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Division of Engineering
BROWN UNIVERSITY
PROVIDENCE, R. I.

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DIELECTRIC COATED ANTENNA

BY

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ABSTRACT

Propagation of electromagnetic waves in a coaxial system is investigated. The configuration is that of an idealized dielectric coated antenna standing perpendicularly on a perfectly conducting plane being fed coaxially at its base. The method used involves integral transforms and leads to an infinite set of simultaneous linear equations relating an infinite number of unknowns. By proper choice of transverse dimensions only a finite number of equations and unknowns need be considered. It is therefore possible to obtain expressions for the amplitude of the surface wave and the admittance of the antenna.

INTRODUCTION

The purpose of this paper is to solve a boundary value problem related to wave propagation in a coaxial system. The system consists of a dielectric-covered cylindrical conductor protruding from an infinite flange and extending to infinity. The center conductor and its dielectric shell are extensions of a coaxial transmission line which is terminated at the flange. The surfaces of the flange, the coaxial line and the center conductor are assumed to be perfectly conducting. This problem is the idealization of an antenna problem in which a vertical dielectric-coated cylinder, standing on a horizontal, perfectly conducting ground is fed by a coaxial line.

A general solution for this boundary value problem is found in terms of an infinite number of linear equations with an infinite number of unknowns. These unknowns are related to the scattering coefficients at the end of the transmission line. The solution is obtained by generalizing the method used by N. V. Zernov to solve the problem of radiation of electromagnetic waves from a circular wave guide [1]. The field components are found in terms of an infinite series whose coefficients are determined by solving the infinite set of linear equations. However by a suitable choice of the transverse dimensions of the line, the infinite set may be approximated by just one equation. From this equation one can find the reflection and transmission coefficients of the dominant mode of the transmission line.

The geometry of the problem suggests that a surface wave of the Sommerfeld-Goubau type will propagate along the semi-infinite dielectric-coated conductor in the half-space. Expressions are found for the amplitude of the surface wave, the radiated energy at great distances, the coefficient of reflection and the ratio of energy carried by the surface wave to the total energy delivered by the line.

The final expression for the ratio of energy in the surface wave to the input energy can be employed in choosing the dimensions of the system so that the dielectric-coated rod will transfer energy efficiently. Such information can be useful in the design of television systems or high frequency links [2].

Problems similar to this one have been treated by Papadopoulos [3], Schelkunoff [4] and Zernov [1]. This particular problem is an extension of work done by Papadopoulos, the difference lying in the addition of the dielectric coating.

I. GENERAL DISCUSSION

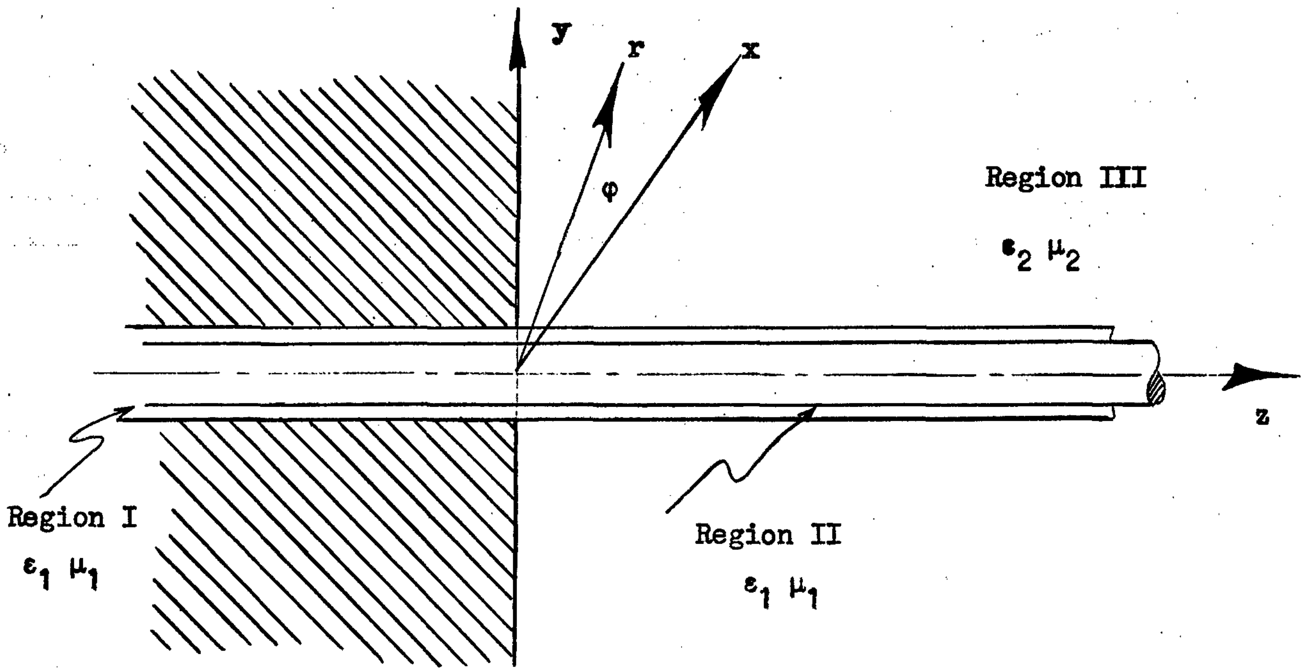

Fig. (1)

In the problem to be considered cylindrical coordinates (r, φ, z) will be used, as shown in Fig. 1. An infinitely long, conducting cylinder of radius b, covered by a lossless coating of radius a $(a > b)$ with dielectric constant ϵ_1 and permeability μ_1 is placed coaxially in the region $-\infty < z \leq 0$ within a circular pipe of radius a, and extends to infinity in the half space $z \geq 0$. The plane $z = 0$ terminates the circular guide and is perfectly conducting in the region $r \geq a$. A symmetrical transverse electro-magnetic wave (TEM) is assumed to propagate within the guide with a time dependence of the form $\exp(-i\omega t)$. The azimuthal angular dependence of the field is taken to be zero; i.e. $\partial/\partial\varphi = 0$. The field components Hr, Hz and E_φ are therefore null.

Since the longitudinal component of the magnetic field is zero and since the medium is isotropic and homogeneous, the entire field can be derived from a scalar wave function Ψ ; or in this case from the axial component of the electric Hertz vector [5]. The field components of these symmetrical modes are given by the following relations:

$$E_z = \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi , \tag{1}$$

$$E_r = \frac{\partial^2 \psi}{\partial z \partial r} , \tag{2}$$

$$H_\varphi = i\omega\varepsilon \frac{\partial \psi}{\partial r} . \tag{3}$$

The function ψ satisfies the wave equation,

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + k^2\right) \psi = 0 , \tag{4}$$

subject to appropriate boundary conditions.

To solve the problem we consider three distinct regions as shown in Fig. 1, namely:

Region 1, $b \le r \le a$, $-\infty < z \le 0$

Region 2, $b \le r \le a$, $0 \le z < \infty$;

Region 3, $a \le r < \infty$, $0 \le z < \infty$.

Potential functions are obtained by finding the solution of (4) appropriate to each of the above three regions and satisfying the necessary boundary and continuity conditions. The boundary conditions which ψ must satisfy are:

$$\frac{\partial^2 \psi_1}{\partial z^2} + k_1^2 \psi_1 = 0 \quad \text{at} \quad r=b,\ r=a,\ z \le 0 \qquad \text{Reg. 1;}$$

$$\frac{\partial^2 \psi_2}{\partial z^2} + k_1^2 \psi_2 = 0 \quad \text{at} \quad r=b,\ z \ge 0 \qquad \text{Reg. 2;}$$

$$\frac{\partial^2 \psi}{\partial z \partial r} = 0 \qquad \text{at} \quad z = 0 \quad r > a \qquad \text{Reg. 3.}$$

The continuity conditions are:

$$\varepsilon_1 E_{z1} = \varepsilon_1 E_{z2} , \quad H_{\varphi 1} = H_{\varphi 2} \quad \text{and} \quad E_{r1} = E_{r2}$$

at the boundary between regions 1 and 2;

$$E_{z2} = E_{z3} , \quad H_{\varphi 2} = H_{\varphi 3} \quad \text{and} \quad \varepsilon_1 E_{r2} = \varepsilon_2 E_{r3}$$

at the boundary between regions 2 and 3.

II. POTENTIAL FUNCTION FOR REGION 1.

The solution of (4) in region 1 which satisfies the required boundary conditions has the form [6],

$$\psi_1 = \varphi_{oo}\, e^{i\gamma_o z} + \sum_{n=o}^{\infty} a_n\, \varphi_{on}\, (\alpha_n r) e^{-i\gamma_n z} \qquad , \qquad (5)$$

where $$\varphi_{on}(\alpha_n r) = J_o(\alpha_n r)\, Y_o(\alpha_n b) - J_o(\alpha_n b)\, Y_o(\alpha_n r) \qquad , \qquad (6)$$

and $\varphi_{oo} = \log_e r$. The constant γ_n is the propagation constant of the n-th mode, a_n is the amplitude coefficient of the n-th mode, and $\alpha = \alpha_n$ is the n-th positive zero in order of magnitude of the function,

$$J_o(\alpha a)\, Y_o(\alpha b) - J_o(\alpha b)\, Y_o(\alpha a) = 0 \qquad . \qquad (7)$$

The potential function ψ_1 is made up of a forward going TEM wave of amplitude one, a reflected TEM wave and an infinite number of circularly symmetric transverse magnetic modes which propagate in the back-ward direction and form the scattered field. Each of these modes satisfies the required boundary conditions. Their collective presence is necessary to satisfy the required continuity conditions at $z = 0$. By symmetry the TEM mode excites only those higher modes which have no φ dependence.

If we substitute ψ_1 in (4), the following relations between the constants α_n and the propagation constant γ_n are found to hold:

$$\gamma_n = \sqrt{k_1^2 - \alpha_n^2} \qquad ;$$

$$\gamma_o = k_1 \ , \ \alpha_o = 0 \qquad .$$

We take that branch of the square root for which $0 \leq \arg\gamma_n \leq \pi$* for $n > 0$, and with the appropriate coax geometry we can assure that $R_e k_1 < \alpha_n$ for $n \geq 1$. This choice allows the dominant mode which has zero cut off frequency to propagate; all other modes are cut off and decay exponentially at a rate that increases with n. For large values of n the propagation constant γ_n of the n-th mode can be accurately approximated by,

$$\gamma_n^2 \simeq k_1^2 - \frac{n^2\pi^2}{(a - b)^2} \qquad . \tag{9}$$

For $n \geq 1$, γ_n^2 is a negative number.

*The choice is made so that the radiation condition at $-\infty$ will hold. For the n-th mode $\varphi_{on} e^{-i\gamma_n z} = \varphi_{on} \exp[-iz(R_e\gamma_n + i I_m \gamma_n)] = \varphi_{on} \exp[z I_m \gamma_n - iz R_e \gamma_n]$ if $I_m \gamma_n > 0$ $\varphi_{on} \exp(-i\gamma_n z) \to 0$ as $z \to -\infty$.

III. POTENTIAL FUNCTION FOR REGION 2.

To find the potential function for region 2 of dielectric constant ε_1 and permeability μ_1, we use the Fourier integral cosine transform pair which is usually written in the form,

$$S(\lambda,r) = \int_0^\infty \psi(r,z) \cos\lambda z dz \quad , \tag{10}$$

and

$$\psi(r,z) = \frac{2}{\pi}\int_0^\infty S(\lambda,r) \cos\lambda z d \quad . \tag{11}$$

If we multiply (4) by cosλzdz and integrate between zero and infinity, we obtain the following result:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + k_1^2 - \lambda^2\right) S(\lambda,r) = \left.\frac{\partial\psi}{\partial z}\right|_{z=0} ; \; b \leq r \leq a \; . \tag{12}$$

Representing the differential operator in (12) by L we have,

$$L\, S(\lambda,r) = \left.\frac{\partial\psi}{\partial z}\right|_{z=0} \quad . \tag{13}$$

The solution of (13), involving the Hermitian operator **L**, can be represented by,

$$S(\lambda,r) = f(\lambda,r) + g(\lambda,r), \tag{14}$$

where f is the inhomogeneous solution and g is the homogeneous solution (i.e. $Lg = 0$).

In region 2, the functions $\varphi_{oi}(\lambda_i r)$, $i = 1, 2, 3, \ldots$ are the eigen vectors of the differential operator L with eigen values λ_i ($\lambda_i \neq \lambda_j$, for $j \neq i$) and form a denumerably infinite dimensional orthogonal basis. Any function in this space can be uniquely represented in terms of its components along this orthognal system. We can therefore express the inhomogeneous solution f

$$f = \sum_{i=o}^{\infty} f_i \, \varphi_{oi}(\alpha_i r) \quad . \tag{15}$$

Let,

$$\left.\frac{\partial\psi}{\partial z}\right|_{z=o} = \sum_{o}^{\infty} g_i \, \varphi_{oi}(\alpha_i r) \ , \tag{16}$$

where $f_i = \frac{(f, \varphi_{oi})}{(\varphi_{oi}, \varphi_{oi})}$, $g_i = \frac{(\partial_z \psi|_{z=o}, \varphi_{oi})}{(\varphi_{oi}, \varphi_{oi})}$ and the bracket notation is defined by $(p,q) = \int_b^a r\,p\,q\,dr$. Substituting for f and $\left.\frac{d\psi}{dz}\right|_{z=o}$ from (15) and (16) in (13), we find that,

$$\sum_{o}^{\infty} f_i \lambda_i \, \varphi_{oi} = \sum_{i=o}^{\infty} g_i \, \varphi_{oi} \quad . \tag{17}$$

From (17) we have $f_i = \frac{g_i}{\lambda_i}$ and hence,

$$f = \sum_{i=o}^{\infty} \frac{g_i}{\lambda_i} \, \varphi_{oi}\,(\alpha_i r). \tag{18}$$

The eigenvalues λ_i can be found by direct substitution of $\varphi_{oi}(\alpha_i r)$ in (12), $\lambda_i = \gamma_i^2 - \lambda^2$. Substituting for λ_i in (18), we get:

$$f = \sum_{i=o}^{\infty} \frac{g_i}{\gamma_i^2 - \lambda^2} \, \varphi_{oi}(\alpha_i r) \quad . \tag{19}$$

If we substitute (19) in (14) and take the inverse Fourier transform we get,

$$\psi_2 = \frac{2}{\pi} \sum_{i=o}^{\infty} \int_0^{\infty} \frac{g_i}{\gamma_i^2 - \lambda^2} \varphi_{oi}(\alpha_{ir}) \cos\lambda z d\lambda + \frac{2}{\pi} \int_0^{\infty} g(\lambda, r) \cos\lambda z d\lambda,$$

or after integration,

$$\psi_2 = \sum_{i=o}^{\infty} b_i \varphi_{oi}(\alpha_{ir}) e^{i\gamma_i z} + \int_0^{\infty} G(\lambda, r) \cos\lambda z d\lambda , \qquad (20)$$

where the constant $\frac{2}{\pi}$ has been absorbed by the unknown functions, b_i and $G(\lambda,r)$ ($G(\lambda,r) = \frac{2}{\pi} g$). It should be noticed that the unknown homogeneous solution $g(\lambda,r)$ is nondenumerably infinite or $g(\lambda,r)$ is a continuous function while the b_i - s are countable.

The homogeneous solution $G(\lambda,r)$ is chosen as follows,

$$G(\lambda, r) = \frac{\Delta_1(k_1' r)}{\Delta_1(k_1' a)} \frac{\xi(\lambda)}{k_1^2 - \lambda^2} , \qquad (20)a$$

where

$$\Delta_1(k_1' r) = J_o(k_1' r)\, Y_o(k_1' b) - J_o(k_1' b)\, Y_o(k_1' r), \qquad (21)$$

and $k_1' = \sqrt{k_1^2 - \lambda^2}$ with $0 \leq \arg \sqrt{k_1^2 - \lambda^2} \leq \pi$ . .

The choice for $G(\lambda,r)$ in (20)a is made to facilitate the matching of the boundary conditions.

In region 2, coaxial modes of the type $\varphi_{oi} e^{i\gamma_i z}$ cannot exist, so the residues of the integral in the complex λ-plane of (20) must cancel all such modes, and it will later on be shown that such modes are cancelled. ψ_2 satisfies the boundary condition at $r=b$, $z > 0$ and the integral converges uniformly for large λ if $\xi(\lambda)$ is well behaved. We will see in sec. V that $\xi(\lambda)$ in fact behaves nicely when λ is greater than a constant λ' and converges uniformly to zero as $O(\lambda^{-2})$ when $\lambda \to \infty$.

IV. POTENTIAL FUNCTION FOR REGION 3:

For region 3 of dielectric constant ε_2 and permeability μ_2, following a line of argument similar to that of ψ_2, we can get ψ_3 in terms of an integral function, i.e.

$$\psi_3 = \int_0^\infty F(\lambda,r) \cos\lambda z \lambda d\lambda \,, \tag{22}$$

where $F(\lambda,r) = \zeta(\lambda) \dfrac{H_o^{(1)}(k_2' r)}{H_o^{(1)}(k_2' a)} \times \dfrac{1}{k_2^2 - \lambda^2}$ and $k_2' = \sqrt{k_2^2 - \lambda^2}$.

We take again that branch of the square root for which $0 \leq \arg(\sqrt{k_2^2 - \lambda^2}) \leq \pi$, and to satisfy the radiation condition at infinity we assume the free space propagation constant k_2 to be $k_2 = R_e k_2 + i\delta$ where δ is a small positive real number. $F(\lambda,r)$ is associated with the cosine Fourier transform of ψ_3. ψ_3 satisfies the boundary condition at $z=0$, $r \geq a$ and the radiation condition at infinity. At large distances ψ_3 behaves as,

$$\psi_3 \simeq \int_{-\infty}^{\infty} d\lambda f(\lambda)\, e^{iR\left[\sqrt{k_2^2 - \lambda^2}\cos\theta + \sin\theta\right]}, \tag{23}$$

where θ is the polar angle and $R^2 = z^2 + r^2$.

V. THE CONTINUITY CONDITIONS

The continuity of the tangential and normal components of electric and magnetic fields across the boundaries of the three regions give the following relations:

$$\frac{\partial^2\psi_3}{\partial z^2} + k_1^2\psi_3 = \frac{\partial^2\psi_2}{\partial z^2} + k_2^2\psi_2\,, \qquad r=a,\ 0\leq z<\infty\ ; \qquad (24)$$

$$\varepsilon_1\,\frac{\partial^2\psi_2}{\partial z\partial r} = \varepsilon_2\,\frac{\partial^2\psi_3}{\partial z\partial r}\,, \qquad r=a,\ 0\leq z<\infty\ ; \qquad (25)$$

$$i\omega\varepsilon_1\,\frac{\partial\psi_2}{\partial r} = i\omega\varepsilon_2\,\frac{\partial\psi_3}{\partial r}\,, \qquad r=a,\ 0\leq z<\infty\ ; \qquad (26)$$

$$\frac{\partial^2\psi_1}{\partial z\partial r} = \frac{\partial^2\psi_2}{\partial r\partial z}\,, \qquad b\leq r\leq a,\ z=0\ ; \qquad (27)$$

$$\varepsilon_1\,\frac{\partial^2\psi_1}{\partial z^2} + \varepsilon_1 k_1^2\psi_1 = \varepsilon_1\,\frac{\partial^2\psi_2}{\partial z^2} + \varepsilon_1 k_1^2\psi_2\,, \qquad b\leq r<a,\ z=0\ ; \qquad (28)$$

and

$$i\omega\varepsilon_1\,\frac{\partial^2\psi_1}{\partial r} = i\omega\varepsilon_2\,\frac{\partial\psi_2}{\partial r}\,, \qquad b\leq r\leq a,\ z=0\ . \qquad (29)$$

In equations (5), (20) and (22) for ψ, ψ_2 and ψ_3 there are four sets of unknowns, namely the b_n, a_n, $\xi(\lambda)$ and $\zeta(\lambda)$. The unknowns can be found from four of the six continuity conditions given above, since two of them are not independent. For example (25) is a consequence of (24) and (26); similarly (28) follows from (27) and (29). The four unknowns are therefore uniquely determined by conditions (24), (26), (28) and (29).

The functions $\xi(\lambda)$ and $\zeta(\lambda)$ are determined by substitution of ψ_3 and ψ_2 in (24) and (26).

The first gives,

$$\int_0^\infty \xi(\lambda)\cos\lambda z d\lambda = \int_0^\infty \zeta(\lambda)\cos\lambda z d\lambda ,$$

and thus,

$$\xi(\lambda) = \zeta(\lambda) . \qquad (30)$$

Substituting the above relation in (26), we obtain,

$$\varepsilon_1 \frac{b_o}{a} e^{i\gamma_o z} + \varepsilon_1 \sum_{n=1}^{\infty} b_n \alpha_n \varphi'_{on}(\alpha_n a) e^{i\gamma_n z} + \qquad (31)$$

$$\varepsilon_1 \int_0^\infty \frac{\Delta_2(k_1' a)}{\Delta_1(k_1' a)} \frac{\xi(\lambda)\cos\lambda z}{k_1'} d\lambda = -\varepsilon_2 \int_0^\infty \xi(\lambda) \frac{H_1^{(1)}(k_2' a)}{H_o^{(1)}(k_2' a)} \frac{\cos\lambda z}{k_2'} d\lambda ,$$

where $k_2' = \sqrt{k_2^2 - \lambda^2}$,

$$-\Delta_2(k_1' a) = -J_1(k_1' a)\, Y_o(k_1' b) + J_o(k_1' b)\, Y_1(k_1' a)$$

and $\Delta_1(k_1' a)$ has already been defined by (21); the prime in the expression $\varphi'_{on}(\alpha_n a)$ denotes differentiation with respect to the argument. Equation (31) can further be simplified by making use of the Wronskian relation, which for the pair $J_o(\alpha_n r)$ and $Y_o(\alpha_n r)$ is,

$$J_1(\alpha_n r)\ Y_o(\alpha_n r) - J_o(\alpha_n r)\ Y_1(\alpha_n r) = \frac{2}{\pi\alpha_n r} \quad . \tag{32}$$

With the help of (32) and (7) we can write $\varphi'_{on}(\alpha_n r)$ in its equivalent form,

$$-\varphi'_{on}(\alpha_n r) = J_1(\alpha_n a)\ Y_o(\alpha_n a) - Y_1(\alpha_n a)\ J\ (\alpha_n a),$$

$$= (J_1(\alpha_n a)\ Y_o(\alpha_n a) - J_o(\alpha_n a)\ Y_1(\alpha_n a))\ \frac{J_o(\alpha_n b)}{J_o(\alpha_n a)}\ ,$$

$$= \frac{2}{\pi\alpha_n a}\ \frac{J_o(\alpha_n b)}{J_o(\alpha_n a)} \quad .$$

Making use of the above expression and rearranging (31), we get,

$$\frac{b_o}{a}\, e^{i\gamma_o z} - \frac{2}{\pi a} \sum_{n=1}^{\infty} b_n \frac{J_o(\alpha_n b)}{J_o(\alpha_n a)}\, e^{i\gamma_n z} = \tag{33}$$

$$-\int_0^{\infty} \xi(\lambda) \left[\frac{\epsilon_2}{\epsilon_1 k'_2}\ \frac{H_1^{(1)}(k'_2 a)}{H_o^{(1)}(k'_2 a)} - \frac{1}{k'_1}\ \frac{\Delta_2(k'_1 a)}{\Delta_1(k'_1 a)} \right] \cos\lambda z d\lambda .$$

The argument of the integral in (33) is the cosine Fourier integral transform of the left hand side. Using (11), we obtain,

$$\left[\frac{\epsilon_2}{\epsilon_1 k'_2}\ \frac{H_1^{(1)}(k'_2 a)}{H_o^{(1)}(k'_2 a)} - \frac{1}{k'_1}\ \frac{\Delta_2(k'_1 a)}{\Delta_1(k'_1 a)} \right] \xi(\lambda) =$$

$$\frac{2}{\pi} \int_0^{\infty} \left(\frac{b_o}{a}\, e^{i\gamma_o z} - \frac{2}{\pi a} \sum_{n=1}^{\infty} b_n \frac{J_o(\alpha_n b)}{J_o(\alpha_n a)}\, e^{i\gamma_n z}\right) \cos\lambda z \lambda dz,$$

and after simplification the resulting equation for $\xi(\lambda)$ is,

$$\xi(\lambda) = \frac{4i\varepsilon_1}{a\pi^2} \frac{\sum_{n=1}^{\infty} b_n \frac{J_0(\alpha_n b)}{J_0(\alpha_n a)} \frac{\gamma_n}{\gamma_n^2 - \lambda^2} - \frac{\pi}{2} \frac{\gamma_o b_o}{\gamma_o^2 - \lambda^2}}{\frac{\varepsilon_2 H_1^{(1)}(k_2' a)}{k_2' H_o^{(1)}(k_2' a)} - \frac{\varepsilon_1}{k_1'} \frac{\Delta_2(k_1' a)}{\Delta_1(k_1' a)}} . \qquad (34)$$

Equation (34) gives $\xi(\lambda)$ in terms of the unknowns b_n; to find the b_n, we shall make use of the continuity conditions (27) and (29) between ψ_1 and ψ_3 at $z=o$. From (27),

$$(1-a_o-b_o)\frac{b_o}{r} - \sum_{n=1}^{\infty} \alpha_n \gamma_n (a_n+b_n)\varphi'_{on}(\alpha_n r) = 0, \qquad (35)$$

where again prime indicates differentiation with respect to the argument of the cylindrical functions. Equating the coefficients of the independent functions $\varphi'_{on}(\alpha_n r)$ to zero, we get,

$$-a_n + \delta_{no} = b_n , \qquad (36)$$

where δ_{no} is the Kroneker delta symbol having the following property:

$$\delta_{mn} = \begin{cases} 0 \text{ for } m \neq n , \\ 1 \text{ for } m = n . \end{cases}$$

Similarly, substitution of ψ_1 and ψ_2 from (5) and (20) in to relation (29) results in,

$$\frac{1}{r} + \frac{a_o}{r} + \sum_{n=1}^{\infty} a_n \alpha_n \varphi'_{on}(\alpha_n r) = \frac{b_o}{r} + \sum_{n=1}^{\infty} b_n \alpha_n \varphi'_{on}(\alpha_n r)$$

$$- \int_0^{\infty} \frac{\Delta_2(k_1')}{\Delta_1(k_1'a)} \frac{\xi(\lambda)}{k_1'} d\lambda \tag{37}$$

Eliminating the a_n by means of (36) we obtain

$$\int_0^{\infty} \frac{\Delta_2(k_1'r)\xi(\lambda)}{\Delta_1(k_1'r)\, k_1'} d\lambda = 2(1-b_o)\frac{1}{r} - \sum_{n=1}^{\infty} b_n \alpha_n \varphi'_{on}(\alpha_n r) \tag{38}$$

Equation (38) has the form of an expansion in terms of the orthogonal cylindrical harmonics $\varphi'_{on}(\alpha_n r)$ where,

$$\varphi'_{on}(\alpha_n r) = -J_1(\alpha_n r)\, Y_o(\alpha_n b) + J_o(\alpha_n b)\, Y_1(\alpha_n r), \quad n \neq 0 \tag{39}$$

and $\qquad \varphi'_{oo} = \frac{1}{r}, \quad \alpha_o = 0 .$

If we multiply (38) by $r\varphi'_{on}(\alpha_n r)\, dr$ and make use of the orthogonality relations, assuming that changing the order of integration and summation is permissible, we shall have an infinite set of linear equations for b_n. Following the above discussions, we get,

$$\int_0^{\infty} \frac{\xi(\lambda)}{k_1'\Delta_1(k_1'a)} T_j(\lambda) d\lambda = 2(1-b_o)\, I_{oj} - \sum_{n=1}^{\infty} b_n \alpha_n I_{nj} , \tag{40}$$

where $T_j(\lambda) = \int_b^a r\,\Delta_2(k_2' r)\,\varphi_{oj}(\alpha_j r)\,dr,$

and $I_{jn} = \int_b^a r\varphi_{oj}(\alpha_j r)\,\varphi_{on}(\alpha_n r)\,dr.$

After simplification these integrals reduce to,

$$T_j(\lambda) = \begin{cases} -\dfrac{2\,k_1'}{k_1' - \alpha_j^2}\,\dfrac{1}{\alpha_j\pi}\,\dfrac{J_o(\alpha_j b)}{J_o(\alpha_j a)}\,\Delta_1(k_1' a), & j \neq 0, \\[2ex] \dfrac{\Delta_2(k_1' a)}{k_1'}, & j = 0, \end{cases} \tag{41}$$

and

$$I_{jn} = \begin{cases} 0, & j \neq n; \\[1ex] \dfrac{2}{\pi^2\alpha_n^2}\left[\left(\dfrac{J_o(\alpha_n b)}{J_o(\alpha_n a)}\right)^2 - 1\right], & j = n \neq 0, \\[2ex] \log_e \dfrac{a}{b} & j = n = 0. \end{cases} \tag{42}$$

Substituting I_{nj}, T_j and $\xi(\lambda)$ in (40) gives the linear set of equations for b_n, i.e.

$$b_o = 1 + \frac{i\,b_o\gamma_o c_{oo}}{\pi a \log \frac{a}{b}} - \frac{2i}{\pi^2 a \log \frac{a}{b}} \sum_{n=1}^{\infty} b_n\gamma_n \frac{J_o(\alpha_n b)}{J_o(\alpha_n a)}\,c_{on}, \tag{43}$$

$$b_n = \frac{2i}{\pi^2\alpha_n^2 I_{nn}}\left(-b_o\gamma_o c_{no} + \frac{2}{\pi}\sum_{m=1}^{\infty} b_m\gamma_m \frac{J_o(\alpha_m b)}{J_o(\alpha_m a)}\,c_{nm}\right)\frac{J_o(\alpha_n b)}{J_o(\alpha_n a)} \tag{44}$$

where,

$$c_{mn} = \int_0^{\infty} \frac{d\lambda}{(\gamma_n^2-\lambda^2)(\gamma_m^2-\lambda^2)} \times \frac{\varepsilon_1}{\left(\frac{\varepsilon_2 H_1^{(1)}(k_2'a)}{k_2' H_o^{(1)}(k_2'a)} - \frac{\varepsilon_1}{k_1'} \frac{\Delta_2(k_1'a)}{\Delta_1(k_1'a)}\right)} . \qquad (45)$$

If the coefficients of the unknowns in the linear set of equations (43), (44) are made small the infinite set may be approximated by a finite set and b_o the transmission coefficient can be found accurately.* The coefficients include the integral expression c_{mn} which is bounded and whose integrand converges as $O(\lambda^{-3})$ for $\lambda >> 1$.

If in (45) we substitute ρ for b/a, σ for $k_1 a = a/\lambda\!\!\!^{-}$ † and $\beta = k_2/k_1$, the integrand can be made non-dimensional. It can be shown** that when $\varepsilon = \frac{a - b}{\lambda\!\!\!^{-}} = (1 - \rho)\sigma$ approaches zero, occuring when $\rho \rightarrow 1$ and $\sigma \neq 0$, c_{mn} is of the following order:

$$c_{mn} \rightarrow O(\varepsilon^5), \quad m, n \neq 0 ;$$

$$c_{om} = c_{mo} \rightarrow O(\varepsilon^3), \ m \neq 0 ;$$

$$c_{oo} \rightarrow O(\varepsilon) .$$

For the case when $\sigma \rightarrow 0$ and $\rho \neq 1$,

$$c_{mn} \rightarrow O(\varepsilon^3), \text{ for all m and n} .$$

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*Appendix (i)

† $\lambda\!\!\!^{-}$ = wavelength/2π

**Appendix (ii)

Consider the case when $\rho \to 1$, $\sigma = 0$, and let $\varepsilon \to 0$.

If we neglect all b_n for $n \geq 1$ the coefficients of reflection and transmission become:

$$b_o = \frac{1}{1 - \beta_o} \quad , \tag{46}$$

and

$$a_o = \frac{\beta_o}{\beta_o - 1} \quad , \tag{47}$$

where

$$\beta_o = \frac{i\,\gamma_o c_{oo}}{\pi a \log\frac{a}{b}} \quad .$$

If we take one more term and neglect b_n for $n \geq 2$, we obtain,

$$b_o = \frac{1}{1 - \beta_o}\left(1 - \frac{O(\varepsilon^2)}{1 - \beta_o}\right) , \tag{48}$$

and

$$b_1 = O(\varepsilon) b_o \quad .$$

Continuing in this manner we see that the inclusion of higher members of the sequence b_n has little effect upon b_o .

VI. SURFACE WAVES AND THE RADIATION FIELD.

The potential function ψ_3 can now be put in its final form by substituting in (22) $\zeta(\lambda)$ from (30) and (34). Thus we obtain,

$$\psi_3 = \frac{4i\varepsilon_1}{a\pi^2}\int_0^{\infty} d\lambda \, \frac{H_o^{(1)}(k_2' r)}{H_o^{(1)}(k_2' a)} \, \frac{\cos\lambda z}{k^2 - \lambda^2} \, \frac{1}{I(\lambda)} \quad x \tag{49}$$

$$\left[\sum_{n=1}^{\infty} b_n \frac{J_o(\alpha_n b)}{J_o(\alpha_n a)} \frac{\gamma_n}{\gamma_n^2 - \lambda^2} - \frac{\pi}{2} \frac{b_o \gamma_o}{\gamma_o^2 - \lambda^2} \right],$$

where

$$I(\lambda) = \frac{\varepsilon_2}{k_2'} \frac{H_1^{(1)}(k_2' a)}{H_o^{(1)}(k_2' a)} - \frac{\varepsilon_1}{k_1'} \frac{\Delta_2(k_1' a)}{\Delta_1(k_1' a)} \quad . \tag{50}$$

Since the integrand of (49) is an even function of λ, we can represent $\cos\lambda z$ by its exponential form and combine the two terms into a single integral from $-\infty$ to $+\infty$. The resulting integral may be simplified by substituting non-dimensional quantities ρ, σ and β for b/a, $k_1 a$, and k_2/k_1 in the integrand. As the result of the above manipulations we obtain,

$$\psi_3 = \int_{-\infty}^{\infty} \frac{H_o^{(1)}(v\frac{r}{a})}{H_o^{(1)}(v)} \, \frac{e^{ix\frac{z}{a}}}{v} \, \frac{dx}{I(u,v)} \quad x \tag{51}$$

$$\left[\frac{2ia}{\pi^2} \sum_{n=1}^{\infty} b_n \frac{J_o(\alpha_n b)}{J_o(\alpha_n a)} \frac{\gamma_n}{\gamma_n^2 a^2 - x^2} - \frac{b_o}{\pi} \frac{\sigma}{\sigma^2 - x^2} \right],$$

where

$$I(u,v) = \frac{\beta^2 H_1^{(1)}(v)}{v H_o^{(1)}(v)} - \frac{1}{u} \frac{\delta_1(u)}{\delta_o(u)} \, , \tag{52}$$

$$\delta_1(u) = J_1(u)\, Y_o(\rho u) - J_o(\rho u)\, Y_1(u)\ ,$$

$$\delta_o(u) = J_o(u)\, Y_o(\rho u) - J_o(\rho u)\, Y_o(u)\ ,$$

$$x = \lambda a,\ u = \sqrt{\sigma^2 - x^2} \quad \text{and} \quad v = \sqrt{\beta^2\sigma^2 - x^2}\ .$$

The integral along the real axis (51), may now be transformed to a contour integral in the complex x-plane. The path of integration will be chosen to guarantee the convergence of the integral; but before determining the actual path, we must investigate the poles and the branch points of the integrand.

A - THE SIGNIFICANCE OF THE POLES OF THE INTEGRAND.

The poles of the integrand in the complex x-plane that give physically significant results are those that are situated on the real or the imaginary axis. The residues of any poles on the imaginary axis would yield terms of the form

$$\text{const}\ H_o^{(1)}\left(\sqrt{\beta^2\sigma^2 + x_o^2}\,\frac{r}{a}\right) e^{-x_o\frac{z}{a}}\ .$$

These are Zeneck surface waves which propagate radially outward along the surface $z=0$ but attenuate in the z-direction [8]. The residues of the poles on the real axis $(|x_1|>\beta\sigma)$ give terms of the form

$$\text{const}\ H_o^{(1)}\left(i\sqrt{x_1^2 - \beta^2\sigma^2}\,\frac{r}{a}\right) e^{ix_1\frac{z}{a}}\ ,$$

which unlike the former waves propagate unattenuated in the positive z-direction and are guided by the dielectric coating. These surface waves decay exponentially in the radial direction and are of the Sommerfeld-Goubau type. The residues of the complex poles of the integrand give waves which decay exponentially both in the z and r direction and have no physical interest.

B - POSITION OF POLES.

Although poles of integrand seem to be situated at the points $x=\mp a\gamma_n$ where $n=o, 1, 2, \ldots\ldots$, a Laurent series expansion about these points shows that $I(u,v)$ has simple poles there. Thus these points $(x=\mp\gamma_n a)$ are in fact ordinary points of the integrand. To locate the significant zeros of $I(u,v)$, we can make the following classification $(x=s+it)$:

(i) The imaginary axis $s=o$, $o\leq|t|<\infty$;

(ii) The real axis $t=o$, $o<|s|\leq\beta\sigma$;

(iii) The real axis $t=o$, $\beta\sigma\leq|s|\leq\sigma$,

(iv) The real axis $t=o$, $\sigma\leq|s|<\infty$.

(i) For $s=o$, $o\leq|t|<\infty$.

If we set $x=it$ in (51) and equate the result to zero we obtain,

$$\frac{H_o^{(1)}(\sqrt{\beta^2\sigma^2+t^2})}{\sqrt{\beta^2\sigma^2+t^2}\,H_o^1(\sqrt{\beta^2\sigma^2+t^2})}=-\frac{J_1(\sqrt{\sigma^2+t^2})Y_o(\rho\sqrt{\sigma^2+t^2})-J_o(\rho\sqrt{\sigma^2+t^2})Y_1(\sqrt{\sigma^2+t^2})}{J_o(\sqrt{\sigma^2+t^2})Y_o(\rho\sqrt{\sigma^2+t^2})-J_o(\rho\sqrt{\sigma^2+t^2})Y_o(\sqrt{\sigma^2+t^2})} \tag{53}$$

If there are poles on the imaginary axis, (53) must be satisfied for some real t . However the **L.H.S.** is a complex number; more over the imaginary part of the L.S.H. is given by

$$\beta^2 \frac{-J_1(\sqrt{\beta^2\sigma^2+t^2})\, Y_o(\sqrt{\beta^2\sigma^2+t^2}) + Y_1(\sqrt{\beta^2\sigma^2+t^2})\, J_o(\sqrt{\beta^2\sigma^2+t^2})}{\sqrt{(\beta^2\sigma^2+t^2)}\, \left|H_o^1(\sqrt{\beta^2\sigma^2+t^2})\right|^2}$$

or

$$\frac{\beta^2}{2\pi\ (\beta^2\sigma^2+t^2)\, \left|H_o^{(1)}(\sqrt{\beta^2\sigma^2+t^2})\right|^2} ,$$

which is positive for all real values of t. Thus for all finite values of x $I(u,v)$ is different from zero, and there are no poles on the imaginary axis. This is an expected result since the surface, $z=o$, has no reactive component, i.e. is not covered by a dielectric coating which is necessary to sustain radial surface wave [9].

(ii) For $t=o$, $0 \leq |s| \leq \beta\sigma$,

To find the zeros of $I(u,v)$ on this portion of the real axis we replace in (52) x by s, obtaining

$$\frac{\beta^2 H_o^{(1)}(\sqrt{\beta^2\sigma^2-s^2})}{\sqrt{\beta^2\sigma^2-s^2}\, H_o^{(1)}(\sqrt{\beta^2\sigma^2-s^2})} = \frac{1}{\sqrt{\sigma^2-s^2}} \; \frac{J_1(\sqrt{\sigma^2-s^2})\, Y_o(\rho\sqrt{\sigma^2-s^2}) - J_o(\rho\sqrt{\sigma^2-s^2})\, Y_1(\sqrt{\sigma^2-s^2})}{J_o(\sqrt{\sigma^2-s^2})\, Y_o(\rho\sqrt{\sigma^2-s^2}) - J_o(\rho\sqrt{\sigma^2-s^2})\, Y_o(\sqrt{\sigma^2-s^2})} \tag{54}$$

For $o \leq s \leq \beta\sigma$ we have the same situation as in (i); the R.H.S. is always real while the L.H.S. remains always a complex quantity whose imaginary part is non zero and so $I(u,v)$ has no zero in this interval.

(iii) For $t=o$ $\beta\sigma \leq |s| \leq \sigma$;

In this region the argument of the Hankel functions becomes a pure imaginary number and the L.H.S. of (54) changes to,

$$\frac{\beta^2 K_1(\sqrt{s^2-\beta^2\sigma^2})}{\sqrt{s^2-\beta^2\sigma^2}\, K_o(\sqrt{s^2-\beta^2\sigma^2})} \quad , \tag{55}$$

where K_o and K_1 are modified Bessel functions of the second kind of zero and first order;

$$K_o(z) = i\,\frac{\pi}{2}\,H_o^{(1)}(iz) \ ,$$

or

$$K_1(z) = -\,\frac{\pi}{2}\,H_1^{(1)}(iz) \ .$$

(55) is a rapidly decreasing function of s with infinite value at $s=\mp\beta\sigma$ and decays to zero as $|s| \to \infty$. To show the existence of a zero in this region, let us take the special case $\rho \to 1$ and expand every term of the R.H.S. in a Taylor series in the neighborhood of $\rho = 1$. Taking the dominant terms we have,

$$\text{R.H.S.} = \frac{1}{(1-\rho)(\sigma^2-s^2)} \ .$$

Both sides of (54) are shown graphically in Fig. (2) and we notice that a zero exists at the intersection of the two curves.

(iv) For $t=o$ $\sigma \leq |s| < \infty$;

In this region the arguments of all Bessel functions in (54) become pure imaginary numbers. Replacing these functions by their modified forms we have,

$$\frac{\beta^2 K_1(\sqrt{s^2-\sigma^2\beta^2})}{\sqrt{s^2-\beta^2\sigma^2}\, K_0(\sqrt{s^2-\beta^2\sigma^2})} = -\frac{I_1(\sqrt{s^2-\sigma^2})\, K_0(\rho\sqrt{s^2-\sigma^2}) + K_1(\sqrt{s^2-\sigma^2})\, I_0(\rho\sqrt{s^2-\sigma^2})}{\sqrt{s^2-\sigma^2}\,[I_0(\sqrt{s^2-\sigma^2})K_0(\rho\sqrt{s^2-\sigma^2}) - K_0(\sqrt{s^2-\sigma^2})\, I_0(\rho\sqrt{s^2-\sigma^2})]}, \tag{55}$$

where

$$I_0(z) = J_0(iz) ,$$

and

$$I_1(z) = -i\, J_1(iz) .$$

The R.H.S. is always negative* while the L.H.S. is positive and thus in the finite part of this region no zero can exist.

C - PHYSICAL CONSEQUENCE OF THE POLE OF REGION (iii),

From the above discussions we can see that the antenna excites a single surface wave mode of Sommerfeld-Goubau type which is sustained by the geometry. The φ component of the magnetic field of this surface wave has the form,

$$\text{const } K_1(\sqrt{s_0^2 - \beta^2\sigma^2}\, r)e^{is_0 \frac{z}{a}} , \tag{57}$$

where $\frac{s_0}{a}$ is the propagation constant of the surface wave and $s_0 = a\lambda_0$ is the real zero of $I(u,v)$ in the interval $\beta\sigma \leq s_0 \leq \sigma$. If we draw a graph of

---

*The numerator is always positive. The denominator can be written as follows:

$$I_0(\rho\sqrt{s^2-\sigma^2})\, K_0(\rho\sqrt{s^2-\sigma^2}) \left[\frac{I_0(\sqrt{s^2-\sigma^2})}{I_0(\rho\sqrt{s^2-\sigma^2})} - \frac{K_0(\sqrt{s^2-\sigma^2})}{K_0(\rho\sqrt{s^2-\sigma^2})} \right].$$

The first term in the bracket is always greater than unity while the second term is smaller than unity.

s_o vs. σ for small values of ε, $\varepsilon = \sigma(1-\rho)$, we see that we have a linear relationship between s_o and σ, Fig. (3). The ratio

$$\frac{\sigma}{s_o} = \frac{ak_1}{a\lambda_o} = \frac{\omega}{c_1\lambda_o} = \frac{v_p}{c_1} = \text{const.} \quad ,$$

where c_1 is the velocity of light in the dielectric medium. This shows that the phase velocity of the surface wave is constant for all exciting frequencies and the surface wave has no cutoff frequency. We note that for all frequencies the geometry sustains only one principal mode whereas in the excitation of a dielectric rod with no inner conductor one obtains modes whose number increases proportionally to the frequency [10].

D - THE BRANCH POINTS.

Because of the logarithmic singularity exhibited by the Bessel functions of the second kind, branch points might be expected at points corresponding to $u = o$ and $v = o$. It turns out however that the function $h(u) = \frac{1}{u}\frac{\delta_1(u)}{\delta_o(u)}$ is even in u when $u \to o$ and in that vicinity behaves as $\frac{\text{const}}{u^2}$. Thus although u is multivalued near the points $x = \mp\sigma$, $h(u)$ has only simple poles at these points. The integrand does have branch points at $x = \mp\beta\sigma$ $(v = 0)$.

E - THE PATH OF INTEGRATION.

Knowing the position of the singularities of the real line we choose the path of integration as shown in Fig. (4), where the branch cuts are taken parallel to the imaginary axis. The integral over the real line becomes,

$$\int_{-\infty}^{\infty} = 2\pi i \sum_{\text{Residues}} - \int_{R} - \int_{\Gamma} - \sum_{\substack{\text{sum over small}\\ \text{semi-circles}}} \cdot \int_{r_i}$$

In the upper half plane $\int_R$ vanishes as $R \rightarrow \infty$ and the integral over small semi-circles can be easily evaluated. The integrals along the branch cuts give us the radiation field and in the radiation zone these integrals can be evaluated approximately using the method of saddle point integration.

VII THE RADIATION PATTERN

The φ component of magnetic field can be found by using (3), i.e.

$$H_\varphi = \frac{2\omega\varepsilon_2}{a\pi^2} \int_{-\infty}^{\infty} \frac{H_1^{(1)}(k_2'r)}{H_o^{(1)}(k_2'a)} \frac{e^{i\lambda z}}{k_2'} \frac{d\lambda}{I(\lambda)} \times \tag{58}$$

$$\left[\sum_{n=1}^{\infty} b_n \frac{J_o(\alpha_n b)}{J_o(\alpha_n a)} \frac{\gamma_n}{\gamma_n^2 - \lambda^2} - \frac{\pi}{2} \frac{b_o \gamma_o}{\gamma_o^2 - \lambda^2} \right] ,$$

Since we are interested in the field at very large distances from the source, we are justified in replacing the Hankel function in the numerator by its asymptotic representation. Thus we obtain,

$$H_\varphi \cong \frac{2\omega\varepsilon}{a\pi^2} \sqrt{\frac{2}{\pi r}}\, e^{-\frac{3\pi}{4}i} \int_{-\infty}^{\infty} \frac{e^{i(k_2'r + \lambda z)}}{H_o^{(1)}(k_2'a)\, k_2'^{\,3/2}\, I(\lambda)} \times$$

$$\left[\sum_{n=1}^{\infty} b_n \frac{J_o(\alpha_n b)}{J_o(\alpha_n a)} \frac{\gamma_n}{\gamma_n^2 - \lambda^2} - \frac{\pi}{2} \frac{b_o \gamma_o}{\gamma_n^2 - \lambda^2} \right] d\lambda . \tag{59}$$

The cylindrical coordinate system in which the fields are expressed is not suitable for obtaining the radiation pattern at large distances. For this reason we shall use spherical coordinates shown in Fig. (5) where,

$$r = R \sin\theta ,$$

and

$$z = R \cos\theta .$$

The electric field components in this system are found from $-i\omega\varepsilon \underline{E} = \nabla_\wedge \underline{H}$, i.e.

$$E_r = \frac{i}{\omega \varepsilon R \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta\, H_\varphi), \tag{60}$$

and

$$E_\theta = \frac{i}{i\omega\varepsilon R} \frac{\partial}{\partial R} (RH_\varphi) \quad . \tag{61}$$

In spherical coordinates (59) becomes,

$$H_\varphi = \frac{\frac{2\omega\varepsilon}{2}}{a\pi^2} \sqrt{\frac{2}{\pi R \sin\theta}}\, e^{\frac{-3\pi}{2}i} \int_{-\infty}^{\infty} \frac{e^{iR[\sin\ k_2' + \lambda\cos]}}{H_o^{(1)}(k_2'a)\, k_2'^{\,3/2} I(\lambda)} \quad x$$

$$\sum_{n=1}^{\infty} b_n \frac{J_o(\alpha_n b)}{J_o(\alpha_n b)} \frac{\gamma_n}{\gamma_n^2 - \lambda^2} - \frac{\pi}{2} \frac{b_o \gamma_o}{\gamma_o^2 - \lambda^2} \quad d\lambda \tag{62}$$

In equation (61) the integral can be approximated by means of the method of steepest descent since R is large. The saddle point is situated at that value of λ for which,

$$\frac{d}{d\lambda} (\sin\theta\ k_2' + \lambda\cos\theta) = 0, \tag{63}$$

or

$$\lambda = k_2 \cos\theta \,.$$

The contribution to the integral from the neighborhood of the saddle point turns out to be

$$H_\varphi \cong \sqrt{\frac{\varepsilon_2}{\mu_2}}\ \frac{e^{ikR}}{R}\ g_1\,(k_2 \cos\theta) \quad , \tag{64}$$

where,

$g_1(k_2\cos\theta) =$

$$\frac{\dfrac{2b_o}{\pi}\dfrac{\gamma_o}{\gamma_o^2-k_2^2\cos^2} - \dfrac{4}{\pi^2 a}\sum_{n=1}^{\infty} b_n\gamma_n \dfrac{J_o(\alpha_n b)/J_o(\alpha_n a)}{(\gamma_n^2-k_2^2\cos^2\theta)}}{\sin\theta H_o^{(1)}(k_2 a\sin\theta)\left(\dfrac{\beta^2 H^{(1)}(k_2 a\sin\theta)}{k_2\sin\theta H_o^{(1)}(k_2 a\sin\theta)} - \dfrac{1}{\nu}\dfrac{J_1(\nu)Y_o(\rho\nu)-J_o(\rho\nu)Y_1(\nu)}{J_o(\nu)Y_o(\rho\nu)-J_o(\rho\nu)Y_o(\nu)}\right)}$$

and $\nu = \sqrt{k_1^2-k_2^2\cos^2\theta}$. Up to terms of order $0(\frac{1}{R})$. The Poynting vector is

$$P = \frac{1}{2}\,\mathrm{Re}(E_{\theta}\wedge H_{\varphi}^{*}) \sim |H_{\varphi}|^2 \tag{65}$$

Fig. (5) is a drawing of the radiation pattern, where for small ε only the first term of the series has been taken. It can be seen that as the ratio $\frac{b}{a} \to 1$, the radiation becomes more and more into the forward direction.

VIII EFFICIENCY

With a suitable choice of the transverse dimensions a, b and the parameter $\beta = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$ the geometry may be used as a transmission line to carry surface wave energy from one point to another. This line has no cutoff frequency. Single wire transmission lines of this kind have already been put into use to feed television antennas at a carrier frequency of 200 MC [2].

For this application we define the efficiency to be the ratio of the energy carried as surface wave to the total energy delivered by the coaxial line, i.e.

$$E = \frac{\text{Power Propagated as S.W.}}{\text{Total Power}} \tag{66}$$

The total power is found by integrating the Poynting vector on a surface in the coaxial region and at a distance great enough for the higher modes to have decayed

$$P_{total} = \frac{1}{2} \text{Re} \int_s (\underline{E}_\wedge \underline{H}^*) . \, ds. \tag{67}$$

Replacing $\underline{E}$ and $\underline{H}$ from equations (2), (3) and (5) in (67) we get,

$$P_{total} = \pi\omega^2 \sqrt{\frac{\varepsilon_1}{\mu_1}} \log_e \frac{a}{b} (1 - |a_o|^2) \; . \tag{68}$$

Similarly, the energy propagated as a surface wave along the z-axis is found by integrating the Poynting vector over a transverse plane since the surface wave has no radiation in the r direction.

Following the discussion in sec. VII, the potential function in region (3) responsible for the surface wave is,

$$\varphi_{s\omega 3} = 2\pi i \text{ Res. of } \psi_3\Big|_{\lambda=\lambda_o} ,$$

or

$$\varphi_{s\omega 3} = \frac{-4}{\pi a \nu_3^2} \frac{K_o(\nu_3 r)e^{i\lambda_o z}}{K_o(\nu_3 a)\, S} \left(\sum_{n=1} b_n \frac{J_o(\alpha_n b)}{J_o(\alpha_n a)} \frac{\gamma_n}{\gamma_n^2-\lambda_o^2} - \frac{\pi}{2} \frac{b_o\gamma_o}{\gamma_o^2-\lambda_o^2} \right), \tag{69}$$

where

$$S = \frac{d}{d\lambda} I(\lambda)\Big|_{\lambda=\lambda_o} , \tag{70}$$

and

$$\nu_3 = \sqrt{\lambda_o^2-k_2^2} \quad .$$

The field components $H_{\varphi 3}$ and E_{r3} are,

$$H_{\varphi 3} = -M_3 K_1(\nu_3 r)e^{i\lambda_o z} , \tag{71}$$

and

$$H_{r3} = -\frac{\lambda_o}{\omega \varepsilon_2} M_3 K_1(\nu_3 r)e^{i\lambda_o z} , \tag{72}$$

where

$$M_3 = \frac{4}{\pi a} \frac{i\omega\, \varepsilon_2 \nu_3}{K_o'(\nu_3 a)} \frac{\sum_{n=1}^{\infty} b_n \frac{J_o(\alpha_n b)}{J_o(\alpha_n a)} \frac{\gamma_n}{\gamma_n^2-\lambda_o^2} - \frac{\pi}{2} \frac{b_o\gamma_o}{\gamma_o^2-\lambda_o^2}}{S(k_2^2-\lambda_o^2)} . \tag{73}$$

The surface wave power propagating in region (3) is

$$P_{s3} = \pi \frac{\lambda_o}{\omega\varepsilon_2} \left|M_3\right|^2 J_3 \tag{74}$$

where

$$J_3 = \frac{a^2}{2} [K_1'^2(\nu_3 a) - (1 + \frac{1}{\nu_3^2 a^2}) K_1(\nu_3 a)]$$

In a very similar manner we can find the field components of the surface wave in region 2, i.e.

$$H_{\varphi 2} = M_2 \Delta_2(\nu_2 r) e^{i\lambda_o z} ,$$

and

$$E_{r2} = \frac{\lambda_o}{\omega \varepsilon_1} M_2 \Delta_2 (\nu_2 r) e^{i\lambda_o z}$$

where

$$\Delta_2(\nu_2 r) = J_1(\nu_2 r)\, Y_o(\nu_2 b) - Y_1(\nu_2 r)\, J_o(\nu_2 b) ,$$

$$M_2 = \frac{4i\omega\varepsilon_1 \nu_2}{a\pi} \frac{\sum_1 b_n \frac{J_o(\alpha_n b)}{J_o(\alpha_n a)} \frac{\gamma_n}{\gamma_n^2 - \lambda^2} - \frac{\pi}{2} \frac{b_o \gamma_o}{\gamma_o^2 - \lambda^2}}{S\Delta_1(\nu_2 a)\ (k_1^2 - \lambda_o^2)} , \tag{75}$$

and

$$\nu_2 = \sqrt{k_1^2 - \lambda_o^2} .$$

The power propagating in the z direction due to this field is,

$$P_2 = \frac{\pi\lambda_o}{\omega\varepsilon_1} \left| M_2 \right|^2 J_2 , \tag{76}$$

where

$$J_2 = \frac{a^2}{2} \left(\begin{vmatrix} J_1(\nu_2 a) & Y_1(\nu_2 a) \\ J_o(\nu_2 b) & Y_o(\nu_2 b) \end{vmatrix}^2 - \begin{vmatrix} J_o(\nu_2 a) & Y_o(\nu_2 a) \\ J_o(\nu_2 a) & Y_o(\nu_2 b) \end{vmatrix} \begin{vmatrix} J_2(\nu_2 a) & Y_2(\nu_2 a) \\ J_o(\nu_2 b) & Y_o(\nu_2 b) \end{vmatrix} \right) - \frac{2}{\pi^2 \nu_2^2} . \tag{77}$$

For the case when $\varepsilon \to 0$, $\rho \to 1$ and $\sigma \neq 0$ M_2 and M_3 can be approximated by the first term of the series, or,

$$M_2 = -\frac{2}{a}\frac{i\omega\varepsilon_1}{\nu_2^4} \times \frac{b_o\gamma_o}{\Delta_1(\nu_2 a)S} + O(\varepsilon^2)$$

and

$$M_3 = \frac{2}{a}\frac{i\omega\varepsilon_2}{\nu_2^2\nu_3^2}\;\frac{b_o\gamma_o}{K_o(\nu_3 a)S} + O(\varepsilon^2)$$

The total power propagating in the two media due to the surface wave is found to be,

$$P_{s\omega} = \pi\omega\lambda_o \left|\frac{b_o}{S}\right|^2 \frac{4\gamma_o}{a^2\nu_2^4}\left[\frac{\varepsilon_1 J_2}{\nu_2^2\Delta_1^2(\nu_2 a)} + \frac{\varepsilon_2 J_3}{\nu_3^2 K_o^2(\nu_3 a)}\right] + O(\varepsilon^2) \tag{78}$$

and here the efficiency is

$$E = \frac{|1-a_o|^2}{1-|a_o|^2}\,\frac{4\lambda_o\gamma_o}{a^2\nu_2^4 S^2}\;\frac{\dfrac{J_2\varepsilon_1}{\nu_2^2\Delta_1^2(\nu_2 a)} + \dfrac{J_3\varepsilon_2}{\nu_3^2K_o^2(\nu_3 a)}}{\varepsilon_1\log\frac{a}{b}} \tag{79}$$

The value of S is

$$S = \frac{a^3\beta^2\lambda_o}{r_3^3K_o^2(r_3)}\left[r_3K_o^2(r_3) + r_3K_1^2(r_3) + 2K_o(r_3)\,K_1(r_3)\right]$$

$$+ \frac{a^3\lambda_o}{r_2r_1^3\delta_{oo}^2}\left[r_1r_2(\delta_{oo}^2 + \delta_{10}^2) - 2r_1\delta_{oo}\delta_{10}\right.$$

$$\left. + r_1^2\delta_{01}\delta_{10} - r_1^2\delta_{11}\delta_{oo}\right], \tag{80}$$

where,

$$\delta_{nm} = J_n(r_1)\, Y_m(r_2) - J_m(r_2)\, Y_n(r_1) ,$$

$$r_1 = a \sqrt{k_1^2 - \lambda_o^2} ,$$

$$r_2 = b \sqrt{k_1^2 - \lambda_o^2} ,$$

$$r_3 = a \sqrt{\lambda_o^2 - k_2^2} .$$

IX. ADMITTANCE

The driving point admittance of this radiating system at $z = 0$ may be defined as,

$$Y = \frac{I(o)}{V(o)} , \tag{82}$$

where

$$V(o) = \int_b^a E_r dr , \tag{83}$$

is the voltage at the base of the antenna and

$$I(o) = \oint_{r=b} H_\varphi \, dr \tag{84}$$

is the current being fed into the half space. From (2) and (5) we have

$$E_{r3} = \frac{1-a_o e^{-2i\gamma_o}}{r} i\gamma_o e^{i\gamma_o z} - i \sum_{n=1}^{\infty} \gamma_n a_n \varphi'_{on}(\alpha_n r) e^{-i\gamma_n z} , \tag{85}$$

where the prime denotes differentiation with respect to r . With the help of (85), (83) becomes,

$$V(o) = i\gamma_o [1-a_o] \log \frac{a}{b} \tag{86}$$

Similarly from (3) and (5) we obtain,

$$\frac{1}{i\omega\varepsilon} H_\varphi = \frac{1+a_o e^{-2i\gamma_o z}}{r} e^{i\gamma_o z} + \Sigma\, a_n \varphi'_{on}(\alpha_n r) e^{-i\gamma_n z} , \tag{87}$$

and thus (84) becomes

$$I(o) = 2\pi i\omega\varepsilon \left[(1 + a_o) + b \sum_{n=1}^{\infty} a_n \varphi'_{on}(\alpha_n b)\right] . \tag{89}$$

Using (32), (89) reduces to,

$$I(o) = 2\pi i\omega\varepsilon \left[(1 + a_o) - \frac{2}{\pi} \sum_{1}^{\infty} a_n\right] , \tag{90}$$

and so

$$Y = Y_o \frac{1+a_o}{1-a_o} - \frac{2Y_o}{\pi(1-a_o)} \sum_{n=1}^{\infty} a_n \tag{91}$$

where Y_o is the characteristic impedance of the coaxial system,

$$Y_o = \frac{2\pi\omega\varepsilon}{\Upsilon_o \log \frac{a}{b}} . \tag{92a}$$

The first term on the R.H.S. of (91) gives the admittance that would be determined from standing wave measurements made at large enough distances in the coaxial line such that all the higher modes could be neglected. Figs. (6) and (7) show graphs of real and imaginary parts of $\frac{Y}{Y_o} = \frac{1+a_o}{1-a_o}$ against the ratio $\rho = \frac{b}{a}$.

Appendix (i): Convergence of the Integral ρ_{mn} .

In (45) let us substitute the following quantities, ρ , σ , x and β for b/a, ka, λa and k_2/k_1 . As a result the integral becomes dimensionless and we obtain

$$C_{mn} = \frac{\sigma^2}{k_1^2} \int_0^\infty U_{mn} dx \quad (92)$$

where

$$U_{mn} = \frac{1}{(a^2\gamma_n^2 - x^2)(a^2\gamma_m^2 - x^2) I(u,v)} . \quad (93)$$

The integral can be transformed into a contour integral in the complex x-plane covering a path similar to that of Fig. (4). For m and $n \neq 0$, on the real axis the integrand has a branch point $|x| = \beta\sigma$, a surface wave pole at $|x| = x_o$ $(\beta\sigma \leq x_o \leq \sigma)$ and a zero at $|x| = \sigma$. For $m = n = 0$, U_{oo} has a pole at $|x| = \sigma$. The integral is bounded at its upper limit since, for $x_1 \gg \delta > \sigma$ we have,

$$\left| \int_{x_1}^\infty U_{mn} dx \right| \leq \int_{x_1}^\infty O\left(\frac{1}{x^3} \right) dx . \quad (94)$$

In the complex plane, since the contribution about the branch points and the large semi-circle vanish, we have,

$$2C_{mn} = \Sigma \text{Residues on the upper half plane} - \int_\Gamma U_{mn} . \quad (95)$$

As long as $\beta = k_2/k_1 < 1$, the number of residues will remain finite* and along the branch cuts it can be seen that U_{mn} is finite and bounded. Thus the second term in R.H.S. of (95) will be a finite number and so C_{mn} is finite.

For fixed $m, n \neq 0$ and if $\rho \to 1$, $\sigma \neq 0$ such that $\varepsilon = (1 - \rho) \to 0$ we have

$$a^2\gamma_n^2 \sim \sigma^2 - \frac{\sigma^2 n^2 \pi^2}{\varepsilon^2} . \qquad (96)$$

Substituting (96) in (92) gives the following results:

$$C_{mn} \sim O(\varepsilon^5) \qquad m, n \neq 0$$

$$C_{mn} \sim O(\varepsilon^3) \qquad m \neq 0, n = 0$$

$$C_{mn} \sim O(\varepsilon) \qquad m = n = 0$$

Similarly for $\sigma \to 0$, $\rho \neq 1$ and $\varepsilon \to 0$,

$$C_{mn} \sim O(\varepsilon^3) \text{ for all } m \text{ and } n.$$

Furthermore, for large n $U_{mn} \sim O(\frac{1}{n^2})$ for either one of the indices while the other is fixed or $U_{mn} \sim O(\frac{1}{n^2} \times \frac{1}{m^2})$ when both m and n are large.

---

*For large x, $U_{mn} \sim \dfrac{1}{x^3[\coth(\rho-1)x - \beta^2]}$. The roots of the quantity in bracket are located at $x = (\frac{1}{\rho-1} \coth^{-1}\beta^2 + i\frac{n\pi}{\rho-1})$. $n = 0, \mp 1, \mp 2 \ldots$. If $\beta < 1$ it can be seen the bracketed term will never be zero.

Appendix (ii): The Solution of the Infinite Set of Equations.

The infinite set of equations in (43), (44) can be written as,

$$b_o = 1 + r_{oo}c_{oo}b_o - i \sum_{n=1}^{\infty} \gamma_n r_{on} c_{on} b_n \quad , \tag{97}$$

$$b_n = i\omega_n \left[- \gamma_o b_o c_{no} + \frac{2}{\pi} \sum_{m=1}^{\infty} \gamma_m v_m b_m c_{nm} \right] , \tag{98}$$

where

$$\omega_n = \frac{v_m}{v_m^2 - 1} \cdot \frac{1}{A} \qquad v_m = \frac{J_o(\alpha_n b)}{J_o(\alpha_n a)}$$

$$r_{oo} = \frac{i\gamma_o}{\pi a \log \frac{a}{b}} \qquad r_{on} = \frac{2}{\pi^2 a \log \frac{a}{b}} v_n \ .$$

This infinite set of equations may be solved by iteration. For the case when $\rho \to 1$, $\sigma \neq 0$, $\varepsilon \to 0$, and if in the first approximation we neglect terms with coefficients greater than $O(\varepsilon)$, we have,

$$b_o = \frac{1}{1 - r_{oo}c_{oo}} \ . \tag{99}$$

For the second approximation we neglect all terms with coefficients greater than $O(\varepsilon^3)$ and using (99) we get,

$$b_n = -i\omega_n \frac{c_{no}\gamma_o}{1 - r_{oo}c_{oo}} \quad , \tag{100}$$

where $\omega_n \sim O(\varepsilon^{-1})$, $n \geq 1$ and $c_{no} \sim O(\varepsilon^3)$. Replacing (100) in (97) we obtain the second approximate value for b_o ,

$$b_o = \frac{1}{(1 - r_{oo}c_{oo})} + \frac{\gamma_o}{(1 - r_{oo}c_{oo})^2} \sum_{n=1}^{\infty} \gamma_n r_{on} \omega_n c_{on}^2 \ . \tag{101}$$

In the second term of equation (101), if it is permissible to change the order of summation and integration we obtain,

$$\sum_{n=1}^{\infty} \gamma_n r_{on} \omega_n c_{on}^2 = \frac{2}{\pi^2 a^2 \log \frac{a}{b}} \int_0^{\infty}\int_0^{\infty} \frac{d\lambda d\mu}{I(\lambda)I(\mu)} \sum_{n=1}^{\infty} F_n(\lambda,\mu) \quad ,$$

where

$$F_n = \frac{v_n^2}{v_n^2-1} \frac{\gamma_n}{(\gamma_n^2-\lambda^2)(\gamma_n^2-\mu^2)(\gamma_o^2-\lambda^2)(\gamma_o^2-\mu^2)} \quad . \tag{102}$$

For $n \gg 1$, $\gamma_n \sim ik \frac{n\pi}{\varepsilon}$ and thus for a fixed ε, $F_n \sim O(\frac{1}{n^3})$. We can see that the series converges quite rapidly. Having obtained b_o and b_n we can use these values in (98) and get a more accurate value for b_n,

$$b_n = \frac{i\omega_n \gamma_o}{1-r_{oo}c_{oo}} \left[-c_{no} + \frac{2}{\pi} \sum_{m=1}^{\infty} \gamma_m \frac{v_m^2}{v_m^2-1} \left(\frac{c_{on}c_{om}^2}{\pi a \log \frac{a}{b}} \frac{\gamma_o}{1-r_{oo}c_{oo}} - i\, c_{nm} c_{om} \right) \right]$$

we can repeat the whole process until the difference between successive approximations become negligibly small.

Appendix (iii): An Alternative Method to Obtain b_n and $\xi(\lambda)$.

If we substitute (41) and (42) in (40) we get,

$$b_o = 1 - \frac{1}{2 \log \frac{a}{b}} \int_0^\infty \frac{\xi(\lambda)}{k_1^2 - \lambda^2} \, d\lambda \, , \qquad (103)$$

and

$$b_n = \frac{1}{\pi \alpha_n^2 I_{nn}} \frac{J_o(\alpha_n b)}{J_o(\alpha_n a)} \int_0^\infty \frac{\xi(\lambda)}{k_1^2 - \alpha_n^2} \, d\lambda \, . \qquad (104)$$

Replacing these in (34), we obtain an integral equation for $\xi(\lambda)$, i.e.

$$\xi(\lambda) \, I(\lambda) = d\zeta_o(\lambda) + \int_0^\infty \xi(\lambda') \, Q(\lambda, \lambda') \, d\lambda' \quad , \qquad (105)$$

where

$$Q(\lambda, \lambda') = \sum_{n=0}^{\infty} d_n \zeta_n(\lambda) \, \zeta_n(\lambda') \quad ,$$

$$d_o = \frac{i\gamma_o}{a\pi \log \frac{a}{b}} \quad ,$$

$$d_n = \frac{2i}{a\pi} \gamma_n \frac{v_m^2}{v_m^2 - 1} \quad ,$$

$$d = -\frac{2i\gamma_o}{a\pi} \quad ,$$

and

$$\zeta_n = \frac{1}{\gamma_n^2 - \lambda^2} \, .$$

For the case when $\rho \to 1$, $\sigma \neq 0$ and $\varepsilon \to 0$, if we neglect terms of order greater than $O(\varepsilon^o)$, $Q(\lambda, \lambda')$ becomes

$$Q(\lambda, \lambda') = d_o \zeta_o(\lambda) \, \zeta_o(\lambda') + O(\varepsilon^2) \quad . \qquad (106)$$

Putting (106) in (105) we obtain,

$$\xi(\lambda) = \frac{\zeta_o(\lambda)}{I(\lambda)} \left[d + d_o \int_0^{\infty} \xi(\lambda') \zeta_o(\lambda') \, d\lambda'\right] \quad . \tag{107}$$

Multiply (107) by $\zeta_o(\lambda)$ and integrate it from zero to infinity, gives

$$\int_0^{\infty} \xi(\lambda') \zeta_o(\lambda') \, d\lambda' = \frac{d \int_0^{\infty} \zeta_o^2(\lambda)/I(\lambda) d\lambda}{1 - d_o \int_0^{\infty} \zeta_o^2(\lambda)/I(\lambda) d\lambda} , \tag{108}$$

and hence,

$$\xi(\lambda) = d \frac{J_o(\lambda)}{I(\lambda)} + \frac{d d_o \zeta_o(\lambda)}{I(\lambda)} \frac{\int_0^{\infty} \zeta_o^2(\lambda)/I(\lambda) \, d\lambda}{1 - d_o \int_0^{\infty} \zeta_o^2(\lambda)/I(\lambda) d\lambda} . \tag{109}$$

If we use (109) to find b_o , we get the same result as was found in (46).

Appendix (iv): Solution of this Problem in the absence of Dielectric.

If in Fig. (1), we let $\varepsilon_1 = \varepsilon_2$ and $\mu_1 = \mu_2$, the problem reduces to that solved by V. M. Papadopoulos [3], and under the above conditions we can show that the results of this problem agree with Papadopoulos. For this purpose we will try to derive equation (28) of [3] from our equation (63) by letting $\varepsilon_1 = \varepsilon_2$. The only function undergoing appreciable change is the function $I(\lambda)$ given by (50). When $\varepsilon_1 = \varepsilon_2$, $k_1' = k_2' = k' = \sqrt{k^2-\lambda^2}$ and $I(\lambda)$ becomes,

$$I(\lambda) = \frac{H_o^{(1)}(k'a)}{k' H_o^{(1)'}(k'a)} - \frac{c_o'(k'a)}{k' c_o(k'a)} , \tag{110}$$

where

$$c_o(k'a) = J_o(k'a)\, Y_o(k'b) - J_o(k'b)\, Y_o(k'a) ,$$

and

$$c_o'(k'a) = J_o'(k'a)\, Y_o(k'b) - J_o(k'b)\, Y_o'(k'a) .$$

The Wronskian relation for the pair $H_o^{(1)}(k'r)$ and $c_o(k'r)$ is,

$$H_o^{(1)}(k'r)C_o'(k'r) - H_o^{(1)'}(k'r)C_o(k'r) = -\frac{4i}{\pi k' r} H_o^{(1)}(k'r) . \tag{111}$$

Using (111), (110) becomes,

$$I(\lambda) = \frac{2}{\pi k'^2 a} \frac{H_o^{(1)}(k'b)}{H_o^{(1)}(k'a)} \frac{1}{C_o(k'a)} . \tag{112}$$

In (112), replace k' by $k\sin\theta$ and substitute this in the expression for $g_1(k_2\cos\theta)$, then (64) becomes,

$$H_\varphi \simeq \text{const}\ \frac{e^{ikR}}{R}\ \frac{\sin\theta}{H_o^{(1)}(kb\ \sin\theta)} \left\{ J_o(ka\ \sin\theta)\ Y_o(kb\ \sin\theta) - J_o(kb\ \sin\theta)\ Y_o(ka\ \sin\theta) \right\} \Sigma\ \frac{b_n\gamma_n\ J_o(\alpha_n b)/J_o(\alpha_n a)}{\gamma_n^2 - k^2\ \cos\theta} \tag{113}$$

This result corresponds to that derived by Papadopoulos. Instead of $\exp(-ikR)/H_o^{(1)}(kb\ \sin\theta)$ we have $\exp(ikR)/H_o^{(1)}(kb\ \sin\theta)$ since we have assumed $\exp(-i\omega t)$ time dependence instead of $\exp(i\omega t)$.

Appendix (v): Non Existance of Coaxial Modes in Region (2).

In Sec. III, the Function F in equation (15) was expanded in terms of the coaxial eigen vectors $\varphi_{oi}(\alpha_n r)$. We can now show that these modes are cancelled by the residues of the integral in (20). Substitution of the expression for $\xi(\lambda)$ into (20) results in,

$$\psi_2 = \Sigma\, b_n \varphi_{on} e^{i\gamma_n z} + \frac{2i}{a\pi^2} \int_{-\infty}^{\infty} \frac{\Delta_1(k_1' r)}{\Delta_1(k_1' a)} \frac{e^{i\gamma z}}{I(\lambda)(k_1^2-\lambda^2)} \times \left(\sum_n b_n \frac{J_o(\alpha_n b)}{J_o(\alpha_n b)} \frac{\gamma_n}{\gamma_n^2-\lambda^2} - \frac{\pi}{2} \frac{\gamma_o b_o}{\gamma_o^2-\lambda^2} \right) . \tag{114}$$

The terms in brackets have poles at $\lambda = \gamma_n$ for $n = 0, 1, 2,$ $\Delta_1(k_1' a)$ has a zero while $I(\lambda)$ has poles at these points. As a result the integrand has simple poles at $\lambda = \gamma_n$ for $n = 0, 1, 2, \ldots$. Transforming the integral to a contour integral similar to Fig. (4), we obtain,

$$\int_{-\infty}^{+\infty} = \sum_{n=0}^{\infty} R_n + \Sigma \text{ Residue of other Points} - \int_\Gamma , \tag{115}$$

where R_n is the residue due to pole at γ_n. By simple manipulation we find R_n to be,

$$R_n = - \frac{a\pi^2}{2i} b_n \varphi_{on}(\alpha_n r) e^{i\gamma_n z} . \tag{116}$$

Putting (115) in (114) cancels all modes $\varphi_{oi}(\alpha_i r)$.

Appendix (vi) Calculation for Y/Y_o.

In Sec. IX we found the admittance of the antenna looking into the half space to be,

$$\frac{Y}{Y_o} = \frac{1 + a_o}{1 - a_o} - \frac{2}{\pi} \sum_{n=1}^{\infty} a_n .$$

For the first approximation neglecting all terms with index $n \geq 1$ we get,

$$\frac{Y}{Y_o} = \frac{1 + a_o}{1 - a_o}, \tag{117}$$

or substituting for a_o from (47), we obtain

$$\frac{Y}{Y_o} = 1 - 2i \frac{\gamma_o c_{oo}}{\pi a \log_e \frac{a}{b}} , \tag{118}$$

or

$$\frac{Y}{Y_o} = \left(1 + \frac{2\sigma \text{ Im } c_{oo}}{\pi \log \frac{a}{b}} - 2i \frac{\sigma \text{ Re } c_{oo}}{\pi \log \frac{a}{b}}\right). \tag{119}$$

To find $\frac{Y}{Y_o}$ we have to calculate the value of c_{oo} from the expression in (45), The integral c_{oo} is divided into three ranges viz: $0 \leq \lambda \leq \beta\sigma$, $\beta\sigma \leq \lambda \leq \sigma$ and $\sigma \leq \lambda < \infty$, since in each of the regions the integrand is completely different from the other regions. For this calculation the real and the imaginary parts of the integrand of c_{oo} (45) were programmed on the IBM 650 computer, the results plotted on graph paper, and the area under the curves measured by a planimeter. At the two simple poles of the integrand, the integrand was expanded in Laurent series and the principal part was calculated. The accuracy of the graphs $\frac{Y}{Y_o}$ in Figs. (6) and (7) greatly depend upon the accuracy of the measurement of the area by the planemeter. However the curve for $\mathrm{Re}\,\frac{Y}{Y_o}$ against ρ is quite accurate.

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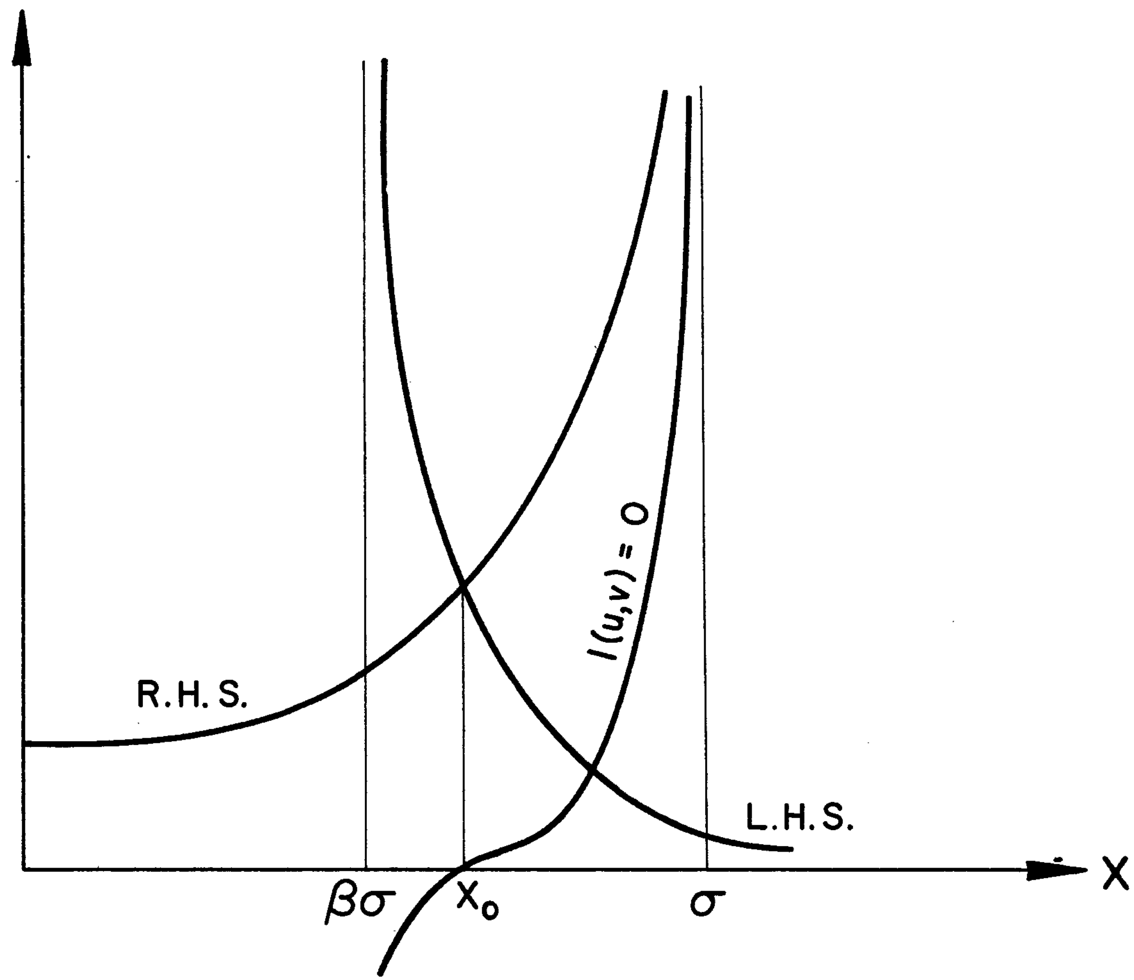


FIG. 2

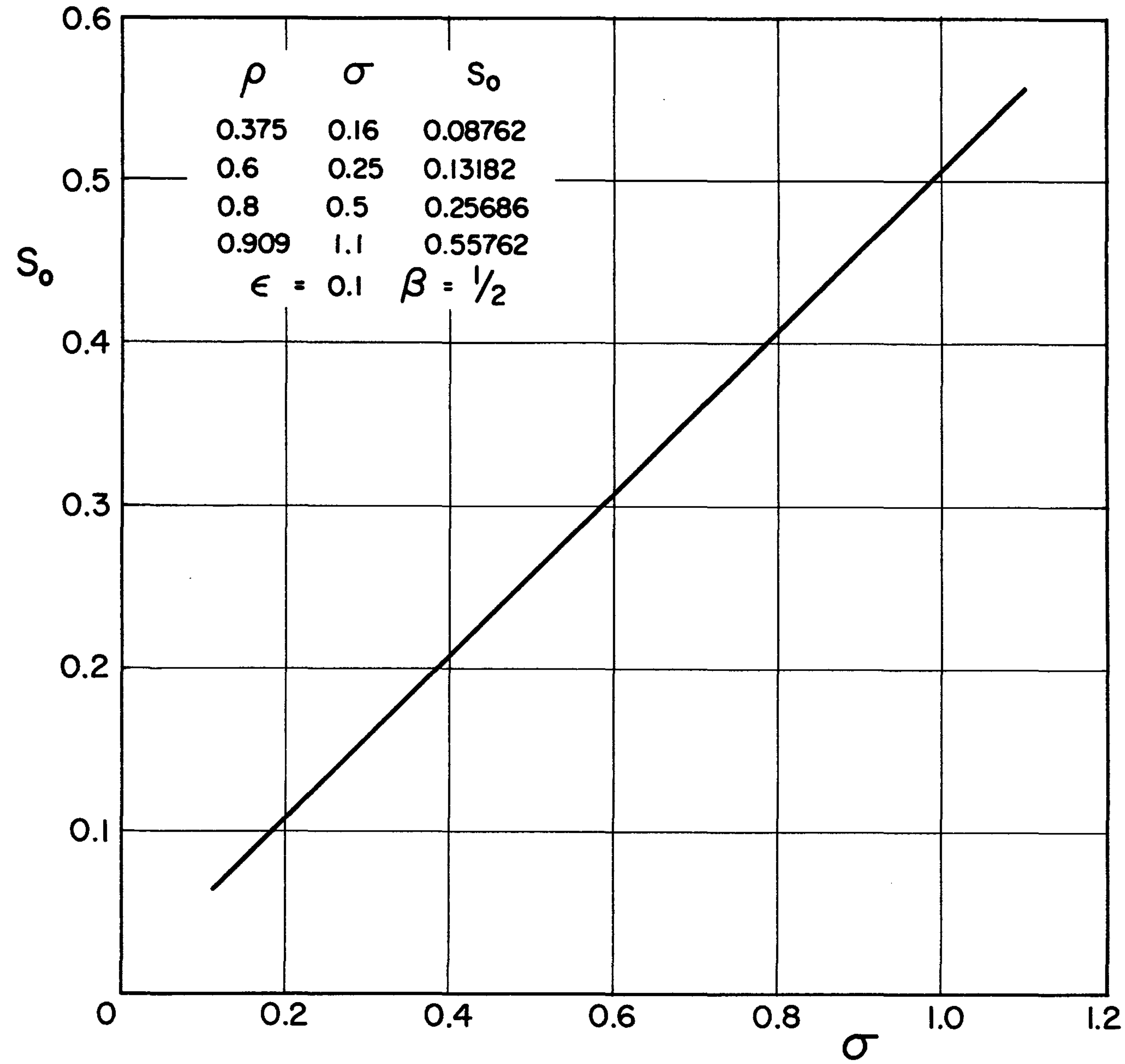

ρ σ S₀
0.375 0.16 0.08762
0.6 0.25 0.13182
0.8 0.5 0.25686
0.909 1.1 0.55762
ϵ = 0.1 β = 1/2
0.6
0.5
0.4
0.3
0.2
0.1
0
0.2
0.4
0.6
0.8
1.0
1.2
S₀
σ

FIG. 3

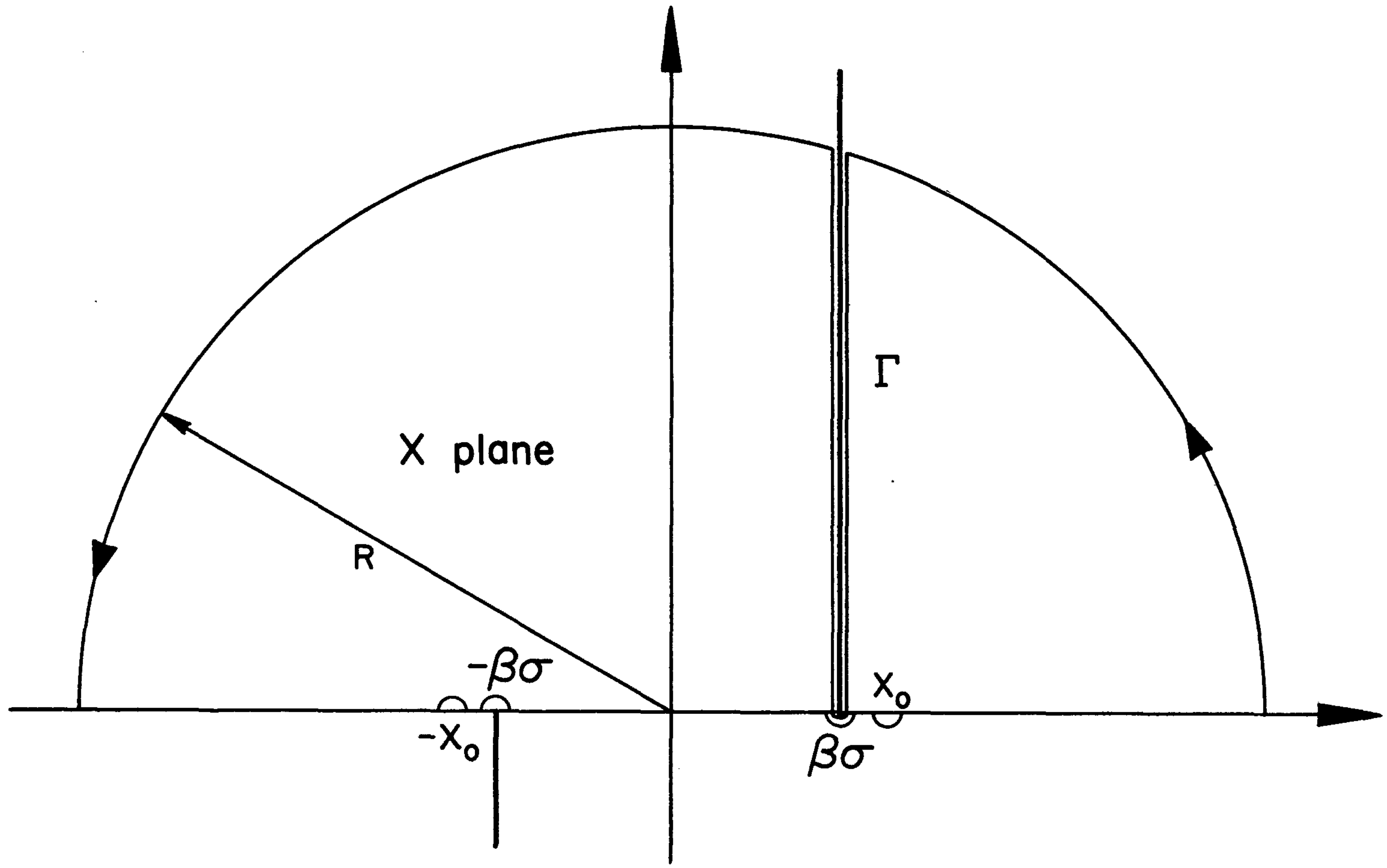


FIG. 4

PATH OF THE INTEGRATION IN THE X-PLANE

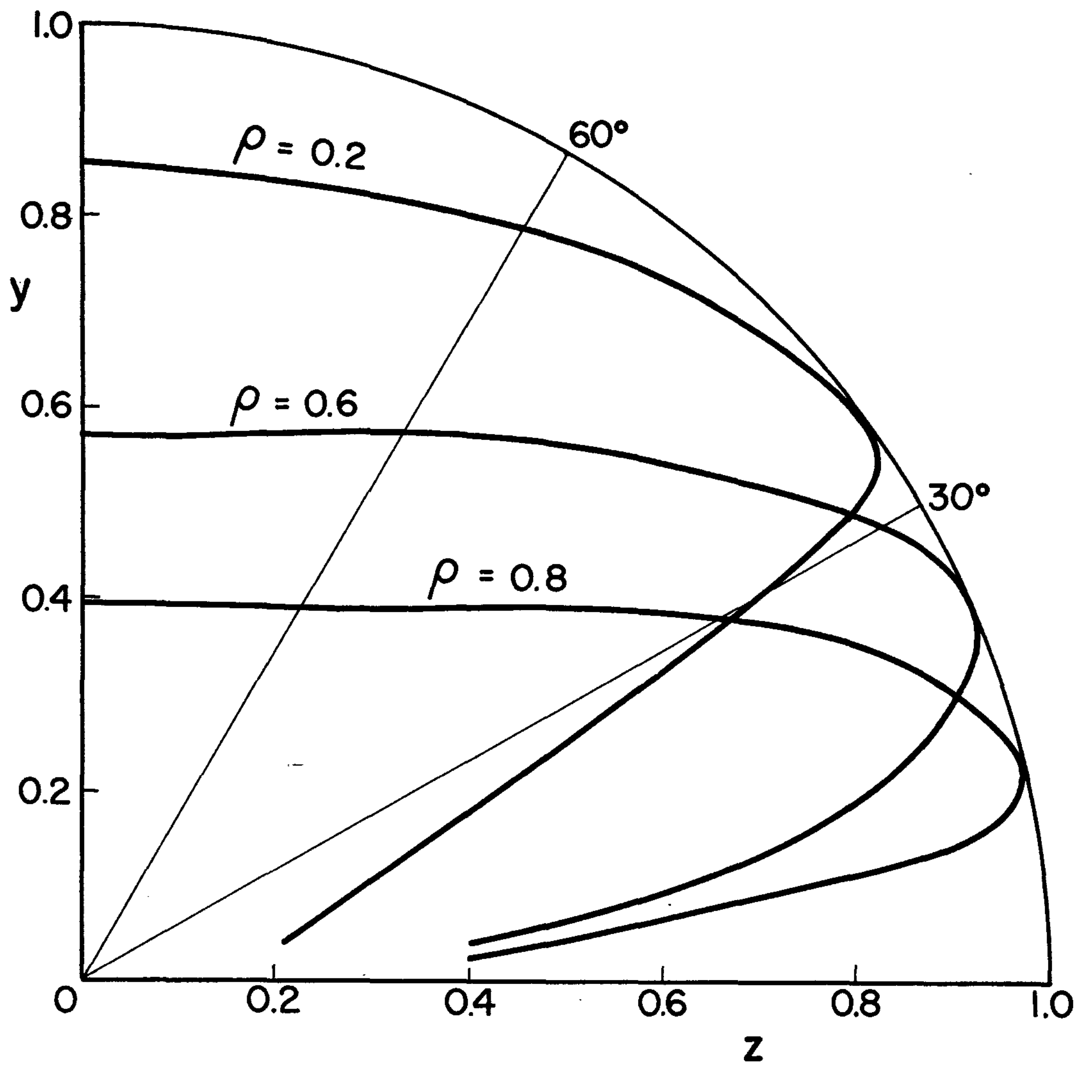


FIG. 5 NORMALIZED RADIATION PATTERN

$\frac{(H\phi)^2}{(H\phi)^2_{max.}}$ for different $\rho = \frac{b}{a}$, $\epsilon = \sigma(1-\rho)$, $\beta = \frac{1}{2}$

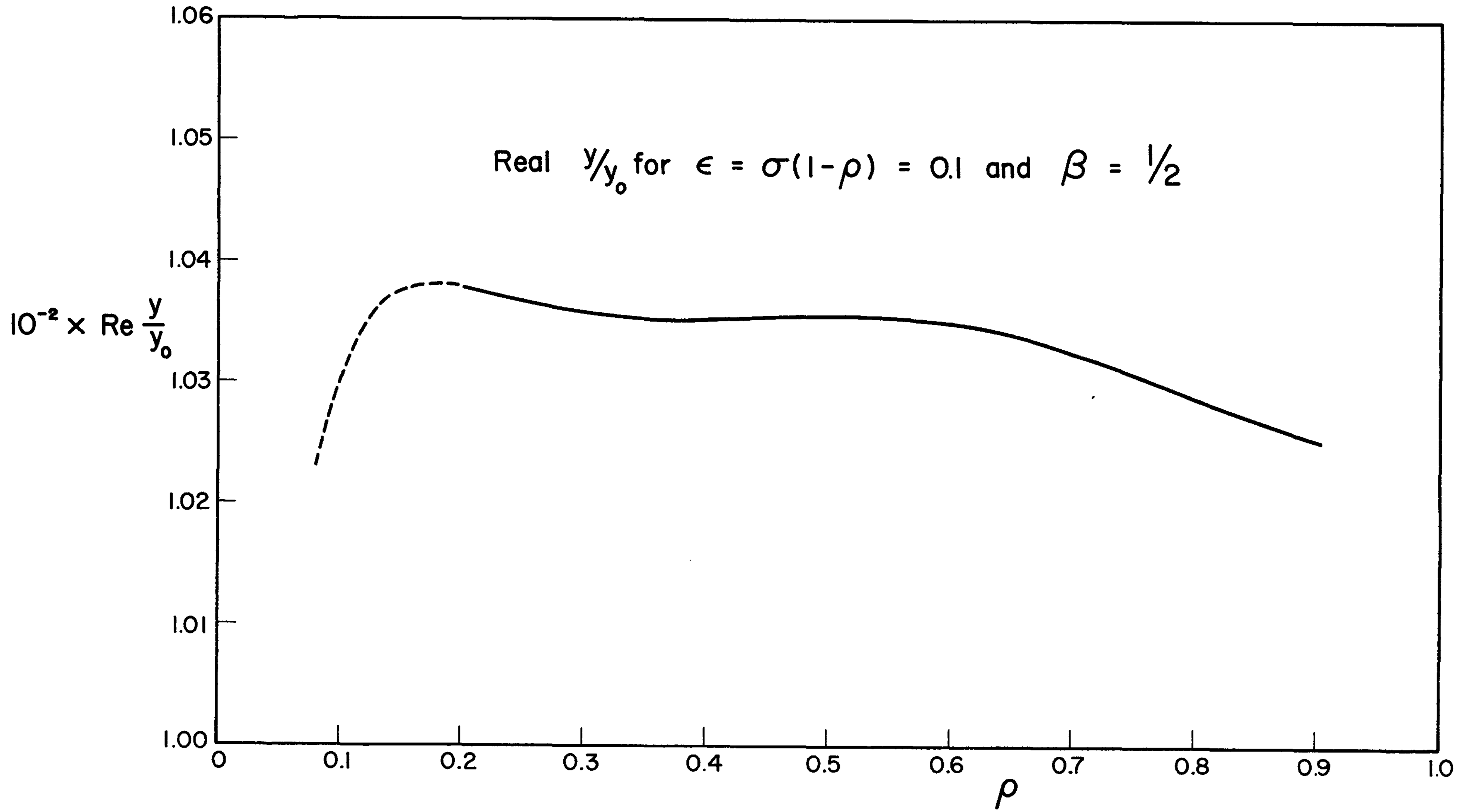
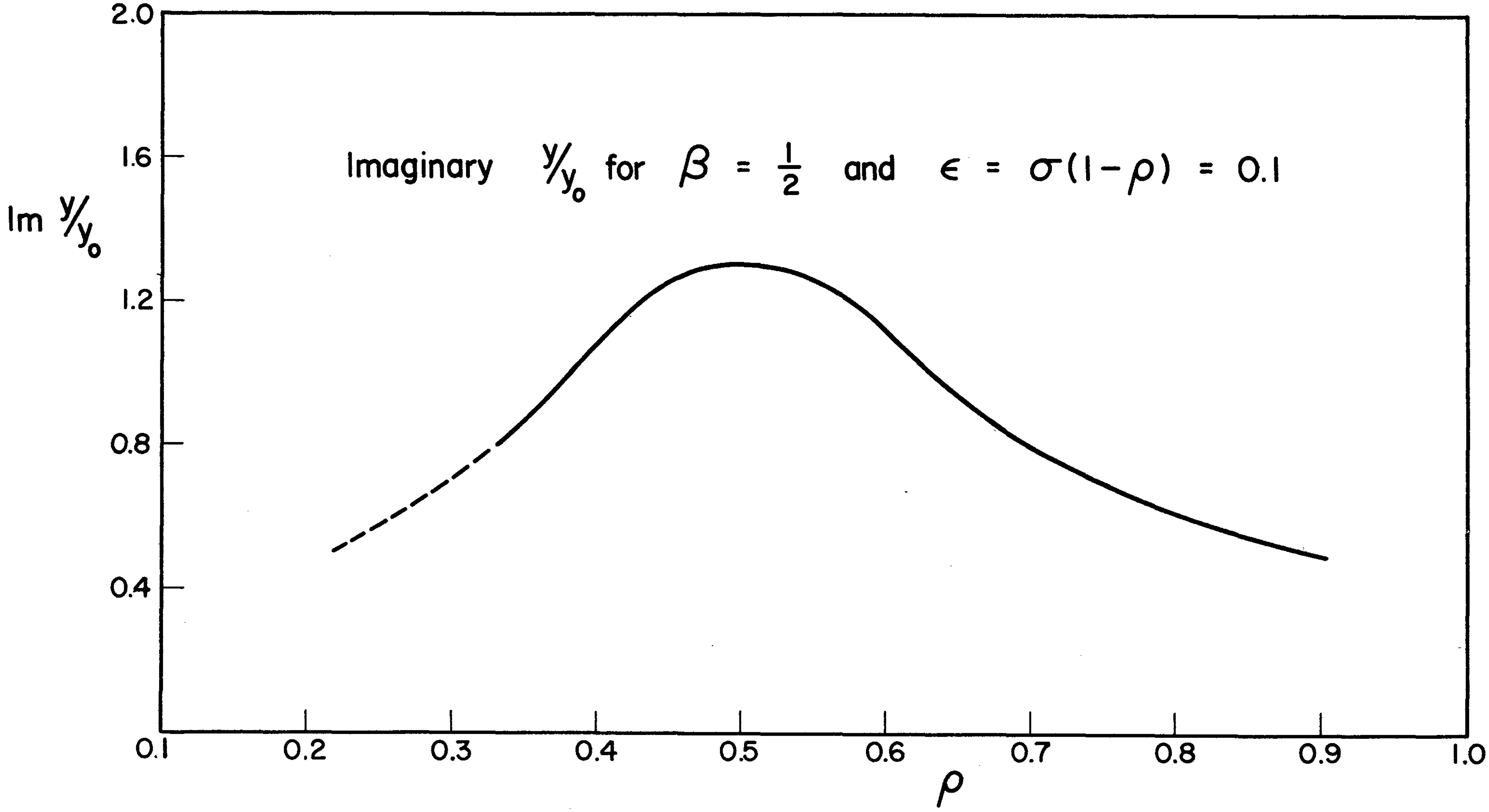

Real y/y_o for $\epsilon = \sigma(1-\rho) = 0.1$ and $\beta = 1/2$
$10^{-2} \times \mathrm{Re}\,\frac{y}{y_o}$
1.06
1.05
1.04
1.03
1.02
1.01
1.00
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
ρ

FIG. 6



Imaginary y/y_o for $\beta = \frac{1}{2}$ and $\epsilon = \sigma(1-\rho) = 0.1$
Im y/y_o
2.0
1.6
1.2
0.8
0.4
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
ρ

FIG. 7

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