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HYPERSONIC FLOW OVER CONES

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ABSTRACT

The Taylor-Maccoll equation for the supersonic flow of an ideal gas about a right circular cone with an attached shock wave is solved for the two cases of infinite and finite free stream Mach numbers. The solution for the former is given in terms of an expansion in the limiting density ratio across the shock, ϵ_ℓ , and the solution to the latter in terms of a double expansion in the actual density ratio, ϵ , and ϵ_ℓ. The results are compared with a Taylor series solution to the Taylor-Maccoll equation and also with the exact values of Kopal for air and Mueller for helium.

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NOMENCLATURE

a	isentropic speed of sound
h	specific enthalpy
k	entropy function in adiabatic pressure-density relationship
p	pressure
$\bar{p}$	dimensionless pressure, p/p_s
$\bar{\bar{p}}$	dimensionless pressure, $p/\rho_\infty U^2$
q_o	maximum speed obtainable by expanding to zero absolute temperature
r	length measured along any ray from apex of cone
u	velocity along a ray
$\bar{u}$	dimensionless radial component of velocity, u/U
v	velocity normal to a ray
$\bar{v}$	dimensionless polar component of velocity, v/U
M	Mach number
U	free stream velocity
γ	ratio of specific heats
ϵ	density ratio across shock, ρ_∞/ρ_s
ϵ_ℓ	limiting density ratio for strong shocks
θ	polar angle
θ_c	half-angle of cone
ρ	density
$\bar{\rho}$	dimensionless density, ρ/ρ_s
σ	shock inclination angle
ψ	reduced angular variable, $(\theta - \sigma)/\epsilon$

Subscripts:

c	cone surface
s	shock
∞	conditions upstream of shock

HYPERSONIC FLOW OVER CONES

I. INTRODUCTION

The problem of determining the steady supersonic flow of an ideal, compressible fluid about a circular cone when the shock wave is attached was completely treated by Taylor and Maccoll in a paper in 1933. Since the flow is both isentropic and irrotational, and depends only on one variable, the entire problem can be reduced to the solution of a second order non-linear ordinary differential equation for one of the velocity components; from a knowledge of this velocity component all the other variables can be obtained. Taylor and Maccoll solved this differential equation numerically for a number of particular cases.

The Taylor and Maccoll analysis has been the basis for a large volume of work on the cone problem which appeared subsequently. Kopal integrated the Taylor-Maccoll equation for air for a wide range of conditions and presented his results in tabular form. Hence, in a sense, the problem of the supersonic flow of air about a cone may be said to be solved. However, in recent years, the interest in high Mach number flows has led to the need for these basic solutions over a wide range of Mach numbers and specific heat ratios. Since the work required for the tabulation of these solutions over the entire range of these parameters would be prohibitive, the need for approximate solutions giving analytic relations among the variables is greatly increased. Even in cases where the exact numerical results are available, an approximate analytic description of the flow is very useful in indicating the effect of the various parameters upon the resulting flow. Hence, much work has gone into finding approximate solutions to the supersonic cone problem.

With the advent of concern for flow problems at hypersonic speeds, an assumption made in many problems is that of small density ratio, where this is defined as the ratio of free stream density to density immediately behind the shock wave. The assumption of small density ratio, in general, leads to the concept of a thin shock layer. Applying this assumption to the cone problem the Taylor-Maccoll equation may be reduced to a very simple linear equation which can be solved by two simple quadratures. If the full implications of the small density ratio assumption are not made in simplifying the Taylor-Maccoll equation, that is, one additional term is kept which in terms of density ratio is of the same order of smallness as those terms already dropped, then the reduced Taylor-Maccoll equation is still linear, but the analysis becomes more difficult. This reduced equation is a Legendre equation and so the solution may be explicitly set down. This has been done independently by Hayes and Probstein, and Feldman. If one traces back through the analysis one then finds that this linearized Taylor-Maccoll equation is exactly the equation that would result making no assumption about the density ratio, but assuming that the density in the shock layer is constant. As clearly pointed out by Hayes and Probstein this assumption of constant density is not equivalent to the assumption of incompressibility. The latter is a statement about the equation of state of the fluid, while the former is a statement that the pressure changes experienced by the fluid are small so that the flow is at nearly constant density. The assumption of constant density yields a solution to the cone problem which is exact;

however, this assumption is approximately valid only if the density ratio ϵ is sufficiently small, and in this case it yields a solution which is much more complicated than the analysis indicated above, but which is no more accurate in terms of order in ϵ.

In an alternate, much simpler analysis, Hayes starts with the assumption that the shock layer is thin and obtains an approximate picture of the structure of the layer on the basis of continuity arguments; from this he derives results of equal validity as those obtained by the Legendre function analysis, except for the isolated case of the almost-normal shock. This exception holds in all theories in which the concept of a thin shock layer is used as a consequence of the assumption of small density ratio, for in the case of a nearly normal shock the density ratio may be small without the shock layer being thin.

On the assumption of the shock layer being thin, it is possible to show that, to a first approximation, the streamlines in the layer are hyperbolas with the body surface as asymptote. Using this fact as a starting point, Hord has obtained an approximate solution for the cone in a perfect gas of constant γ. He compares the results of his theory with various other approximations and the exact results of Kopal. His expression relating shock angle and cone angle agrees with the result of Hayes and Probstein.

In a paper on flow about cones at very high speeds, Zienkiewicz develops an approximate analytic solution for the cone problem. He starts with the full Taylor-Maccoll equation and drops one of the velocity components from the equation, an approximation which is valid as long as ϵ is small and the shock layer is thin. Zienkiewicz's approximation occupies an intermediate point between the approximation of the complete constant density theory and the simplest equally valid approximation indicated above. Zienkiewicz does not derive any geometrical or pressure relationships based upon his approximation, but if one uses his results to obtain the relationship between shock and cone angle it agrees with the results of the analyses mentioned above to first order. The fact that all these various approximations give results which are identical within their range of validity is due to the fact, as expressed by Hayes and Probstein, that "within the limits of validity of constant density theory a number of almost equivalent approximations of slightly different form are equally valid except in the case of the almost-normal shock". In the exceptional case the constant density assumption is still valid as long as ϵ is small, even though the shock layer may be anything but thin. In this case, the complete theory involving Legendre functions is essential.

An entirely different method of analysis was proposed by Maccoll, who used the Taylor-Maccoll equation to determine the first few coefficients in a Taylor series expansion about the cone surface. This expansion was then employed in the earlier numerical method of Taylor and Maccoll to determine the location of the shock wave and thus to give the solution to the problem. Hammitt and Murthy extended the usefulness of this approach by allowing the Taylor series to represent the entire flow between the shock and cone and matching this flow analytically to the free stream flow by means of the shock relations. Since the Taylor series involves terms in powers of $(\theta-\theta_c)$ this analysis is limited to this quantity being small, and so is also a thin shock layer analysis.

All the approximations given above, except for the Legendre function treatment of Hayes and Probstein, require that as a consequence of ϵ being small that the shock layer must be thin; they do not put any requirement on the magnitude of the cone angle. There are several other approximate solutions which have proven very useful, but these, in general, do make an assumption about the cone angle. The linearized and second order solutions are useful for small cone angles at supersonic Mach numbers, while Lees' solution is good at high Mach numbers and small angles, and that of Feldman at large angles and high Mach numbers.

Some of the approximate solutions given above, while yielding important results, either do not indicate clearly the order to which the approximation is valid, or do not indicate the method of obtaining higher order solutions. It is the purpose of this paper to develop a systematic method of solution of the cone problem by an expansion in density ratio. This will be done first for infinite Mach number and then for finite, but large, Mach number. First, it is hoped in this way to obtain a solution which will clearly indicate the order of accuracy of approximate solutions which depend on the idea of a thin shock layer. Secondly, since nearly all the solutions mentioned above are first order solutions in density ratio, it is hoped to develop a solution accurate to a considerably higher order, which may itself be used in place of the numerical solution for high Mach numbers, and which will indicate the procedure for obtaining even higher order approximations.

II. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

As coordinates we choose a polar coordinate system r, θ, where $\theta = 0$ is the axis of the cone and r is the length measured along any ray from the apex of the cone (see Figure 1). We assume the shock wave is attached. The geometry of the problem indicates the existence of a flow for which all the variables are functions of θ only. In this case the flow must be irrotational, which implies the condition

$$\frac{du}{d\theta} - v = 0 \tag{1}$$

The equation of continuity is

$$\frac{\partial}{\partial r}(\rho u r^2 \sin\theta) + \frac{\partial}{\partial \theta}(\rho v r \sin\theta) = 0$$

Since $\partial u/\partial\varphi = 0$ this may also be written as

$$2\rho u \sin\theta + \frac{d}{d\theta}(\rho v \sin\theta) = 0 \tag{2}$$

Substituting $v = du/d\theta$, obtained from equation (1), into equation (2), we obtain the following equation

$$\frac{d^2u}{d\theta^2} + \cot\theta \frac{du}{d\theta} + 2u = -\frac{v}{\rho}\frac{d\rho}{d\theta} \tag{3}$$

The condition that the flow is of constant total energy is expressed by the equation

$$h + \frac{1}{2}(u^2 + v^2) = \text{constant} = \frac{1}{2}q_0^2 \tag{4}$$

Assuming the fluid is a perfect gas with the adiabatic relation $p = k\rho^{\gamma}$ the enthalpy may be written

$$h = \frac{\gamma p}{(\gamma-1)\rho} = \frac{\gamma K}{\gamma-1}\rho^{\gamma-1}$$

Equation (4) then becomes

$$\frac{\gamma K}{\gamma-1}\rho^{\gamma-1} + \frac{1}{2}(u^2+v^2) = \frac{1}{2}q_0^2 \tag{5}$$

where q_0, the maximum velocity, is a constant throughout the flow field. Differentiating equation (5) with respect to θ to obtain $d\rho/d\theta$ and substituting the resulting expression into equation (3) we obtain

$$\frac{d^2u}{d\theta^2} + \cot\theta\frac{du}{d\theta} + 2u = v\frac{\left(u\frac{du}{d\theta} + v\frac{dv}{d\theta}\right)}{\gamma K\rho^{\gamma-1}} \tag{6}$$

From equation (5)

$$\gamma K\rho^{\gamma-1} = \frac{\gamma-1}{2}(q_0^2 - u^2 - v^2) = a^2$$

Substituting this into equation (6) and using $v = du/d\theta$ we obtain, after some rearrangement of terms

$$\left(\frac{du}{d\theta}\right)^2\left(u + \frac{d^2u}{d\theta^2}\right) = \frac{\gamma-1}{2}\left[q_0^2 - u^2 - \left(\frac{du}{d\theta}\right)^2\right]\left(\frac{d^2u}{d\theta^2} + \cot\theta\frac{du}{d\theta} + 2u\right) \tag{7}$$

This equation, a second order non-linear ordinary differential equation for u as a function of θ is the basic equation for ideal fluid flow about unyawed cones. The solution of equation (7) determines the complete solution to a particular problem since the pressure and density can then be determined from the energy equation (equation (4)), the equation of state and the shock conservation conditions. Equation (7) is called the Taylor-Maccoll equation.

Boundary Conditions

Boundary conditions are given both at the cone surface and immediately behind the shock wave.

At the cone surface we have only one boundary condition, the vanishing of the normal component of the velocity. Using equation (1) this condition is

$$V_c = \left(\frac{du}{d\theta}\right)_c = 0 \qquad (8)$$

At the shock wave we have the following four conservation conditions

$$\rho_\infty U \sin\sigma = -\rho_s V_s \qquad (9) \text{ conservation of mass}$$

$$p_\infty + \rho_\infty U^2 \sin^2\sigma = p_s + \rho_s V_s^2 \qquad (10) \text{ conservation of normal momentum}$$

$$\frac{1}{2}U^2 + h_\infty = \frac{1}{2}(u_s^2 + v_s^2) + h_s \qquad (11) \text{ conservation of energy}$$

$$U\cos\sigma = u_s \qquad (12) \text{ conservation of tangential momentum}$$

Since the Taylor-Maccoll equation is a second order ordinary differential equation only two boundary conditions are needed to obtain the complete solution. One of these is supplied by the condition at the cone surface, equation (8). The other boundary condition must come from the shock relations. However, since the shock wave is a free boundary whose location is not known until the problem is solved, we must employ an additional condition to determine this additional unknown quantity. Hence, two boundary conditions are required at the shock wave. The two appropriate boundary conditions are given by the conditions for conservation of mass and conservation of tangential momentum (which reduces to continuity of tangential velocity by virtue of mass conservation). From (12) and (9) these boundary conditions are

$$u_s = U \cos \sigma \tag{12}$$

$$v_s = \left(\frac{du}{d\theta}\right)_s = -\frac{\rho_\infty}{\rho_s} U \sin \sigma = -\epsilon U \sin \sigma \tag{13}$$

The entire problem is now reduced to finding a solution to equation (7) together with the boundary conditions, equations (12), (13) and (8). In the actual physical problem the cone is given and the shock wave location is to be determined; however, in all the following work we shall regard the shock as the fixed boundary and determine the location of the cone surface from the third boundary condition. This latter procedure is chosen because two of the three boundary conditions are given at the shock and only one at the cone surface. Use of this procedure allows us to treat the problem as a one-point rather than a two-point boundary value problem.

III. ORDERS OF MAGNITUDE OF VARIABLES AND SIMPLE APPROXIMATE SOLUTION OF PROBLEM

Since equation (7) is highly non-linear we cannot expect to obtain a complete solution satisfying all the boundary conditions. (Note that $u = A\cos\theta$, where A is a constant, is a particular solution of (7), but does not satisfy the boundary conditions. Since (7) is non-linear the knowledge of a particular solution of the equation does not simplify the problem of finding the complete solution.) In order to obtain an approximate solution we shall look into the case where $M_\infty \gg 1$. For a perfect gas of constant specific heats we obtain from the shock relations

$$\epsilon = \frac{\rho_\infty}{\rho_s} = \frac{(\gamma-1)M_\infty^2 \sin^2\sigma + 2}{(\gamma+1)M_\infty^2 \sin^2\sigma} \tag{14}$$

when $M_\infty \gg 1$ (i.e. $M_\infty \longrightarrow \infty$) this becomes

$$\epsilon = \frac{\rho_\infty}{\rho_s} \approx \frac{\gamma-1}{\gamma+1} \tag{15}$$

Thus, in the limiting case of infinite M_∞ the boundary condition (13) becomes

$$\left(\frac{du}{d\theta}\right)_s = -\epsilon U \sin\sigma = -\frac{\gamma-1}{\gamma+1} U \sin\sigma \tag{16}$$

Now, if $\gamma - 1 \ll 1$ then $\epsilon \ll 1$ and so $\left(\frac{du}{d\theta}\right)_s = O(\epsilon) \ll 1$.

For an oblique shock we readily find that

$$\tan(\sigma - \theta_s) = \frac{v_s}{u_s} = \epsilon \tan\sigma \tag{17}$$

This relation shows that if ϵ is small (which means $M_\infty \gg 1$ and $\gamma \approx 1$) and the shock is not nearly normal, then the streamlines behind the shock must lie close to the shock. That is, since

$$\tan(\sigma - \theta_s) = O(\epsilon) \ll 1$$

we may write

$$\tan(\sigma - \theta_s) \approx (\sigma - \theta_s) = O(\epsilon) \ll 1 \tag{18}$$

Thus we expect the angular width of the shock layer to be $O(\epsilon)$.

We are now in a position to estimate orders of magnitude for u and its derivatives appearing in equation (7). At the shock $u_s = U\cos\sigma = O(1)$ and $\left(\frac{du}{d\theta}\right)_s = O(\epsilon)$. Since $\sigma - \theta = O(\epsilon)$, where $\theta_c \leq \theta \leq \sigma$, we have

$$u \approx u_s + \left(\frac{du}{d\theta}\right)_s (\theta - \sigma) = O(1) + O(\epsilon) \cdot O(\epsilon) \tag{19}$$

or

$$u = O(1) + O(\epsilon^2) \tag{20}$$

Since $\left(\frac{du}{d\theta}\right)_s = O(\epsilon)$ and $\left(\frac{du}{d\theta}\right)_c = 0$ we expect $\frac{du}{d\theta} = O(\epsilon)$ throughout the shock layer. The second derivative of u may be analyzed as follows, since the shock layer is thin

$$\frac{d^2u}{d\theta^2} \approx \frac{\left(\frac{du}{d\theta}\right)_s - \left(\frac{du}{d\theta}\right)_c}{\sigma - \theta_c} = \frac{O(\epsilon) - 0}{O(\epsilon)} = O(1) \tag{21}$$

To summarize, we have obtained the following orders of magnitude for u and its first two derivatives

$$\left.\begin{aligned} u &= O(1) \\ \frac{du}{d\theta} &= O(\epsilon) \\ \frac{d^2u}{d\theta^2} &= O(1) \end{aligned}\right\} \tag{22}$$

If we now examine equation (7) using the results of (22) we see that the left hand side of (7) is $O(\epsilon^2) \lll 1$ (for M_∞ very large and γ near 1, which will be assumed throughout this section). The expression in parentheses on the right hand side of (7) is seen to be O(1). The rest of the right hand side is equal to a^2 (from equation (5)). At the shock

$$p_s - p_\infty = \rho_\infty U^2 \sin^2\sigma (1-\epsilon) \tag{23}$$

or

$$\frac{p_s}{\rho_\infty U^2} - \frac{p_\infty}{\rho_\infty U^2} = \sin^2\sigma (1-\epsilon)$$

Since

$$\frac{p_\infty}{\rho_\infty U^2} = \frac{a_\infty^2}{\gamma U^2} = \frac{1}{\gamma M_\infty^2} \tag{24}$$

we may now write

$$\frac{p_s}{\rho_\infty U^2} - \frac{1}{\gamma M_\infty^2} = \sin^2\sigma (1-\epsilon)$$

For $M_\infty \longrightarrow \infty$ the second term on the left goes to zero and we are left with

$$p_s \approx \rho_\infty U^2 \sin^2\sigma (1-\epsilon) \tag{25}$$

or

$$\frac{p_s}{\rho_s} \approx \epsilon U^2 \sin^2\sigma (1-\epsilon)$$

or

$$a_s^2 = \frac{\gamma p_s}{\rho_s} \approx \epsilon\gamma U^2 \sin^2\sigma (1-\epsilon) \tag{26}$$

Thus

$$a_s^2 = O(\epsilon) \tag{27}$$

and we would expect $a^2 = O(\epsilon)$ throughout the shock layer. Hence

$$\frac{\gamma-1}{2}\left[q_\infty^2 - u^2 - \left(\frac{du}{d\theta}\right)^2\right] = a^2 = O(\epsilon) \tag{28}$$

Combining equation (28) with our previous results shows that the left hand side of equation (7) is $O(\epsilon^2)$ while the right hand side is $O(\epsilon)$. To lowest order in ϵ then, the right hand side is equal to zero, and since the first factor is not zero this becomes

$$\frac{d^2u}{d\theta^2} + \cot\theta\,\frac{du}{d\theta} + 2u = 0 \tag{29}$$

Equation (29), unlike equation (7) from which it came, is linear. From equation (3) we see that this is the same equation one would obtain under the assumption of constant density in the shock layer. The complete solution of equation (29) subject to the boundary conditions can be obtained in terms of Legendre functions (see Hayes and Probstein, p. 143). However, this is not really necessary if one is making a thin layer analysis based on ϵ being small and yields no more information than does a much simpler analysis. This follows from the fact that, if $\cot\theta = O(1)$ at most in the shock layer, then

$$\cot\theta\,\frac{du}{d\theta} = O(\epsilon)$$

so that the middle term in equation (29) is $O(\epsilon)$ while the two other terms in this equation are $O(1)$. Hence to lowest order in ϵ equation (7) reduces not to equation (29) but to

$$\frac{d^2u}{d\theta^2} + 2u = 0 \tag{30}$$

The middle term cannot be consistently retained for more accuracy since there are other terms in equation (7) which are of the same order as this term and are not included in (29). (The correct first order equation will be derived later.) Hence, while equation (29) together with the boundary conditions yields an exact solution for constant density flow about a cone it does not consistently represent the lowest order formulation of the problem in a hypersonic analysis.

Returning to equation (30), the solution of this equation satisfying the boundary conditions (12) and (13) is given by

$$u = U\cos\sigma\cos\sqrt{2}(\sigma-\theta) + \frac{1}{\sqrt{2}}\epsilon U\sin\sigma\sin\sqrt{2}(\sigma-\theta) \tag{31}$$

To find the cone angle, θ_c, we differentiate this result and set it equal to zero, according to (8). We obtain in this manner the relation

$$\sqrt{2}\tan\sqrt{2}(\sigma-\theta_c) = \epsilon\tan\sigma \tag{32}$$

Since ϵ is assumed small we may write this as

$$2(\sigma-\theta_c) \approx \epsilon\tan\sigma \tag{33}$$

or

$$\sigma-\theta_c \approx \frac{1}{2}\epsilon\tan\sigma \tag{34}$$

Equation (31) can be somewhat simplified by introducing the flow velocity at the cone surface. Using equation (32) to replace ϵ in the second term on the right hand side of (31) we obtain

$$u = \frac{U\cos\sigma}{\cos\sqrt{2}(\sigma-\theta_c)}\left[\cos\sqrt{2}(\sigma-\theta)\cos\sqrt{2}(\sigma-\theta_c) + \sin\sqrt{2}(\sigma-\theta_c)\sin\sqrt{2}(\sigma-\theta)\right]$$

or

$$u = \frac{U\cos\sigma\,\cos\sqrt{Z}(\theta-\theta_c)}{\cos\sqrt{Z}(\sigma-\theta_c)} \tag{35}$$

When $\theta = \theta_c$, $u = u_c$, so (35) gives

$$u_c = \frac{U\cos\sigma}{\cos\sqrt{Z}(\sigma-\theta_c)} \tag{36}$$

Substituting (36) into (35) we obtain

$$u = u_c\cos\sqrt{Z}(\theta-\theta_c) \tag{37}$$

which gives u directly in terms of u_c and θ_c. The solution in this form was first given by Zienkiewicz.

Referring back to (31) again, we may expand the cos and sin terms since $(\sigma-\theta) \ll 1$, to obtain

$$u \approx U\cos\sigma - U\cos\sigma(\sigma-\theta)^2 + (\epsilon U\sin\sigma)(\sigma-\theta) \tag{38}$$

Since $(\sigma-\theta) = O(\epsilon)$ we may write

$$u = U\cos\sigma + O(\epsilon^2) = O(1) + O(\epsilon^2) \tag{39}$$

which shows that for fixed U and σ that u is constant to $O(\epsilon)$, a result we found earlier when estimating orders of magnitude for u and its derivatives.

IV. SOLUTION FOR INFINITE MACH NUMBER AND $\gamma - 1 \ll 1$

We now wish to obtain the solution to the cone problem in terms of an expansion in density ratio across the shock. For simplification we first treat the case of infinite free stream Mach number so that $\epsilon_\ell = \frac{\rho_\infty}{\rho_s} = \frac{\gamma-1}{\gamma+1}$. We assume that the shock location, given by σ, is known along with the upstream conditions. The generic angle θ will be measured relative to σ.

The shock conditions and our first order solution have already shown that $(\sigma - \theta_c) = O(\epsilon)$. Our series expansion for any one of the dependent variables will be of the form

$$f_0(\theta) + f_1(\theta)\epsilon_\ell + f_2(\theta)\epsilon_\ell^2 + f_3(\theta)\epsilon_\ell^3 + \ldots$$

Since the value of θ is itself unimportant, the important quantity being $(\sigma - \theta)$ we would want the f_i to be functions of this angular difference. However $(\sigma - \theta_c) = O(\epsilon)$, while a necessary requirement of the f_i is that they be $O(1)$. Thus, it seems natural that in the above expansion we should introduce a new angular variable

$$\frac{\theta - \sigma}{\epsilon_\ell} = \psi \text{, say} \tag{40}$$

(since $\sigma > \theta$ always, ψ will be negative throughout the shock layer). That is, all the expansions of dependent variables will be of the form

$$f_0(\psi) + f_1(\psi)\epsilon_\ell + f_2(\psi)\epsilon_\ell^2 + f_3(\psi)\epsilon_\ell^3 + \ldots \tag{41}$$

where all the $f_i(\psi)$ are assumed $O(1)$.

Since the T-M equation (Taylor-Maccoll, an abbreviation we shall use henceforth) involves u only, for the moment we need expand only u in a series in ϵ_ℓ. Since we already know u is constant to $O(\epsilon_\ell)$ we write this series as

$$\frac{u}{U} = u_0 + \epsilon_\ell^2 \underline{u} = u_0 + \epsilon_\ell^2 \left(u_1 + u_2 \epsilon_\ell + u_3 \epsilon_\ell^2 + u_4 \epsilon_\ell^3 + \ldots \right) \tag{42}$$

where u_0 = const. and all the u_i for $i \geq 1$ are functions of ψ and are non-dimensionalized with respect to free stream velocity U. We assume all the $u_i(\psi)$ and their derivatives are O(1). The derivatives of u are

$$\left.\begin{aligned} \frac{1}{U}\frac{du}{d\psi} &= \epsilon_\ell^2 \left(u_1' + u_2' \epsilon_\ell + u_3' \epsilon_\ell^2 + \ldots \right) = \epsilon_\ell^2 \underline{u}' \\ \frac{1}{U}\frac{d^2u}{d\psi^2} &= \epsilon_\ell^2 \left(u_1'' + u_2'' \epsilon_\ell + u_3'' \epsilon_\ell^2 + \ldots \right) = \epsilon_\ell^2 \underline{u}'' \end{aligned}\right. \tag{43}$$

To transform these to θ derivatives, we use, from (40)

$$d\theta = \epsilon_\ell \, d\psi \tag{44}$$

The derivatives of u then become

$$\left.\begin{aligned} \frac{1}{U}\frac{du}{d\theta} &= \frac{du}{d\psi}\frac{1}{\epsilon_\ell} = \epsilon_\ell \left(u_1' + u_2' \epsilon_\ell + u_3' \epsilon_\ell^2 + \ldots \right) = \epsilon_\ell \underline{u}' \\ \frac{1}{U}\frac{d^2u}{d\theta^2} &= \frac{d^2u}{d\psi^2}\frac{1}{\epsilon_\ell^2} = u_1'' + u_2'' \epsilon_\ell + u_3'' \epsilon_\ell^2 + \ldots = \underline{u}'' \end{aligned}\right. \tag{45}$$

Equations (42) and (45) give the following orders of magnitude

$$\left.\begin{aligned} u &= O(1) \\ \frac{du}{d\theta} &= O(\epsilon_\ell) \\ \frac{d^2u}{d\theta^2} &= O(1) \end{aligned}\right\}$$

which agree with those given in equation (22).

Dividing both sides of the T-M equation by U^3 and denoting u/U by $\overline{u}$ we obtain

$$\left(\frac{d\bar{u}}{d\theta}\right)^2\left(\bar{u}+\frac{d^2\bar{u}}{d\theta^2}\right)=\frac{\gamma-1}{2}\left[\left(\frac{q_0}{U}\right)^2-\bar{u}^2-\left(\frac{d\bar{u}}{d\theta}\right)^2\right]\left(\frac{d^2\bar{u}}{d\theta^2}+\cot\theta\frac{d\bar{u}}{d\theta}+2\bar{u}\right)$$

Using equations (40) and (45) to change θ to ψ and equation (42) we obtain

$$\epsilon_\ell^2\,\underline{u}'^2\left(u_0+\epsilon_\ell^2\underline{u}+\underline{u}''\right)=\frac{\gamma-1}{2}\left[\left(\frac{q_0}{U}\right)^2-\left(u_0+\epsilon_\ell^2\underline{u}\right)^2-\epsilon_\ell^2\underline{u}'^2\right]\times$$

$$\left[\underline{u}''+\epsilon_\ell\cot(\sigma+\epsilon_\ell\psi)\underline{u}'+2u_0+2\epsilon_\ell^2\underline{u}\right] \tag{46}$$

For a perfect gas the following relation holds

$$\left(\frac{q_0}{U}\right)^2=1+\frac{2}{(\gamma-1)M_\infty^2}$$

Since $\frac{\gamma-1}{2}=\frac{\epsilon_\ell}{1-\epsilon_\ell}$ this may be written

$$\left(\frac{q_0}{U}\right)^2=1-\frac{1}{M_\infty^2}+\frac{1}{\epsilon_\ell M_\infty^2} \tag{47}$$

Substituting for γ and $\left(\frac{q_0}{U}\right)^2$ with these relations in equation (46) we obtain

$$\epsilon_\ell^2\,\underline{u}'^2\left(\underline{u}''+u_0+\epsilon_\ell^2\underline{u}\right)=\frac{\epsilon_\ell}{1-\epsilon_\ell}\left[1-\frac{1}{M_\infty^2}+\frac{1}{\epsilon_\ell M_\infty^2}-u_0^2-2\epsilon_\ell^2u_0\underline{u}-\epsilon_\ell^4\underline{u}^2-\epsilon_\ell^2\underline{u}'^2\right]\times$$

$$\left[\underline{u}''+\epsilon_\ell\cot(\sigma+\epsilon_\ell\psi)\underline{u}'+2u_0+2\epsilon_\ell^2\underline{u}\right] \tag{48}$$

Since we are now developing an infinite Mach number theory it may seem odd having M_∞^2 appear in the above equation. Since ϵ_ℓ multiplies M^2_∞ in (48) it is not permitted simply to drop this term. We proceed in the following manner: Equation (48) may be written

$$\epsilon_\ell^2(1-\epsilon_\ell)\underline{u}'^2(\underline{u}''+u_0+\epsilon_\ell^2\underline{u}) = \left[\epsilon_\ell\left(1-\frac{1}{M_\infty^2}+\frac{1}{\epsilon_\ell M_\infty^2}-u_0^2\right)-\epsilon_\ell^3(2u_0\underline{u}+\underline{u}'^2)-\epsilon_\ell^5\underline{u}^2\right]\times$$

$$\left[\underline{u}''+\epsilon_\ell\cot(\sigma+\epsilon_\ell\psi)\underline{u}'+2u_0+2\epsilon_\ell^2\underline{u}\right]$$

(49)

Now consider the first expression in the first bracket on the right hand side. Since u_0 is a constant it must equal the value of u/U behind the shock, or $\cos\sigma$. Thus, we may write

$$\epsilon_\ell\left(1-\frac{1}{M_\infty^2}+\frac{1}{\epsilon_\ell M_\infty^2}-u_0^2\right) = \sin^2\sigma\left(\epsilon_\ell-\frac{\epsilon_\ell}{M_\infty^2\sin\sigma}+\frac{1}{M_\infty^2\sin^2\sigma}\right) \tag{50}$$

From the shock relations we obtain

$$\epsilon_\ell = \frac{\rho_\infty}{\rho_s} = \frac{(\gamma-1)M_\infty^2\sin^2\sigma+2}{(\gamma+1)M_\infty^2\sin^2\sigma} \tag{51}$$

which may also be written

$$\epsilon_\ell = \frac{\gamma-1}{\gamma+1}+\frac{2}{(\gamma+1)M_\infty^2\sin^2\sigma}$$

Since $\epsilon_\ell = \frac{\gamma-1}{\gamma+1}$ this can now be written

$$\epsilon = \epsilon_\ell-\frac{\epsilon_\ell}{M_\infty^2\sin^2\sigma}+\frac{1}{M_\infty^2\sin^2\sigma} \tag{52}$$

Substituting equation (52) into (50) we obtain

$$\epsilon_\ell\left(1-\frac{1}{M_\infty^2}+\frac{1}{\epsilon_\ell M_\infty^2}-u_0^2\right) = \epsilon\sin^2\sigma \tag{53}$$

We now substitute this result into equation (49), at the same time letting $\epsilon = \epsilon_\ell$. We obtain

$$\epsilon_2(1-\epsilon_2)\underline{u}'^2(\underline{u}''+u_0+\epsilon_2^2\underline{u}) = \left[\sin^2\sigma - \epsilon_2^2(2u_0\underline{u}+\underline{u}'^2) - \epsilon_2^4\underline{u}^2\right]\times$$
$$\left[\underline{u}''+\epsilon_2\cot(\sigma+\epsilon_2\psi)\underline{u}'+2u_0+2\epsilon_2^2\underline{u}\right] \tag{54}$$

Equation (54) is the basic equation we will use in developing our series solution.

To obtain the boundary conditions on $\underline{u}$ we consider the shock boundary conditions, equations (12) and (13). These give, since $\psi = 0$ when $\theta = \sigma$,

$$\frac{u}{U} = u_0 + \epsilon_2^2\underline{u}(0) = \cos\sigma$$
$$\frac{1}{U}\left(\frac{du}{d\psi}\right)_{\psi=0} = -\epsilon_2^2\sin\sigma$$

or

$$\left.\begin{aligned} &\underline{u}(0) = 0 \\ &\left(\frac{d\underline{u}}{d\psi}\right)_{\psi=0} = -\sin\sigma \end{aligned}\right\} \tag{55}$$

From the boundary condition at the cone, $V_c = 0$, we obtain

$$\left(\frac{d\underline{u}}{d\psi}\right)_{\psi=\psi_c} = 0 \tag{56}$$

where $\psi_c = \dfrac{\theta_c-\sigma}{\epsilon_2}$.

Before we substitute the series for $\underline{u}$ into equation (54) we must expand cot $(\sigma+\epsilon_2\psi)$ in a series in ϵ_2. This is done by expanding this function in a Taylor series about the angle σ. In this way we obtain

$$\cot(\sigma+\epsilon_\ell\psi) = \cot\sigma - (\psi\csc^2\sigma)\,\epsilon_\ell + (\psi^2\csc^2\sigma\cot\sigma)\,\epsilon_\ell^2 - \frac{1}{3}\left[\psi^3\csc^4\sigma(1+2\cos^2\sigma)\right]\epsilon_\ell^3 + \ldots \tag{57}$$

The series for $\underline{u}$ is

$$\underline{u} = u_1 + u_2\epsilon_\ell + u_3\epsilon_\ell^2 + u_4\epsilon_\ell^3 + \ldots \tag{58}$$

Substituting the series (57) and (58) into equation (54) we obtain

$$\begin{aligned}
&\epsilon_\ell(1-\epsilon_\ell)\left[u_1' + u_2'\epsilon_\ell + u_3'\epsilon_\ell^2 + u_4'\epsilon_\ell^3 + \ldots\right]^2\Big[u_1'' + u_2''\epsilon_\ell + u_3''\epsilon_\ell^2 + u_4''\epsilon_\ell^3 + \quad + u_0 \\
&\qquad + \epsilon_\ell^2(u_1 + u_2\epsilon_\ell + u_3\epsilon_\ell^2 + u_4\epsilon_\ell^3 + \ldots)\Big] \\
&= \Big[\sin^2\sigma - \epsilon_\ell^2\left\{2u_0(u_1 + u_2\epsilon_\ell + u_3\epsilon_\ell^2 + u_4\epsilon_\ell^3 + \ldots) + (u_1' + u_2'\epsilon_\ell + u_3'\epsilon_\ell^2 + u_4'\epsilon_\ell^3 + \ldots)^2\right\} \\
&\qquad - \epsilon_\ell^4(u_1 + u_2\epsilon_\ell + u_3\epsilon_\ell^2 + \quad)^2\Big] \times \Big[u_1'' + u_2''\epsilon_\ell + u_3''\epsilon_\ell^2 + u_4''\epsilon_\ell^3 + \ldots \\
&\qquad + \epsilon_\ell\left\{\cot\sigma - (\psi\csc^2\sigma)\epsilon_\ell + (\psi^2\csc^2\sigma\cot\sigma)\epsilon_\ell^2 + \quad\right\}\left\{u_1' + u_2'\epsilon_\ell + u_3'\epsilon_\ell^2 + u_4'\epsilon_\ell^3 + \ldots\right\} \\
&\qquad + 2u_0 + 2\epsilon_\ell^2(u_1 + u_2\epsilon_\ell + u_3\epsilon_\ell^2 + u_4\epsilon_\ell^3 + \ldots)\Big]
\end{aligned} \tag{59}$$

We are now in a position to equate coefficients of like powers of ϵ_ℓ on both sides of (59). Doing this we obtain:

Coefficient of ϵ_ℓ^0

$$\sin^2\sigma\,(u_1'' + 2u_0) = 0 \tag{60}$$

Coefficient of ϵ_ℓ

$$u_1'^2(u_1'' + u_0) = \sin^2\sigma\,(u_2'' + u_1'\cot\sigma) \tag{61}$$

Coefficient of ϵ_ℓ^2

$$u_1'^2 u_2'' + 2u_1' u_2'(u_1''+u_0) - u_1'^2(u_1''+u_0)$$
$$= \sin^2\sigma\left[u_3'' - \psi\csc^2\sigma\, u_1' + \cot\sigma\, u_2' + 2u_1\right]$$
$$- (2u_0 u_1 + u_1'^2)(u_1'' + 2u_0) \qquad (62)$$

We have written down these equations only as far as the coefficient of ϵ_ℓ^2, but in principle we can go as far as we please. Note that the equations are all linear and must be solved in sequential order. Note also that the determining equation for any u_i contains that quantity in only one position, so for each equation the u_i is given immediately in terms of a quadrature. The circumstance that the equations for the u_i are all linear whereas the original T-M equation is non-linear is due to the fact that the non-linear terms in the T-M equation are of higher order in ϵ_ℓ than the linear terms, so their effect is consequently felt in succedding equations where they act as linear terms.

In order to solve the above series of equations, we need boundary conditions on the u_i. These are supplied by the shock conditions on $\underline{u}$, equations (55). Substituting the series for $\underline{u}$ into these equations, we obtain the following conditions on the u_i

$$\left.\begin{aligned} u_i(0) &= 0, \quad i \geq 1 \\ u_1'(0) &= -\sin\sigma \\ u_i'(0) &= 0, \quad i > 1 \end{aligned}\right\} \qquad (63)$$

We are now in a position to solve equations (60), (61), ...
We solve them in turn:

Coefficient of ϵ_1°

Equation (60) reduces to

$$u_1'' = -2u_0$$

Since $u_0 = \cos\tau$ = const. this may be integrated twice to give

$$u_1 = -\psi^2\cos\tau + c\psi + d$$

Using the boundary conditions, equation (63), this becomes

$$u_1 = -\psi\sin\tau\,(\psi\cot\tau + 1) \tag{64}$$

Coefficient of ϵ_1

This is the first equation in which the cotθ on the right hand side of the T-M equation appears. Equation (61) shows clearly that when this term is taken into account so must terms on the left hand side of the T-M equation. In fact, equation (61) shows that to this order the left hand side of the T-M equation need only be approximated by its lowest order value.

Substituting $u_0 = \cos\tau$ and for u_1 from (64) we may write (61) as

$$-(2\psi\cot\tau + 1)^2\cos\tau = u_2'' - \cos\tau\,(2\psi\cot\tau + 1)$$

which when solved for u_2'' gives

$$u_2'' = -2\psi\,(2\psi\cot\tau + 1)\cos\tau\cot\tau$$

Integrating twice and using the boundary conditions (63) we obtain

$$u_2'' = -\tfrac{1}{3}\cos\tau\cot\tau\,(\psi^4\cot\tau + \psi^3) \tag{65}$$

Coefficient of ϵ_ℓ^2

If we substitute $u_0 = \cos\sigma$ into equation (62) and for u_1 and u_2 from equations (64) and (65), and integrate twice using the boundary conditions, we obtain, after a good deal of manipulation, the following expression for u_3

$$u_3 = -\frac{32}{45}\psi^6\cos\sigma\cot^4\sigma - \frac{22}{15}\psi^5\cos\sigma\cot^3\sigma - \frac{11}{12}\psi^4\cos\sigma\cot^2\sigma + \frac{1}{6}\psi^3\sin\sigma(1+\cot^2\sigma) + \frac{1}{2}\psi^2\cos\sigma \tag{66}$$

Due to the increasing complexity of the algebra involved we shall not proceed any further in determining the u_i. We now wish to gather together the terms in the series for u we have already obtained. Substituting for u_0, u_1, u_2 and u_3 in equation (42) this series becomes

$$\frac{u}{U} = \cos\sigma + \epsilon_\ell^2\Big\{-\psi\sin\sigma(\psi\cot\sigma+1) - \frac{1}{3}\cos\sigma\cot\sigma(\psi^4\cot\sigma + \psi^3)\epsilon_\ell + \Big[-\frac{32}{45}\psi^6\cos\sigma\cot^4\sigma - \frac{22}{15}\psi^5\cos\sigma\cot^3\sigma - \frac{11}{12}\psi^4\cos\sigma\cot^2\sigma + \frac{1}{6}\psi^3\sin\sigma(1+\cot^2\sigma) + \frac{1}{2}\psi^2\cos\sigma\Big]\epsilon_\ell^2 + \ldots\Big\}$$

If we factor out $\cos\sigma$ this equation may be more neatly written as

$$\frac{u}{U\cos\sigma} = 1 + \epsilon_\ell^2\Big\{-\psi\tan\sigma(\psi\cot\sigma+1) - \frac{1}{3}\psi^3\cot\sigma(\psi\cot\sigma+1)\epsilon_\ell + \Big[-\frac{32}{45}\psi^6\cot^4\sigma - \frac{22}{15}\psi^5\cot^3\sigma - \frac{11}{12}\psi^4\cot^2\sigma + \frac{1}{6}\psi^3\tan\sigma(1+\cot^2\sigma) + \frac{1}{2}\psi^2\Big]\epsilon_\ell^2 + \ldots\Big\} \tag{67}$$

Equation (67) gives u up to fourth order in ϵ_ℓ ; the number one on the right hand side of this equation represents the Newtonian value, while the terms in curled brackets represent higher order corrections to this value.

To find the cone angle, θ_c, we must differentiate equation (67) with respect to ψ and set the resulting expression equal to zero. Before doing this let us rearrange the right hand side of (67) in terms of powers of ψ. The equation may be written

$$\frac{u}{U\cos\sigma} = 1 + \epsilon_l \Big\{ -\psi\tan\sigma + \left(\tfrac{1}{2}\epsilon_l^2 - 1\right)\psi^2 + \left[\tfrac{1}{6}\tan\sigma(1+\cot^2\sigma)\epsilon_l^2 - \tfrac{1}{3}\epsilon_l\cot\sigma\right]\psi^3$$
$$- \left(\tfrac{11}{12}\epsilon_l^2\cot^2\sigma + \tfrac{1}{3}\epsilon_l\cot^2\sigma\right)\psi^4 - \left(\tfrac{22}{15}\epsilon_l^2\cot^3\sigma\right)\psi^5$$
$$- \left(\tfrac{32}{45}\epsilon_l^2\cot^4\sigma\right)\psi^6 + \ldots \Big\} \tag{68}$$

Differentiating equation (68) and setting the resulting expression equal to zero we obtain

$$-\tan\sigma + (\epsilon_l^2 - 2)\psi_c + \left[\tfrac{1}{2}\tan\sigma(1+\cot^2\sigma)\epsilon_l^2 - \epsilon_l\cot\sigma\right]\psi_c^2$$
$$-\left(\tfrac{11}{3}\epsilon_l^2 + \tfrac{4}{3}\epsilon_l\right)\cot^2\sigma\,\psi_c^3 - \tfrac{22}{3}\epsilon_l^2\cot^3\sigma\,\psi_c^4 - \tfrac{64}{15}\epsilon_l^2\cot^4\sigma\,\psi_c^5 = 0 \tag{69}$$

Dividing through by $\tan\sigma$ and letting $S = \psi_c\cot\sigma$, we may write this equation as

$$\tfrac{64}{15}\epsilon_l^2 S^5 + \tfrac{22}{3}\epsilon_l^2 S^4 + \left(\tfrac{11}{3}\epsilon_l^2 + \tfrac{4}{3}\epsilon_l\right)S^3 - \left(\tfrac{1}{2}\epsilon_l^2\sec^2\sigma - \epsilon_l\right)S^2$$
$$+ (2 - \epsilon_l^2)S + 1 = 0 \tag{70}$$

This equation, a fifth degree polynomial in S, involves terms of O(1), O(ϵ_l) and O(ϵ_l^2). To lowest order, dropping all terms involving ϵ_l, it reduces to

$$S = -\tfrac{1}{2} \tag{71}$$

Using the definitions of S and ψ_c this becomes

$$\sigma - \theta_c = \frac{1}{2}\epsilon_\ell \tan\sigma \tag{72}$$

which agrees with a result we obtained earlier, equation (34), and with the constant density value (Hayes and Probstein, p. 146, equation 4.2.20).

If we retain only those terms in (70) which are O(1) or O(ϵ_ℓ) the equation reduces to

$$\frac{4}{3}\epsilon_\ell S^3 + \epsilon_\ell S^2 + 2S + 1 = 0 \tag{73}$$

Since this is a cubic equation in S it can be solved exactly to give S in terms of ϵ_ℓ . However, since this equation will give S accurate to only O(ϵ_ℓ) due to our having dropped O(ϵ_ℓ^2) terms, it is not necessary to go to this lengthy detail. The lowest order equation in S having given $S = -\frac{1}{2}$, let us therefore assume $S = -\frac{1}{2}(1+P)$, where P is to be no larger than O(ϵ_ℓ), and obtain a corresponding equation for P. Substituting this expression into equation (73) we obtain

$$\frac{1}{6}\epsilon_\ell(1+P)^3 - \frac{1}{4}\epsilon_\ell(1+P)^2 + P = 0 \tag{74}$$

Since P = O(ϵ_ℓ) and we want only the leading term in the expression for P let us now expand $(1+P)^3$ and $(1+P)^2$ and retain only the first two terms in each expression. We obtain

$$\frac{1}{6}\epsilon_\ell(1+3P) - \frac{1}{4}\epsilon_\ell(1+2P) + P = 0$$

which gives

$$P = \frac{1}{12}\epsilon_\ell$$

Thus

$$S = -\frac{1}{2}\left(1 + \frac{1}{12}\epsilon_\ell\right) \tag{75}$$

and hence

$$\sigma - \epsilon_c = \frac{1}{2}\epsilon_\ell\left(1 + \frac{1}{12}\epsilon_\ell\right)\tan\sigma \tag{76}$$

This result is in agreement to O(ϵ_ℓ^2) with Hayes and Probstein (p. 150, equation 4.2.36), who obtain their result by correcting the constant density expression, equation (72), using an argument based on the approximate constancy of the Howarth-Dorodnitsyn variable $\int_0 \rho\, dy$ to take into account the small density variations in the shock layer.

So far we have obtained S to O(ϵ_ℓ). To proceed to the next highest order in ϵ_ℓ we must return to equation (70). As before we introduce a new variable T where

$$S = -\tfrac{1}{2}(1+T) \tag{77}$$

where we assume T = O(ϵ_ℓ). Substituting (77) into the full equation for S we obtain

$$-\frac{2}{15}\epsilon_\ell^2(1+T)^5 + \frac{11}{24}\epsilon_\ell^2(1+T)^4 - \frac{1}{24}(11\epsilon_\ell^2 + 4\epsilon_\ell)(1+T)^3$$
$$-\frac{1}{4}\left(\frac{1}{2}\epsilon_\ell^2\sec^2\sigma - \epsilon_\ell\right)(1+T)^2 - \left(1 - \frac{1}{2}\epsilon_\ell^2\right)(1+T) + 1 = 0$$

Now, letting T = $\epsilon_\ell W$ and dividing through by ϵ_ℓ , we obtain

$$-\frac{2}{15}\epsilon_\ell(1+\epsilon_\ell W)^5 + \frac{11}{24}\epsilon_\ell(1+\epsilon_\ell W)^4 - \frac{1}{24}(11\epsilon_\ell + 4)(1+\epsilon_\ell W)^3$$
$$-\frac{1}{4}\left(\frac{1}{2}\sec^2\sigma\,\epsilon_\ell - 1\right)(1+\epsilon_\ell W)^2 + \frac{1}{2}\epsilon_\ell(1+\epsilon_\ell W) - W = 0 \tag{78}$$

Since we want to obtain W to O(ϵ_ℓ) we must retain those terms in the above equation which are O(1) or O(ϵ_ℓ) and may drop all higher order terms.

Equation (78) then becomes

$$-\frac{2}{15}\epsilon_\ell + \frac{11}{24}\epsilon_\ell - \frac{1}{24}(11\epsilon_\ell + 4) - \frac{1}{2}\epsilon_\ell W - \frac{1}{4}\left(\frac{1}{2}\sec^2\sigma\,\epsilon_\ell - 1\right)$$
$$+ \frac{1}{2}\epsilon_\ell W + \frac{1}{2}\epsilon_\ell - W = 0$$

which when solved for W gives

$$W = \frac{1}{12} + \left(\frac{11}{30} - \frac{1}{8}\sec^2\sigma\right)\epsilon_\ell \qquad (79)$$

which can also be written

$$W = \frac{1}{12} + \frac{1}{8}\left(\frac{29}{15} - \tan^2\sigma\right)\epsilon_\ell \qquad (80)$$

Then, since $S = \psi_c \cot\sigma$, where

$$S = -\frac{1}{2}(1+T) = -\frac{1}{2}(1+\epsilon_\ell W) \qquad (81)$$

we finally obtain

$$\sigma - \epsilon_c = \frac{1}{2}\epsilon_\ell\left[1 + \frac{1}{12}\epsilon_\ell + \frac{1}{8}\epsilon_\ell^2\left(\frac{29}{15} - \tan^2\sigma\right)\right]\tan\sigma \qquad (82)$$

which gives the relationship between cone angle and shock angle to third order in ϵ_ℓ .

(In considering the application to a particular gas of a theory based upon the assumption that ϵ_ℓ may be used in place of ϵ , our first question must be the degree to which this assumption is met for this gas. The replacement of ϵ by ϵ_ℓ requires infinite Mach number, a requirement that cannot be met in practice. We should then ask for what range of Mach numbers is ϵ sufficiently close to ϵ_ℓ so that the theory may be reasonably applied. We know that ϵ decreases from 1 to the value ϵ_ℓ as the normal Mach number, $M_\infty \sin\sigma$, increases from 1 to infinity. For the application of the theory to the widest range of Mach numbers for a particular gas it would be best if the greatest rate of decrease of ϵ with normal Mach number should occur as near $M_\infty \sin\sigma = 1$ as possible. From equation (57) we have

$$\frac{1}{\epsilon} = \frac{\rho_s}{\rho_\infty} = \frac{(\gamma+1) M_n^2}{(\gamma-1) M_n^2 + 2}$$

in which we have let $M_n = M_\infty \sin\sigma$. If we differentiate this equation twice we obtain

$$\frac{d^2(\frac{1}{\epsilon})}{d M_n^2} = \frac{4(\gamma+1)\left[2 - 3(\gamma-1) M_n^2\right]}{\left[(\gamma-1) M_n^2 + 2\right]^3}$$

Setting this equal to zero we obtain

$$M_n^2 = \frac{2}{3(\gamma-1)}$$

If we set γ = 1.4 in this expression we obtain

$$M_n = \sqrt{5/3} \simeq 1.3$$

Setting γ = 5/3 there results

$$M_n = 1$$

This shows that helium is the gas for which the rate of decrease of ϵ with Mach number has a maximum at a Mach number of one. It is not true, however, that among different gases it has the greatest rate of decrease at M_n = 1. At M_n = 1 it turns out that

$$\frac{d(\frac{1}{\epsilon})}{d M_n} = \frac{4}{\gamma+1}$$

which shows that the maximum rate of decrease of ϵ at M_n = 1 for different gases (different γ) occurs for the minimum γ , that is, for γ = 1.)

V. SOLUTION FOR LARGE MACH NUMBER AND $\gamma - 1 \ll 1$

In the last section we developed a method of solution of the cone problem in terms of an expansion in the limiting density ratio, $(\gamma-1)/(\gamma+1)$. Our analysis was then limited to infinite Mach number. In this section, we want to analyze the problem for the case of large but not infinite free stream Mach number.

The most obvious approach now might be a double expansion in the limiting density ratio, ϵ_ℓ, and $1/M_\infty^2$ or $1/M_\infty^2 \sin^2\sigma$. However, a consideration of the governing T-M equation and the boundary conditions indicates the best approach is a double expansion in $\epsilon_\ell = \frac{\gamma-1}{\gamma+1}$, the limiting density ratio, and $\epsilon = \rho_\infty/\rho_s$, the actual density ratio across the shock. The latter parameter takes into account essentially the variation of the Mach number while the former accounts for the variation of γ. Since the basic approach of this entire paper is the smallness of the actual density ratio across the shock, it must, of course, be assumed that $\epsilon \ll 1$, but this condition can only hold if both $\gamma - 1 \ll 1$ and $M_\infty^2 \sin^2\sigma \gg 1$ (see equation (52)). Thus, the condition that the density ratio be small implies two other completely independent conditions (neglecting effects such as dissociation and ionization), one on the particular gas (γ variation) and the other on the particular flow conditions of this gas (M_∞ variation). So, while we may be making a double expansion there is really only one independent parameter, ϵ, whose range we may specify. Once this is done the range of ϵ_ℓ is determined (equation (52)). (According to equation (52) one of the requirements for ϵ to be small is not $M_\infty^2 \gg 1$ but $M_\infty^2 \sin^2\sigma \gg 1$, that is, the normal Mach number be large. Therefore, in places in this paper where note is made of the Mach number being large, as for example in the title of this section, it should be understood that it is actually the normal Mach number to which reference is being made. No distinction need be made between these two values in the case where $\sin\sigma = O(1)$, which taken together with our thin shock layer assumption means for large cone angles.)

Instead of equation (42) we now let

$$\frac{u}{U} = u_0 + \epsilon^2 \underline{u} \tag{83}$$

where

$$\underline{u} = \underline{u}_1 + \underline{u}_2\epsilon + \underline{u}_3\epsilon_\ell + \underline{u}_4\epsilon\epsilon_\ell + \underline{u}_5\epsilon^2 + \underline{u}_6\epsilon_\ell^2 + \quad . \; . \tag{84}$$

and u_0 is constant ($= \cos\sigma$) while the $\underline{u}_i$ for $i > 0$, are functions of $\psi = \frac{\theta - \sigma}{\epsilon}$. All the $\underline{u}_i(\psi)$ are assumed to be O(1). The derivatives of u are now

$$\left.\begin{aligned} \frac{1}{U}\frac{du}{d\psi} &= \epsilon^2 \underline{u}' = \epsilon\frac{du}{d\theta} \\ \frac{1}{U}\frac{d^2u}{d\psi^2} &= \epsilon^2 \underline{u}'' = \epsilon^2\frac{d^2u}{d\theta^2} \end{aligned}\right\} \tag{85}$$

The orders of magnitude of u and its derivatives agree with our earlier analysis.

If we now go through the same analysis as earlier on the T-M equation, equation (7), but carefully distinguish between ϵ and ϵ_ℓ we obtain, instead of equation (54), the following very similar equation

$$\epsilon(1-\epsilon_\ell)\underline{u}'^2(\underline{u}'' + u_0 + \epsilon^2\underline{u}) = \left[\sin^2\sigma - \epsilon\epsilon_\ell(2u_0\underline{u} + \underline{u}'^2) - \epsilon_\ell\epsilon^3\underline{u}^2\right] \times \left[\underline{u}'' + \epsilon\cot(\sigma + \epsilon\psi)\underline{u}' + 2u_0 + 2\epsilon^2\underline{u}\right] \tag{86}$$

The boundary conditions remain the same and are:

at the shock

$$\left.\begin{aligned} \underline{u}(0) &= 0 \\ \left(\frac{d\underline{u}}{d\psi}\right)_{\psi=0} &= -\sin\sigma \end{aligned}\right\} \tag{55}$$

at the cone surface

$$\left(\frac{d\underline{u}}{d\psi}\right)_{\psi=\psi_c} = 0 \tag{56}$$

Equation (86) is the basic equation in this section; we seek its solution subject to the boundary conditions, equations (55) and (56). As before, we shall assume the shock angle is known, so (86) need be solved subject to the boundary conditions (55) only. Equation (56) is then used to determine the cone angle. Note that equation (86) involves the two parameters ϵ and ϵ_ℓ and hence is in the correct form for our assumed form of expansion.

From equation (57) we have

$$\cot(\sigma+\epsilon\psi) = \cot\sigma - (\psi\csc^2\sigma)\epsilon + (\psi^2\csc^2\sigma\cot\sigma)\epsilon^2 - \frac{1}{3}\left[\psi^3\csc^4\sigma\,(1+2\cos^2\sigma)\right]\epsilon^3 + \ldots \tag{87}$$

Substituting equations (84) and (87) into equation (86) we obtain

$$\epsilon(1-\epsilon_\ell)\left[\underline{u}_1' + \underline{u}_2'\epsilon + \underline{u}_3'\epsilon_\ell + \underline{u}_4'\epsilon\epsilon_\ell + \underline{u}_5'\epsilon^2 + \underline{u}_6'\epsilon_\ell^2 + \ldots\right]^2\Big\{\underline{u}_1'' + \underline{u}_2''\epsilon + \underline{u}_3''\epsilon_\ell + \underline{u}_4''\epsilon\epsilon_\ell$$

$$+ \underline{u}_5''\epsilon^2 + \underline{u}_6''\epsilon_\ell^2 + \ldots + u_0 + \epsilon^2(\underline{u}_1 + \underline{u}_2\epsilon + \underline{u}_3\epsilon_\ell + \underline{u}_4\epsilon\epsilon_\ell + \underline{u}_5\epsilon^2 + \underline{u}_6\epsilon_\ell^2 + \ldots)\Big\}$$

$$= \Big\{\sin^2\sigma - \epsilon_\ell\epsilon\Big[2u_0(\underline{u}_1 + \underline{u}_2\epsilon + \underline{u}_3\epsilon_\ell + \underline{u}_4\epsilon\epsilon_\ell + \underline{u}_5\epsilon^2 + \underline{u}_6\epsilon_\ell^2 + \ldots)$$

$$+ (\underline{u}_1' + \underline{u}_2'\epsilon + \underline{u}_3'\epsilon_\ell + \underline{u}_4'\epsilon\epsilon_\ell + \underline{u}_5'\epsilon^2 + \underline{u}_6'\epsilon_\ell^2 + \ldots)^2\Big]$$

$$- \epsilon_\ell \epsilon^3 \left(\underline{u}_1 + \underline{u}_2 \epsilon + \underline{u}_3 \epsilon_\ell + \underline{u}_4 \epsilon \epsilon_\ell + \underline{u}_5 \epsilon^2 + \underline{u}_6 \epsilon_\ell^2 + \ldots \right)^2 \Big\} \times$$

$$\Big\{ \underline{u}_1'' + \underline{u}_2'' \epsilon + \underline{u}_3'' \epsilon_\ell + \underline{u}_4'' \epsilon \epsilon_\ell + \underline{u}_5'' \epsilon^2 + \underline{u}_6'' \epsilon_\ell^2 + \ldots + 2u_c$$

$$+ \epsilon \left[\cot\sigma - (\psi \csc^2\sigma)\epsilon + (\psi^2 \csc^2\sigma \cot\sigma)\epsilon^2 - \ldots \right] \left[\underline{u}_1' + \underline{u}_2' \epsilon + \underline{u}_3' \epsilon_\ell + \underline{u}_4' \epsilon \epsilon_\ell + \underline{u}_5' \epsilon^2 + \right.$$

$$\left. + \underline{u}_6' \epsilon_\ell^2 + \ldots \right] + 2\epsilon^2 \left(\underline{u}_1 + \underline{u}_2 \epsilon + \underline{u}_3 \epsilon_\ell + \underline{u}_4 \epsilon \epsilon_\ell + \underline{u}_5 \epsilon^2 + \underline{u}_6 \epsilon_\ell^2 + \ldots \right) \Big\}$$

(88)

We may now equate coefficients of like terms in $\epsilon^i \epsilon_\ell^j$ on both sides of equation (88). Doing this we obtain the following set of equations

zero order term: $$\sin^2\sigma \left(\underline{u}_1'' + 2u_c \right) = 0 \tag{89}$$

ϵ term: $$\underline{u}_1'^2 \left(\underline{u}_1'' + u_c \right) = \sin^2\sigma \left(\underline{u}_2'' + \underline{u}_1' \cot\sigma \right) \tag{90}$$

ϵ_ℓ term: $$\sin^2\sigma \left(\underline{u}_3'' \right) = 0 \tag{91}$$

$\epsilon \epsilon_\ell$ term: $$- \underline{u}_1'^2 \left(\underline{u}_1'' + u_o \right) = \sin^2\sigma \, \underline{u}_4'' - \left(2u_c \underline{u}_1 + \underline{u}_1'^2 \right) \left(\underline{u}_1'' + 2u_c \right) \tag{92}$$

ϵ^2 term: $$\underline{u}_1'^2 \underline{u}_2'' + 2\underline{u}_1' \underline{u}_2' \left(\underline{u}_1'' + u_o \right) = \sin^2\sigma \left(\underline{u}_5'' + \underline{u}_2' \cot\sigma - \psi \csc^2\sigma \, \underline{u}_1' + 2\underline{u}_1 \right) \tag{93}$$

ϵ_2^2 term: $\sin^2\sigma\,(u_6'') = 0$ (94)

$$\vdots \qquad\qquad \vdots$$

This procedure can be continued to obtain equations determining all the coefficients in the series for $\underline{u}$. We have set down only those giving the coefficients $\underline{u}_1$ through $\underline{u}_6$, that is, up to second order in $\epsilon^i\epsilon_2^j$ (the order being equal to n, where n = i+j, i, j > 0).

In order to solve this set of equations we need the appropriate boundary conditions on the $\underline{u}_i$. Substituting the series (84) into equations (55) we obtain the following shock boundary conditions on the $\underline{u}_i$

$$\left.\begin{aligned} u_i(0) &= 0, \quad i \geq 1 \\ u_1'(0) &= -\sin\sigma \\ u_i'(0) &= 0, \quad i > 1 \end{aligned}\right\} \tag{95}$$

Since $\sin^2\sigma \neq 0$, equation (89) becomes

$$u_1'' + 2u_0 = 0 \tag{96}$$

Setting $u_0 = \cos\sigma$ and integrating twice taking into account the boundary conditions (95) we obtain

$$u_1 = -\psi\sin\sigma(\psi\cot\sigma + 1) \tag{97a}$$

Equations (90) onwards can be solved in succession by simple quadratures, in each equation using the results of the preceeding ones. Since the solution of these equations involves only extensive algebra and simple calculus we omit the intermediate steps and present only the results. The solutions of equations (90) to (94) subject to the boundary conditions, equations (95), are

$$\underline{u}_2 = -\frac{1}{3}\cos\sigma\cot\sigma\left(\psi^4\cot\sigma + \psi^3\right) \tag{97b}$$

$$\underline{u}_3 \equiv 0 \tag{97c}$$

$$\underline{u}_4 = \left(\frac{1}{3}\psi^4\cot^2\sigma + \frac{2}{3}\psi^3\cot\sigma + \frac{1}{2}\psi^2\right)\cos\sigma \tag{97d}$$

$$\begin{aligned}\underline{u}_5 = &-\frac{32}{45}\psi^6\cos\sigma\cot^4\sigma - \frac{22}{15}\psi^5\cos\sigma\cot^3\sigma \\ &-\frac{5}{4}\psi^4\cos\sigma\cot^2\sigma + \frac{1}{6}\psi^3\sin\sigma\left(1-3\cot^2\sigma\right)\end{aligned} \tag{97e}$$

$$\underline{u}_6 \equiv 0 \tag{97f}$$

(N.B. Although we have not written down the equations for any additional terms in the series for $\underline{u}$, we see immediately from the form of equation (88) that all the coefficients of terms of the form ϵ_ℓ^i, $i > 0$, will be zero.)

Since the algebra involved in determining further coefficients in the series for u becomes lengthy, we shall not go further than those already obtained. From equations (83) and (84) we have

$$\frac{u}{U} = u_0 + \epsilon^2\underline{u} = \cos\sigma + \epsilon^2\left(\underline{u}_1 + \underline{u}_2\epsilon + \underline{u}_3\epsilon_\ell + \underline{u}_4\epsilon\epsilon_\ell + \underline{u}_5\epsilon^2 + \underline{u}_6\epsilon_\ell^2 + \ldots\right) \tag{98}$$

Substituting equations (97) into the above equation we obtain

$$\begin{aligned}\frac{u}{U} = \cos\sigma + \epsilon^2\Big[&-\psi\sin\sigma\left(\psi\cot\sigma + 1\right) - \frac{1}{3}\cos\sigma\cot\sigma\left(\psi^4\cot\sigma + \psi^3\right)\epsilon \\ &+ \cos\sigma\left(\frac{1}{3}\psi^4\cot^2\sigma + \frac{2}{3}\psi^3\cot\sigma + \frac{1}{2}\psi^2\right)\epsilon\epsilon_\ell \\ &+ \Big\{-\frac{32}{45}\psi^6\cos\sigma\cot^4\sigma - \frac{22}{15}\psi^5\cos\sigma\cot^3\sigma - \frac{5}{4}\psi^4\cos\sigma\cot^2\sigma \\ &+ \frac{1}{6}\psi^3\sin\sigma\left(1 - 3\cot^2\sigma\right)\Big\}\epsilon^2 + \ldots\Big]\end{aligned} \tag{99}$$

Gathering like powers of ψ in the brackets this may also be written

$$\frac{u}{U} = \cos\sigma + \epsilon^2\Big[-\psi\sin\sigma + \psi^2\left(-1+\tfrac{1}{2}\epsilon\epsilon_\ell\right)\cos\sigma$$
$$+\psi^3\left\{-\tfrac{1}{3}\epsilon\cos\sigma\cot\sigma + \tfrac{2}{3}\epsilon\epsilon_\ell\cos\sigma\cot\sigma + \tfrac{1}{6}\epsilon^2\sin\sigma\left(1-3\cot^2\sigma\right)\right\}$$
$$+\psi^4\left(-\tfrac{1}{3}\epsilon\cos\sigma\cot^2\sigma + \tfrac{1}{3}\epsilon\epsilon_\ell\cos\sigma\cot^2\sigma - \tfrac{5}{4}\epsilon^2\cos\sigma\cot^2\sigma\right)$$
$$-\frac{22}{15}\epsilon^2\psi^5\cos\sigma\cot^3\sigma - \frac{32}{45}\epsilon^2\psi^6\cos\sigma\cot^4\sigma + \ldots\Big] \tag{100}$$

Substituting $\psi = \frac{\theta-\sigma}{\epsilon}$ into this equation we obtain

$$\frac{u}{U} = \cos\sigma - (\epsilon\sin\sigma)(\theta-\sigma) - \left(1-\tfrac{1}{2}\epsilon\epsilon_\ell\right)\cos\sigma(\theta-\sigma)^2$$
$$+\left\{\tfrac{1}{6}\epsilon\sin\sigma + \cos\sigma\cot\sigma\left(-\tfrac{1}{3}+\tfrac{2}{3}\epsilon_\ell - \tfrac{1}{2}\epsilon\right)\right\}(\theta-\sigma)^3$$
$$+\frac{1}{\epsilon}\cos\sigma\cot^2\sigma\left(-\tfrac{1}{3}+\tfrac{1}{3}\epsilon_\ell - \tfrac{5}{4}\epsilon\right)(\theta-\sigma)^4$$
$$-\frac{22}{15}\,\frac{\cos\sigma\cot^3\sigma}{\epsilon}(\theta-\sigma)^5 - \frac{32}{45}\,\frac{\cos\sigma\cot^4\sigma}{\epsilon^2}(\theta-\sigma)^6 + \ldots \tag{101}$$

Equations (99), (100), (101) are our final equations for u. They give u to fourth order in $\epsilon^i\epsilon_\ell^j$. If we set $\epsilon = \epsilon_\ell$ in equation (101) this equation becomes identical with equation (67), which is the result for u in the case of infinite Mach number.

To find the cone angle, θ_c, we must now differentiate equation (100) with respect to ψ and set the resulting expression equal to zero. If we do this and set $S = \psi_c \cot\sigma$ as before, we obtain, in place of equation (70) which holds only for infinite Mach number, the following fifth degree polynomial in S

$$\frac{64}{15}\epsilon^2 S^5 + \frac{22}{3}\epsilon^2 S^4 + \left(\frac{4}{3}\epsilon - \frac{4}{3}\epsilon\epsilon_\ell + 5\epsilon^2\right)S^3$$

$$- \left[-\epsilon + 2\epsilon\epsilon_\ell + \frac{1}{2}\epsilon^2(\tan^2\sigma - 3)\right]S^2 + (2 - \epsilon\epsilon_\ell)S + 1 = 0 \tag{102}$$

Since this equation can be treated in the same way as equation (70) was for the infinite Mach number case, we shall omit the details. We obtain for the cone angle the following expression

$$\sigma - \theta_c = \frac{1}{2}\epsilon\left[1 + \frac{1}{12}\epsilon + \frac{1}{2}\epsilon\left\{\frac{1}{20}\epsilon(3 - 5\tan^2\sigma) + \frac{1}{3}\epsilon_\ell\right\} + O(\epsilon^3)\right]\tan\sigma \tag{103}$$

If we put $\epsilon = \epsilon_\ell$ in this equation it agrees with equation (82), the result we obtained earlier for infinite Mach number.

If we drop the third order term equation (103) becomes

$$\sigma - \theta_c = \frac{1}{2}\epsilon\left(1 + \frac{1}{12}\epsilon\right)\tan\sigma \tag{103a}$$

For given values of ϵ and σ equations (103) and (103a) can be used to determine θ_c. Denoting the value of θ_c given by (103a) by $(\theta_c)_1$ and that given by (103) by $(\theta_c)_2$ we have compared in Figures 2 to 7 the computed values $\sigma - (\theta_c)_1$ and $\sigma - (\theta_c)_2$ with the exact values of Kopal for air ($\epsilon_\ell = 1/6$) for cone angles of 5^0, 10^0, 20^0, 30^0, 40^0 and 50^0. The agreement of both computed values with the exact values improves as the cone angle increases for fixed M_∞ . This is so because the normal Mach number is larger and hence the density ratio is smaller. For cone angles of 20^0 and higher, as shown in the figures, both the second and the third order values of $\sigma - \theta_c$ lie so close to the exact values over a very wide Mach number range that separate curves have not been drawn. For these large cone angles the agreement is very good even for free stream Mach numbers as low as 2 and 3. For the cone angles of 5° and 10° the overall agreement between approximate and exact values is not so good, the largest discrepancy, as expected, occurring for low supersonic free stream Mach numbers. For both these cone angles the third order value predicts the width of the shock layer considerably more accurately than does the second order value.

VI. EXPANSIONS FOR PRESSURE AND DENSITY

Having obtained the first few terms in the series expansion for u we now wish to obtain the series expansions for p and ρ. The two relations giving p and ρ in terms of u are the energy equation

$$\frac{\gamma p}{(\gamma-1)\rho} + \frac{1}{2}(u^2+v^2) = \frac{1}{2}q_0^2 \tag{104}$$

and the equation of state for the isentropic flow behind the shock

$$p = \left(\frac{p_s}{\rho_s^{\gamma}}\right)\rho^{\gamma} \tag{105}$$

The values of p_s and ρ_s are determined from free stream conditions by the shock relations. In terms of ϵ we have for p_s and ρ_s

$$p_s = p_\infty + \rho_\infty U^2 \sin^2\sigma(1-\epsilon) \tag{106}$$

$$\rho_s = \frac{\rho_\infty}{\epsilon} \tag{107}$$

We now wish to solve equations (104) and (105) for p and ρ. First we non-dimensionalize the quantities appearing in equation (104). Since the velocities are non-dimensionalized relative to free stream velocity, U, we write (104) as

$$\frac{\gamma}{\gamma-1}\,\frac{\epsilon p}{\frac{\rho}{\rho_s}(\rho_\infty U^2)} = \frac{1}{2}\left(\frac{q_0^2}{U^2} - \frac{u^2+v^2}{U^2}\right)$$

Now define the non-dimensional variables

$$\left.\begin{aligned} \bar{u} &= \frac{u}{U} \\ \bar{v} &= \frac{v}{U} \\ \bar{\rho} &= \frac{\rho}{\rho_s} \\ \bar{\bar{p}} &= \frac{p}{\rho_\infty U^2} \end{aligned}\right\} \tag{108}$$

The energy equation may now be written in terms of these variables as

$$\frac{\gamma}{\gamma-1}\,\frac{\epsilon\bar{\bar{p}}}{\bar{\rho}} = \frac{1}{2}\left(\frac{q_0^2}{U^2} - \bar{u}^2 - \bar{v}^2\right)$$

From equation (105) we have

$$\bar{\bar{p}} = \frac{p_s}{\rho_\infty U^2}\,\bar{\rho}^{\gamma} \tag{109}$$

Substituting this into the energy equation we obtain

$$\frac{\gamma}{\gamma-1}\,\frac{\epsilon p_s}{\rho_\infty U^2}\,\bar{\rho}^{\gamma-1} = \frac{1}{2}\left(\frac{q_0^2}{U^2} - \bar{u}^2 - \bar{v}^2\right)$$

Substituting for $p_s/\rho_\infty U^2$ in this equation using equation (106) we obtain

$$\frac{\gamma}{\gamma-1}\,\epsilon\left[\frac{p_\infty}{\rho_\infty U^2} + \sin^2\sigma(1-\epsilon)\right]\bar{\rho}^{\gamma-1} = \frac{1}{2}\left(\frac{q_0^2}{U^2} - \bar{u}^2 - \bar{v}^2\right) \tag{110}$$

Now

$$\left.\begin{aligned} \frac{\gamma}{\gamma-1} &= \frac{1+\epsilon_\ell}{2\epsilon_\ell} \\ \gamma-1 &= \frac{2\epsilon_\ell}{1-\epsilon_\ell} \end{aligned}\right\} \tag{111}$$

Also,

$$\frac{p_\infty}{\rho_\infty U^2} = \frac{1}{\gamma M_\infty^2} \tag{112}$$

but from equation (52) we find

$$\frac{1}{M_\infty^2} = \frac{\epsilon-\epsilon_\ell}{1-\epsilon_\ell}\sin^2\sigma$$

so we may now write, noting also that $\gamma = \dfrac{1+\epsilon_\ell}{1-\epsilon_\ell}$,

$$\frac{p_\infty}{\rho_\infty U^2} = \frac{\epsilon - \epsilon_\ell}{1+\epsilon_\ell} \sin^2\sigma \tag{113}$$

Substituting equations (111) and (113) into equation (110) we obtain

$$\frac{\epsilon(1+\epsilon_\ell)}{\epsilon_\ell}\left[\frac{\epsilon-\epsilon_\ell}{1+\epsilon_\ell} + (1-\epsilon)\right]\sin^2\sigma\,\bar{\rho}^{\frac{2\epsilon_\ell}{1-\epsilon_\ell}} = \left(\frac{q_0^2}{U^2} - \bar{u}^2 - \bar{v}^2\right)$$

which can be simplified to

$$\frac{\epsilon \sin^2\sigma}{\epsilon_\ell}(1-\epsilon\epsilon_\ell)\,\bar{\rho}^{\frac{2\epsilon_\ell}{1-\epsilon_\ell}} = \frac{q_0^2}{U^2} - \bar{u}^2 - \bar{v}^2 \tag{114}$$

We now let $\bar{u}(\psi) = u_0 + \epsilon^2 \underline{u}(\psi)$ from which it follows that $\bar{v}(\psi) = \epsilon\underline{u}'(\psi)$.

We may now write the right hand side of equation (114) as

$$\frac{q_0^2}{U^2} - \bar{u}^2 - \bar{v}^2 = \frac{q_0^2}{U^2} - u_0^2 - \epsilon^2\left(\underline{u}'^2 + 2u_0\underline{u}\right) - \epsilon^4\underline{u}^2$$

Since $u_0 = \cos\sigma$ we have

$$\frac{q_0^2}{U^2} - u_0^2 = \frac{q_0^2}{U^2} - \cos^2\sigma = \frac{\epsilon}{\epsilon_\ell}\sin^2\sigma \tag{115}$$

so we may also write

$$\frac{q_0^2}{U^2} - \bar{u}^2 - \bar{v}^2 = \frac{\epsilon}{\epsilon_\ell}\left[\sin^2\sigma - \epsilon\epsilon_\ell(\underline{u}'^2 + 2u_0\underline{u}) - \epsilon^3\epsilon_\ell\underline{u}^2\right] \tag{116}$$

Substituting this into equation (114) we obtain

$$\sin^2\sigma(1-\epsilon\epsilon_\ell)\,\bar{\rho}^{\frac{2\epsilon_\ell}{1-\epsilon_\ell}} = \sin^2\sigma - \epsilon\epsilon_\ell(\underline{u}'^2 + 2u_0\underline{u}) - \epsilon^3\epsilon_\ell\underline{u}^2 \tag{117}$$

Solving for $\bar{\rho}$ this may be written

$$\bar{\rho} = \left\{1 - \frac{\epsilon\epsilon_\ell}{(1-\epsilon\epsilon_\ell)\sin^2\sigma}\left[\underline{u}'^2 + 2u_0\underline{u} + \epsilon^2\underline{u}^2 - \sin^2\sigma\right]\right\}^{\frac{1-\epsilon_\ell}{2\epsilon_\ell}} \tag{118}$$

Since the second term in parentheses is $O(\epsilon\epsilon_\ell)$ we now expand the right hand side of equation (118) to obtain

$$\bar{\rho} = 1 - \frac{(1-\epsilon_\ell)\epsilon}{2(1-\epsilon\epsilon_\ell)\sin^2\sigma}\left[\underline{u}'^2 + 2u_0\underline{u} + \epsilon^2\underline{u}^2 - \sin^2\sigma\right]$$

$$+ \frac{1}{8}\frac{\epsilon^2(1-\epsilon_\ell)(1-3\epsilon_\ell)}{(1-\epsilon\epsilon_\ell)^2\sin^4\sigma}\left[\underline{u}'^2 + 2u_0\underline{u} + \epsilon^2\underline{u}^2 - \sin^2\sigma\right]^2 + \ldots$$

If into this equation we now substitute the series for $\underline{u}$

$$\underline{u} = \underline{u}_1 + \underline{u}_2\epsilon + \underline{u}_4\epsilon\epsilon_\ell + \underline{u}_5\epsilon^2 + \ldots$$

omitting the $\underline{u}_3$ and $\underline{u}_6$ terms which are both zero, and retain only enough terms so that $\bar{\rho}$ is given to second order, we obtain the following

$$\bar{\rho} = 1 - \frac{\epsilon}{2\sin^2\sigma}\left(\underline{u}_1'^2 + 2u_0\underline{u}_1 - \sin^2\sigma\right) + \frac{\epsilon\epsilon_\ell}{2\sin^2\sigma}\left(\underline{u}_1'^2 + 2u_0\underline{u}_1 - \sin^2\sigma\right)$$

$$+ \frac{\epsilon^2}{8\sin^4\sigma}\left[\left(\underline{u}_1'^2 + 2u_0\underline{u}_1 - \sin^2\sigma\right)^2 - 8\sin^2\sigma\left(\underline{u}_1'\underline{u}_2' + u_0\underline{u}_2\right)\right] + O(\epsilon^3) \tag{119}$$

Equation (119) gives the first few terms in expansion of $\bar{\rho}$ as

$$\bar{\rho} = 1 + \bar{\rho}_1\epsilon + \bar{\rho}_2\epsilon_\ell + \bar{\rho}_3\epsilon\epsilon_\ell + \bar{\rho}_4\epsilon^2 + \bar{\rho}_5\epsilon_\ell^2 + \ldots \tag{120}$$

Comparing equations (119) and (120) we note that $\bar{\rho}_1 = -\bar{\rho}_3$, also that $\bar{\rho}_2 \equiv \bar{\rho}_5 \equiv 0$. Since

$$\underline{u}_1'^2 + 2u_0\underline{u}_1 - \sin^2\sigma = -2\bar{\rho}_1\sin^2\sigma$$

we may write the expression for $\bar{\rho}_4$ as

$$\bar{\rho}_4 = \frac{1}{8\sin^4\sigma}\left[4\bar{\rho}_1^2\sin^4\sigma - 8\sin^2\sigma\left(\underline{u}_1'\underline{u}_2' + u_0\underline{u}_2\right)\right]$$

or

$$\left(\bar{\rho}_4 - \tfrac{1}{2}\bar{\rho}_1^2\right)\sin^2\sigma = -\left(\underline{u}_1'\underline{u}_2' + u_0\underline{u}_2\right)$$

If we now substitute the known values of the $\underline{u}_i$ and $\underline{u}_i'$ into the expressions for the $\bar{\rho}_i$ we obtain for the $\bar{\rho}$ series the following result

$$\bar{\rho} = \frac{\rho}{\rho_s} = 1 - \psi\cot\sigma\,(\psi\cot\sigma + 1)\epsilon + \psi\cot\sigma(\psi\cot\sigma + 1)\epsilon\epsilon_\ell - \tfrac{1}{6}\psi^2\cot^2\sigma\,(11\psi^2\cot^2\sigma + 12\,\psi\cot\sigma + 3)\epsilon^2 + \ldots \tag{121}$$

which may also be written

$$\bar{\rho} = \frac{\rho}{\rho_s} = 1 - \epsilon\psi\cot\sigma\Big\{(\psi\cot\sigma+1)\big[1-\epsilon_\ell + \tfrac{1}{6}\psi\cot\sigma(11\psi\cot\sigma+1)\epsilon\big] + \tfrac{1}{3}\psi\cot\sigma\,\epsilon + \ldots\Big\} \tag{122}$$

These series give $\bar{\rho}$ to $O(\epsilon^2)$.

To obtain the corresponding series for $\bar{p} = p/p_s$ we use the relation

$$\bar{p} = \bar{\rho}^{\,\gamma} \tag{123}$$

Let us represent the series for $\bar{\rho}$ as

$$\bar{\rho} = 1 + \rho^*$$

Since $\rho^* = O(\epsilon)$ we may expand $\bar{p}$ as

$$\bar{p} = (1+\rho^*)^\gamma = 1 + \gamma\rho^* + \tfrac{1}{2}\gamma(\gamma-1)\rho^{*2} + \ldots \tag{124}$$

As $\bar{\rho}$ is known only to $O(\epsilon^2)$ equation (123) can be used to determine $\bar{p}$ only to this order. Since $(\gamma - 1) = O(\epsilon_\ell)$ the third term on the right hand side of equation (124) is $O(\epsilon^3)$ and may be dropped. Substituting for ρ^* from the series for $\bar{\rho}$, equation (120), putting $\gamma = (1+\epsilon_\ell)/(1-\epsilon_\ell)$, and retaining only terms of $O(\epsilon^2)$ and larger, we may write $\bar{p}$ as

$$\bar{p} = 1 + \bar{\rho}_1 \epsilon + (\bar{\rho}_3 + 2\bar{\rho}_1)\epsilon\epsilon_\ell + \bar{\rho}_4 \epsilon^2 + O(\epsilon^3) \tag{125}$$

Since $\bar{\rho}_1 = -\bar{\rho}_3$ the coefficient of the $\epsilon\epsilon_\ell$ term in the above equation becomes

$$\bar{\rho}_3 + 2\bar{\rho}_1 = -\bar{\rho}_3$$

so equation (125) may now be written

$$\bar{p} = 1 + \bar{\rho}_1 \epsilon - \bar{\rho}_3 \epsilon\epsilon_\ell + \bar{\rho}_4 \epsilon^2 + O(\epsilon^3) \tag{126}$$

which gives $\bar{p}$ as a series in $\epsilon^i \epsilon_\ell^j$. Comparing equations (120) and (126) we see that to second order the series expansions for $\bar{\rho}$ and $\bar{p}$ are identical, except that the sign of the term in $\epsilon\epsilon_\ell$ is different in the two series. Substituting the expressions for the $\bar{\rho}_i$ into equation (126) we obtain for $\bar{p}$ the following series

$$\bar{p} = \frac{p}{p_3} = 1 - \Psi\cot\sigma(\Psi\cot\sigma + 1)\epsilon - \Psi\cot\sigma(\Psi\cot\sigma + 1)\epsilon\epsilon_\ell$$
$$- \frac{1}{6}\Psi^2\cot^2\sigma(11\Psi^2\cot^2\sigma + 12\Psi\cot\sigma + 3)\epsilon^2 + \ldots \tag{127}$$

VII. SOLUTION OF THE TAYLOR-MACCOLL EQUATION BY A TAYLOR SERIES

Maccoll has solved the T-M equation in terms of a Taylor series about the angle corresponding to the cone surface. At the cone surface we have

$$u = u_c$$

$$\frac{du}{d\theta} = 0$$

where u_c is the as yet unknown velocity on the cone surface. Substituting these values into the T-M equation we find the value of $d^2u/d\theta^2$ at the cone surface to be

$$\frac{d^2u}{d\theta^2} = -2u_c \tag{128}$$

Successive differentiation of the T-M equation then gives a series of equations from which the higher order derivatives of u at the cone surface may be evaluated. Maccoll has gone as far as the fifth derivative. His final result for the Taylor series is the following

$$\frac{u}{q_0} = \frac{u_c}{q_0} - \frac{u_c}{q_0}(\theta-\theta_c)^2 + \frac{u_c}{q_0}\cot\theta_c\,\frac{(\theta-\theta_c)^3}{3} - \left\{\frac{u_c}{q_0}\cot^2\theta_c + \frac{8}{3(\gamma-1)}\,\frac{\left(\frac{u_c}{q_0}\right)^3}{1-\left(\frac{u_c}{q_0}\right)^2}\right\}\frac{(\theta-\theta_c)^4}{4}$$
$$+ \left\{\frac{u_c}{q_0}\cot^3\theta_c + \frac{\frac{7}{12}\frac{u_c}{q_0} + \frac{87-78\gamma}{12(\gamma-1)}\left(\frac{u_c}{q_0}\right)^3}{1-\left(\frac{u_c}{q_0}\right)^2}\cot\theta_c\right\}\frac{(\theta-\theta_c)^5}{5} + \ldots \tag{129}$$

This series gives u in terms of q_0, θ_c and u_c. In general, q_0 and θ_c are known quantities, while u_c is an unknown. Maccoll gives some information about the convergence of this series.

Since the Taylor series contains the unknown u_c it cannot be used directly as the solution to the problem. If we now apply the two boundary conditions at the shock, equations (12) and (13), to the Taylor series we will obtain two equations in the four quantities σ , u_c, q_0, θ_c, which we can solve to give any two of them in terms of the other two. In this way, we may obtain u_c in terms of q_o and θ_c; the use of this result along with the Taylor series then gives the solution to the problem. This has been done by Hammitt and Murthy, who compare their results for certain ranges of the parameters with the exact values from Kopal and find very good agreement.

Since our previous work involved assuming the shock to be given and required the finding of the cone surface we now wish to obtain u as a Taylor series taken about the shock angle rather than the cone angle. The velocity and its first derivative evaluated behind the shock are given by

$$u = U\cos\sigma \tag{12}$$

$$\frac{du}{d\theta} = -\epsilon U \sin\sigma \tag{13}$$

The evaluation of the derivatives in the previous Taylor expansion was comparatively simple due to the fact that $du/d\theta$ at the cone was zero; this made many terms drop out of the resulting expressions. This simplification, unfortunately, is not possible now since neither u nor $du/d\theta$ is zero at the shock.

Substituting equations (12) and (13) into the T-M equation, we find, after some simplification of terms, the following expression for the second derivative of the velocity evaluated at the shock

$$\left(\frac{d^2u}{d\theta^2}\right)_s = -2U\cos\sigma\left[\frac{\left(\frac{q_o^2}{U^2}-\cos^2\sigma\right)\left(1-\frac{\epsilon}{2}\right)-\epsilon^2\sin^2\sigma\left(\frac{\gamma}{\gamma-1}-\frac{\epsilon}{2}\right)}{\left(\frac{q_o^2}{U^2}-\cos^2\sigma\right)-\frac{\gamma+1}{\gamma-1}\epsilon^2\sin^2\sigma}\right] \tag{130}$$

This expression is quite complicated but it can be very much simplified. We have already obtained the following relation

$$\frac{q_0^2}{U^2} - \cos^2\sigma = \frac{\epsilon}{\epsilon_\ell}\sin^2\sigma \tag{115}$$

Substituting this relation into equation (130) and replacing $\frac{\gamma+1}{\gamma-1}$ by $1/\epsilon_\ell$, equation (130) becomes

$$\left(\frac{d^2u}{d\theta^2}\right)_s = -2U\cos\sigma\left[\frac{\left(1-\frac{1}{2}\epsilon\right) - \epsilon\epsilon_\ell\left(\frac{\gamma}{\gamma-1} - \frac{1}{2}\epsilon\right)}{1-\epsilon}\right] \tag{131}$$

Since $\frac{\gamma}{\gamma-1} = \frac{1+\epsilon_\ell}{2\epsilon_\ell}$ this may now be reduced to

$$\left(\frac{d^2u}{d\theta^2}\right)_s = -2U\cos\sigma\left(1-\frac{1}{2}\epsilon\epsilon_\ell\right) \tag{132}$$

which is considerably simpler than equation (130).

To obtain the third derivative we must differentiate the T-M equation. Doing this, solving for $d^3u/d\theta^3$, and gathering like terms, we obtain

$$\begin{aligned}\frac{d^3u}{d\theta^3} = \Bigg[&\left\{\frac{\gamma+1}{\gamma-1}\frac{1}{2q_0^2}\left(\frac{du}{d\theta}\right)^2 - \frac{1}{2}\left(1-\frac{u^2}{q_0^2}\right)\right\}\left\{\frac{du}{d\theta}\left[1-\frac{3u^2}{q_0^2} - \frac{1}{2}\left(1-\frac{u^2}{q_0^2}\right)\frac{1}{\sin^2\theta}\right]\right.\\ &+\frac{d^2u}{d\theta^2}\left[\frac{1}{2}\left(1-\frac{u^2}{q_0^2}\right)\cot\theta - \frac{2\gamma u}{(\gamma-1)q_0^2}\frac{du}{d\theta}\right] + \frac{1}{q_0^2}\left(\frac{du}{d\theta}\right)^2\left[-u\cot\theta - \frac{\gamma}{\gamma-1}\frac{du}{d\theta} - \frac{3}{2}\cot\theta\frac{d^2u}{d\theta^2}\right]\\ &\left.+\frac{1}{2q_0^2\sin^2\theta}\left(\frac{du}{d\theta}\right)^3\right\} - \left\{u\left(1-\frac{u^2}{q_0^2}\right) + \frac{1}{2}\left(1-\frac{u^2}{q_0^2}\right)\cot\theta\frac{du}{d\theta} - \frac{\gamma}{\gamma-1}\frac{u}{q_0^2}\left(\frac{du}{d\theta}\right)^2\right.\\ &\left.-\frac{1}{2q_0^2}\cot\theta\left(\frac{du}{d\theta}\right)^3\right\}\left\{\frac{\gamma+1}{\gamma-1}\frac{1}{q_0}\frac{d^2u}{d\theta^2} + \frac{u}{q_0}\right\}\frac{1}{q_0}\frac{du}{d\theta}\Bigg]\Bigg/\left[\frac{\gamma+1}{\gamma-1}\frac{1}{2q_0^2}\left(\frac{du}{d\theta}\right)^2 - \frac{1}{2}\left(1-\frac{u^2}{q_0^2}\right)\right]^2\end{aligned} \tag{133}$$

To obtain $(d^3u/d\theta^3)_s$ we now substitute σ for θ and for u, $du/d\theta$, and $d^2u/d\theta^2$ from the shock boundary conditions and equation (132), respectively. If we also employ equation (115) and use ϵ_ℓ to eliminate γ we can reduce equation (133), after much algebraic manipulation, to the following

$$\left(\frac{d^3u}{d\theta^3}\right)_s = \epsilon U \sin\sigma + \frac{U \cot\sigma \cos\sigma}{1-\epsilon}\left(-2 + 4\epsilon_\ell + \epsilon\epsilon_\ell + \epsilon^2\epsilon_\ell^2 - 4\epsilon\epsilon_\ell^2 - \epsilon + \epsilon_\ell\epsilon^2\right) \tag{134}$$

This may also be written

$$\left(\frac{d^3u}{d\theta^3}\right)_s = \epsilon U \sin\sigma + U\cot\sigma\cos\sigma\left[1 - \epsilon_\ell(1+\epsilon_\ell)\epsilon + \epsilon_\ell(3\epsilon_\ell - 2) - 3\frac{(1-\epsilon_\ell)^2}{1-\epsilon}\right] \tag{135}$$

Equation (135) is the exact expression for the third derivative evaluated at the shock. For the case of infinite Mach number, that is, when $\epsilon = \epsilon_\ell$, it reduces to

$$\left(\frac{d^3u}{d\theta^3}\right)_s = \epsilon_\ell U \sin\sigma + U\cot\sigma\cos\sigma\left[-2 + \epsilon_\ell + 2\epsilon_\ell^2 - \epsilon_\ell^3\right] \tag{136}$$

If, however, the Mach number is not infinite, but $\epsilon \ll 1$, then we may expand $1/(1-\epsilon)$ in a power series in ϵ, in which case equation (135) becomes approximately

$$\left(\frac{d^3u}{d\theta^3}\right)_s = \epsilon U \sin\sigma + U\cot\sigma\cos\sigma\left(-2 + 4\epsilon_\ell - 3\epsilon + 5\epsilon\epsilon_\ell - 3\epsilon^2 - 4\epsilon\epsilon_\ell^2 + 6\epsilon_\ell\epsilon^2 - 3\epsilon^3 + \ldots\right) \tag{137}$$

in which only third order terms in $\epsilon^i\epsilon_\ell^j$ have been retained.

Before continuing further, we must digress a moment to consider the philosophy behind this procedure. In seeking a Taylor series as the solution to the problem, we are obtaining a series in which the successive terms involve powers of $(\theta - \sigma)$. The fundamental assumption of this whole paper, however, has been the assumption of small density ratio. Thus, any series solution of the problem should be examined in the light of the order in density ratio to which it represents the true solution. In other words, each term in the Taylor series should be examined as to its order in density ratio. It may turn out that, although $(\theta-\sigma) = O(\epsilon)$, so that adding successive terms in powers of $(\theta-\sigma)$ appears to add higher order terms, we may actually be adding on terms which are as large or larger than those already contained in the series. This would occur if any of the coefficients of the powers of $(\theta-\sigma)$ should be $O(\epsilon^{-n})$, where $n > 0$. It is this state of affairs which makes the Taylor series approach difficult in a small parameter analysis and leads to the procedure of direct expansion in terms of this parameter.

For our particular problem, we see from equations (132) and (137) that

$$\left(\frac{d^2u}{d\theta^2}\right)_s = O(1)$$

$$\left(\frac{d^3u}{d\theta^3}\right)_s = O(1) \tag{138}$$

From the shock boundary conditions we know

$$u_s = O(1)$$

$$\left(\frac{du}{d\theta}\right)_s = O(\epsilon) \tag{139}$$

If the fourth and all higher order derivatives evaluated at the shock are no larger than O(1), as is true for the first three derivatives, then the Taylor series taken up to the term in $(\theta-\sigma)^3$ will approximate the exact solution to $O(\epsilon^3)$.

Unfortunately, the supposition made above about the behavior of higher order derivatives is not correct. In fact, the fourth derivative itself is

$O(\epsilon^{-1})$; we will now show this and also obtain the term in $(d^4u/d\theta^4)_s$ of this order. To actually differentiate equation (133) in order to analyze each of the resulting terms would be quite an undertaking in view of the complexity of this equation. Fortunately, this arduous task is not necessary and a more cursory examination will suffice. Since the denominator in equation (133) is $O(1)$, we see from examining the numerator that the only terms in the equation which will be $O(\epsilon^{-1})$ when differentiated are those of the form

$$\frac{A}{\gamma-1}\,\frac{du}{d\theta}$$

where $A = O(1)$, for when this is differentiated one of the resulting terms will be

$$\frac{A}{\gamma-1}\,\frac{d^2u}{d\theta^2}$$

and since $A_s = O(1)$, $\gamma - 1 = O(\epsilon_\ell) = O(\epsilon)$, and $(d^2u/d\theta^2)_s = O(1)$ we have

$$\left[\frac{A}{\gamma-1}\,\frac{d^2u}{d\theta^2}\right]_s = O(\epsilon^{-1})$$

Thus, retaining only those terms $O(\epsilon^{-1})$ when evaluated at the shock we find for $(d^4u/d\theta^4)_s$, after some simplification,

$$\left(\frac{d^4u}{d\theta^4}\right)_s = \frac{-4u_s}{(\gamma-1)(q_0^2-u_s^2)}\left(\frac{d^2u}{d\theta^2}\right)_s^2 + O(1) \tag{140}$$

Since

$$q_0^2 - u_s^2 = q_0^2 - U^2\cos^2\sigma = \frac{\epsilon}{\epsilon_\ell}U^2\sin^2\sigma$$

and

$$\gamma - 1 = \frac{2\epsilon_\ell}{1-\epsilon_\ell}$$

we may write equation (140), also using our values for u_s and $(d^2u/d\theta^2)_s$, as

$$\left(\frac{d^4u}{d\theta^4}\right)_s = -\frac{8\bar{U}}{\epsilon}(1-\epsilon_\ell)\cot^2\sigma\cos\sigma + O(1) \tag{141}$$

(The term in ϵ_ℓ/ϵ in this equation is really O(1) and should not be retained, but as we shall see later it does exactly represent the term in ϵ_ℓ/ϵ in the series solution.)

An examination of equation (133) to prophesy the behavior of higher derivatives than the fourth reveals that all further terms in the Taylor series will be at least O(ϵ^4). Hence the terms we have already obtained should give the solution correct to O(ϵ^3).

The Taylor series for u may be written

$$\frac{u}{U} = \frac{u_s}{U} + \frac{1}{U}\left(\frac{du}{d\theta}\right)_s(\theta-\sigma) + \frac{1}{2!}\frac{1}{U}\left(\frac{d^2u}{d\theta^2}\right)_s(\theta-\sigma)^2 + \frac{1}{3!}\frac{1}{U}\left(\frac{d^3u}{d\theta^3}\right)_s(\theta-\sigma)^3 + \frac{1}{4!}\frac{1}{U}\left(\frac{d^4u}{d\theta^4}\right)_s(\theta-\sigma)^4 + \ldots \tag{142}$$

Substituting for these derivatives from our results we obtain

$$\begin{aligned}\frac{u}{U} = {} & \cos\sigma - (\epsilon\sin\sigma)(\theta-\sigma) - \cos\sigma\left(1-\tfrac{1}{2}\epsilon\epsilon_\ell\right)(\theta-\sigma)^2 \\ & + \Big[\tfrac{1}{6}\epsilon\sin\sigma + \cot\sigma\cos\sigma\Big(-\tfrac{1}{3} + \tfrac{2}{3}\epsilon_\ell - \tfrac{1}{2}\epsilon + \tfrac{5}{6}\epsilon\epsilon_\ell - \tfrac{1}{2}\epsilon^2 - \tfrac{2}{3}\epsilon\epsilon_\ell^2 + \epsilon_\ell\epsilon^2 \\ & - \tfrac{1}{2}\epsilon^3 + \ldots\Big)\Big](\theta-\sigma)^3 + \Big[\frac{1}{3\epsilon}\cos\sigma\cot^2\sigma(-1+\epsilon_\ell) + O(1)\Big](\theta-\sigma)^4 + \ldots\end{aligned} \tag{143}$$

which gives u as a Taylor series to third order in $\epsilon^i\epsilon_\ell^j$ (although some higher order terms have also been included).

We are now in a position to compare this result for u with our previous results. Comparing equation (143) with equation (101), obtained by our small parameter expansion analysis, we see that both expressions for u agree to third order in $\epsilon^{i} \epsilon_{\ell}^{j}$; in fact, the terms in $(\theta - \sigma)^2$ and $(\theta - \sigma)^3$ agree to fourth order, and the term in $\epsilon_{\ell} / \epsilon$ in the $(\theta - \sigma)^4$ term, while also of fourth order, is the same in both.

In summary, while the Taylor series method of solution serves as a valuable check on our previous expansion technique, it suffers from the defect of its successive terms being of varying orders in the small parameter ϵ , the only method of determining these orders being the examination of all the derivatives of u.

VIII. ADDITIONAL RESULTS FROM SERIES SOLUTION

Thus far we have the solution to the problem in terms of a double series expansion for the variables u, p, and ρ. We are now in a position to deduce certain results from these series.

The location of the cone surface is determined from the condition $(du/d\theta)_c = 0$. This implies immediately that the radial velocity u is stationary at the cone surface. Since $(d^2u/d\theta^2)_c < 0$ (see equation (128)), it follows that this value is a maximum. From the Bernoulli equation we then find that the density has a maximum at the cone surface and hence also the pressure. Hence, there is an isentropic compression from the shock to the cone surface. From our series for u, ρ and p we may find the values of these variables at the cone surface. These turn out to be

$$\frac{u_c}{U} = \cos\sigma + \frac{1}{4}\epsilon^2\tan^2\sigma\cos\sigma\left(1 + \frac{1}{12}\epsilon + \frac{5}{12}\epsilon\epsilon_\ell + \frac{11}{120}\epsilon^2\right) - \frac{5}{96}\epsilon^4\tan^3\sigma\sin\sigma + O(\epsilon^5) \tag{144}$$

$$\frac{\rho_c}{\rho_s} = 1 + \frac{1}{4}\epsilon\left(1 - \epsilon_\ell + \frac{1}{24}\epsilon\right) + O(\epsilon^3) \tag{145}$$

$$\frac{p_c}{p_s} = 1 + \frac{1}{4}\epsilon\left(1 + \epsilon_\ell + \frac{1}{24}\epsilon\right) + O(\epsilon^3) \tag{146}$$

An important feature of these equations is that for a given gas (ϵ_ℓ fixed) the value of the velocity at the cone surface depends upon both ϵ and σ, while to second order in ϵ both the surface density and pressure depend only on ϵ when non-dimensionalized with respect to their values behind the shock.

In Figures 8 to 11 are plotted ρ_c/ρ_s and p_c/p_s against ϵ for the case of air and helium (for helium $\epsilon_\ell = 1/4$). For a given ϵ_ℓ the curves corresponding to equations (145) and (146) are parabolas. However, since the ϵ term in parentheses in both these equations appears divided by 24 this term is very small for ϵ small and hence the resulting curves are nearly straight lines in the figures. In both of the figures for air we have plotted exact values from Kopal. For cone angles of 5° to 40°, the agreement between the approximate and exact values for the case of air is seen to be very good for values of ϵ up to 1/2; beyond this value of ϵ the exact values for the 20° and 30° cones begin to fall below the appropriate curve.

For helium for cone angles from 10° to 40° the agreement between approximate and exact values is seen to be not so good as for air. This is partly due to the value of ϵ_ℓ being greater for helium than for air. It is also in large part due to the incompleteness of the tables for helium flow about cones. The exact values plotted on the figures for air were taken directly from the tables of Kopal. To obtain the equivalent values for helium required, in addition to some computation, the reading of two graphs from Mueller. It is from the reading of these graphs that the greatest error has come.

We have not yet set down the final result for v. Differentiating equation (101) with respect to θ we obtain

$$\begin{aligned}\frac{v}{U} = &-\epsilon\sin\sigma - 2\left(1-\tfrac{1}{2}\epsilon\epsilon_\ell\right)\cos\sigma(\theta-\sigma) + \Big\{\tfrac{1}{2}\epsilon\sin\sigma \\ &+ \cos\sigma\cot\sigma\left(-1+2\epsilon_\ell-\tfrac{3}{2}\epsilon\right)\Big\}(\theta-\sigma)^2 + 4\,\frac{\cos\sigma\cot^2\sigma}{\epsilon}\left(-\tfrac{1}{3}+\tfrac{1}{3}\epsilon_\ell-\tfrac{5}{4}\epsilon\right)(\theta-\sigma)^3 \\ &- \frac{22}{3}\,\frac{\cos\sigma\cot^3\sigma}{\epsilon}(\theta-\sigma)^4 - \frac{64}{15}\,\frac{\cos\sigma\cot^4\sigma}{\epsilon^2}(\theta-\sigma)^5 + \ldots\end{aligned} \tag{147}$$

Since u is known to fourth order in $\epsilon^i \epsilon_\ell^j$, v is obtained only up to third order in this quantity.

It was mentioned earlier that to a first approximation the streamlines are hyperbolas with the body surface as asymptote. In deriving the equation of the streamlines now we shall show that this is so. Referring to Figure I we see that

$$\tan\phi = \tan(\theta - \theta_s) = -\frac{v}{u} \tag{148}$$

so that

$$\theta - \theta_s = \tan^{-1}\left(-\frac{v}{u}\right) = -\tan^{-1}\left(\frac{v}{u}\right) \tag{149}$$

If in this last equation we now substitute for v and u from equations (147) and (101) respectively we shall obtain an expression giving the streamline angle θ_s in terms of the angular variable θ and σ , ϵ and ϵ_ℓ . This equation is then the equation of the streamlines of the flow.

Since $v = O(\epsilon)$ while $u = O(1)$ the argument of the arc tangent in equation (149) is $O(\epsilon)$. Using the series expansion of arc tangent we may write equation (149) as

$$\theta - \theta_s = -\left(\frac{v}{u}\right) + \frac{1}{3}\left(\frac{v}{u}\right)^3 - \frac{1}{5}\left(\frac{v}{u}\right)^5 + \frac{1}{7}\left(\frac{v}{u}\right)^7 + \ldots \tag{150}$$

Substituting for v and u in equation (150) and retaining only those terms of $O(\epsilon^3)$ and larger we obtain

$$\begin{aligned}\theta - \theta_s = {} & \left[\epsilon\tan\sigma + 2(\theta-\sigma)\right] - \epsilon\epsilon_\ell(\theta-\sigma) - \left\{\frac{1}{2}\epsilon\tan\sigma + \cot\sigma\left(-1 + 2\epsilon_\ell - \frac{3}{2}\epsilon\right)\right\}(\theta-\sigma)^2 \\ & - \frac{4\cot^2\sigma}{\epsilon}\left(-\frac{1}{3} + \frac{1}{3}\epsilon_\ell - \frac{5}{4}\epsilon\right)(\theta-\sigma)^3 + \frac{22}{3}\frac{\cot^3\sigma}{\epsilon}(\theta-\sigma)^4 + \frac{64}{15}\frac{\cot^4\sigma}{\epsilon^2}(\theta-\sigma)^5 \\ & + \epsilon(\theta-\sigma)\tan\sigma\left(\frac{\theta-\sigma}{\epsilon}\cot\sigma + 1\right)\left[\epsilon\tan\sigma + 2(\theta-\sigma)\right] \\ & - \frac{1}{3}\left[\epsilon\tan\sigma + 2(\theta-\sigma)\right]^3 + O(\epsilon^4)\end{aligned} \tag{151}$$

We note that the particular combination

$$\epsilon \tan\sigma + 2(\theta - \sigma)$$

occurs in three places in equation (151). Using the relation between shock and cone angle, equation (103), we find that

$$\epsilon \tan\sigma + 2(\theta-\sigma) = 2(\theta - \theta_c) - \frac{1}{2}\epsilon^2 \tan\sigma \left[\frac{1}{6} + \left\{\frac{1}{20}\epsilon(3 - 5\tan^2\sigma) + \frac{1}{3}\epsilon_\ell\right\} + O(\epsilon^2)\right]$$

Substituting this result into equation (151) and dropping those terms which are smaller than $O(\epsilon^3)$ we obtain

$$\begin{aligned}
\theta - \theta_s = {} & 2(\theta-\theta_c) - \frac{1}{2}\epsilon^2 \tan\sigma\left[\frac{1}{6} + \left\{\frac{1}{20}\epsilon(3-5\tan^2\sigma) + \frac{1}{3}\epsilon_\ell\right\}\right] \\
& - \epsilon\epsilon_\ell(\theta-\sigma) - \left\{\frac{1}{2}\epsilon\tan\sigma + \cot\sigma\left(-1 + 2\epsilon_\ell - \frac{3}{2}\epsilon\right)\right\}(\theta-\sigma)^2 \\
& - \frac{4\cot^2\sigma}{\epsilon}\left(-\frac{1}{3} + \frac{1}{3}\epsilon_\ell - \frac{5}{4}\epsilon\right)(\theta-\sigma)^3 + \frac{22}{3}\frac{\cot^3\sigma}{\epsilon}(\theta-\sigma)^4 \\
& + \frac{64}{15}\frac{\cot^4\sigma}{\epsilon^2}(\theta-\sigma)^5 + 2\epsilon(\theta-\sigma)(\theta-\theta_c)\tan\sigma\left[\frac{\theta-\sigma}{\epsilon}\cot\sigma + 1\right] \\
& - \frac{8}{3}(\theta-\theta_c)^3 + O(\epsilon^4)
\end{aligned} \tag{152}$$

Equation (152) represents the equation of the streamlines to $O(\epsilon^3)$. The right hand side of this equation involves, apart from the polar variable θ, the four parameters θ_c, σ, ϵ, and ϵ_ℓ. However, these parameters are not independent, being related by the shock-cone angle relation, equation (103), so the number of independent parameters is actually only three.

Since equation (152) is quite complicated, we shall consider in detail only those terms of $O(\epsilon^2)$, that is, those terms which represent the first correction to the constant density or lowest order solutions. The terms of $O(\epsilon^2)$ on the right hand side of equation (152) are the following

$$-\frac{1}{12}\epsilon^2 \tan\sigma + \cot\sigma(\theta-\sigma)^2 + \frac{4}{3}\frac{\cot^2\sigma}{\epsilon}(\theta-\sigma)^3 \tag{153}$$

Since we are only retaining terms of $O(\epsilon^2)$ in this expression we may substitute into it the lowest order result for the relation between shock and cone angle, which is

$$\sigma - \theta_c = \frac{1}{2}\epsilon \tan\sigma$$

In this way, we obtain from (153)

$$\frac{\cot\sigma}{3(\sigma-\theta_c)}\left[-(\sigma-\theta_c)^3 + 3(\sigma-\theta_c)(\sigma-\theta)^2 - 2(\sigma-\theta)^3\right] \tag{154}$$

We note that the expression in brackets in (154) is zero when $\theta = \theta_c$. It turns out that $\theta = \theta_c$ is a double zero of this expression. Hence, (154) may be written

$$\frac{\cot\sigma}{3(\sigma-\theta_c)}(\theta-\theta_c)^2\left\{-3\sigma + 2\theta + \theta_c\right\}$$

Let us denote this expression by $A(\theta;\ \sigma,\ \theta_c)$ and also note that it may be written

$$A(\theta;\sigma;\theta_c) = -\frac{(\theta-\theta_c)^2}{3(\sigma-\theta_c)}\left\{2(\sigma-\theta) + (\sigma-\theta_c)\right\}\cot\sigma \tag{155}$$

We are now interested in considering σ and θ_c as fixed and treating A as a function of θ only, which we shall denote by $A(\theta)$. First, it is immediately evident upon examining (155) that, since $\sigma \gg \theta \gg \theta_c$, A is never positive. Upon examining (155) in greater detail we find that A has a maximum at $\theta = \theta_c$ (at which A = 0) and decreases monotonically to a minimum at $\theta = \sigma$, with a point of inflection halfway between θ_c and σ. At its minimum the value of A is

$$A(\sigma) = -\frac{1}{3}(\sigma-\theta_c)^2 \cot\sigma \tag{156}$$

A graph of the function A is given in Figure IA.

Returning to equation (152) we may now write this as

$$\theta - \theta_s = z(\theta-\theta_c) + A(\theta;\sigma,\theta_c) + O(\epsilon^3)$$

which may also be written as

$$\theta_c - \theta_s = \theta - \theta_c + A(\theta;\sigma,\theta_c) + O(\epsilon^3) \tag{157}$$

where $A(\theta;\ \sigma,\ \theta_c)$ is given by equation (155), and is of $O(\epsilon^2)$. If we drop terms of higher order in ϵ than first in this expression, it reduces to

$$\theta_c-\theta_s = \theta-\theta_c \tag{158}$$

This relation says that the cone bisects the angle between the ray and the streamline passing through that point. This is equivalent to the statement that the slope of a streamline with respect to the body surface is equal to minus the distance from the body surface divided by the distance from the vertex. Such streamlines are hyperbolas with the body surface as asymptote.

If we now return to equation (157) and drop terms of higher order than second we have

$$\theta_c-\theta_s = \theta-\theta_c+A(\theta;\ \sigma,\ \theta_c) \tag{159}$$

The information we have already obtained about the function A shows that instead of the cone bisecting the angle between ray and streamline, the angle between the cone and the streamline is smaller than that between ray and cone. This means that the effect of the term A on the slope of the streamlines is to incline them less to the cone and hence to push them away from the cone. Since A decreases monotonically from zero to a minimum value as θ goes from θ_c to σ we see that this effect increases from zero at the cone surface to a maximum at the shock wave. Also, since A is also a function of σ, it follows that the shape of the streamlines now depends on the shock angle explicitly, whereas to lowest order the streamlines can be drawn in as soon as the cone is given, without a knowledge of the shock angle. In other words, in the lowest order solution the location and shape of the streamlines are frozen and independent of the particular flow conditions, whereas in the next higher order the streamline pattern does depend on the particular problem under consideration.

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11. Distribution limitations: None

12. Abstract: The Taylor-Maccoll equation for the supersonic flow of an ideal gas about a right circular cone with an attached shock wave is solved for the two cases of infinite and finite free stream Mach numbers. The solution for the former is given in terms of an expansion in the limiting density ratio across the shock, ϵ_ℓ and the solution to the latter in terms of a double expansion in the actual density ratio, ϵ , and ϵ_ℓ . The results are compared with a Taylor series solution to the Taylor-Maccoll equation and also with the exact values of Kopal for air and Mueller for helium.

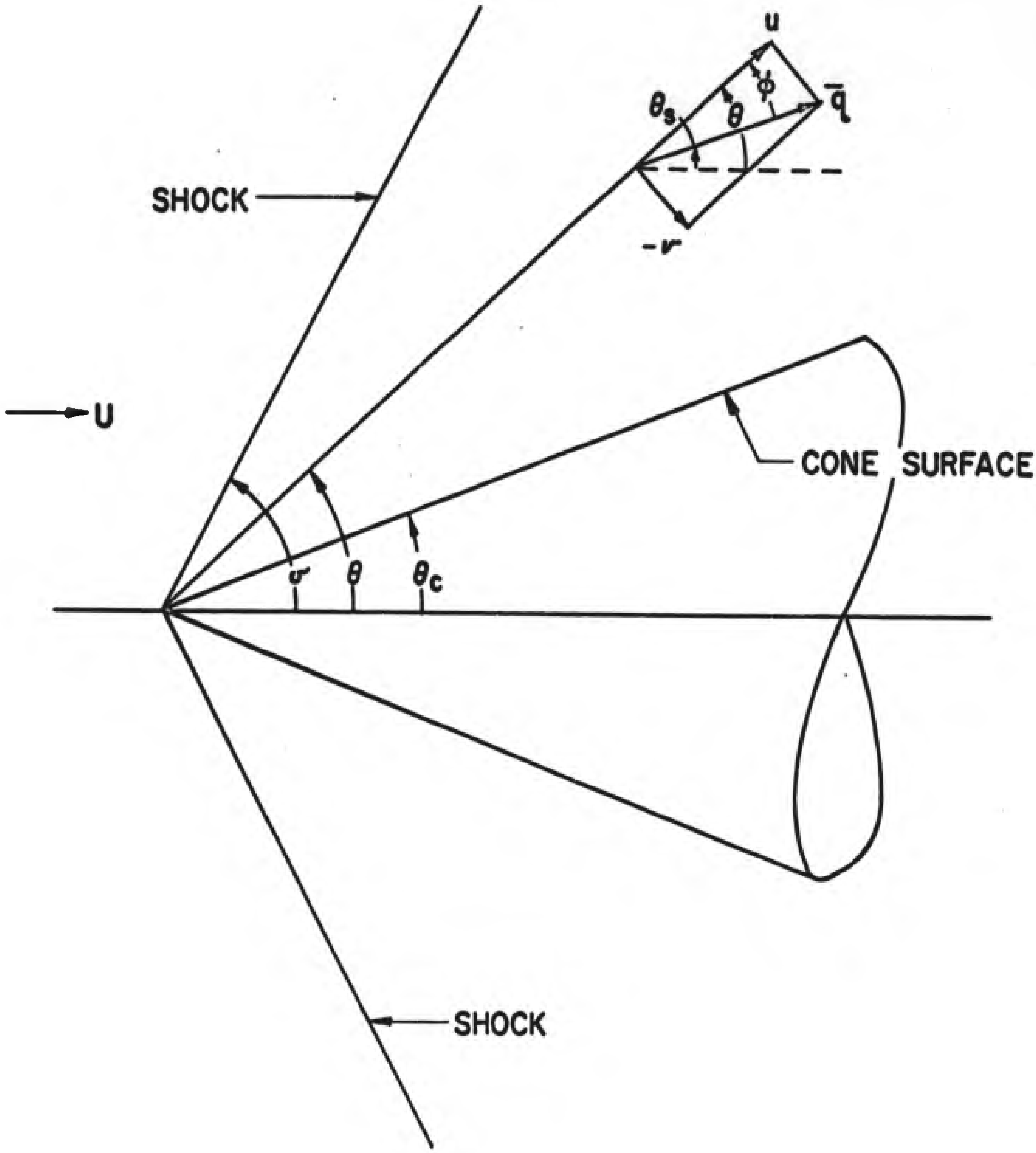

FIG. I

GENERAL GEOMETRY OF THE CONICAL FLOW FIELD

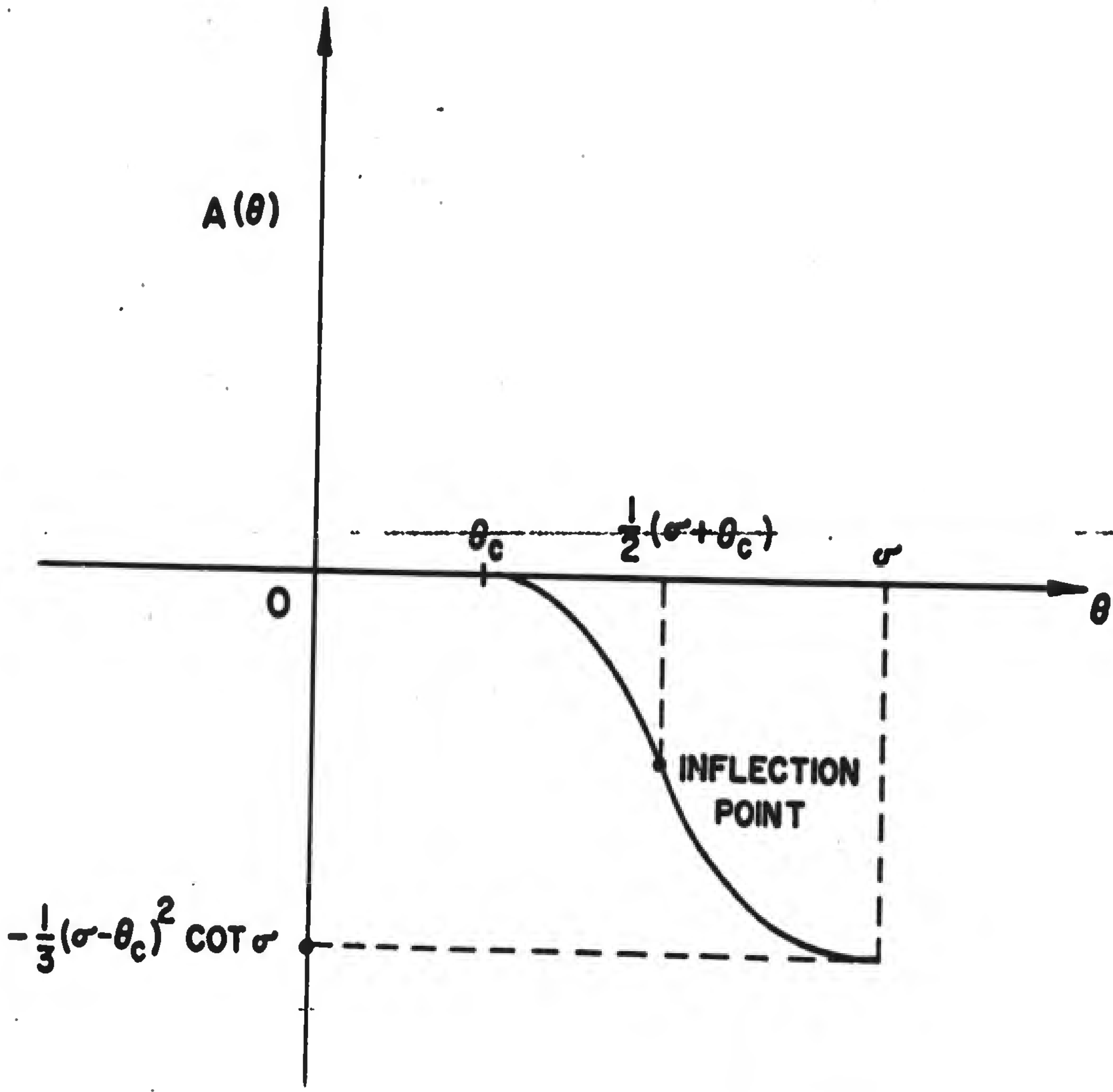

FIG. 1A

GRAPH OF A $(\theta;\sigma,\theta_c)$ AS A FUNCTION OF θ

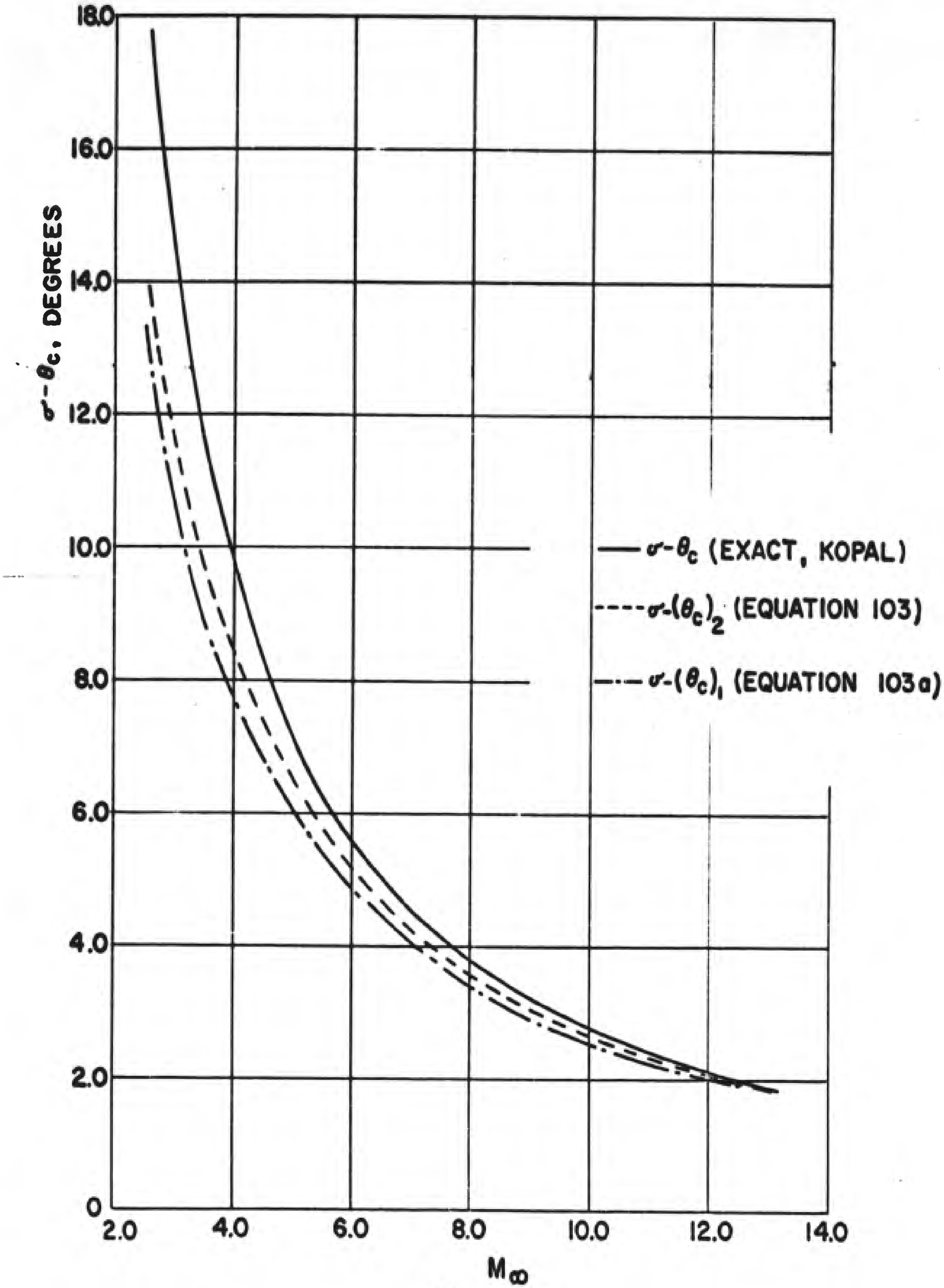

FIG. 2

VARIATION OF $\sigma - \theta_c$ WITH MACH NUMBER FOR $\theta_c = 5°$

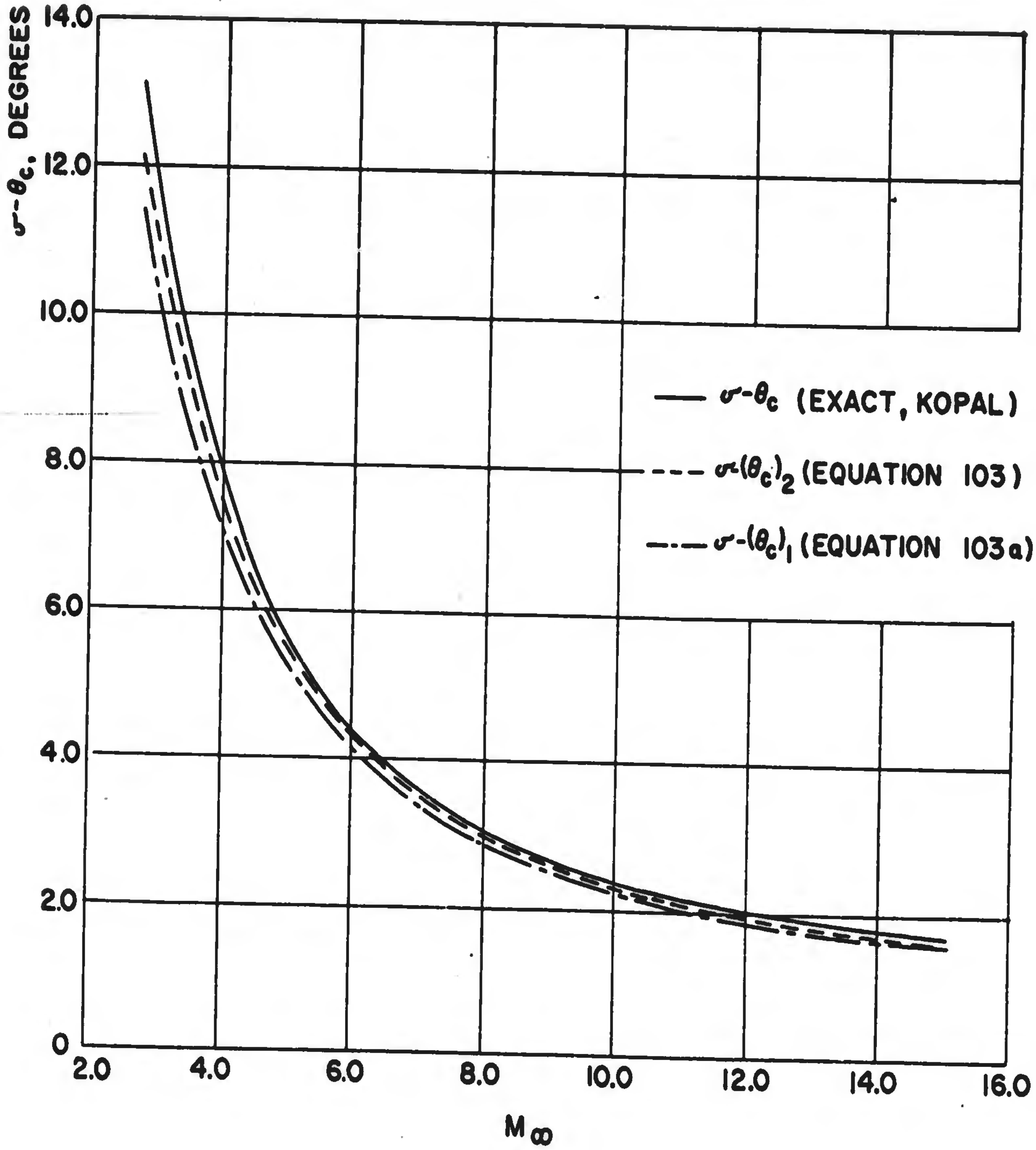

FIG. 3

VARIATION OF $\sigma-\theta_c$ WITH MACH NUMBER FOR $\theta_c = 10°$

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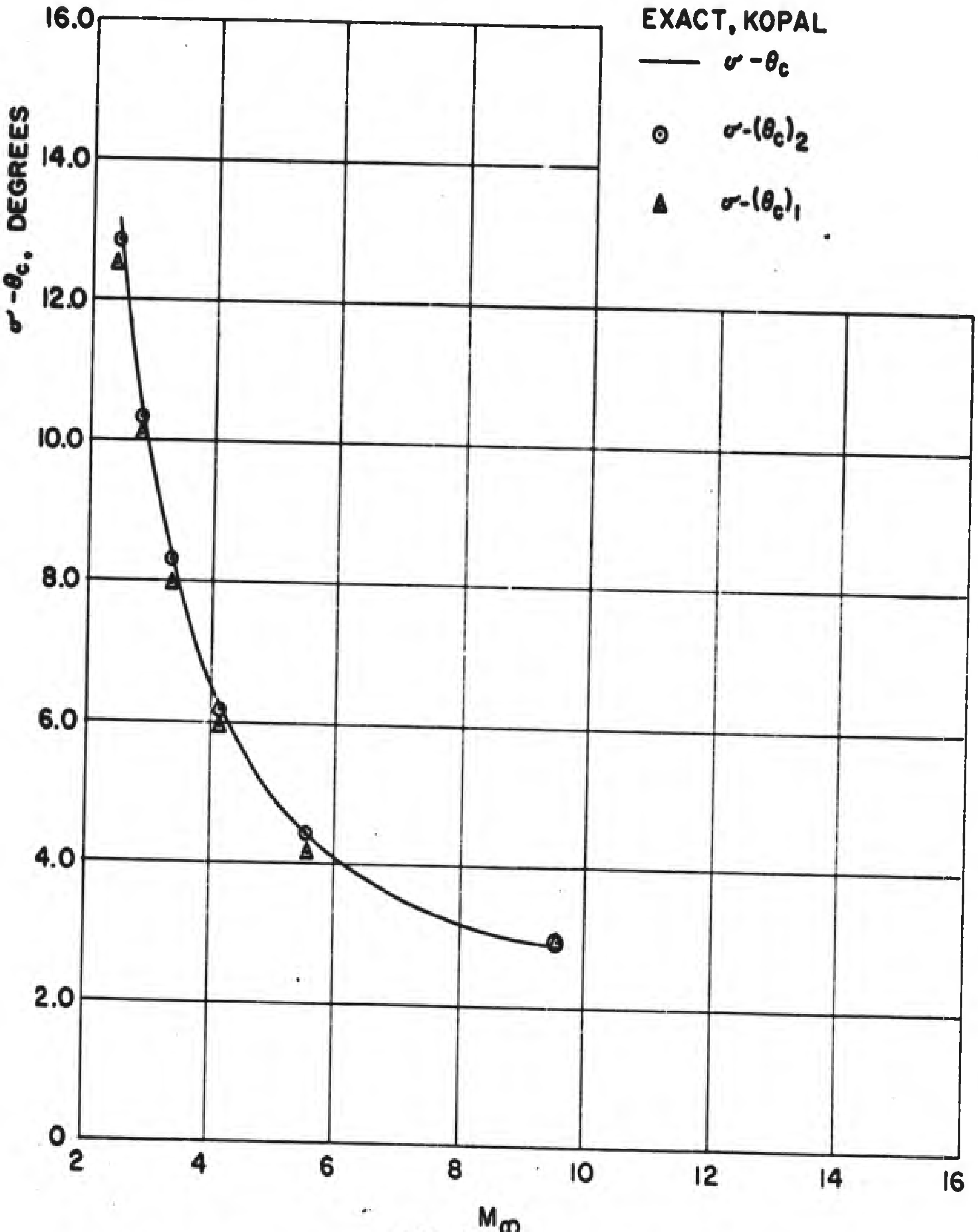

FIG. 4
VARIATION OF $\sigma - \theta_c$ WITH MACH NUMBER FOR $\theta_c = 20°$

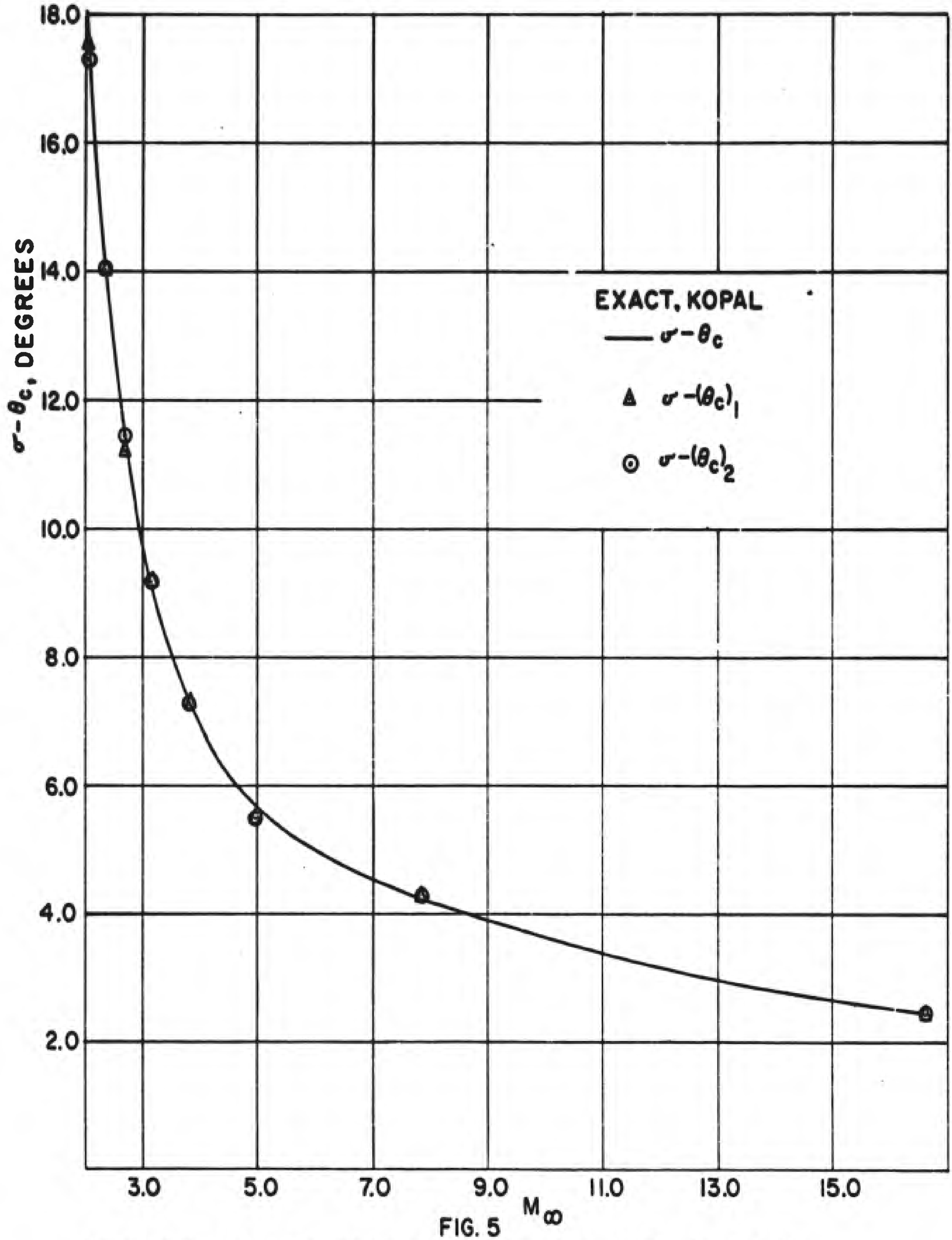

FIG. 5

VARIATION OF $\sigma - \theta_c$ WITH MACH NUMBER FOR $\theta_c = 30°$

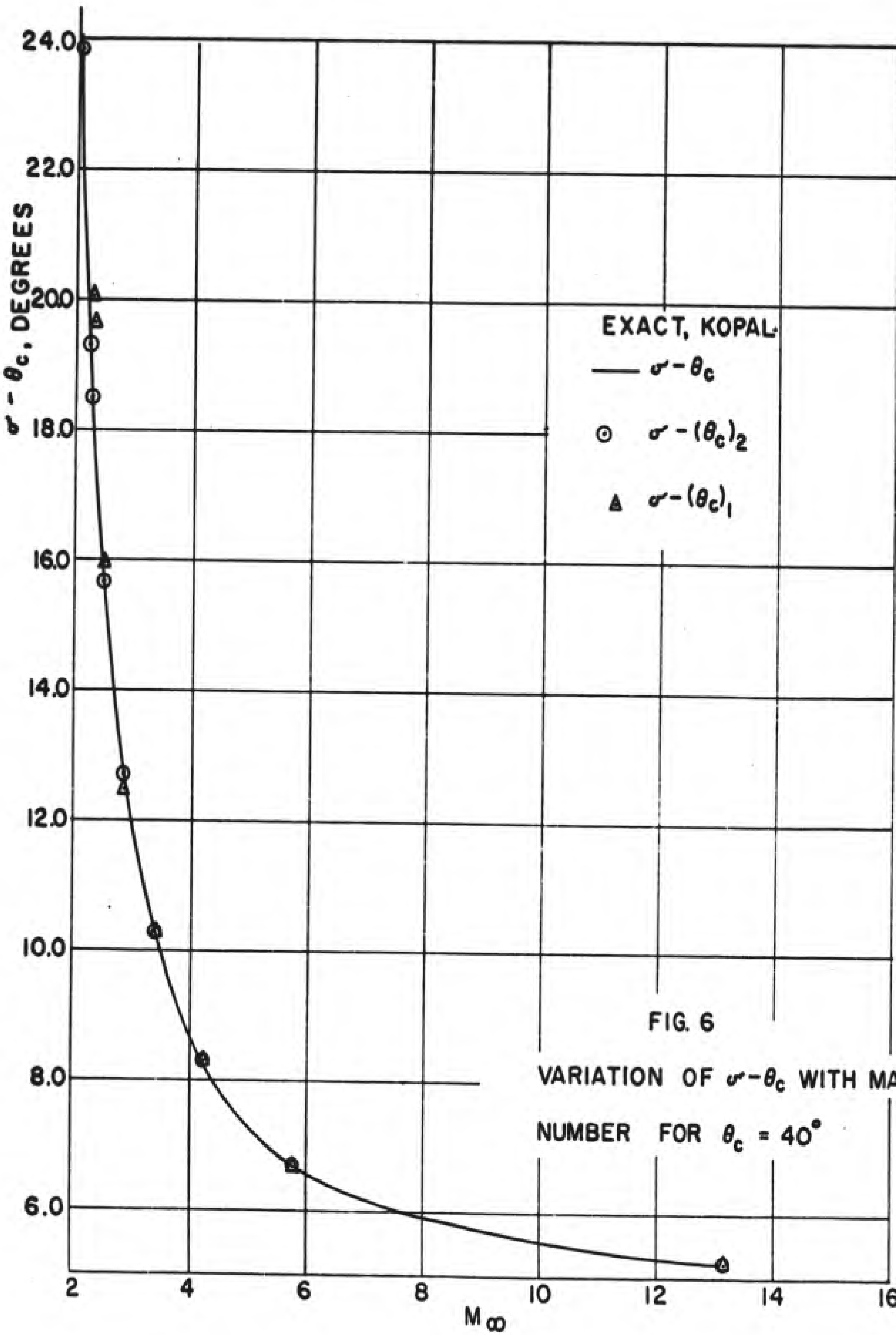

FIG. 6

VARIATION OF $\sigma - \theta_c$ WITH MACH NUMBER FOR $\theta_c = 40°$

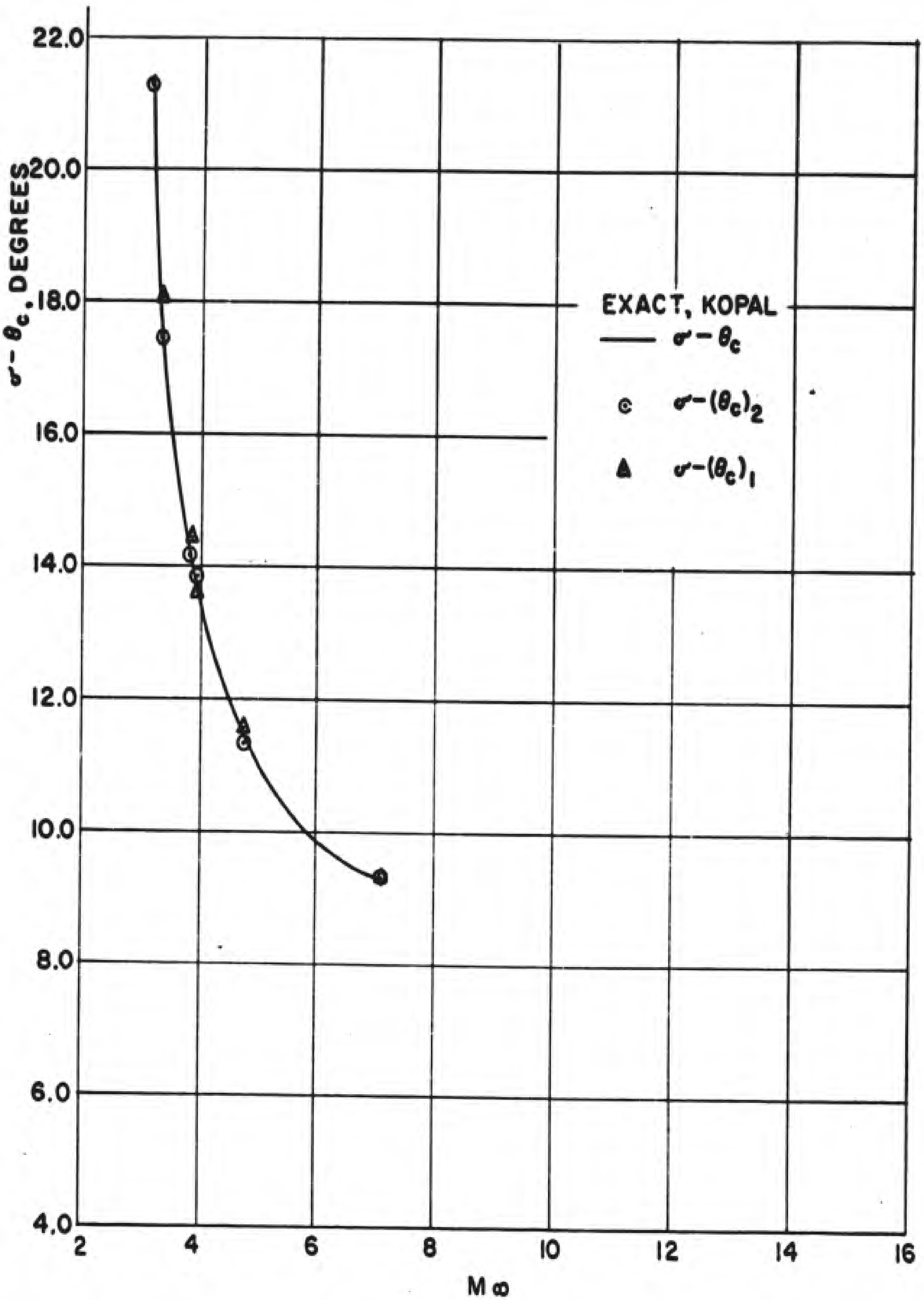

FIG. 7

VARIATION OF $\sigma - \theta_c$ WITH MACH NUMBER FOR $\theta_c = 50°$

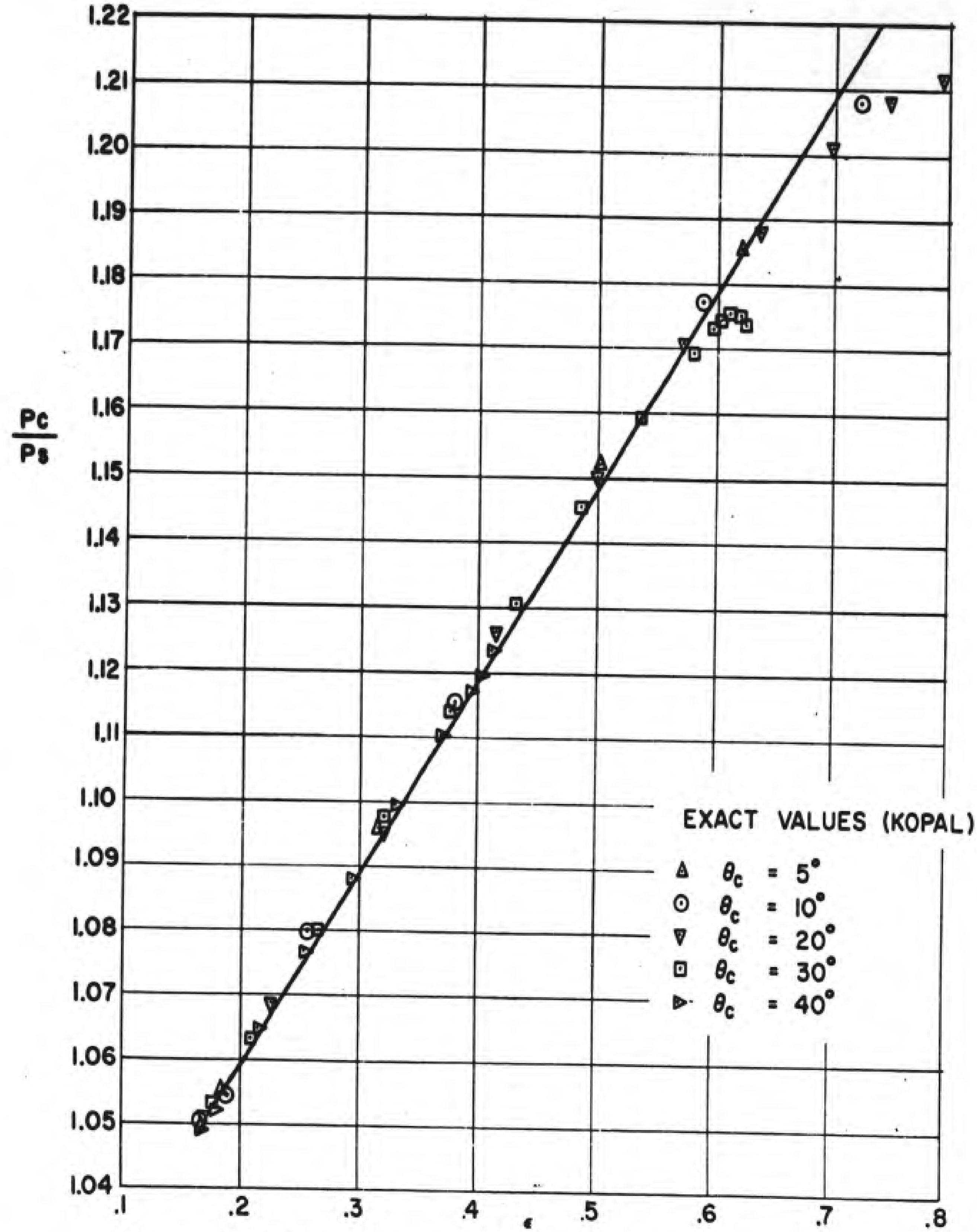

FIG. 8 VARIATION OF P_c / P_s WITH ε FOR AIR

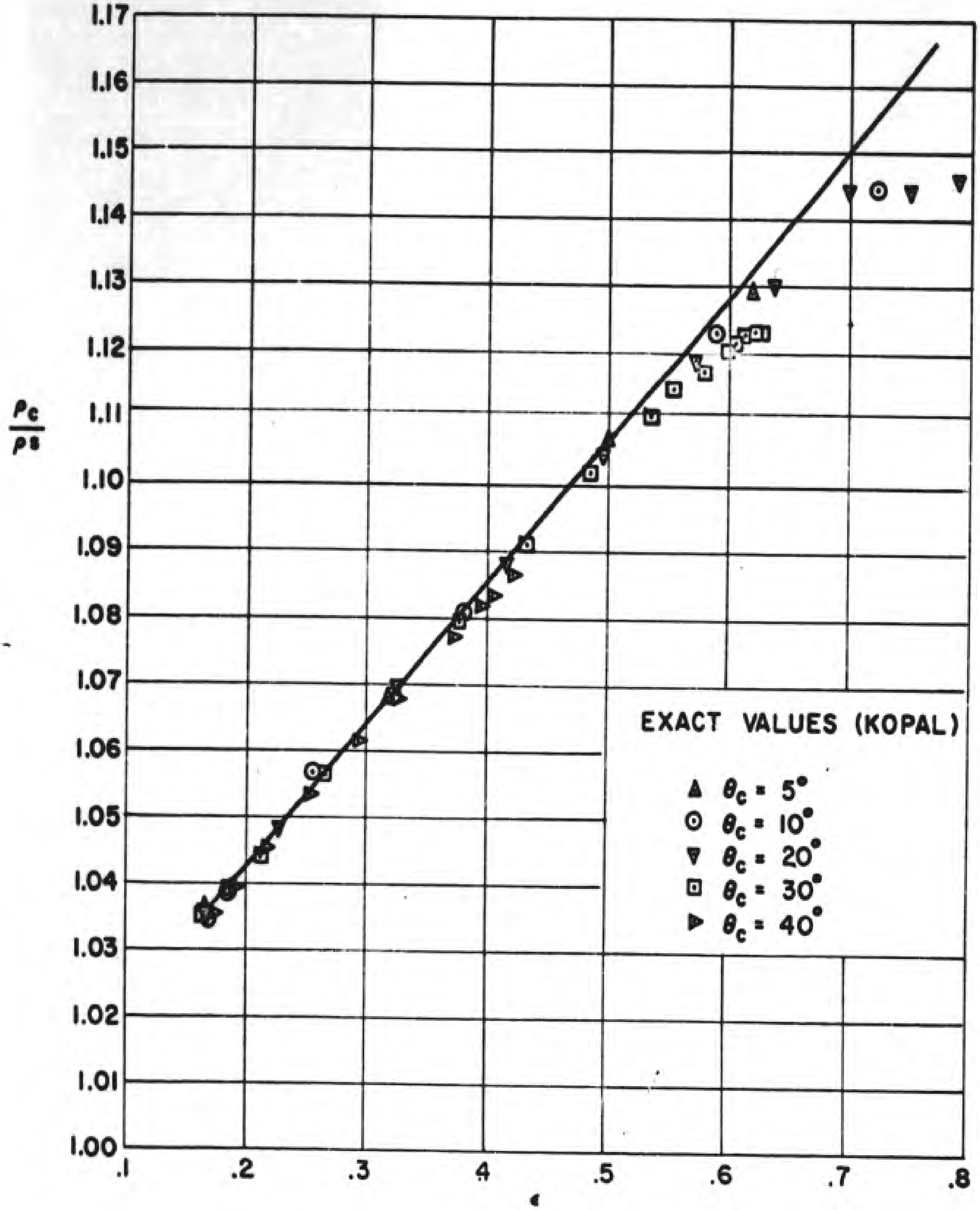

FIG. 9

VARIATION OF ρ_c/ρ_s WITH ϵ FOR AIR

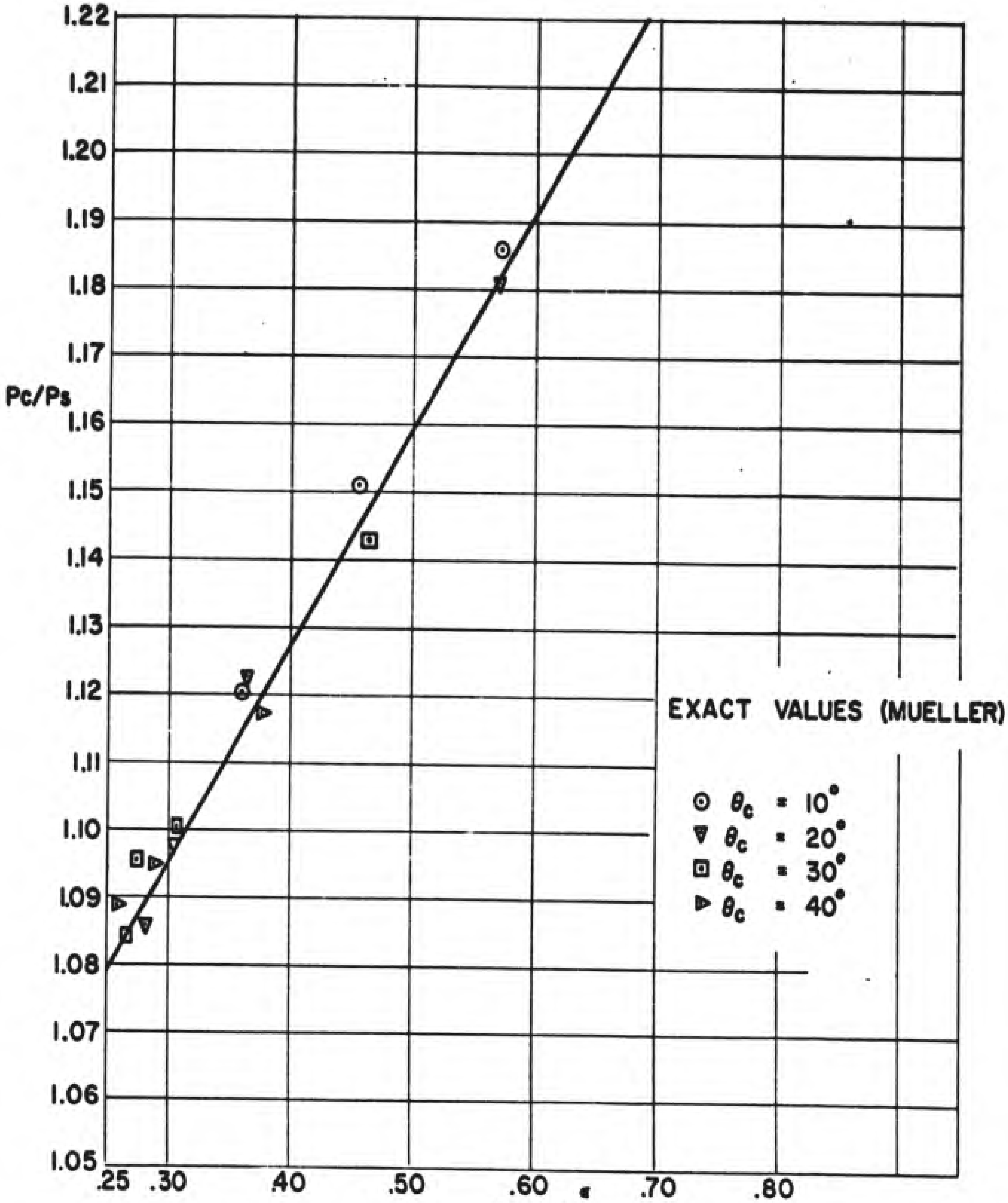

FIG. 10

VARIATION OF P_c / P_s WITH ε FOR HELIUM

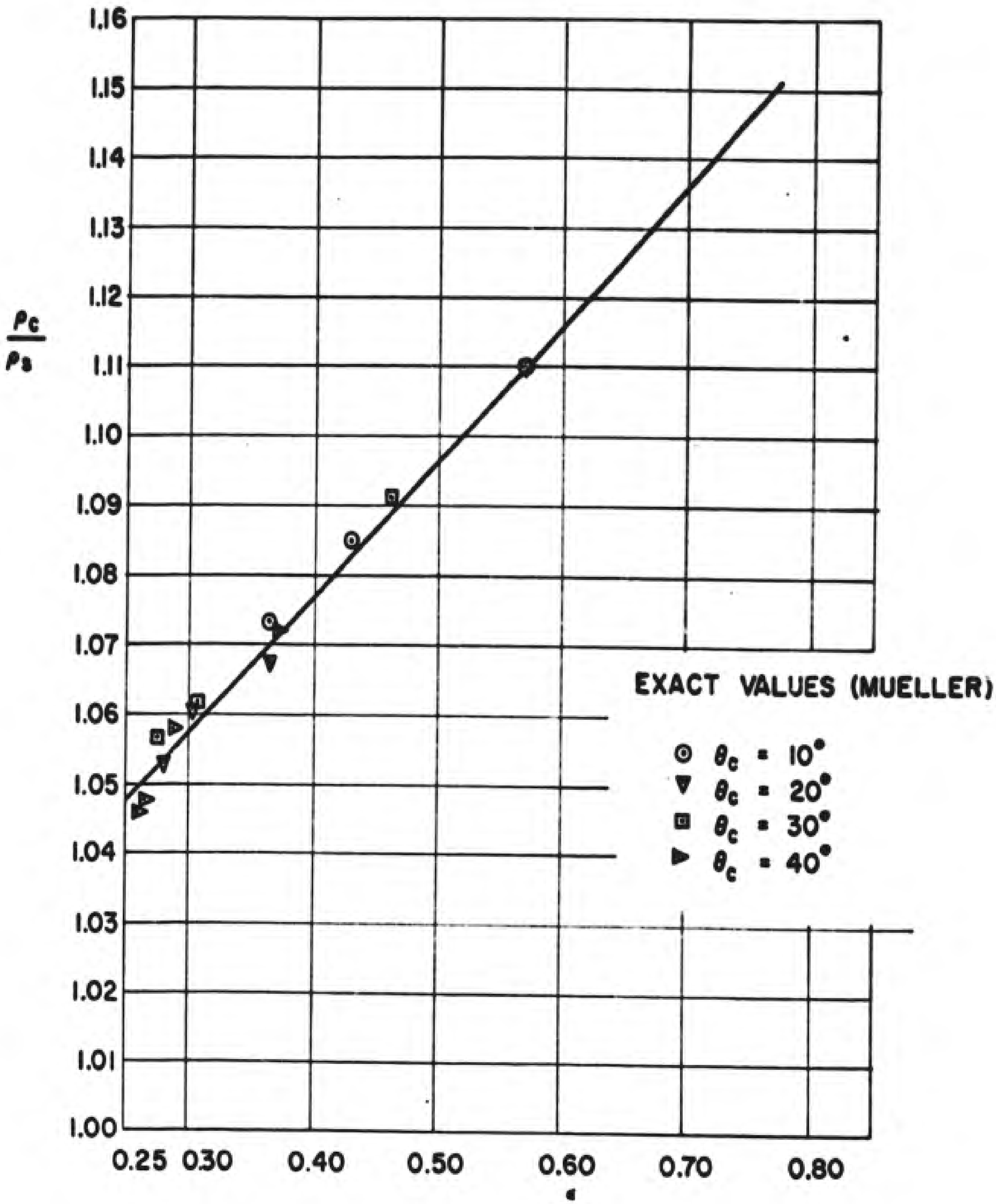

FIG. 11

VARIATION OF ρ_c/ρ_s WITH ϵ FOR HELIUM

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