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TECHNICAL REPORT

FREE MOLECULE FLOW THEORY AND ITS APPLICATION TO THE DETERMINATION OF AERODYNAMIC FORCES

By

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ABSTRACT

The purpose of this report is twofold. It is designed as an introduction to free molecule flow theory for those with no background in rarefied gas dynamics or kinetic theory, and it presents an exact method for the determination of the forces and moments on bodies in a free molecule flow. Approximate methods are evaluated.

Comparison with the exact method of this report shows that for molecular speed ratios at satellite altitudes the approximate theories, which do not completely account for the random motion of the impinging molecules, are satisfactory for the prediction of C_N and C_M. For elongated bodies at small angles of attack, the approximate methods are shown to be inadequate for the prediction of C_A. Tables of certain functions requisite to the determination of free molecule aerodynamic coefficients are presented.

NOTATION

A	- area
$\bar{C}$	- average speed of a molecule
C_A	- axial force coefficient
C_D	- drag coefficient
C_L	- lift coefficient
C_M	- pitching moment coefficient
C_N	- normal force coefficient
c_m	- most probable random speed of the molecules, $C_m = \sqrt{2RT}$
I_1^F, I_2^F, I_3^F	- Free molecule flow functions for a flat plate with both sides exposed to the flow
$I_1^{CF}, I_2^{CF}, I_3^{CF}$	- Free molecule flow functions for a flat plate with one side exposed to the flow
$I_1^{CY}, I_2^{CY}, I_3^{CY}, I_4^{CY},$	- Free molecule flow functions for a cylinder
$I_1^{CO}, I_2^{CO}, I_3^{CO}, I_4^{CO}, I_5^{CO}, I_6^{CO}$	- Free molecule flow functions for a cone
$I_1^{SP}, I_2^{SP}, I_3^{SP}, I_4^{SP}, I_5^{SP},$	- Free molecule flow functions for a spherical segment
$\underline{i}, \underline{j}, \underline{k}$	- unit vectors
Kn	- Knudsen number
k	- direction cosine between the local x axis and the direction in which the force is desired
l	- direction cosine between the local y axis and the direction in which the force is desired

NOTATION (Continued)

t - direction cosine between the local z axis and the direction in which the force is desired

L - characteristic length of the problem under consideration

m - mass of a molecule

n - number density of molecules in space

N - total number of molecules that impinge on unit area per unit time

q - mass velocity of the gas

$\underline{\bar{q}}$ - unit vector in the direction of the mass velocity vector

R - gas constant for a particular gas

S - molecular speed ratio

t - time

T - temperature

u, v, w - components of the total velocity of a molecule

$\bar{u}, \bar{v}, \bar{w}$ - components of the mass velocity, q, of the gas

U, V, W - components of the random thermal motion of a molecule

x, y, z - local axis system

X, Y, Z - body axis system

$\boldsymbol{\alpha}$ - angle of attack defined as the angle between the mass velocity and the longitudinal axis of the vehicle

α - thermal accommodation coefficient

NOTATION (Continued)

β - quantity defined as $\beta \equiv \frac{1}{2RT}$, or a general geometric angle

δ - semi-vertex angle of a cone

ϵ - direction cosine between the mass velocity vector and the local x axis

γ - direction cosine between the mass velocity vector and the local y axis

η - direction cosine between the mass velocity vector and the local z axis

θ_1 - one half the included angle of a spherical segment

Θ_{ij} - collision frequency of one type i molecule with type j molecules

λ - mean free path of a molecule

ρ - mass density of the gas

SUBSCRIPTS:

i - incident molecules

r - reflected molecules

w - wall condition

$\underline{(\ )}$ - an underlined symbol denotes a vector

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SECTION I
INTRODUCTION

Analysis of the trajectory and attitude control of near earth satellites requires a knowledge of the forces and moments caused by molecular impingement. A detailed analysis of these molecular impingement forces and moments which act on the satellite must be carried out and incorporated into the vehicle design.

The determination of the aerodynamic force acting on a satellite is a free molecule flow problem. The first extension of aerodynamic considerations to free molecule flow was make by Tsien (Ref. 1) in which he discussed the method of defining the various flow regimes, continuum, transition and free molecule, by means of the Knudsen number. Expressions for the lift and drag coefficients of a surface element were derived. Ashley (Ref. 2) applied free molecule theory to the determination of lift and drag coefficients for several shapes and discussed the possibility of aerodynamic flight in the free molecule regime. Detailed studies by Stalder and associates, (Ref. 3) developed basic free molecule flow theory and applied it to determine lift and drag coefficients for several shapes. However, certain specific assumptions regarding the temperature (and hence the velocity and momenta) of the re-emitted molecules which are not applicable in satellite environments tend to limit the practical usefulness of these results. In this report, any pertinent value of the temperature of the reflected molecules can be used.

A closed form solution for the free molecule aerodynamic coefficients can be obtained only for a flat plate and the cylinder. For this reason, several approximate methods were developed by Gustafson (Ref. 4) and Schrello (Ref. 5). Unfortunately both of these solutions break down at small molecular speed ratios, and at small angles of attack for all values of molecular speed ratio. An exact analysis is necessary in many cases and is presented in this report.

The interaction of molecules with each other and with container walls falls in the realm of the kinetic theory of gases. The fundamentals of this subject are contained in several readily available texts. However, to facilitate the reader without a background in this field, a development of the basic concepts is contained in Appendix A of this report.

SECTION II
FUNDAMENTALS OF FREE MOLECULE FLOW

A. ASSUMPTIONS ON WHICH FREE MOLECULE FLOW THEORY RESTS

Depending on the governing physical phenomena, the subject of fluid mechanics can be broken into three general categories. The first is the ordinary regime of continuum fluid mechanics in which intermolecular collisions are the important physical phenomena. The second is called the "transition regime" in which molecule-molecule and molecule-surface interactions are of equal importance. The third is the free molecule regime in which molecule-surface interactions are the governing phenomena. The regime in which a particular problem falls is determined by a consideration of a non-dimensional parameter called the Knudsen number, Kn. This is defined as the ratio of an appropriate mean free path (λ) of the molecules to an appropriate characteristic dimension (L) of problem under consideration, i. e., $Kn \equiv \frac{\lambda}{L}$. A detailed discussion of the determination of λ and L is given in Section V.

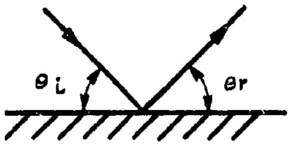
Based on Knudsen number, the flow regimes are arbitrarily divided as follows. If the Knudsen number is less than 0.01, the flow is in the continuum regime; if the Knudsen number is between 0.01 and 10, the flow is in the transition regime; and if the Knudsen number is greater than 10, the flow is in the free molecule regime. However, there is some experimental evidence which seems to indicate that free molecule flow may occur at Knudsen numbers as low as 3 in certain cases.

That the Knudsen number is actually an indication of the flow regime of a particular problem can be seen by the following consideration. The characteristic length, L, is determined by the problem of interest. If the Knudsen number is small, the distance a molecule travels on the average before it suffers a collision with another molecule is small compared to the geometry of the problem. Since $\lambda \propto \frac{1}{\rho}$ where ρ is the density of the fluid, this implies that the fluid is dense and thus may be treated as a continuum. If, however, the Knudsen number is large, this means that the distance a molecule travels between collisions with other molecules is large compared to the geometry of the problem; and since $\lambda \propto \frac{1}{\rho}$, this implies that the density is small and the gas is rarefied. Thus, the fluid may no longer be considered a continuum but must be treated as an assembly of discrete particles. For intermediate values of the Knudsen number, the fluid can not be considered to be a continuum or a rarefied gas. The flow in this transition regime will exhibit some of the characteristics of both continuum and rarefied gas flow.

The theory for predicting the aerodynamic forces on a body in continuum and free molecule flow is available. However, at the present time there is no adequate theory for predicting the aerodynamic forces on a body in the transition regime.

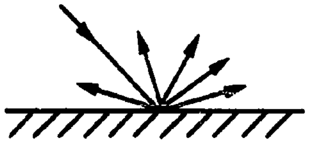
In a very rarefied gas, which is the main concern of this report, the mean free path of the molecules is much greater than a characteristic body dimension. The basic assumptions of free molecule flow are: 1. that incident molecule-surface collisions are much more numerous than incident molecule-incident molecule collisions and 2. that incident molecule-surface collisions are much more numerous than incident molecule-reflected molecule collisions, i.e., the incident flow is undisturbed by the presence of the body and the equilibrium velocity distribution of the incident molecules is changed only by collision with the body. Since we have assumed that the incident and reflected molecules do not affect each other, the effect on the surface of the incident and reflected molecules can be computed separately. Thus, the flow phenomena are governed by the molecule-surface interactions.

In order to compute the force on a body in a free molecule flow, the molecule-surface interaction must be specified. Most surfaces are rough as far as a molecule is concerned. The impinging molecule hits the surface, bounces around in the spaces between the molecules or atoms of the surface and then at some later time leaves the surface. This interaction is usually considered in two parts, the momentum transferred to the surface and the energy transferred to the surface. The momentum which is imparted to the surface depends on the type of reflection that occurs. There are two types of reflection that may occur. They are: specular in which the molecule hits the surface and is then reflected like a billiard ball, i. e., the angle of incidence equals the angle of reflection (Fig. II-1); and diffuse in which the molecule hits the surface and is then re-emitted in a random direction, i. e., there are as many molecules with a particular velocity leaving the surface in one direction as there are leaving the surface in the opposite direction and the angle of incidence of a molecule is in no way related to its angle of reflection (Fig. II-2). In both of these reflection processes, the velocity with which a molecule leaves the surface depends on the amount of energy which the molecule transfers to the surface before it is re-emitted. The degree of equilibrium attained between the molecule and the surface before the molecule is re-emitted is measured by the energy accommodation coefficient, α, which is defined as



SPECULAR REFLECTION
$\theta_i = \theta_r$ V_r IS
NOT NECESSARILY EQUAL TO V_i

Figure II-1



DIFFUSE REFLECTION
θ_r IS NOT RELATED TO θ_i
V_r IS NOT NECESSARILY EQUAL TO V_i

Figure II-2

$$\alpha \equiv \frac{E_i - E_r}{E_i - E_w}$$

where E_i is the energy carried to unit area of the surface by the incident molecules, E_r is the energy carried away from the unit area by the reflected molecules and E_w is the energy the reflected molecules would carry away from the surface if they were re-emitted at the temperature of the surface, T_w. If $E_r = E_i$, there is no energy exchange between the incident molecules and the surface and $\alpha = 0$. If the incident molecules and the surface reach thermal equilibrium before the molecules are re-emitted, $E_r = E_w$ and $\alpha = 1$. It is implicity assumed in the definition of α that all the energies associated with those molecular degrees of freedom which enter into an energy exchange with the surface are accommodated to the same degree. Experimental evidence seems to indicate that this is true for the translational and rotational degrees of freedom, while the vibrational degrees of freedom are practically not affected by a surface collision. If necessary, it is possible to introduce separate accommodation coefficients for each degree of freedom.

The momentum transferred to the surface is specified by momentum accommodation coefficients. Maxwell introduced the momentum accommodation coefficient, g, defined as the fraction of molecules that reflect diffusely, the remainder, $(1-g)$, being reflected specularly. At the present time, however, the trend is toward the use of two momentum accommodation coefficients, one for tangential momentum, σ, and one for normal momentum, σ'. They are defined as

$$\sigma \equiv \frac{\tau_i - \tau_r}{\tau_i - \tau_w} = \frac{\tau_i - \tau_r}{\tau_i}$$

$$\sigma' \equiv \frac{P_i - P_r}{P_i - P_w}$$

where τ_i is the tangential momentum carried to unit area of the surface by the incident molecules, τ_r is the tangential momentum carried away from the unit area by the specularly reflected molecules, τ_w is the tangential momentum which would be carried away from the surface by the diffusely reflected molecules if they were in thermal equilibrium with the surface. By the definition of diffuse reflection, τ_w is equal to zero. P_i is the normal momentum carried to unit area of the surface by the incident molecules, P_r is the normal momentum carried away by the reflected molecules and P_w is the normal momentum which would be carried away by the diffusely reflected molecules if they were in thermal equilibrium with the surface.

For completely diffuse reflection, $\tau_r = \tau_W = 0$ and $\sigma = 1$ regardless of the degree of thermal accommodation. If any specular reflection occurs, the value of σ depends on the degree of thermal accommodation through τ_r. If there is complete specular reflection and no thermal accommodation, $\tau_r = \tau_i$ and $\sigma = 0$. If completely diffuse reflection and complete thermal accommodation occur, $\sigma' = 1$; if completely specular reflection and no thermal accommodation occur, $\sigma' = 0$. For any type of reflection between these two limits, σ' depends on the degree of thermal accommodation that occurs. This is different from σ which is equal to one for completely diffuse reflection regardless of the degree of thermal accommodation which occurs. σ depends on the degree of thermal accommodation which occurs only when there is some specular reflection taking place.

Available experimental evidence seems to indicate that for the type of surfaces used in most satellites, the assumption of completely diffuse reflection is legitimate (Ref. 6). This assumption has been used in the development presented in this report. However, no assumption has been made concerning the temperature of the re-emitted molecules, T_r. Thus, the results presented here are only subject to the assumption of completely diffuse reflection.

B. DERIVATION OF THE EQUATION FOR THE FORCE ON AN ELEMENT OF AREA IN A FREE MOLECULE FLOW

We now compute the force on an arbitrarily oriented element of area in a free molecule flow. Since it has been assumed that the incident and reflected molecules do not interact with each other, we first compute the force due to the incident molecules and then the force due to the reflected molecules and add them together. This gives the total force on the element of area. There is one additional assumption which we make at this point concerning the geometry of the surface under consideration, i. e., it is assumed that the element of area is located on a convex surface. If we do not make this assumption, we must take account of the force on the element of area due to the molecules that have been reflected from other parts of the body.

In order to compute the force which the molecules exert on a surface, it is necessary to know the velocity of each of the molecules in the gas. We assume that both the incident and reflected molecules are in equilibrium with themselves. Kinetic theory then states that their velocities are specified by the Maxwellian velocity distribution function f. The velocity distribution function f is the probability that the velocity of a molecule selected at random will be in the range u to u + du, v to v + dv and w to w + dw, where u, v and w are the components of the total velocity of the molecule. Then, the number of molecules in unit volume of space that have velocities in the range u to u + du, v to v + dv, w to w + dw is

$$n f(u, v, w)\, du\, dv\, dw$$

where n is the number of molecules per unit volume. A discussion and derivation of the Maxwellian velocity distribution function is given in Appendix A.

If a gas is at rest macroscopically, the average velocity of its molecules in any direction is zero. If a gas has a macroscopic velocity, the average velocity of its molecules in three mutually perpendicular directions will be the components in those directions of the macroscopic velocity of the gas. The macroscopic velocity of the gas is usually referred to as the mass velocity. Thus, in a gas which has a mass velocity, the mass velocity can be subtracted from the total velocity of each molecule. The resulting velocity is called the random thermal motion of the molecule, that is,

$$\begin{aligned} u - \bar{u} &= U \\ v - \bar{v} &= V \\ w - \bar{w} &= W \end{aligned}$$

where $\bar{u}$, $\bar{v}$, $\bar{w}$ are the components of the mass velocity and U, V, W are the components of the random thermal motion of the molecule.

Consider an element of a surface which is fixed with respect to a body axes coordinate system X Y Z, Figure II-3. The coordinate system X Y Z is stationary with the gas molecules moving relative to it with a mass velocity, $\underline{q}$.

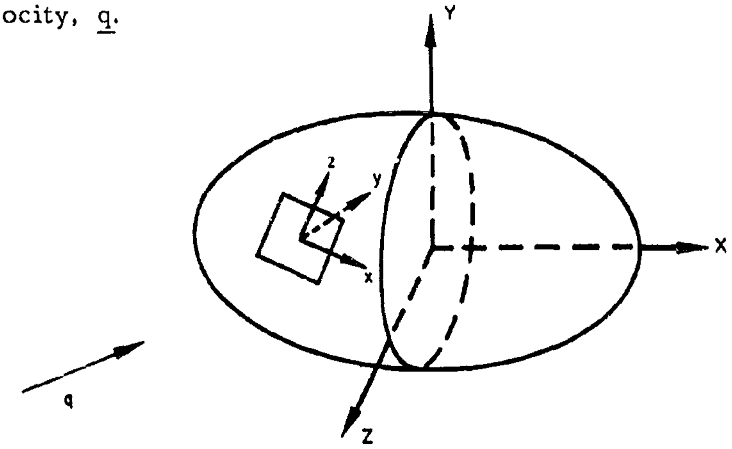


Figure II-3 Definition of the Coordinate Systems

We now chose a local coordinate system x, y, z on the element of surface area with positive y axis being the inward directed normal to the surface, the x and z axes being tangent to the surface element. u, v, w are the components of the total velocity of the molecules in the local x, y, z directions respectively.

Consider the force exerted on the element of area by the incident molecules. Force is defined as the rate of change of momentum. We have assumed that when the molecules impinge on the surface, they lose all their momentum relative to the surface. Thus, the rate of change of momentum of the incident molecules is obtained by integrating over all possible velocities the rate at which the molecules in a particular velocity range transport momentum to the surface.

Only molecules with a velocity component normal to the surface, i. e in the positive y direction, will hit the element of area. The number of molecules in the velocity range u to u + du, v to v + dv, w to w + dw which strike the element of area in time t are contained in a cylinder of height vt, Figure II-4.

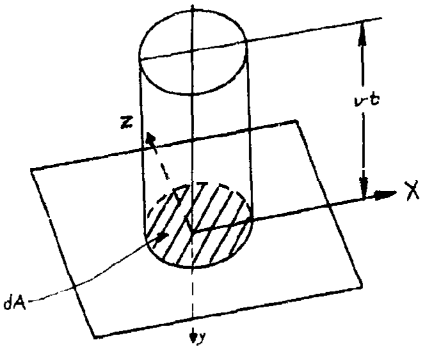


Figure II-4

The number of molecules in the velocity range u to u + du, v to v + dv, w to w + dw per unit volume is

$$n_i f\, du\, dv\, dw \qquad (1)$$

where n_i is the number density of incident molecules. Therefore, the number of molecules in the velocity range u to u + du, v to v + dv, w to w + dw that strike unit area of the surface in time t is

$$v t\, n_i f\, du\, dv\, dw \qquad (2)$$

The number of molecules striking the element of area per unit time is obtained by dividing equation (2) by t. The number of molecules in the velocity range u to u + du, v to v + dv, w to w + dw that strike unit area of the surface per unit time is

$$n_i v f\, du\, dv\, dw \qquad (3)$$

Each molecule transports momentum mu in the x direction, mv in the y direction and mw in the z direction to the element of area. The force is given by the product of the number of molecules in a given velocity range that hit the surface per unit time multiplied by the momentum they carry to the surface. Thus, the force in the x direction on the element of area due to molecules in the velocity range u to u + du, v to v + dv, w to w + dw is

$$m u n_i v f \, du \, dv \, dw \tag{4}$$

The total force in the x direction on the element of area due to molecules in all velocity ranges is

$$m n_i \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} u v f \, du \, dv \, dw \tag{5}$$

m and n_i have been factored out of the integral because m, the mass of each molecule, is a constant over all the molecules and n_i, the number density of molecules, is assumed to be uniform locally in space. f is the Maxwell velocity distribution function given in Appendix A as

$$f = \left(\frac{1}{2\pi RT}\right)^{3/2} e^{-\frac{1}{2RT}\left[(u-\bar{u})^2+(v-\bar{v})^2+(w-\bar{w})^2\right]} \tag{6}$$

where R is the gas constant for the particular gas of interest and $\bar{u}$, $\bar{v}$, $\bar{w}$ are the components of the mass velocity $\underline{q}$ in the x, y, z directions respectively. The limits of the v integration are 0 to + ∞ because only molecules with a velocity component in the positive y direction will hit the surface. Molecules with velocity component in the y direction between 0 and - ∞ are moving away from the element of area and do not strike it.

A similar procedure yields the expressions for the force in the y and z directions. The total force in the y direction on the element of area due to molecules in all velocity ranges is

$$m n_i \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} v^2 f \, du \, dv \, dw \tag{7}$$

The total force in the z direction on the element of area due to molecules in all velocity ranges is

$$mn_i \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} w v f \, du \, dv \, dw \qquad (8)$$

The vector sum of eqs. (5), (7) and (8) gives the total force on the element of area due to molecules in all velocity ranges. However, we are interested in the component of each of these forces in a particular direction, for example the normal or axial force. Let k be the direction cosine between the direction in which the force is desired and the local x axis; let ℓ be the direction cosine between the direction in which the force is desired and the local y axis; and let t be the direction cosine between the direction in which the force is desired and the local z axis. Then, the component in a particular direction of the total force on an element of area due to the incident molecules is given as

$$\frac{dF_{INCIDENT}}{dA} = mn_i \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} (ku + \ell v + tw) v f \, du \, dv \, dw \qquad (9)$$

Next we compute the force on the element of area due to the re-emitted molecules. Since we have assumed that the molecules are reflected diffusely, they are re-emitted from the surface randomly with a Maxwellian velocity distribution corresponding to some temperature T_r. The reflected molecules are thought of as issuing from a fictitious gas which is at rest relative to the surface at a temperature T_r. The number of molecules which leave the surface per unit area per unit time is the same as the number of molecules of this fictitious gas that hit unit area of the surface per unit time. Thus, the number of molecules leaving the element of area per unit time in the velocity range u to u + du, v to v + dv, w to w + dw is

$$n_r(-v) f \, du \, dv \, dw \qquad (10)$$

where (-v) is used because, in order to leave the surface, a molecule must have a velocity component in the negative y direction. n_r is the number density of the fictitious gas from which the reflected molecules are assumed to issue. Since force is defined as the rate of change of momentum, the force on an element of area due to the reflected molecules is given as

$$\text{Force} = (\text{momentum reflected})_{\text{initial}} - (\text{Momentum reflected})_{\text{final}}.$$

Since the reflected molecules are initially at rest on the surface, $(\text{Momentum reflected})_{\text{initial}} = 0$. Thus, the force in the x direction on the element of area due to re-emitted molecules in all velocity ranges is

$$-\left[-mn_r\int_{-\infty}^{+\infty}\int_{-\infty}^{0}\int_{-\infty}^{+\infty} u\,v\,f\,du\,dv\,dw\right] \tag{11}$$

The force in the y direction on the element of area due to re-emitted molecules in all velocity ranges is

$$-\left[-mn_r\int_{-\infty}^{+\infty}\int_{-\infty}^{0}\int_{-\infty}^{+\infty} v^2 f\,du\,dv\,dw\right] \tag{12}$$

The force in the z direction on the element of area due to re-emitted molecules in all velocity ranges is

$$-\left[-mn_r\int_{-\infty}^{+\infty}\int_{-\infty}^{0}\int_{-\infty}^{+\infty} w\,v\,f\,du\,dv\,dw\right] \tag{13}$$

The vector sum of eqs. (11), (12) and (13) gives the total force on the element of area due to the re-emitted molecules. However, we are interested in the components of these forces in a particular direction. Using the direction cosines k, l and t, the component in the direction of interest of the total force on an element of area due to the re-emitted molecules is

$$\frac{dF_{REFLECTED}}{dA} = mn_r \int_{-\infty}^{+\infty}\int_{-\infty}^{0}\int_{-\infty}^{+\infty} (ku + lv + tw)\, v f \, du\, dv\, dw \tag{14}$$

The component in a particular direction of the total force on an element of area due to both incident and reflected molecules is given by the sum of eqs. (9) and (14). Thus

$$\frac{dF}{dA} = mn_i \int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty} (ku + lv + tw)\, v f \, du\, dv\, dw$$
$$+ mn_r \int_{-\infty}^{+\infty}\int_{-\infty}^{0}\int_{-\infty}^{+\infty} (ku + lv + tw)\, v f \, du\, dv\, dw \tag{15}$$

These integrals are evaluated in Appendix B. It should be noted that u, v, and w are the total velocity components of the molecules while the distribution function, f, is a function of the random thermal motion of the molecules only. Thus, in order to perform the integrations, the velocities u, v and w must be written in terms of the mass velocity and the random thermal velocity of the molecules.

Using the results of Appendix B, Eq. (15) becomes

$$\frac{dF}{dA} = mn_i \left\{ \frac{1}{2\sqrt{\pi}} \left(k\bar{u} + l\bar{v} + t\bar{w}\right) \left[\bar{v}\sqrt{\pi}\left(1 + \operatorname{erf} \bar{v}\sqrt{\beta}\right) + \frac{1}{\sqrt{\beta}} e^{-\beta \bar{v}^2} \right] \right.$$
$$\left. + \frac{l}{4\beta}\left(1 + \operatorname{erf} \bar{v}\sqrt{\beta}\right) \right\} + mn_r \frac{l}{4\beta_r} \tag{16}$$

where $\beta = \frac{1}{2RT_i}$, $\beta_r = \frac{1}{2RT_r}$, n_r is the number density of the fictitious gas from which the reflected molecules are assumed to issue, n_i is the number density of incident molecules, T_i is the temperature of the incident molecules, and T_r is the temperature of the reflected molecules. $\bar{u}$, $\bar{v}$, $\bar{w}$ are the components of the mass velocity, q, in the local x, y, z directions respectively.

Equation (16) was obtained by applying Newton's second law of motion to the molecules impinging on and reflecting from the surface. We now need an expression for n_r, the fictitious number density of reflected molecules. This is obtained from continuity considerations. Since we assume that molecules are not being created or destroyed, the total number of molecules that impinge on an element of area per unit time must be equal to the total number of molecules re-emitted from the surface per unit time. We now compute these two quantities, set them equal to each other, and solve for n_r. The mechanics of the computation are carried out in Appendix B, the result of which is

$$n_r = n_i\sqrt{\beta_r}\left[\bar{v}\sqrt{\pi}\,(1+\operatorname{erf}\bar{v}\sqrt{\beta}) + \frac{1}{\sqrt{\beta}}\,e^{-\beta\bar{v}^2}\right] \tag{17}$$

Substituting Eq. (17) into Eq. (16) for n_r gives

$$\frac{dF}{dA} = mn_i\left\{\frac{1}{2\sqrt{\pi}}(k\bar{u}+l\bar{v}+t\bar{w})\left[\bar{v}\sqrt{\pi}\,(1+\operatorname{erf}\bar{v}\sqrt{\beta})+\frac{1}{\sqrt{\beta}}\,e^{-\beta\bar{v}^2}\right]\right.$$
$$\left.+\frac{l}{4\beta}(1+\operatorname{erf}\bar{v}\sqrt{\beta})+\frac{l}{4\sqrt{\beta_r}}\left[\bar{v}\sqrt{\pi}\,(1+\operatorname{erf}\bar{v}\sqrt{\beta})+\frac{1}{\sqrt{\beta}}\,e^{-\beta\bar{v}^2}\right]\right\} \tag{18}$$

This is the equation for the component in a particular direction of the total force on an element of area in a free molecule flow assuming diffuse reflection.

At this point it is convenient to introduce a dimensionless quantity, S, called the molecular speed ratio. This is defined as the ratio of the magnitude of the mass velocity of the gas to the most probable random speed of the molecules, c_m, that is

$$S \equiv \frac{q}{C_m} \qquad \text{where} \qquad C_m = \sqrt{2RT_i}$$

or using the notation $\beta = \frac{1}{2RT_i}$,

$$S = q\sqrt{\beta}$$

The molecular speed ratio, S, is a measure of the relative importance of the mass velocity and the random thermal motion of the molecules. If S is small, the random thermal motion is important in determining the force on an element of area, while if S is large, the effect of the random thermal motion is small.

Let ϵ, γ, η be the direction cosines between the local x, y, z axes and the mass velocity vector $\underline{q}$ respectively. Then,

$$\bar{u} = \epsilon q \;, \quad \bar{v} = \gamma q \;, \quad \bar{w} = \eta q$$

$$\bar{v}\sqrt{\beta} = \gamma q \sqrt{\beta} = \gamma S$$

Introducing these notations, eq. (18) becomes

$$\frac{dF}{dA} = mn_i \left\{ \frac{q}{2\sqrt{\pi}}(k\epsilon + \ell\gamma + t\eta)\left[\gamma q\sqrt{\pi}(1+\mathrm{erf}\,\gamma S) + \frac{1}{\sqrt{\beta}}e^{-\gamma^2 S^2}\right] \right.$$
$$\left. + \frac{\ell}{4\beta}(1+\mathrm{erf}\,\gamma S) + \frac{\ell}{4\sqrt{\beta_r}}\left[\gamma q\sqrt{\pi}(1+\mathrm{erf}\,\gamma S) + \frac{1}{\sqrt{\beta}}e^{-\gamma^2 S^2}\right]\right\} \qquad (19)$$

Multiply through by $\frac{\beta}{\beta}$. Then

$$\frac{dF}{dA} = \frac{\rho_i}{2\beta}\left\{\frac{1}{\sqrt{\pi}}(k\epsilon + l\gamma + t\eta)\left[\gamma s^2\sqrt{\pi}(1+\operatorname{erf}\gamma s) + s e^{-\gamma^2 s^2}\right]\right.$$
$$\left. + \frac{l}{2}(1+\operatorname{erf}\gamma s) + \frac{l}{2}\sqrt{\frac{T_r}{T_i}}\left[\gamma s\sqrt{\pi}(1+\operatorname{erf}\gamma s) + e^{-\gamma^2 s^2}\right]\right\} \qquad (20)$$

where $\rho_i = m n_i$.

Since we are interested in nondimensional coefficients, we will divide through by $\frac{1}{2}\rho_i q^2 A_{ref}$.

Eq. (20) now becomes

$$\frac{dC}{dA} = \frac{1}{A_{ref}}\left\{(\epsilon k + \gamma l + \eta t)\left[\gamma(1+\operatorname{erf}\gamma s) + \frac{1}{s\sqrt{\pi}} e^{-\gamma^2 s^2}\right]\right.$$
$$\left. + \frac{l}{2s^2}(1+\operatorname{erf}\gamma s) + \frac{l}{2}\sqrt{\frac{T_r}{T_i}}\left[\frac{\gamma\sqrt{\pi}}{s}(1+\operatorname{erf}\gamma s) + \frac{1}{s^2} e^{-\gamma^2 s^2}\right]\right\} \qquad (21)$$

when we note that $q^2\beta = S^2$. This is the equation for the component in a particular direction of the total force on an element of area assuming diffuse reflection. This equation is exact within the physical assumptions of kinetic theory, free molecule flow and diffuse reflection.

All the quantities in Eq. (21) are known except for T_r, the temperature of the reflected molecules. This can be determined from an energy balance for the surface. To simplify the computations, T_r will not be specified. However, T_r will be assumed to be a constant over the surface under consideration.

To remove the restriction of diffuse reflection, the accommodation coefficients as defined in Section II A are introduced.

$$\sigma = \frac{\tau_i - \tau_r}{\tau_i} \qquad (22)$$

$$\sigma' = \frac{P_i - P_r}{P_i - P_w} \qquad (23)$$

Solving these expression for τ_r and P_r gives

$$\tau_r = (1-\sigma)\,\tau_i \qquad (24)$$

$$P_r = (1-\sigma')P_i + \sigma' P_w \qquad (25)$$

The total shear force on an element of area is

$$\tau = \tau_i - \tau_r = \tau_i - (1-\sigma)\tau_i = \sigma\,\tau_i \qquad (26)$$

The total normal force on an element of area is

$$P = P_i + P_r = P_i + (1-\sigma')P_i + \sigma' P_w$$

$$P = (2-\sigma')P_i + \sigma' P_w \qquad (27)$$

Using these results equation (21) becomes

$$\frac{dC}{dA} = \frac{1}{A_{ref}}\left\{\left[\sigma(\epsilon k + \eta t) + (2-\sigma')\gamma \ell\right]\left[\gamma(1+\operatorname{erf}\gamma S) + \frac{1}{S\sqrt{\pi}}\,e^{-\gamma^2 S^2}\right]\right.$$
$$\left. + \frac{(2-\sigma')\ell}{2\,S^2}(1+\operatorname{erf}\gamma S) + \frac{\sigma'\ell}{2}\sqrt{\frac{T_w}{T_i}}\left[\frac{\gamma\sqrt{\pi}}{S}(1+\operatorname{erf}\gamma S) + \frac{1}{S^2}\,e^{-\gamma^2 S^2}\right]\right\} \qquad (28)$$

An alternate form of the expression for the force on an element of area for the case of diffuse reflection sometimes appears in the literature. For diffuse reflection $\sigma = 1$, σ' is unspecified, and EQ. (28) becomes

$$\frac{dC}{dA} = \frac{1}{A_{ref}} \left\{ \left[\epsilon k + \eta t + (2 - \sigma')\gamma \ell \right] \left[\gamma(1 + \operatorname{erf} \gamma s) + \frac{1}{s\sqrt{\pi}} e^{-\gamma^2 s^2} \right] \right.$$

$$\left. + \frac{(2 - \sigma')\ell}{2\, s^2} (1 + \operatorname{erf} \gamma s) + \frac{\sigma' \ell}{2} \sqrt{\frac{T_w}{T_i}} \left[\frac{\gamma \sqrt{\pi}}{s} (1 + \operatorname{erf} \gamma s) + \frac{1}{s^2} e^{-\gamma^2 s^2} \right] \right\} \quad (29)$$

Comparing equations (21) and (29) we see that they appear different even though they are both for the case of diffuse reflection. This is because, in EQ. (21), use is made of the fact that for diffuse reflection, P_r may be computed directly instead of using EQ. (25) to obtain P_r. EQ. (21) depends on the degree of thermal accommodation that occurs through T_r, the temperature of the relected molecules. EQ. (29) depends on σ' which in turn depends on the degree of thermal accommodation occurring.

SECTION III

DETERMINATION OF AERODYNAMIC COEFFICIENTS FOR SIMPLE SHAPES IN A FREE MOLECULE FLOW

The results of the preceding section will be used to determine the normal force, axial force and moment coefficients for several different bodies in a free molecule flow. Since the variation of T_r is a complicated and, in general, an unknown function of location on the surface, we will assume that it is a constant over the surface.

Since a body in a free molecule flow does not disturb the flow, it is permissible to resolve a complicated shape into simple parts (cones, cylinders, etc.) and to compute the contributions of each part and then add them together to obtain the total coefficient for the entire vehicle. Thus, there is no need to consider a complicated shape in general. We have only considered simple shapes, i. e., flat plates, cylinder, cone and spherical segment, because these are the most common configurations of which a vehicle is composed. One must be careful to consider the shielding of one part by another when the body is divided into simple shapes. The equations presented here take into account only shielding such as the shielding of the back surface of a cylinder by its front surface, and thus care must be exercised in their use.

A. FLAT PLATE, BOTH SIDES EXPOSED TO THE FLOW.

Consider a flat plate oriented as follows.

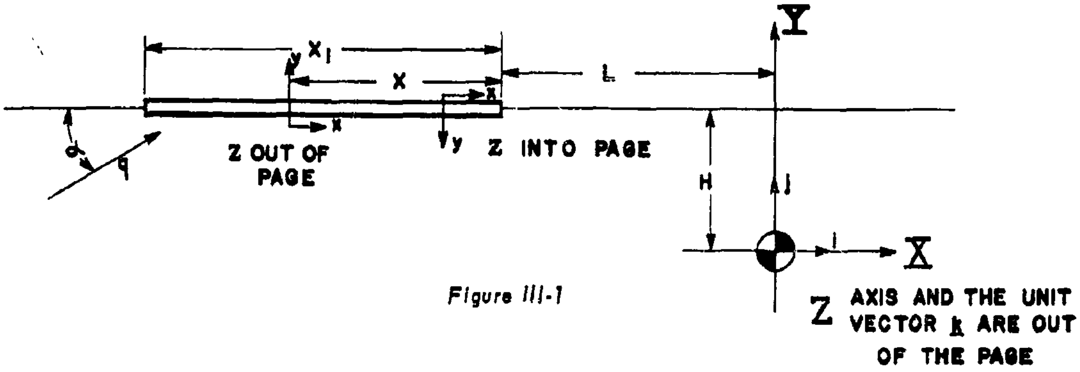


Figure III-1

We must always use the inward normal to the surface under consideration since eq. (21) Section II, Part B was derived using the inward normal. From Section II, Part B,

$$dC = \Big\{ (\epsilon k + \gamma \ell + \eta t)\left[\gamma(1+\operatorname{erf}\gamma s) + \frac{1}{s\sqrt{\pi}} e^{-\gamma^2 s^2}\right] + \frac{\ell}{2s^2}(1+\operatorname{erf}\gamma s) + \frac{\ell}{2}\sqrt{\frac{T_r}{T_i}}\left[\frac{\gamma\sqrt{\pi}}{s}(1+\operatorname{erf}\gamma s) + \frac{1}{s^2} e^{-\gamma^2 s^2}\right]\Big\} \frac{dA}{A_{ref}} \qquad (1)$$

The unit vector in the direction of the mass velocity, q, is

$$\underline{\bar{q}} = \underline{i}\cos\alpha + \underline{j}\sin\alpha$$

Unit vectors in the directions of the local axis systems are as follows.

For the bottom surface	For top surface
$\underline{x} = \underline{i}$	$\underline{x} = \underline{i}$
$\underline{y} = \underline{j}$	$\underline{y} = -\underline{j}$
$\underline{z} = \underline{k}$	$\underline{z} = -\underline{k}$
$\epsilon = \bar{\underline{q}} \cdot \underline{x} = \cos\alpha$	$\epsilon = \cos\alpha$
$\gamma = \bar{\underline{q}} \cdot \underline{y} = \sin\alpha$	$\gamma = -\sin\alpha$
$\eta = \bar{\underline{q}} \cdot \underline{z} = 0$	$\eta = 0$

Normal Force:

For the normal force, the direction cosines k, l and t are

For bottom surface	For top surface
$k = \underline{x} \cdot \underline{j} = 0$	$k = 0$
$\ell = \underline{y} \cdot \underline{j} = 1$	$\ell = -1$
$t = \underline{z} \cdot \underline{j} = 0$	$t = 0$

Substituting these values of $\epsilon, \gamma, \eta, k, \ell, t$ into the expression for dc gives

$$dC_N = \frac{1}{A_{ref}} \Big\{ \sin\alpha \Big[\gamma (1 + \operatorname{erf} \gamma s) + \frac{1}{s\sqrt{\pi}} e^{-\gamma^2 s^2} \Big] + \frac{\ell}{2s^2} (1 + \operatorname{erf} \gamma s)$$
$$+ \frac{\ell}{2}\sqrt{\frac{T_r}{T_i}} \Big[\frac{\gamma\sqrt{\pi}}{s} (1 + \operatorname{erf} \gamma s) + \frac{1}{s^2} e^{-\gamma^2 s^2} \Big] + \sin\alpha \Big[-\gamma \Big(1 + \operatorname{erf}(-\gamma s)\Big)$$
$$+ \frac{1}{s\sqrt{\pi}} e^{-\gamma^2 s^2} \Big] - \frac{\ell}{2s^2} \Big[1 + \operatorname{erf}(-\gamma s) \Big] - \frac{\ell}{2}\sqrt{\frac{T_r}{T_i}} \Big[-\frac{\gamma\sqrt{\pi}}{s} \Big(1 + \operatorname{erf}(-\gamma s)\Big) + \frac{1}{s^2} e^{-\gamma^2 s^2} \Big] \Big\} dx\,dz$$

where the relations $\gamma_{top} = -\gamma_{bottom}$ and $\ell_{top} = -\ell_{bottom}$ were used to express everything in terms of γ_{bottom} and ℓ_{bottom}.

Assume $\left(\sqrt{\frac{T_r}{T_i}}\right)_{BOTTOM} = \left(\sqrt{\frac{T_r}{T_i}}\right)_{TOP}$. Using the fact that erf $(-x) = -$erf x, dC_N becomes

$$dC_N = \frac{1}{A_{ref}}\left\{\sin\alpha\left[2\gamma\,\mathrm{erf}\,\gamma S + \frac{2}{S\sqrt{\pi}}e^{-\gamma^2 S^2}\right] + \frac{\ell}{S^2}\,\mathrm{erf}\,\gamma S + \sqrt{\frac{T_r}{T_i}}\,\frac{\ell\gamma\sqrt{\pi}}{S}\right\}dx\,dz$$

Integrating over the area of the plate gives

$$C_N = \frac{A}{A_{ref}}\left\{\left(2\sin^2\alpha + \frac{1}{S^2}\right)\mathrm{erf}\,S\sin\alpha + \frac{2\sin\alpha}{S\sqrt{\pi}}e^{-S^2\sin^2\alpha} + \sqrt{\frac{T_r}{T_i}}\,\frac{\sqrt{\pi}}{S}\sin\alpha\right\} \qquad (2)$$

Axial Force:

For the axial force, the direction cosines k, ℓ and t are

Bottom surface	Top surface
$k = \underline{x}\cdot\underline{i} = 1$	$k = 1$
$\ell = \underline{y}\cdot\underline{i} = 0$	$\ell = 0$
$t = \underline{z}\cdot\underline{i} = 0$	$t = 0$

Expressing everything in terms of ϵ_{bottom}, γ_{bottom}, k_{bottom}, dc_A is

$$dc_A = \frac{1}{A_{ref}}\left\{\cos\alpha\left[\gamma(1+\mathrm{erf}\,\gamma S) + \frac{1}{S\sqrt{\pi}}e^{-\gamma^2 S^2}\right] + \cos\alpha\left[-\gamma\left(1+\mathrm{erf}(-\gamma S)\right) + \frac{1}{S\sqrt{\pi}}e^{-\gamma^2 S^2}\right]\right\}dx\,dz$$

Integrating over the area of the plate gives

$$C_A = \frac{A}{A_{ref}}\left\{2\cos\alpha\sin\alpha\,\mathrm{erf}\,S\sin\alpha + \frac{2\cos\alpha}{S\sqrt{\pi}}e^{-S^2\sin^2\alpha}\right\} \qquad (3)$$

PITCHING MOMENT:

The pitching moment is the moment about the Z axis. Nose up moments are positive. Thus,

$$dC_M = \left\{ H\, dC_A + (L+X)\, dC_N \right\} \frac{1}{A_{ref} L_{ref}}$$

Substitute in for dC_A and dC_N and integrate. This gives

$$C_M = \frac{H}{L_{ref}} C_A + \frac{\left(L + \frac{X_1}{2}\right)}{L_{ref}} C_N \qquad (4)$$

Let

$$\left.\begin{aligned} I_1^F(S,\alpha) &= 2\cos\alpha \sin\alpha \operatorname{erf} S\sin\alpha + \frac{2\cos\alpha}{S\sqrt{\pi}} e^{-S^2\sin^2\alpha} \\ I_2^F(S,\alpha) &= \left(2\sin^2\alpha + \frac{1}{S^2}\right)\operatorname{erf} S\sin\alpha + \frac{2\sin\alpha}{S\sqrt{\pi}} e^{-S^2\sin^2\alpha} \\ I_3^F(S,\alpha) &= \frac{\sqrt{\pi}}{S}\sin\alpha \end{aligned}\right\} \qquad (5)$$

Then the expressions for C_N, C_A and C_M become

$$\left.\begin{aligned} C_A &= \frac{A}{A_{ref}} I_1^F \\ C_N &= \frac{A}{A_{ref}} \left\{ I_2^F + \sqrt{\frac{T_r}{T_i}}\, I_3^F \right\} \\ C_M &= \frac{H}{L_{ref}} C_A + \frac{\left(L + \frac{X_1}{2}\right)}{L_{ref}} C_N \end{aligned}\right\} \qquad (6)$$

If the edge of the plate to which L is measured is located to the right of the Y axis, L is negative; if the plate is located below the X axis, H is negative.

B. FLAT PLATE, ONE SIDE EXPOSED TO THE FLOW

We will now consider the case of a flat plate on the end of a cylinder or a cone. In this case, one surface of the plate is inside the body and thus is never exposed to the flow. The plate is located as follows.

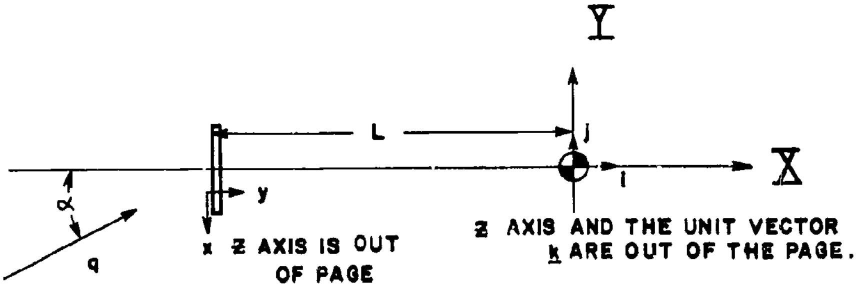


Figure III-2

The unit vector in the direction of the mass velocity is

$\bar{q} = \underline{i}\cos\alpha + \underline{j}\sin\alpha$.

The unit vectors in the direction of the local axis system are

$\underline{x} = -\underline{j}, \quad \underline{y} = \underline{i}, \quad \underline{z} = \underline{k}$

The direction cosines ϵ, γ, η are

$\epsilon = \bar{q} \cdot \underline{x} = -\sin\alpha, \quad \gamma = \bar{q} \cdot \underline{y} = \cos\alpha, \quad \eta = \bar{q} \cdot \underline{z} = 0$

Normal Force:

For the normal force, the direction cosines k, l and t are

$k = \underline{x} \cdot \underline{j} = -1$

$l = \underline{y} \cdot \underline{j} = 0$

$t = \underline{z} \cdot \underline{j} = 0$

Substituting into the expression for dc gives

$$dC_N = \frac{1}{A_{ref}}\left\{\sin\alpha\left[\cos\alpha\,(1+\operatorname{erf} s\cos\alpha) + \frac{1}{s\sqrt{\pi}}\,e^{-s^2\cos^2\alpha}\right]\right\} dA$$

Integrating over the area of the plate we obtain

$$C_N = \frac{A}{A_{ref}}\left\{\sin\alpha\cos\alpha\,(1+\operatorname{erf} s\cos\alpha) + \frac{\sin\alpha}{s\sqrt{\pi}}\,e^{-s^2\cos^2\alpha}\right\} \qquad (7)$$

AXIAL FORCE:

For the axial force, the direction cosines k, l and t are

$$k = \underline{x}\cdot\underline{i} = 0$$

$$l = \underline{y}\cdot\underline{i} = 1$$

$$t = \underline{z}\cdot\underline{i} = 0$$

Substituting into the expression for dc gives

$$dC_A = \frac{1}{A_{ref}}\left\{\cos\alpha\left[\cos\alpha\,(1+\operatorname{erf} s\cos\alpha) + \frac{1}{s\sqrt{\pi}}\,e^{-s^2\cos^2\alpha}\right]\right.$$
$$\left. + \frac{1}{2s^2}(1+\operatorname{erf} s\cos\alpha) + \frac{1}{2}\sqrt{\frac{T_r}{T_i}}\left[\frac{\sqrt{\pi}}{s}\cos\alpha\,(1+\operatorname{erf} s\cos\alpha) + \frac{1}{s^2}\,e^{-s^2\cos^2\alpha}\right]\right\} dA$$

Integrating over the surface we obtain

$$C_A = \frac{A}{A_{ref}}\left\{\left(\cos^2\alpha + \frac{1}{2S^2}\right)\left(1+\operatorname{erf} S\cos\alpha\right) + \frac{\cos\alpha}{S\sqrt{\pi}}\, e^{-S^2\cos^2\alpha} + \sqrt{\frac{T_r}{T_i}}\left[\frac{\sqrt{\pi}}{2S}\cos\alpha\left(1+\operatorname{erf} S\cos\alpha\right) + \frac{1}{2S^2}\, e^{-S^2\cos^2\alpha}\right]\right\} \tag{8}$$

Pitching Moment:

The pitching moment is the moment about the Z axis. Since the plate is symmetric about the Z axis, the axial force does not contribute to the pitching moment. Nose up moments are taken as positive. Thus

$$dC_M = \frac{L}{L_{ref}}\, dC_N$$

Integrating we obtain

$$C_M = \frac{L}{L_{ref}}\, C_N$$

Let

$$\left.\begin{aligned} I_1^{CF}(S,\alpha) &= \sin\alpha\cos\alpha\left(1+\operatorname{erf} S\cos\alpha\right) + \frac{\sin\alpha}{S\sqrt{\pi}}\, e^{-S^2\cos^2\alpha} \\ I_2^{CF}(S,\alpha) &= \left(\cos^2\alpha + \frac{1}{2S^2}\right)\left(1+\operatorname{erf} S\cos\alpha\right) + \frac{\cos\alpha}{S\sqrt{\pi}}\, e^{-S^2\cos^2\alpha} \\ I_3^{CF}(S,\alpha) &= \frac{\sqrt{\pi}}{2S}\cos\alpha\left(1+\operatorname{erf} S\cos\alpha\right) + \frac{1}{2S^2}\, e^{-S^2\cos^2\alpha} \end{aligned}\right\} \tag{9}$$

Then C_N, C_A and C_M are

$$\left.\begin{aligned} C_N &= \frac{A}{A_{ref}} \mathcal{I}_1^{CF} \\ C_A &= \frac{A}{A_{ref}} \left\{ \mathcal{I}_2^{CF} + \sqrt{\frac{T_r}{T_i}}\, \mathcal{I}_3^{CF} \right\} \\ C_M &= \frac{L}{L_{ref}} C_N \end{aligned}\right\} \qquad (10)$$

If the plate is to the right of the $\mathcal{Y}$ axis, L is negative.

C. CIRCULAR CYLINDER

Next we will consider a right circular cylinder oriented as shown below.

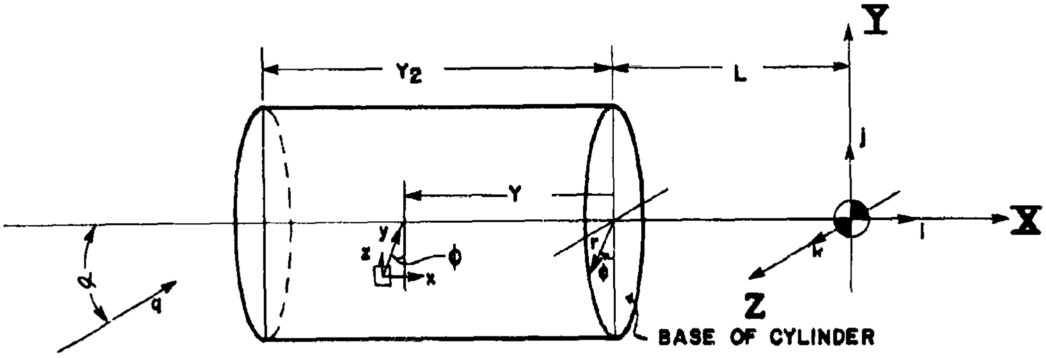


Figure III-3

We will neglect end effects in computing the forces on the cylinder. Since the cylinder is axially symmetric, it will always be possible to orient the body axis system so that the velocity vector is in the $\mathcal{X}$ - $\mathcal{Y}$ plane.

The element of area on the surface of the cylinder is

$$dA = r\,d\phi\,dy$$

The unit vectors in the direction of the local axis system are

$$\underline{x} = \underline{i}$$

$$\underline{y} = \underline{j}\cos\phi - \underline{k}\sin\phi$$

$$\underline{z} = \underline{j}\sin\phi + \underline{k}\cos\phi$$

The unit vector in the direction of the mass velocity is

$$\underline{\bar{g}} = \underline{i}\cos\alpha + \underline{j}\sin\alpha$$

The direction cosines ϵ, γ, η are

$$\epsilon = \underline{\bar{g}}\cdot\underline{x} = \cos\alpha$$

$$\gamma = \underline{\bar{g}}\cdot\underline{y} = \sin\alpha\cos\phi$$

$$\eta = \underline{\bar{g}}\cdot\underline{z} = \sin\alpha\sin\phi$$

The integration limits for ϕ are 0 to 2π, and for y they are 0 to y_2.

Normal force:

For the normal force, the direction cosines k, l and t are

$$k = \underline{x}\cdot\underline{j} = 0$$

$$l = \underline{y}\cdot\underline{j} = \cos\phi$$

$$t = \underline{z}\cdot\underline{j} = \sin\phi$$

Substituting these expressions for $\epsilon, \gamma, \eta, k, \ell, t$ into the equation for dc gives

$$dC_N = \frac{1}{A_{ref}} \Big\{ (\sin\alpha \cos^2\phi + \sin\alpha \sin^2\phi)\Big[\gamma(1+\operatorname{erf}\gamma s) + \frac{1}{s\sqrt{\pi}} e^{-\gamma^2 s^2}\Big]$$
$$+ \frac{\cos\phi}{2s^2}(1+\operatorname{erf}\gamma s) + \frac{\cos\phi}{2}\sqrt{\frac{T_r}{T_i}}\Big[\frac{\gamma\sqrt{\pi}}{s}(1+\operatorname{erf}\gamma s) + \frac{1}{s^2} e^{-\gamma^2 s^2}\Big]\Big\} r\,d\phi\,dy$$

Integrating over the surface of the cylinder we get

$$C_N = \frac{r y_2}{A_{ref}} \Big\{ \sin^2\alpha \Big(\int_0^{2\pi} \cos\phi\, d\phi + \int_0^{2\pi} \cos\phi \operatorname{erf}\gamma s\, d\phi \Big) + \frac{\sin\alpha}{s\sqrt{\pi}} \int_0^{2\pi} e^{-\gamma^2 s^2} d\phi$$
$$+ \frac{1}{2s^2}\Big(\int_0^{2\pi} \cos\phi\, d\phi + \int_0^{2\pi} \cos\phi \operatorname{erf}\gamma s\, d\phi \Big) + \sqrt{\frac{T_r}{T_i}} \Big[\frac{\sqrt{\pi}}{2s} \sin\alpha \Big(\int_0^{2\pi} \cos^2\phi\, d\phi$$
$$+ \int_0^{2\pi} \cos^2\phi \operatorname{erf}\gamma s\, d\phi \Big) + \frac{1}{2s^2} \int_0^{2\pi} \cos\phi\, e^{-\gamma^2 s^2} d\phi \Big] \Big\}$$

Simplifying and combining terms gives

$$C_N = \frac{r y_2}{A_{ref}} \Big\{ \Big(\sin^2\alpha + \frac{1}{2s^2}\Big) \int_0^{2\pi} \cos\phi \operatorname{erf}\gamma s\, d\phi + \frac{\sin\alpha}{s\sqrt{\pi}} \int_0^{2\pi} e^{-\gamma^2 s^2} d\phi$$
$$+ \sqrt{\frac{T_r}{T_i}}\, \frac{(\pi)^{3/2}}{2s} \sin\alpha \Big\}$$

The integrals are evaluated as follows

$$\int_0^{2\pi} \cos\phi \operatorname{erf}\gamma s\, d\phi = 2\int_0^{\pi} \cos\phi \operatorname{erf}\gamma s\, d\phi = 4\int_0^{\frac{\pi}{2}} \cos\phi \operatorname{erf}\gamma s\, d\phi$$
$$= 2s\sqrt{\pi} \sin\alpha\, e^{-\frac{s^2 \sin^2\alpha}{2}} \Big[I_0\Big(\frac{s^2 \sin^2\alpha}{2}\Big) + I_1\Big(\frac{s^2 \sin^2\alpha}{2}\Big) \Big] \qquad (11)$$

$$\int_0^{2\pi} e^{-\gamma^2 s^2} d\phi = 2\int_0^{\pi} e^{-\gamma^2 s^2} d\phi = 4\int_0^{\pi/2} e^{-\gamma^2 s^2} d\phi = 2\pi e^{-\frac{s^2 \sin^2\alpha}{2}} I_0\Big(\frac{s^2 \sin^2\alpha}{2}\Big) \qquad (12)$$

where I_o and I_1 are modified bessel functions of the first kind of degree zero and one respectively. Using these results C_N becomes

$$C_N = \frac{r y_2}{A_{ref}} \left\{ S\sqrt{\pi}\, \sin\alpha \left(2\sin^2\alpha + \frac{1}{S^2}\right) e^{-\frac{S^2\sin^2\alpha}{2}} \left[I_o\left(\frac{S^2\sin^2\alpha}{2}\right) + I_1\left(\frac{S^2\sin^2\alpha}{2}\right)\right] + \frac{2\sqrt{\pi}}{S}\sin\alpha\, e^{-\frac{S^2\sin^2\alpha}{2}} I_o\left(\frac{S^2\sin^2\alpha}{2}\right) + \sqrt{\frac{T_r}{T_i}}\,\frac{(\pi)^{3/2}}{2S}\sin\alpha \right\} \qquad (13)$$

AXIAL FORCE:

For the axial force, the direction cosines k, l and t are

$$k = \underline{x} \cdot i = 1$$
$$l = \underline{y} \cdot i = 0$$
$$t = \underline{z} \cdot i = 0$$

Using these values for k, l, t,

$$dC_A = \frac{1}{A_{ref}} \left\{ \cos\alpha \left[\gamma(1 + \mathrm{erf}\,\gamma S) + \frac{1}{S\sqrt{\pi}}\, e^{-\gamma^2 S^2} \right] \right\} r\, d\phi\, dy$$

Integrating over the surface we obtain

$$C_A = \frac{r y_2}{A_{ref}} \left\{ \sin\alpha\cos\alpha \left(\int_0^{2\pi} \cos\phi\, d\phi + \int_0^{2\pi} \cos\phi\, \mathrm{erf}\,\gamma s\, d\phi \right) + \frac{\cos\alpha}{s\sqrt{\pi}} \int_0^{2\pi} e^{-\gamma^2 s^2} d\phi \right\}$$

Substituting in the results of EQS. (11) and (12) gives

$$C_A = \frac{r y_2}{A_{ref}} \left\{ 2 S\sqrt{\pi} \sin^2\alpha \cos\alpha \, e^{-\frac{S^2 \sin^2\alpha}{2}} \left[I_0\left(\frac{S^2\sin^2\alpha}{2}\right) + I_1\left(\frac{S^2\sin^2\alpha}{2}\right) \right] \right.$$
$$\left. + \frac{2\sqrt{\pi}}{S} \cos\alpha \, e^{-\frac{S^2\sin^2\alpha}{2}} I_0\left(\frac{S^2\sin^2\alpha}{2}\right) \right\} \tag{14}$$

PITCHING MOMENT:

The pitching moment is the moment about the Z axis. Moments about the X and Y axes are zero due to symmetry. Nose up moments are taken as positive. Thus,

$$dC_M = \frac{1}{A_{ref} L_{ref}} \left\{ (L + y) dC_N - r\cos\phi \, dC_A \right\}$$

Integrating over the surface we get

$$C_M = \frac{1}{A_{ref} L_{ref}} \left\{ r y_2 \left(L + \frac{y_2}{2}\right) \left[S\sqrt{\pi} \sin\alpha \left(2\sin^2\alpha + \frac{1}{S^2}\right) e^{-\frac{S^2\sin^2\alpha}{2}} \left(I_0\left(\frac{S^2\sin^2\alpha}{2}\right) \right. \right. \right.$$
$$\left. + I_1\left(\frac{S^2\sin^2\alpha}{2}\right) \right) + \frac{2\sqrt{\pi}}{S} \sin\alpha \, e^{-\frac{S^2\sin^2\alpha}{2}} I_0\left(\frac{S^2\sin^2\alpha}{2}\right)$$
$$\left. + \sqrt{\frac{T_r}{T_i}} \frac{(\pi)^{3/2}}{2S} \sin\alpha \right] - r^2 y_2 \cos\alpha \left[\sin\alpha \left(\int_0^{2\pi} \cos^2\phi \, d\phi + \int_0^{2\pi} \cos^2\phi \, \mathrm{erf}\, \gamma S \, d\phi \right) \right.$$
$$\left. \left. + \frac{1}{S\sqrt{\pi}} \int_0^{2\pi} \cos\phi \, e^{-\gamma^2 S^2} d\phi \right] \right\}$$

Performing the indicated integrations gives

$$C_M = \frac{r y_2 \left(L + \frac{y_2}{2}\right)}{A_{ref} L_{ref}} \Bigg\{ s\sqrt{\pi}\sin\alpha\left(2\sin^2\alpha + \frac{1}{s^2}\right) e^{-\frac{s^2\sin^2\alpha}{2}} \left[I_0\left(\frac{s^2\sin^2\alpha}{2}\right) + I_1\left(\frac{s^2\sin^2\alpha}{2}\right)\right] + \frac{2\sqrt{\pi}}{s}\sin\alpha\, e^{-\frac{s^2\sin^2\alpha}{2}} I_0\left(\frac{s^2\sin^2\alpha}{2}\right) + \sqrt{\frac{T_r}{T_i}}\,\frac{(\pi)^{3/2}}{2s}\sin\alpha \Bigg\} - \frac{r^2 y_2}{A_{ref} L_{ref}}\,\pi\sin\alpha\cos\alpha \qquad (15)$$

Let

$$\left.\begin{aligned}
I_1^{cY}(s,\alpha) &= s\sqrt{\pi}\sin\alpha\left(2\sin^2\alpha + \frac{1}{s^2}\right) e^{-\frac{s^2\sin^2\alpha}{2}} \left[I_0\left(\frac{s^2\sin^2\alpha}{2}\right) + I_1\left(\frac{s^2\sin^2\alpha}{2}\right)\right] \\
&\quad + \frac{2\sqrt{\pi}}{s}\sin\alpha\, e^{-\frac{s^2\sin^2\alpha}{2}} I_0\left(\frac{s^2\sin^2\alpha}{2}\right) \\
I_2^{cY}(s,\alpha) &= \frac{(\pi)^{3/2}}{2s}\sin\alpha \\
I_3^{cY}(s,\alpha) &= \pi\sin\alpha\cos\alpha \\
I_4^{cY}(s,\alpha) &= 2s\sqrt{\pi}\sin^2\alpha\cos\alpha\, e^{-\frac{s^2\sin^2\alpha}{2}} \left[I_0\left(\frac{s^2\sin^2\alpha}{2}\right) + I_1\left(\frac{s^2\sin^2\alpha}{2}\right)\right] \\
&\quad + \frac{2\sqrt{\pi}}{s}\cos\alpha\, e^{-\frac{s^2\sin^2\alpha}{2}} I_0\left(\frac{s^2\sin^2\alpha}{2}\right)
\end{aligned}\right\} \quad (16)$$

Then C_N, C_A and C_M become

$$C_N = \frac{r y_2}{A_{ref}} \left\{ I_1^{cY} + \sqrt{\frac{T_r}{T_i}}\, I_2^{cY} \right\}$$

$$\left.\begin{aligned} C_A &= \frac{r y_2}{A_{ref}} \mathcal{I}_4^{cY} \\ C_M &= \frac{\left(L + \frac{y_2}{2}\right)}{L_{ref}} C_N - \frac{r^2 y_2}{A_{ref} L_{ref}} \mathcal{I}_3^{cY} \end{aligned}\right\} (17)$$

If the point about which the moment is taken is to the left of the base of the cylinder, L is negative. L as shown in the figure is positive.

D. CONE FRUSTUM

We will now compute the forces and moments on a cone frustum of circular cross section in a free molecule flow. The cone frustum is oriented as shown below. We will neglect end effects on the cone.

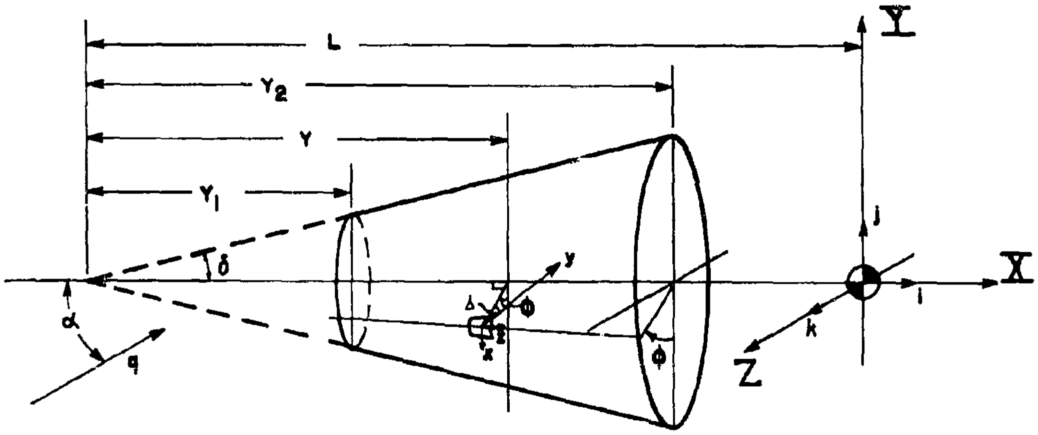


Figure III-4

From the above figure we see that

$$r = y \tan\delta$$

$$dr = \tan\delta \, dy$$

The element of surface area is

$$dA = r\,d\phi\,ds$$

$$(ds)^2 = (dy)^2 + (dr)^2 = (dy)^2(1+\tan^2\delta) = (dy)^2\sec^2\delta$$

$$ds = \sec\delta\,dy$$

$$\therefore \quad dA = y\tan\delta\sec\delta\,dy\,d\phi = y\frac{\tan\delta}{\cos\delta}\,dy\,d\phi$$

From the preceding figure, we see that the unit vectors in the direction of the local axis system are

$$\underline{x} = -\underline{j}\sin\phi - \underline{k}\cos\phi$$

$$\underline{y} = \underline{i}\sin\delta + \underline{j}\cos\delta\cos\phi - \underline{k}\cos\delta\sin\phi$$

$$\underline{z} = \underline{i}\cos\delta - \underline{j}\sin\delta\cos\phi + \underline{k}\sin\delta\sin\phi$$

The unit vector in the direction of the mass velocity is

$$\underline{\bar{g}} = \underline{i}\cos\alpha + \underline{j}\sin\alpha$$

Since the cone is axially symmetric, it will always be possible to orient the body axis system so that the velocity vector will be in the X-Y plane. From the definition of the direction cosines ϵ, γ, η, we find that

$$\epsilon = \underline{x}\cdot\underline{\bar{g}} = -\sin\alpha\sin\phi$$

$$\gamma = \underline{y}\cdot\underline{\bar{g}} = \cos\alpha\sin\delta + \sin\alpha\cos\delta\cos\phi$$

$$\eta = \underline{z}\cdot\underline{\bar{g}} = \cos\alpha\cos\delta - \sin\alpha\sin\delta\cos\phi$$

Normal Force:

For the normal force, the direction cosines k, l and t are

$$k = \underline{x} \cdot \underline{j} = -\sin\phi$$

$$l = \underline{y} \cdot \underline{j} = \cos\delta\cos\phi$$

$$t = \underline{z} \cdot \underline{j} = -\sin\delta\cos\phi$$

Substituting into eq. (1) for k, l, t, ϵ, γ, η gives for the normal force coefficient

$$\begin{aligned} dC_N = \frac{1}{A_{ref}} \Big\{ & \big(\sin\alpha\sin^2\phi + \cos\alpha\sin\delta\cos\delta\cos\phi + \sin\alpha\cos^2\delta\cos^2\phi \\ & - \cos\alpha\cos\delta\sin\delta\cos\phi + \sin\alpha\sin^2\delta\cos^2\phi\big)\left[\gamma(1+\operatorname{erf}\gamma S) + \frac{1}{S\sqrt{\pi}}e^{-\gamma^2S^2}\right] \\ & + \frac{\cos\delta\cos\phi}{2S^2}(1+\operatorname{erf}\gamma S) + \frac{\cos\delta\cos\phi}{2}\sqrt{\frac{T_r}{T_i}}\left[\frac{\gamma\sqrt{\pi}}{S}(1+\operatorname{erf}\gamma S) + \frac{1}{S^2}e^{-\gamma^2S^2}\right]\Big\} dA \end{aligned}$$

Simplifying this expression we get

$$\begin{aligned} dC_N = \frac{\tan\delta}{A_{ref}\cos\delta}\, y\,dy \Big\{ & \sin\alpha\left[\gamma(1+\operatorname{erf}\gamma S) + \frac{1}{S\sqrt{\pi}}e^{-\gamma^2S^2}\right] \\ & + \frac{\cos\delta\cos\phi}{2S^2}(1+\operatorname{erf}\gamma S) + \frac{\cos\delta\cos\phi}{2}\sqrt{\frac{T_r}{T_i}}\left[\frac{\gamma\sqrt{\pi}}{S}(1+\operatorname{erf}\gamma S) + \frac{1}{S^2}e^{-\gamma^2S^2}\right]\Big\} d\phi \end{aligned}$$

The integration limits on ϕ are 0 to 2π and on y they are y_1 to y_2. Performing the indicated integrations we obtain

$$C_N = \frac{\tan\delta}{\cos\delta}\,\frac{y_2^2 - y_1^2}{2A_{ref}}\left\{\sin\alpha\left[\int_0^{2\pi}\gamma\,d\phi + \int_0^{2\pi}\gamma\,\mathrm{erf}\,\gamma s\,d\phi + \frac{1}{s\sqrt{\pi}}\int_0^{2\pi}e^{-\gamma^2 s^2}d\phi\right]\right.$$

$$+\frac{\cos\delta}{2s^2}\left(\int_0^{2\pi}\cos\phi\,d\phi + \int_0^{2\pi}\cos\phi\,\mathrm{erf}\,\gamma s\,d\phi\right)$$

$$\left.+\sqrt{\frac{T_r}{T_i}}\left[\frac{\sqrt{\pi}}{2s}\cos\delta\left(\int_0^{2\pi}\gamma\cos\phi\,d\phi + \int_0^{2\pi}\gamma\cos\phi\,\mathrm{erf}\,\gamma s\,d\phi\right) + \frac{\cos\delta}{2s^2}\int_0^{2\pi}\cos\phi\,e^{-\gamma^2 s^2}d\phi\right]\right\}$$

Investigation of the above integrals gives two results. First, the integrals involving $\mathrm{erf}\,\gamma s$ and $e^{-\gamma^2 s^2}$ cannot be evaluated in closed form, and second, these integrals are symmetric about $\phi = \pi$, i.e. the integral from 0 to 2π is equal to two times the integral from 0 to π. Then performing the integrations that can be accomplished gives

$$C_N = \frac{\tan\delta}{\cos\delta}\,\frac{y_2^2 - y_1^2}{2A_{ref}}\left\{2\pi\sin\alpha\cos\alpha\sin\delta + 2\sin\alpha\cos\alpha\sin\delta\int_0^{\pi}\mathrm{erf}\,\gamma s\,d\phi\right.$$

$$+\cos\delta\left(2\sin^2\alpha + \frac{1}{s^2}\right)\int_0^{\pi}\cos\phi\,\mathrm{erf}\,\gamma s\,d\phi + \frac{2\sin\alpha}{s\sqrt{\pi}}\int_0^{\pi}e^{-\gamma^2 s^2}d\phi$$

$$+\sqrt{\frac{T_r}{T_i}}\left[\frac{(\pi)^{3/2}}{2s}\sin\alpha\cos^2\delta + \frac{\sqrt{\pi}\cos\delta}{s}\left(\cos\alpha\sin\delta\int_0^{\pi}\cos\phi\,\mathrm{erf}\,\gamma s\,d\phi\right.\right.$$

$$\left.\left.\left.+\sin\alpha\cos\delta\int_0^{\pi}\cos^2\phi\,\mathrm{erf}\,\gamma s\,d\phi\right) + \frac{\cos\delta}{s^2}\int_0^{\pi}\cos\phi\,e^{-\gamma^2 s^2}d\phi\right]\right\} \qquad (18)$$

AXIAL FORCE:

For the axial force, the direction cosines k, l and t are

$k = \underline{x}\cdot\underline{i} = 0$

$l = \underline{y}\cdot\underline{i} = \sin\delta$

$t = \underline{z}\cdot\underline{i} = \cos\delta$

Substituting into eq. (1) for k, l, t, ϵ , γ , η gives

$$dC_A = \frac{\tan\delta}{\cos\delta}\,\frac{y\,dy}{A_{ref}}\Big\{\Big(\cos\alpha\sin^2\delta+\sin\alpha\cos\delta\sin\delta\cos\phi+\cos\alpha\cos^2\delta$$
$$-\sin\alpha\sin\delta\cos\delta\cos\phi\Big)\Big[\gamma(1+\operatorname{erf}\gamma s)+\frac{1}{s\sqrt{\pi}}\,e^{-\gamma^2s^2}\Big]+\frac{\sin\delta}{2s^2}(1+\operatorname{erf}\gamma s)$$
$$+\frac{\sin\delta}{2}\sqrt{\frac{T_r}{T_i}}\Big[\frac{\gamma\sqrt{\pi}}{s}(1+\operatorname{erf}\gamma s)+\frac{1}{s^2}\,e^{-\gamma^2s^2}\Big]\Big\}\,d\phi$$

Integrating over the surface we get

$$C_A = \frac{\tan\delta}{\cos\delta}\,\frac{y_2^2-y_1^2}{2A_{ref}}\Big\{\cos\alpha\Big[\int_0^{2\pi}\gamma\,d\phi+\int_0^{2\pi}\gamma\operatorname{erf}\gamma s\,d\phi+\frac{1}{s\sqrt{\pi}}\int_0^{2\pi}e^{-\gamma^2s^2}\,d\phi\Big]$$
$$+\frac{\sin\delta}{2s^2}\Big(2\pi+\int_0^{2\pi}\operatorname{erf}\gamma s\,d\phi\Big)+\frac{\sin\delta}{2}\sqrt{\frac{T_r}{T_i}}\Big[\frac{\sqrt{\pi}}{s}\Big(\int_0^{2\pi}\gamma\,d\phi+\int_0^{2\pi}\gamma\operatorname{erf}\gamma s\,d\phi\Big)$$
$$+\frac{1}{s^2}\int_0^{2\pi}e^{-\gamma^2s^2}\,d\phi\Big]\Big\}$$

Simplifying further we obtain

$$C_A = \frac{\tan\delta}{\cos\delta}\,\frac{y_2^2-y_1^2}{2A_{ref}}\Big\{\pi\sin\delta\Big(2\cos^2\alpha+\frac{1}{s^2}\Big)+\sin\delta\Big(2\cos^2\alpha+\frac{1}{s^2}\Big)\int_0^{\pi}\operatorname{erf}\gamma s\,d\phi$$
$$+2\sin\alpha\cos\alpha\cos\delta\int_0^{\pi}\cos\phi\operatorname{erf}\gamma s\,d\phi+\frac{2\cos\alpha}{s\sqrt{\pi}}\int_0^{\pi}e^{-\gamma^2s^2}\,d\phi$$
$$+\sqrt{\frac{T_r}{T_i}}\Big[\frac{(\pi)^{3/2}}{s}\cos\alpha\sin^2\delta+\frac{\sqrt{\pi}\sin\delta}{s}\Big(\cos\alpha\sin\delta\int_0^{\pi}\operatorname{erf}\gamma s\,d\phi$$
$$+\sin\alpha\cos\delta\int_0^{\pi}\cos\phi\operatorname{erf}\gamma s\,d\phi\Big)+\frac{\sin\delta}{s^2}\int_0^{\pi}e^{-\gamma^2s^2}\,d\phi\Big]\Big\} \qquad (19)$$

PITCHING MOMENT:

The pitching moment is the moment about the Z axis. Nose up moments are positive. Due to symmetry, the moments about the X and Y axes are zero. The moment due to the force on an element of area is

$$dC_M = \frac{(L-y)}{L_{ref}} dC_N - \frac{y}{L_{ref}} \tan\delta \cos\phi \, dC_A$$

Substitute in for dC_N and dC_A. This gives

$$dC_M = \frac{\tan\delta}{\cos\delta} \frac{y}{L_{ref} A_{ref}} \Bigg[(L-y) \Big\{ \sin\alpha \Big[\gamma(1+\operatorname{erf}\gamma S) + \frac{1}{S\sqrt{\pi}} e^{-\gamma^2 S^2} \Big]$$

$$+ \frac{\cos\delta\cos\phi}{2S^2}(1+\operatorname{erf}\gamma S) + \frac{\cos\delta\cos\phi}{2}\sqrt{\frac{T_r}{T_i}} \Big[\frac{\gamma\sqrt{\pi}}{S}(1+\operatorname{erf}\gamma S) + \frac{1}{S^2} e^{-\gamma^2 S^2} \Big] \Big\}$$

$$- y \tan\delta\cos\phi \Big\{ \cos\alpha \Big[\gamma(1+\operatorname{erf}\gamma S) + \frac{1}{S\sqrt{\pi}} e^{-\gamma^2 S^2} \Big] + \frac{\sin\delta}{2S^2}(1+\operatorname{erf}\gamma S)$$

$$+ \frac{\sin\delta}{2}\sqrt{\frac{T_r}{T_i}} \Big[\frac{\gamma\sqrt{\pi}}{S}(1+\operatorname{erf}\gamma S) + \frac{1}{S^2} e^{-\gamma^2 S^2} \Big] \Big\} \Bigg] dy\, d\phi$$

Performing the indicated integrations gives

$$C_M = \frac{\tan\delta}{\cos\delta \, L_{ref} A_{ref}} \Bigg\{ \Big[L\Big(\frac{y_2^2 - y_1^2}{2}\Big) - \Big(\frac{y_2^3 - y_1^3}{3}\Big) \Big] \Big[\sin\alpha \Big(\int_0^{2\pi} \gamma \, d\phi$$

$$+ \int_0^{2\pi} \gamma \operatorname{erf}\gamma S \, d\phi + \frac{1}{S\sqrt{\pi}} \int_0^{2\pi} e^{-\gamma^2 S^2} d\phi \Big) + \frac{\cos\delta}{2S^2} \int_0^{2\pi} \cos\phi \operatorname{erf}\gamma S \, d\phi$$

$$+ \frac{\cos\delta}{2}\sqrt{\frac{T_r}{T_i}} \Big(\frac{\sqrt{\pi}}{S} \int_0^{2\pi} \gamma\cos\phi \, d\phi + \frac{\sqrt{\pi}}{S} \int_0^{2\pi} \gamma \cos\phi \operatorname{erf}\gamma S \, d\phi + \frac{1}{S^2} \int_0^{2\pi} \cos\phi \, e^{-\gamma^2 S^2} d\phi \Big) \Big]$$

$$
\begin{aligned}
&-\tan\delta\left(\frac{y_2^3-y_1^3}{3}\right)\Bigg[\cos\alpha\Big(\int_0^{2\pi}\gamma\cos\phi\,d\phi+\int_0^{2\pi}\gamma\cos\phi\,\mathrm{erf}\,\gamma S\,d\phi\\
&+\frac{1}{S\sqrt{\pi}}\int_0^{2\pi}\cos\phi\,e^{-\gamma^2S^2}d\phi\Big)+\frac{\sin\delta}{2S^2}\Big(\int_0^{2\pi}\cos\phi\,d\phi+\int_0^{2\pi}\cos\phi\,\mathrm{erf}\,\gamma S\,d\phi\Big)\\
&+\frac{\sin\delta}{2}\sqrt{\frac{T_r}{T_i}}\Big(\frac{\sqrt{\pi}}{S}\int_0^{2\pi}\gamma\cos\phi\,d\phi+\frac{\sqrt{\pi}}{S}\int_0^{2\pi}\gamma\cos\phi\,\mathrm{erf}\,\gamma S\,d\phi+\frac{1}{S^2}\int_0^{2\pi}\cos\phi\,e^{-\gamma^2S^2}d\phi\Big)\Bigg]\Bigg\}
\end{aligned}
$$

Simplifying and combining terms gives

$$
\begin{aligned}
C_M=&\frac{1}{A_{ref}L_{ref}}\frac{\tan\delta}{\cos\delta}\Bigg\{\Big[L\left(\frac{y_2^2-y_1^2}{2}\right)-\left(\frac{y_2^3-y_1^3}{3}\right)\Big]\Big[2\pi\sin\alpha\cos\alpha\sin\delta\\
&+2\sin\alpha\cos\alpha\sin\delta\int_0^{\pi}\mathrm{erf}\,\gamma S\,d\phi+\cos\delta\left(2\sin^2\alpha+\frac{1}{S^2}\right)\int_0^{\pi}\cos\phi\,\mathrm{erf}\,\gamma S\,d\phi\\
&+\frac{2\sin\alpha}{S\sqrt{\pi}}\int_0^{\pi}e^{-\gamma^2S^2}d\phi+\sqrt{\frac{T_r}{T_i}}\Big[\frac{(\pi)^{3/2}}{2S}\sin\alpha\cos^2\delta+\frac{\sqrt{\pi}}{S}\cos\delta\Big(\cos\alpha\sin\delta\int_0^{\pi}\cos\phi\,\mathrm{erf}\,\gamma S\,d\phi\\
&+\sin\alpha\cos\delta\int_0^{\pi}\cos^2\phi\,\mathrm{erf}\,\gamma S\,d\phi\Big)+\frac{\cos\delta}{S^2}\int_0^{\pi}\cos\phi\,e^{-\gamma^2S^2}d\phi\Big]\Big]\\
&-\tan\delta\left(\frac{y_2^3-y_1^3}{3}\right)\Big[\pi\cos\alpha\sin\alpha\cos\delta+\sin\delta\left(2\cos^2\alpha+\frac{1}{S^2}\right)\int_0^{\pi}\cos\phi\,\mathrm{erf}\,\gamma S\,d\phi\\
&+2\cos\alpha\sin\alpha\cos\delta\int_0^{\pi}\cos^2\phi\,\mathrm{erf}\,\gamma S\,d\phi+\frac{2\cos\alpha}{S\sqrt{\pi}}\int_0^{\pi}\cos\phi\,e^{-\gamma^2S^2}d\phi\\
&+\sqrt{\frac{T_r}{T_i}}\Big[\frac{(\pi)^{3/2}}{2S}\sin\alpha\cos\delta\sin\delta+\frac{\sqrt{\pi}}{S}\sin\delta\Big(\cos\alpha\sin\delta\int_0^{\pi}\cos\phi\,\mathrm{erf}\,\gamma S\,d\phi\\
&+\sin\alpha\cos\delta\int_0^{\pi}\cos^2\phi\,\mathrm{erf}\,\gamma S\,d\phi\Big)+\frac{\sin\delta}{S^2}\int_0^{\pi}\cos\phi\,e^{-\gamma^2S^2}d\phi\Big]\Big]\Bigg\} \qquad (20)
\end{aligned}
$$

We will now define the following functions. Let

$$\left.\begin{aligned}
I_1^{co}(s,\alpha,\delta) &= 2\pi\sin\alpha\cos\alpha\sin\delta + 2\sin\alpha\cos\alpha\sin\delta\int_0^\pi \operatorname{erf}\gamma s\,d\phi \\
&\quad + \cos\delta\left(2\sin^2\alpha+\frac{1}{s^2}\right)\int_0^\pi \cos\phi\operatorname{erf}\gamma s\,d\phi + \frac{2\sin\alpha}{s\sqrt{\pi}}\int_0^\pi e^{-\gamma^2 s^2}\,d\phi \\
I_2^{co}(s,\alpha,\delta) &= \frac{(\pi)^{3/2}}{2s}\sin\alpha\cos^2\delta + \frac{\sqrt{\pi}}{s}\cos\delta\left[\cos\alpha\sin\delta\int_0^\pi \cos\phi\operatorname{erf}\gamma s\,d\phi\right. \\
&\quad \left.+ \sin\alpha\cos\delta\int_0^\pi \cos^2\phi\operatorname{erf}\gamma s\,d\phi\right] + \frac{\cos\delta}{s^2}\int_0^\pi \cos\phi\, e^{-\gamma^2 s^2}\,d\phi \\
I_3^{co}(s,\alpha,\delta) &= \pi\cos\alpha\sin\alpha\cos\delta + \sin\delta\left(2\cos^2\alpha+\frac{1}{s^2}\right)\int_0^\pi \cos\phi\operatorname{erf}\gamma s\,d\phi \\
&\quad + 2\cos\alpha\sin\alpha\cos\delta\int_0^\pi \cos^2\phi\operatorname{erf}\gamma s\,d\phi + \frac{2\cos\alpha}{s\sqrt{\pi}}\int_0^\pi \cos\phi\, e^{-\gamma^2 s^2}\,d\phi \\
I_4^{co}(s,\alpha,\delta) &= \frac{(\pi)^{3/2}}{2s}\sin\alpha\cos\delta\sin\delta + \frac{\sqrt{\pi}}{s}\sin\delta\left[\cos\alpha\sin\delta\int_0^\pi \cos\phi\operatorname{erf}\gamma s\,d\phi\right. \\
&\quad \left.+ \sin\alpha\cos\delta\int_0^\pi \cos^2\phi\operatorname{erf}\gamma s\,d\phi\right] + \frac{\sin\delta}{s^2}\int_0^\pi \cos\phi\, e^{-\gamma^2 s^2}\,d\phi \\
I_5^{co}(s,\alpha,\delta) &= \pi\left(2\cos^2\alpha+\frac{1}{s^2}\right)\sin\delta + \sin\delta\left(2\cos^2\alpha+\frac{1}{s^2}\right)\int_0^\pi \operatorname{erf}\gamma s\,d\phi \\
&\quad + 2\cos\alpha\sin\alpha\cos\delta\int_0^\pi \cos\phi\operatorname{erf}\gamma s\,d\phi + \frac{2\cos\alpha}{s\sqrt{\pi}}\int_0^\pi e^{-\gamma^2 s^2}\,d\phi \\
I_6^{co}(s,\alpha,\delta) &= \frac{(\pi)^{3/2}}{s}\cos\alpha\sin^2\delta + \frac{\sqrt{\pi}}{s}\sin\delta\left[\cos\alpha\sin\delta\int_0^\pi \operatorname{erf}\gamma s\,d\phi\right. \\
&\quad \left.+ \sin\alpha\cos\delta\int_0^\pi \cos\phi\operatorname{erf}\gamma s\,d\phi\right] + \frac{\sin\delta}{s^2}\int_0^\pi e^{-\gamma^2 s^2}\,d\phi
\end{aligned}\right\} \quad (21)$$

where $\gamma = \cos\alpha\sin\delta + \sin\alpha\cos\delta\cos\phi$

Using these relations, the expressions for C_A, C_N, and C_M for the cone become

$$\left.\begin{aligned}
C_N &= \frac{y_2^2-y_1^2}{2A_{ref}}\,\frac{\tan\delta}{\cos\delta}\left\{I_1^{co}+\sqrt{\frac{T_r}{T_i}}\,I_2^{co}\right\}\\
C_A &= \frac{y_2^2-y_1^2}{2A_{ref}}\,\frac{\tan\delta}{\cos\delta}\left\{I_5^{co}+\sqrt{\frac{T_r}{T_i}}\,I_6^{co}\right\}\\
C_M &= \frac{1}{A_{ref}L_{ref}}\,\frac{\tan\delta}{\cos\delta}\left\{\left[L\left(\frac{y_2^2-y_1^2}{2}\right)-\left(\frac{y_2^3-y_1^3}{3}\right)\right]\left[I_1^{co}+\sqrt{\frac{T_r}{T_i}}\,I_2^{co}\right]\right.\\
&\qquad\left.-\tan\delta\left(\frac{y_2^3-y_1^3}{3}\right)\left[I_3^{co}+\sqrt{\frac{T_r}{T_i}}\,I_4^{co}\right]\right\}
\end{aligned}\right\}(22)$$

If the point about which the moment is being taken is to the left of the vertex of the frustum of the cone, L is negative.

E. SPHERICAL SEGMENT

We will now compute the forces and moments on a spherical segment in a free molecule flow. Since the spherical segment is axially symmetric, it is always possible to consider the mass velocity vector in a plane of symmetry. We will neglect the effect of molecules impinging on the "back" side of the spherical segment. The spherical segment is oriented as shown below.

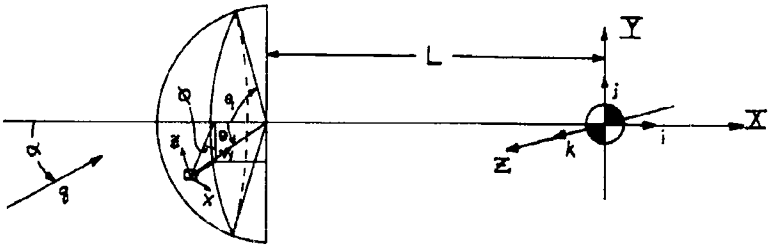


Figure III-5

The element of area on a sphere is

$$dA = r^2 \sin\theta \, d\phi \, d\theta$$

From the above figure we see that the unit vector in the direction of the mass velocity is

$$\bar{\underline{g}} = \underline{i}\cos\alpha + \underline{j}\sin\alpha$$

The unit vectors in the direction of the local axis system are

$$\underline{x} = \underline{i}\sin\theta - \underline{j}\cos\theta\cos\phi + \underline{k}\cos\theta\sin\phi$$

$$\underline{y} = \underline{i}\cos\theta + \underline{j}\sin\theta\cos\phi - \underline{k}\sin\theta\sin\phi$$

$\underline{x}$ can be arbitarily chosen in the plane containing $\underline{y}$ and the X axis such that it makes a right angle with $\underline{y}$. The orientation of $\underline{z}$ is then fixed. To find the orientation of $\underline{z}$, we use the fact that $\underline{z}$ is normal to both $\underline{x}$ and $\underline{y}$ and that the dot product of $\underline{z}$ with itself is one. Thus, let $\underline{z} = a\underline{i} + b\underline{j} + c\underline{k}$
Then

$$\underline{z}\cdot\underline{x} = 0 = a\sin\theta - b\cos\theta\cos\phi + c\cos\theta\sin\phi$$
$$\underline{z}\cdot\underline{y} = 0 = a\cos\theta + b\sin\theta\cos\phi - c\sin\theta\sin\phi$$
$$\underline{z}\cdot\underline{z} = 1 = a^2 + b^2 + c^2$$

This gives us three equations in the three unknowns a, b, and c which we can now solve. The solution of this system of equations gives for $\underline{z}$,

$$\underline{z} = \underline{j}\sin\phi + \underline{k}\cos\phi$$

From the definitions of the direction cosines ϵ , γ , η we find that

$$\epsilon = \underline{\bar{q}} \cdot \underline{x} = \cos\alpha \sin\theta - \sin\alpha \cos\theta \cos\phi$$

$$\gamma = \underline{\bar{q}} \cdot \underline{y} = \cos\alpha \cos\theta + \sin\alpha \sin\theta \cos\phi$$

$$\eta = \underline{\bar{q}} \cdot \underline{z} = \sin\alpha \sin\phi$$

The integration limits on ϕ are 0 to 2π and on θ they are 0 to θ_1.

NORMAL FORCE:

For the normal force, the direction cosines k, l and t are

$$k = \underline{x} \cdot \underline{j} = -\cos\theta \cos\phi$$

$$l = \underline{y} \cdot \underline{j} = \sin\theta \cos\phi$$

$$t = \underline{z} \cdot \underline{j} = \sin\phi$$

Substituting into Eq. (1) the above values of k, l, t, ϵ , γ , η gives

$$\begin{aligned} dC_N = \frac{r^2 \sin\theta}{A_{ref}} \Big\{ \Big(& -\cos\alpha \sin\theta \cos\theta \cos\phi + \sin\alpha \cos^2\theta \cos^2\phi + \cos\alpha \cos\theta \sin\theta \cos\phi \\ & + \sin\alpha \sin^2\theta \cos^2\phi + \sin\alpha \sin^2\phi \Big) \Big[\gamma (1 + \operatorname{erf} \gamma S) + \frac{1}{S\sqrt{\pi}} e^{-\gamma^2 S^2} \Big] \\ & + \frac{\sin\theta \cos\phi}{2 S^2} (1 + \operatorname{erf} \gamma S) + \frac{\sin\theta \cos\phi}{2} \sqrt{\frac{T_r}{T_i}} \Big[\frac{\gamma_1 \sqrt{\pi}}{S} (1 + \operatorname{erf} \gamma S) \\ & + \frac{1}{S^2} e^{-\gamma^2 S^2} \Big] \Big\} d\phi \, d\theta \end{aligned}$$

Simplifying we get

$$\begin{aligned} dC_N = \frac{r^2}{A_{ref}} \Big\{ & \sin\alpha \sin\theta \Big[\gamma (1 + \operatorname{erf} \gamma S) + \frac{1}{S\sqrt{\pi}} e^{-\gamma^2 S^2} \Big] + \frac{\sin^2\theta \cos\phi}{2 S^2} (1 + \operatorname{erf} \gamma S) \\ & + \frac{\sin^2\theta \cos\phi}{2} \sqrt{\frac{T_r}{T_i}} \Big[\frac{\gamma \sqrt{\pi}}{S} (1 + \operatorname{erf} \gamma S) + \frac{1}{S^2} e^{-\gamma^2 S^2} \Big] \Big\} d\phi \, d\theta \end{aligned}$$

Integrating the preceding equation we obtain

$$\begin{aligned}C_N = \frac{r^2}{A_{ref}}\Big\{\sin\alpha\Big[&\int_0^{\theta_1}\int_0^{2\pi}\gamma\sin\theta\,d\phi\,d\theta + \int_0^{\theta_1}\int_0^{2\pi}\gamma\sin\theta\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta \\ &+ \frac{1}{S\sqrt{\pi}}\int_0^{\theta_1}\int_0^{2\pi}\sin\theta\,e^{-\gamma^2S^2}\,d\phi\,d\theta\Big] + \frac{1}{2S^2}\Big(\int_0^{\theta_1}\int_0^{2\pi}\sin^2\theta\cos\phi\,d\phi\,d\theta \\ &+ \int_0^{\theta_1}\int_0^{2\pi}\sin^2\theta\cos\phi\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta\Big) + \sqrt{\frac{T_r}{T_i}}\Big[\frac{\sqrt{\pi}}{2S}\Big(\int_0^{\theta_1}\int_0^{2\pi}\gamma\sin^2\theta\cos\phi\,d\phi\,d\theta \\ &+ \int_0^{\theta_1}\int_0^{2\pi}\gamma\sin^2\theta\cos\phi\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta\Big) + \frac{1}{2S^2}\int_0^{\theta_1}\int_0^{2\pi}\sin^2\theta\cos\phi\,e^{-\gamma^2S^2}\,d\phi\,d\theta\Big]\Big\}\end{aligned}$$

Investigations of the resulting integrals shows two things. First, the integrals involving erf γS and $e^{-\gamma^2 S^2}$ cannot be evaluated in closed form, and second, the ϕ part of these double integrals is symmetric about $\phi = \pi$, that is, the integral over ϕ from 0 to 2π is equal to two times the integral over ϕ from 0 to π. Performing the indicated integrations that yield closed form solutions and using the symmetry property of the ϕ integrals gives

$$\begin{aligned}C_N = \frac{r^2}{A_{ref}}\Big\{&\pi\sin\alpha\cos\alpha\sin^2\theta_1 + 2\sin\alpha\cos\alpha\int_0^{\theta_1}\int_0^{\pi}\sin\theta\cos\theta\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta \\ &+ \Big(2\sin^2\alpha + \frac{1}{S^2}\Big)\int_0^{\theta_1}\int_0^{\pi}\sin^2\theta\cos\phi\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta + \frac{2\sin\alpha}{S\sqrt{\pi}}\int_0^{\theta_1}\int_0^{\pi}\sin\theta\,e^{-\gamma^2S^2}\,d\phi\,d\theta \\ &+ \sqrt{\frac{T_r}{T_i}}\Big[\frac{(\pi)^{3/2}}{2S}\sin\alpha\Big(\frac{\cos^3\theta_1}{3} - \cos\theta_1 + \frac{2}{3}\Big) + \frac{\sqrt{\pi}}{S}\Big(\cos\alpha\int_0^{\theta_1}\int_0^{\pi}\cos\theta\sin^2\theta\cos\phi\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta \\ &+ \sin\alpha\int_0^{\theta_1}\int_0^{\pi}\sin^3\theta\cos^2\phi\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta\Big) + \frac{1}{S^2}\int_0^{\theta_1}\int_0^{\pi}\sin^2\theta\cos\phi\,e^{-\gamma^2S^2}\,d\phi\,d\theta\Big]\Big\} \quad (23)\end{aligned}$$

AXIAL FORCE:

For the axial force, the direction cosines k, l and t are

$$k = \underline{x} \cdot \underline{i} = \sin\theta$$

$$l = \underline{y} \cdot \underline{i} = \cos\theta$$

$$t = \underline{z} \cdot \underline{i} = 0$$

Substituting these values of k, l, t into eq. (1) gives

$$dC_A = \frac{r^2}{A_{ref}} \sin\theta \Big\{ (\cos\alpha \sin^2\theta - \sin\alpha \cos\theta \sin\theta \cos\phi + \cos\alpha \cos^2\theta + \sin\alpha \sin\theta \cos\theta \cos\phi)\Big[\gamma(1 + \operatorname{erf}\gamma S) + \frac{1}{S\sqrt{\pi}} e^{-\gamma^2 S^2}\Big] + \frac{\cos\theta}{2S^2}(1 + \operatorname{erf}\gamma S) + \frac{\cos\theta}{2}\sqrt{\frac{T_r}{T_i}}\Big[\frac{\gamma\sqrt{\pi}}{S}(1 + \operatorname{erf}\gamma S) + \frac{1}{S^2} e^{-\gamma^2 S^2}\Big]\Big\} d\phi\, d\theta$$

Simplifying the above expression and integrating over the surface gives

$$C_A = \frac{r^2}{A_{ref}} \Big\{ \cos\alpha \Big[\int_0^{\theta_1}\int_0^{2\pi} \gamma \sin\theta\, d\phi\, d\theta + \int_0^{\theta_1}\int_0^{2\pi} \gamma \sin\theta \operatorname{erf}\gamma S\, d\phi\, d\theta + \frac{1}{S\sqrt{\pi}} \int_0^{\theta_1}\int_0^{2\pi} \sin\theta\, e^{-\gamma^2 S^2} d\phi\, d\theta \Big] + \frac{1}{2S^2}\Big(\int_0^{\theta_1}\int_0^{2\pi} \sin\theta \cos\theta\, d\phi\, d\theta + \int_0^{\theta_1}\int_0^{2\pi} \sin\theta \cos\theta \operatorname{erf}\gamma S\, d\phi\, d\theta \Big) + \sqrt{\frac{T_r}{T_i}} \Big[\frac{\sqrt{\pi}}{2S} \Big(\int_0^{\theta_1}\int_0^{2\pi} \gamma \sin\theta \cos\theta\, d\phi\, d\theta + \int_0^{\theta_1}\int_0^{2\pi} \gamma \sin\theta \cos\theta \operatorname{erf}\gamma S\, d\phi\, d\theta \Big) + \frac{1}{2S^2} \int_0^{\theta_1}\int_0^{2\pi} \sin\theta \cos\theta\, e^{-\gamma^2 S^2} d\phi\, d\theta \Big] \Big\}$$

Performing the indicated integrations gives

$$C_A = \frac{r^2}{A_{ref}}\left\{\pi\cos^2\alpha\sin^2\theta_1 + \frac{\pi}{2S^2}\sin^2\theta_1 + \left(2\cos^2\alpha + \frac{1}{S^2}\right)\int_0^{\theta_1}\int_0^{\pi}\sin\theta\cos\theta\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta\right.$$

$$+ 2\sin\alpha\cos\alpha\int_0^{\theta_1}\int_0^{\pi}\sin^2\theta\cos\phi\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta + \frac{2\cos\alpha}{S\sqrt{\pi}}\int_0^{\theta_1}\int_0^{\pi}\sin\theta\,e^{-\gamma^2S^2}\,d\phi\,d\theta$$

$$+ \sqrt{\frac{T_r}{T_i}}\left[\frac{(\pi)^{3/2}}{3S}\cos\alpha\left(1-\cos^3\theta_1\right) + \frac{\sqrt{\pi}}{S}\left(\cos\alpha\int_0^{\theta_1}\int_0^{\pi}\sin\theta\cos^2\theta\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta\right.\right.$$

$$\left.\left.\left. + \sin\alpha\int_0^{\theta_1}\int_0^{\pi}\sin^2\theta\cos\theta\cos\phi\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta\right) + \frac{1}{S^2}\int_0^{\theta_1}\int_0^{\pi}\sin\theta\cos\theta\,e^{-\gamma^2S^2}\,d\phi\,d\theta\right]\right\} \quad (24)$$

PITCHING MOMENT:

The pitching moment is the moment about the Z axis. Due to symmetry, the moments about the X and Y axes are zero. Nose up moments are positive. The pitching moment is given by the following equation.

$$dC_M = \frac{1}{L_{ref}}\left(L + r\cos\theta\right)dC_N - \frac{1}{L_{ref}}\,r\sin\theta\cos\phi\,dC_A$$

Expanding this expression we obtain

$$C_M = \frac{L}{L_{ref}}C_N + \frac{r^3}{A_{ref}L_{ref}}\left\{\sin\alpha\left[\int_0^{\theta_1}\int_0^{2\pi}\gamma\sin\theta\cos\theta\,d\phi\,d\theta\right.\right.$$

$$+ \int_0^{\theta_1}\int_0^{2\pi}\gamma\sin\theta\cos\theta\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta + \frac{1}{S\sqrt{\pi}}\int_0^{\theta_1}\int_0^{2\pi}\sin\theta\cos\theta\,e^{-\gamma^2S^2}\,d\phi\,d\theta$$

$$+ \frac{1}{2S^2}\left(\int_0^{\theta_1}\int_0^{2\pi}\sin^2\theta\cos\theta\cos\phi\,d\phi\,d\theta + \int_0^{\theta_1}\int_0^{2\pi}\sin^2\theta\cos\theta\cos\phi\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta\right)$$

$$+ \sqrt{\frac{T_r}{T_i}}\left[\frac{\sqrt{\pi}}{2S}\left(\int_0^{\theta_1}\int_0^{2\pi}\gamma\sin^2\theta\cos\theta\cos\phi\,d\phi\,d\theta + \int_0^{\theta_1}\int_0^{2\pi}\gamma\sin^2\theta\cos\theta\cos\phi\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta\right)\right.$$

$$+ \frac{1}{2S^2} \int_0^{\theta_1}\int_0^{2\pi} \sin^2\theta \cos\theta \cos\phi \, e^{-\gamma^2 S^2} d\phi \, d\theta \Bigg]\Bigg\}$$

$$- \frac{r^3}{L_{ref} A_{ref}} \Bigg\{ \cos\alpha \Bigg[\int_0^{\theta_1}\int_0^{2\pi} \gamma \sin^2\theta \cos\phi \, d\phi \, d\theta + \int_0^{\theta_1}\int_0^{2\pi} \gamma \sin^2\theta \cos\phi \, \mathrm{erf}\, \gamma s \, d\phi \, d\theta$$

$$+ \frac{1}{S\sqrt{\pi}} \int_0^{\theta_1}\int_0^{2\pi} \sin^2\theta \cos\phi \, e^{-\gamma^2 S^2} d\phi \, d\theta \Bigg] + \frac{1}{2S^2} \Bigg(\int_0^{\theta_1}\int_0^{2\pi} \sin^2\theta \cos\theta \cos\phi \, d\phi \, d\theta$$

$$+ \int_0^{\theta_1}\int_0^{2\pi} \sin^2\theta \cos\theta \cos\phi \, \mathrm{erf}\, \gamma S \, d\phi \, d\theta \Bigg)$$

$$+ \sqrt{\frac{T_r}{T_i}} \Bigg[\frac{\sqrt{\pi}}{2S} \Bigg(\int_0^{\theta_1}\int_0^{2\pi} \gamma \sin^2\theta \cos\theta \cos\phi \, d\phi \, d\theta$$

$$+ \int_0^{\theta_1}\int_0^{2\pi} \gamma \sin^2\theta \cos\theta \cos\phi \, \mathrm{erf}\, \gamma s \, d\phi \, d\theta + \frac{1}{2S^2} \int_0^{\theta_1}\int_0^{2\pi} \sin^2\theta \cos\theta \cos\phi \, \bar{e}^{\gamma^2 S^2} d\phi \, d\theta \Bigg]\Bigg\}$$

Simplifying the above equation and performing the integrations that yield closed form solutions gives

$$C_M = \frac{L}{L_{ref}} C_N + \frac{r^3}{A_{ref} L_{ref}} \Bigg\{ \pi \sin\alpha \cos\alpha (\cos\theta_1 - \cos^3\theta_1)$$

$$+ 2 \cos\alpha \sin\alpha \Bigg(\int_0^{\theta_1}\int_0^{\pi} \sin\theta \cos^2\theta \, \mathrm{erf}\, \gamma S \, d\phi \, d\theta - \int_0^{\theta_1}\int_0^{\pi} \sin^3\theta \cos^2\phi \, \mathrm{erf}\, \gamma s \, d\phi \, d\theta \Bigg)$$

$$+ 2 (\sin^2\alpha - \cos^2\alpha) \int_0^{\theta_1}\int_0^{\pi} \sin^2\theta \cos\theta \cos\phi \, \mathrm{erf}\, \gamma S \, d\phi \, d\theta$$

$$+ \frac{2}{S\sqrt{\pi}} \Bigg(\sin\alpha \int_0^{\theta_1}\int_0^{\pi} \sin\theta \cos\theta \, e^{-\gamma^2 S^2} d\phi \, d\theta$$

$$- \cos\alpha \int_0^{\theta_1}\int_0^{\pi} \sin^2\theta \cos\phi \, e^{-\gamma^2 S^2} d\phi \, d\theta \Bigg) \Bigg\} \qquad (25)$$

Now let us introduce the following notations.

$$
\left.
\begin{aligned}
I_1^{SP}(S,\alpha,\theta_1) &= \pi \sin\alpha\cos\alpha\sin^2\theta_1 + 2\sin\alpha\cos\alpha\int_0^{\theta_1}\int_0^{\pi}\sin\theta\cos\theta\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta \\
&\quad + \left(2\sin^2\alpha + \frac{1}{S^2}\right)\int_0^{\theta_1}\int_0^{\pi}\sin^2\theta\cos\phi\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta \\
&\quad + \frac{2\sin\alpha}{S\sqrt{\pi}}\int_0^{\theta_1}\int_0^{\pi}\sin\theta\, e^{-\gamma^2 S^2}\,d\phi\,d\theta \\
I_2^{SP}(S,\alpha,\theta_1) &= \frac{(\pi)^{3/2}}{2S}\sin\alpha\left(\frac{\cos^3\theta_1}{3} - \cos\theta_1 + \frac{2}{3}\right) \\
&\quad + \frac{\sqrt{\pi}}{S}\left(\cos\alpha\int_0^{\theta_1}\int_0^{\pi}\cos\theta\sin^2\theta\cos\phi\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta + \sin\alpha\int_0^{\theta_1}\int_0^{\pi}\sin^3\theta\cos^2\phi\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta\right) \\
&\quad + \frac{1}{S^2}\int_0^{\theta_1}\int_0^{\pi}\sin^2\theta\cos\phi\, e^{-\gamma^2 S^2}\,d\phi\,d\theta \\
I_3^{SP}(S,\alpha,\theta_1) &= \left(2\cos^2\alpha + \frac{1}{S^2}\right)\left(\frac{\pi}{2}\sin^2\theta_1 + \int_0^{\theta_1}\int_0^{\pi}\sin\theta\cos\theta\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta\right) \\
&\quad + 2\sin\alpha\cos\alpha\int_0^{\theta_1}\int_0^{\pi}\sin^2\theta\cos\phi\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta + \frac{2\cos\alpha}{S\sqrt{\pi}}\int_0^{\theta_1}\int_0^{\pi}\sin\theta\, e^{-\gamma^2 S^2}\,d\phi\,d\theta \\
I_4^{SP}(S,\alpha,\theta_1) &= \frac{(\pi)^{3/2}}{3S}\cos\alpha(1-\cos^3\theta_1) + \frac{\sqrt{\pi}}{S}\left(\cos\alpha\int_0^{\theta_1}\int_0^{\pi}\sin\theta\cos^2\theta\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta\right. \\
&\quad \left. + \sin\alpha\int_0^{\theta_1}\int_0^{\pi}\sin^2\theta\cos\theta\cos\phi\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta\right) + \frac{1}{S^2}\int_0^{\theta_1}\int_0^{\pi}\sin\theta\cos\theta\, e^{-\gamma^2 S^2}\,d\phi\,d\theta \\
I_5^{SP}(S,\alpha,\theta_1) &= \pi\sin\alpha\cos\alpha(\cos\theta_1 - \cos^3\theta_1) \\
&\quad + 2\cos\alpha\sin\alpha\left(\int_0^{\theta_1}\int_0^{\pi}\sin\theta\cos^2\theta\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta - \int_0^{\theta_1}\int_0^{\pi}\sin^3\theta\cos^2\phi\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta\right) \\
&\quad + 2(\sin^2\alpha - \cos^2\alpha)\int_0^{\theta_1}\int_0^{\pi}\sin^2\theta\cos\theta\cos\phi\,\mathrm{erf}\,\gamma S\,d\phi\,d\theta \\
&\quad + \frac{2}{S\sqrt{\pi}}\left(\sin\alpha\int_0^{\theta_1}\int_0^{\pi}\sin\theta\cos\theta\, e^{-\gamma^2 S^2}\,d\phi\,d\theta - \cos\alpha\int_0^{\theta_1}\int_0^{\pi}\sin^2\theta\cos\phi\, e^{-\gamma^2 S^2}\,d\phi\,d\theta\right)
\end{aligned}
\right\} \quad (26)
$$

where $\gamma = \cos\alpha\cos\theta + \sin\alpha\sin\theta\cos\phi$

Using the notations (26), the expressions for C_A, C_N and C_M become

$$\left.\begin{aligned} C_N &= \frac{r^2}{A_{ref}}\left\{ I_1^{SP} + \sqrt{\frac{T_r}{T_i}}\, I_2^{SP} \right\} \\ C_A &= \frac{r^2}{A_{ref}}\left\{ I_3^{SP} + \sqrt{\frac{T_r}{T_i}}\, I_4^{SP} \right\} \\ C_M &= \frac{L}{L_{ref}} C_N + \frac{r^3}{A_{ref} L_{ref}} I_5^{SP} \end{aligned}\right\} \quad (27)$$

If the point about which the moment is taken is to the left of the center of the sphere, L is negative.

The foregoing analysis resulted in closed form solutions for the forces and moments on a flat plate and a cylinder. However, it was impossible to obtain analytic expressions for the cone and spherical segment. Since there are several approximate solutions available (4), (5), it was felt that another approximate solution would not serve a useful purpose. Further, both approximate solutions fail at low molecular speed ratios, and at small angles of attack for all values of the molecular speed ratio. Thus, it was decided to evaluate the exact expressions numerically so that the approximate solutions could be compared with the exact solution.

All of the expressions for the force and moment coefficients of the shapes investigated were able to be written in the following functional form,

$$C = g(L, r)\left[I_1(S, \alpha, \beta) + \sqrt{\frac{T_r}{T_i}}\, I_2(S, \alpha, \beta) \right]$$

where $g(L, r)$ is a function of the length dimensions of the shape only, $I_1(S, \alpha, \beta)$ and $I_2(S, \alpha, \beta)$ are functions only of the molecular speed ratio, S, angle of attack, α, and in the case of the cone and spherical segment a geometric angle, β . For the cone, the geometric angle β is the semi-vertex angle, δ , and for the spherical segment, β is half the included angle of the segment, Θ_1. Since the I's are functions of S , α , and β only, they may be considered as general free molecule flow functions which are independent of the length, radius, etc. of the particular shape under consideration. Once they are known, the aerodynamic coefficients for the particular shape can be computed. In general, the I's were seen to be complicated integrals which can be evaluated in closed form only for the flat plate and cylinder. For this reason, they were evaluated on the IBM 7090 computer and are presented in tabular form for the following ranges of the parameters S , α , δ , and Θ_1: $1 \leq S \leq 13$, $0 \leq \alpha \leq 180^\circ$, $5^\circ \leq \delta \leq 85^\circ$, $5^\circ \leq \Theta_1 \leq 180^\circ$ where δ and Θ_1 increase in increments of 5°.

The α range was chosen to be 0 to π because the magnitude of the I's in the range of α from 0 to $-\pi$ is the same as in the range 0 to π. It was felt that values of S from 1 to 13 would cover the range of molecular speed ratios likely to be encounted in practice. The use of the tables of the I's for computing the free molecule aerodynamic coefficients of a particular body will be discussed in Section V.

SECTION IV

COMPARISON OF THE EXACT AND APPROXIMATE FREE MOLECULE THEORIES

Two approximate theories (refs. 4 and 5) for determining free molecule aerodynamic coefficients have been presented in the literature. Gustafson's theory (ref. 4) consists of an approximation of the physical phenomena which occur in a free molecule flow. In this theory it is assumed that the random thermal motion of the molecules can be neglected in comparison with their mass motion. This physical assumption results in a infinite molecular speed ratio, $S = \infty$ and is known as the hyperthermal free molecule theory. The kinetic theory should give the same results as this hyperthermal theory when S is large. In the kinetic theory free molecule expressions, S is always multiplied by a function of the angle of attack which goes to zero as $\alpha \rightarrow 0$. Thus, even for large S, there will be a discrepancy between this approximate theory and the kinetic theory approach for small angles of attack.

In the hyperthermal theory, the incident molecules, which have mass motion only, impinge on the surface and give up all their momentum to it. They are then assumed to be reemitted diffusely with a temperature corresponding to the wall temperature T_w.

For a convex body, the force on an element of area in this type of flow is given as follows. By Newton's second law of motion, the force is equal to the rate of change of momentum.

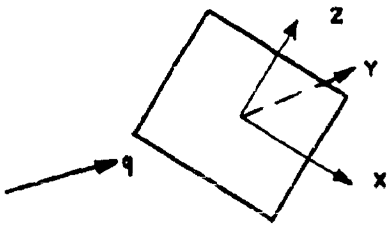


Figure IV-1

x, y, z is a local coordinate system on the element of area with y in the direction of the inward normal to the surface. Then

$$dF = \frac{d}{dt}(mq) = q\,\frac{dm}{dt} = q\rho\gamma q\,dA$$

$$dF = \rho q^2 \gamma\,dA$$

where γ is the direction cosine between the mass velocity, q, and the inward normal to the element of area, y. Nondimensionalizing we obtain for the force coefficient

$$dC = 2\gamma \frac{dA}{A_{ref}} \qquad (1)$$

Since the molecules have mass motion only, the total force due to the incident molecules is in the direction of the velocity q which by definition is the drag. Thus

$$dC_D = 2\gamma \frac{dA}{A_{ref}} \qquad (2)$$

If we take the projection of A normal to the velocity as the reference area, then C_D becomes

$$C_D = 2 \qquad (3)$$

based on the projected area normal to the velocity q.

Since the total force due to the incident molecules is drag, we see that there is no component of force normal to the drag, i.e. there is no lift due to the incident molecules. This is true only for this theory which is based on the assumption of no random thermal motion of the molecules.

To find the force on the element of area due to the reflected molecules, we take the limit as $S \rightarrow \infty$ of the kinetic theory expression for the force on an element of area due to diffusely reflected molecules. From part B of section II,

$$dF_{REFLECTED} = \frac{\rho}{2\beta}\frac{1}{2}\sqrt{\frac{T_W}{T_i}}\left[\gamma s\sqrt{\pi}(1+\operatorname{erf}\gamma s)+e^{-\gamma^2 s^2}\right]dA \qquad (4)$$

Taking the limit of this expression as S → ∞ gives

$$dF_{REFLECTED} = \frac{\rho}{2\beta}\sqrt{\pi}\, s\gamma\sqrt{\frac{T_W}{T_i}}\, dA \qquad (5)$$

Nondimensionalizing gives

$$\frac{dC_{REFLECTED}}{dA} = \frac{\sqrt{\pi}\,\gamma}{S\, A_{ref}}\sqrt{\frac{T_W}{T_i}} \qquad (6)$$

This force is normal to the surface because the diffusely reflected molecules have a net charge in momentum normal to the surface only. The shear components cancel each other out. To find the total force in a particular direction due to both the incident and reflected molecules, Eqs. (2) and (6) must each be multiplied by the appropriate direction cosine and then added together. The total force on a body in a particular direction is then obtained by integrating the total force equation over the surface of the body. The results for a cylinder, cone and spherical segment are presented in reference (4).

The approximation by Schrello (ref. 5) is a mathematical assumption which was introduced in the kinetic theory expressions for the force and moment coefficients in order to obtain a closed form result. This approach makes no assumptions about the physical processes occurring in a free molecule flow. The mathematical assumption does result in a criteria which a body must fulfill before the results can be used to compute the force and moment coefficients. This criteria is that the average normal component of the molecular speed ratio over the exposed body surface must be greater than or equal to unity. This approximation also fails at small molecular speed ratios, and at small angles of attack for all values of the molecular speed ratio.

The exact theory includes co[illegible] motion of the molecules whereas the approximate theori[illegible] effect of the inclusion of the random motions [illegible] the number of molecules impinging [illegible] per unit time. This is illustrated by considering the [illegible] the number of molecules impinging on unit area per unit time, [illegible] by the exact theory and N_i as given by the theory of reference 4. From Appendix [illegible], the exact theory gives N_i, when the molecular speed ratio is introduced,

$$N_i = n_i q \left[\frac{\gamma}{2} \left(1 + \operatorname{erf} \gamma S \right) + \frac{1}{2 S \sqrt{\pi}} e^{-\gamma^2 S^2} \right] \qquad (7)$$

According to the hyperthermal theory (ref. 4),

$$N_i = n_i q \gamma \qquad (8)$$

Then,

$$\eta_i = \frac{N_{i_{EXACT}} - N_{i_{APPROX.}}}{n_i q} = \frac{\gamma}{2} \left(\operatorname{erf} \gamma S - 1 \right) + \frac{1}{2 S \sqrt{\pi}} e^{-\gamma^2 S^2} \qquad (9)$$

In Figure IV-2, η_i is plotted against the angle of attack, α, of the surface for several values of the molecular speed ratio, S. As seen in Figure IV-2, for values of S greater than 5, which are pertinent to a near earth satellite, the random motion effect is confined to small surface inclinations. As such, the primary effect would be an increase in the axial force coefficient, with a relatively small effect on the normal force coefficient. This is shown in Figure IV-3 where the axial and normal force coefficients for a surface element are plotted as a function of surface inclination. From Figure IV-3 we can readily see that the approximate theory is quite adequate for the prediction of the normal force at all surface inclination angles, but is inadequate for the prediction of the axial force when the surface is closely aligned to the free stream.

It is interesting to apply these results to an idealized satellite shape such as that shown in Figure IV-4. The aerodynamic force and moment coefficients are presented in Figure IV-5. The computations were performed for a molecular speed ratio of 7 and, assuming complete thermal accommodation, $\sqrt{T_r/T_i} = 0.425$ which are typical of the conditions at satellite altitudes. In computing the coefficients, the forward end of the cone and aft end of the cylinder were considered as flat plates. The results using the theory of reference 5 are omitted for clarity of presentation since they would fall between the exact and approximate results of reference 4.

As seen in Figure IV-5, the primary difference between the approximate and exact methods lies in the prediction of the axial force coefficient at small angles of attack. [illegible] difference [illegible] as we have previously seen, to the [illegible] the random motion [illegible] the conical surface elements [illegible], 5° to [illegible] and the cylindrical surface elements [illegible] parallel to the [illegible]. The difference between the exact and approximate predictions of C_A [illegible] molecular [illegible] impingement on the surface of the cylinder. This [illegible] of the results shown in Figure IV-2. The magnitude of [illegible] is a linear function of the L/D of the cylindrical [illegible] on the body, [illegible] in Figure IV-6. As noted from the analysis of a un[illegible] surface, the approximate theory appears adequate for the prediction of C_N [illegible] all angles of attack, but is inadequate for the prediction of C_A for an elongated body at small angles of attack.

The effect of surface inclination on the agreement between the exact and approximate theories is illustrated further in Figure IV-7 in which C_A at $\alpha = 0$ is plotted against L/D for a 15° and a 5° cone, and a cylinder. End effects and the contribution of the reflected molecules have been neglected so that the discrepancy between the curves is due entirely to the incident molecules.

In Figure IV-8, C_N and C_A for a spherical segment are presented versus angle of attack. From the preceding discussion, the close agreement between the exact and approximate theories for this configuration is to be expected.

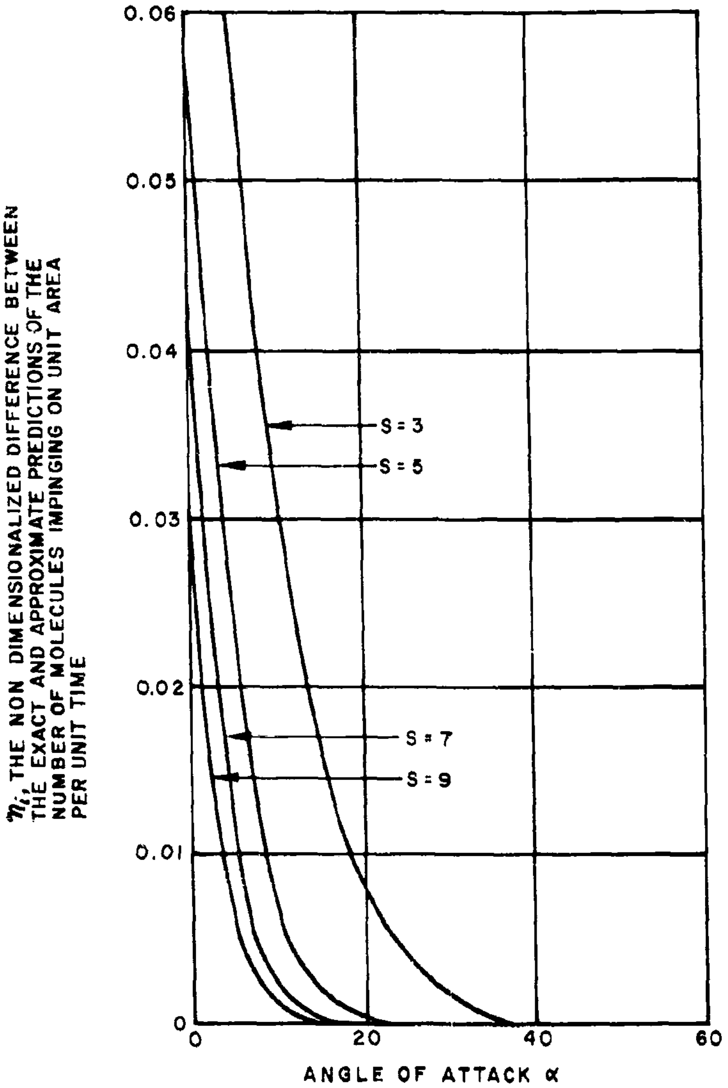


Figure IV-2 Difference in the Number of Molecules Impinging on Unit Area Per Unit Time

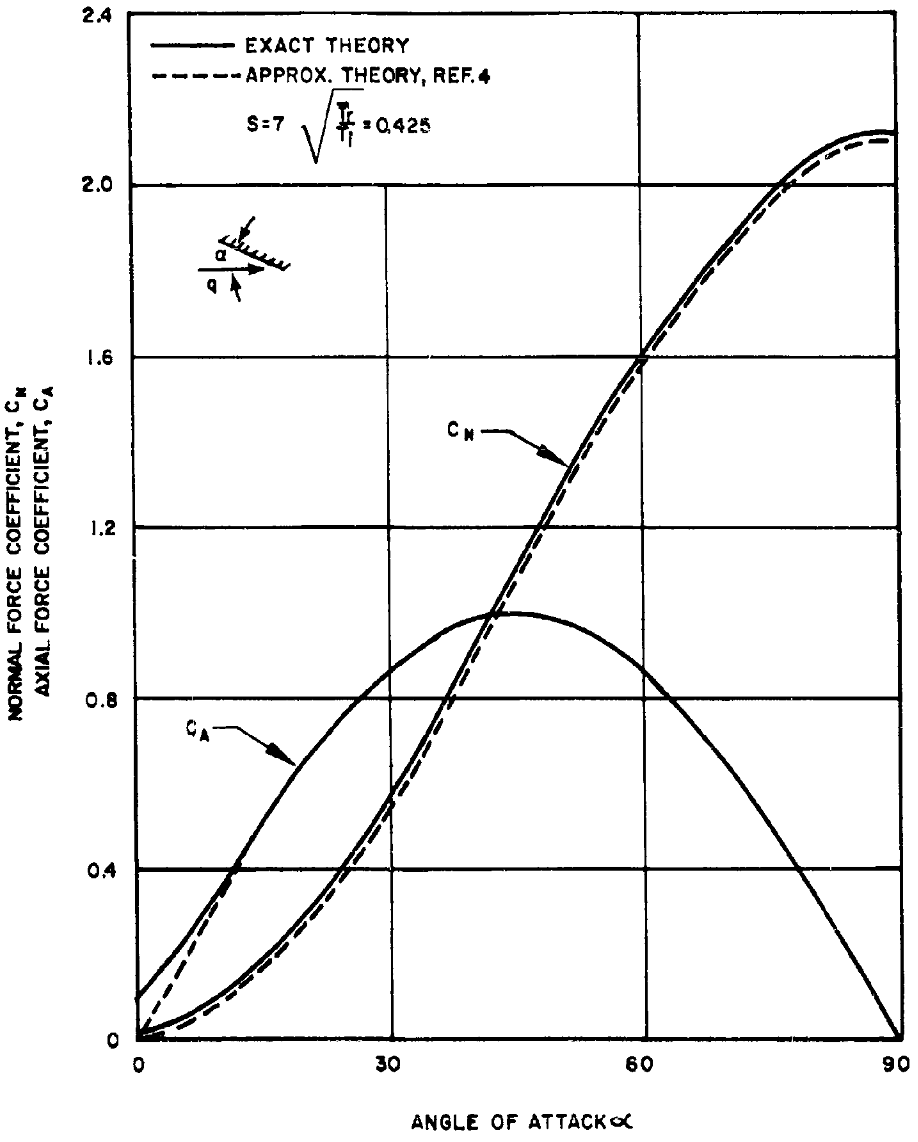
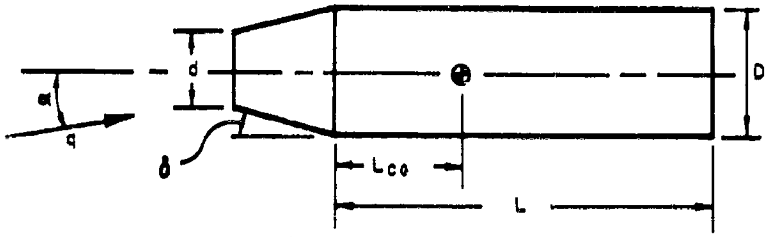


Figure IV-3 Aerodynamic Coefficients for a Flat Plate with One Side Exposed to the Flow Assuming Complete Thermal Accommodation

A_{ref} = Plan Form Area of the Plate



$\delta = 15°$ $\frac{d}{D} = 0.834$ $\frac{L_{co}}{D} = 1.0$ $\frac{L}{D} = 3.0$

Figure IV-4 Idealized Satellite Shape

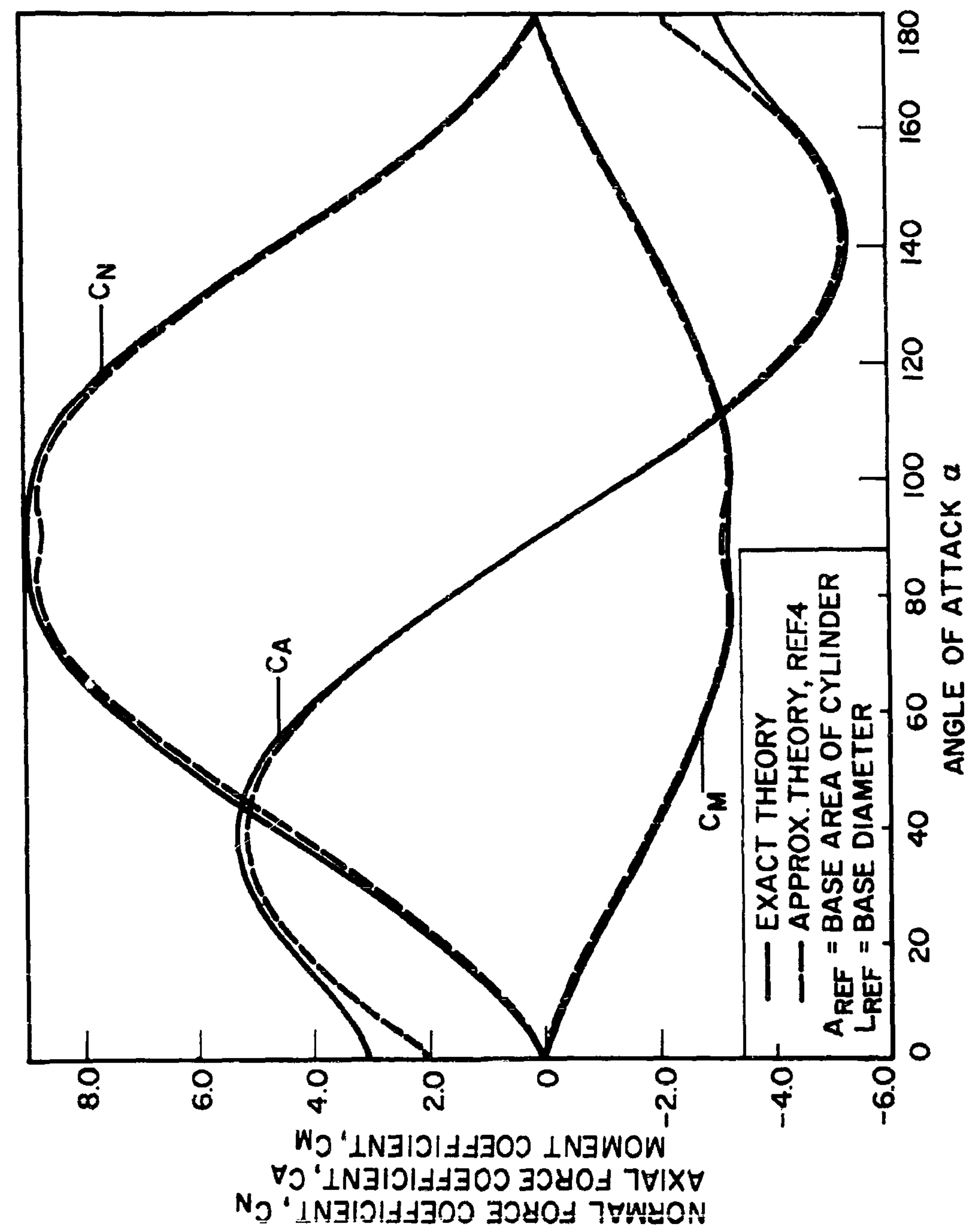


Figure IV-5 Aerodynamic Coefficients for Idealized Satellite Shape

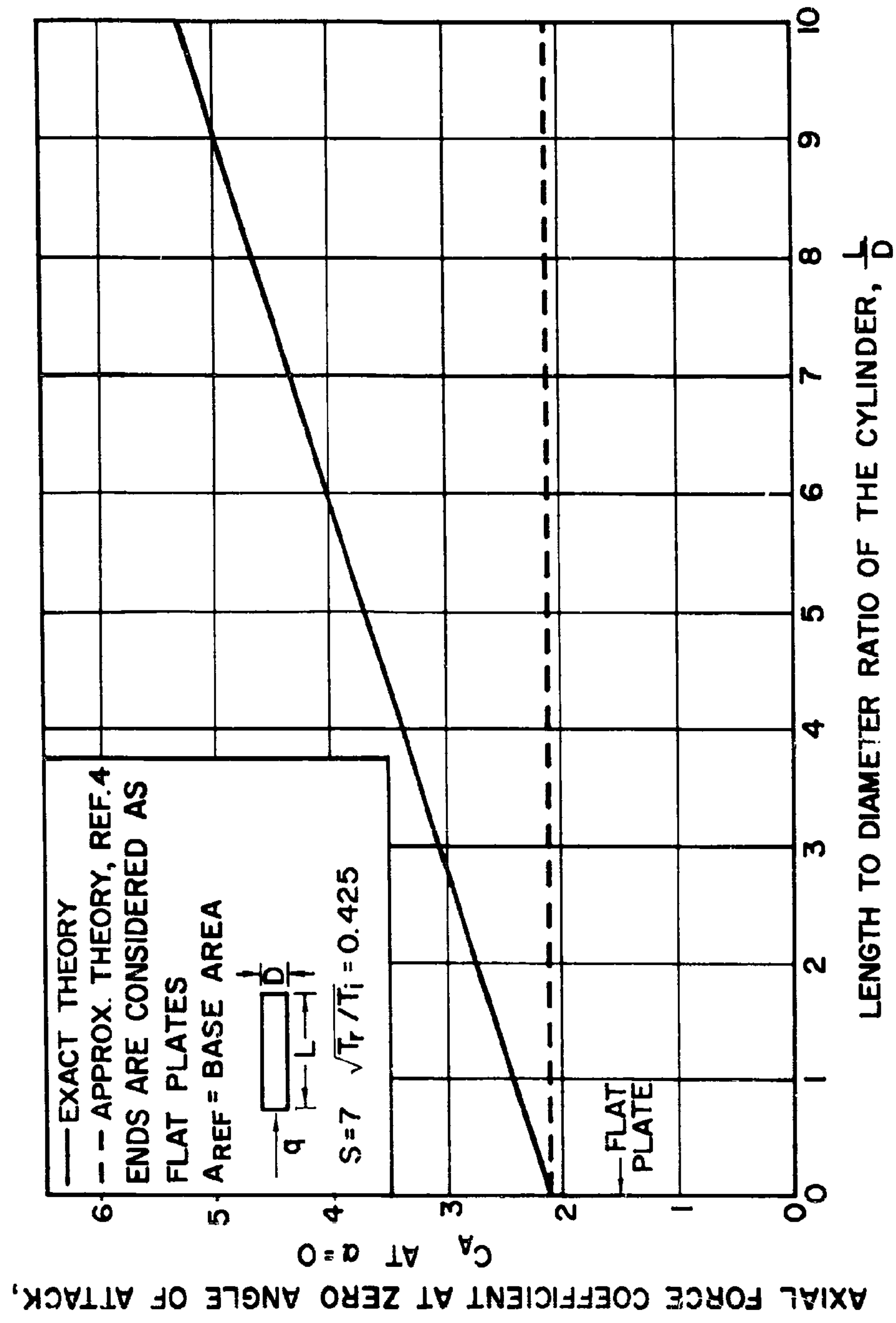


Figure IV-6 Variation of Axial Force at Zero Angle of Attack With L/D For a Cylinder

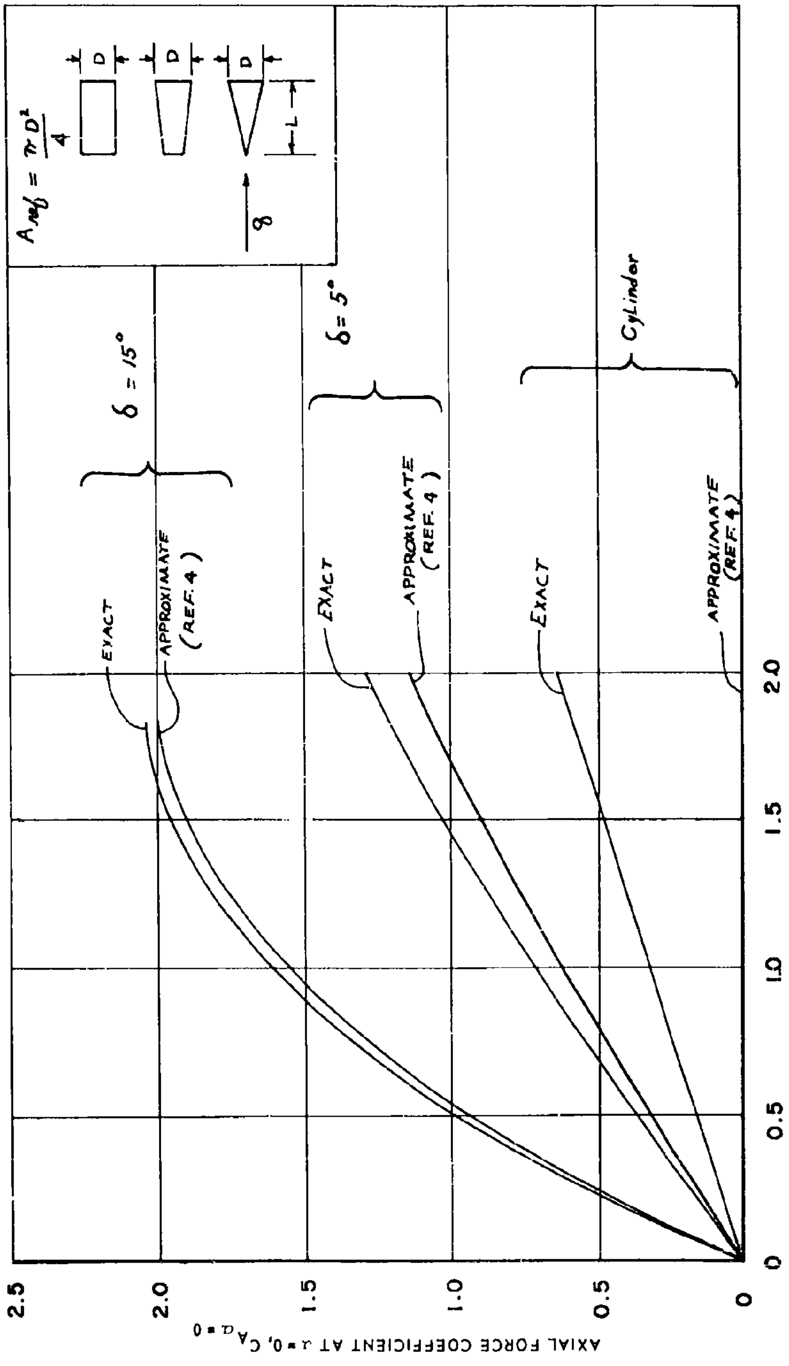


Figure IV-7 C_A AT $\alpha = 0$ Vs. $\frac{L}{D}$ For a 15° and a 5° Cone and a Cylinder Neglecting End Effects and Contributions of the Reflected Molecules

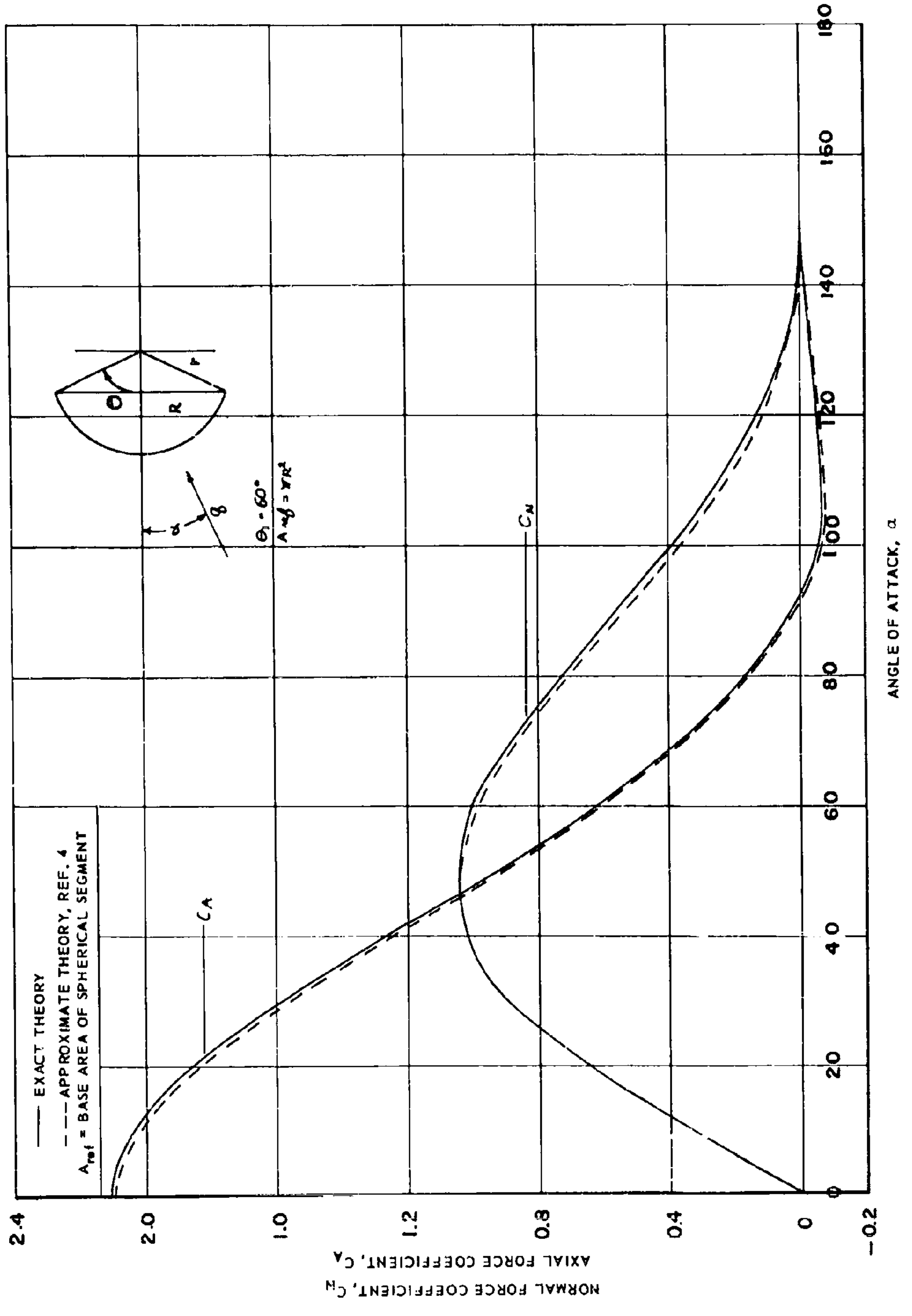


Figure IV-8 **Aerodynamic Coefficients for a Spherical Segment**

SECTION V

USE OF THE FREE MOLECULE FLOW TABLES FOR DETERMINING THE AERODYNAMIC COEFFICIENTS OF A PARTICULAR VEHICLE

The purpose of this section is to discuss some of the difficulties which might be encountered in analyzing a vehicle in a free molecule flow.

A. DETERMINATION OF THE KNUDSEN NUMBER

The first step in analyzing a vehicle is to determine if a free molecule flow exists. This question is answered by computing the Knudsen number, $Kn = \lambda/L$, where λ is an appropriate mean free path of the molecules and L is an appropriate dimension of the vehicle. If Kn is greater than 10 ($Kn > 10$), a free molecule flow exists. For purposes of determining whether or not an entire vehicle is in a free molecule flow, the largest dimension of the vehicle is the appropriate L to use in computing the Knudsen number.

The determination of the appropriate mean free path is more difficult. (A discussion similar to the following is given in reference 7.) The magnitude of the mean free path depends on the coordinate system in which the motion of the molecules is observed. Since we are interested in a body which is moving with constant velocity through a gas at rest, there are two coordinate systems with respect to which the motion may be observed. We may refer the motion to a coordinate system that is stationary with respect to the gas or we may use a coordinate system that is fixed in the body. Since the expressions for the aerodynamic coefficients have been written with respect to a coordinate system fixed in the body, this is the coordinate system that will be used to compute the mean free path of the molecules.

There are four mean free paths that can be defined: The mean free path of the incident molecules relative to themselves, λ_{ii}, the mean free path of the incident molecules relative to the reflected molecules, λ_{ir}, the mean free path of the reflected molecules relative to the incident molecules, λ_{ri}, and the mean free path of the reflected molecules relative to themselves, λ_{rr}.

Since the Knudsen number is used to determine when a free molecule flow exists, it must indicate when the assumptions on which free molecule flow is based are satisfied. These assumptions are: 1. it is assumed that molecule-surface collisions are the governing physical phenomena, i.e. incident molecule-surface collisions are more numerous than incident molecule-incident molecule collisions near the body; and 2. it is assumed that the incident and reflected molecules do not interact (collide) with each other near the body, i.e. that collisions of the incident molecules with the surface are much more numerous than incident molecule-reflected molecule collisions. Since the mean free path by definition is the average distance travelled by a molecule between collisions with other molecules, a Knudsen number based on λ_{ii} will indicate when the first assumption is satisfied and a Knudsen number based on λ_{ir} or λ_{ri} will indicate when the second assumption is satisfied. To obtain a conservative estimate of when the second assumption is fulfilled, the smaller of the two mean free paths, λ_{ir} and λ_{ri}, will be used. It will be shown that λ_{ir} is the smaller of the two mean free paths, and that, for a near earth satellite, the Knudsen number based on λ_{ir} is an order of magnitude smaller than the Knudsen number based on λ_{ii} when complete thermal accommodation is assumed. Thus, the criteria for the existence of a free molecule flow is that the Knudsen number based on λ_{ir} must be greater than 10.

The mean free path is defined as the average distance a molecule travels in unit time divided by the number of collisions it undergoes in unit time. Relative to a coordinate system fixed in the body, the preceding mean free paths are given by the following expressions,

$$\lambda_{ii} = \frac{g}{\theta_{ii}}$$

$$\lambda_{ir} = \frac{g}{\theta_{ir}}$$

$$\lambda_{ri} = \frac{\bar{c}_r}{\Theta_{ri}}$$

$$\lambda_{rr} = \frac{\bar{c}_r}{\Theta_{rr}}$$

where Θ_{jk} is the collision frequency of one type j molecule with all the type k molecules and $\bar{c} = \left(\frac{8kT}{\pi m}\right)^{1/2}$ is the average speed of a molecule. At this point it is convenient to introduce a quantity which we will need in the future, λ_∞, the mean free path of the incident molecules relative to themselves referred to a coordinate system that is fixed in the gas. This is the mean free path that is tabulated in most atmosphere tables, for example, the 1959 ARDC atmosphere. From Eq. (12A) Appendix A, this is given as

$$\lambda_\infty = \frac{1}{\sqrt{2}\, n_i \pi d^2} \qquad (1)$$

The collision frequency is independent of the coordinate system in which the motion of the molecules is observed. It can be shown that the collision frequency for a gas with mass motion is the same as that for a gas with no mass motion. From eqn. (10A) Appendix A,

$$\Theta_{ii} = \pi n_i d^2 \sqrt{2} \left(\frac{8kT_i}{\pi m}\right)^{1/2} = \sqrt{2}\, n_i \pi d^2 \bar{c}_i \qquad (2)$$

$$\Theta_{rr} = \sqrt{2}\, \bar{n}_r \pi d^2 \bar{c}_r \qquad (3)$$

Unfortunately, it is difficult to evaluate exactly the collision frequency for a mixture of two gases which are at different temperatures and one of which has a mass motion while the other does not. This situation occurs only in the vicinity of a body in a free molecule flow. The collision frequencies Θ_{ir} and Θ_{ri} are given approximately as

$$\Theta_{ir} = \pi d^2 \bar{n}_r g \qquad (4)$$

$$\Theta_{ri} = \pi d^2 n_i g \qquad (5)$$

Substituting into the expressions for the λ's the appropriate Θ's gives

$$\begin{aligned} \lambda_{ii} &= \frac{g}{\sqrt{2}\, n_i \bar{c}_i \pi d^2} \\ \lambda_{ir} &= \frac{1}{\bar{n}_r \pi d^2} \\ \lambda_{ri} &= \frac{\bar{c}_r}{n_i g \pi d^2} \\ \lambda_{rr} &= \frac{1}{\sqrt{2}\, \bar{n}_r \pi d^2} \end{aligned} \qquad (6)$$

The $\bar{n}_r$ which appears in these expressions is the true number density of reflected molecules in the vicinity of the body whereas the n_r used in Section II B was a fictitious number density which does not exist anywhere in the flow.

Before we can decide which of the two meanfree paths, λ_{ir} or λ_{ri}, is the smaller, $\bar{n}_r$ must be computed. This will, in general, be a function of location on the surface of the body and distance away from the surface of the body. Thus, we will compute an average $\bar{n}_r$. From continuity considerations, we know that $N_i = N_r$. To determine $\bar{n}_r$, we need to know the average volume that the number of molecules reflected from unit area per unit time occupy. This is given by an average velocity of the reflected molecules times unit time, times unit area. Thus,

$$\bar{n}_r = \frac{N_r t\, dA}{\bar{V} t\, dA} = \frac{N_r}{\bar{V}} \qquad (7)$$

To see that the total number of molecules that will be contained in this average volume is N_r, consider the following. By the assumption of diffuse reflection, the same number of molecules with a particular velocity will leave an element of area in one direction as leave it in the opposite direction. Thus, it would be expected that not all N_r molecules would be in the average volume $\bar{V}t\,dA$ (see figure V-1). However, since each element of area is surrounded by other elements of area from which molecules are leaving in all directions, it seems probable that as many molecules will enter the average volume above dA in unit time from the surrounding dA's as the number of molecules which impinge on dA leave it (see figure V-2).

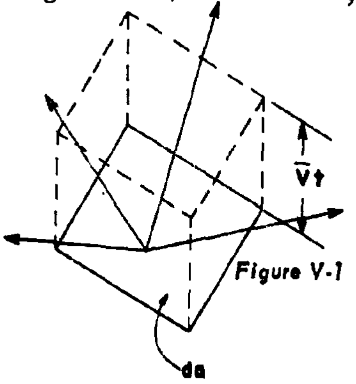


Figure V-1

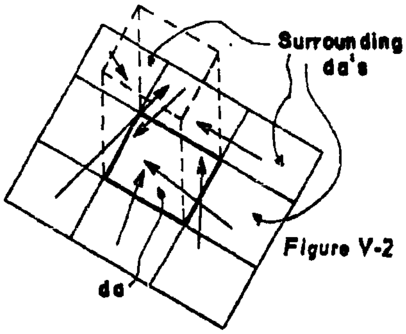


Figure V-2

Since $N_r = N_i$, and introducing the molecular speed ratio, we obtain from Eq. (15B), appendix B

$$N_r = \frac{n_i}{2\sqrt{\pi}\sqrt{\beta}}\left[\sqrt{\pi}\,\gamma s\left(1+\operatorname{erf}\gamma s\right)+e^{-\gamma^2 s^2}\right] \qquad (8)$$

There are many average velocities that may be defined. There are the average speed of a molecule which may be moving in any direction in space, the average speed of a molecule which can only move in the space above the element of area, and the average component of velocity normal to the element of area. It turns out that these velocities differ from each other by less than 25%. Since the purpose of these considerations is to determine if a free molecule flow exists and since it is desirable to be conservative in our estimate, we want to choose the smallest reasonable value of λ_{ir} which means that $\bar{n}_r$ should be as large as possible, and thus the average velocity we use should be the smallest one. The smallest average velocity is also the one which it seems reasonable to use, the average velocity of the reflected molecules normal to the surface. To compute this, we proceed as follows. v is the velocity component normal to the element of area where the positive normal is directed inward. The number of molecules in a given velocity range that leave unit area of the surface in unit time is

$$(-v)\, n_r f \, du \, dv \, dw$$

where n_r is the number density of the fictitious gas from which the reflected molecules are assumed to issue. The velocity normal to the surface that the number of molecules in a given velocity range that leave unit area in unit time carry with them is

$$v(-v)\, n_r f \, du \, dv \, dw$$

The average normal velocity of the reflected molecules is then

$$\overline{V} = -\frac{1}{N_r}\left(\frac{1}{2\pi R T_r}\right)^{3/2} \int_{-\infty}^{+\infty}\int_{-\infty}^{0}\int_{-\infty}^{+\infty} n_r v^2 e^{-\frac{1}{2RT_r}(u^2+v^2+w^2)} \, du \, dv \, dw$$

Performing the indicated integrations we find that

$$\overline{V} = -\frac{n_r}{N_r}\,\frac{2RT_r}{4} \qquad (9)$$

From Section II, Part B, Eq. (17) gives n_r when the molecular speed ratio is introduced as

$$n_r = n_i \sqrt{\frac{T_i}{T_r}}\left[\sqrt{\pi}\,\gamma S\,(1+\operatorname{erf}\gamma S) + e^{-\gamma^2 S^2}\right] \qquad (10)$$

Substituting Eqs. (8) and (10) into (9) gives

$$\bar{V} = -\frac{1}{2}\sqrt{2\pi R T_r} \qquad (11)$$

The minus sign means that this $\bar{V}$ is in the negative y direction, i.e. it is directed outward, normal to the surface. Thus, we will use only the magnitude of $\bar{V}$,

$$\bar{V} = \frac{1}{2}\sqrt{2\pi R T_r} \qquad (12)$$

Substituting Eqs. (8) and (12) into Eq. (7) gives for the average number density of reflected molecules

$$\bar{n}_r = \frac{n_i}{\pi}\sqrt{\frac{T_i}{T_r}}\left[\sqrt{\pi}\,\gamma S(1+\operatorname{erf}\gamma S) + e^{-\gamma^2 S^2}\right] \qquad (13)$$

Since we are interested in the greatest density of reflected molecules which can occur, we want the maximum value of the above expression for $\bar{n}_r$. This occurs for large S when $\gamma \to 1$ which occurs in the stagnation region of the body. Thus, in the stagnation region

$$\bar{n}_r = \frac{2 n_i S}{\sqrt{\pi}}\sqrt{\frac{T_i}{T_r}} \qquad (14)$$

Substitute Eq. (14) into the expression for λ_{ir}. This gives

$$\lambda_{ir} = \frac{\sqrt{\pi}}{2 n_i S \pi d^2}\sqrt{\frac{T_r}{T_i}} \qquad (15)$$

Forming the ratio $\lambda_{ir}/\lambda_{ri}$ and introducing the molecular speed ratio in λ_{ri} gives

$$\frac{\lambda_{ir}}{\lambda_{ri}} = \frac{\pi}{4} < 1$$

Thus, $\lambda_{ir} < \lambda_{ri}$ and λ_{ir} will be used to compute the Knudsen number which indicates when the second assumption of free molecule flow is satisfied. Introduce λ_∞ into Eq. (15). This gives for λ_{ir},

$$\lambda_{ir} = \sqrt{\frac{\pi}{2}}\,\frac{1}{S}\sqrt{\frac{T_r}{T_i}}\,\lambda_\infty \tag{16}$$

Now form the ratio $\lambda_{ir}/\lambda_{ii}$.

$$\frac{\lambda_{ir}}{\lambda_{ii}} = \frac{\sqrt{2}}{S^2}\sqrt{\frac{T_r}{T_i}} \tag{17}$$

From this expression, we see that for $S > 1$, $T_w < T_i$ (which is the case for a satellite), and nearly complete energy accommodation, $\lambda_{ir} < \lambda_{ii}$ and the critical Knudsen number is based on λ_{ir}. When this is greater than ten, the Knudsen number based on λ_{ii} will also be greater than ten. If there is an appreciable lack of energy accommodation, the critical Knudsen number may be based on λ_{ii}. At the time this is written, it appears reasonable to assume nearly complete energy accommodation and to require that the Knudsen number based on λ_{ir} be greater than ten for a free molecule flow to exist, i.e.,

$$Kn = \frac{\lambda_{ir}}{L} > 10$$

For large molecular speed ratios, λ_{ir} becomes a severe criteria for determining whether or not a free molecule flow exists. In marginal cases, the physical processes that are occurring should be given a more detailed examination.

The preceding choice of λ_{ir} as the critical mean free path differs from Reference 7 in which λ_{ri} was chosen. The result of Reference 7 was based on the fact that in this article it was shown that the actual number density of reflected particles falls off very rapidly away from the body, and thus, only very near the surface will the number density approach n_r and λ_{ir} be less than λ_{ri}. Since this work was based on the hyperthermal assumption and n_r is a fictitious density that does not exist anywhere in the flow, it was felt that the use of an average number density as presented here provides a more appropriate criteria for determining when a free molecule flow exists. It should be pointed out, however, that the ratio of λ_{ir} to λ_{ri} is $\frac{\pi}{4}$ which is almost one.

B. COMPILATION OF THE AVAILABLE EQUATIONS

The equations for the aerodynamic coefficients which are available are:

FLAT PLATE WITH BOTH SIDES EXPOSED TO THE FLOW.

The orientation of this plate is given in Figure III-1.

$$C_A = \frac{A}{A_{ref}} I_1^F$$

$$C_N = \frac{A}{A_{ref}} \left\{ I_2^F + \sqrt{\frac{T_r}{T_i}}\, I_3^F \right\} \qquad (18)$$

$$C_M = \frac{H}{L_{ref}} C_A + \frac{\left(L + \frac{X_1}{2}\right)}{L_{ref}} C_N$$

FLAT PLATE WITH ONE SIDE EXPOSED TO THE FLOW

The orientation of this plate is given in Figure III-2.

$$C_N = \frac{A}{A_{ref}} I_1^{CF}$$

$$C_A = \frac{A}{A_{ref}} \left\{ I_2^{CF} + \sqrt{\frac{T_r}{T_i}}\, I_3^{CF} \right\} \qquad (19)$$

$$C_M = \frac{L}{L_{ref}} C_N$$

It should be noted that the center of pressure of a flat plate is always the centroid of the area of the plate. It is usually more convenient to use this fact to compute the moment than to use the moment equations which are given for the flat plates.

RIGHT CIRCULAR CYLINDER

The orientation of the cylinder is given in Figure III-3.

$$C_N = \frac{r y_2}{A_{ref}} \left\{ I_1^{CY} + \sqrt{\frac{T_r}{T_i}}\, I_2^{CY} \right\}$$

$$C_A = \frac{r y_2}{A_{ref}} I_4^{CY} \qquad (20)$$

$$C_M = \frac{\left(L + \frac{y_2}{2}\right)}{L_{ref}} C_N - \frac{r^2 y_2}{A_{ref} L_{ref}} I_3^{CY}$$

FRUSTUM OF A RIGHT CIRCULAR CONE

The orientation of the cone is given in Figure III-4.

$$C_N = \frac{y_2^2 - y_1^2}{2 A_{ref}} \frac{\tan\delta}{\cos\delta} \left\{ I_1^{CO} + \sqrt{\frac{T_r}{T_i}}\, I_2^{CO} \right\}$$

$$C_A = \frac{y_2^2 - y_1^2}{2 A_{ref}} \frac{\tan\delta}{\cos\delta} \left\{ I_5^{CO} + \sqrt{\frac{T_r}{T_i}}\, I_6^{CO} \right\} \qquad (21)$$

$$C_M = \frac{1}{A_{ref} L_{ref}} \frac{\tan\delta}{\cos\delta} \left\{ \left[L\left(\frac{y_2^2 - y_1^2}{2}\right) - \left(\frac{y_2^3 - y_1^3}{3}\right) \right] \left[I_1^{CO} + \sqrt{\frac{T_r}{T_i}}\, I_2^{CO} \right] - \tan\delta \left(\frac{y_2^3 - y_1^3}{3}\right) \left[I_3^{CO} + \sqrt{\frac{T_r}{T_i}}\, I_4^{CO} \right] \right\}$$

SPHERICAL SEGMENT

The orientation of the spherical segment is given in Figure III-5.

$$C_N = \frac{r^2}{A_{ref}} \left\{ I_1^{sp} + \sqrt{\frac{T_r}{T_i}}\, I_2^{sp} \right\}$$

$$C_A = \frac{r^2}{A_{ref}} \left\{ I_3^{sp} + \sqrt{\frac{T_r}{T_i}}\, I_4^{sp} \right\} \quad . \qquad (22)$$

$$C_M = \frac{L}{L_{ref}} C_N + \frac{r^3}{A_{ref} L_{ref}} I_5^{sp}$$

The preceding equations are valid for angles of attack which are positive as defined in the figures giving the orientation of the different configurations. For negative angles of attack,

$$C_N(-\alpha) = -C_N(\alpha)$$

$$C_M(-\alpha) = -C_M(\alpha)$$

but

$$C_A(-\alpha) = C_A(\alpha)$$

where $|(-\alpha)| = \alpha$.

C. RESOLUTION OF A VEHICLE INTO SIMPLE SHAPES

The next question which arises is that of shielding. The equations which have been presented do not take into account the shielding of one part of a vehicle by another part. The only shielding they consider is that of an exposed part shielding a portion of itself such as the shielding of the back side of a cylinder by its front side. The shielding at a given angle of attack is best determined graphically using the techniques of descriptive geometry. For each angle of attack, the exposed parts of the vehicle can then be divided into flat plates, cylinders, cones and spherical segments. The aerodynamic coefficients are computed at each angle of attack for each piece of the vehicle and are then added together to give the total coefficients for the vehicle at each angle of attack.

D. DETERMINATION OF THE PARAMETERS S, T_i AND T_r.

The molecular speed ratio is given by $S = q/\sqrt{2RT_i}$ where q is the vehicle velocity, T_i the temperature of the incident molecules, and R the gas constant for the particular gas of interest.

T_i is the temperature of the incident molecules and is obtained from an atmosphere table.

The value to be used for T_r depends on the degree of thermal accommodation that occurs. The scarcity of experimental data for large molecular speed ratios makes it difficult to specify the value of the thermal accommodation coefficient. In the results presented in this report, T_r was assumed constant over the surface. Thus, if T_r actually varies over the surface, an average T_r should be used. If complete thermal accommodation is assumed, $T_r = T_w$ and the average wall temperature over the surface should be used. In this case the value of $\sqrt{T_r/T_i}$ is not very sensitive to T_r because T_i is of the order of 2200° R and T_w is of the order of 500°R and a ±50° variation in T_w does not affect $\sqrt{T_r/T_i}$ very much. Further, the value of the $I(S, \alpha, \beta)$ due to the reflected molecules which is multiplied by $\sqrt{T_r/T_i}$ is usually an order of magnitude smaller than the $I(S, \alpha, \beta)$ due to the incident molecules. Thus, for complete thermal accommodation, the error introduced into the aerodynamic coefficients by using an average T_w is not very large. At the present time, it appears reasonable to assume nearly complete energy accommodation.

If, however, very little thermal accommodation occurs, the value of T_r may be very large due to the large mass velocity of the incident molecules (of the order of 25,000 ft/sec) which gives them a large kinetic energy. Then, if there is little thermal accommodation, the energy of the reflected molecules, which for diffuse reflection is measured by T_r, will be very nearly equal to the energy of the incident molecules and thus T_r will be large. In fact, under these conditions $T_r > T_i$ and the value of T_w will significantly affect the value of the aerodynamic coefficients. In this case, the value of T_r must be obtained from an energy balance for the surface. Thus, it is important that the latest results of work on the problem of thermal accommodation be incorporated as soon as possible into the free molecule aerodynamic analysis.

E. PITCH AND YAW COMPONENTS OF THE AERODYNAMIC COEFFICIENTS

The equations presented here give the aerodynamic coefficients in the plane containing the velocity vector and the longitudinal axis of the vehicle, the moment being about the axis normal to this plane. Thus, the forces in and the moment normal to the plane containing the velocity vector and the longitudinal axis can be determined for an arbitrarily oriented vehicle in a uniform free molecule stream. It is sometimes convenient to consider the pitch and yaw components of the forces and moments on a vehicle. Since the equation for the force on an element of area is non linear, this situation must be handled by computing the forces in and moment about an axis normal to the plane formed by the mass velocity vector and the longitudinal axis of the vehicle. The C_N and C_M are then resolved into a pitch component and a yaw component. C_A is the total C_A and remains unchanged.

The values of the free molecule flow functions, $I(S, \alpha, \beta)$ are obtained from the tables once the molecular speed ratio S, angle of attack α, and, in the case of the cone and spherical segment, the semi vertex angle δ or the sphere half angle θ_1 are known.

SECTION VI
CONCLUDING REMARKS

At the present time there has not been an experimental determination of the free molecule aerodynamic coefficients of any bodies for Knudsen numbers greater than 10 and molecular speed ratios between 5 and 10. The only experimental work which has been done in the free molecule regime is the heat transfer and drag measurements of Reference (8) for a cylinder transverse to the flow. In this experiment the Knudsen number varied from 4 to 185 and the molecular speed ratio varied from 0.5 to 2.3. The high Knudsen numbers were obtained by placing a 0.0031 inch diameter wire in a low density wind tunnel.

The lack of experimental data is due primarily to two things. First, it is very difficult to obtain a free molecule flow for which the Knudsen number is greater than 10 and still use a model which is large enough to be used for force and moment measurements, and second, even if a large model can be used, the magnitude of the forces to be measured is within the limits of accuracy of available equipment. Thus, there is essentially no experimental data available that can be used to check the theoretical predictions in the Knudsen number and molecular speed ratio range of interest.

To further complicate the picture, the molecule-surface interaction phenomena and its variation with angle of incidence, molecular speed ratio, temperature, etc. is not very well known. There is also the problem of the shielding of one surface by another, and the reflection of molecules from one surface onto another. This latter problem has been discussed and results given for the sphere (references 9 & 10) and infinite cylinder, reference 9.

It is evident that there is a large amount of work that must be done before we will have a complete understanding of the processes occurring in a free molecule flow. For these reasons, it is not possible at this time to estimate the accuracy of the theoretical free molecule aerodynamic coefficient predictions. The only thing that can be said is that they are as accurate as we can predict theoretically.

APPENDIX A

FUNDAMENTAL ASSUMPTIONS AND IDEAS OF THE KINETIC THEORY OF GASES

The kinetic theory of gases rests upon two closely related hypotheses: The first is that of the molecular structure of matter and the second is that the degree of molecular motion is determined by the heat content of the body, that is, the internal energy of the gas is contained in the random translational motion of the molecules. The physical model upon which kinetic theory is based is that of an assembly of gas molecules in a container in random motion, bumping into themselves and the walls of the container. Kinetic theory is a statistical theory, that is, it is based on the laws of chance.

Theoretically it is possible from the laws of dynamics to completely describe the motion of all the molecules of a gas in a container. However, this is of little interest. Complete knowledge of the history of every molecule in the gas is not what we are seeking; we need to know only so much about the molecular motions as will suffice to explain and predict quantitatively the observable macroscopic properties of the gas such as pressure or viscosity. Kinetic theory attempts to explain these properties in terms of the molecular structure and the forces between the molecules.

To say anything quantitative, we must say something about the geometry of the molecules. We will picture a molecule as a rigid, perfectly elastic, spherical body, that is, a billiard ball. This assumption means that no momentum or energy will be transferred to the internal degrees of freedom of the molecule and thus momentum and energy will be conserved during collision. The justification for this assumption of a billiard ball molecule is in its success in predicting the properties of a real gas. One of the most striking features of the kinetic theory is the extent to which it is possible to predict the behavior of a gas as a whole, while remaining in almost complete ignorance of the behavior and properties of the molecules of which it is composed. The reason for this is that many properties of gases depend only on general dynamical principles such as the conservation of energy and momentum.

Four things are required to specify the properties of a gas: 1. The diameter of each molecule, 2. the mass of each molecule, 3. the number of molecules per unit volume and 4. the distribution of the random motion of the molecules. Once the particular gas of interest is specified, the molecular diameter and mass are known. The number of molecules per unit volume is determined once the density is specified. That leaves only the distribution of the random motion of the molecules to be determined.

A distribution function may be defined as follows. A distribution function gives some sort of value (i.e. density, velocity, etc.) associated with a "volume" in some "space". We want to find the velocity distribution function for a steady state distribution of velocities. In the steady state, the distribution function will be independent of time.

A molecule may be said to have six coordinates, three spacial coordinates, and three velocity coordinates. Let the spacial coordinates be x, y, z and the velocity coordinates be u, v, w corresponding to velocities in the x, y, and z directions respectively. Let n be the number of molecules per unit volume. We want to know how many molecules have velocities which lie between u and $u + du$, v and $v + dv$, w and $w + dw$. If we define the density of velocities with coordinates u, v, w as τ, then $\tau(u, v, w)$ is the distribution function for velocity in the velocity space. It gives the number of points in an element of volume, du dv dw, in velocity space. Since each point corresponds to one molecule in x, y, z space, τ gives the number of molecules with velocities in du dv dw. Then the number of molecules per unit volume with velocities between u and $u + du$, v and $v + dv$, w and $w + dw$ is

$$dn = \tau\, du\, dv\, dw$$

We will define the normalized distribution function as

$$f(u, v, w) = \frac{\tau(u, v, w)}{n}$$

Using this notation, dn becomes

$$dn = nf(u, v, w)\,du\,dv\,dw$$

Since n is the number of molecules per unit volume, the probability that the velocity of a molecule selected at random from the element of volume dx dy dz will be in the range u to u + du, v to v + dv and w to w + dw is

$$\frac{dn}{n} = f(u, v, w)\,du\,dv\,dw.$$

We now search for a steady state law for the distribution of velocity in a gas. A rigorous derivation of the Maxwell velocity distribution function is a tedious procedure. Since its existence has been verified experimentally, it may be considered a known physical law and the following discussion may be considered as a plausibility argument for its existence. Since it is assumed that the molecules in a gas do not effect each other except during collision, the only way the velocity of a molecule can be changed is by a collision with another molecule. Consider two molecules in collision, molecules A and B with velocities before and after collision as indicated below.

	Before Collision	After Collision
Molecule A	$u,\ v,\ w$	$\bar{u},\ \bar{v},\ \bar{w}$
Molecule B	$u',\ v',\ w'$	$\bar{u}',\ \bar{v}',\ \bar{w}'$

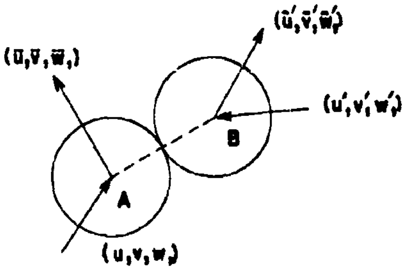


Figure A-1

In order to determine the velocities after collision, the direction of the line of centers as well as the initial velocities must be known. Let type α collisions be collisions of molecules with given velocities (u, v, w) and (u′, v′, w′) and given direction of line of centers. The number of collisions of this type will be proportional to the number of molecules of class (u, v, w) and of class (u′, v′, w′) which are present.

$$\left.\begin{array}{l}\text{The number of molecules of class } (u, v, w)\\ \text{per unit volume.}\end{array}\right\} = nf(u, v, w)\,du\,dv\,dw$$

$$\left.\begin{array}{l}\text{The number of molecules of class } (u', v', w')\\ \text{per unit volume.}\end{array}\right\} = nf(u', v', w')\,du'\,dv'\,dw'$$

The number of collisions of type α will be proportional to the product of the number of molecules of classes A and B that are present. Thus the number of collisions of type α per unit volume per unit time is

$$A_\alpha\, n^2 f(u,v,w) f(u',v',w')\,du\,dv\,dw\,du'\,dv'\,dw'$$

where A_α is the function of proportionality. A_α is a function of:

(1) the diameter of the molecules - the larger the molecules, the more likely they are to have a collision.

(2) the relative velocity of the molecules - molecules moving parallel to each other are not likely to run into each other while if they are moving toward each other, they are very likely to have a collision.

(3) the direction of the line of centers.

To be completely rigorous, we should determine the functional form of A_α. However, this is a detailed question and we will not go into it since it is not necessary to know A_α in order to determine the distribution function. During each collision of type α, one molecule of class (u, v, w) and (u′, v′, w′) is lost and one molecule of class $(\bar{u}, \bar{v}, \bar{w})$ and $(\bar{u}', \bar{v}', \bar{w}')$ is gained.

Now let a type β collision be the inverse of a type α collision, that is, it is a collision with velocity $(\bar{u}, \bar{v}, \bar{w})$ and $(\bar{u}', \bar{v}', \bar{w}')$ before collision and velocity (u, v, w) and (u′, v′, w′) after collision. Using reasoning similar to that for a type α collision, we find that the number of collisions of type β per unit volume per unit time $= A_\beta\, n^2 f(u, v, w) f(\bar{u}', \bar{v}', \bar{w}')\, d\bar{u}\,d\bar{v}\,d\bar{w}\,d\bar{u}'\,d\bar{v}'\,d\bar{w}'$.

The relative velocities of the molecules in type α and β collisions are the same. The diameters of the molecules are the same because all the molecules are physically the same. From the dynamics of the collisions and the definition of a type β collision, the direction of the line of centers is the same for type α and β collisions. Therefore,

$$A_\alpha = A_\beta$$

It can be shown that $du dv dw \, du' dv' dw' = d\bar{u} d\bar{v} d\bar{w} \, d\bar{u}' d\bar{v}' d\bar{w}'$. Using these facts,

$$\left.\begin{array}{l}\text{the number of collisions of type } \beta \\ \text{per unit volume per unit time}\end{array}\right\} = A_\alpha n^2 f(\bar{u}, \bar{v}, \bar{w}) f(\bar{u}', \bar{v}', \bar{w}') du dv dw du' dv' dw'$$

The net gain of molecules of class (u, v, w) per unit volume per unit time as a result of collisions of types α and β is equal to the number of molecules gained by type β collisions minus the number of molecules lost by type α collisions

$$= A_\alpha n^2 \left[f(\bar{u}, \bar{v}, \bar{w}) f(\bar{u}', \bar{v}', \bar{w}') - f(u, v, w) f(u', v', w') \right] du dv dw \, du' \, dv' \, dw'$$

If we set the term in brackets equal to 0, we have that

$$f(\bar{u}, \bar{v}, \bar{w}) f(\bar{u}', \bar{v}', \bar{w}') - f(u, v, w) f(u', v', w') = 0. \quad (2\text{-A})$$

This means that there will be no net gain or loss of molecules of class (u, v, w). If this term is zero, then since A_α which depends on the direction of the line of centers does not appear in the brackets, it is zero regardless of the direction of the line of centers. Further, if we require that the term in brackets be zero for all velocities, then the gas is in equilibrium. Equation (2-A) is a sufficient condition for equilibrium. Boltzmann showed that it is also a necessary condition for equilibrium. Thus, for a steady state to exist, every collision and its inverse must be in equilibrium. This is a special case of the principle of detailed balance which states that if a substance is in macroscopic equilibrium, every microscopic process occurring inside the medium must be in equilibrium.

In order to determine the equilibrium velocity distribution function, we must solve Equation (2-A) subject to the restraints placed on u, v, w, u', v', w', $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{u}'$, $\bar{v}'$, $\bar{w}'$ by the energy and momentum conservation equations for a molecular collision. Take logarithms of Equation (2A) and rearrange. This gives

$$\ln f(u, v, w) + \ln f(u', v', w') = \ln f(\bar{u}, \bar{v}, \bar{w}) + \ln f(\bar{u}', \bar{v}', \bar{w}')$$

Let $F = \ln f$. Then, $F(u, v, w) + F(u', v', w') = F(\bar{u}, \bar{v}, \bar{w}) + F(\bar{u}', \bar{v}', \bar{w}')$ (3-A)

To solve equation (3A) we need some functions of velocity which have the same form before and after collision. We have four such functions, three from the conservation of momentum and one from the conservation of energy (mu, mv, mw and $m(u^2 + v^2 + w^2)$). Since each of these is a solution of the equation, a linear combination of them will also be a solution. Thus

$$F(u, v, w) = \alpha_1 m(u^2 + v^2 + w^2) + \alpha_2 mu + \alpha_3 mv + \alpha_4 mw + \alpha_5 .$$

It can be shown that this is the only solution of this equation. Rearranging the constants we obtain

$$\ln f(u, v, w) = \alpha_1 m \left[(u-u_o)^2 + (v-v_o)^2 + (w-w_o)^2 \right] + \alpha_6$$

or

$$f(u, v, w) = Ae^{-\beta m \left[(u-u_o)^2 + (v-v_o)^2 + (w-w_o)^2 \right]} \tag{4-A}$$

We now have to determine the constants which appear in equation (4-A). The probability of finding a molecule for which $(u-u_o)$ is negative is the same as the probability of finding a molecule for which $(u-u_o)$ is positive. Thus, the average value of $(u-u_o)$ is zero. Since

$$\overline{u-u_o} = 0, \quad \bar{u} - \bar{u}_o = 0$$

and $\bar{u} = \bar{u}_o = u_o$ because u_o is a constant. This indicates that u_o is the mass velocity of the gas. Thus, u_o, v_o and w_o are the components of the mass motion of the gas. For no mass motion,

$$f(u, v, w) = Ae^{-\beta m(u^2 + v^2 + w^2)}$$

where in this case, u, v and w are the components of the thermal motion of the molecules.

Now, the number of molecules with a particular velocity is $nf\,du\,dv\,dw$. The integral of this over all possible velocities will give the total number of molecules in a unit volume. Thus

$$\int\int\int_{-\infty}^{+\infty} n f \, du dv dw = n$$

Since n is a constant, it can be factored out of the integral. Then

$$\int\int\int_{-\infty}^{+\infty} f \, du dv dw = 1$$

Substituting in for f in the case of no mass motion gives

$$A \int_{-\infty}^{+\infty} e^{-m\beta u^2} du \int_{-\infty}^{+\infty} e^{-m\beta v^2} dv \int_{-\infty}^{+\infty} e^{-m\beta w^2} dw = 1$$

Performing the integrations we obtain

$$A \left(\frac{\pi}{\beta m} \right)^{3/2} = 1$$

$$A = \left(\frac{\pi}{\beta m} \right)^{-3/2} \qquad (5-A)$$

It can be shown that the average value of u^2 is

$$\overline{u^2} = RT = \frac{k}{m} T$$

where k is the universal gas constant per molecule.

The average value of u^2 can be determined by

$$\overline{u^2} = \int\int\int_{-\infty}^{+\infty} u^2 f(u, v, w) du dv dw$$

Equating these two expressions for $\overline{u^2}$ gives

$$\overline{u^2} = \frac{k}{m}T = \left(\frac{\pi}{\beta m}\right)^{-3/2}\int_{-\infty}^{+\infty} u^2 e^{-m\beta u^2}\,du\int_{-\infty}^{+\infty} e^{-m\beta v^2}\,dv\int_{-\infty}^{+\infty} e^{-m\beta w^2}\,dw$$

$$\frac{k}{m}T = \left(\frac{\pi}{\beta m}\right)^{-\frac{1}{2}}\left[2\,\frac{1}{4}\left(\frac{\pi}{\beta^3 m^3}\right)^{\frac{1}{2}}\right]$$

$$\frac{k}{m}T = \frac{1}{2\beta m}$$

$$\therefore\ \beta = \frac{1}{2kT} \tag{6-A}$$

Substituting this value for β into the expression for A gives

$$A = \left(\frac{2\pi kT}{m}\right)^{-3/2} \tag{7-A}$$

Using these values for β and A, f becomes

$$f(u,v,w) = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m}{2kT}\left(u^2+v^2+w^2\right)} \tag{8-A}$$

This is the Maxwell velocity distribution function for a gas in equilibrium with itself where u, v, and w are the components of the random thermal motion of the molecules.

Since we must know the mean free path of the molecules in order to determine the flow regime, we will now derive an expression for this quantity. Consider the frequency of collisions between molecules of type 1 and 2.

We have a volume V with N_1 molecules of type 1 with velocity denoted by $c_1 = c_1(u_1, v_1, w_1)$ and N_2 molecules of type 2 with velocity denoted by $c_2 = c_2(u_2, v_2, w_2)$. The frequency of collisions will depend on the relative velocity between molecules 1 and 2. If the relative velocity is zero, the molecules will not collide. The relative velocity c_r is defined as

$$c_r = \sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2 + (w_1 - w_2)^2}$$

We want the average c_r over all molecules of types 1 and 2. First we will average c_r for a given type 1 molecule over all type 2 molecules. Then we will average this over all type 1 molecules to obtain the average over all molecules of types 1 and 2.

The number of molecules of type 2 with a given velocity c_2 is

$$N_2 f_2(c_2)\, du_2 dv_2 dw_2 .$$

The average speed of one molecule of type 1 with velocity c_1 relative to all molecules of type 2 is

$$\frac{\iiint_{-\infty}^{+\infty} c_r N_2 f_2(c_2)\, du_2 dv_2 dw_2}{N_2}$$

$$\left.\begin{array}{l}\text{The number of molecules of type 1}\\ \text{with given velocity } c_1\end{array}\right\} = N_1 f_1(u_1, v_1, w_1)\, du_1 dv_1 dw_1 .$$

$$\left.\begin{array}{l}\text{The average velocity of all molecules}\\ \text{of type 1 relative to all molecules}\\ \text{of type 2}\end{array}\right\} = \bar{c}_r$$

$$\bar{c}_r = \frac{\iiint_{-\infty}^{+\infty}\left[\iiint_{-\infty}^{+\infty} c_r f_2(u_2 v_2 w_2)\, du_2 dv_2 dw_2\right] N_1 f_1(u_1 v_1 w_1)\, du_1 dv_1 dw_1}{N_1}$$

$$\bar{c}_r = \iiiint\!\!\iint_{-\infty}^{+\infty} c_r f_1(u_1 v_1 w_1) f_2(u_2 v_2 w_2)\, du_1 dv_1 dw_1 du_2 dv_2 dw_2$$

It can be shown that in a mixture of gases in equilibrium, each gas has a maxwellian velocity distribution. Thus

$$\bar{c}_r = \frac{(m_1 m_2)^{3/2}}{(2\pi kT)^3} \int_{-\infty}^{+\infty}\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int c_r \, e^{-\frac{1}{2kT}(m_1 c_1^2 + m_2 c_2^2)} \, du_1 dv_1 dw_1 du_2 dv_2 dw_2$$

Performing the indicated integrations we obtain.

$$\bar{c}_r = \left(\frac{8kT}{\pi m_{12}^*}\right)^{\frac{1}{2}} \qquad (9\text{-}A)$$

where $m_{12}^* = \dfrac{m_1 m_2}{m_1 + m_2}$ is the reduced mass.

We now wish to find the frequency of collision between type 1 and type 2 molecules. Consider a type 1 molecule moving through type 2 molecules.

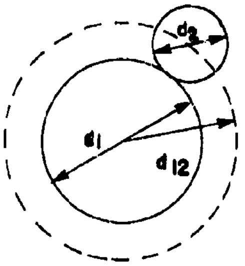


Figure A-2.

The radius of the sphere of influence is $d_{12} = \dfrac{d_1 + d_2}{2}$.

The type 1 molecule is moving relative to the type 2 molecules with the average relative velocity $\bar{c}_r$.

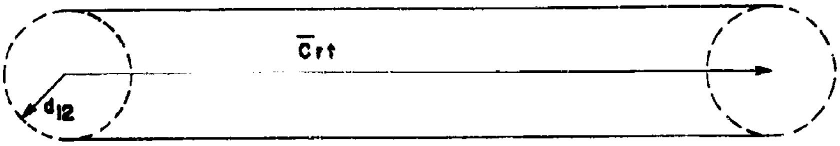


Figure A-3

The distance traveled by a type 1 molecule in time t is $\bar{c}_r t$.

The volume swept out by the sphere of influence per unit time $\Big\} = \bar{c}_r \pi d_{12}^2$

The average rate of collision of one type 1 molecule with type 2 molecules is Θ_{12}. Then

$$\Theta_{12} = n_2 \pi d_{12}^2 \bar{C}_r = \pi n_2 d_{12}^2 \left(\frac{8KT}{\pi m_{12}^*}\right)^{\frac{1}{2}}$$

This holds for any number of molecules 1, 2, 3, ... S.

Thus, the average rate of collision of one type 1 molecule with molecules of types 1, 2, 3, ... S is

$$\Theta_1 = \sum_S \Theta_{1S} = (8\pi KT)^{1/2} \sum_S \frac{n_S d_{1S}^2}{(m_{1S}^*)^{1/2}}$$

$$\Theta_1 = \sqrt{8\pi KT} \sum_S n_S d_{1S}^2 \sqrt{\frac{1}{m_1} + \frac{1}{m_S}} \qquad (10\text{-}A)$$

The mean free path of type 1 molecules is the average distance traveled between collisions. It is given by the ratio of the distance traveled per unit time to the average number of collisions per unit time. Thus,

$$\lambda_1 = \frac{\bar{C}_1}{\Theta_1} = \frac{\left(\frac{8KT}{\pi m_1}\right)^{\frac{1}{2}}}{\sqrt{8\pi KT} \sum_S n_S d_{1S}^2 \sqrt{\frac{1}{m_1} + \frac{1}{m_S}}}$$

where $\bar{c}_1$ is the average velocity of the type 1 molecules.

$$\therefore \lambda_1 = \frac{1}{\pi \sum_S n_S d_{1S}^2 \sqrt{1 + \frac{m_1}{m_S}}} \qquad (11\text{-}A)$$

This expression is rigorously accurate for a mixture of molecules in thermodynamic equilibrium. If all the molecules are the same, λ_1, becomes

$$\lambda = \frac{1}{\sqrt{2}\,\pi n d^2} \qquad (12\text{-}A)$$

APPENDIX B

INTEGRATION OF THE EQUATION FOR THE FORCE ON AN ELEMENT OF AREA

We want to evaluate the integrals in the equation

$$\frac{dF}{dA} = mn_i \int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty} (ku + lv + tw)\, v f \, du\, dv\, dw$$

$$+ mn_r \int_{-\infty}^{+\infty}\int_{-\infty}^{0}\int_{-\infty}^{+\infty} (ku + lv + tw)\, v f \, du\, dv\, dw \qquad (1-B)$$

where

$$f = \left(\frac{1}{2\pi RT}\right)^{3/2} e^{-\frac{1}{2RT}\left[(u-\bar{u})^2 + (v-\bar{v})^2 + (w-\bar{w})^2\right]}$$

$$= \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta\left[(u-\bar{u})^2 + (v-\bar{v})^2 + (w-\bar{w})^2\right]} \qquad (2-B)$$

and $\beta = \frac{1}{2RT}$

These integrations were carried out with the aid of appendix one in reference (14). Consider the first term of Eq. (1-B). It can be written as

$$mn_i\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty}(ku+lv+tw)v f\,du\,dv\,dw$$

$$= mn_i\Big\{k\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty}uvf\,du\,dv\,dw + l\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty}v^2 f\,du\,dv\,dw$$

$$+ t\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty}wvf\,du\,dv\,dw\Big\} \qquad (3-B)$$

We will evaluate the integrals of Eq. (3-B) separately.

$$\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty}uvf\,du\,dv\,dw = \left(\frac{\beta}{\pi}\right)^{3/2}\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty}uv\,e^{-\beta[(u-\bar{u})^2+(v-\bar{v})^2+(w-\bar{w})^2]}\,du\,dv\,dw \qquad (4-B)$$

Let the components of the thermal motions of the molecules be represented by H_1 in x direction, H_2 in the y direction and H_3 in the z direction. Then

$u = \bar{u} + H_1 \; , \quad u - \bar{u} = H_1$

$v = \bar{v} + H_2 \; , \quad v - \bar{v} = H_2$

$w = \bar{w} + H_3 \; , \quad w - \bar{w} = H_3$

$du = dH_1$

$dv = dH_2$

$dw = dH_3$

because ū, v̄, and w̄, the components of the mass velocity, are constants. Making this substitution eq. (4-B) becomes

$$\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty} uvf\,du\,dv\,dw = \left(\frac{\beta}{\pi}\right)^{3/2}\int_{-\infty}^{+\infty}\int_{-\bar{v}}^{+\infty}\int_{-\infty}^{+\infty}(\bar{u}+H_1)(\bar{v}+H_2)e^{-\beta(H_1^2+H_2^2+H_3^2)}\,dH_1\,dH_2\,dH_3$$

Since the random motions are independent of each other, the above equation can be written as

$$\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty} uvf\,du\,dv\,dw = \left(\frac{\beta}{\pi}\right)^{3/2}\left[\int_{-\infty}^{+\infty}(\bar{u}+H_1)e^{-\beta H_1^2}dH_1\right]\left[\int_{-\bar{v}}^{+\infty}(\bar{v}+H_2)e^{-\beta H_2^2}dH_2\right]\left[\int_{-\infty}^{+\infty}e^{-\beta H_3^2}dH_3\right]$$

$$=\left(\frac{\beta}{\pi}\right)^{3/2}\left[\bar{u}\sqrt{\frac{\pi}{\beta}}+0\right]\left[\frac{\bar{v}}{2}\sqrt{\frac{\pi}{\beta}}\left(1+\operatorname{erf}\bar{v}\sqrt{\beta}\right)+\frac{1}{2\beta}e^{-\beta\bar{v}^2}\right]\sqrt{\frac{\pi}{\beta}}$$

$$\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty} uvf\,du\,dv\,dw = \frac{\bar{u}}{2\sqrt{\pi}}\left[\bar{v}\sqrt{\pi}\left(1+\operatorname{erf}\bar{v}\sqrt{\beta}\right)+\frac{1}{\sqrt{\beta}}e^{-\beta\bar{v}^2}\right] \qquad (5\text{-}B)$$

The second integral is evaluated as follows.

$$\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty} v^2 f\,du\,dv\,dw = \left(\frac{\beta}{\pi}\right)^{3/2}\int_{-\infty}^{+\infty}e^{-\beta H_1^2}dH_1\int_{-\bar{v}}^{+\infty}(\bar{v}^2+2\bar{v}H_2+H_2^2)e^{-\beta H_2^2}dH_2\int_{-\infty}^{+\infty}e^{-\beta H_3^2}dH_3$$

$$= \left(\frac{\beta}{\pi}\right)^{3/2}\sqrt{\frac{\pi}{\beta}}\left[\frac{\bar{v}^2}{2}\sqrt{\frac{\pi}{\beta}}\left(1+\operatorname{erf}\bar{v}\sqrt{\beta}\right)+\frac{\bar{v}}{\beta}e^{-\beta\bar{v}^2}\right.$$

$$\left.+\frac{1}{2\beta}\left\{\frac{1}{2}\sqrt{\frac{\pi}{\beta}}\left(1+\operatorname{erf}\bar{v}\sqrt{\beta}\right)-\bar{v}e^{-\beta\bar{v}^2}\right\}\right]\sqrt{\frac{\pi}{\beta}}$$

$$=\left(\frac{\beta}{\pi}\right)^{\frac{1}{2}}\left[\frac{\bar{v}^2}{2}\sqrt{\frac{\pi}{\beta}}\left(1+\operatorname{erf}\bar{v}\sqrt{\beta}\right)+\frac{\bar{v}}{2\beta}e^{-\beta\bar{v}^2}+\frac{1}{4\beta}\sqrt{\frac{\pi}{\beta}}\left(1+\operatorname{erf}\bar{v}\sqrt{\beta}\right)\right]$$

$$\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty} v^2 f\,du\,dv\,dw = \frac{\bar{v}}{2\sqrt{\pi}}\left[\bar{v}\sqrt{\pi}\left(1+\operatorname{erf}\bar{v}\sqrt{\beta}\right)+\frac{1}{\sqrt{\beta}}e^{-\beta\bar{v}^2}\right]+\frac{1}{4\beta}\left(1+\operatorname{erf}\bar{v}\sqrt{\beta}\right) \qquad (6\text{-}B)$$

The third integral is evaluated as follows,

$$\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty} w v f\,du\,dv\,dw = \left(\frac{\beta}{\pi}\right)^{3/2}\int_{-\infty}^{+\infty} e^{-\beta H_1^2}\,dH_1 \int_{-\bar{v}}^{+\infty}(\bar{v}+H_2)\,e^{-\beta H_2^2}\,dH_2 \int_{-\infty}^{+\infty}(\bar{w}+H_3)\,e^{-\beta H_3^2}\,dH_3$$

$$= \left(\frac{\beta}{\pi}\right)^{3/2}\sqrt{\frac{\pi}{\beta}}\left[\frac{\bar{v}}{2}\sqrt{\frac{\pi}{\beta}}\left(1+\operatorname{erf}\bar{v}\sqrt{\beta}\right)+\frac{1}{2\beta}e^{-\beta\bar{v}^2}\right]\left[\bar{w}\sqrt{\frac{\pi}{\beta}}+0\right]$$

$$\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty} w v f\,du\,dv\,dw = \frac{\bar{w}}{2\sqrt{\pi}}\left[\bar{v}\sqrt{\pi}\left(1+\operatorname{erf}\bar{v}\sqrt{\beta}\right)+\frac{1}{\sqrt{\beta}}e^{-\beta\bar{v}^2}\right] \qquad (7\text{-}B)$$

Substituting into Eq. (3-B), equations (5-B), (6-B) and (7-B) we get

$$mn_i\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty}(ku+lv+tw)\,v f\,du\,dv\,dw$$

$$= mn_i\left\{\frac{1}{2\sqrt{\pi}}\left(k\bar{u}+l\bar{v}+t\bar{w}\right)\left[\bar{v}\sqrt{\pi}\left(1+\operatorname{erf}\bar{v}\sqrt{\beta}\right)+\frac{1}{\sqrt{\beta}}e^{-\beta\bar{v}^2}\right] + \frac{l}{4\beta}\left(1+\operatorname{erf}\bar{v}\sqrt{\beta}\right)\right\} \qquad (8\text{-}B)$$

Now consider the second term of Eq. (1-B). It can be written as

$$mn_r\int_{-\infty}^{+\infty}\int_{-\infty}^{0}\int_{-\infty}^{+\infty}(ku+\ell v+tw)vf\,du\,dv\,dw$$

$$= mn_r\left\{k\int_{-\infty}^{+\infty}\int_{-\infty}^{0}\int_{-\infty}^{+\infty}uvf\,du\,dv\,dw+\ell\int_{-\infty}^{+\infty}\int_{-\infty}^{0}\int_{-\infty}^{+\infty}v^2f\,du\,dv\,dw\right.$$

$$\left.+t\int_{-\infty}^{+\infty}\int_{-\infty}^{0}\int_{-\infty}^{+\infty}wvf\,du\,dv\,dw\right\} \qquad (9\text{-}B)$$

This term gives the force due to the re-emitted molecules which we assumed reflected diffusely, i. e. they are assumed to issue randomly from a fictitious gas which has a maxwellian velocity distribution corresponding to T_r and no mass motion relative to the element of area. Thus, the only velocity the reflected molecules possess is their random thermal motion corresponding to T_r. Thus,

$$u = H_1, \qquad v = H_2, \qquad w = H_3 \qquad \text{and}$$

$$du = dH_1, \qquad dv = dH_2, \qquad dw = dH_3.$$

We will evaluate these integrals one at a time.

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{0}\int_{-\infty}^{+\infty}uvf\,du\,dv\,dw = \left(\frac{\beta_r}{\pi}\right)^{3/2}\int_{-\infty}^{+\infty}H_1e^{-\beta_rH_1^2}dH_1\int_{-\infty}^{0}H_2e^{-\beta_rH_2^2}dH_2\int_{-\infty}^{+\infty}e^{-\beta_rH_3^2}dH_3 = 0 \qquad (10\text{-}B)$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{0}\int_{-\infty}^{+\infty}v^2f\,du\,dv\,dw = \left(\frac{\beta_r}{\pi}\right)^{3/2}\int_{-\infty}^{+\infty}e^{-\beta_rH_1^2}dH_1\int_{-\infty}^{0}H_2^2e^{-\beta_rH_2^2}dH_2\int_{-\infty}^{+\infty}e^{-\beta_rH_3^2}dH_3 \ldots$$

$$= \left(\frac{\beta_r}{\pi}\right)^{3/2}\sqrt{\frac{\pi}{\beta_r}}\sqrt{\frac{\pi}{\beta_r}}\int_{-\infty}^{0}H_2^2e^{-\beta_rH_2^2}dH_2$$

Let $H_2 = -X \qquad H_2 = -\infty,\ X = \infty$

$dH_2 = -dX \qquad H_2 = 0,\ X = 0$

$$\int_{-\infty}^{0} H_2^2 e^{-\beta_r H_2^2} dH_2 = -\int_{\infty}^{0} X^2 e^{-\beta_r X^2} dX = \int_{0}^{\infty} X^2 e^{-\beta_r X^2} dX = \frac{1}{4\beta_r}\sqrt{\frac{\pi}{\beta_r}}$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{0}\int_{-\infty}^{+\infty} v^2 f\,du\,dv\,dw = \frac{1}{4\beta_r} \qquad (11-B)$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{0}\int_{-\infty}^{+\infty} wvf\,du\,dv\,dw = \left(\frac{\beta_r}{\pi}\right)^{3/2}\int_{-\infty}^{+\infty} e^{-\beta_r H_1^2} dH_1 \int_{-\infty}^{0} H_2 e^{-\beta_r H_2^2} dH_2 \int_{-\infty}^{+\infty} H_3 e^{-\beta_r H_3^2} dH_3$$

$$= 0 \qquad (12-B)$$

Substituting eqs. (10-B), (11-B) and (12-B) into eq. (9-B) gives

$$mn_r \int_{-\infty}^{+\infty}\int_{-\infty}^{0}\int_{-\infty}^{+\infty} (ku + \ell v + tw) vf\,du\,dv\,dw = mn_r \frac{\ell}{4\beta_r} \qquad (13-B)$$

where $\beta_r = \frac{1}{2RT_r}$

At first it might seem surprising that the reflected molecules contribute only a force normal to the surface. However, if it is remembered that when we assumed that the molecules where diffusely reflected, it was assumed that they were reemitted in a random manner, that is, as many molecules with a particular velocity leave the surface in one direction as in the opposite direction. Thus, the tangential momentum components cancel each other out. Therefore, this result is to be expected.

Substituting equations (8-B) and (13-B) into equation (1-B) gives for the component of the total force on a element of area in a particular direction

$$\frac{dF}{dA} = mn_i \left\{ \frac{1}{2\sqrt{\pi}} (k\bar{u} + \ell\bar{v} + t\bar{w}) \left[\bar{v}\sqrt{\pi} \left(1 + \operatorname{erf} \bar{v}\sqrt{\beta}\right) + \frac{1}{\sqrt{\beta}} e^{-\beta\bar{v}^2} \right] + \frac{\ell}{4\beta}\left(1 + \operatorname{erf} \bar{v}\sqrt{\beta}\right) \right\} + mn_r \frac{\ell}{4\beta_r} \qquad (14\text{-}B)$$

Determination of n_r:

The number of molecules in the velocity range u to u + du, v to v + dv, w to w + dw that impinge on the element of area in unit time is

$$n_i v f \, du \, dv \, dw$$

The total number of molecules in all velocity ranges that impinge on the element of area in unit time is

$$N_i = n_i \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} v f \, du \, dv \, dw$$

This integral is evaluated in a manner similar to the previous integrals.

$$N_i = n_i\left(\frac{\beta}{\pi}\right)^{3/2}\int_{-\infty}^{+\infty} e^{-\beta H_1^2}\,dH_1 \int_{-\bar{v}}^{+\infty}(\bar{v}+H_2)\,e^{-\beta H_2^2}\,dH_2 \int_{-\infty}^{+\infty} e^{-\beta H_3^2}\,dH_3$$

$$= n_i\left(\frac{\beta}{\pi}\right)^{3/2}\sqrt{\frac{\pi}{\beta}}\left[\frac{\bar{v}}{2}\sqrt{\frac{\pi}{\beta}}\left(1+\operatorname{erf}\bar{v}\sqrt{\beta}\right)+\frac{1}{2\beta}e^{-\beta\bar{v}^2}\right]\sqrt{\frac{\pi}{\beta}}$$

$$N_i = \frac{n_i}{2\sqrt{\pi}}\left[\bar{v}\sqrt{\pi}\left(1+\operatorname{erf}\bar{v}\sqrt{\beta}\right)+\frac{1}{\sqrt{\beta}}e^{-\beta\bar{v}^2}\right] \qquad (15\text{-}B)$$

We proceed in a similar manner for the reflected molecules.

The number of molecules in the velocity range u to u + du, v to v + dv, w to w + dw that leave the element of area per unit time is

$$n_r(-v)f\,du\,dv\,dw$$

The total number of molecules in all velocity ranges that leave the element of area in unit time is

$$N_r = n_r\int_{-\infty}^{+\infty}\int_{-\infty}^{0}\int_{-\infty}^{+\infty}(-v)f\,du\,dv\,dw$$

Since the fictitious gas from which the reflected molecules are assumed to issue has no mass motion relative to the surface, the integral becomes

$$N_r = n_r\left(\frac{\beta_r}{\pi}\right)^{3/2}\int_{-\infty}^{+\infty} e^{-\beta_r H_1^2}\,dH_1 \int_{-\infty}^{0}(-H_2)\,e^{-\beta_r H_2^2}\,dH_2 \int_{-\infty}^{+\infty} e^{-\beta_r H_3^2}\,dH_3$$

$$N_r = n_r\left(\frac{\beta_r}{\pi}\right)^{3/2}\sqrt{\frac{\pi}{\beta_r}}\sqrt{\frac{\pi}{\beta_r}}\int_{-\infty}^{0}(-H_z)e^{-\beta_r H_z^2}dH_z$$

LET $H_z = -X$, $dH_z = -dX$ and $H_z = -\infty$, $X = \infty$; $H_z = 0$, $X = 0$

$$-\int_{-\infty}^{0} H_z e^{-\beta_r H_z^2} dH_z = -\int_{\infty}^{0} X e^{-\beta_r X^2} dX = \int_{0}^{\infty} X e^{-\beta_r X^2} dX = \frac{1}{2\beta_r}$$

$$\therefore \quad N_r = \frac{n_r}{2\sqrt{\pi\beta_r}} \qquad (16-B)$$

From continuity, since molecules cannot be created or destroyed,

$$N_i = N_r \qquad (17-B)$$

Substituting in eq. (17-B) for N_i and N_r and solving for n_r gives

$$n_r = n_i\sqrt{\beta_r}\left[\bar{v}\sqrt{\pi}\left(1+\operatorname{erf}\bar{v}\sqrt{\beta}\right)+\frac{1}{\sqrt{\beta}}e^{-\beta\bar{v}^2}\right] \qquad (18-B)$$

The function, erf x, which appears in the preceding equations is defined as

$$\operatorname{erf} X \equiv \frac{2}{\sqrt{\pi}}\int_{0}^{X} e^{-y^2} dy \qquad (19-B)$$

APPENDIX C

TABLES OF THE FREE MOLECULE FLOW FUNCTIONS

Due to their bulk, the tables of the free molecule flow functions have been published under separate cover and will be supplied to interested persons upon request. Requests should be sent to:

Project Publications and Services
Department 61-62
Lockheed Missiles and Space Company
Sunnyvale, California

The tables have been published under the title "Tables of Free Molecule Flow Functions, Appendix C to LMSC Technical Report 448514" and has been given the number LMSC 448514-1.

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