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by

E. Eisenberg

RESEARCH REPORT 8

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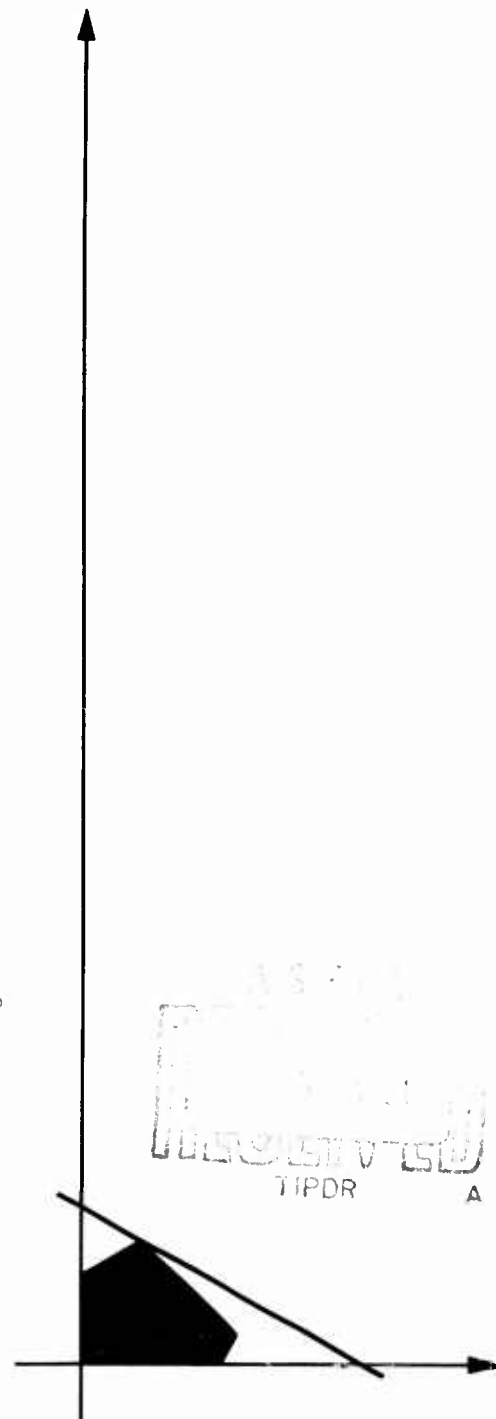
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# DUALITY IN HOMOGENEOUS PROGRAMMING

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Research Report 8

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## DUALITY IN HOMOGENEOUS PROGRAMMING

The problem of maximizing a concave function subject to linear constraints does not have a dual, as is the case in linear programming, in which primal optimizing variables do not appear. As a special case of our principal result it will follow that such a dual does indeed exist whenever the objective function is also homogeneous.

In the linear case we are given an  $m \times n$  matrix  $A$  and vectors<sup>\*</sup>  $a \in R^n$ ,  $b \in R^m$ . The feasibility sets  $X$  and  $Y$  are defined by:  $X = R_+^m \cap \{x \mid xA \leq a\}$ ,  $Y = R_+^n \cap \{y \mid Ay \geq b\}$ . Since  $xA \leq a$  if and only if  $xAy \leq ay$  for all  $y \in R_+^n$  (and similarly for  $Ay \geq b$ ), we may write:

$$(1) \quad \begin{aligned} X &= R_+^m \cap \left\{ x \mid xAy \leq \psi(y) \quad \text{all } y \in R_+^n \right\} \\ Y &= R_+^n \cap \left\{ y \mid xAy \geq \phi(x) \quad \text{all } x \in R_+^m \right\} \end{aligned}$$

where  $\psi(y) = ay$  and  $\phi(x) = bx$ .

A fundamental theorem of linear programming (see, e.g., [3] and [5]) states that if  $X$  and  $Y$  are both non-empty then

$$(2) \quad \max_{x \in X} \phi(x), \min_{y \in Y} \psi(y) \text{ exist and are equal.}$$

We propose to demonstrate that (2) holds for larger class of triples  $(A, \phi, \psi)$ .

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\*  $R^m$  denotes the set of all real  $m$ -tuples. If  $u, v \in R^m$  then  $u \leq v$  means that the inequality holds for each component. In particular,  $R_+^m = R^m \cap \{x \mid x \geq 0\}$ . If  $M$  is a  $p \times q$  matrix and  $N$  is a  $q \times t$  matrix then  $MN$  represents the usual matrix product. To simplify notation, the same symbol is used for both a column vector and its transpose; the meaning will, in any case, be clear from the context.

Assumption  $A_1$ . Let  $\phi : R_+^m \rightarrow R$ ,  $\psi : R_+^n \rightarrow R$  be positively homogeneous\*, continuous, concave and convex respectively.

Let us first show that  $A_1$  does not guarantee that (2) holds when  $X$  and  $Y$  are non-empty. If  $m = 2$ ,  $n = 1$  and  $A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ,  $\phi(x) = \phi(\xi, \eta) = \frac{\xi \eta}{\xi + \eta}$ ,  $[\phi(0) = 0]$ ,  $\psi(y) = y$  then  $A_1$  is satisfied and  $X = R_+^2 \cap \{(\xi, \eta) \mid \eta \leq 1\}$ ,  $Y = R_+ \cap \{y \mid y \geq 1\}$  are non-empty. Thus  $\min_{y \in Y} \psi(y) = 1$ , but if  $\eta \leq 1$  then  $\phi(\xi, \eta) < 1$ , although  $\sup_{x \in X} \phi(x) = 1$ , hence  $\max_{x \in X} \phi(x)$  does not exist.

The situation just illustrated cannot occur if the following holds:

Assumption  $A_2$ .

- i) If  $x \in R_+^m$ ,  $xA \leq 0$ ,  $\phi(x) \geq 0$  then  $x = 0$
- ii) If  $y \in R_+^n$ ,  $Ay \geq 0$ ,  $\psi(y) \leq 0$  then  $y = 0$

One sees immediately that (i) is violated in the preceding example, for let  $x = (1, 0)$  then  $xA = 0$  and  $\phi(x) = 0$ .

Before proving that if  $A_1$  and  $A_2$  hold then so does (2), we require the following lemma which specializes to homogeneous functions the well-known fact that a concave function is the infimum of its supports. The proof is presented here for the sake of completeness.

Lemma 1. Let  $\phi$  be as in assumption  $A_1$ , consider  $T = R^m \cap \{t \mid tx \geq \phi(x) \text{ all } x \in R_+^m\}$ , then  $T$  is non-empty, and  $\phi(x) = \inf_{t \in T} tx$ , for all  $x \in R_+^m$ .

Proof. Let  $C = \{(x, \lambda) \mid x \in R_+^m, \lambda \leq \phi(x)\}$  then  $C$  is a closed convex cone. Now if  $x_0 \in R_+^m$ ,  $\mu > 0$ , then  $(x_0, \mu + \phi(x_0)) \notin C$ , whence (see [2], Theorem 1)

\* A function  $f : C \rightarrow R^q$ , where  $C \subset R^p$  is a cone, is positively homogeneous providing  $f(\lambda x) = \lambda f(x)$  for all  $x \in C$  and  $\lambda \in R_+$ .

there exist  $t \in R^m$  and  $\alpha \in R$  such that  $tx_0 - \alpha[\mu + \phi(x_0)] < 0 \leq tx - \alpha\lambda$  all  $(x, \lambda) \in C$ .

It then follows that  $\alpha > 0$ , so that (dividing by  $\alpha$ ) we may assume  $\alpha = 1$ , but then  $t \in T$ . Reiterating, if  $x_0 \in R_+^m$ ,  $\mu > 0$  then  $\exists t \in T$  such that:

$$tx_0 - \mu \leq \phi(x_0) \leq tx_0$$

giving the desired result. We are now able to prove:

Theorem 1. If assumptions  $A_1$  and  $A_2$  hold then (2) holds.

Proof. Let

$$(3) \quad \begin{aligned} S &= R^n \cap \left\{ s \mid sy \leq \psi(y), \quad \text{all } y \in R_+^n \right\} \\ T &= R^m \cap \left\{ t \mid tx \geq \phi(x), \quad \text{all } x \in R_+^m \right\} \end{aligned}$$

Then  $S$  and  $T$  are convex sets; now consider the system of inequalities:

$$(4) \quad \begin{aligned} (x, y, s, t) &\in R_+^m \times R_+^n \times S \times T \\ s - xA &> 0 \\ -t + Ay &> 0 \\ \phi(x) - \psi(y) &> 0 \end{aligned}$$

If (4) has a solution  $x, y, s, t$  then

$$\psi(y) < \phi(x) \leq tx \leq xAy \leq sy \leq \psi(y)$$

which is a contradiction. Thus (see [2], Theorem 1) there exist  $x_0 \in R_+^m$ ,  $y_0 \in R_+^n$ ,  $\lambda \in R_+$ , not all zero and such that  $(s - xA)y_0 + x_0(Ay - t) + \lambda[\phi(x) - \psi(y)] \leq 0$  for all  $x \in R_+^m$ ,  $y \in R_+^n$ ,  $s \in S$ ,  $t \in T$ . From the homogeneity and continuity of  $\phi$  and  $\psi$  it then follows that:

$$x Ay_0 \geq \lambda \phi(x) \quad , \quad \text{all } x \in R_+^m$$

$$x_0 Ay \leq \lambda \psi(y) \quad , \quad \text{all } y \in R_+^n$$

$$s y_0 \leq t x_0 \quad , \quad \text{all } s \in S, t \in T$$

The last condition together with Lemma 1 imply:

$$\psi(y_0) \leq \phi(x_0)$$

Now if  $\lambda = 0$  then either  $x_0 \neq 0$  or  $y_0 \neq 0$  and  $Ay_0 \geq 0$ ,  $x_0 A \leq 0$ . Suppose  $x_0 \neq 0$ , if  $y_0 \neq 0$  the argument is analogous, then by  $A_2(i)$  we have  $\phi(x_0) < 0$ , whence  $\psi(y_0) < 0$  and  $y_0 \neq 0$ , contradicting  $A_2(ii)$ . Thus  $\lambda > 0$  and, dividing all inequalities by  $\lambda$ , we may assume  $\lambda = 1$ . This tells us that  $x_0 \in X$ ,  $y_0 \in Y$  and  $\phi(x_0) \leq x_0 Ay_0 \leq \psi(y_0) \leq \phi(x_0)$ . So that if  $x \in X$ ,  $y \in Y$  then

$$\phi(x) \leq x Ay_0 \leq \psi(y_0) = \phi(x_0)$$

$$\psi(y) \geq x_0 Ay \geq \phi(x_0) = \psi(y_0)$$

proving the theorem.

In case  $\phi$  and  $\psi$  are linear-homogeneous then it is true that  $\max_{x \in X} \phi(x)$  exists if and only if  $\min_{y \in Y} \psi(y)$  exists, in which case they are equal. As above, this statement is not always true under assumption  $A_1$ ; however, we show:

Theorem 2.

- I) If  $A_1$  and  $A_2(ii)$  hold, and  $\max_{x \in X} \phi(x)$  exists then (2) holds.
- II) If  $A_1$  and  $A_2(i)$  hold, and  $\min_{y \in Y} \psi(y)$  exists then (2) holds.

We prove (I), the proof of (II) is similar. Suppose that  $x_0 \in X$  and  $\phi(x_0) = \max_{x \in X} \phi(x)$ , then the system:

$$\begin{aligned}
(6) \quad & (x, s) \in R_+^m \times S \\
& s - xA > 0 \\
& \phi(x) - \phi(x_0) > 0
\end{aligned}$$

has no solution. Thus (see [2], Theorem 1) there exist  $y_0 \in R_+^n$ ,  $\lambda \in R_+$ , not both zero and such that

$sy_0 - xAy_0 + \lambda[\phi(x) - \phi(x_0)] \leq 0$  for all  $x \in R_+^m$ ,  $s \in S$ . From the homogeneity of  $\phi$  and Lemma 1 it then follows that

$$\begin{aligned}
(7) \quad & xAy_0 \geq \lambda\phi(x), \quad \text{for all } x \in R_+^m \\
& \psi(y_0) \leq \lambda\phi(x_0)
\end{aligned}$$

Now, if  $\lambda = 0$  then  $y_0 \neq 0$  and  $Ay_0 \geq 0$ ,  $\psi(y_0) \leq 0$ , contradicting  $A_2(ii)$ . It may then be assumed that  $\lambda > 0$  and, in fact, that  $\lambda = 1$  (replacing  $y_0$  by  $\lambda y_0$ ). Thus, from (7),  $y_0 \in Y$ , and for any  $y \in Y$  we have:

$$\psi(y_0) \leq \phi(x_0) \leq x_0 Ay \leq \psi(y),$$

$$\text{i.e.,} \quad \psi(y_0) = \min_{y \in Y} \psi(y) = \phi(x_0). \quad \text{q.e.d.}$$

It should be remarked that if  $A_1$  holds then (i) and (ii) of assumption  $A_2$  are equivalent to (i)' and (ii)' respectively of:

Assumption  $A_2'$ .

- i)'  $\exists y_0 \in R_+^n \ni xAy_0 > \phi(x) \quad \text{all } x \in R_+^m, x \neq 0$
- ii)'  $\exists x_0 \in R_+^m \ni x_0 Ay < \psi(y) \quad \text{all } y \in R_+^n, y \neq 0$

These in turn are equivalent to the familiar conditions that  $X$ ,  $Y$  have non-empty interiors. To see, for instance, that (i) and (i)' are equivalent it

suffices to show that (i) implies (i)' since the implication in the other direction is trivial. Assuming (i)' false, the system

$$(8) \quad (y, t) \in R_+^n \times T$$

$$Ay - t > 0$$

has no solution, whence (see [2], Theorem 1) there is an  $x \in R_+^m$ ,  $x \neq 0$ , and such that  $xAy \leq tx$  for all  $y \in R_+^n$  and  $t \in T$ . Thus  $xA \leq 0$  and (using Lemma 1)  $\phi(x) \geq 0$ , contradicting (i). To return to our remark about maximizing a concave homogeneous and continuous function  $\phi: R_+^m \rightarrow R$ , subject to the inequalities  $x \geq 0$  and  $xA \leq a$ , the dual is then: minimize  $ay$  subject to  $y \in Y$ . Conditions (i) and (ii)' become:

$$x \in R_+^m, x \neq 0, xA \leq 0, \phi(x) \geq 0 \quad \text{has no solution; and}$$

$$x \in R_+^m, xA < a \quad \text{has a solution; respectively.}$$

Also, since  $y \in Y$  providing  $y \geq 0$  and  $Ay \geq t$  for some support  $t$  of  $\phi$ , we may characterize  $Y$  by means of the gradient of  $\phi$ .

Results similar to Theorems 1 and 2 can be shown to hold under other and somewhat less restrictive assumptions; the duality theorems of linear programming then turn out to be special cases of these theorems (3, 4 and 5).

Henceforth we assume that  $A_1$  holds and consider the sets:

$$K_1 = R^{m+n+1} \cap \left\{ (\bar{x}, \bar{y}, \lambda) \mid \exists s \in S, t \in T, x \in R_+^m, y \in R_+^n \text{ and} \right.$$

$$\left. x \geq t - Ay, \bar{y} \leq s - xA, \lambda \leq \phi(x) - \psi(y) \right\},$$

$$K_2 = R^{n+1} \cap \left\{ (y, \lambda) \mid \exists s \in S, x \in R_+^m, \text{ and } \bar{y} \leq s - xA, \lambda \leq \phi(x) \right\},$$

$$K_3 = R^{m+1} \cap \left\{ (x, \lambda) \mid \exists t \in T, y \in R_+^n, \text{ and } \bar{x} \geq t - Ay, \lambda \geq \psi(y) \right\}.$$

The sets  $K_1$ ,  $K_2$ , and  $K_3$  are readily seen to be convex (because  $\psi$  and  $-\phi$  are convex); furthermore, if  $\phi$  and  $\psi$  are linear then  $K_1$ ,  $K_2$  and  $K_3$  are also closed sets. Of course any (or all) of  $K_1$ ,  $K_2$ ,  $K_3$  may be closed without either  $\phi$  or  $\psi$  being linear. Thus it is important to know the following:

Theorem 3.

If  $K_1$  is closed and  $X, Y$  are both non-empty, then (2) holds.

Theorem 4.

If  $K_3$  is closed and  $\max_{x \in X} \phi(x)$  exists, then (2) holds.

Theorem 5.

If  $K_2$  is closed and  $\min_{y \in Y} \psi(y)$  exists, then (2) holds.

Proof of Theorem 3: If the point  $(\bar{x}, \bar{y}, \lambda) = 0$  is in  $K_1$  then (2) obviously holds, suppose  $0 \notin K_1$ . Since  $K_1$  is convex and closed, there exist (see [3])  $x_0 \in R^m$ ,  $y_0 \in R^n$ ,  $\lambda_0 \in R$ ,  $\alpha \in R$ , such that:

$$0 < \alpha \leq x_0 \bar{x} - y_0 \bar{y} - \lambda_0 \lambda, \quad \text{all } (\bar{x}, \bar{y}, \lambda) \in K_1.$$

Since  $S$  and  $T$  are non-empty (see Lemma 1), and since  $(\bar{\bar{x}}, \bar{\bar{y}}, \bar{\lambda}) \in K_1$  whenever there exist  $(\bar{x}, \bar{y}, \lambda) \in K_1$  such that  $\bar{\bar{x}} \geq \bar{x}$ ,  $\bar{\bar{y}} \leq \bar{y}$  and  $\bar{\lambda} \leq \lambda$ , it follows that  $x_0 \geq 0$ ,  $y_0 \geq 0$ , and  $\lambda_0 \geq 0$ . Also,

$$0 < \alpha \leq x_0 (t - Ay) - (s - xA)y_0 - \lambda_0 [\phi(x) - \psi(y)],$$

$$\text{all } (s, t, x, y) \in S \times T \times R_+^m \times R_+^n$$

From the homogeneity of  $\phi$  and  $\psi$  and Lemma 1 it then follows that:

$$0 < \alpha \leq \phi(x_0) - \psi(y_0)$$

$$x_0 Ay \leq \lambda_0 \psi(y) \quad \text{all } y \in R_+^n$$

$$xAy_0 \geq \lambda_0 \phi(x) \quad \text{all } x \in R_+^m$$

Thus,  $\lambda_0 \psi(y_0) \geq x_0 A y_0 \geq \lambda_0 \phi(x_0) \geq \lambda_0 \alpha + \lambda_0 \psi(y_0)$ , and  $\lambda_0 = 0$ . But then  $x_0 A \leq 0$  and  $A y_0 \geq 0$ . Now  $X, Y$  we assumed non-empty, let  $x \in X, y \in Y$ . For any  $\lambda \in R_+$  we then have:

$$\begin{aligned} \phi(x) + \lambda \phi(x_0) &\leq \phi(x + \lambda x_0) \leq (x + \lambda x_0) A y \leq \\ &\leq (x + \lambda x_0) A (y + \lambda y_0) \leq x A (y + \lambda y_0) \\ &\leq \psi(y + \lambda y_0) \leq \psi(y) + \lambda \psi(y_0). \end{aligned}$$

Thus,  $\psi(y) - \phi(x) \geq \lambda [\phi(x_0) - \psi(y_0)] \geq \lambda \alpha$  for all  $\lambda \in R_+$  which contradicts  $\alpha > 0$ . Thus  $0 \in K_1$  and (2) holds. q. e. d.

Proof of Theorems 4 and 5: We prove Theorem 4, the proof of Theorem 5 is analogous. By hypothesis  $X$  is non-empty and  $\phi$  is bounded above on  $X$ , let:

$$M = \sup_{x \in X} \phi(x)$$

If  $(x, \lambda) = (0, M)$  is in  $K_3$  then, trivially, (2) holds. We show that the contrary assumption leads to a contradiction. If  $(0, M) \notin K_3$  then, as in the proof of Theorem 4, it follows from the various properties of  $K_3$  that there exist  $x_0 \in R_+^m$ ,  $\lambda_0 \in R_+$  and  $\alpha \in R$  such that:

$$\lambda_0 M < \alpha \leq x_0(t - Ay) + \lambda_0 \psi(y), \quad \text{all } (t, y) \in T \times R_+^n$$

Hence, as before,

$$x_0 A y \leq \lambda_0 \psi(y), \quad \text{all } y \in R_+^n$$

$$\text{and } \phi(x_0) > \lambda_0 M.$$

If  $\lambda_0$  is positive then  $x = \lambda^{-1} x_0 \in X$  and  $\phi(x) > M$ , contradicting the definition of  $M$ . Thus  $\lambda_0 = 0$ , and  $x_0 A \leq 0$ ,  $\phi(x_0) > 0$ ; the last contradicts the fact that  $X$  is non-empty and that  $\phi$  is bounded above on  $X$ . q.e.d.

As a final result we demonstrate that if  $\phi$  and  $\psi$  are both linear (homogeneous) then  $K_1$  is closed. That  $K_2$  and  $K_3$  are closed, under the same linearity hypothesis, follows in a similar manner.

Suppose  $\phi(x) = bx$ ,  $\psi(y) = ay$  ( $b \in R^m$ ,  $a \in R^n$ ), first note that in this case:

$$S = \{s \mid s \in R^n \text{ and } s \leq a\}$$

$$T = \{t \mid t \in R^m \text{ and } t \geq b\}.$$

Next, suppose we have a sequence  $(\bar{x}_k, \bar{y}_k, \lambda_k) \in K_1$  ( $k = 1, 2, \dots$ ) which converges to  $(\bar{x}, \bar{y}, \lambda) \in R^{m+n+1}$ . Thus there exist  $(s_k, t_k, x_k, y_k) \in S \times T \times R_+^m \times R_+^n$  such that:

$$\begin{aligned} \bar{x}_k &\geq t_k - Ay_k \geq b - Ay_k \\ (9) \quad \bar{y}_k &\leq s_k - x_k A \leq a - x_k A \quad k = 1, 2, \dots \\ \lambda_k &\leq bx_k - ay_k \end{aligned}$$

and

$$(10) \quad \bar{x}_k \rightarrow \bar{x}, \quad \bar{y}_k \rightarrow \bar{y}, \quad \lambda_k \rightarrow \lambda.$$

Now, suppose  $x \in R_+^m$ ,  $y \in R_+^n$ ,  $a \in R_+$  are such that  $Ay - ab \geq 0$ ,  $xA - aa \leq 0$ .

From (9) it then follows that for each  $k$  we have:

$$\begin{aligned} x\bar{x}_k &\geq bx - xAy_k \geq bx - aay_k \geq bx - abx_k + a\lambda_k \geq \\ &\geq bx + a\lambda_k - x_k Ay \geq bx + a\lambda_k + \bar{y}_k y - ay. \end{aligned}$$

i. e.,  $x(\bar{x}_k - b) + y(a - \bar{y}_k) - a\lambda_k \geq 0$ ,  $k = 1, 2, \dots$  and, by (10),  
 $x(\bar{x} - b) + y(a - \bar{y}) - a\lambda \geq 0$ .

In summary, then, the system:

$$x \in R_+^m, \quad y \in R_+^n, \quad a \in R_+$$

$$Ay - ab \geq 0, \quad xA - aa \leq 0$$

$$x(\bar{x} - b) + y(a - \bar{y}) - a\lambda < 0$$

has no solution. It follows then from the ordinary feasibility theorem for linear inequalities (see e. q. [5]) that there is an  $x \in R_+^m$  and  $y \in R_+^n$  such that:

$$(x, y) \begin{bmatrix} A & 0 & -b \\ 0 & -A^T & a \end{bmatrix} \leq (a - \bar{y}, \bar{x} - b, -\lambda),$$

i. e.,  $xA \leq a - \bar{y}$ ,  $-Ay \leq \bar{x} - b$  and  $ay - bx \leq -\lambda$ . But, as noted before,  $a \in S$  and  $b \in T$ , thus  $(\bar{x}, \bar{y}, \lambda) \in K$ , and  $K_1$  is closed.

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