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Giannini Controls Corporation
Astromechanics Research Division

DECAY-DAMPING RELATIONSHIPS FOR HIGHLY COUPLED SYSTEMS WITH MANY DEGREES OF FREEDOM

by A. G. FONDA

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Similarly, Mr. M. B. Zisfein's direct assistance in the historical background has provided an authoritative introductory viewpoint for the report.

INTRODUCTION

This is a report of investigations sponsored by the Directorate of Aerospace Sciences of the Air Force Office of Scientific Research, under Contract AF49(638)-1015, "Advanced Aeroelastic System Studies." The AFOSR research coordinator has been Mr. Howard Wolko (SRHM).

The guiding motive of this research has been the desire and the need for rigorous generalization of our already fruitful previous work, References 1 through 4, and the intuitive conviction that such generalization, although difficult, was obtainable. This was an ambitious objective in itself, because of the great algebraic complexity always encountered in highly-coupled systems having many degrees of freedom. Yet, in the final analysis, we seem to have accomplished not only our original objective, but possibly much more. We have developed a rigorous methodology which sidesteps the barriers of algebraic complexity, and not only yields the desired generalizations but may well be equally effective in treating other mathematically similar, highly complex systems. Such further applications lie, at this time, in the speculative future.

Our previous AFOSR work demonstrated, in References 1 through 4, that for two representative aeroelastic systems having two degrees of freedom, and for three having three degrees of freedom, certain approximate but very simple and very useful relationships could be found between so-called "flutter damping" (the ordinarily computed value of "required structural damping") and so-called "system damping" (the value of motion decay coefficient observable experimentally for the same system). These new relationships were applied numerically to hypothetical systems of two and three degrees of freedom and, without further proof, to more practical systems of many degrees of freedom. The resulting predictions of motion decay coefficient were excellent in all cases.

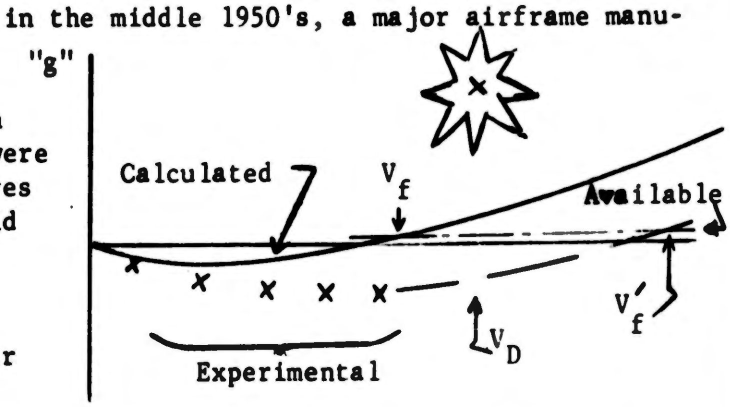
The present report makes no attempt to demonstrate similar numerical applications of the method. That is left to the preceding reports, which fully demonstrate the utility of the method, and to succeeding reports, which will demonstrate the application of the method in how-to-do-it fashion. The reader is accordingly cautioned that the present report does not embody any "tricks of the trade" which, although they are theoretical trivialities, are quite necessary to a sophisticated application of the method. Rather, this is a tightly constructed, rigorous theoretical report which lays a solid foundation for the confident development of a practical working technique. The actual step-by-step, how-to-do-it technique is left for a succeeding report.

Before even quoting the previously derived and published predictive relationships, it is worthwhile to take even more of a backward look at the history of the problem. We have found that many a practicing flutter analyst is (or was) under the impression that twice the motion decay ratio observed after striking a wing or other configuration in the wind tunnel or in flight, is a value of "g" which will agree with and is supposed to be synonymous with the g (which we prefer to call g_R) evaluated theoretically from an AMC type of flutter analysis. (Even the

Air Materiel Command ancestry of the method is often unknown to the younger analyst; it is simply "the" theoretical method.) He believes that the "theoretical g" is a prediction of the "experimental g," and that failure to agree is an indication of a poor prediction for the experiment performed. Prior to our work, his thought in the matter was no more profound than that.

A competitive school of thought which was more profound (but often less useful) did also exist. Many intelligent flutter analysts were aware that the computed g (our g_R) was a measure of the energy dissipation needed for exactly neutral stability, and these "enlightened" analysts did argue that the experimentally observed motion decay coefficient bore no acceptable theoretical relationship to the required-for-neutral-stability g, except at the flutter point, when the required damping equalled the available and the resulting decay was zero. The result was purely a negative one, in that these investigators would refuse to recognize any attempted prediction of the decay coefficient from g_R, except when g_R equalled the available g, and sometimes not even then. The fact that g_R and 2γ (where γ is our symbol for the motion decay coefficient) did sometimes show good agreement was regarded as an interesting but useless coincidence, despite the fact that the textbooks (e.g. Scanlan and Rosenbaum, Reference 5, pages 87-88) did show such a correlation for a simple and non-aerodynamic system.

The consequences of this state of affairs have often been as spectacular as the situation is ludicrous. For instance, in the middle 1950's, a major airframe manufacturer ran a wind tunnel program on an advanced delta-wing vehicle, using a very expensive complete-configuration flutter model. Calculated V-g curves were furnished to the wind-tunnel team, curves which they interpreted to signify a mild onset of flutter of the vertical tail, because of the low slope of the calculated V-g curve (see the adjacent sketch) as it passed through the flutter point.



As they found and plotted experimental values of doubled decay coefficient for the vertical tail (shown by x's), they decided that the calculated curve shape was good but that the calculated flutter speed (V_f) was quite conservative, the true flutter speed being V_f'. Their next test was therefore performed at the airspeed V_D, spaced nicely along toward V_f' but beyond the calculated flutter speed V_f. The result was a disastrous flutter of the vertical tail, and their expensive model was severely damaged.

The "enlightened" comment on this fiasco was merely that the g_R curve shouldn't have been trusted except to indicate V_f (which it did). Yet, in current retrospect, the rapid onset of violent flutter was quite predictable, and could even have been plotted in advance by proper transformation of the calculated data which was available at that time.

How many times this and related incidents have occurred is a matter purely for conjecture. Based on various discussions with other people in the field, it appears to have been a frequent mistake. We can only guess how many flight-test disasters have been caused in whole or in part by over-optimistic use of g_R curves - nor can we know how many perfectly useful g_R curves have been abandoned because of the converse attitude, because of over-pessimism.

Members of both of these schools of thought may be surprised to see, at the proper point in our text (specifically, at page 24), a proof which refutes both of their arguments - and yet is based on the mere assumption of a four-parameter system having just one degree of freedom! The conclusion which can be drawn even from this elementary example, a conclusion which has every indication of validity for systems of any complexity, is that

> The simple relationship $2\gamma = g_R$ is valid at zero airspeed, and does continue to hold at higher airspeeds for some systems, but ceases to be valid when, as indicated by sensitivity of mode frequency to airspeed, the effective aerodynamic stiffness becomes appreciable compared with the basic structural stiffness.

This broadly stated rule is still mostly negative in its utility, but is solidly supported by and given positive application by our derived equation

$$\frac{\gamma_o}{g_R} = \frac{1}{2} - \frac{V_o/\omega}{dV_o/d\omega} \tag{41}$$

This equation, which will be thoroughly explained as well as rigorously derived in our text, specifically states the variation of the damping-to-decay transfer function from its zero-airspeed value, namely 1/2, solely as a function of the relative airspeed sensitivity of the natural frequency involved. Along with a related equation,

$$\gamma = \gamma_o \left[1 - \frac{g_s}{g_R} - \frac{\gamma_s}{\gamma_R} \right] \tag{16}$$

(which defines the influence of actual material damping of two types upon the decay coefficient), this equation provides a remarkably rapid, convenient, and effective procedure for the truly "enlightened" interpretation of the outputs of AMC-type flutter solutions.

Forms of both equations were first derived and published in our earlier work, References 1 through 4. (The reader is specifically referred to Reference 4 for a broader background in the subject, and for further discussions of concepts used here, such as our "base curve" concept.) In our present work, all of the seeming restrictions of our previous work are removed, although two actual restrictions do remain. For practical aeroelastic use, neither of these is a severe or prohibitive restriction.

One of these two restrictions is the restriction to "small" values of material damping coefficient and motion decay coefficient, values small enough that the products and powers of these coefficients can be neglected. The usual coefficient values which are of interest to the aeroelastician are within this limit of validity.

The other restriction results from the assumption, in our derivation, of aerodynamic theory such that the effect of airspeed is proportional only to the product of airspeed and modal displacement. This assumption suits first-order piston theory in particular, and suits some of the older quasi-steady theories as well. Now, first-order piston theory is quite respectable in itself; but as if that were not enough, there are good indications that our results would be fairly insensitive to the use of some other aerodynamic theory. We are not, after all, trying to predict a good g_R and a good γ. We are merely trying to predict a good ratio of γ to g_R. If the effect of a change of aerodynamic theory upon γ resembles its effect upon g_R, then the ratio of γ to g_R is insensitive to this change in theory, and our method of correlating γ with g_R can be applied regardless. We already have good reasons to believe that such insensitivity does actually occur; thus the assumption of piston theory in this report should not be taken as rigidly imposing the same restriction upon the results reached from this assumption. One theory may be as good as another, in this regard.

It might be remarked that, if interpreted with the proper level of confidence, only good can come from the application of our equations to an available g_R calculation. We do not lose the flutter point; thus, we offer all that the former "pessimists" could offer. Neither do we lose the relation $2\gamma = g_R$, except where that relation is almost indubitably in error; thus, we offer all that the former "optimists" could validly offer. And where we do reject $2\gamma = g_R$, we replace it with a varying transfer function which can be used with considerable confidence, confidence which will, of course, continue to improve with use.

In future reports, we hope to derive the similar (if not identical) relationships which must exist for aerodynamic regimes other than that of piston theory. Our basic methodology will still apply. We hope also to numerically investigate and illustrate the utility of all these results; to introduce, as we have mentioned, a detailed practical how-to-do-it guide; to actually furnish currently useful interpretations and to furnish guidance for the attainment of practical design solutions; and we hope to pursue, on occasion, the application of our new mathematical methodology to other problems, even if these problems are physically unrelated to aeroelasticity.

I. THE LOW-DECAY THEOREM

In this section, we will begin the use of a special notation, and we will use it to show the linear dependence of any s polynomial upon its decay coefficient.

First, let a single symbol, E, denote any general polynomial in s:

$$E = E_o + E_1 s + E_2 s^2 + E_3 s^3 + \dots E_n s^n$$

$$= \sum_{i=o}^{n} E_i s^i \qquad (1)$$

where E_i is repeatedly the coefficient of s^i, with i having values from zero to n. Generally, an array of values of E_i describes a system configuration, and the polynomial represents the ratio of input to output for that system. This polynomial becomes the characteristic equation of the system for E = 0 (denoting zero input).

A general substitution for s is $s = (\gamma + j)\omega$, denoting the complex roots of the characteristic equation. The decay coefficient γ is the ratio of the real to the imaginary part of the root. (The classical "damping coefficient" is the sine of the angle whose tangent is γ, and approaches γ as they both approach zero.)

A special substitution for s is $s = j\omega$. This denotes a condition of no decay ($\gamma = 0$), that is, neither growth nor decay of the output oscillation which occurs at the frequency ω. If we now let the single symbol E_γ denote the value of E which is obtained for $\gamma = 0$, we have the definition

$$E_\gamma = \sum_{i=o}^{n} E_i (j\omega)^i \qquad (2)$$

This forms a special case of E, wherein E_i is the coefficient of $(j\omega)^i$ from i=o to i=n. The result is a frequency polynomial having readily separable real and imaginary parts, with all of the even values of i in the real part and all of the odd values of i in the imaginary part. Because of this easy separability, it is much easier to solve $E_\gamma = 0$ for two unknowns, than to solve E = 0 for two unknowns. This is exactly the basis of the AMC method (in which companion values of frequency and structural damping are evaluated from $E_\gamma = 0$) and also of British methods (in which companion values of frequency and viscous material damping are evaluated from $E_\gamma = 0$).

But the "classical" method is to evaluate companion values of frequency and motion decay coefficient (ω and γ) directly from E = 0 (Equation 1), losing the advantage of readily separable real and imaginary parts. However, if we confine our interest to small values of γ, we can regain much of that advantage by making a binomial expansion of $(\gamma + j)$, giving

$$s^i = \left(j^i + ij^{i-1}\gamma + \ldots \frac{i!}{(i-m)!m!} j^{i-m}\gamma^m + \ldots\right)\omega^i \qquad (3)$$

and by then neglecting all but the first two terms. This is permissible for small values of $i\gamma$, since the third term is approximately $(i\gamma)^2/2$ times the first, and succeeding terms are even smaller. Thus, we have

$$s^i \cong (1 - \gamma ji)(j\omega)^i \qquad (4)$$

If we substitute this approximation for s^i into Equation (1), we will alter the associated value of E by some small amount, an amount depending on the various coefficients E_i as well as upon the magnitude of $i\gamma$ (which itself ranges from γ to $n\gamma$). Therefore no definite limit can be stated regarding the validity of this approximation, although $|\gamma| = .2$ has been found to be a fair rule-of-thumb limit.

By substituting Equation (4) into Equation (1), we find that each $E_i(j\omega)^i$ term of E_γ will now appear twice, once with a unity coefficient and once with a $-\gamma ji$ coefficient. By "factoring out" these coefficients we find

$$\boxed{E \cong E_\gamma - \gamma jiE_\gamma} \qquad (5)$$

where we must define the "special polynomial" which is formed by multiplying each term of E_γ by its frequency exponent,

$$iE_\gamma = \left[E_1 s + 2E_2 s^2 + 3E_3 s^3 + 4E_4 s^4 + \ldots nE_n s^n\right]_{s = j\omega} \qquad (6)$$

Thus, the "i" of "iE_γ" is a term-to-term variable, not an ordinary coefficient. This is a convenient notation which simplifies Equation (5) and will also simplify our remaining derivation.

Equation (5) is an approximation which indicates the <u>linearity of E with γ</u> for small γ. This is a dependency not ordinarily recognized. The rate of variation of E with γ is, specifically,

$$\frac{\partial E}{\partial \gamma} = -jiE_\gamma \qquad (7)$$

where iE_γ will have a fixed complex value at a fixed ω, a vector which when rotated clockwise 90° and diminished in proportion to γ will indicate the change in the complex value of E for that γ. We could evaluate iE_γ by knowing each E_i and each ω^i, but (as we will see in Section III) it is also possible to evaluate this γ partial derivative of E by proper interpretation of numerical-graphical results obtained for $\gamma = 0$.

An algebraically useful version of these results is obtained if we note that the frequency derivative of ω^i is $i\,\omega^{i-1}$. This gives us

$$(\omega\partial/\partial\omega)(\omega^i) = i(\omega^i)$$

so that Equation (6) becomes simply

$$iE_\gamma = \omega\partial E_\gamma/\partial\omega \tag{8}$$

where the operator $\omega\partial/\partial\omega$ is implicitly applied to each term of E_γ in succession, and develops the coefficient i for each. Then, Equation (5) becomes

$$\boxed{E \cong E_\gamma - \gamma\,(j\omega\partial E_\gamma/\partial\omega)} \tag{9}$$

This allows us to state the "low-decay theorem" as follows: "For low values of the decay coefficient γ, the complete differential equation of any system is approximated by its own zero-decay (e.g. AMC) value, minus the product of γ times j times ω times the frequency derivative of that same zero-decay characteristic equation." The rate of change of E with γ is thus, corresponding to Equation (7),

$$\frac{\partial E}{\partial\gamma} = -\, j\omega\,\frac{\partial E_\gamma}{\partial\omega} \tag{10}$$

which is minus $j\omega$ times the rate of change of E_γ with ω. This states the linear variation of E with γ, and its magnitude. No assumptions as to the physical nature of the system have been made.

This theorem is a key to relationships between the no-decay behavior of any dynamic system, and its low-decay behavior. It forms a transition between the classical assumption of "possibly large" γ and the AMC-type assumption of "necessarily zero" γ.

II. THE DECAY COEFFICIENT FOR LOW MATERIAL DAMPING

In this section, we will assume that the polynomial E is linear not only with γ (as proven in Section 1) but also with the material damping coefficients g_s and ζ_s. This means that any products and powers of g_s, ζ_s, and γ will be neglected; this restricts us to small values of g_s, ζ_s, and γ. The resulting statement of linearity, valid for low damping and low decay, is

$$E \cong E_{g\zeta\gamma} + g_s \frac{\partial E}{\partial g} + \zeta_s \frac{\partial E}{\partial \zeta} + \gamma \frac{\partial E}{\partial \gamma} \tag{11}$$

The term $E_{g\zeta\gamma}$ is (in accordance with our prior E_γ notation) the value of E for $g = 0$, $\zeta = 0$, and $\gamma = 0$. Of these three zero conditions, two can also apply for each of the partial derivatives; that is,

$$\partial E/\partial g \cong \partial E_{\zeta\gamma}/\partial g$$

$$\partial E/\partial \zeta \cong \partial E_{g\gamma}/\partial \zeta$$

$$\partial E/\partial \gamma \cong \partial E_{g\zeta}/\partial \gamma$$

meaning that the various products of g_s, ζ_s, and γ can be neglected here also.

As a means of utilizing this statement of linearity, let us now assume special conditions at which reference values of g_s, ζ_s, and γ may be defined. The first special condition is $E = 0$ at $g_s = g_R$, $\zeta_s = 0$, and $\gamma = 0$, giving:

$$E_{g\zeta\gamma} + g_R \frac{\partial E}{\partial g} \cong 0 \tag{12a}$$

Then, the solution for g_R is

$$g_R \cong - \frac{E_{g\zeta\gamma}}{\partial E/\partial g} \tag{12b}$$

This is the amount of structural damping required for neutral stability ($E = 0$, $\gamma = 0$) at $\zeta_s = 0$.

The second special condition is $E = 0$ at $g_s = 0$, $\zeta_s = \zeta_R$, and $\gamma = 0$, giving:

$$E_{g\zeta\gamma} + \zeta_R \frac{\partial E}{\partial \zeta} \cong 0 \tag{13a}$$

Then, the solution for ζ_R is :

$$\zeta_R \cong -\frac{E_{g\zeta\gamma}}{\partial E/\partial \zeta} \qquad (13b)$$

This is the amount of material viscous damping required for neutral stability ($E = 0$, $\gamma = 0$) at $g_s = 0$.

The third special condition is $E = 0$ at $g_s = 0$, $\zeta_s = 0$, and $\gamma = \gamma_o$, giving:

$$E_{g\zeta\gamma} + \gamma_o \frac{\partial E}{\partial \gamma} \cong 0 \qquad (14a)$$

Then, the solution for the "basic decay coefficient" γ_o is

$$\gamma_o = -\frac{E_{g\zeta\gamma}}{\partial E/\partial \gamma} \qquad (14b)$$

This is the decay coefficient obtained for the case of zero material damping.

Using the three reference values so established, Equation (11) takes the form

$$E \cong E_{g\zeta\gamma} \left[1 - \frac{g_s}{g_R} - \frac{\zeta_s}{\zeta_R} - \frac{\gamma}{\gamma_o} \right] \qquad (15)$$

This is a restatement of the original assumption of linearity, using arbitrary reference values. It reduces to the identity $E_{g\zeta\gamma} = E_{g\zeta\gamma}$ for the condition $g_s = 0$, $\zeta_s = 0$, and $\gamma = 0$. It reduces also to $E = 0$ for the conditions $g_s = g_R$, $\zeta_s = \zeta_R$, and $\gamma = \gamma_o$ taken separately.

But the zero input condition $E = 0$ is also satisfied by various suitable mixtures of g_s, ζ_s, and γ. If we assume $E = 0$ (so that E is the "characteristic" equation) and solve Equation (15) for γ, we find

$$\boxed{\gamma = \gamma_o \left[1 - \frac{g_s}{g_R} - \frac{\zeta_s}{\zeta_R} \right]} \qquad (16)$$

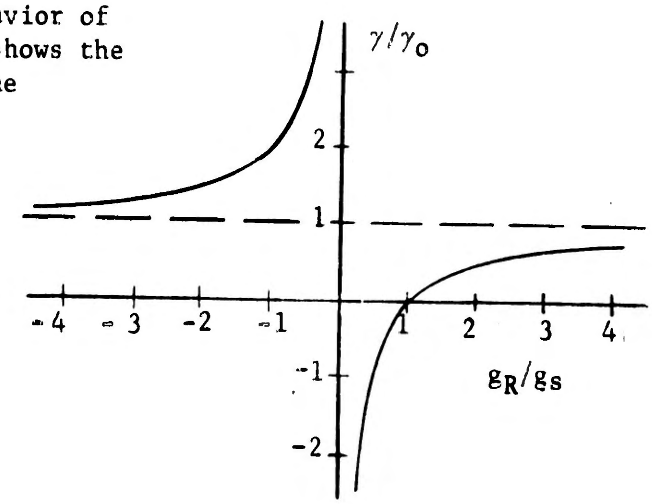
In that ζ has been included, this equation is a generalization of our previous results such as Equation (14) of Reference 4. These previous results were not only less general, but were obtained less concisely, namely by manipulation of the particular polynomials defining $\partial E/\partial g$ and $\partial E/\partial \gamma$ for several particular systems having two or three degrees of freedom. Our new, shorter derivation is fully as valid, and yet does not depend upon the number of degrees of freedom nor upon any of the configuration parameters. It is purely a useful statement of the linear dependence of a decay coefficient upon a material damping coefficient, for small values of those coefficients. This linear dependence has not been generally recognized because the linear variation of E with small γ (Equation 5) is not generally recognized.

Equation (16) may be regarded from two points of view. Either we fix g_R and ζ_R while we vary g_s and/or ζ_s, or vice versa. In both cases, variations of γ/γ_o result. The first viewpoint is the viewpoint we took in deriving Equation (16), and it also corresponds to the case of varying the construction or the material of a system so as to change g_s and/or ζ_s, evaluating the resulting effect upon γ/γ_o. This obviously is a fairly rare point of view to assume, though it does sometimes apply.

Much more commonly, we would hold g_s and ζ_s fixed while varying g_R and/or ζ_R, the required material dampings. This may be a result, for instance, of varying the assumed airspeed in an aeroelastic system. If g_R and/or ζ_R can be computed for each consecutive airspeed, then Equation (16) will yield the resulting consecutive values of γ/γ_o. This would be the usual method of utilizing Equation (16).

By including ζ_s as well as g_s in Equation (16), we have made it possible to evaluate the simultaneous effects upon γ of both kinds of material damping. The inputs for this evaluation would be g_R, ζ_R, and γ_o. The value of g_R is the conventional end product of aeroelastic analysis in American industry. The value of ζ_R could be found by similar procedures (and indeed seems to be the output of some British studies). The value of γ_o can be found by proper manipulation of the g_R data (as we will show in Section III) or, we believe, by similar manipulation of ζ_R data. The result will be a considerably refined evaluation of system stability, obtained through our approximations with considerably less effort than by any other method.

To illustrate the numerical behavior of Equation (16), the accompanying sketch shows the variation of the parameter γ/γ_o with the inverted parameter g_R/g_s.



These curves are antisymmetric about the (0, + 1) point. Their small slope at large g_R/g_s indicates the small effect of g_s when g_R is large, that is, when the system is heavily stable or unstable.

The converse situation occurs at small g_R/g_s, when the system is almost neutrally stable. At exactly neutral stability ($g_R = 0$), the value of γ/γ_o is infinite. But of course, γ_o is then also zero. Thus, the product of γ/γ_o and γ_o is numerically indeterminant at $g_R = 0$. We can avoid this indeterminancy of γ by algebraic rearrangement of Equation (16), for instance into

$$\gamma = \frac{\gamma_o}{g_R} (g_R - g_s) \qquad (17)$$

This equation, obtained from (16) for the case of $\zeta_s = 0$, avoids the former indeterminancy at $g_R = 0$ because (as we will see in Section III) the value of γ_o/g_R is determinant even when both γ_o and g_R are zero*. As another point entirely, Equation (17) also shows that the function γ_o/g_R is useable as a transfer function from $g_R - g_s$ to γ, as well as from g_R to γ_o. Thus, the expressions for this transfer function, which we will develop in Section III, are not limited in applicability to the $g_s = 0$ case alone; we will be able to account for any g_s without needing an additional step from γ_o to γ.

A brief discussion of physical significance is also in order here. If γ_o and g_R are both positive (denoting respectively a "growth" of oscillation, and a need for dissipative damping) then any increase of g_s from zero (denoting some dissipative damping) signals a proportional decrease in the oscillation "growth" rate. Or, if g_s is fixed and g_R increases, this means that the need for energy dissipation has become larger, so that the growth rate will be greater at the same g_s. Thus γ increases with g_R and decreases with g_s.

A negative γ_o and a negative g_R would mean, conversely, that an increase of g_s would signal more decay (where it previously signalled less growth). A positive increment (to a smaller negative g_R) would signal less decay.

On rare occassions, γ_o and g_R are of opposite sign. This occurs when the addition of "dissipative" damping to a highly coupled system makes it less stable, rather than more stable. The resulting "loop-back" phenomenon was described in our earlier work, and will be discussed further in future reports.

Note that realistic values of g_s may run around .01 whereas our approximations are good for g_R past .1. Thus, it is not unlikely that the effect of g_s in Equation (17) will be trivial. In that case, we will be able to study γ_o as if it were γ. The transfer function γ_o/g_R, for use with or without consideration of g_s, will be developed in Section III, following now.

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* Equation (17) may not avoid indeterminancy at some other airspeed, if it happens that the numerator and the denominator both pass through zero for some other reason. Such regions must be treated with care; our future reports will cover this in detail.

III. THE DAMPING-TO-DECAY TRANSFER FUNCTION

In this section, we will develop successive expressions for the transfer function γ_o/g_R. We will clarify, in the process, the inherent logic which relates γ_o to g_R, and we will show the influence of several successive special assumptions. The end result will be a simple key expression utilizing the outputs of a conventional g_R solution to predict γ_o.

The "artificial intermediate system" created by the AMC assumption $g_s = g_R$ (for $\gamma = 0$) is, to some degree, a distortion of the true system because the "true" system has some fixed value of g_s. A generally lesser distortion is produced by assuming $g_s = 0$; for this second "artificial intermediate system" the value of γ is γ_o. These two distinct but related systems are described, respectively, by our previous equations

$$E_{g\varphi\gamma} + g_R\, \partial E/\partial g = 0 \qquad (12a)$$

which is one special case of Equation (11), and by

$$E_{g\varphi\gamma} + \gamma_o\, \partial E/\partial \gamma = 0 \qquad (14a)$$

which is another special case of Equation (11). Since each equation has both real and imaginary parts, two unknowns can be evaluated, nominally g_R and ω from (12a), or γ_o and ω from (14a), at a given system configuration (fixed airspeed). As will be shown later, the real parts are largely independent of damping effects and essentially determine ω without knowing g_R or γ_o. Thereafter, the frequency being known, the imaginary part of (12a) would be used to find g_R as a function of the damping of the system, and the imaginary part of (14a) would be used to find γ_o as a function of that same system damping.

Since g_R and γ_o are thus both functions of the same imaginary part of $E_{g\varphi\gamma}$, we can find the direct relationship of γ_o to g_R, namely (using imaginary parts only)

$$\frac{\gamma_o}{g_R} = \frac{\partial E/\partial g}{\partial E/\partial \gamma} \qquad (18)$$

This states that the ratio of γ_o to g_R is the ratio of the rates of variation of the polynomial E with g and with γ. This equation may seem too simple to be useful; but by developing numerator and denominator separately, we can obtain very useful expressions for γ_o/g_R.

So long as we utilize Equation (18) algebraically, its accuracy depends only upon the "small g_R" assumption used in Equation (12a), and the "small γ_o" assumption similarly used in Equation (14a). For convenience, however, we may choose to substitute numerically a value of frequency which almost, but not quite, satisfies the real parts of both (12a) and (14a) simultaneously. To keep the resulting error small, we must avoid the regions in which the g_R solutions for frequency differ markedly from the γ_o solutions for frequency. This restriction, fortunately, must largely coincide with our prior restriction to small g_R and γ_o, since only if either g_R or γ_o is large can the real parts of (12a) and (14a) fail to give almost identical values of frequency.

In due course, we will replace the denominator $\partial E/\partial\gamma$ of Equation (18) by using, consecutively, Equations (7) and (10) of Section I. However, let us defer those substitutions until we can simultaneously make similar substitutions for the numerator. These substitutions will depend upon assumptions which typify the nature of structural damping and further assumptions which typify the results obtained using first-order piston theory.

Effect of Conventional Structural Damping

First, the polynomial E will be assumed to be the determinant of the matrix which would be formed for a p-degree-of-freedom system (a "p by p" matrix),

$$E = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A & \cdots & A_{2p} \\ \vdots & & & \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{vmatrix} \qquad (19)$$

The expanded form of such a determinant consists of the sum of many products of A terms. The above representation is the conventional abbreviation for this lengthy sum, but for our purposes even this abbreviation is too bulky and too unwieldy Therefore, we denote each one of the matrix products by A^p, where p is a superscript denoting the fact that A^p is a product of p differently subscripted A's, such as

$$A^p = A_{11}\,A_{22}\,A_{33} \quad \text{or} \quad A_{11}\,A_{23}\,A_{32}$$

for p = 3, or

$$A^p = A_{11}\,A_{22}\,A_{33}\,A_{44}\,A_{55}\,A_{66}\,A_{77}\,A_{88}\,A_{99}$$

for p = 9. Then, with A^p so defined, we can replace Equation (19) by

$$E = \sum_{11}^{pp} A^p \tag{20}$$

This denotes the summation of all possible A^p products using the subscripts 11 to pp. We can merely assume that the proper rules of determinant expansion, implicit in Equation (19), are also implicit in Equation (20).* The number of A^p terms which results is p!, or (for instance) 362,880 for p = 9. Equation (20) represents all these terms simultaneously.

To illustrate for a ternary (p = 3) system, we have

$$\begin{aligned} E &= \sum_{11}^{33} A^3 \\ &= \sum (A_{11} A_{22} A_{33}, \text{ etc.}) \\ &= A_{11} A_{22} A_{33} - A_{11} A_{23} A_{32} - A_{12} A_{21} A_{33} \\ &\quad - A_{12} A_{23} A_{31} - A_{13} A_{21} A_{32} + A_{13} A_{22} A_{31} \end{aligned}$$

obtained by noting that with the first subscripts fixed in 1-2-3 sequence, the second subscripts can be arranged in six different sequences, four of which include an inversion making the product negative. This is identical to the expanded determinant of a 3 by 3 matrix. For higher-order systems, Equation (20) changes only by changing p, rather than by writing (as with Equation 19) a larger array of terms.

The next step depends on the fact that, in any system, a typical A term always can be postulated which will algebraically encompass all the actual A's of that system. For instance, in a high-supersonic aeroelastic system, we have, typically,

$$A_{aa} = M_{aa} s^2 + B_{aa} s + (1 + g \frac{s}{\omega}) K_{aa} + Y_{aa} V \tag{21}$$

---

*These rules specify that each product can contain only one term from each row and each column, and that the sign of the product is negative when the number of vertical inversions of sequence is odd while proceeding horizontally.

as a general matrix term. This contains inertia, damping, phase-lead structural stiffness, and aerodynamic stiffness. Some of these coefficients (e.g. the structural stiffness of off-diagonal terms) may be zero in certain instances; this does not affect our derivation. In fact, until we reach Equation (32), we are concerned only with the existence of the phase-lead stiffness*; the other terms could have any form whatsoever, the terms shown being merely representative.

If we substitute Equation (21) into our abbreviated determinant equation, (20), we find

$$E = \sum_{11}^{pp} \left[Ms^2 + Bs + (1 + g\frac{s}{\omega})K + YV \right]^P \tag{22}$$

where the subscripts of M, B, K, and Y are now implicit, since they vary within each A^P product.

Our next step is to algebraically introduce a logical but unconventional "psuedo-binomial expansion" as a means of isolating the K-dependent terms of (22) from the K-independent terms. Consider that, if the product A^P were a product of binomials, variously subscripted, then the expansion of this product of binomials would be the sum of all the possible combinations of the first terms times second terms. The more first terms were chosen, the fewer second terms would remain, so that if we had (for example)

$$A^P = (a_{11} + b_{11})(a_{22} + b_{22})(a_{33} + b_{33})$$

which is a p = 3 case of

$$A^P = (a + b)^P$$

then we would have the expansion shown on the next page,

---

* A parallel derivation using viscous damping (ζ) instead of phase-lead stiffness must, we presume, be possible. The result would be a transfer function γ_o/ζ_R, useable if a ζ_R solution were to be interpreted.

$$
\begin{aligned}
A^p = \; & a_{11}\; a_{22}\; a_{33} && (m=o) \\
& + a_{11}\; a_{22}\; b_{33} && \\
& + a_{11}\; b_{22}\; a_{33} && (m=1) \\
& + b_{11}\; a_{22}\; a_{33} && \\
& + a_{11}\; b_{22}\; b_{33} && \\
& + b_{11}\; a_{22}\; b_{33} && (m=2) \\
& + b_{11}\; b_{22}\; a_{33} && \\
& + b_{11}\; b_{22}\; b_{33} && (m=3)
\end{aligned}
$$

These eight terms have been arranged in four groups according to the number of appearances ("m") of the second term "b." The number of appearances of "a" is, of course, "p-m," decreasing as m increases.* This is a p = 3 case of

$$A^p = \sum_{m=o}^{p} a^{p-m}\, b^m \tag{23}$$

which is our general representation of a pseudo-binomial expansion. This expansion of A^p is to be formed by summing all the possible products of (p-m) "a" terms and (m) "b" terms, in the manner already illustrated. The number of terms in this summation will be 2^p, which is eight for p = 3 and 512 for p = 9. Equation (23) represents all these terms simultaneously.

If we introduce the notation of Equation (23) into Equation (22), using $(Ms^2 + Bs + YV)$ in place of "a" and $(1 + g\, s/\omega)K$ in place of "b," we obtain

$$E = \sum_{11}^{pp} \sum_{m=o}^{p} (Ms^2 + Bs + YV)^{p-m} \left(1 + g\frac{s}{\omega}\right)^m K^m \tag{24}$$

This is only a partial expansion of E, because we have not yet separated the terms of $Ms^2 + Bs + YV$ from each other, nor the terms of $1 + g\, s/\omega$ from each other. Still, this is a symbolic representation of $p!(2^p)$ terms, or more than 185 million terms for p = 9. Equation (24) represents all these terms simultaneously, and yet is sufficiently brief as well as sufficiently meaningful for our purposes.

From Equation (24), we can define both the g = 0 value of E and its rate of variation with g. First, note that we did not define any subscript for g in Equation (21). Thus, we have assumed that the same value of g applies in all modes (wherever a K appears). This assumption is

---

*Without subscripts, this becomes $a^3 + 3a^2b + 3ab^2 + b^3$ (a true binomial expansion).

a) Common in the industry, so that our results will apply to the usual g_R solution on which we may wish to operate;

b) Physically realistic, at least as a first approximation; and

c) Algebraically convenient, because it allows us the simplification which follows now.

Because g bears no subscript, the term $(1 + g\, s/\omega)$ is a true binomial, and "m" is hence a true exponent. This permits us a now-familiar type of approximation, in which we expand conventionally and (by assuming small g and small γ) neglect all but the first two terms. The result is

$$(1 + g \frac{s}{\omega})^{m} \cong 1 + g\, j\, m \tag{25}$$

This shows that each term of E will increase the relative amount gm as g increases, and at 90° to its g = 0 value. This obviously resembles the γi effect of γ upon E. If we define the g = 0 value of E as E_g, where from (24) we have

$$E_g = \sum_{11}^{pp} \sum_{m=o}^{p} (Ms^2 + Bs + YV)^{p-m} K^m \tag{26}$$

then we see that Equation (24) is the product of Equations (25) and (26), or

$$\boxed{E \cong E_g + g\, j\, m\, E_g} \tag{27}$$

This corresponds to Equation (5) which expressed the effect of γ upon E. In the case of g, the "special polynomial" is

$$mE_g = \sum_{11}^{pp} \left[(Ms^2+Bs+YV)^{p-1} K + 2\,(Ms^2+Bs+YV)^{p-2} K^2 + \ldots . pK^p \right] \tag{28}$$

corresponding to Equation (6) for iE_{γ}. The coefficient "m" in (28) is the pseudo-exponent of K, a term-to-term variable resembling "i."

Equation (27) is an approximation which indicates the linearity of E with g for small values of g and γ. Furthermore, it specifies the rate of variation of E with g as

$$\frac{\partial E}{\partial g} = j\, m\, E_g \tag{29}$$

corresponding to Equation (7) for $\partial E/\partial\gamma$. This is valid for any system having a uniformly distributed structural damping coefficient, regardless of the other effects present in the system.

By using Equation (29) in Equation (12a), and Equation (7) in Equation (14a), we obtain

$$E_{g\zeta\gamma} + g_R\, j\, m\, E_{g\zeta\gamma} = 0 \qquad (30a)$$

and

$$E_{g\zeta\gamma} - \gamma_o\, j\, i\, E_{g\zeta\gamma} = 0 \qquad (30b)$$

where (for convenience) we have replaced E_g and E_γ by $E_{g\zeta\gamma}$, which is equivalent to neglecting the small products $g_R \zeta_s$, $g_R\gamma$, $\zeta_o g_s$, and $\gamma_o \zeta_s$.

Because $E_{g\zeta\gamma}$ is a complex quantity, the two equations (30a) and (30b) represent four useful relationships: two real equations and two imaginary equations. Although simultaneous solution of each real-imaginary pair for two unknowns is possible, further study also shows that an iterative approach will be rapidly convergent. This iterative approach (which should not be confused with a digital-computer approach) may be described as follows.

1. To solve (30a), first assume $g_R = 0$ in the real part of (30a). This leaves $RE_{g\zeta\gamma} = 0$ (where "R" denotes "real part of"). Solve this for a frequency at a given system configuration; or solve for some configuration parameter, such as airspeed, at a given frequency. (As in our earlier work, we will use the symbol V_o to represent the airspeed which thus satisfies $RE_{g\zeta\gamma} = 0$ at a given frequency. A plot of V_o versus ω is called a "base curve" in References 1 through 4.)

2. Use this initial result in the imaginary part of (30a), to find a value for g_R at that frequency-airspeed condition. This completes the first cycle of iteration.

3. Use the initial g_R in the real part of (30a) to refine the initial solution. However, the refinement will generally be small, indeed trivial, because the real part of $g_R\, j\, m\, E_{g\zeta\gamma}$ can result only from the imaginary, or damping, part of $E_{g\zeta\gamma}$; this damping part of $E_{g\zeta\gamma}$ must be small, so that when multiplied by g_R there will be hardly any effect upon the result reached in step (1).

4. Repeat step (2) using the results of step (3), and so on.

When the corresponding procedure is also applied to (30b), the first step yields the identical value of frequency, and the succeeding steps will then yield the joint values of γ_0 and ω. But in both instances, the convergence is so rapid, for small g_R and small γ_0, that the first two steps alone are sufficient for practical purposes.

Thus, $RE_{g\delta\gamma} \cong 0$ defines the approximate relationship of frequency to airspeed, both for the g_R case and for the γ_0 case, while the imaginary parts of (30a) and (30b) determine g_R and γ_0, respectively, from the imaginary (damping) part of $E_{g\delta\gamma}$.

Since g_R and γ_0 both are measures of the same damping influence, their relation to each other may now be defined. When we solve the imaginary parts of (30a) and (30b) for the transfer function γ_0/g_R, we will be utilizing only the real parts of $mE_{g\delta\gamma}$ and $iE_{g\delta\gamma}$. These real parts denote the rates of change $\partial E/\partial g$ and $\partial E/\partial\gamma$ (which are not functions of the damping of the system). Thus, the transfer function from g_R to γ_0 is

$$\frac{\gamma_0}{g_R} = - \frac{mRE_{g\delta\gamma}}{iRE_{g\delta\gamma}} \tag{31}$$

where only the real part of $E_{g\delta\gamma}$ is to be considered. This version of Equation (18) is valid for any system having uniformly distributed structural damping. It is interesting to note that the relation of γ_0 to g_R is to be determined from an equation in which there isn't any g and there isn't any γ!

——————— Example A

We may illustrate Equations (21), (24), (26), (29), (7), and (31) for the case of a very simple aeroelastic system, the binary panel of Reference 1 (page 14 ff.). In that instance, Equation (21) takes the forms

$$A_{11} = s^2 + \frac{\omega_A}{2} s + (1 + g\frac{s}{\omega})\,\omega_1^2$$

$$A_{22} = s^2 + \frac{\omega_A}{2} s + (1 + g\frac{s}{\omega})\,\omega_2^2$$

$$A_{12} = \frac{4}{3}\,\omega_A V/1$$

$$A_{21} = -\frac{4}{3}\,\omega_A V/1$$

Evidently, the coefficients Y_{11}, Y_{22}, M_{12}, B_{12}, K_{12}, M_{21}, B_{21}, and K_{21} are zero in this instance. Thus, eight of the sixteen possible binary coefficients are zero, simplifying our illustration still further.

The corresponding dynamic equation will be

$$E = A_{11}A_{22} - A_{12}A_{21}$$
$$= \left[s^2 + \frac{\omega_A}{2} s + (1+g\frac{s}{\omega})\omega_1^2\right]\left[s^2 + \frac{\omega_A}{2} s + (1+g\frac{s}{\omega})\omega_2^2\right] + (\frac{4}{3}\,\omega_A V/1)^2$$

Expanding partially according to m, we have, corresponding to Equation (24)

$$E = (s^2 + \frac{\omega_A}{2} s)^2 + (\frac{4}{3} \omega_A V/1)^2 \quad \} m=0$$

$$+ (s^2 + \frac{\omega_A}{2} s)(1 + g \frac{s}{\omega})(\omega_1^2 + \omega_2^2) \quad \} m=1$$

$$+ (1 + g\frac{s}{\omega})^2 \omega_1^2 \omega_2^2 \quad \} m=2$$

Then, for g = 0 and γ = 0 (and with ζ = 0 already) we find, corresponding to Equation (26), and arranged laterally according to the exponent of frequency,

$$E_{g\zeta\gamma} = \underbrace{\omega^4}_{i=4} \underbrace{-2 \frac{\omega_A}{2} j \omega^3}_{i=3} \underbrace{-(\frac{\omega_A}{2})^2 \omega^2}_{i=2} \underbrace{+ 0}_{i=1} \underbrace{+(\frac{4}{3} \omega_A V/1)^2}_{i=0} \quad \} m=0$$

$$-(\omega_1^2 + \omega_2^2)\omega^2 + \frac{\omega_A}{2}(\omega_1^2 + \omega_2^2) j\omega + 0 \quad \} m=1$$

$$+ \omega_1^2 \omega_2^2 \quad \} m=2$$

The g derivative of E will be, as per Equation (29),

$$\frac{\partial E}{j\partial g} = mE_{g\zeta\gamma}$$

$$= (\omega_1^2 + \omega_2^2)(-\omega^2 + \frac{\omega_A}{2} j \omega) + 2\omega_1^2 \omega_2^2$$

which can be verified directly, neglecting the g^2 effect.

Similarly, the γ derivative of E will be, as per Equation (7),

$$-\frac{\partial E}{j\partial \gamma} = iE_{g\zeta\gamma}$$

$$= 4 \omega^4 - 3 \omega_A j \omega^3 - 2(\frac{\omega_A}{2})^2 \omega^2 + (\omega_1^2 + \omega_2^2)(-2\omega^2 + \frac{\omega_A}{2} j \omega)$$

which can be verified directly by substituting $s = (\gamma + j)\omega$ into $E_{g\gamma}$, neglecting th γ^2, γ^3, and γ^4 effects.

Then, taking real parts only, the γ_o/g_R transfer function will be, corresponding to (31),

$$\frac{\gamma_o}{g_R} \approx - \frac{(\omega_1^2 + \omega_2^2)(-\omega^2) + 2\,\omega_1^2\,\omega_2^2}{4\,\omega^4 - 2\left[\left(\frac{\omega_A}{2}\right)^2 - (\omega_1^2 + \omega_2^2)\right]\omega^2}$$

As per our iterative procedure, we would evaluate this transfer function from the frequency which satisfies $RE_{g\gamma\gamma} = 0$ at a given airspeed.

———————(Example A will be continued)

Equation (31) is an expression for γ_o/g_R which would be very useful if we had the complete algebraic expression for $E_{g\gamma\gamma}$ available, so that we could inspect each individual term for the individual values of m and i. Unfortunately, this full knowledge of $E_{g\gamma\gamma}$ is often unavailable, especially for complex systems where the expanded determinant is often not explicit. Our salvation lies in the fact that both $mE_{g\gamma\gamma}$ and $iE_{g\gamma\gamma}$ can be evaluated indirectly.

We have already seen, in Equation (8) of Section I, that $iE_{g\gamma\gamma}$ is equal to $\partial\,\omega E_{g\gamma\gamma}/\partial\omega$, because $\omega\partial/\partial\omega$ brings out the i of each term as a coefficient. Thus, if we had a plot of $E_{g\gamma\gamma}$ versus ω, we could evaluate its slope, and multiply each such value by its abscissa, to obtain a plot of $iE_{g\gamma\gamma}$ versus ω. We will show now that a similar indirect evaluation of $mE_{g\gamma\gamma}$ can be made, and that these two indirect evaluations can be combined into one graphical interpretation of a conventional AMC data plot.

Effect of First-Order Piston-Theory Aerodynamics

In order to indirectly evaluate $mE_{g\gamma\gamma}$, we must utilize the form of the $(Ms^2 + Bs + YV)$ portion of $E_{g\gamma\gamma}$. Thus, we now enter a phase of the derivation which is dependent on first-order piston theory (represented by the YV term). In some future, parallel derivation, any other type of aerodynamics in which airspeed or any power of airspeed has a linear effect might be utilized.

Our first step is to complete the pseudo-binomial expansion of $E_{g\gamma\gamma}$ by treating first YV and then Bs as the second term of the pseudo-binomial. The result, from Equation (26), is

$$E_{g\gamma\gamma} = \sum_{11}^{pp} \sum_{m=o}^{p} \sum_{q=o}^{p-m} \sum_{r=o}^{p-m-q} (-M\omega^2)^{p-m-q-r}(jB\omega)^r (YV)^q K^m \qquad (32)$$

The assigned pseudo-exponents are q for YV and r for $jB\omega$, leaving p-m-q-r as the necessary pseudo-exponent of $-M\omega^2$. Each of the three pseudo-binomial expansions multiplies the total number of terms by 2^p, so for p = 9, the number of terms resulting is $512^3(362,880) \cong 48$ trillion terms. <u>Equation (32) represents all of these terms simultaneously and manageably.</u>

The advantage of Equation (32) is that we can now relate the exponents of ω, V, and K to each other. <u>This will allow us to evaluate m indirectly.</u> By inspecting Equation (32), we see that the exponent of frequency, which is "i" by definition, is also

$$\begin{aligned} i &= 2(p-m-q-r) + r \\ &= 2(p-m-q) - r \end{aligned}$$

for the case of piston-theory aerodynamics. This expression for i in terms of p, m, q, and r can be rearranged to give m in terms of p, q, i, and r:

$$m = p-q - \frac{i + r}{2} \tag{33}$$

or, more to the point,

$$mE_{g\zeta\gamma} = (p-q - \frac{i + r}{2})\, E_{g\zeta\gamma} \tag{34}$$

where a group of "special polynomials" is summed to give the desired indirect evaluation of $mE_{g\zeta\gamma}$.

———————— Example A (Continued)

To demonstrate the validity of Equation (34) for the binary panel example, we may arrange our previous $E_{g\zeta\gamma}$ in the form

$$\begin{array}{cccccc}
 & r=0,q=0 & r=1,q=0 & r=2,q=0 & r=0,q=2 & \\
E_{g\zeta\gamma} = & \omega^4 & -2\,\frac{\omega_A}{2}\,j\,\omega^3 & -(\frac{\omega_A}{2})\omega^2 & +(\frac{4}{3}\,\omega_A V/1)^2 & \} \; m=o \\
 & -(\omega_1^2+\omega_2^2)\omega^2 & +\frac{\omega_A}{2}\,(\omega_1^2+\omega_2^2)\,j\omega & & & \} \; m=1 \\
 & +\omega_1^2\,\omega_2^2 & & & & \} \; m=2 \\
 & r=0,q=0 & r=1,q=0 & & &
\end{array}$$

Then, the separate terms of Equation (34) are respectively

$$
\begin{array}{rcl}
pE_{g\zeta\gamma} & = & 2\omega^4 - 4\omega^2\left(\frac{\omega_A}{2}j\omega\right) + 2\left(\frac{\omega_A}{2}j\omega\right)^2 + 2\left(\frac{4}{3}\omega_A\frac{V}{1}\right) - 2\omega^2(\omega_1^2+\omega_2^2) + 2\frac{\omega_A}{2}j\omega(\omega_1^2+\omega_2^2) + 2\omega_1^2\omega_2^2 \\
-\frac{i}{2}E_{g\zeta\gamma} & = & -2\omega^4 + 3\omega^2\left(\frac{\omega_A}{2}j\omega\right) - \left(\frac{\omega_A}{2}j\omega\right)^2 + 0 + \omega^2(\omega_1^2+\omega_2^2) - \frac{1}{2}\frac{\omega_A}{2}j\omega(\omega_1^2+\omega_2^2) + 0 \\
-qE_{g\zeta\gamma} & = & 0 + 0 + 0 - 2\left(\frac{4}{3}\omega_A\frac{V}{1}\right) + 0 + 0 + 0 \\
-\frac{r}{2}E_{g\zeta\gamma} & = & 0 + \omega^2\left(\frac{\omega_A}{2}j\omega\right) - \left(\frac{\omega_A}{2}j\omega\right) + 0 + 0 - \frac{1}{2}\frac{\omega_A}{2}j\omega(\omega_1^2+\omega_2^2) + 0 \\
\hline
\text{Total} & = & 0 + 0 + 0 + 0 - \omega^2(\omega_1^2+\omega_2^2) + \frac{\omega_A}{2}j\omega(\omega_1^2+\omega_2^2) + 2\omega_1^2\omega_2^2 \\
 & = & mE_{g\zeta\gamma}
\end{array}
$$

Certainly, it would have been impossible to recognize such a relationship as Equation (34) from mere inspection of the equations for even this simple binary example. In our Reference 4, this relationship was established only through a lengthy series of steps, and of course never appeared in the form of Equation (34) even then. Our new methodology not only abbreviates the correlation for our binary and ternary examples, but also vastly abbreviates the detailed correlation for any other number of degrees of freedom as well.

——————(End of Example A)

Of the four terms of Equation (34), two can be expressed indirectly and the other two can be neglected. The indirect expressions are

$$iE_{g\zeta\gamma} = \omega\,\partial E_{g\zeta\gamma}/\partial\omega \qquad (35)$$

from Equation (8), and

$$qE_{g\zeta\gamma} = V\,\partial E_{g\zeta\gamma}/\partial V \qquad (36)$$

which is similar to Equation (8), but is based instead on the fact that q is the exponent of V throughout $E_{g\zeta\gamma}$. Then, $\partial/\partial V$ brings out q as a coefficient of each term, and the V multiplier restores the power of V.

The two negligible terms of Equation (34) are $pE_{g\gamma\gamma}$ and $rE_{g\gamma\gamma}$, or more properly the real parts thereof, since we want only the real part of $mE_{g\gamma\gamma}$. It is permissible to assume $pRE_{g\gamma\gamma} \cong 0$ because we have already assumed $RE_{g\gamma\gamma} = 0$ in our first iterative step. (Alternatively, we need not neglect $pRE_{g\gamma\gamma}$; the resulting effect on γ_o/g_R is shown in the Appendix, but reduces to zero for the region of small γ_o and small g_R).

It is permissible to assume $rRE_{g\gamma\gamma} \cong 0$ because it is composed only of products of the damping terms of $E_{g\gamma\gamma}$, such as $(Bs)^2$, $(Bs)^4$ and so on, and will have essentially no effect on the rate of change of E with g.

Using these four substitutions, Equation (34) becomes

$$mRE_{g\gamma\gamma} = -\left(V\frac{\partial}{\partial V} + \frac{\omega}{2}\frac{\partial}{\partial\omega}\right) RE_{g\gamma\gamma} \tag{37}$$

and hence, Equation (31) becomes

$$\frac{\gamma_o}{g_R} \cong \frac{\left(V\frac{\partial}{\partial V} + \frac{\omega}{2}\frac{\partial}{\partial\omega}\right) RE_{g\gamma\gamma}}{\omega\partial RE_{g\gamma\gamma}/\partial\omega}$$

or

$$\boxed{\frac{\gamma_o}{g_R} \cong \frac{1}{2} + \frac{V\partial RE_{g\gamma\gamma}/\partial V}{\omega\partial RE_{g\gamma\gamma}/\partial\omega}} \tag{38}$$

——————— Example B

To illustrate, consider the simplest possible system, a single-degree-of-freedom system. According to Equation (21), the characteristic equation will be

$$E = Ms^2 + Bs + (1 + g\frac{s}{\omega})K + YV = 0$$

For this equation, the exact rate of change of E_γ with g is, from Equation (29) or by inspection,

$$\frac{\partial E_\gamma}{\partial g} = j\,K$$

showing that K controls the rate of effectiveness of structural damping. For the same equation, the approximate rate of change of E_g with γ will be, from Equation (7) or by inspection,

$$\frac{\partial E_g}{\partial\gamma} \cong 2\,j\,M\omega^2$$

where only a γ^2 term has been omitted. (This value of $\partial E_g/\partial\gamma$ is excellent as long as γ is small.)

Combining these two values according to Equation (18), we have

$$\frac{\gamma_o}{g_R} \cong \frac{jK}{2jM\omega^2}$$

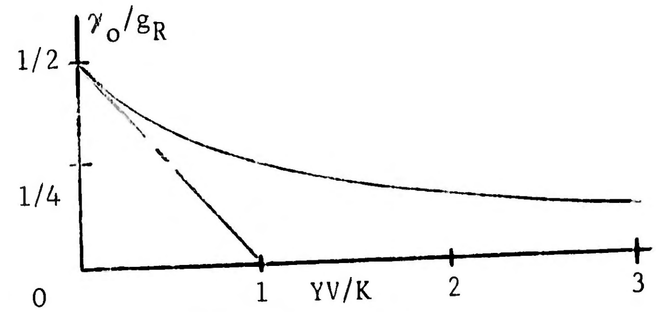
At zero airspeed, for $s \cong j\omega$ the real part of the characteristic equation is $-M\omega^2 + K \cong 0$. This defines ω^2, and thereby yields

$$\frac{\gamma_o}{g_R} \cong \frac{1}{2}$$

This is the classical textbook evaluation of γ_o/g_R, and is frequently mistakenly applied as a factor which will predict a valid γ_o from a calculated g_R without regard for airspeed. But this fixed classical value is derivable only at zero airspeed, or at airspeeds where structural stiffness (K) is heavily predominant over aerodynamic stiffness (YV). Otherwise, even in our simple example, we have $-M\omega^2 + K + YV \cong 0$, indicating that the natural frequency must increase as the airspeed increases. This dependency of frequency on airspeed also makes γ_o/g_R depend upon airspeed, as shown by

$$\frac{\gamma_o}{g_R} \cong \frac{j(M\omega^2 - YV)}{2jM\omega^2}$$

$$= \frac{1}{2} - \frac{YV}{2(K + YV)}$$



This relationship, plotted here, has an initial slope of -Y/2K (which extrapolates to $\gamma_o/g_R = 0$ at $YV = K$). The actual curve approaches zero asymptotically, passing through $\gamma_o/g_R = 1/4$ at $YV = K$. Thus, as the aerodynamic stiffness becomes appreciable compared with the structural stiffness, the natural frequency and the γ_o/g_R transfer function both depart appreciably from their zero-airspeed values. Thus we see, even for this extremely simple system, that $2\gamma = g_R$ cannot be trusted when there is appreciable dependency of frequency upon airspeed. This result applies, as we shall see, regardless of system complexity.

To obtain the same result using Equation (38), we would note

$$V \frac{\partial RE_{g\gamma\gamma}}{\partial V} = YV$$

and

$$\omega \frac{\partial RE_{g\gamma\gamma}}{\partial \omega} = - 2M\omega^2$$

yielding

$$\frac{\gamma_o}{g_R} = \frac{1}{2} - \frac{YV}{2M\omega^2} = \frac{1}{2} - \frac{YV}{2(K + YV)}$$

as before.

______________(End of Example B)

In practice, Equation (38) still is difficult to utilize because $RE_{g\gamma\gamma}$ is not always available in a form which permits evaluation of its V and ω partial derivatives. But, from elementary differential-equation theory we know that the full derivative of $RE_{g\gamma\gamma}$ relative to one of its variables is

$$\frac{dRE_{g\gamma\gamma}}{d\omega} = \frac{\partial RE_{g\gamma\gamma}}{\partial \omega} + \frac{\partial RE_{g\gamma\gamma}}{\partial V}\frac{dV}{d\omega} \qquad (39)$$

(see e.g. page 50 of Phillips' Differential Equations). This relates the two partial derivatives to each other in terms of the full derivatives $dRE_{g\gamma\gamma}/d\omega$ and $dV/d\omega$. But since we are maintaining the condition $RE_{g\gamma\gamma} = 0$ in our first iterative step, the rate of change of $RE_{g\gamma\gamma}$ with frequency must be zero. The result is that the ratio of the partial derivatives must be the negative of the full derivative $dV/d\omega$, or

$$\frac{\partial RE_{g\gamma\gamma}/\partial\omega}{\partial RE_{g\gamma\gamma}/\partial V} = - \frac{dV_o}{d\omega} \qquad (40)$$

The replacement of V by V_o, as shown, is concomittant with the assumption $RE_{g\gamma\gamma} = 0$ of our first iterative step. That is, for a given frequency, the associated airspeed is not exactly V, but V_o, because V_o exactly satisfies $RE_{g\gamma\gamma} = 0$. The rate of change of this airspeed with frequency is then compatible with the condition of $dRE_{g\gamma\gamma}/d\omega = 0$ which led to Equation (40). The difference is, of course, trivial for small γ_o and small g_R.

When Equation (40) is substituted into Equation (38) we find, finally,

$$\frac{\gamma_o}{g_R} \cong \frac{1}{2} - \frac{V_o/\omega}{dV_o/d\omega} \qquad (41)$$

which is an equation which does not require any numerical knowledge of the differential equation of the system. It requires only numerical knowledge of the "base curve" of the system (a plot of ω versus V_o) or, what is practically identical, a plot of the frequency-airspeed relationship for the g_R condition so long as g_R is small. Thus, in the region in which we plan to apply our prediction of γ_o from g_R, we need only inspect a frequency-airspeed plot prepared from conventional AMC data; we do not need any knowledge of the configuration of the system.

The ratio $(V_o/\omega)/(dV_o/d\omega)$ is a slope ratio, since V_o/ω is the slope of a line from the origin to a point (V_o, ω), and $dV_o/d\omega$ is the slope of the base curve at the point (V_o, ω). The same quantity also may be called the airspeed sensitivity of the mode frequency, since if ω is constant with airspeed then $dV_o/d\omega$ is infinite and the slope ratio is zero, while if the mode frequency ω varies rapidly with V_o then $dV_o/d\omega$ is small and the slope ratio is large. Indeed, if we followed this logic we would write the slope ratio as

$$\frac{d\omega}{\omega} \Big/ \frac{dV_o}{V_o}$$

to show that it is the rate of change of the relative frequency with the relative airspeed which affects γ_o/g_R. However, the form shown in Equation (41) is usually more convenient, being a ratio of two graphically or numerically evaluated slopes.

It is possible to program Equation (41) into an AMC-type digital solution, and we have done this in some of our investigative work. However, Equation (41) is such a simple equation that automatic programming is not necessary, and a rapid reinterpretation of a given AMC output is possible. In neither case do we abandon the AMC data (despite the fact that when we first started to study this problem we were quite prepared to abandon the AMC approach in favor of a better one). We have found such back-tracking to be unnecessary. The AMC output is both valid and useful, if it is properly interpreted; our equations and concepts are the missing link, and are valid even for highly coupled systems of many degrees of freedom.

This completes the present report, except for the Appendix, which shows the effect of retaining $pRE_{g\gamma\gamma}$ in $mRE_{g\gamma\gamma}$. As we have mentioned, the pure and rigorous theory of our present report will be supplemented by a later, how-to-do-it report.

REFERENCES

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APPENDIX

As a supplement to the body of this report, we will show now the effect of retaining the previously neglected term $pRE_{g\zeta\gamma}$ of Equation (34), so that instead of Equation (37) we have

$$mRE_{g\zeta\gamma} \cong \left(p - V\frac{\partial}{\partial V} - \frac{\omega}{2}\frac{\partial}{\partial \omega}\right) RE_{g\zeta\gamma} \tag{A-1}$$

and, instead of Equation (38), we have

$$\frac{\gamma_o}{g_R} = \frac{1}{2} - \frac{(p - V\partial / \partial V)RE_{g\zeta\gamma}}{\omega \partial RE_{g\zeta\gamma} / \partial \omega} \tag{A-2}$$

In order to obviate the seeming need for detailed knowledge of the differential equation, we now need to define algebraically the distinction between the base-curve airspeed, V_o, and the general airspeed V.

The airspeed dependencies of the polynomial $E_{g\zeta\gamma}$ may be defined by

$$E_{g\zeta\gamma} = E_{Vg\zeta\gamma} + VF_1 + V^2F_2 + V^3F_3 + \ldots.$$

$$= E_{Vg\zeta\gamma} + \sum V^qF_q \tag{A-3}$$

where $E_{Vg\zeta\gamma}$ is the zero-airspeed value of $E_{g\zeta\gamma}$, and each "airspeed coefficient" F_q is the coefficient of V^q, one of the powers of airspeed in the characteristic equation. If we were to define F_q from Equation (32), for instance, we would have

$$F_q = \sum_{11}^{pp} \sum_{m=o}^{p-q} \sum_{r=o}^{p-q-m} (-M\,\omega^2)^{p-m-q-r} \quad (jB\,\omega)^r \; Y^q \; K^m \tag{A-4}$$

but we do not need this for our present derivation.

Since we already have defined V_o as the airspeed which satisfies $RE_{g\zeta\gamma} = 0$, we can now set $V = V_o$ and the real part of Equation (A-3) equal to zero, giving

$$RE_{Vg\zeta\gamma} = -\sum V_o{}^q \, F^q \tag{A-5}$$

This equation defines V_o. Since $E_{Vg\zeta\gamma}$ is the part of $E_{g\zeta\gamma}$ which does not depend upon airspeed, we see that V_o is an expression of the airspeed-independent variation of $E_{g\zeta\gamma}$. Of course, V_o does definitely depend upon frequency.

If we evaluate $RE_{g\zeta\gamma}$ when it is not zero, that is, when the value of V differs from V_o, then we find

$$RE_{g\zeta\gamma} = \sum (V^q - V_o^{\,q})\,F_q \tag{A-6}$$

This was obtained by substituting (A-5) into (A-3) and collecting terms with like airspeed coefficients. It does, of course, reach zero for $V = V_o$, whereupon $pRE_{g\zeta\gamma}$ can be neglected in (A-2). But for $V \neq V_o$, Equation (A-2) now becomes

$$\frac{\gamma_o}{g_R} \cong \frac{1}{2} - \frac{(p - V\partial/\partial V)\sum(V^q - V_o^{\,q})F_q}{(\omega\partial/\partial\omega)\sum(V^q - V_o^{\,q})F_q} \tag{A-7}$$

The airspeed derivative of any given term of the indicated summation will be

$$\frac{\partial}{\partial V}(V^q - V_o^{\,q})F_q = qV^{q-1}F_q \tag{A-8}$$

since we know V_o and F_q are airspeed independent. The frequency derivative of the same term will be

$$\frac{\partial}{\partial V}(V^q - V_o^{\,q})F_q = -F_q\frac{\partial V_o^{\,q}}{\partial\omega} + (V^q - V_o^{\,q})\frac{\partial F_q}{\partial\omega}$$

$$= -q\,V_o^{\,q-1}F_q\frac{dV_o}{d\omega} + (V^q - V_o^{\,q})\frac{dF_q}{d\omega} \tag{A-9}$$

where full frequency derivatives are now appropriate, since V_o and F_q are airspeed independent.

Then, Equation (A-7) becomes

$$\frac{\gamma_o}{g_R} = \frac{1}{2} + \frac{\sum\left[p(V^q - V_o^{\,q}) - qV^q\right]F_q}{\frac{\omega}{V_o}\frac{dV_o}{d\omega}\sum q\,V_o^{\,q}F_q - \omega\sum(V^q - V_o^{\,q})\frac{dF_q}{d\omega}} \tag{A-10}$$

This is a useful equation for the $V \neq V_o$ condition if the expressions for F_q are known, which requires only partial knowledge of the original differential equation. Furthermore, there are several progressive simplifications of Equation (A-10), for special circumstances.

If the products of $V^q - V_o^q$ and $dF_q/d\omega$ are small, then even for $V \neq V_o$ the second term of the denominator can be neglected. This will be assumed to be true throughout the remaining discussion.

If one F_q predominates over all others, then there is only one term in each summation of Equation (A-10). As a result, that predominant F_q will cancel top and bottom, leaving

$$\frac{\gamma_o}{g_R} \cong \frac{1}{2} + \frac{p(V^q - V_o^q) - qV^q}{\frac{\omega}{V_o}\frac{dV_o}{d\omega} q V_o^q}$$

or, by rearranging,

$$\frac{\gamma_o}{g_R} \cong \frac{1}{2} - \frac{V_o/\omega}{dV_o/d\omega}\left[\frac{V^q}{V_o^q} - \frac{p}{q}\left(\frac{V^q}{V_o^q} - 1\right)\right] \qquad \text{(A-11)}$$

Only the predominant value of q (the subscript of the predominant F_q) would be used in Equation (A-11). If that predominant q happened to be unity, then (A-11) would become

$$\frac{\gamma_o}{g_R} \cong \frac{1}{2} - \frac{V - p(V - V_o)}{\omega dV_o/d\omega}$$

or, by rearranging,

$$\frac{\gamma_o}{g_R} \cong \frac{1}{2} - \frac{V_o - (p - 1)(V - V_o)}{\omega dV_o/d\omega} \qquad \text{(A-12)}$$

Thus, the term $(p - 1)(V - V_o)$ acts as a correction for the V_o numerator which would have been used in the $V \cong V_o$ region. If we had a binary system ($p = 2$), then (A-12) would become

$$\frac{\gamma_o}{g_R} \cong \frac{1}{2} - \frac{V_o - (V - V_o)}{\omega dV_o/d\omega} \qquad \text{(A-13)}$$

This corresponds to Equation B-13 of Reference 4, for the pitch-plunge wing. Similarly, if we had a ternary system (p = 3), then Equation (A-12) would become

$$\frac{\gamma_o}{g_R} = \frac{1}{2} - \frac{V_o - 2(V - V_o)}{\omega dV_o/d\omega} \qquad \text{(A-14)}$$

This applies to the ternary (h_1, h_2, α) wing of Reference 4 (see discussion following Equation (B-25) thereof).

If the predominant q is more than unity but less than p, we must resort to Equation (A-11). The ternary panel of Reference 4 was such a case, a p = 3, q = 2 case (see discussion following Equation (B-16) thereof).

If the predominant q is equal to p, as for the simply-supported binary panel of Reference 4 and of our present text, then Equation (A-11) becomes simply

$$\frac{\gamma_o}{g_R} = \frac{1}{2} - \frac{V_o/\omega}{dV_o/d\omega} \qquad \text{(A-15)}$$

which is the same as Equation (41). However, the reasoning behind Equation (A-15) is different from that behind (41). In Equation (41) we used $V = V_o$, so that the first two terms of $p(V^q - V_o^q) - qV_o^q$ would drop out even for $p \neq q$. But in Equation (A-15) we used $p = q$, so that the first and third terms of the same $p(V^q - V_o^q) - qV_o^q$ would drop out even for $V \neq V_o$. Thus, Equation (A-15) represents the sole special case in which omission of $pRE_g\gamma$ from $mRE_g\gamma$ is of no consequence even for $V \neq V_o$.

If there is no one predominant Fq, then we must utilize Equation (A-10) with or without the second denominator term. The uniform cantilever wing with control (h, α, β) of Reference 4 is, at least algebraically, such a case, as both q = 1 and q = 2 (that is, V and V^2) terms are present in its characteristic equation. (See discussion following Equation (B-32) thereof). Preferably, one of the two airspeed coefficients (F_1 and F_2) could be neglected on a numerical basis, thereby avoiding this extra complexity.

For systems having even more than two possible airspeed coefficients, Equation (A-10) still applies. Preferably, all but one of these could be neglected on a numerical basis, again avoiding the extra complexity of Equation (A-10). For these more complex systems, Equations (A-11) through (A-14) may still provide adequate approximations for the effect of $V \neq V_o$ upon the transfer function γ_o/g_R. In any given instance, some one of these simpler expressions may be found to be adequate for our purposes; by providing this set of logical choices, (A-11) through (A-14), such results may be reached intelligently rather than groped for quite blindly.

But any utilization of Equations (A-10) through (A-14) presumes one fact: that airspeed-frequency combinations differing appreciably from those of the base curve are of interest. For the most part, this is not the case; roughly 90% of the decay-coefficient information of interest to the analyst can be obtained using analysis restricted to the $V = V_o$ assumption. After all, the flutter speed itself can only occur within the base-curve region, as the flutter condition $g_R = 0 = \gamma_o$ does designate exactly $V = V_o$. It follows that the γ_o/g_R given by Equation (41) must be valid for some distance each side of the flutter speed, as long as V remains close to V_o. This is a region of great practical utility, as it will indicate the rate of onset of flutter in terms of decay coefficient and airspeed.

At airspeeds further above the flutter speed, practical interest diminishes rapidly, as the structure would not survive tests at such airspeeds. Thus, even where V does depart from V_o in this region, the application of Equations (A-10) through (A-14) would be of largely academic interest.

At airspeeds below the flutter speed, better use of the equations of this appendix is possible. A region of large $V-V_o$ will occur below the flutter speed whenever the flutter speed is near the nose of the base curve but the large g_R departures are not near the nose. (This means that loop-back occurs in the unstable branch). Then a region of large $V-V_o$ does occur at practical airspeeds, whereupon Equations (A-10) through (A-14) do apply and may yield meaningful values of γ_o from seemingly meaningless values of g_R. It is in such regions that one can expect to profit from the results found in this appendix.

One should not assume that these equations yield an indefinite extension of the γ_o/g_R transfer function. We have merely removed the assumption of $V = V_o$ from Equation (41), not the assumption of small powers and products of g_R and γ_o. Fortunately, in our region of interest, such effects may be either trivial compared with, or empirically reproducible by means of, our allowance for $V \neq V_o$. Thus, this appendix does provide a tool which penetrates the $V \neq V_o$ barrier.

Another function of this appendix is to firmly verify Equation (41) for the condition $V \cong V_o$. When we assume $V = V_o$ in Equation (A-10), the first numerator term and the second denominator term drop out. Then the remaining series cancels out regardless of F_q, whereupon Equation (A-10) reduces exactly to Equation (41) of the text.

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