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AERODYNAMICS OF DEFLECTED SLIPSTREAMS

PART I

Formulation of the Integral Equations

Project 9-38-01-000, ST902

Contract DA44-177-TC-439

December 1960

prepared by :

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AERODYNAMICS OF DEFLECTED SLIPSTREAMS
PART I
FORMULATION OF THE INTEGRAL EQUATIONS

OCTOBER 1961

PROJECT 9-38-01-000, ST902
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U.S. ARMY TRANSPORTATION RESEARCH COMMAND
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FORT EUSTIS, VIRGINIA

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FOREWORD

The research on which this report is based was performed by the author at the Cornell Aeronautical Laboratory, Inc., Buffalo, New York, under Army Contract DA 44-177-TC-439, Project Number 9-38-01-000, ST 902. The Transportation Corps, U. S. Army Transportation Research Command, Fort Eustis, Virginia, is the monitoring agency. This report represents part of a research program, which is devoted to the investigation of several specific problems associated with STOL/VTOL flight.

LIST OF SYMBOLS

$\mathbf{v}$ a vector field; disturbance velocity field

$\mathbf{v}_1$ lamellar component field of $\mathbf{v}$

$\mathbf{v}_2$ solenoidal component field of $\mathbf{v}$

ϕ scalar potential of $\mathbf{v}_1$

M, N functions defined with Eq. (6)

$d\tau$ volume element

$d\sigma$ surface element

$\mathbf{r}$ position vector of ϕ

$\mathbf{r}'$ position vector of $d\tau$

$\bar{\mathbf{r}}$ vector distance from $\mathbf{r}$ to $\mathbf{r}'$ $(\bar{\mathbf{r}} = \mathbf{r} - \mathbf{r}')$

x, y, z Cartesian coordinates of point $\mathbf{r}$

ξ, η, ζ Cartesian coordinates of point $\mathbf{r}'$

R radius of large sphere on which $\mathbf{v}$ vanishes

ϵ radius of small sphere in which ϕ is not analytic

S_1, S_2 surfaces enclosing all points of field $\mathbf{v}$ where $\nabla \cdot \mathbf{v}$ and $\nabla \times \mathbf{v}$ are not zero, respectively

R_1 radius of smallest sphere enclosing S_1

$\mathbf{A}$ vector potential of $\mathbf{v}_2$

$\mathbf{V}_\infty$ uniform free-stream velocity vector

$\mathbf{V}$ velocity vector $(\mathbf{V} = \mathbf{V}_\infty + \mathbf{v})$

p pressure

ρ density

LIST OF SYMBOLS (CONT.)

$\mathbf{F}$ body force vector (per unit volume)

$\boldsymbol{\Omega}$ rotation vector $(\boldsymbol{\Omega} = \nabla \times \mathbf{v})$

$d\mathbf{s}$ vector element of streamline

$\boldsymbol{\Omega}_T$ component of $\boldsymbol{\Omega}$ along streamline

$\boldsymbol{\Omega}_N$ component of $\boldsymbol{\Omega}$ normal to streamline

$\mathbf{n}$ unit vector in direction of $\boldsymbol{\Omega}_N$

n absolute value of $\mathbf{n}$

C_1 First Curvature or flexure of streamlines in jet rotational layer

$\mathbf{k}$ First radius of curvature; reciprocal of C_1

$\mathbf{c}$ unit vector in direction of $\mathbf{k}$, a perpendicular to vector streamline element $d\mathbf{s}$

S wing surface as given by equation $S(x, y, z) = 0$

$\mathbf{r}_W$ position vector of points on wing surface

$\bar{\mathbf{r}}_W$ vector distance from $\mathbf{r}'$ to $\mathbf{r}_W$ $(\mathbf{r}'_W = (\mathbf{r}_W - \mathbf{r}')$

$\boldsymbol{\Omega}_W$ rotation distribution in wing surface region

$\boldsymbol{\Omega}_{WW}$ rotation distribution in wing wake region

$\boldsymbol{\Omega}_J$ rotation distribution in jet boundary region

$f'(x)$ local slope of wing surface

c wing chord

b wing span

γ_x vorticity of the x-component of rotation; trailing vorticity in the wing wake and on the jet boundary

LIST OF SYMBOLS (CONT.)

γ_y vorticity of the y-component of rotation; wing bound vorticity

γ_z vorticity of the z-component of rotation

ξ, r', φ cylindrical coordinates of point P'

$(\eta = r' \cos \varphi \, , \, \zeta = r' \sin \varphi)$

x, r, ϑ cylindrical coordinates of point P

$(y = r \cos \vartheta, \; z = r \sin \vartheta)$

R_0 jet radius

γ_ϑ vorticity of the ϑ component of rotation; that component of vorticity on the jet boundary

γ_{ϑ_0} jet vorticity far upstream

V_{J0} disturbance speed due to jet vorticity far upstream, $\left(V_{J0} = \gamma_{\vartheta_0}\right)$

γ_T component of the jet vorticity tangent to the local streamline element

$\overline{\frac{dy}{dx}}, \overline{\frac{dz}{dx}}$ mean values across the jet rotational layer of the streamline deflections

v_w, w_w disturbance velocities due to wing vortex system

v_J, w_J disturbance velocities due to jet vortex system

V_J jet velocity $V_J = V_\infty + V_{J0}$

$\tilde{b}^2$ $(\tilde{b}^2 = R_0^2 - 2 R_0 \eta \cos \vartheta + \eta^2)$

I. INTRODUCTION

There is at present considerable interest in the aerodynamic performance of the vectored slipstream type of STOL aircraft. To a large degree the performance of this arrangement depends on the aerodynamic interaction of two systems, the wing and the propeller, and as such it constitutes a formidable problem in flow analysis. In the interest of getting wing data, one simplifying approach consists of neglecting the propeller, per se, and substituting for its system a jet or slipstream of indefinite extent. The analysis is thereby reduced to one of an isolated wing in a nonuniform or rotational flow.

It is usual in this kind of analysis to effect a further important simplification by specifying an ideal jet and postulating, on the basis of small disturbances, that the cross-sectional shape will remain essentially unchanged. Under such conditions the problem is related in some respects to that of a wing in an open throat wind tunnel - - a scheme that has been examined extensively by the method of images.

The latter approach has been applied to the wing-jet problem by a number of investigators, whose contributions can be conveniently grouped according to the parameter, jet to main-stream velocity ratio. For values of this parameter close to unity, the problem was investigated by Koning[1], Franke and Weinig[2], Ruden[3], Bausch[4], Kyewski and Vandrey[5], Vandrey[6], etc. For large values we have the developments of Glauert[7], Stuper[8], Pistolesi[9], Baranoff[10], and others. The more recent contributions, namely those of Rethorst[11], and Ribner[12], seem to apply for all values of the jet-velocity ratio as does the work of Graham et al[13].

The present study was initiated to investigate the effects of jet curvature (are investigated) on the premise that at higher angles of flow deflection, this factor should have an important influence on wing lift. The jet boundary is ~~here~~ regarded as a vortical layer, as is also the wing surface and its wake. The jet vorticity distribution is determined subject to the dynamic equations of rotational flow which are applied in a thin region at the jet boundary. This approach leads to the consideration of jet trailing vorticity or ~~the so-called~~ "secondary vorticity" which occurs, as shown by Squire and Winter[14] and also by Hawthorne[15], in certain curved nonuniform flows.

II. INTEGRAL REPRESENTATION OF THE GENERAL VELOCITY FIELD

To show clearly the present approach, it will be convenient, first, to recall certain general properties of vector fields, and how they relate to steady-flow fields of an incompressible, inviscid fluid when the flow field includes free vorticity. We begin with the following theorem which is proved in Ref. 16. Any continuous vector field, $\mathbf{v}$, defined everywhere in space and vanishing at infinity together with its first derivatives, can be represented as the sum of an irrotational vector field, $\mathbf{v}_1$, and a solenoidal field, $\mathbf{v}_2$. The representation is unique except for a vectorial constant. Thus,

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 \tag{1}$$

where

$$\nabla \times \mathbf{v}_1 = 0 \tag{2}$$

and

$$\nabla \cdot \mathbf{v}_2 = 0 \tag{3}$$

The application of this important theorem to the study of wing-jet flow fields is not precisely correct. The steady-flow field about any body which sustains a resultant force due to fluid pressure will include a vortical wake, which, along with the velocity field, extends indefinitely if the fluid is inviscid. Such flow fields apparently do not meet the requirement imposed in the theorem, namely, that the vector vanish at infinity. Nevertheless, an important insight into flow problems involving free vorticity, as is characteristic of jet flows, can be attained from the following considerations, based on the more restricted vector fields for which the theorem strictly applies.

Lamellar Vector Fields

It follows from Eq. (2) that v_1 can be represented by a scalar potential, ϕ such that

$$v_1 = -\nabla\phi + \text{const. vector} \tag{4}$$

and hence ϕ satisfies the inhomogeneous potential equation

$$\nabla^2\phi = -\nabla\cdot v_1 = -\nabla\cdot v \tag{5}$$

Consider Green's Second Formula

$$\int (M\nabla^2 N - N\nabla^2 M)\, d\tau = \int \left(M\frac{\partial N}{\partial n} - N\frac{\partial M}{\partial n}\right) d\sigma \tag{6}$$

where M and N are two scalar point functions, continuous in the first partial derivatives and possessing second partial derivatives, and n is the outward normal seen from the region of integration. Let

$$N = \frac{1}{r} \tag{7}$$

where

$$r = |\bar{r}| = |r - r'|$$

In rectangular coordinates

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$$

where x , y , z are the coordinates of point r , and ξ , η , ζ are the coordinates of point r' . So defined, N is harmonic, i. e., satisfies

$$\nabla^2 N = 0$$

everywhere except at $r = 0$ and hence, except at this point, satisfies all the necessary differentiability conditions everywhere. The volume integral in Eq. (6)

is taken in the r' system over the region enclosed between a very large sphere of radius R and a very small sphere of radius ϵ, with centers at the point $r = 0$. Now let M be identified as ϕ of Eq. (5) subject to all the necessary differentiability conditions.

Equation (6) then becomes, on the basis of Eqs. (5) and (7),

$$\int_R \frac{1}{r}(\nabla \cdot v)\,d\tau = \int_R \left[\phi \frac{\partial(\frac{1}{r})}{\partial r} - \frac{1}{r}\frac{\partial \phi}{\partial r}\right] d\sigma - \int_\epsilon \left[\phi \frac{\partial \frac{1}{r}}{\partial r} - \frac{1}{r}\frac{\partial \phi}{\partial r}\right] d\sigma \tag{8}$$

The second integral on the right of Eq. (8) reduces to $4\pi\phi$ by virtue of the assumed continuity of ϕ. Hence, Eq. (8) can be written

$$\phi = \frac{1}{4\pi}\int_R \frac{1}{r}(\nabla \cdot v)\,d\tau - \frac{1}{4\pi}\int_R \left[\phi \frac{\partial \frac{1}{r}}{\partial r} - \frac{1}{r}\frac{\partial \phi}{\partial r}\right] d\sigma \tag{9}$$

Equation (9) is an expression for the potential at points within R in terms of the divergence distribution and the values of both ϕ and its normal derivative on the boundary.

For our purpose we now specify the vector field v to be such that the region, or all the regions, in which $\nabla \cdot v$ is not zero, are finite in extent and can be enclosed within a surface S_1.

In Equation (9) consider R to be large enough so that if R_1 is the radius of the smallest sphere that encloses S_1, then

$$R \gg R_1$$

In the region between the spheres R and R_1.

$$\nabla \cdot v = 0$$

and therefore ϕ in this region satisfies

$$\nabla^2 \phi = 0$$

Since ϕ represents the potential throughout space of a suitably restricted spacial source distribution, it will vanish at least as

$$\frac{1}{R-R_1} \sim \frac{1}{R} \text{ for } R \gg R_1$$

This situation assures that the second integral on the right-hand side of Eq. (9) vanishes for sufficiently large R. It should be noted that the corresponding velocity field also vanishes at infinity. Thus,

$$\phi = \frac{1}{4\pi} \int \frac{1}{r} (\nabla \cdot \mathbf{v}) \, d\tau \tag{10}$$

It is shown in Ref. 17 that ϕ, as given by Eq. (10), and its first derivatives are in fact both defined and continuous everywhere in the field if $\nabla \cdot \mathbf{v}$ is bounded in extent, as was assumed, and is at least piecewise continuous. The higher derivatives are also shown to exist provided the divergence distribution satisfies a Holder condition.

Solenoidal Vector Fields

Let

$$\mathbf{v}_2 = \nabla \times \mathbf{A} + \text{const. vector} \tag{11}$$

with the condition

$$\nabla \cdot \mathbf{A} = 0 \tag{12}$$

By forming the curl of Eq. (11) and applying the vector identity

$$\nabla \times \nabla \times \mathbf{A} = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) \tag{13}$$

for which the extreme right term vanishes by Eq. (12), the following is obtained

$$\nabla^2 \mathbf{A} = -\nabla \times \mathbf{w}_2 = \nabla \times \mathbf{w} \tag{14}$$

Hence, like ϕ the components of $\mathbf{A}$ satisfy the inhomogeneous potential equation.

Again, if the region designated S_2, in which $\nabla \times \mathbf{w}$ is not zero, is finite in extent, then it can be shown on the basis of Green's Second Formula and the harmonicity of the components of $\mathbf{A}$ exterior to S_2 that

$$\mathbf{A} = \frac{1}{4\pi} \int \frac{1}{r} (\nabla \times \mathbf{w})\, d\tau \tag{15}$$

From Eq. (15) one may observe that each vector element of rotation, $\nabla \times \mathbf{w}$, contributes a vector element parallel to $\mathbf{A}$. However, it can be shown, since

$$\nabla \cdot (\nabla \times \mathbf{w}) = 0 \tag{16}$$

that the net effect of all radial components of $\nabla \times \mathbf{w}$ (that is, radial from the position of $\mathbf{A}$) vanish. Hence, more concisely it is the tangential component of rotation-element that contributes a net vector element parallel to $\mathbf{A}$.

Evidently all the conditions concerning the necessary functional continuity of the distribution $\nabla \cdot \mathbf{w}$ apply to the components of $\nabla \times \mathbf{w}$ as well, and in addition the condition imposed by Eq. (16) must also be met. The latter condition implies that the rotation distribution itself must comprise a solenoidal field. Further consideration will be given this fact later.

Equation (1) can be expressed in terms of spacial distributions of the divergence and the rotation on the basis of Eqs. (4), (10), (11), and (15).

$$
\begin{aligned}
v &= -\frac{1}{4\pi}\,\nabla\cdot\int\frac{\nabla\cdot v}{r}\,d\tau + \frac{1}{4\pi}\,\nabla\times\int\frac{\nabla\times v}{r}\,d\tau + \text{vector const.} \qquad (17)\\
&= -\frac{1}{4\pi}\int\frac{\bar{r}(\nabla\cdot v)}{r^{3}}\,d\tau - \frac{1}{4\pi}\int\frac{\bar{r}\times\Omega}{r^{3}}\,d\tau + \text{vector const.}
\end{aligned}
$$

where

$$\Omega = \nabla \times v$$

For any given vector field, v , which satisfies the conditions of the theorem, the distributions $\nabla\cdot v$ and Ω are furthermore unique. (See Ref. 18: Chapter IV; Section 20.)

Solenoidal Flow Fields

In steady-flow problems only certain information concerning the flow field is given; e.g., the conditions far upstream, the geometry of solid boundaries, and the trailing-edge conditions. The flow field must be determined from these data alone, and its establishment involves, in essence, the determination of appropriate distributions of the divergence and the rotation. Very little is known concerning distributions, however, except on surfaces. Equation (17) offers a means by which surface distributions can be established for certain practical problems.

As pertains to steady-flow fields of an inviscid, incompressible fluid, Eq. (17) requires some further consideration in view of the conditions imposed on the vector field in connection with the derivation.

First, the distribution of rotation in the steady inviscid flow field about a lifting body, in contradistinction to the condition imposed in the preceding development, extends indefinitely in the wake. For our purpose, Eq. (15) is presumed to apply generally, provided, of course, the integral exists.

Second, since the fluid is taken to be incompressible, the entity of flow divergence never enters into the actual field of flow. Rotation, on the other hand, will certainly be present in the wake if there exists a resultant force on the closed stream surface of a body. Provided the fluid is incompressible, the former statement permits us to formulate any steady-flow field which possesses a continuous velocity vector (hence one which vanishes on the body surface) on the basis of solenoidal field concepts alone.

On the other hand, if the fluid is also inviscid, in which case the velocity will not vanish at the body surface, then the flow field must be continued analytically into the region within the body surface in order that it meet the requirement of a continuous vector field. Under such conditions it may be observed that the flow field as a whole does not comprise a definite vector field. But this situation is precisely the one which again permits us to treat such flow problems on the basis of purely solenoidal flows by so defining the analytically continued flow field.

Since we will be dealing exclusively with solenoidal flows, it is essential to recall that the rotation distribution is subject to certain basic constraints. First, the divergence of the rotation is everywhere identically zero

$$\nabla \cdot \left(\nabla \times w \right) = 0 \tag{18}$$

Therefore, the rotation vector field itself comprises a solenoidal field. The same can be said of the rotation of the rotation vector, etc. Second, rotation is a convective property of the fluid particles subject to a dynamical requirement which can be derived from Newton's second law. Thus,

$$\frac{D\mathbb{V}}{Dt} = -\frac{1}{\rho}\nabla p + \frac{\mathbb{F}}{\rho} \tag{19}$$

where $\mathbb{V}$ is the flow velocity $(\mathbb{V}_\infty + w)$

p is pressure

$\mathbb{F}$ is extraneous force (per unit volume)

ρ is density

By operating with the curl on Eq. (19) and applying Eq. (18), the following form of the Helmholtz equation is obtained

$$\frac{D\Omega}{Dt} = \frac{\nabla \times \mathbb{F}}{\rho} + (\Omega \cdot \nabla)\mathbb{V}$$

where

$$\Omega = \nabla \times \mathbb{V}$$

For steady conditions the preceding equation becomes

$$(\mathbb{V} \cdot \nabla)\Omega = (\Omega \cdot \nabla)\mathbb{V} + \frac{\mathbb{V} \times \mathbb{F}}{\rho} \tag{20}$$

Equation (20) governs the rotation distribution. A few elementary solutions of this nonlinear system have been determined for the condition of force-free flow: one relates to a circular jet in a parallel stream. Another obvious solution for trailing vorticity is

$$\Omega = const. \cdot \mathbb{V}$$

On the other hand, integration of this equation along an arbitrary streamline leads to a relationship between the normal and tangential component of the rotation vector which will prove to be of considerable value in determining the vorticity distribution in the vortical layer of a deflected jet.

Consider the line integral of Eq. (20) along a streamline whose vector element is designated $d\mathbf{s}$.

$$\int [(\mathbb{V}\cdot\nabla)\boldsymbol{\Omega}]\cdot d\mathbf{s} = \int [(\boldsymbol{\Omega}\cdot\nabla)\mathbb{V}]\cdot d\mathbf{s} + \int \frac{\nabla\times\mathbb{F}}{\rho}\cdot d\mathbf{s} \tag{21}$$

Since $\mathbb{V}$ is parallel to $d\mathbf{s}$, the first term on the left may be rearranged as follows

$$\begin{aligned}\int [(\mathbb{V}\cdot\nabla)\boldsymbol{\Omega}]\cdot d\mathbf{s} &= \int \left[V\left(\frac{d\mathbf{s}}{ds}\cdot\nabla\right)\boldsymbol{\Omega}\right]\cdot d\mathbf{s}\\ &= \int V\frac{d\boldsymbol{\Omega}}{ds}\cdot d\mathbf{s}\\ &= \int V\frac{d\boldsymbol{\Omega}_T}{ds}\cdot d\mathbf{s} + \int V\frac{d\boldsymbol{\Omega}_N}{ds}\cdot d\mathbf{s}\\ &= \int V\,d\Omega_T - \int V C_1\left(\boldsymbol{\Omega}_N\cdot\mathbb{C}\right)ds\end{aligned}$$

where $\boldsymbol{\Omega}_T$ and $\boldsymbol{\Omega}_N$ are respectively the tangential and normal components of $\boldsymbol{\Omega}$ referred to the direction of the streamline element, $d\mathbf{s}$. The operation involving the vector derivatives along the streamline, which lead to the last expression, is carried out with the aid of Frenet's formulas (see Ref. 19). Here, C_1, is the First Curvature or Flexure of the streamline; its reciprocal is equal to the first radius of curvature, k . The vector $\mathbb{c}$ is a unit vector perpendicular to the streamline element ds and in the direction of k .

The first term on the right-hand side of Eq. (21) can be developed as follows:

$$\int [(\boldsymbol{\Omega} \cdot \nabla)\mathbb{V}] \cdot ds = \int [(\boldsymbol{\Omega}_T \cdot \nabla)\mathbb{V}] \cdot ds + \int [(\boldsymbol{\Omega}_N \cdot \nabla)\mathbb{V}] \cdot ds$$

$$= \int \Omega_T \, dV + \int \left[\Omega_N \left(\frac{d\boldsymbol{n}}{dn} \cdot \nabla\right)\mathbb{V}\right] \cdot ds$$

$$= \int \Omega_T \, dV + \int \Omega_N \frac{d\mathbb{V}}{dn} \cdot ds$$

$$= \int \Omega_T \, dV + \int \Omega_N \frac{dV}{dn} \, ds$$

where $d\boldsymbol{n}/dn$ is the unit vector in the direction $\boldsymbol{\Omega}_N$.

The preceding developments substituted into Eq. (21) give, when $\mathbb{F}$ is taken to be zero

$$\int V \, d\Omega_T - \int V C_1 \Omega_N (\boldsymbol{n} \cdot \mathbb{c}) \, ds = \int \Omega_T \, dV + \int \Omega_N \frac{dV}{dn} \, ds$$

or

$$\int d\left(\frac{\Omega_T}{V}\right) = \int \frac{\Omega_N}{V^2} \frac{dV}{dn} \, ds + \int \frac{V C_1 \Omega_N}{V^2} (\boldsymbol{n} \cdot \mathbb{c}) \, ds$$

Therefore

$$\frac{\Omega_T}{V} = \int \frac{\Omega_N}{V^2} \left[\frac{dV}{dn} + C_1 V (\boldsymbol{n} \cdot \mathbb{c})\right] ds + const. \tag{22}$$

Equation (22) is an expression for the tangential component of the rotation along a streamline, in terms of the integral along the streamline of the normal component of rotation. The tangential rotation, Ω_T , includes both the initial value of this component and the accumulation of the "secondary" rotation which is known to occur in certain curved nonuniform flows. The latter arises when a fluid element, which has a component of rotation normal to its projected streamline, moves along a streamline whose geodesic curvature on the Bernoulli surface is other than zero.

III. DEVELOPMENT OF THE WING-JET LIFTING SURFACE EQUATIONS

We now apply the preceding developments to the formulation of the wing-jet problem for a uniform, circular jet under the conditions of small deflections. The integral relationship for the disturbance flow field is given by the solenoidal part of Eq. (17)

$$v = -\frac{1}{4\pi} \int \frac{\bar{r} \times \Omega}{\bar{r}^3} \, d\tau \tag{23}$$

Let $S = 0$ define the wing surface. The unit vector, normal to the surface, will then be

$$\frac{\nabla S}{|\nabla S|}$$

and the condition that $S = 0$ be a stream surface is

$$\left(\mathbb{V}_\infty + v\right) \cdot \frac{\nabla S}{|\nabla S|} = 0 \tag{24}$$

where $\mathbb{V}_\infty$ is the uniform free-stream velocity.

The condition at the wing surface is, therefore,

$$\mathbb{V}_\infty \cdot \nabla S - \nabla S \cdot \frac{1}{4\pi} \int \frac{\bar{r}_W \times \Omega}{\bar{r}_W^3} = 0 \tag{25}$$

where

$$\bar{r}_W = r_W - r'$$

and r_W are points on the wing surface. Equation (25) can be arranged so that the integral is taken over three regions where the rotation is known to exist; the wing region, the wing wake region, and the jet boundary region. Thus

$$4\pi \mathbb{V}_\infty \cdot \nabla S = \nabla S \cdot \int_W \frac{\bar{r}_W \times \Omega_W}{r_W^3}\, d\tau + \nabla S \cdot \int_{WW} \frac{\bar{r}_W \times \Omega_{WW}}{r_W^3}\, d\tau$$

$$+ \nabla S \cdot \int_J \frac{\bar{r}_W \times \Omega_J}{r_W^3}\, d\tau \tag{26}$$

Equation (26) may be developed into a more useful form on the basis that all regions of rotation are surface-like and hence can be represented by vortex layers. For this purpose, it is convenient to introduce a Cartesian coordinate system with x in the direction of $\mathbb{V}_\infty$, y along the wing span, and z vertical, as shown in Fig. 1.

The wing is considered of arbitrary planform, but without spanwise twist. In the range

$$-\frac{c}{2} \leq x \leq \frac{c}{2} \qquad\qquad -\frac{b}{2} \leq y^* \leq \frac{b}{2}$$

let its surface be represented by

$$S = 0 = z + f(x) \tag{27}$$

Then

$$\frac{\partial S}{\partial x} = f'(x), \quad \frac{\partial S}{\partial y} = 0, \quad \frac{\partial S}{\partial x} = 1 \tag{28}$$

With Eq. (28), the term on the left-hand side of Eq. (26) becomes

$$4\pi \mathbb{V}_\infty \cdot \nabla S = 4\pi V_\infty f'(x) \tag{29}$$

and each of the terms on the right-hand side can be expressed in the following form

$$\nabla S \cdot \int \frac{\bar{\pi}_W \times \Omega}{\bar{\pi}_W^3} d\tau = f'(x) \cdot \iiint \left\{ \frac{(y-\eta)\Omega_z - (z-\zeta)\Omega_y}{[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{3/2}} \right\} d\xi\, d\eta\, d\zeta$$

$$+ \iiint \left\{ \frac{(x-\xi)\Omega_y - (y-\eta)\Omega_x}{[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{3/2}} \right\} d\xi\, d\eta\, d\zeta \tag{30}$$

If the local surface inclination, $f'(x)$, is everywhere small, then the wing and wing wake terms can be simplified and combined to give the usual surface integral relationship for downwash from lifting surface theory. Thus, for the wing and wing wake terms, the first triple integral term on the right-hand side of Eq. (30) can be neglected in comparison with the second, the individaul terms being of higher order in $|f'(x)|_{max}$ than those of the second triple integral.

Therefore, the first two terms on the right-hand side of Eq. (26) can be represented approximately as follows

$$\nabla S \cdot \int_W \frac{\bar{\pi}_W \times \Omega}{\bar{\pi}_W^3} d\tau + \nabla S \cdot \int_{WW} \frac{\pi_W \times \Omega_{WW}}{\pi_W^3} d\tau$$

$$= \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{c}{2}}^{\frac{c}{2}} \left\{ \frac{(x-\xi)\gamma_y - (y-\eta)\gamma_x}{[(x-\xi)^2 + (y-\eta)^2]^{3/2}} \right\} d\xi\, d\eta \tag{31}$$

$$- \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{+\frac{c}{2}}^{\infty} \frac{(y-\eta)\gamma_x}{[(x-\xi)^2 + (y-\eta)^2]} d\xi\, d\eta$$

where

$$\gamma_y = \int \Omega_y\, d\zeta$$

$$\gamma_x = \int \Omega_x\, d\zeta$$

The terms on the right-hand side of Eq. (31) can be rearranged into the following form

$$\int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{c}{2}}^{\frac{c}{2}} \frac{(x-\xi)\gamma_y \, d\xi \, d\eta}{\left[(x-\xi)^2+(y-\eta)^2\right]^{3/2}} - \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{c}{2}}^{\infty} \frac{(y-\eta)\gamma_x \, d\xi \, d\eta}{\left[(x-\xi)^2+(y-\eta)^2\right]^{3/2}}$$

Partial differentiation of the first term above with respect to η and the second term with respect to ξ, together with the divergence relationship

$$\frac{\partial \gamma_x}{\partial \xi} = - \frac{\partial \gamma_y}{\partial \eta} \tag{32}$$

yields the following expression for Eq. (31)

$$\nabla S \cdot \int_W \frac{\bar{\pi}_W \times \Omega_W}{\bar{\pi}_W^3} \, d\tau + \nabla S \cdot \int_{WW} \frac{\bar{\pi}_W \times \Omega_{WW}}{\bar{\pi}_W^3} \, d\tau$$

$$= \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{c}{2}}^{\frac{c}{2}} \left\{ 1 + \frac{\sqrt{(x-\xi)^2+(y-\eta)^2}}{x-\xi} \right\} \left(\frac{1}{y-\eta}\right) \frac{\partial \gamma_y}{\partial \eta} \, d\xi \, d\eta \tag{33}$$

$$+ \int_{-\frac{c}{2}}^{\frac{c}{2}} \left\{ \frac{\left[\gamma_y(-R_o+\epsilon) - \gamma_y(-R_o-\epsilon)\right](y+R_o)}{(x-\xi)\sqrt{(x-\xi)^2+(y+R_o)^2}} + \frac{\left[\gamma_y(R_o+\epsilon) - \gamma_y(R_o-\epsilon)\right](y-R_o)}{(x-\xi)\sqrt{(x-\xi)^2+(y-R_o)^2}} \right\} d\xi$$

It should be stated concerning Eq. (31) that we anticipate a discontinuity in γ_y at the wing-jet boundary. Equation (32) therefore does not apply at such points and Eq. (33) is developed in that light.

Consider now the third term on the right-hand side of Eq. (26). Its form as given by Eq. (30) can be expressed in cylindrical coordinates as follows, with the x-axis of the two systems coinciding.

Let

$$
\begin{aligned}
y &= r \cos \vartheta \qquad & \eta &= r' \cos \varphi \\
z &= r \sin \vartheta \qquad & \zeta &= r' \sin \varphi
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
\Omega_z &= \Omega_\vartheta \cos \varphi + \Omega_r \sin \varphi \\
\Omega_y &= -\Omega_\vartheta \sin \varphi + \Omega_r \cos \varphi
\end{aligned}
$$

and therefore

$$
\nabla S \cdot \int \frac{\bar{r}_W \times \Omega}{\bar{r}_W^{3}} \, d\tau = f'(x) \iiint \frac{\left[\left(r \cos(\vartheta - \varphi) - r'\right) \Omega_\vartheta - r \sin(\vartheta - \varphi) \Omega_r\right] r' \, dr' \, d\varphi \, d\xi}{\left[(x-\xi)^2 + r^2 + (r')^2 - 2 r r' \cos(\vartheta - \varphi)\right]^{3/2}}
$$

$$
+ \iiint \frac{\left[-(x-\xi)\left(\Omega_\vartheta \sin \varphi - \Omega_r \cos \varphi\right) - r\left(\cos \vartheta - r' \cos \varphi\right)\Omega_x\right]}{\left[(x-\xi)^2 + r^2 + (r')^2 - 2 r r' \cos(\vartheta - \varphi)\right]^{3/2}} \, r' \, dr' \, d\varphi \, d\xi \tag{35}
$$

Equation (35) will be simplified under the conditions that the jet is initially circular and undergoes only small deflection and cross-sectional distortion. The jet rotation, Ω_J, will in that case be essentially in the region r' equals a constant, R_0, and terms containing Ω_r can be neglected. We may write

$$
\begin{aligned}
\gamma_\vartheta &= \int \Omega_\vartheta \, dr' \\
\gamma_x &= \int \Omega_x \, dr'
\end{aligned}
\tag{36}
$$

On this basis, the expression, for the downwash on the wing surface, due to the jet vorticity, becomes

$$\nabla S \cdot \int \frac{\bar{r}_W \times \Omega_J}{\bar{r}_W^3} \, d\tau = R_0 f'(x) \int_{-\infty}^{+\infty}\int_0^{2\pi} \frac{(y\cos\varphi - R_0)\gamma_\vartheta \, d\varphi \, d\xi}{[(x-\xi)^2 + y^2 + R_0^2 - 2yR_0\cos\varphi]^{3/2}}$$

$$+ R_0 \int_{-\infty}^{+\infty}\int_0^{2\pi} \frac{[(x-\xi)(-\gamma_\vartheta \sin\varphi) - (y - R_0\cos\varphi)\gamma_x]}{[(x-\xi)^2 + y^2 + R_0^2 - 2yR_0\cos\varphi]^{3/2}} \, d\varphi \, d\xi \tag{37}$$

where the wing surface is taken to lie in the plane $\vartheta = 0$.

To a first approximation, γ_ϑ, in the first term on the right-hand side of Eq. (37) may be taken to be the constant value of jet vorticity far upstream, γ_{ϑ_0}, in which case the expression can be integrated at once to give

$$R_0 f'(x)\gamma_{\vartheta_0} \int_{-\infty}^{+\infty}\int_0^{2\pi} \frac{(y\cos\varphi - R_0)\, d\varphi \, d\xi}{[(x-\xi)^2 + y^2 + R_0^2 - 2yR_0\cos\varphi]^{3/2}}$$

$$= -f'(x)\gamma_{\vartheta_0} \begin{cases} +4\pi & \text{for } y < R_0 \\ +2\pi & \text{for } y = R_0 \\ 0 & \text{for } y > R_0 \end{cases} \tag{38}$$

The other surface integral on the right-hand side of Eq. (37) can be developed by integration by parts allowing for the fact that γ_ϑ will be discontinuous at the wing-jet boundary. Thus, partial integration of the γ_ϑ term with respect to φ yields

$$R_0 \int_{-\infty}^{+\infty}\int_0^{2\pi} \frac{-(x-\xi)\gamma_\vartheta \sin\varphi \, d\varphi \, d\xi}{[(x-\xi)^2 + y^2 + R_0^2 - 2yR_0\cos\varphi]^{3/2}}$$

$$= -\int_{-\frac{c}{2}}^{\frac{c}{2}} \frac{(x-\xi)}{y} \left\{ \frac{\gamma_\vartheta(\pi+\epsilon) - \gamma_\vartheta(\pi-\epsilon)}{\sqrt{(x-\xi)^2 + (y+R_0)^2}} + \frac{\gamma_\vartheta(\epsilon) - \gamma(2\pi-\epsilon)}{\sqrt{(x-\xi)^2 + (y-R_0)^2}} \right\} d\xi \tag{39}$$

$$+ R_0 \int_{-\infty}^{+\infty}\int_0^{2\pi} \frac{\partial \gamma_x}{\partial \xi} \frac{(x-\xi)}{y} \left\{ \frac{1}{\sqrt{(x-\xi)^2 + y^2 + R_0^2 - 2yR_0\cos\varphi}} - \frac{1}{\sqrt{(x-\xi)^2 + (y-R_0)^2}} \right\} d\varphi \, d\xi$$

where use has been made of the fact that

$$\gamma_v(\pi-\epsilon) = \gamma_v(\epsilon)$$
$$\gamma_v(2\pi-\epsilon) = \gamma(\pi+\epsilon)$$

and also of the condition

$$\frac{\partial \gamma_x}{\partial \xi} = -\frac{1}{R_0}\frac{\partial \gamma_v}{\partial \varphi}$$

at appropriate points.

Integration by parts of the γ_x term on the variable ξ gives

$$R_0\int_{-\infty}^{+\infty}\int_0^{2\pi}\frac{-(y-R_0\cos\varphi)\gamma_x\,d\varphi\,d\xi}{\left[(x-\xi)^2+y^2+R_0^2-2yR_0\cos\varphi\right]^{3/2}}$$

$$= -R_0\int_0^{2\pi}\frac{(y-R_0\cos\varphi)\gamma_x(\infty)\,d\varphi}{\left[y^2+R_0^2-2yR_0\cos\varphi\right]} \tag{40}$$

$$-R_0\int_{-\infty}^{+\infty}\int_0^{2\pi}\frac{(y-R_0\cos\varphi)(x-\xi)\frac{\partial\gamma_x}{\partial\xi}\,d\varphi\,d\xi}{(y^2+R_0^2-2yR_0\cos\varphi)\sqrt{(x-\xi)^2+y^2+R_0^2-2yR_0\cos\varphi}}$$

The total contributions of the jet vorticity to downwash at the wing surface, as given by Eq. (37), consists of the sum of Eqs. (38), (39), (40). These relationships, together with the wing vorticity effect as given by Eq. (33) and also Eq. (29), substituted into Eq. (26) yield the following expression for the total downwash on the lifting surface.

$$4\pi f_W'(x)\left[V_\infty + \begin{cases} V_{J0} & y < R_0 \\ V_{J0/2} & y = R_0 \\ 0 & y > R_0 \end{cases}\right]$$

$$= \int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{c}{2}}^{\frac{c}{2}} \left[1 + \frac{\sqrt{(x-\xi)^2+(y-\eta)^2}}{(x-\xi)}\right]\frac{1}{y-\eta}\frac{\partial \gamma_y}{\partial \eta}\, d\xi\, d\eta - R_0 \int_0^{2\pi} \frac{(y - R_0\cos\varphi)\gamma_x(\infty)\, d\varphi}{(y^2 + R_0^2 - 2y R_0 \cos\varphi)}$$

$$+ R_0 \int_{-\infty}^{+\infty}\int_0^{2\pi} \frac{\partial \gamma_x}{\partial \xi} \frac{(x-\xi)}{y\sqrt{(x-\xi)^2 + y^2 + R_0^2 - 2y R_0\cos\varphi}} \left\{ 1 \right. \tag{41}$$

$$\left. - \sqrt{\frac{(x-\xi)^2 + y^2 + R_0^2 - 2y R_0\cos\varphi}{(x-\xi)^2 + (y-R_0)^2}} - \frac{y\,(y - R_0\cos\varphi)}{y^2 + R_0^2 - 2y R_0 \cos\varphi} \right\} d\varphi\, d\xi$$

$$+ \int_{-\frac{c}{2}}^{\frac{c}{2}} \left\{ \frac{\Delta\gamma_y(-R_0)\,(y+R_0)}{(x-\xi)\sqrt{(x-\xi)^2 + (y+R_0)^2}} + \frac{\Delta\gamma_y(R_0)\,(y-R_0)}{(x-\xi)\sqrt{(x-\xi)^2 + (y-R_0)^2}} \right.$$

$$\left. - \frac{\Delta\gamma_v(\pi)\,(x-\xi)}{y\sqrt{(x-\xi)^2 + (y+R_0)^2}} - \frac{\Delta\gamma_v(0)\,(x-\xi)}{y\sqrt{(x-\xi)^2 + (y-R_0)^2}} \right\} d\xi$$

where

$$\Delta\gamma_y(-R_0) = \gamma_y(-R_0+\epsilon) - \gamma_y(-R_0-\epsilon)$$
$$\Delta\gamma_y(+R_0) = \gamma_y(+R_0+\epsilon) - \gamma_y(+R_0-\epsilon)$$
$$\Delta\gamma_v(\pi) = \gamma_v(\pi+\epsilon) - \gamma_v(\pi-\epsilon)$$
$$\Delta\gamma_v(0) = \gamma_v(\epsilon) - \gamma_v(2\pi-\epsilon)$$

The jet trailing vorticity distribution designated γ_X in the above equations can be determined approximately by the use of Eq. (22), which applies along streamlines within the jet vortical layer. Along any particular streamline not subject to body forces, we have

$$\frac{\Omega_T}{V} = \int_{-\infty}^{s} \frac{\Omega_N}{V^2}\left[\frac{dV}{dn} + C_1 V (n\cdot c)\right] ds \tag{42}$$

It can be shown that, in the absence of body forces, the velocity derivative in the direction of n, which is both normal to the streamline and tangent to the Bernoulli surface is also equal to $V \cdot C_1 (m \cdot c)$, where $C_1 (m \cdot c)$ is the geodesic curvature of the streamline relative to its Bernoulli surface. In Cartesian coordinates, the flexure, C_1, may be expressed as follows

$$C_1 = \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2}$$

For streamlines curved only slightly from their original X direction, ds is approximately dx, and the above expression can be represented by

$$C_1 = \sqrt{\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{d^2z}{dx^2}\right)^2} \qquad (43)$$

Furthermore, for small values of jet deflection and cross-sectional distortion, $m \cdot c$ can be evaluated on the cylinder $R_0 = const.$ in which case it can be expressed as follows

$$m \cdot c = \cos\left[\frac{\pi}{2} + \varphi - \tan^{-1}\left(\frac{\frac{d^2z}{dx^2}}{\frac{d^2y}{dx^2}}\right)\right] \qquad (44)$$

where φ is the azimuthal coordinate introduced previously. Under these conditions, Eq. (42) may be written as follows

$$\frac{\Omega_T}{V} = 2 \int_{-\infty}^{x} \frac{\Omega_N}{V}\left[-\sin\varphi \frac{d^2y}{dx^2} + \cos\varphi \frac{d^2z}{dx^2}\right] dx \qquad (45)$$

If, in addition, V and Ω_N are assumed not to vary much along streamlines, the integration can be performed and one obtains

$$\Omega_T = 2\Omega_N \left[-\sin\varphi \frac{dy}{dx} + \cos\varphi \frac{dz}{dx} \right] \tag{46}$$

Integration of Eq. (46) across the jet vortical layer yields an expression for the secondary vorticity. Here, we apply the Second Theorem of the mean for integrals to obtain

$$\gamma_T = 2\gamma_N \left[-\sin\varphi \frac{\overline{dy}}{dx} + \cos\varphi \frac{\overline{dz}}{dx} \right] \tag{47}$$

where $\overline{dy}/dx$ and $\overline{dz}/dx$ are appropriate mean values of the streamline deflections across the vortical layer.

The γ_X component of the jet vorticity can now be determined for the condition of small deflections by taking appropriate components of γ_T and γ_N ; the result is

$$\gamma_X = \gamma_{v_0} \left[-\sin\varphi \frac{\overline{dy}}{dx} + \cos\varphi \frac{\overline{dz}}{dx} \right] \tag{48}$$

It may be noted that the magnitude of γ_X in the jet vortical layer depends on the product of $\gamma_{v_0} \equiv V_J - V_\infty$ and the mean deflections of the stream-lines which are of order $f'(x)$. For small values of $V_J - V_\infty$, i. e. of order $V_\infty f'(x)$, γ_X is therefore of higher order than the wing vorticity terms retained in Eq. (41) and accordingly such terms can be neglected in determining first-order effects on the wing. On the other hand, the lift distribution, to a first order in small values of $\frac{V_J - V_\infty}{V_\infty}$, as for example Koning, Ref. (1), must necessarily be second order in terms of wing angle - of-attack.

The mean deflections $\overline{dy}/dx$ and $\overline{dz}/dx$ may be expressed in terms of the normal disturbance velocities evaluated on the jet vortex sheet. In that case

$$\frac{\overline{dy}}{dx} = \frac{2\left(v_\omega + v_J\right)}{V_\infty + V_J} \qquad \frac{\overline{dz}}{dx} = \frac{2\ w_\omega + w_J}{V_\infty + V_J} \tag{49}$$

where v_J and w_J are due to the jet vorticity, and v_ω and w_ω are due to the wing vortex system.

$$v_J = -\frac{R_0}{4\pi}\int_{-\infty}^{+\infty}\int_0^{2\pi}\left[\frac{R_0(\sin\vartheta - \sin\varphi)\,\gamma_x - (x-\xi)\cos\varphi\,\gamma_\vartheta}{\left\{(x-\xi)^2 + 2R_0^2\left[1-\cos(\vartheta-\varphi)\right]\right\}^{3/2}}\right] d\varphi\, d\xi$$

$$w_J = -\frac{R_0}{4\pi}\int_{-\infty}^{+\infty}\int_0^{2\pi}\left[\frac{-(x-\xi)\sin\varphi\,\gamma_\vartheta - R_0(\cos\vartheta - \cos\varphi)\,\gamma_x}{\left\{(x-\xi)^2 + 2R_0^2\left[1-\cos(\vartheta-\varphi)\right]\right\}^{3/2}}\right] d\varphi\, d\xi \tag{50}$$

$$v_\omega = -\frac{R_0\sin\vartheta}{4\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{c}{2}}^{\infty}\frac{\gamma_x\, d\xi\, d\eta}{\left[(x-\xi)^2 + R_0^2 - 2R_0\eta\cos\vartheta + \eta^2\right]^{3/2}}$$

$$w_\omega = -\frac{1}{4\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{c}{2}}^{\infty}\frac{\left[(x-\xi)\gamma_y - (R_0\cos\vartheta - \eta)\gamma_x\right] d\xi\, d\eta}{\left[(x-\xi)^2 + R_0^2 - 2R_0\eta\cos\vartheta + \eta^2\right]^{3/2}}$$

Substitution of Eqs. (49) and the first two of Eqs. (50) into Eq. (48) yields the following expression

$$\gamma_x(x,\vartheta) = \frac{\gamma_{\vartheta_0} R_0^2}{2\pi(V_\infty + V_J)} \int_{-\infty}^{+\infty}\int_0^{2\pi} \frac{[1-\cos(\vartheta-\varphi)]\gamma_x(\xi,\varphi)\,d\varphi\,d\xi}{\{(x-\xi)^2 + 2R_0^2[1-\cos(\vartheta-\varphi)]\}^{3/2}}$$

$$-\frac{\gamma_{\vartheta_0} R_0}{2\pi(V_\infty + V_J)} \int_{-\infty}^{+\infty}\int_0^{2\pi} \frac{(x-\xi)\sin(\vartheta-\varphi)\gamma_\vartheta(\xi,\varphi)\,d\varphi\,d\xi}{\{(x-\xi)^2 + 2R_0^2[1-\cos(\vartheta-\varphi)]\}^{3/2}} \tag{51}$$

$$\frac{-2\gamma_{\vartheta_0} v_\omega \sin\vartheta}{V_\infty + V_J} + \frac{2\gamma_{\vartheta_0} w_\omega \cos\vartheta}{V_\infty + V_J}$$

Let

$$\gamma_x(\xi,\varphi) = \gamma_x(x,\varphi) + \left.\frac{\partial\gamma_x}{\partial\xi}\right|_{x,\varphi}(\xi - x) + ---$$

$$\gamma_\vartheta(\xi,\varphi) = \gamma_\vartheta(x,\varphi) + \left.\frac{\partial\gamma_\vartheta}{\partial\xi}\right|_{x,\varphi}(\xi - x) + ---$$

Since $\gamma_x(x,\varphi)$ is symmetrical about the jet periphery, the contribution of the first term on the right hand side of Eq. (51) depends on the higher differential coefficients of the Taylor expansion than the first, and therefore, according to Eq. (48) on the derivatives of the streamline deflection higher than curvature. The contribution of the second term, however, depends on the first differential coefficient of the $\gamma_\vartheta(\xi,\varphi)$ expansion. It may be noted that even this coefficient already depends on the product of streamline deflection and curvature. For small values of deflection, curvature, etc., Eq. (51) may therefore be approximated by

$$\gamma_x = \frac{-2\gamma_{\vartheta_0} v_\omega \sin\vartheta}{V_J + V_\infty} + \frac{2\gamma_{\vartheta_0} w_\omega \cos\vartheta}{V_\infty + V_J} \tag{52}$$

The last two of Eqs. (50) can be expressed as follows

$$v_w = \frac{R_o \sin\vartheta}{4\pi} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{c}{2}}^{\frac{c}{2}} \frac{\partial \gamma_y}{\partial \eta} \frac{1}{[\tilde{b}^2]} \left\{ 1 + \frac{(x-\xi)}{[(x-\xi)^2 + \tilde{b}^2]^{1/2}} \right\} d\xi \, d\eta$$

$$\begin{aligned} w_w = & -\frac{1}{4\pi} \int_{-\frac{c}{2}}^{\frac{c}{2}} \frac{(x-\xi)}{[(x-\xi)^2 + R_o^2 \sin^2\vartheta]} \left\{ \frac{[\gamma_y(-R_o+\epsilon) - \gamma_y(-R_o-\epsilon)] R_o (1+\cos\varphi)}{\sqrt{(x-\xi)^2 + 2R_o^2(1+\cos\varphi)}} \right. \\ & \left. - \frac{[\gamma_y(R_o+\epsilon) - \gamma_y(R_o-\epsilon)] R_o (1-\cos\varphi)}{\sqrt{(x-\xi)^2 + 2R_o^2(1-\cos\varphi)}} \right\} d\xi \\ & - \frac{1}{4\pi} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{c}{2}}^{\frac{c}{2}} \frac{\partial \gamma_y}{\partial \eta} \frac{R_o \cos\vartheta - \eta}{\tilde{b}^2} \left\{ 1 + \right. \\ & \left. \frac{(x-\xi)[(x-\xi)^2 + \tilde{b}^2 + R_o^2 \sin\vartheta]}{[(x-\xi)^2 + R_o^2 \sin^2\vartheta] \sqrt{(x-\xi)^2 + b^2}} \right\} d\xi \, d\eta \end{aligned} \tag{53}$$

where $\tilde{b}^2 \equiv R_o^2 - 2R_o \eta \cos\vartheta + \eta^2$

Upon substitution of the above expressions into Eq. (52) one obtains

$$\begin{aligned} \gamma_x \left(\frac{V_\infty + V_J}{2\gamma_{v_o}} \right) = & -\frac{1}{4\pi} \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\partial \Gamma}{\partial \eta} \frac{(R_o - \eta \cos\vartheta)}{\tilde{b}^2} d\eta \\ & - \frac{1}{4\pi} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{c}{2}}^{\frac{c}{2}} \frac{\partial \gamma_y}{\partial \eta} \frac{(x-\xi)}{\tilde{b}^2[(x-\xi)^2 + \tilde{b}^2]^{1/2}} \left\{ R_o - \eta \cos\vartheta \right. \end{aligned} \tag{54}$$

$$+ \frac{(\tilde{b}^2)(R_o \cos^2 \vartheta - \eta \cos \vartheta)}{(x-\xi)^2 + R_o^2 \sin^2 \vartheta} \Bigg\} d\xi \, d\eta$$

$$- \frac{R_o \cos \vartheta}{4\pi} \int_{-\frac{c}{2}}^{\frac{c}{2}} \frac{(x-\xi)}{(x-\xi)^2 + R_o^2 \sin^2 \vartheta} \Bigg\{ \frac{[\gamma_y(-R_o+\epsilon) - \gamma_y(-R_o-\epsilon)](1+\cos \vartheta)}{\sqrt{(x-\xi)^2 + 2R_o^2(1+\cos \vartheta)}}$$

$$- \frac{[\gamma_y(R_o+\epsilon) - \gamma_y(R_o-\epsilon)](1-\cos \vartheta)}{\sqrt{(x-\xi)^2 + 2R_o^2(1-\cos \vartheta)}} \Bigg\} d\xi$$

For large values of x , Eq. (54) becomes

$$\left(\frac{V_\infty + V_J}{2\gamma_{\vartheta_o}}\right) \gamma_x(\infty) = - \frac{1}{2\pi} \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\partial \Gamma}{\partial \eta} \; \frac{(R_o - \eta \cos \vartheta)}{\tilde{b}^2} \, d\eta \tag{55}$$

and the derivative of Eq. (54) with respect to x yields

$$\frac{V_\infty + V_J}{(2\gamma_{\vartheta_o})} \; \frac{\partial \gamma_x}{\partial x} = - \frac{1}{\pi} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{c}{2}}^{\frac{c}{2}} \frac{\partial \gamma_y}{\partial \eta} \; \frac{1}{\tilde{b}^2}$$

$$\Bigg\{ \left[R_o - \eta \cos \vartheta + \frac{\tilde{b}^2 (R_o \cos^2 \vartheta - \eta \cos \vartheta)}{(x-\xi)^2 + R_o^2 \sin^2 \vartheta} \right] \frac{\tilde{b}^2}{\left[(x-\xi)^2 + \tilde{b}^2\right]^{3/2}} \tag{56}$$

$$- \frac{2\tilde{b}^2 (x-\xi)^2 (R_o \cos^2 \vartheta - \eta \cos \vartheta)}{\sqrt{(x-\xi)^2 + \tilde{b}^2} \left[(x-\xi)^2 + R_o^2 \sin^2 \vartheta\right]^2} \Bigg\} d\xi \, d\eta + - - - - -$$

By considering once more the rotational flow aspects of the vortex sheets, it can be shown with Eq. (20) that if the wing lift is considered to be a constant across the jet rotational layer, there will be no shed vorticity arising at the wing and jet-boundary intersections. Therefore

$$\Delta \gamma_y (+R_o) = -\Delta \gamma_\nu (0)$$

$$\text{and} \quad \Delta \gamma_y (-R_o) = -\Delta \gamma_\nu (\pi) \tag{57}$$

Equations (41), (55), (56), and (57) complete the formulation of the lifting surface problem for a wing-jet combination for the condition of small flow deflection. Since the system of equations appears quite formidable in its present form, one must, as in the case of ordinary lifting surface theory, resort to approximate methods, employing expansions that are suitable for special classes of configurations in order to obtain numerical results. It appears that wing aspect ratio offers a suitable expansion parameter for this purpose, since the limiting cases of both high and low aspect ratio yield great simplification in ordinary lifting surface theory which is implicit in the present problem.

This course has been pursued for the case of a high aspect ratio wing and is presented in the Appendix. For simplicity, jet curvature terms have been neglected. The lifting surface equations given here in that approximation, reduce to a couplet of integral equations, the same as given by Graham et al. in Reference 13 but with somewhat different expressions for the image strength factors. In place of the quadratic expressions $(V_J^2 - V_\infty^2)/(V_J^2 + V_\infty^2)$ and $2V_J V_\infty/(V_J^2 + V_\infty^2)$ which occur in basic image analysis of a perturbed circular jet, the present development obtains linear expressions $(V_J - V_\infty)/(V_J + V_\infty)$ and $2V_\infty/V_J + V_\infty$ respectively. Numerical results, showing the effect of these differences are given in Figure 2 where a comparison of the wing lift increment due to the jet is made with other existing theories and experiment for one set of conditions. The present developments yield values of wing lift within the jet region considerably higher than previous theories and available test data for a nonuniform jet. As discussed in the Appendix, this difference is due to the approximations used in determining the mean deflection of the streamlines on the jet vortex sheet.

IV. CONCLUDING REMARKS

A formulation of the lifting surface equations for a wing-jet configuration is given for the case of a circular jet experiencing small deflections. Jet curvature effects are included but jet cross-sectional distortion is neglected.

It is shown that any steady flow of an incompressible, inviscid fluid can be represented as a solenoidal field with the corresponding rotation distribution subject to the dynamical equations of rotational flow. A useful streamline integral of the latter system is derived that gives the distribution of the streamwise component of the rotation along curved streamlines within the jet rotational layer. This is identical with Hawthorne's expression for secondary vorticity in nonuniform flows. The flow field of the wing-jet combination is first represented as a general solenoidal field. It is then specialized, for the case of a lifting surface and an initially uniform circular jet, to one involving only vortex sheets.

On the jet sheet both the normal and the streamwise components of vorticity enter into consideration. However, the downwash effects of the jet on the lifting surface, with exception of discontinuity effects at the wing and jet-boundary intersections, can be expressed entirely in terms of the trailing component of jet vorticity and its streamwise derivative.

With the aid of the aforementioned streamline integral of the rotational flow equations, an approximate expression for the jet trailing vorticity is obtained in terms of the wing vortex system. For small values, the jet trailing vorticity depends on the product of the initial jet vorticity far upstream and that component of the local flow deflection which occurs due to geodesic curvature of the streamlines on the jet vortex sheet. Its value is a maximum on the sides of a downward deflected jet and vanishes at top and bottom with sign such as to cause upwash within the jet and downwash on the outside.

For simplicity, jet curvature effects have been neglected in that development. The resulting system then reduces to a couplet of integral equations the same as given by Graham etc. in Reference 13 except with image strength factors altered. Numerical results, showing the effect of these differences, yield values of wing lift within the jet region considerably higher than previous theories and available test data for a nonuniform jet.

The approach taken here, and basic developments, lend themselves to the investigation of other problems involving jet curvature effects. Thus whenever a jet is turned toward alignment by a main flow, there is as a consequence, and in accordance with the general prescription, a redistribution of its vorticity, the immediate effect of which is a self-consistent flow field both within and exterior to the jet. Through the extended influence of this flow field, jet curvature effects manifest themselves on solid boundaries of wings, propellers, and other similar parts.

APPENDIX

LIFTING LINE APPROXIMATION

If the high aspect ratio approximation, i. e., $|x-\xi| << |y-\eta|$, by which ordinary lifting surface theory is reduced to lifting line theory, is adopted and applied to Eq. (41) with the jet curvature effects term neglected, an integral equation is obtained for the spanwise circulation distribution from which numerical results can be derived quite readily. Under such conditions, Eq. (41) with $\gamma_x(\infty)$ given by Eq. (54), becomes

$$4\pi f_W'(x)\left[V_\infty + \begin{cases} V_{J0}, & y<R_0 \\ 0, & y>R_0 \end{cases}\right] = \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\partial\Gamma}{\partial y}\,\frac{1}{y-h}\,d\eta + 2\int_{-\frac{c}{2}}^{\frac{c}{2}} \frac{\gamma_y}{x-\xi}\,d\xi$$

$$+\frac{R_0}{\pi}\left(\frac{V_J-V_\infty}{V_J+V_\infty}\right)\int_0^{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\partial\Gamma}{\partial\eta}\,\frac{(y-R_0\cos\varphi)(R_0-\eta\cos\varphi)\,d\eta\,d\varphi}{(y^2-2yR_0\cos\varphi+R_0^2)(R_0^2-2\eta R_0\cos\varphi+\eta^2)} \qquad (1\text{-A})$$

The above equation, for the case of a flat plate wing, i. e., $f_W'(x)=\alpha$, can be written as follows

$$\frac{1}{\pi}\int_{-\frac{c}{2}}^{\frac{c}{2}} \frac{\gamma_y}{(x-\xi)}\,d\xi = F(y)$$

where (2-A)

$$F(y) = 2\alpha\left[V_\infty + \begin{cases} V_{J0}, & y<R_0 \\ 0, & y>R_0 \end{cases}\right] - \frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\partial\Gamma}{\partial y}\left(\frac{1}{y-\eta}\right)d\eta$$

$$+\frac{R_0}{2\pi^2}\left(\frac{V_J-V_\infty}{V_J+V_\infty}\right)\int_0^{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\partial\Gamma}{\partial\eta}\,\frac{(y-R_0\cos\varphi)(R_0-\eta\cos\varphi)}{(y^2-2yR_0\cos\varphi+R_0^2)(R_0^2-2\eta R_0\cos\varphi+\eta^2)}\,d\eta\,d\varphi$$

Inversion of Eq. (2-A) yields

$$\gamma_y = \frac{1}{\pi}\sqrt{\frac{\frac{c}{2}-x}{x+\frac{c}{2}}}\int_{-\frac{c}{2}}^{\frac{c}{2}}\sqrt{\frac{\xi+\frac{c}{2}}{\frac{c}{2}-\xi}}\;\frac{F(y)}{(\xi-x)}\,d\xi = \sqrt{\frac{\frac{c}{2}-x}{x+\frac{c}{2}}}\;F(y) \tag{3-A}$$

Now

$$\Gamma(y) = \int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma_y \, dx$$

which, with γ_y as given by Eq. (3-A), becomes, upon integration

$$\Gamma(y) = \frac{\pi c}{2}\;F(y) \tag{4-A}$$

Equation (4-A) with $F(y)$ as defined following Eq. (2-A), is an integral equation for the wing spanwise circulation distribution, $\Gamma(y)$.

In the expression for $F(y)$ the term which is a surface integral can be reduced to a single integral by performing the integration on ζ. Equation (4-A) becomes therefore for $0 < y < R_o$.

$$\begin{aligned}\Gamma(y) = {} & \pi c \alpha\left(V_\infty + V_{Jo}\right) - \frac{c}{4}\int_{-\frac{b}{2}}^{\frac{b}{2}}\frac{\partial\Gamma}{\partial\eta}\,\frac{1}{y-\eta}\,d\eta \\ & + \frac{c}{4y}\left(\frac{V_J - V_\infty}{V_J + V_\infty}\right)\left\{\int_{+\frac{b}{2}}^{-R_o} + \int_{+R_o}^{\frac{b}{2}}\right\}\frac{\partial\Gamma}{\partial\eta}\,\frac{y}{y-\eta}\,d\eta \\ & + \frac{c}{8y}\left(\frac{V_J - V_\infty}{V_J + V_\infty}\right)\int_{-R_o}^{+R_o}\frac{\partial\Gamma}{\partial\eta}\left(\frac{R_o^{\,2}+y\eta}{R_o^{\,2}-y\eta}\right)d\eta\end{aligned} \tag{5-A}$$

and for $R_o < y < \frac{b}{2}$

$$\Gamma(y) = \pi c \alpha V_\infty - \frac{c}{4}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\partial \Gamma}{\partial \eta}\frac{1}{y-\eta}\,d\eta$$

$$- \frac{c}{8y}\left(\frac{V_J - V_\infty}{V_J + V_\infty}\right)\left\{\int_{+\frac{b}{2}}^{-R_o} + \int_{+R_o}^{\frac{b}{2}}\right\}\frac{\partial \Gamma}{\partial \eta}\left(\frac{R_o^2 + y\eta}{R_o^2 - y\eta}\right) d\eta \qquad (6\text{-A})$$

$$- \frac{c}{4y}\left(\frac{V_J - V_\infty}{V_J + V_\infty}\right)\int_{-R_o}^{+R_o}\frac{\partial \Gamma}{\partial \eta}\left(\frac{y}{y-\eta}\right) d\eta$$

Equations (5-A) and (6-A) are equivalent to Eqs. (3, 11) of Reference 13, which were developed on the basis of simple image theory. In fact, if the present system is integrated partially one obtains Eqs. (3, 11), but with different expressions for the image strength factors that precede the integrals. Instead of $\left(V_J^2 - V_\infty^2\right)/\left(V_J^2 + V_\infty^2\right)$ and $2\,V_J V_\infty/\left(V_J^2 + V_\infty^2\right)$ the present expressions turn out to be $\left(V_J - V_\infty\right)/\left(V_J + V_\infty\right)$ and $2\,V_\infty/(V_J + V_\infty)$.

The procedure used here for solving Eqs. (5-A) and (6-A) makes use of a single sine series expansion for Γ. The leading terms of the series were determined in the usual manner by solving m simultaneous equations which apply at m discreet points along the semi-span of the lifting line. For this purpose it was convenient to let Γ be expressed as follows

$$\Gamma(y) = \pi c \alpha V_J \sum_m A_{2m-1} \sin\left(2m-1\right)\theta$$

The section lifts coefficient, defined as

$$C_\ell = \frac{\rho \tilde{V} \Gamma}{\frac{1}{2}\rho V_\infty^2 c}$$

can then be written as follows

$$C_\ell = \frac{2\pi \tilde{V} V_J \alpha}{V_\infty^2} \sum_m A_{2m-1} \sin (2m-1)\theta$$

where

$$\tilde{V} = \begin{Bmatrix} V_J \text{ inside the jet} \\ V_\infty \text{ outside the jet} \end{Bmatrix}$$

Numerical results for the spanwise lift distribution obtained with a five point collocation solution of the present systems of equations are shown in Fig. 2. For these computations, the five points were taken according to the following formula

$$y = \frac{b}{2} \cos m\frac{\pi}{10} \qquad m = 1, 2, 3, 4, 5$$

and the remaining parameters were

$$\frac{V_\infty}{V_J} = .735$$

$$\text{Wing aspect ratio} = 5$$

$$\frac{2R_0}{b} = .12$$

The theoretical curves in Fig. 2 show, in essence, a comparison of the lift increments due to the jet as computed by the various existing theories. In the present analysis, this increment was computed by subtracting out the theoretical lift for the wing in a uniform flow obtained by using Glauert's lifting line theory.

The unusually high value for the maximum lift predicted with the present theory as compared with that of Reference 13 reflects the effect of the difference in the constants cited above, as well as a possible difference which may arise from the respective methods of treating the equation. In Reference 13 two sine series were employed in the expansion of Γ, one applying outside the jet and the other inside the jet with a matching of the lift at the wing-jet boundary. A single sine series expansion was assumed for Γ in the present computations.

It should be noted further that the present lifting line analysis neglects jet curvature effects, and, in addition, is based on the condition that V_J is not close to V_∞, more specifically, that $V_{J0} = V_J - V_\infty$ is of order V_∞. The numerical computations, however, have been carried out for

$$\frac{V_{J0}}{V_\infty} = .360$$

which is close to the range where wing effects to the second order in angle of attack may be of importance.

In view of seemingly poor agreement with experimental data, it is at this time difficult to justify the present method over previous analyses of the wing-jet interaction problems. Nevertheless, it is felt that the present approach, with the jet boundary regarded as a free vortex sheet, affords a more realistic model than does the image technique, as well as a better understanding of the effects of the controlling parameters, i.e., jet deflection, curvature, etc.

If the results of simple image theory of Reference 13 are to be recovered, one is led to conclude that either Eq. (49), or the ensuing developments leading to Eq. (52), must be modified. This follows from the fact that the expression for γ_x on the jet boundary as given by Eq. (48) corresponds exactly to that which is obtained with the flow fields from simple image analysis if $\frac{dy}{dx}$ and $\frac{dz}{dx}$ are taken to be the mean deflections of the streamlines on the two sides of an elementary jet boundary. Equation (52) on the other hand cannot be so reconciled. The need for further analysis is suggested in this area. In addition, it is recommended that a more complete development in the lifting line approximation be carried out to include the jet curvature effects terms.

It should be stated that experimental data, more consistent with the theoretical flow model than is at present available, is required in the final correlation. The ideal jet, which is usually assumed for simplicity of analysis, can perhaps best be simulated in a wind tunnel by employing drag screens[20] rather than a propeller whose wake, due to its characteristic loading, is generally nonuniform. This is, of course, not to lose track of the ultimate problem which pertains to a real propeller and wing combination, but merely to test the ideal jet model. It could well be that the realistic problem requires an even more elaborate model for the propeller wake such as, for example, a series of concentric ideal jets.

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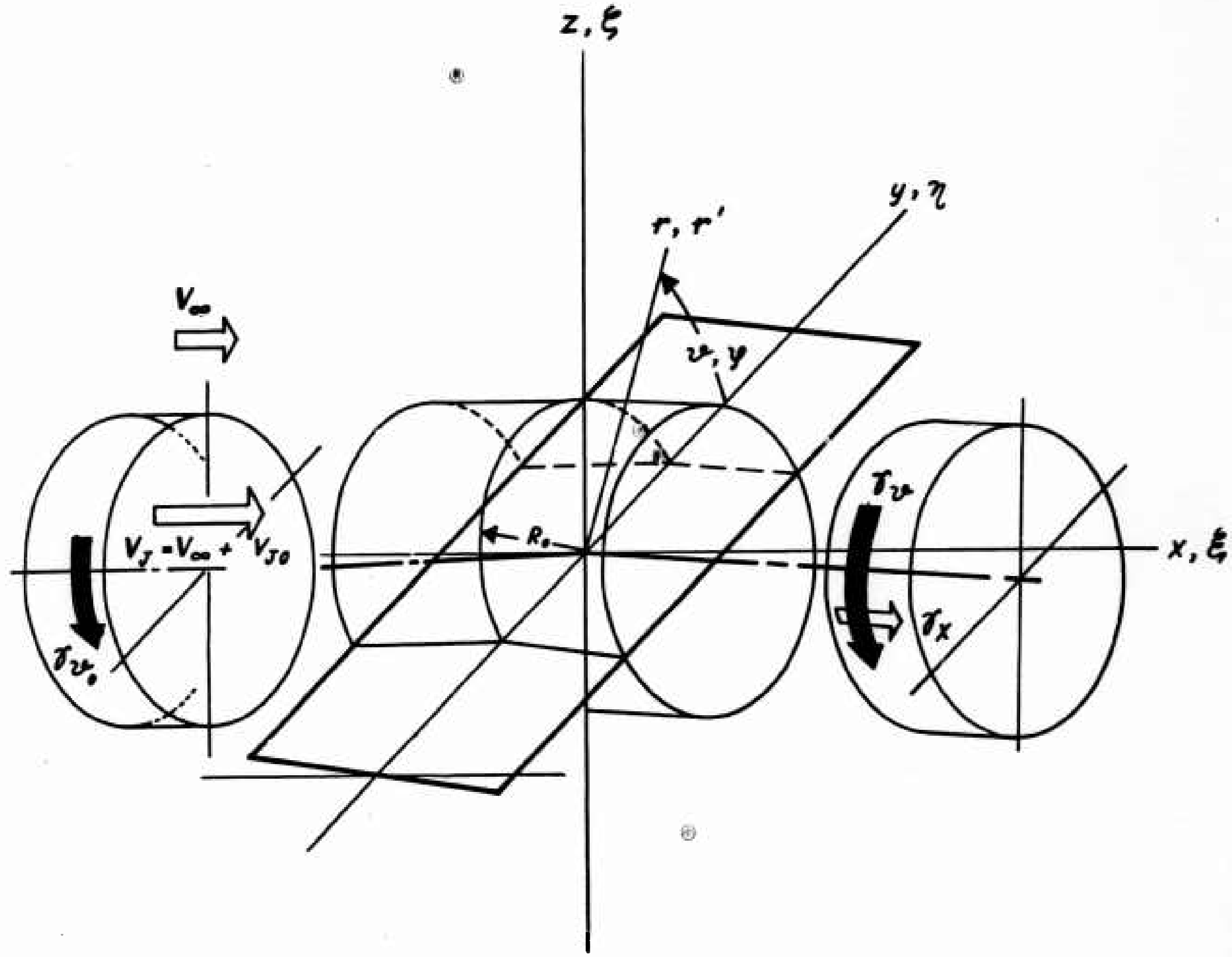

Figure I WING-SLIPSTREAM ARRANGEMENT SHOWING COORDINATE SYSTEMS

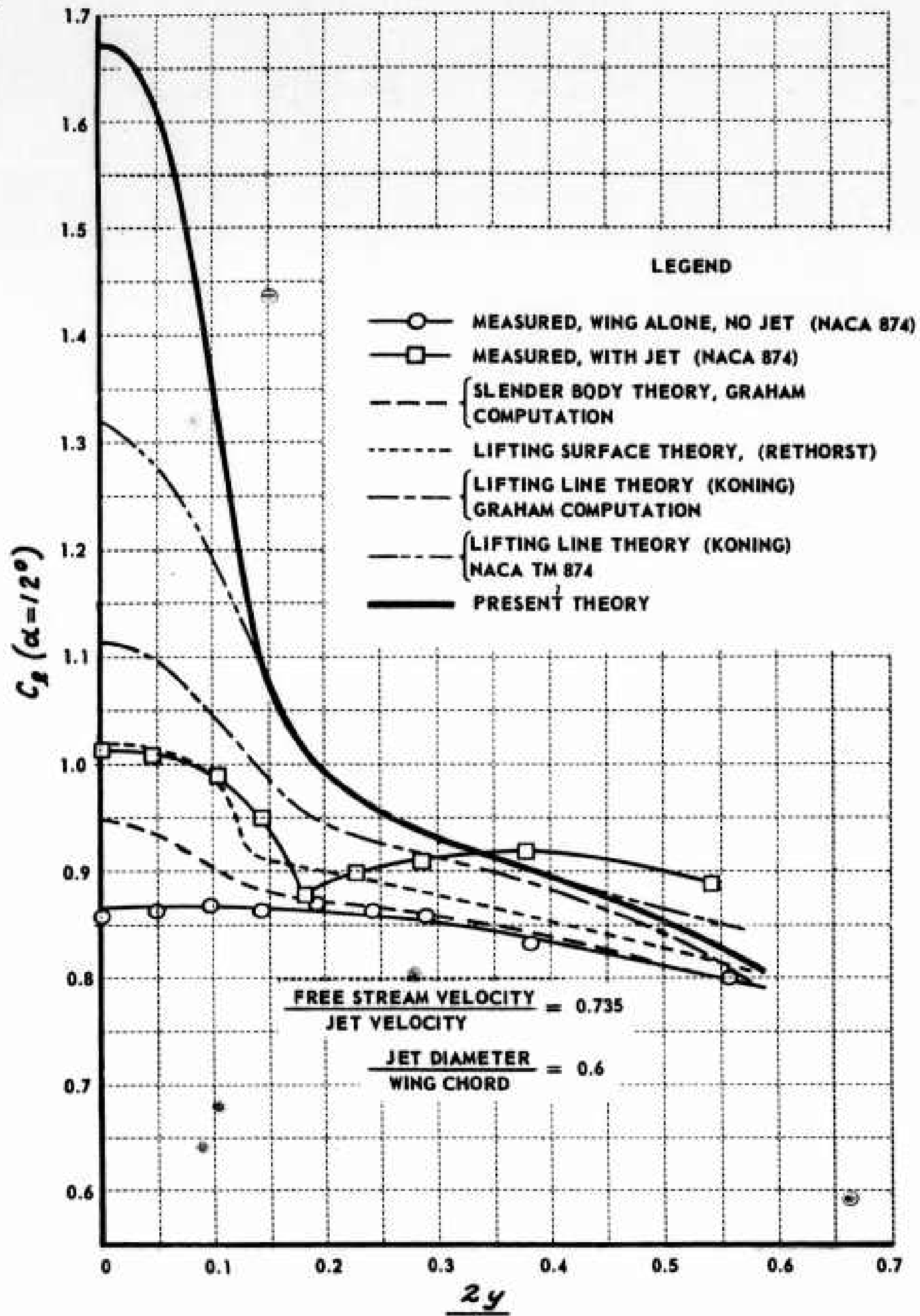

Figure 2 COMPARISON OF VARIOUS THEORIES WITH EXPERIMENT

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CAL Report AI-1190-A-6, October 1961, 49 pp. illus. (Contract DA 44-177-TC-439) Project 9-38-01-000, ST 902

The lifting surface equations for a wing-jet combination are derived for small flow deflections. In the present formulation both the jet and the wing, together with its wake, are represented by vortex sheets as the limit of a thin layer of rotational flow. The vorticity distributions are accordingly subject to the dynamical equations governing rotational flows.

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By employing a general streamline integral of the rotational flow equations, it is shown that jet deflection gives rise to trailing vorticity in the jet rotational layer analogous to secondary vorticity in curved nonuniform flows. For small values, the jet trailing vorticity depends on the produce of the initial jet vorticity far upstream and that component of the local flow deflection which occurs due to geodesic curvature of the streamlines on the jet vortex sheet. This relationship permits the jet vorticity distribution to be established for an initially straight, uniform, circular jet in terms of the wing vortex system, and provides an expression, which together with the general downwash equation for a lifting surface completes the formulation.

A treatment of the lifting surface equations for the case of a jet in combination with a high-aspect ratio wing is appended.

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