar transformations of the coefficients A_{nm} and B_{nm}. Since the array of disks has reflection symmetry in both the x and y axis it follows that solutions of the field equations obtained by means of the above transformations will also satisfy all the conditions of the problem.

If the wave normal is now chosen so that $\ell = \pi/b$ and $h = \pi/a$ then all the modes given by n=m=0; n=-1, m=0; n=0, m=-1; n=m=-1 in (34) correspond to modes propagating energy away, from the plane of disks, in the -z direction. Thus there are a total of 8 unknown coefficients A_{nm}, B_{nm} corresponding to propagating modes. (It is assumed that all other modes are evanescent, which will be the case if $(\pi/a)^2 + (\pi/b)^2 < k_o^2 < k^2$ where k^2 is the smaller of $(3\pi/a)^2 + (\pi/b)^2$ and $(\pi/a)^2 + (3\pi/b)^2$.)

By a suitable linear combination of the fields determined by (34), (35), (36) and (37) the four waveguide problems illustrated in Fig. 9 can be obtained.

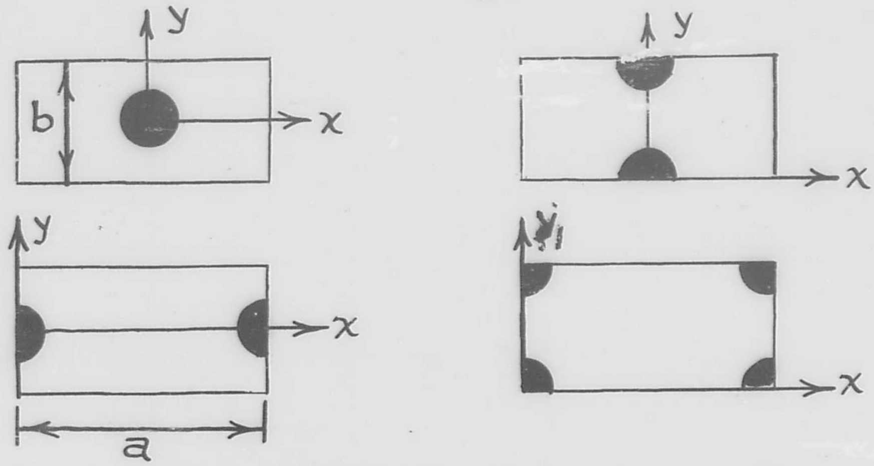

Fig. 9, Waveguide problems equivalent to free space problem in Fig. 8.

A complete solution of these four waveguide problems provides a complete solution of the free space problem and, in particular, determines all the amplitude coefficients of the reflected propagating waves. In a similar way a complete solution of the free space problem provides a solution to four different but related waveguide problems. These solutions will not be detailed here since the analysis is only intended to establish the connection between the free space and waveguide problems. It is clear from the analysis that a single waveguide problem is not equivalent to the free space problem unless, of course, more than one basic two dimensional cell of the free space structure is contained within the waveguide walls. Actually the minimum number of cells to be contained within the waveguide walls is four in order that a single waveguide problem be equivalent to the free space problem (it is now assumed that only the n=m=0 modes propagate). It should also be noted that when a single waveguide problem is said to be equivalent to the free space problem the equivalence is meant only to apply to the determination of the dominant mode amplitudes. Solution for all the mode amplitudes still requires the solution of all four waveguide problems.

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