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**VISCOELASTIC BEHAVIOR OF SURFACES OF
REVOLUTION UNDER COMBINED
MECHANICAL AND THERMAL LOADS**

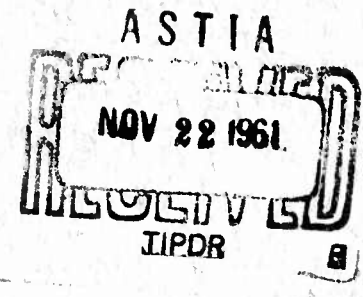
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MAY 1961

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AERONAUTICAL RESEARCH LABORATORY
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE



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OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

FOREWORD

This report was prepared by the University of Minnesota, Minneapolis, Minnesota, under Air Force Contract No. AF 33(616)-5723. The research program was initiated by the Aeronautical Research Laboratory, Directorate of Laboratories, Air Force Research Division, Wright-Patterson Air Force Base, Ohio, under Project 7063, "Mechanics of Flight", and Task 70524, "Structures Research at Elevated Temperatures". Dr. Robert Mayerjak represented ARL as task scientist.

The work was done under direction Dr. L. Albert Scipio and experimental study was supervised by Mr. James A. Moses.

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ABSTRACT

This paper presents the results of an analytical and experimental study of thin conical and hemispherical shells subjected to combined mechanical and thermal loads.

The analytical solutions are applicable for shells constructed from a linear viscoelastic material with temperature dependent properties.

Experimental models were subjected to various constant radiant heating rates to various steady-state temperatures. In some cases a constant normal pressure was combined with the thermal load.

Theoretical values for the meridional and circumferential stress distributions based on a viscoelastic analysis (both temperature independent and temperature dependent material properties) and the elastic analysis are compared with experimental results for typical time intervals during the transient and steady-state periods.

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LIST OF SYMBOLS

- α Thermal expansion coefficient
- β Rate of variation of the shear modulus of elasticity with respect to temperature evaluated between a datum temperature and a prescribed temperature
- γ Angle between axis of the conical shell and an element of the cone, i.e., half-angle
- δ Thickness of the shell
- δ_{ij} Kronecker delta
- θ Angle measured about axis of shell in a plane normal to axis
- ϕ Angle between axis of the shell and a normal to a meridian
- η Viscosity
- η_0 Viscosity at a datum temperature
- ϵ_{ij} Components of strain tensor
- $\epsilon_{\phi M}$ Strain component in ϕ direction at the shell middle-surface
- $\epsilon_{\theta M}$ Strain component in θ direction at the shell middle-surface
- σ_{ij} Components of stress tensor
- ξ Distance from the shell middle-surface, positive inward; displacement
- ρ_1 Radius of meridian curvature
- ρ_2 Distance from the shell to axis of rotation
- κ Temperature - viscosity coefficient, evaluated between a datum temperature and a prescribed temperature
- ν Poisson's ratio
- ν_i Poisson's ratio at the initial time
- ψ Change of curvature
- l Distance along cone element, measured from vertex
- l_0 Slant height of cone

p	Pressure, positive inward
r	Radius of parallel circle
t	Time
u	Component of displacement along the meridian, positive in the direction of increasing ϕ
w	Component of displacement perpendicular to the surface, positive inward
E	Modulus of elasticity
E_i	Modulus of elasticity at initial time
E_{ij}	Strain deviator
F_i	Components of body force
G	Modulus of elasticity in shear
G_0	Modulus of elasticity in shear at a datum temperature
M_ϕ	Bending moment in meridian plane per unit length of parallel circle
M_θ	Bending moment in plane perpendicular to meridian plane per unit length of meridional section
N_θ	Membrane force per unit length of shell section
N_ϕ	Membrane force tangent to a parallel circle per unit length of the meridian
Q_ϕ	Shearing force per unit in plane perpendicular to meridian plane
Q_θ	Shearing force per unit length tangent to a parallel circle
T	Amount of temperature above that of datum temperature
T_M	Amount of temperature at shell middle surface above that of datum temperature
T_o	Amount of temperature at shell outer surface above that of datum temperature
T_t	Amount of temperature at the tip of the shell outer surface above that of datum temperature
T_i	Temperature of the shell inner surface
S_{ij}	Stress deviator

I. INTRODUCTION

In recent years there has been considerable interest in the class of aircraft problems commonly known as aerodynamic heating. The effects of aerodynamic heating on the structure are: (1) thermal stresses induced in the structure, (2) creep and/or relaxation effects, and (3) development of inhomogeneity since the material properties vary with the temperature distribution.

At elevated temperatures, the thermal stress problem (Condition 1) is but the first of several related problems. The phenomenon of creep (Condition 2) becomes of importance if the temperatures are high enough, even if the exposure times are short. In addition, creep buckling can take place since the stresses are compressive as well as tensile.

The deformational response^{10*} of structural metals at elevated temperatures is a combination of elastic, viscous, and plastic components. The three mechanical constants associated with these components of deformation: the elastic modulus, the coefficient of viscosity, and the yield stress; and the thermal constants which govern the level of the thermal stresses: the coefficients of linear expansion and of thermal conductivity, are temperature dependent (Condition 3).

Classical thermoelasticity²⁴ does not take Conditions 2 and 3 into account and therefore gives only crude approximations at elevated temperatures when these effects are known to be pronounced. A viscoelastic material¹ will satisfy Condition 2, and if in addition its material properties are taken as temperature dependent, then true aerodynamic heating effects are approached. A viscoelastic response, that is, one which exhibits relaxation and heredity effects, can be assumed to represent approximately the behavior of metals either at relatively low stresses or at very high temperatures. If this viscoelastic response is produced by temperatures, the term thermoviscoelasticity has been used. It includes the general theory of heat conduction, thermal stresses, and deformations produced by thermal flow in the viscoelastic media and the reverse effect of temperature distribution produced by the elastic and viscous deformation.

The first concern, therefore, is with the heat conduction in the structure itself. The usual procedure of thermoelastic analysis is to determine the stresses on the assumption that the thermal and the elastic fields are uncoupled and that the temperature field is given. Although the assumption of uncoupled thermal and elastic fields has practical truth in structural problems, the two fields are coupled by the

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* Numbers refer to entries in the bibliography. The bibliography includes references for background information in addition to those references noted in the text.

second law of thermodynamics. The heat conduction equation will contain elastic terms and the equations of elasticity will contain temperature gradients. The conventional formulation in thermoelasticity simply assumes that the thermal field does not contain the small elastic terms but that the temperature gradients are included in the equations of elasticity.

The equations of the classical theory of elasticity may be immediately extended to viscoelasticity by simply replacing the elastic moduli by their corresponding operators. As in the case of thermoelasticity, the reciprocal coupling between the temperature and deformations is neglected. Classical thermodynamics shows that one effect cannot occur without the other but in the theory of structures, its order of magnitude is not significant.

The objective of the program reported herein which has been conducted at the University of Minnesota under Contract No. AF 33(616)-5723, Task 70524, Project 7063, can be stated in general terms as "Development of Analytical Methods to Predict the Structural Behavior of Curved Surfaces Subject to an Environment that Simultaneously Imposes High Heating Rates, Temperatures, Loading and Accelerations".

Specifically, the investigation is planned to study the stress and strain distributions, and buckling characteristics of surfaces of revolution subjected to external pressure and various temperature conditions.

The methods of analysis take into account the influence of time and temperature upon surfaces of revolution with temperature dependent properties. Conical and hemispherical shells are selected for the verification specimens for analytical and experimental simplicity.

The methods of analysis and their verification by experiments are summarized in the body of this report. Items of a secondary nature such as details of the experimental program, are presented in the appendices.

II. FUNDAMENTAL EQUATIONS OF THERMOVISCOELASTICITY

The mathematical formulation of the response of a viscoelastic body to combined mechanical and thermal loads can be approximately represented by the following set of differential equations:

Equations of Equilibrium	$\sigma_{ij,j} + F_i = 0$
Compatibility Equations	$\epsilon_{ij,k} + \epsilon_{kl,ij} - \epsilon_{ik,jl} - \epsilon_{jl,ik} = 0$
Equations of State	$P\{S_{ij}\} = 2Q\{E_{ij}\}, \quad \epsilon = \alpha T$

where P and Q are linear operators representing different viscoelastic materials and $E_{ij} = \epsilon_{ij} - \frac{1}{3} \epsilon_{kk} \delta_{ij}$, $S_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}$, $\epsilon_{ij} = \frac{1}{2} (\dot{x}_{ij} + \dot{x}_{ji})$. Subject to the boundary conditions, the system of nine equations on the set of nine unknown field variables $\sigma_{ij}, \dot{x}_i$ is complete in the sense that solution of the system is unique, if the solution exists.

2.1 The Model and the Equations of Equilibrium

The model of this analysis, as shown in Fig 2.1-1, is a thin shell of revolution generated by any arbitrary curve. By thin, it is implied that the shell thickness is of small order of magnitude as compared with other dimensions. The shell is further assumed to be of uniform thickness. No restrictions are placed on the manner by which the edge is supported. However, the load system is assumed to be axial-symmetric, having both mechanical and thermal origins.

For a shell of revolution, the shear stress tangent to a parallel circle is equal to zero, or $Q_\theta = 0$ ²⁵. Furthermore, the moment and the tensile stress in θ - direction will be independent of θ . The stresses and moments on a small element a b c d defined by two meridian planes and two paralleled circles, are shown in Fig 2.1 - 1.

Three equations of equilibrium²⁵ can be established by equating the force components in the direction tangent to the meridian and normal to meridian to zero and by setting the moment about cd to zero.

$$\frac{\partial(N_\phi r)}{\partial \phi} - N_\theta r \cos \phi - Q_\phi r = 0 \quad (2.1 - 1)$$

$$N_\phi r + N_\theta r \sin \phi + \frac{\partial(Q_\phi r)}{\partial \phi} + p r = 0 \quad (2.1 - 2)$$

$$\frac{\partial(M_\phi r)}{\partial \phi} - M_\theta r \cos \phi - Q_\phi r = 0 \quad (2.1 - 3)$$

Partial differentiations are being used because the independent variables are also functions of time.

In addition to the three equations shown above, one may also establish another equilibrium condition by considering the equilibrium of the portion of the shell above a parallel circle

$$2\pi r N_\phi \sin \phi + 2\pi r Q_\phi \cos \phi + \pi r^2 p = 0$$

from which

$$N_\phi = -Q_\phi \cot \phi - \frac{p r}{2 \sin \phi} \quad (2.1 - 4)$$

One way to simplify the equations of equilibrium is to use equations (2.1 - 2), (2.1 - 3), and (2.1 - 4) and introducing new variables $W = Q_\phi r$ and $\gamma = \frac{1}{r} (v + \frac{\partial W}{\partial \phi})$ as described in Timoshenko²⁵, or to proceed by eliminating $Q_\phi r$ from the equations (2.1 - 1), (2.1 - 2), and (2.1 - 3) to obtain two fundamental equations of equilibrium. This procedure of approach had been used by Stodola²³, Keller¹⁷, and Frankhauser⁹ in their elastic analysis. A brief derivation will be as follows:

From equation (2.1 - 3) one may write:

$$Q_{\phi}^n = \frac{\partial(M_{\phi}^n)}{r_1 \partial \phi} - M_{\theta} \cos \phi \quad (2.1 - 5)$$

Differentiation with respect to ϕ yields

$$\frac{\partial(Q_{\phi}^n)}{\partial \phi} = \frac{\partial}{\partial \phi} \left[\frac{1}{r_1} \frac{\partial(M_{\phi}^n)}{\partial \phi} \right] - \cos \phi \frac{\partial M_{\theta}}{\partial \phi} + M_{\theta} \sin \phi \quad (2.1 - 6)$$

Introducing equations (2.1 - 5) and (2.1 - 6) into equations (2.1 - 1) and (2.1 - 2) gives

$$\frac{\partial(N_{\phi}^n)}{\partial \phi} - N_{\theta} r_1 \cos \phi + M_{\theta} \cos \phi - \frac{1}{r_1} \frac{\partial(M_{\phi}^n)}{\partial \phi} = 0 \quad (2.1 - 7)$$

$$\frac{1}{r_1} \frac{\partial}{\partial \phi} \left[\frac{1}{r_1} \frac{\partial(M_{\phi}^n)}{\partial \phi} \right] - \frac{\cos \phi}{r_1} \frac{\partial M_{\theta}}{\partial \phi} + \frac{M_{\theta} \sin \phi}{r_1} + \frac{N_{\phi}^n}{r_1} + N_{\theta} \sin \phi + p_n = 0 \quad (2.1 - 8)$$

Equations (2.1 - 7) and (2.1 - 8) are now the equations of equilibrium.

2.2 Equations of State

As this analysis deals with linear viscoelastic material, the relation between the stress and strain may be taken as

$$\{S_{ij}\} = 2 \left(\eta \frac{\partial}{\partial t} + G \right) \{E_{ij}\} \quad (2.2 - 1)$$

Upon introduction of the stress and strain deviators, equations (2.2 - 1) take the following form

$$\left. \begin{aligned} \frac{2}{3} \sigma_{\phi\phi} - \frac{1}{3} \sigma_{\theta\theta} &= 2 \left(\eta \frac{\partial}{\partial t} + G \right) (\epsilon_{\phi\phi} - \alpha T) \\ \frac{2}{3} \sigma_{\theta\theta} - \frac{1}{3} \sigma_{\phi\phi} &= 2 \left(\eta \frac{\partial}{\partial t} + G \right) (\epsilon_{\theta\theta} - \alpha T) \end{aligned} \right\} \quad (2.2 - 2)$$

The components of stress may be written as

$$\left. \begin{aligned} \sigma_{\phi\phi} &= 2 \left(\eta \frac{\partial}{\partial t} + G \right) (2 \epsilon_{\phi\phi} + \epsilon_{\theta\theta} - 3 \alpha T) \\ \sigma_{\theta\theta} &= 2 \left(\eta \frac{\partial}{\partial t} + G \right) (2 \epsilon_{\theta\theta} + \epsilon_{\phi\phi} - 3 \alpha T) \end{aligned} \right\} \quad (2.2 - 3)$$

Obviously, both $\sigma_{\phi\phi}$ and $\sigma_{\theta\theta}$ are functions of the physical properties, strains, time and temperature. It will be discussed later in paragraph 2.6 that the physical properties involved here are functions of temperature. Since the temperature may vary across the shell thickness as well as along the surface meridian, and since, the deformations for viscoelastic material are functions of load and time, the resulting stress pattern is extremely complicated. It is therefore necessary to make reasonable assumptions to simplify the problem as the analysis proceeds.

With values of $\sigma_{\phi\phi}$ and $\sigma_{\theta\theta}$ it is possible to write the expression for membrane forces and moments. Taking the shell middle surface as a reference surface for integration, one may express the membrane forces and moments as follows

$$\left. \begin{aligned} N_{\phi} &= \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \sigma_{\phi\phi} \left(1 - \frac{\xi}{f_1}\right) d\xi, & M_{\phi} &= \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \sigma_{\phi\phi} \xi \left(1 - \frac{\xi}{f_1}\right) d\xi \\ N_{\theta} &= \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \sigma_{\theta\theta} \left(1 - \frac{\xi}{f_2}\right) d\xi, & M_{\theta} &= \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \sigma_{\theta\theta} \xi \left(1 - \frac{\xi}{f_2}\right) d\xi \end{aligned} \right\} \quad (2.2 - 4)$$

But for thin shells, the small quantities $\frac{\xi}{f_1}$ and $\frac{\xi}{f_2}$ can be neglected. Thus,

$$\left. \begin{aligned} N_{\phi} &= \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \sigma_{\phi\phi} d\xi, & M_{\phi} &= \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \sigma_{\phi\phi} \xi d\xi \\ N_{\theta} &= \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \sigma_{\theta\theta} d\xi, & M_{\theta} &= \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \sigma_{\theta\theta} \xi d\xi \end{aligned} \right\} \quad (2.2 - 5)$$

These integrations and the values of $\sigma_{\phi\phi}$ and $\sigma_{\theta\theta}$ in equation (2.2 - 3) reveal the importance of simplification in all variables involved in these equations.

2.3 Strains as Functions of Displacement

Considering the shell middle surface, the strain or deformation along the meridian is

$$\epsilon_{\phi\phi} = \frac{1}{f_1} \left(\frac{\partial v}{\partial \phi} - w \right) \quad (2.3 - 1)$$

and the corresponding strain in the circumferential direction of the parallel circle is

$$\epsilon_{\theta\theta} = \frac{1}{f_2} \left(\frac{v}{\tan \phi} - w \right) \quad (2.3 - 2)$$

Partial derivatives are being used in (2.3 - 1) and (2.3 - 2) instead of ordinary derivatives in order to consider displacements as function of both ϕ and time. If the variation of strains across the shell thickness has to be considered, the strain will be a function of distance, ξ , from the shell middle surface.

$$\left. \begin{aligned} \epsilon_{\phi\phi} &= \frac{1}{f_1} \left(\frac{\partial v}{\partial \phi} - w \right) - \frac{1}{f_1} \frac{\partial}{\partial \phi} \left(\frac{v}{f_1} + \frac{\partial w}{f_1 \partial \phi} \right) \xi \\ \epsilon_{\theta\theta} &= \frac{1}{f_2} \left(\frac{v \cot \phi}{\tan \phi} - w \right) - \frac{\cot \phi}{f_2} \left(\frac{v}{f_1} + \frac{1}{f_1} \frac{\partial w}{\partial \phi} \right) \xi \end{aligned} \right\} \quad (2.3 - 3)$$

Obviously, the strains in the form of equations (2.3 - 3) will add complication in the derivation of basic differential equations. In the treatment of thin shells, however, the variation of deformation across the shell is neglected. Appendix AI also shows the justification of this approximation by the order of magnitude consideration. With this approximation, equation (2.2 - 3) may be written

$$\left. \begin{aligned} \sigma_{\theta\theta} &= 2\left(\gamma \frac{\partial}{\partial t} + G\right) \left(2\epsilon_{\phi M} + \epsilon_{\theta M} - 3\alpha T\right) \\ \sigma_{\phi\phi} &= 2\left(\gamma \frac{\partial}{\partial t} + G\right) \left(2\epsilon_{\theta M} + \epsilon_{\phi M} - 3\alpha T\right) \end{aligned} \right\} \quad (2.3 - 4)$$

In other words, in these stress equations, we take the strain of the shell middle-surface as the mean value of the strain across the shell thickness.

2.4 Pressure and Temperature of the Shell

At any point, the shell is subjected to a pressure load p , which could be taken as a function of time or a constant depending upon the status of the loading, steady or transit. In the case of a flight vehicle, the pressure is of dynamic type. The boundary layer immediately outside the body surface and the phenomena of separation⁸ result in a variation of pressure along the meridian of a shell. The pressure is maximum at the stagnation point and varies as one moves downstream on the body. Furthermore, if the angle of attack is not zero, the pressure distribution will be unsymmetrical. However, this analysis is restricted to bodies of zero angle of attack so that the pressure load will be axial-symmetric. Fig 2.4 - 1 shows the pressure distribution along a meridian.

In the case of high speed vehicles, aerodynamic heating effects are experienced in addition to the surface pressure distribution. The resulting surface temperature distribution normally varies along the body surface. The manner of variation depends upon the flight Mach number and the heat transfer conditions. In this analysis, the body has a constant inner surface temperature, with heat conduction on the edge. Thus the resulting outer surface temperature will be shown as in Fig 2.4 - 2. In some cases, the mathematical complication arising from the variation of temperature along the meridian forces one to assume a uniform outer surface temperature. This assumption deviates quite seriously from the actual problem. However, excellent approximation can be made by taking a number of uniform temperatures as shown in Fig 2.4 - 3.

This analysis is primarily concerned with the temperature variation within the shell thickness. For steady state condition, the temperature gradient across the shell thickness is a constant (based on the assumption of constant inner surface temperature and thickness). For the transient state, the temperature gradient varies with the outer surface temperature and with the distance, ξ , from the middle-surface.

2.5 Boundary Conditions

With the pressure and temperature prescribed in Paragraph 2.4, the boundary conditions can be derived. Depending on the choice of the initial conditions, one of the following categories of boundary conditions should apply:

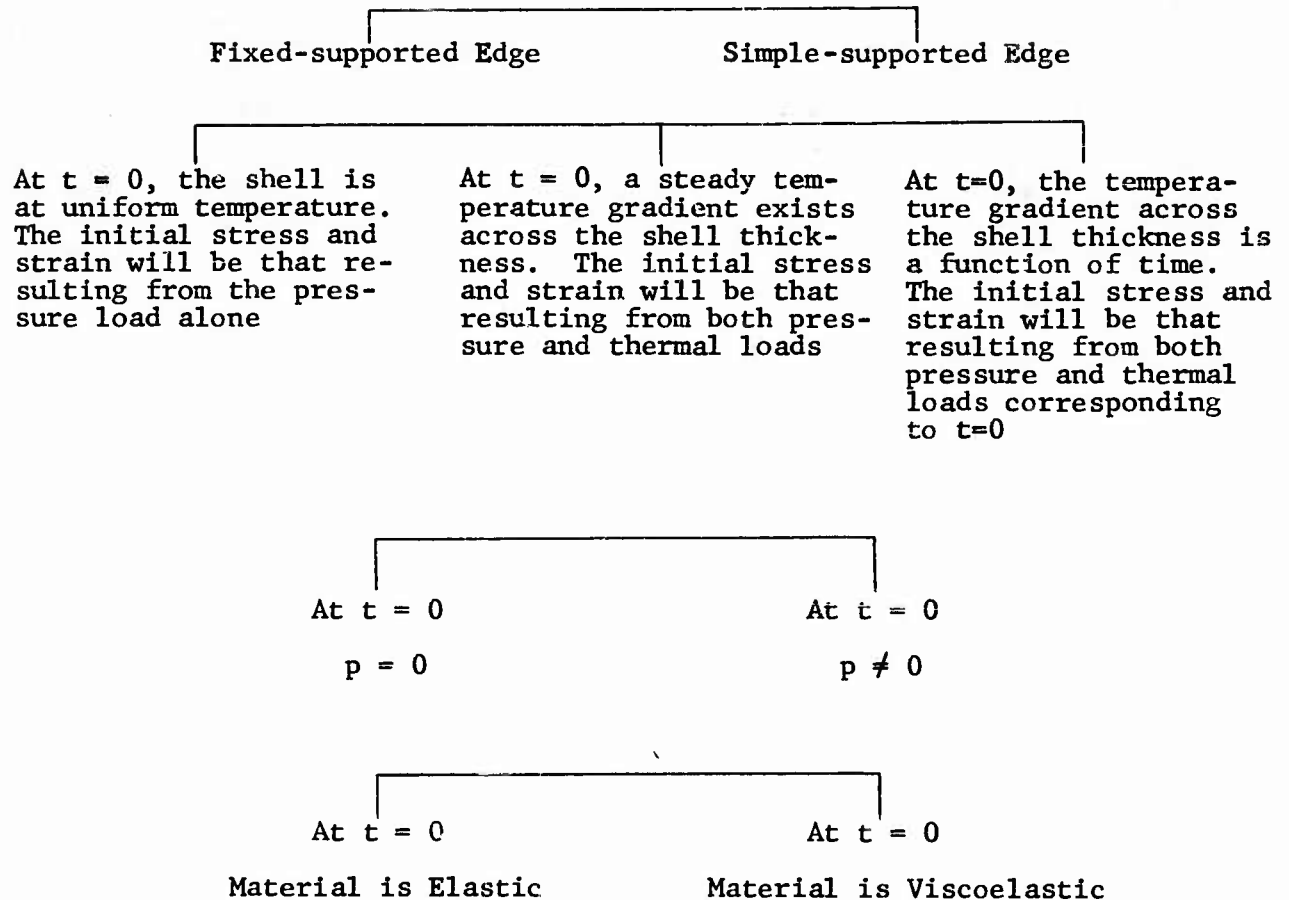
- a) Stress boundary conditions,
- b) Strain boundary conditions,
- c) Temperature boundary conditions,
- d) Displacement and rotational boundary conditions.

It is not intended that a detailed discussion of all possible boundary conditions be presented here. However, in the subsequent methods of analysis, selection of appropriate boundary conditions are illustrated.

For the problem under discussion, the following diagram summarizes the principal boundary conditions applicable for desired solutions:

(see page 8)

BOUNDARY CONDITIONS DIAGRAM



2.6 Physical Properties

The physical properties of shell material involved in this analysis are: viscosity, shear modulus of elasticity and thermal expansion coefficient. For the most metals, viscosity and shear modulus of elasticity decrease with an increase in temperature, while the thermal expansion coefficient increases with temperature. Since we are concerned with the effects of aerodynamic heating on viscoelastic bodies, the variation of material properties with temperature has to be taken into consideration.

Fig 2.6 - 1 (Reference 4) shows the variation of the mean coefficient of thermal expansion with temperature for several typical materials. It

is obvious a linear relationship will closely approximate the actual variation. However, for some materials, the temperature effect on the thermal expansion coefficient is small as compared with other properties. For example, in case of aluminum, the value of thermal expansion coefficient is increased by 8% for a temperature variation from 250°F to 600°F. However, for the same temperature variation, the shear modulus may decrease as much as 50%. In the subsequent analysis, therefore, the thermal expansion coefficient is assumed to be constant.

The shear modulus of elasticity takes a similar trend of variation with temperature as the modulus of elasticity. The value of shear modulus of elasticity sometimes can be approximated by taking one third that of modulus of elasticity. The variation of modulus of elasticity is shown in Fig 2.6 - 2⁴. Similar to the coefficient of thermal expansion, a linear variation may be assumed. The following relationship is used

$$G = G_0(1 - \beta T)$$

where G_0 is the shear modulus at some datum temperature

G is the shear modulus at T degrees above the datum temperature

β is the average rate of change of G with respect to the temperature for a specific temperature range.

The information of the viscosity for metals is very limited. Most liquids follow the Newtonian Law, that is, the viscosity decreases exponentially with the inverse of the absolute temperature. One may use a similar function for metals but the integration of viscosity with respect to temperature T will be extremely difficult. To retain the exponential property of viscosity on one hand and to take the advantage of less definite viscosity theory of metal on the other hand, it is assumed that the viscosity varies according to

$$\eta = \eta_0 e^{-KT}$$

where η_0 is viscosity at some datum temperature

η is viscosity at a temperature T degrees above the datum temperature

K is evaluated between the datum temperature and a prespecified temperature.

2.7 Methods of Analysis

In paragraph 2.1 it is shown that there are two ways of combining the equations of equilibrium which suggest there are different ways of solving the problem. Paragraph 2.3 indicates that the strain-displace-

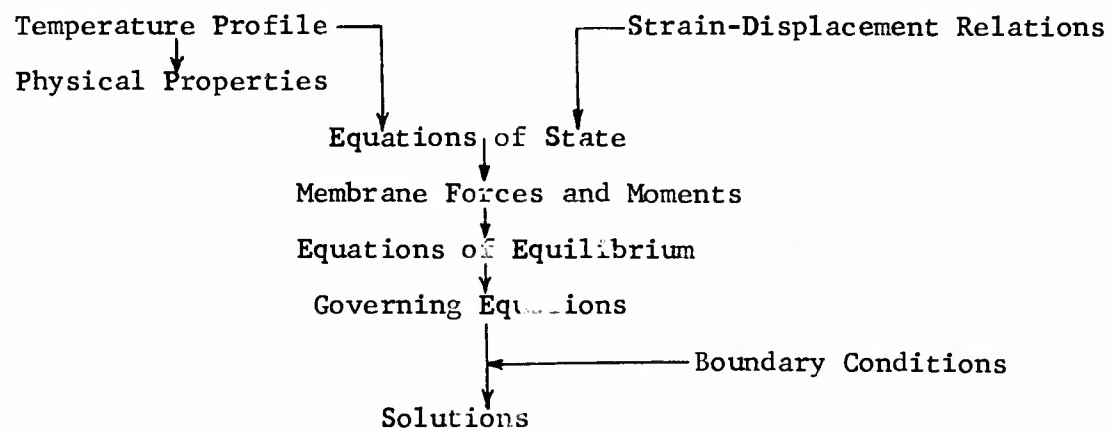
ment relation may take a more simplified form. The physical properties may be taken as functions of temperature, or they may be evaluated at some appropriate mean temperature in which case the problem is treated as being independent of temperature. All of these choices suggest that there are many ways to arrive at a solution. In Section III the conical shell is analyzed by three different approaches. However, from a mathematical point of view, for a given shell geometry one method may be more efficient than the other.

The general procedure of analysis is:

1. To proceed with the given temperature profile which leads to the physical properties and thermal stresses.
2. To make a choice of strain-displacement relation for substitution into the equation of state.
3. To find the values of membrane forces and moments from the stress components and express them in terms of displacements, material properties, and independent variables. In some cases, the use of appropriately chosen new variables may be useful.
4. To introduce the values of membrane forces and moments into the equations of equilibrium to obtain the governing equations. In some cases, instead of using membrane forces as obtained from 3 a fourth equation of equilibrium may be used.
5. To solve the governing equations the boundary conditions are inserted to evaluate arbitrary constants that arise from the integration of the differential equations.

The process of analysis may be summarized with the following diagram

PROCESS OF ANALYSIS DIAGRAM



III. ANALYSIS OF CONICAL SHELL

The critical section of many space flight vehicles is of conical geometry. The conical shell therefore, is taken as one of the examples for demonstrating the analysis.

Three different methods of analysis are used to attack this problem and solutions are illustrated for specific conditions.

3.1 Simplified Strain-Displacement Relation Method

3.1.1 The Governing Equations

For the conical shell, it is convenient to use the shell element length l as an independent variable which is measured from the tip of the shell. The principal radius ρ_1 approaches infinity with $\rho_1 d\phi$ approaching dl and $\rho_2 = l \tan \gamma$, $\rho = l \sin \gamma$, γ being half of the apex angle.

The equations of equilibrium are now written as

$$\left. \begin{aligned} \frac{\partial(M_\theta l)}{\partial l} - M_\theta - Q_\theta l &= 0 \\ \frac{\partial(N_\theta l)}{\partial l} &= N_\theta \\ \frac{\partial(Q_\theta l)}{\partial l} + N_\theta \cot \gamma + p l &= 0 \end{aligned} \right\} \quad (3.1.1 - 1)$$

which may be combined into two equations

$$\left. \begin{aligned} \frac{\partial(N_\theta l)}{\partial l} &= N_\theta \\ \frac{\partial^2(M_\theta l)}{\partial l^2} - \frac{\partial M_\theta}{\partial l} + N_\theta \cot \gamma + p l &= 0 \end{aligned} \right\} \quad (3.1.1 - 1a)$$

For the strain-displacement relations, the simplified functions discussed in paragraph 2.3 and also shown in Appendix A are used

$$\begin{aligned} \epsilon_{\phi\phi} &\doteq \epsilon_{\phi\theta} = \frac{\partial v}{\partial l} \\ \epsilon_{\theta\theta} &\doteq \epsilon_{\theta\theta} = \frac{v - w \cot \gamma}{l} \end{aligned}$$

A uniform outer surface temperature is assumed to be T_t degrees above the inner surface temperature. However, T_t may be considered as a function of time, if desired. The steady-state temperature gradient across the shell thickness will be (T_t/δ) and the temperature at a distance ξ from the middle surface will be T degrees above that of the inner surface temperature, or

$$T = T_t \left(\frac{1}{2} - \frac{\xi}{\delta} \right) \quad (3.1.1 - 2)$$

The temperature dependent properties are written as

$$\left. \begin{aligned} \eta &= \eta_0 e^{-K T_t \left(\frac{1}{2} - \frac{\delta}{\delta} \right)} \\ G &= G_0 \left[1 - \beta \left(\frac{1}{2} - \frac{\delta}{\delta} \right) T_t \right] \\ \alpha &= \alpha_0 \end{aligned} \right\} \quad (3.1.1 - 3)$$

Introducing (3.1.1 - 3) into equation (2.3 - 4), one finds for the stress components

$$\left. \begin{aligned} \sigma_{\phi\phi} &= 2 \left\{ \eta_0 e^{-K T_t \left(\frac{1}{2} - \frac{\delta}{\delta} \right)} \frac{\partial}{\partial t} + G_0 \left[1 - \beta T_t \left(\frac{1}{2} - \frac{\delta}{\delta} \right) \right] \right\} \left[2 \epsilon_{\phi M} + \epsilon_{\phi H} - 3 \alpha T_t \left(\frac{1}{2} - \frac{\delta}{\delta} \right) \right] \\ \sigma_{\theta\theta} &= 2 \left\{ \eta_0 e^{-K T_t \left(\frac{1}{2} - \frac{\delta}{\delta} \right)} \frac{\partial}{\partial t} + G_0 \left[1 - \beta T_t \left(\frac{1}{2} - \frac{\delta}{\delta} \right) \right] \right\} \left[2 \epsilon_{\theta M} + \epsilon_{\theta H} - 3 \alpha T_t \left(\frac{1}{2} - \frac{\delta}{\delta} \right) \right] \end{aligned} \right\} \quad (3.1.1 - 4)$$

Thus the membrane forces and the moments can be integrated as required by equations (2.2 - 5). For example

$$\begin{aligned} N_\theta &= \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \sigma_{\theta\theta} d\zeta = 4 \eta_0 \frac{\delta}{K T_t} e^{-\frac{K T_t}{2}} \sinh \frac{K T_t}{2} \frac{\partial}{\partial t} (2 \epsilon_{\theta M} + \epsilon_{\theta H}) + 2 G_0 \delta \left(1 - \frac{\beta T_t}{2} \right) (2 \epsilon_{\theta M} + \epsilon_{\theta H}) - \\ &\quad - G_0 \alpha e^{-\frac{K T_t}{2}} \frac{\delta}{K T_t} \left[\sinh \frac{K T_t}{2} + \frac{1}{K T_t} \left(2 \sinh \frac{K T_t}{2} - K T_t \cosh \frac{K T_t}{2} \right) \right] \frac{\partial T_t}{\partial t} - \\ &\quad - (3 - 2 \beta T_t) G_0 \alpha T_t \delta \end{aligned} \quad (3.1.1 - 5)$$

$$\begin{aligned} M_\theta &= 2 \eta_0 e^{-\frac{K T_t}{2}} \left(\frac{\delta}{K T_t} \right)^2 \left(K T_t \cosh \frac{K T_t}{2} - 2 \sinh \frac{K T_t}{2} \right) \frac{\partial (2 \epsilon_{\theta M} + \epsilon_{\theta H})}{\partial t} + \frac{G_0 \delta^2 \beta T_t}{6} (2 \epsilon_{\theta M} + \epsilon_{\theta H}) - \\ &\quad - 3 \eta_0 \alpha e^{-\frac{K T_t}{2}} \delta^2 \left\{ \left[\left(\frac{1}{K T_t} \right)^2 + \frac{4}{(K T_t)^3} \right] \left(K T_t \cosh \frac{K T_t}{2} - 2 \sinh \frac{K T_t}{2} \right) - \frac{\sinh \frac{K T_t}{2}}{K T_t} \right\} \frac{\partial T_t}{\partial t} + \\ &\quad + \frac{G_0 \alpha T_t \delta^2}{2} (1 - \beta T_t) \end{aligned} \quad (3.1.1 - 6)$$

Introducing equations (3.1.1 - 5) and (3.1.1 - 6) and the corresponding expressions for N_ϕ and M_ϕ into the equation of equilibrium (3.1.1 - 1a) one obtains

$$\begin{aligned} 2 \eta_0 \frac{1}{K T_t} (1 - e^{-K T_t}) \left[l \frac{\partial^2 (2 \epsilon_{\phi M} + \epsilon_{\phi H})}{\partial t \partial l} + \frac{\partial (\epsilon_{\phi M} - \epsilon_{\phi H})}{\partial t} \right] + \\ + 2 G_0 \left(1 - \frac{\beta T_t}{2} \right) \left[l \frac{\partial (2 \epsilon_{\phi M} + \epsilon_{\phi H})}{\partial l} + (\epsilon_{\phi M} - \epsilon_{\phi H}) \right] = 0 \end{aligned} \quad (3.1.1 - 7)$$

and

$$\begin{aligned}
& 2\gamma_0 e^{-\frac{\kappa T_c}{2}} \left\{ \left(\frac{\delta}{\kappa T_c} \right)^2 \left(\kappa T_c \cosh \frac{\kappa T_c}{2} - 2 \sinh \frac{\kappa T_c}{2} \right) \left[\ell \frac{\partial^2 (2\epsilon_{\phi M} + \epsilon_{\phi M})}{\partial t \partial \ell^2} + 3 \frac{\partial^2 \epsilon_{\phi M}}{\partial t \partial \ell} \right] + \right. \\
& \quad \left. + \frac{2\delta}{\kappa T_c} \cot \gamma \sinh \frac{\kappa T_c}{2} \frac{\partial (2\epsilon_{\phi M} + \epsilon_{\phi M})}{\partial t} \right\} + \frac{G_0 \delta^2 \beta T_c}{6} \left[\ell \frac{\partial^2 (2\epsilon_{\phi M} + \epsilon_{\phi M})}{\partial \ell^2} + 3 \frac{\partial \epsilon_{\phi M}}{\partial \ell} \right] + \\
& \quad + 2G_0 \delta \left(1 - \frac{\beta T_c}{2} \right) \cot \gamma (2\epsilon_{\phi M} + \epsilon_{\phi M}) + p\ell = \\
& = \cot \gamma \left\{ 6\gamma_0 \alpha e^{-\frac{\kappa T_c}{2}} \frac{\partial T_c}{\partial t} \frac{\delta}{\kappa T_c} \left[\sinh \frac{\kappa T_c}{2} - \frac{1}{\kappa T_c} \left(\kappa T_c \cosh \frac{\kappa T_c}{2} - 2 \sinh \frac{\kappa T_c}{2} \right) \right] + (3 - 2\beta T_c) G_0 \alpha \delta T_c \right\}
\end{aligned} \tag{3.1.1 - 8}$$

3.1.2 Method of Solving the Governing Equations

First of all, introducing new variables $U = 2\epsilon_{\phi M} + \epsilon_{\phi M}$ and $V = \epsilon_{\phi M}$ to replace terms involving $\epsilon_{\phi M}$ and $\epsilon_{\phi M}$, results in the following form for the governing equations

$$\frac{2\gamma_0}{\kappa T_c} (1 - e^{-\kappa T_c}) \frac{\partial}{\partial t} \left(\ell \frac{\partial U}{\partial \ell} + 3V - U \right) + G_0 (2 - \beta T_c) \left(\ell \frac{\partial^2 U}{\partial \ell^2} + 3V - U \right) = 0 \tag{3.1.2 - 1}$$

and

$$\begin{aligned}
& \gamma_0 \left(\frac{\delta}{\kappa T_c} \right)^2 \left[\kappa T_c (1 + e^{-\kappa T_c}) - 2(1 - e^{-\kappa T_c}) \right] \frac{\partial}{\partial t} \left(\ell \frac{\partial^2 U}{\partial \ell^2} + 3 \frac{\partial V}{\partial \ell} \right) + 2\gamma_0 \frac{\delta}{\kappa T_c} \cot \gamma (1 - e^{-\kappa T_c}) \frac{\partial}{\partial t} (2U - 3V) + \\
& \quad + \frac{G_0 \delta^2 \beta T_c}{6} \left(\ell \frac{\partial^2 U}{\partial \ell^2} + 3 \frac{\partial V}{\partial \ell} \right) + G_0 \delta (2 - \beta T_c) \cot \gamma (2U - 3V) + p\ell = \\
& = 6\gamma_0 \frac{\alpha \delta}{\kappa T_c} \left(-e^{-\kappa T_c} + \frac{1 - e^{-\kappa T_c}}{\kappa T_c} \right) \cot \gamma \frac{\partial T_c}{\partial t} + G_0 \delta \alpha T_c (3 - 2\beta T_c) \cot \gamma
\end{aligned} \tag{3.1.2 - 2}$$

From equation (3.1.2 - 1) $\frac{\partial}{\partial t} \left(\ell \frac{\partial U}{\partial \ell} + 3V - U \right) = -\frac{G_0 (2 - \beta T_c) \kappa T_c}{2\gamma_0 (1 - e^{-\kappa T_c})} \left(\ell \frac{\partial^2 U}{\partial \ell^2} + 3V - U \right)$

and for smoothly continuous functions of U and V with respect to ℓ and t , it can be written that

$$\ell \frac{\partial^3 U}{\partial t \partial \ell^2} + 3 \frac{\partial^2 V}{\partial t \partial \ell} = \frac{\partial}{\partial \ell} \left(\ell \frac{\partial^2 U}{\partial t \partial \ell} - \frac{\partial U}{\partial t} + 3 \frac{\partial V}{\partial t} \right)$$

and

$$\frac{\partial}{\partial \ell} \left(\ell \frac{\partial U}{\partial \ell} + 3V - U \right) = \ell \frac{\partial^2 U}{\partial \ell^2} + 3 \frac{\partial V}{\partial \ell}$$

Hence

$$\ell \frac{\partial^2 U}{\partial t \partial \ell^2} + 3 \frac{\partial^2 U}{\partial t \partial \ell} = - \frac{G_0 (2 - \beta T_t) K T_t}{2 \gamma_0 (1 - e^{-K T_t})} \left(\ell \frac{\partial^2 U}{\partial \ell^2} + 3 \frac{\partial U}{\partial \ell} \right) \quad (3.1.2 - 3)$$

Using equations (3.1.2 - 1), (3.1.2 - 2), and (3.1.2 - 3), the new governing equations become

$$\frac{\partial}{\partial t} \left(\ell \frac{\partial U}{\partial \ell} + 3U - U \right) + F_1 \left(\ell \frac{\partial U}{\partial \ell} + 3U - U \right) = 0 \quad (3.1.2 - 1a)$$

$$F_2 \left(\ell \frac{\partial^2 U}{\partial \ell^2} + 3 \frac{\partial U}{\partial \ell} \right) + \frac{\partial}{\partial t} \left(\ell \frac{\partial U}{\partial \ell} + U \right) + F_1 \left(\ell \frac{\partial U}{\partial \ell} + U \right) + F_3 \rho \ell = F_4 \quad (3.1.2 - 4)$$

where

$$F_1 = \frac{G_0 K T_t (2 - \beta T_t)}{2 \gamma_0 (1 - e^{-K T_t})}$$

$$F_2 = \frac{G_0 \delta K T_t \tan \gamma}{4 \gamma_0 (1 - e^{-K T_t})} \left[\frac{\beta T_t}{3} - (2 - \beta T_t) \left(\frac{1 + e^{-K T_t}}{1 - e^{-K T_t}} - \frac{2}{K T_t} \right) \right]$$

$$F_3 = \frac{K T_t \tan \delta}{2 \delta \gamma_0 (1 - e^{-K T_t})}$$

$$F_4 = \frac{K T_t}{2 \gamma_0 (1 - e^{-K T_t})} \left[6 \gamma_0 \frac{d}{d T_t} \left(\frac{1 - e^{-K T_t}}{K T_t} - e^{-K T_t} \right) \frac{\partial T_t}{\partial t} + G_0 \alpha T_t (3 - 2 \beta T_t) \right]$$

For cases where T_t is a function of time, then F_1 , F_2 , F_3 , and F_4 are all functions of time. The method of separation of variables are used to solve the governing equations. By trying a solution $(\ell \frac{\partial U}{\partial \ell} + 3U - U)$ of equation (3.1.2 - 1a) in the form $L_1(\ell) T_1(t)$, one obtains

$$\left(\ell \frac{\partial U}{\partial \ell} + 3U - U \right) = L_1(\ell) T_1(t) e^{-\int_0^t F_1 dt} \quad (3.1.2 - 5)$$

The values of $L_1(\ell) T_1(t)$ may be obtained from the boundary conditions, i.e., the value of $(\ell \frac{\partial U}{\partial \ell} - U + 3U)$ at a point ℓ and at an arbitrarily chosen initial time. It is, therefore, the choice of boundary conditions which has an influence on whether the solution of the problem can be obtained. For this example, the boundary conditions are chosen such that

a) A uniform and constant pressure is exerted on the shell at all times.

b) T_t is a function of time, equal to zero, at $t = 0$; increasing to a constant value when a steady state condition is reached. For simplification it is assumed that $T_t = Ct$ for $0 \leq t \leq t_c$ and $T_t = \bar{T}_t$ for $t > t_c$, where C and $\bar{T}_t$ are constants and t_c is the transition period.

c) The edge of the shell is fixed.

These boundary conditions imply that at $t = 0$ the shell has a deformation which is caused by the pressure load alone. The next question is the state of the material conditions at $t = 0$, whether elastic or viscoelastic. This depends on the type of material and the temperature to which the material is subjected at the initial time. In the elastic case, one may evaluate $(\ell \frac{\partial \bar{U}}{\partial \ell} + 3\bar{U} - U)_{\ell=l, t=0}$ from results of elastic analysis as discussed in most applied elasticity textbooks. A simplified version of the elastic analysis of a conical shell with thermal stress is discussed in Appendix A2. For the present application, we will use the results of Appendix A2 but omit the contributions from thermal effects. However, if the boundary conditions are specified as an elastic material with thermal stress at the initial moment, then the complete results should be used.

For elastic analysis

$$\bar{U}|_{\ell=l} = \frac{1}{E_i \delta} (N_\phi - \nu_i N_\theta)_{EL.}, \quad \bar{U}|_{\ell=l} = \frac{1}{E_i \delta} [(2-\nu_i)N_\phi + (1-2\nu_i)N_\theta]_{EL.}$$

and

$$\begin{aligned} (\ell \frac{\partial \bar{U}}{\partial \ell} + 3\bar{U} - U)_{\ell=l} &= (\ell \frac{\partial \bar{U}}{\partial \ell} + 3\bar{U} - U)_{EL.} = \frac{1}{E_i \delta} (1-2\nu_i) \ell \frac{\partial}{\partial \ell} (N_\phi + N_\theta)_{EL.} = \\ &= \frac{(1-2\nu_i)}{E_i \delta} \left\{ E_i \delta \cot \gamma \left[A_1 \left(-\frac{Z_1'}{\gamma} - \frac{2Z_2'}{\gamma^2} \right) + A_2 \left(-Z_2 + \frac{2Z_1'}{\gamma} \right) \right] - \frac{3}{2} p l \tan \gamma \right\} \end{aligned} \quad (3.1.2 - 6)$$

A_1 and A_2 are to be evaluated from the edge conditions.

The equation (3.1.2 - 5) may be written

$$(\ell \frac{\partial \bar{U}}{\partial \ell} + 3\bar{U} - U) = \frac{1}{E_i \delta} (1-2\nu_i) \ell \frac{\partial}{\partial \ell} (N_\phi + N_\theta)_{EL.} e^{-\int_0^t F_i dt} \quad (3.1.2 - 7)$$

From (3.1.2 - 7), the values of $\frac{\partial \bar{U}}{\partial \ell}$ can be expressed in the derivatives of $\bar{U}$ and the known initial values

$$\frac{\partial \bar{U}}{\partial \ell} = -\frac{\ell}{3} \frac{\partial^2 \bar{U}}{\partial \ell^2} + \frac{1}{3} e^{-\int_0^t F_i dt} \frac{(1-2\nu_i)}{E_i \delta} \frac{\partial}{\partial \ell} \left[\ell \frac{\partial}{\partial \ell} (N_\phi + N_\theta) \right]_{EL.} \quad (3.1.2 - 8)$$

where

$$\frac{\partial}{\partial \ell} \left[\ell \frac{\partial}{\partial \ell} (N_\phi + N_\theta) \right]_{EL.} = 4 E_i \cot^2 \gamma \sqrt{3(1-\nu_i)^2} \left[A_1 \left(-\frac{Z_1'}{\gamma} + \frac{2Z_2'}{\gamma^2} + \frac{4Z_2'}{\gamma^3} \right) + A_2 \left(-\frac{Z_2'}{\gamma} + \frac{2Z_2'}{\gamma^2} - \frac{4Z_2'}{\gamma^3} \right) \right] - \frac{3}{2} p \tan \gamma$$

substitution of $\frac{\partial \bar{U}}{\partial \ell}$ from equation (3.1.2 - 8) into equation (3.1.2 - 4) reduces it to a single dependent variable equation.

$$\frac{\partial}{\partial t} (\ell \frac{\partial \bar{U}}{\partial \ell} + U) + F_i (\ell \frac{\partial \bar{U}}{\partial \ell} + U) = -F_3 p l + F_4 - F_2 e^{-\int_0^t F_i dt} \frac{(1-2\nu_i)}{E_i \delta} \frac{\partial}{\partial \ell} \left[\ell \frac{\partial}{\partial \ell} (N_\phi + N_\theta) \right]_{EL.} \quad (3.1.2 - 9)$$

Again using the method of separation of variables and boundary conditions, one finds for $\overline{U}$

$$\overline{U} = \frac{e^{-\int_0^t F_1 dt}}{l} \left(\int_0^l \int_0^t \left\{ (F_4 - F_3 \rho l) e^{\int_0^t F_1 dt} - \frac{F_2(1-2\nu_i)}{E_i \delta} \frac{\partial}{\partial l} \left[l \frac{\partial}{\partial l} (N_\phi + N_\theta)_{EL} \right] \right\} dt dl + \right. \\ \left. + \frac{1}{E_i \delta} \int_0^l \left[3(1-\nu_i) N_\theta + (1-2\nu_i) l \frac{\partial N_\theta}{\partial l} \right]_{EL} dl \right) \quad (3.1.2 - 10)$$

From equation (3.1.2 - 7) the corresponding value of $\overline{V}$ is found to be

$$\overline{V} = \frac{e^{-\int_0^t F_1 dt}}{3} \left(\frac{2}{l} \int_0^l \int_0^t \left\{ (F_4 - F_3 \rho l) e^{\int_0^t F_1 dt} - \frac{F_2(1-2\nu_i)}{E_i \delta} \frac{\partial}{\partial l} \left[l \frac{\partial}{\partial l} (N_\phi + N_\theta)_{EL} \right] \right\} dt dl + \right. \\ \left. + \frac{2}{E_i \delta} \int_0^l \left[3(1-\nu_i) N_\theta + (1-2\nu_i) l \frac{\partial N_\theta}{\partial l} \right]_{EL} dl - \int_0^t \left\{ (F_4 - F_3 \rho l) e^{\int_0^t F_1 dt} - \frac{F_2(1-2\nu_i)}{E_i \delta} \frac{\partial}{\partial l} \left[l \frac{\partial}{\partial l} (N_\phi + N_\theta)_{EL} \right] \right\} dt - \frac{1}{E_i \delta} \left[3(1-\nu_i) N_\theta + (1-2\nu_i) l \frac{\partial N_\theta}{\partial l} \right]_{EL} + \right. \\ \left. + \frac{(1-2\nu_i)}{E_i \delta} l \frac{\partial}{\partial l} (N_\phi + N_\theta)_{EL} \right) \quad (3.1.2 - 11)$$

With values of $\overline{U}$ and $\overline{V}$ the subsequent values of $2\epsilon_{\phi m} + \epsilon_{\theta m}$ and $2\epsilon_{\theta m} + \epsilon_{\phi m}$ their time derivatives, and the membrane forces and moments can be readily determined. The values of N_θ and M_θ are illustrated as follows

$$N_{\theta(l,t)} = 2\delta \left[G_0 \left(1 - \frac{\beta T_e}{2} \right) - \frac{F_1 \eta_0}{K T_e} \left(1 - e^{-K T_e} \right) \right] e^{-\int_0^t F_1 dt} \left(\int_0^t \left\{ (F_4 - F_3 \rho l) e^{\int_0^t F_1 dt} - \frac{F_2(1-2\nu_i)}{E_i \delta} \frac{\partial}{\partial l} \left[l \frac{\partial}{\partial l} (N_\phi + N_\theta)_{EL} \right] \right\} dt - \right. \\ \left. - \frac{(1-2\nu_i)}{E_i \delta} l \frac{\partial}{\partial l} (N_\phi + N_\theta)_{EL} + \frac{1}{E_i \delta} \left[3(1-\nu_i) N_\theta + (1-2\nu_i) l \frac{\partial N_\theta}{\partial l} \right]_{EL} \right) + \\ + \frac{2\delta \eta_0}{K T_e} \left(1 - e^{-K T_e} \right) \left\{ (F_4 - F_3 \rho l) e^{-\int_0^t F_1 dt} - \frac{F_2(1-2\nu_i)}{E_i \delta} \frac{\partial}{\partial l} \left[l \frac{\partial}{\partial l} (N_\phi + N_\theta)_{EL} \right] \right\} - \\ - \frac{2\delta \eta_0 (1 - e^{-K T_e})}{K T_e} F_4 \quad (3.1.2 - 12)$$

and

$$\begin{aligned}
M_{\theta(l,t)} = & \delta^2 e^{-\int_0^t F_1 dt} \left\{ \frac{G_0 \beta T_k}{6} - \frac{F_1 \eta_0}{(K T_k)^2} \left[K T_k (1 + e^{-K T_k}) - 2(1 - e^{-K T_k}) \right] \right\} \left\{ \int_0^t (F_4 - F_3 \rho l) e^{-\int_0^t F_1 dt} \right. \\
& - \frac{F_2 (1 - 2\nu_i)}{E_i \delta} \frac{\partial}{\partial l} \left[l \frac{\partial}{\partial l} (N_\phi + N_\theta)_{EL} \right] \Bigg\} dt - \frac{(1 - 2\nu_i)}{E_i \delta} l \frac{\partial}{\partial l} (N_\phi + N_\theta)_{EL} + \\
& + \frac{1}{E_i \delta} \left[3(1 - \nu_i) N_\theta + (1 - 2\nu_i) l \frac{\partial N_\theta}{\partial l} \right]_{EL} + \eta_0 \left(\frac{\delta}{K T_k} \right)^2 \left[K T_k (1 + e^{-K T_k}) - \right. \\
& \left. - 2(1 - e^{-K T_k}) \right] \left\{ (F_4 - F_3 \rho l) e^{-\int_0^t F_1 dt} - \frac{F_2 (1 - 2\nu_i)}{E_i \delta} \frac{\partial}{\partial l} \left[l \frac{\partial}{\partial l} (N_\phi + N_\theta)_{EL} \right] \right\} - \\
& - 3\eta_0 \alpha \delta^2 \left\{ \left[\frac{1}{(K T_k)^2} + \frac{4}{(K T_k)^3} \right] \left[K T_k \frac{(1 + e^{-K T_k})}{2} - (1 - e^{-K T_k}) \right] - \frac{1 - e^{-K T_k}}{2 K T_k} \right\} \frac{\partial T_k}{\partial t} + \\
& + \frac{G_0 \alpha T_k \delta (1 - \beta T_k)}{2}
\end{aligned} \tag{3.1.2 - 13}$$

From this example, it can be seen that the method of analysis can be extended to apply to other boundary conditions; for example, if pressure is being a function of time and distance along the shell element, $p(t, l)$, or if an initial temperature gradient across the shell thickness exists or if the edge is simply supported.

3.2 Simplified Differential Equation Method

One of the important simplification made in paragraph 3.1.1 was the omission of terms containing ξ in the equations of $\epsilon_{\phi\phi}$ and $\epsilon_{\theta\theta}$. In doing so it was assumed that the middle surface values were the mean values and that the order of magnitude of these terms was small. In this paragraph, the basic governing equations are derived without imposing this limitation, however, ν and ω are assumed not to be functions of ξ .

Applying equation (2.3 - 3) to the conical shell results, one may write

$$\left. \begin{aligned} \epsilon_{\phi\xi} &= \frac{\partial \nu}{\partial l} - \frac{\partial^2 \omega}{\partial l^2} \xi \\ \epsilon_{\theta\xi} &= \frac{\nu - \omega \cot \delta}{l} - \frac{\xi}{l} \frac{\partial \omega}{\partial l} \end{aligned} \right\} \tag{3.2 - 1}$$

Employing the same temperature profile as given in paragraph 3.1.1, the stress components may be written as

$$\sigma_{\phi\phi} = 2 \left\{ \eta_0 e^{-K T_k \left(\frac{l}{2} - \frac{\xi}{\delta} \right)} \frac{\partial}{\partial t} + G_0 \left[-K \left(\frac{l}{2} - \frac{\xi}{\delta} \right) T_k \right] \right\} \left\{ 2 \left(\frac{\partial \nu}{\partial l} - \frac{\partial^2 \omega}{\partial l^2} \xi \right) + \frac{\nu - \omega \cot \delta}{l} - \frac{\xi}{l} \frac{\partial \omega}{\partial l} - 3\alpha T_k \left(\frac{l}{2} - \frac{\xi}{\delta} \right) \right\} \tag{3.2 - 2}$$

$$\sigma_{\theta\theta} = 2 \left\{ \eta_0 e^{-\frac{KT_c}{2} \left(\frac{1}{2} - \frac{\xi}{\delta} \right)} \frac{\partial}{\partial t} + G_0 \left[1 - \beta \left(\frac{1}{2} - \frac{\xi}{\delta} \right) T_c \right] \right\} \left\{ 2 \left(\frac{v - w \cot \delta}{l} - \frac{\xi}{l} \frac{\partial w}{\partial l} \right) + \frac{\partial v}{\partial l} - \xi \frac{\partial^2 w}{\partial l^2} - 3 \alpha l \left(\frac{1}{2} - \frac{\xi}{\delta} \right) \right\} \quad (3.2 - 3)$$

The membrane forces and moments will be

$$\begin{aligned} N_\phi &= \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \sigma_{\phi\phi} d\xi = \\ &= 4\eta_0 e^{-\frac{KT_c}{2}} \frac{\delta}{KT_c} \left[\sinh \frac{KT_c}{2} \frac{\partial}{\partial t} \left(2 \frac{\partial v}{\partial l} + \frac{v - w \cot \delta}{l} \right) - \frac{\delta}{KT_c} \left(\frac{KT_c}{2} \cosh \frac{KT_c}{2} - \sinh \frac{KT_c}{2} \right) \frac{\partial}{\partial t} \left(2 \frac{\partial^2 w}{\partial l^2} + \frac{1}{l} \frac{\partial w}{\partial l} \right) \right] + \\ &\quad + 2G_0 \delta \left(1 - \frac{\beta T_c}{2} \right) \left(2 \frac{\partial v}{\partial l} + \frac{v - w \cot \delta}{l} \right) - \frac{G_0 \beta T_c \delta^2}{6} \left(2 \frac{\partial^2 w}{\partial l^2} + \frac{1}{l} \frac{\partial w}{\partial l} \right) - \\ &\quad - \frac{6 \alpha \delta \eta_0 e^{-\frac{KT_c}{2}}}{6} \left[-e^{-\frac{KT_c}{2}} + \frac{2}{KT_c} \sinh \frac{KT_c}{2} \right] \frac{\partial T_c}{\partial t} - G_0 \alpha T_c \delta (3 - 2\beta T_c) \end{aligned} \quad (3.2 - 4)$$

$$\begin{aligned} N_\theta &= \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \sigma_{\theta\theta} d\xi = \\ &= 4\eta_0 e^{-\frac{KT_c}{2}} \frac{\delta}{KT_c} \left[\sinh \frac{KT_c}{2} \frac{\partial}{\partial t} \left(2 \frac{v - w \cot \delta}{l} + \frac{\partial v}{\partial l} \right) - \frac{\delta}{KT_c} \left(\frac{KT_c}{2} \cosh \frac{KT_c}{2} - \sinh \frac{KT_c}{2} \right) \frac{\partial}{\partial t} \left(\frac{2}{l} \frac{\partial w}{\partial l} + \frac{\partial^2 w}{\partial l^2} \right) \right] + \\ &\quad + 2G_0 \delta \left(1 - \frac{\beta T_c}{2} \right) \left(\frac{\partial v}{\partial l} + 2 \frac{v - w \cot \delta}{l} \right) - \frac{G_0 \beta T_c \delta^2}{6} \left(\frac{2}{l} \frac{\partial w}{\partial l} + \frac{\partial^2 w}{\partial l^2} \right) - \\ &\quad - \frac{6 \alpha \delta \eta_0 e^{-\frac{KT_c}{2}}}{KT_c} \left[\frac{2}{KT_c} \sinh \frac{KT_c}{2} - e^{-\frac{KT_c}{2}} \right] \frac{\partial T_c}{\partial t} - G_0 \alpha T_c \delta (3 - 2\beta T_c) \end{aligned} \quad (3.2 - 5)$$

$$\begin{aligned}
M_\phi = & 2\eta_0 e^{-\frac{KT_t}{2}} \left\{ \left[\left(\frac{\delta}{KT_t} \right)^2 \left(KT_t \cosh \frac{KT_t}{2} - 2 \sinh \frac{KT_t}{2} \right) \frac{\partial}{\partial t} \left(2 \frac{\partial v}{\partial l} + \frac{v - w \cot \gamma}{l} \right) \right] - \right. \\
& - \left[\frac{\delta^3}{2KT_t} \sinh \frac{KT_t}{2} + 2 \left(\frac{\delta}{KT_t} \right)^3 \left(2 \sinh \frac{KT_t}{2} - KT_t \cosh \frac{KT_t}{2} \right) \right] \frac{\partial}{\partial t} \left(2 \frac{\partial^2 w}{\partial l^2} + \frac{1}{l} \frac{\partial w}{\partial l} \right) \Big\} - \\
& - \frac{G_0 \delta^3}{6} \left[\left(1 - \frac{\beta T_t}{2} \right) \left(2 \frac{\partial^2 w}{\partial l^2} + \frac{1}{l} \frac{\partial w}{\partial l} \right) - \frac{\beta T_t}{\delta} \left(2 \frac{\partial v}{\partial l} + \frac{v - w \cot \gamma}{l} \right) \right] + \\
& + \frac{3\alpha \eta_0 \delta^2 e^{-\frac{KT_t}{2}}}{KT_t} \left[-e^{-\frac{KT_t}{2}} - \frac{1}{KT_t} \left(e^{\frac{KT_t}{2}} + 3e^{-\frac{KT_t}{2}} \right) + \frac{8}{(KT_t)^2} \sinh \frac{KT_t}{2} \right] \frac{\partial T_t}{\partial t} + \\
& + \frac{G_0 \alpha \delta^2 T_t}{2} (1 - \beta T_t)
\end{aligned} \tag{3.2 - 6}$$

$$\begin{aligned}
M_\theta = & 2\eta_0 e^{-\frac{KT_t}{2}} \left\{ \left[\left(\frac{\delta}{KT_t} \right)^2 \left(KT_t \cosh \frac{KT_t}{2} - 2 \sinh \frac{KT_t}{2} \right) \frac{\partial}{\partial t} \left(2 \frac{v - w \cot \gamma}{l} + \frac{\partial v}{\partial l} \right) \right] - \right. \\
& - \left[\frac{\delta^3}{2KT_t} \sinh \frac{KT_t}{2} + 2 \left(\frac{\delta}{KT_t} \right)^3 \left(2 \sinh \frac{KT_t}{2} - KT_t \cosh \frac{KT_t}{2} \right) \right] \frac{\partial}{\partial t} \left(2 \frac{\partial w}{\partial l} + \frac{\partial^2 w}{\partial l^2} \right) \Big\} - \\
& - \frac{G_0 \delta^3}{6} \left[\left(1 - \frac{\beta T_t}{2} \right) \left(2 \frac{\partial w}{\partial l} + \frac{\partial^2 w}{\partial l^2} \right) - \frac{\beta T_t}{\delta} \left(2 \frac{v - w \cot \gamma}{l} + \frac{\partial v}{\partial l} \right) \right] + \\
& + \frac{3\alpha \eta_0 \delta^2 e^{-\frac{KT_t}{2}}}{KT_t} \left[-e^{-\frac{KT_t}{2}} - \frac{1}{KT_t} \left(e^{\frac{KT_t}{2}} + 3e^{-\frac{KT_t}{2}} \right) + \frac{8}{(KT_t)^2} \sinh \frac{KT_t}{2} \right] \frac{\partial T_t}{\partial t} + \\
& + \frac{G_0 \alpha \delta^2 T_t}{2} (1 - \beta T_t)
\end{aligned} \tag{3.2 - 7}$$

Introducing equations (3.2 - 4) through (3.2 - 7) into the equations of equilibrium (3.1.1 - 1a), one reduces the governing equations to two 5th order linear partial differential equations. They are

$$\begin{aligned}
\frac{4\eta_0 e^{-\frac{KT_t}{2}}}{KT_t} \left\{ \sinh \frac{KT_t}{2} \frac{\partial}{\partial t} \left[-\frac{2}{l} (v - w \cot \gamma) + 2 \frac{\partial v}{\partial l} + 2l \frac{\partial^2 v}{\partial l^2} - \cot \gamma \frac{\partial w}{\partial l} \right] - \right. \\
& - \frac{2\delta}{KT_t} \left(\frac{KT_t}{2} \cosh \frac{KT_t}{2} - \sinh \frac{KT_t}{2} \right) \frac{\partial}{\partial t} \left[l \frac{\partial^3 w}{\partial l^3} + \frac{\partial^2 w}{\partial l^2} - \frac{1}{l} \frac{\partial w}{\partial l} \right] \Big\} + \\
& + 2G_0 \left(1 - \frac{\beta T_t}{2} \right) \left[2l \frac{\partial^2 v}{\partial l^2} + 2 \frac{\partial v}{\partial l} - \cot \gamma \frac{\partial w}{\partial l} - 2 \frac{v - w \cot \gamma}{l} \right] - \\
& - \frac{G_0 \beta T_t \delta}{3} \left[l \frac{\partial^3 w}{\partial l^3} + \frac{\partial^2 w}{\partial l^2} - \frac{1}{l} \frac{\partial w}{\partial l} \right] = 0
\end{aligned} \tag{3.2 - 8}$$

and

$$\begin{aligned}
& 2\gamma_0 e^{-\frac{\kappa T_c}{2}} \left\{ \frac{\delta^2}{(\kappa T_c)^2} \left(\kappa T_c \cosh \frac{\kappa T_c}{2} - 2 \sinh \frac{\kappa T_c}{2} \right) \frac{\partial}{\partial t} \left(2l \frac{\partial^2 w}{\partial l^2} + 4 \frac{\partial^2 v}{\partial l^2} - \frac{2}{l} \frac{\partial v}{\partial l} + \frac{2v}{l^2} - 2 \cot \gamma \frac{\partial^2 w}{\partial l^2} - \frac{2w \cot \gamma}{l^2} \right) + \right. \\
& + \left[\frac{\delta^3}{2 \kappa T_c} \sinh \frac{\kappa T_c}{2} + \frac{2\delta^3}{(\kappa T_c)^3} \left(2 \sinh \frac{\kappa T_c}{2} - \kappa T_c \cosh \frac{\kappa T_c}{2} \right) \right] \frac{\partial}{\partial t} \left[-2l \frac{\partial^2 w}{\partial l^2} - 4 \frac{\partial^2 v}{\partial l^2} + \frac{1}{l} \frac{\partial^2 w}{\partial l^2} - \frac{1}{l^2} \frac{\partial w}{\partial l} \right] + \\
& + \frac{G_0 \delta^3}{6} \left\{ -2 \left(1 - \frac{\beta T_c}{2} \right) \left(l \frac{\partial^2 w}{\partial l^2} + 2 \frac{\partial^2 w}{\partial l^2} - \frac{1}{2l} \frac{\partial^2 w}{\partial l^2} + \frac{1}{2l^2} \frac{\partial w}{\partial l} \right) + \right. \\
& + \left. \frac{\beta T_c}{\delta} \left[2l \frac{\partial^2 v}{\partial l^2} + 4 \frac{\partial^2 v}{\partial l^2} - \frac{2}{l} \frac{\partial v}{\partial l} + \frac{2v}{l^2} - 2 \cot \gamma \frac{\partial^2 w}{\partial l^2} - \frac{2w \cot \gamma}{l^2} \right] \right\} + \\
& + 4\gamma_0 e^{-\frac{\kappa T_c}{2}} \frac{\delta}{\kappa T_c} \cot \gamma \left\{ \sinh \frac{\kappa T_c}{2} \frac{\partial}{\partial t} \left(\frac{\partial v}{\partial l} + 2 \frac{v - w \cot \gamma}{l} \right) \right\} + 2G_0 \delta \left(1 - \frac{\beta T_c}{2} \right) \left(\frac{\partial v}{\partial l} + 2 \frac{v - w \cot \gamma}{l} \right) \cot \gamma + \\
& + \frac{6\alpha \gamma_0 e^{-\frac{\kappa T_c}{2}}}{\kappa T_c} \delta \cot \gamma \left(e^{-\frac{\kappa T_c}{2}} - \frac{2}{\kappa T_c} \sinh \frac{\kappa T_c}{2} \right) \frac{\partial T_c}{\partial t} - G_0 \alpha T_c \delta \cot \gamma (3 - 2\beta T_c) + p l = 0
\end{aligned}$$

(3.2 - 9)

The order of magnitude of each terms in these two equations is investigated by a method similar to that discussed in Appendix A. With v and w having the order of magnitude of shell thickness δ , the terms involving v and w in equation (3.2 - 8) have the order of magnitude $\frac{1}{l}(\delta/l)$, $\frac{1}{l^2}(\delta/l)^2$, δ/l , and $(\delta/l)^2$, and in equation (3.2 - 9) $\frac{1}{l}(\delta/l)$, $\frac{1}{l^2}(\delta/l)^2$, $\frac{1}{l^3}(\delta/l)^3$, $\delta(\delta/l)$, $\delta(\delta/l)^2$ and $\delta(\delta/l)^3$.

Neglecting the terms having the order of magnitudes of $\frac{1}{l^2}(\delta/l)^2$ and $(\delta/l)^2$ in equation (3.2 - 8) and $\frac{1}{l^3}(\delta/l)^3$, $\delta(\delta/l)^3$ terms in equation (3.2 - 9), the governing equations are written as follows

$$\begin{aligned}
& \frac{2\gamma_0}{\kappa T_c} \left(1 - e^{-\frac{\kappa T_c}{2}} \right) \frac{\partial}{\partial t} \left[2 \left(\frac{\partial v}{\partial l} - \frac{v - w \cot \gamma}{l} \right) + 2l \frac{\partial^2 v}{\partial l^2} - \cot \gamma \frac{\partial w}{\partial l} \right] + \\
& + 2G_0 \left(1 - \frac{\beta T_c}{2} \right) \left[2 \left(\frac{\partial v}{\partial l} - \frac{v - w \cot \gamma}{l} \right) + 2l \frac{\partial^2 v}{\partial l^2} - \cot \gamma \frac{\partial w}{\partial l} \right] = 0
\end{aligned}$$

(3.2 - 10)

and

$$\begin{aligned}
& 2\eta_0 e^{-\frac{K_T}{2}} \left\{ \left(\frac{\delta}{K_T} \right)^2 \left(K_T \cosh \frac{K_T}{2} - 2 \sinh \frac{K_T}{2} \right) \frac{\partial}{\partial t} \left[l \frac{\partial^2}{\partial l^2} \left(2 \frac{\partial v}{\partial l} + \frac{v - w \cot \gamma}{l} \right) + 3 \frac{\partial^2 v}{\partial l^2} - \cot \gamma \left(\frac{\partial^2 w}{\partial l^2} + \frac{2}{l} \frac{\partial w}{\partial l} \right) \right] + \right. \\
& \quad \left. + \frac{2\delta}{K_T} \cot \gamma \sinh \frac{K_T}{2} \frac{\partial}{\partial t} \left(2 \frac{v - w \cot \gamma}{l} + \frac{\partial v}{\partial l} \right) \right\} + \\
& \quad + \frac{G_0 \delta^2 \beta T_c}{6} \left[\frac{\partial^2}{\partial l^2} \left(2 \frac{\partial v}{\partial l} + \frac{v - w \cot \gamma}{l} \right) + 3 \frac{\partial^2 v}{\partial l^2} - \cot \gamma \left(\frac{\partial^2 w}{\partial l^2} + \frac{2}{l} \frac{\partial w}{\partial l} \right) \right] + \\
& \quad + 2 G_0 \delta \left(1 - \frac{\beta T_c}{2} \right) \left(\frac{\partial v}{\partial l} + 2 \frac{v - w \cot \gamma}{l} \right) \cot \gamma + \frac{6 \alpha \eta_0 e^{-\frac{K_T}{2}}}{K_T} \delta \cot \gamma \left(e^{-\frac{K_T}{2}} - \frac{2}{K_T} \sinh \frac{K_T}{2} \right) \frac{\partial T_c}{\partial t} - \\
& \quad - G_0 \delta \alpha T_c \cot \gamma (3 - 2 \beta T_c) + p l = 0
\end{aligned} \tag{3.2 - 11}$$

Comparing equations (3.2 - 10) and (3.2 - 11) with the corresponding equations (3.1.1 - 7) and (3.1.1 - 8) which were derived by applying the order of magnitude consideration before the derivation of the governing equations, one finds the first equations are exactly the same while the second equations differed by the terms

$$-2\eta_0 e^{-\frac{K_T}{2}} \left(\frac{\delta}{K_T} \right)^2 \left(K_T \cosh \frac{K_T}{2} - 2 \sinh \frac{K_T}{2} \right) \cot \gamma \frac{\partial}{\partial t} \left(\frac{\partial^2 w}{\partial l^2} + \frac{2}{l} \frac{\partial w}{\partial l} \right) - \frac{G_0 \delta^2 \beta T_c \cot \gamma}{6} \left(\frac{\partial^2 w}{\partial l^2} + \frac{2}{l} \frac{\partial w}{\partial l} \right)$$

From a mathematical point of view, the solutions of equations (3.2 - 10) and (3.2 - 11) will a better approximation than that of equations (3.1.1 - 7) and (3.1.1 - 8), provided that both sets of equations can be solved analytically. However, it has been found that in this case the governing equations are not as readily solvable as in the former case. Some efforts have been spent to simplify the governing equations. The results are presented here.

$$\frac{\partial}{\partial t} \left[\frac{\partial}{\partial l} (lU) - 2U + 3V \right] + F_1 \left[\frac{\partial}{\partial l} (lU) - 2U + 3V \right] = 0 \tag{3.2 - 12}$$

$$\begin{aligned}
& F_5 \frac{\partial}{\partial t} \left\{ \frac{\partial}{\partial l} \left[l \frac{\partial U}{\partial l} + \frac{\partial l(U-V)}{\partial l} \right] + \frac{U-3V}{l} \right\} + F_6 \left\{ \frac{\partial}{\partial l} \left[l \frac{\partial U}{\partial l} + \frac{\partial l(U-V)}{\partial l} \right] + \frac{U-3V}{l} \right\} + \\
& \quad + \frac{\partial}{\partial t} (2U-3V) + F_1 (2U-3V) + F_3 p l = F_4
\end{aligned} \tag{3.2 - 13}$$

where

$$U = 2 \frac{\partial v}{\partial l} + \frac{1}{l} (v - w \cot \gamma)$$

$$V = \frac{\partial v}{\partial l}$$

$$F_5 = 2\delta \tan \delta \left[\frac{1}{2} \frac{1+e^{-K\bar{T}_k}}{1-e^{-K\bar{T}_k}} - \frac{1}{K\bar{T}_k} \right], \quad F_6 = \frac{G_0 \delta \beta K \bar{T}_k^2 \pi \mu \delta}{6 \eta_0 (1-e^{-K\bar{T}_k})}$$

and F_1, F_3, F_4 are defined in paragraph 3.1.2.

3.3 Constant Property Method

Although the main interest of this analysis is concerned with shells which have temperature dependent material properties, a crude approximation however, can be made by assigning properties at some arbitrary constant temperature, say the middle surface temperature, and then treat the problem as a temperature-independent one.

3.3.1 Governing Equations

The equations of equilibrium will be the same as those in paragraph 3.1.1.

$$\left. \begin{aligned} \frac{\partial(M_\phi l)}{\partial l} - M_\theta - Q_\phi l &= 0 \\ \frac{\partial(N_\phi l)}{\partial l} &= N_\theta \\ \frac{\partial(Q_\phi l)}{\partial l} + N_\theta \cot \delta + P l &= 0 \end{aligned} \right\} \quad (3.3.1 - 1)$$

The strain components remain as

$$\left. \begin{aligned} \epsilon_{\phi\phi} &= 2 \left(\eta \frac{\partial}{\partial l} + G \right) (2 \epsilon_{\phi\phi} + \epsilon_{\theta\theta} - 3\alpha T) \\ \epsilon_{\theta\theta} &= 2 \left(\eta \frac{\partial}{\partial l} + G \right) (2 \epsilon_{\theta\theta} + \epsilon_{\phi\phi} - 3\alpha T) \end{aligned} \right\} \quad (3.3.1 - 2)$$

except, here η, G , and α are constants. Temperature profile across the shell thickness is assumed to be the same as given in paragraph 3.1.1, i.e., $T = (\frac{1}{2} - \frac{z}{\delta}) \bar{T}_k$.

Using the strain-displacement relations as

$$\left. \begin{aligned} \epsilon_{\phi\phi} &= \frac{\partial v}{\partial l} - \frac{\partial^2 w}{\partial l^2} - \xi \\ \epsilon_{\theta\theta} &= \frac{v - w \cot \delta}{l} - \frac{\xi}{l} \frac{\partial w}{\partial l} \end{aligned} \right\} \quad (3.3.1 - 3)$$

In this method, the procedure may follow that given by Timoshenko²⁵.

Introducing

$$\chi = \frac{\partial w}{\partial l}$$

the moments can be written

$$\left. \begin{aligned} M_\phi &= -2\delta^3 \left(\eta \frac{\partial}{\partial t} + G \right) \left[\frac{1}{12} \frac{\chi}{l} + \frac{1}{6} \frac{\partial \chi}{\partial l} - \frac{\alpha T_\epsilon}{4\delta} \right] \\ M_\theta &= -2\delta^3 \left(\eta \frac{\partial}{\partial t} + G \right) \left[\frac{1}{6} \frac{\chi}{l} + \frac{1}{12} \frac{\partial \chi}{\partial l} - \frac{\alpha T_\epsilon}{4\delta} \right] \end{aligned} \right\} \quad (3.3.1 - 4)$$

The value of N_ϕ is derived from equation (2.1 - 4) as follows

$$2\pi l \sin \gamma N_\phi \cot \gamma + 2\pi l Q_\phi \sin^2 \gamma + \pi (l^2 \sin^2 \gamma) p = 0$$

or

$$N_\phi = -Q_\phi \tan \gamma - \frac{Pl \sin \gamma}{2 \cos \gamma} = -\left(Q_\phi + \frac{Pl}{2}\right) \tan \gamma \quad (3.3.1 - 5)$$

From the third equilibrium equation of (3.3.1 - 1), N_θ is:

$$N_\theta = -\left(\frac{\partial(Q_\phi l)}{\partial l} + pl\right) \tan \gamma \quad (3.3.1 - 6)$$

Combination of equations (3.3.1 - 4) through (3.3.1 - 6) and the introduction of the new variable $W = Q_\phi l \tan \gamma$ yields the first governing differential equation

$$\frac{\partial^2 \chi}{\partial t \partial l^2} + \frac{G}{\eta} \frac{\partial^2 \chi}{\partial l^2} + \frac{1}{l} \frac{\partial^2 \chi}{\partial t \partial l} + \frac{G}{\eta l} \frac{\partial \chi}{\partial l} - \frac{1}{l^2} \frac{\partial \chi}{\partial t} - \frac{G}{\eta l^2} \chi = \frac{3 \cot \gamma W}{\eta \delta^3 l} - \frac{3 \alpha}{2 \delta} \frac{\partial^2 T_\epsilon}{\partial t \partial l} - \frac{3 \alpha G}{2 \eta \delta} \frac{\partial T_\epsilon}{\partial l} \quad (3.3.1 - 7)$$

The integrated equation of state in paragraph 2.2

$$E_{ij} = \frac{e^{-\frac{G}{2}t}}{2\eta} \int_0^t S_{ij} e^{\frac{G}{2}t} dt \quad (3.3.1 - 8)$$

may be used to derive the strain components. They are

$$\epsilon_{\phi\phi} = \frac{e^{-\frac{G}{2}t}}{6\eta} \int_0^t (2\sigma_{\phi\phi} - \sigma_{\theta\theta}) e^{\frac{G}{2}t} dt + \alpha T \quad (3.3.1 - 9)$$

and

$$\epsilon_{\theta\theta} = \frac{e^{-\frac{G}{2}t}}{6\eta} \int_0^t (2\sigma_{\theta\theta} - \sigma_{\phi\phi}) e^{\frac{G}{2}t} dt + \alpha T \quad (3.3.1 - 10)$$

The new variable χ may be also written in terms of the strain components.

$$\chi = (\epsilon_{\phi\phi} - \epsilon_{\theta\theta}) \tan \gamma - l \tan \gamma \frac{\partial \epsilon_{\theta\theta}}{\partial l} \quad (3.3.1 - 11)$$

Combining equations (3.3.1 - 5), (3.3.1 - 6), and (3.3.1 - 9) through (3.3.1 - 10), one may write the second governing differential equation as

$$\frac{\partial^2 W}{\partial l^2} + \frac{1}{l} \frac{\partial W}{\partial l} - \frac{W}{l^2} = \frac{3\gamma\delta}{l} \cot \gamma \left(\frac{\partial \chi}{\partial t} + \frac{G}{\gamma} \chi \right) + \frac{3\gamma\delta\alpha}{2} \left(\frac{\partial^2 T_e}{\partial t \partial l} + \frac{G}{\gamma} \frac{\partial T_e}{\partial l} \right) - \frac{3p\mu\nu\gamma}{2} \quad (3.3.1 - 12)$$

Equations (3.3.1 - 7) and (3.3.1 - 12) are now the governing equations of the thermal viscoelastic conical shell of temperature independent properties.

3.3.2 Method of Solving and Solution

From equation (3.3.1 - 7) the variable $\overline{W}$ may be expressed in terms of χ and its derivatives.

$$\overline{W} = -\frac{\delta\gamma}{3} \mu\nu\gamma \left[l \frac{\partial^3 \chi}{\partial t \partial l^2} + \frac{G}{\gamma} l \frac{\partial^2 \chi}{\partial l^2} + \frac{\partial^2 \chi}{\partial t \partial l} + \frac{G}{\gamma} \frac{\partial \chi}{\partial l} - \frac{1}{l} \frac{\partial \chi}{\partial t} + \frac{G}{\gamma} \frac{\chi}{l} - \frac{3\alpha l \partial^2 T_e}{2\delta \partial t \partial l} + \frac{3G\alpha l \partial T_e}{2\gamma \delta \partial l} \right] \quad (3.3.2 - 1)$$

After computing derivatives of $\overline{W}$ with respect to l and substituting the results into equation (3.3.1 - 12), one obtains the fifth order partial differential equation

$$\begin{aligned} l \frac{\partial^5 \chi}{\partial t \partial l^4} + 4 \frac{\partial^4 \chi}{\partial t \partial l^3} + \frac{G}{\gamma} l \frac{\partial^3 \chi}{\partial l^3} + \frac{G}{\gamma} \frac{\partial^3 \chi}{\partial l^2} + \frac{9 \cot^2 \gamma}{\delta^2 l} \left(\frac{\partial \chi}{\partial t} + \frac{G}{\gamma} \chi \right) = \\ = \frac{9p}{2\delta^3 \gamma} - \frac{9\alpha \cot \gamma}{2\delta^2} \left[\frac{\partial^2 T_e}{\partial t \partial l} + \frac{G}{\gamma} \frac{\partial T_e}{\partial l} \right] - \frac{3\alpha l}{2\delta} \frac{\partial^3}{\partial l^3} \left(\frac{\partial T_e}{\partial t} + \frac{G}{\gamma} T_e \right) - \frac{9\alpha}{2\delta} \frac{\partial^2}{\partial l^2} \left(\frac{\partial T_e}{\partial t} + \frac{G}{\gamma} T_e \right) \end{aligned} \quad (3.3.2 - 2)$$

It is easily seen that T_e in this method of analysis has much more flexibility; it may vary with the time t as well as with the shell element length l ; i.e., $T_e = T_e(l, t)$. For uniform outer-surface temperature $\partial T_e / \partial l = 0$; and for steady state thermal load $\partial T_e / \partial t = 0$. Method of the separation of variables may be used to separate equation (3.3.2 - 2) into two ordinary differential equations; a first order equation with respect to time, and a fourth order equation with respect to the shell element length. The complementary equation of the fourth order equation may be further separated into two Bessel equations. Their solutions are the Bessel functions of the first kind and the second kind in complex variables, which can be expressed in ber, bei, kei, ker functions. With the choice of $[(\delta/\gamma) \mu\nu^2 \gamma l]$ as a particular solution, the complete solution for uniform and steady load condition may be readily written as follows

$$\left. \begin{aligned}
\psi &= \left[A_1 \left(Z_1 + \frac{2Z_1'}{\gamma} \right) + A_2 \left(Z_2 - \frac{2Z_2'}{\gamma} \right) + A_3 \left(Z_3 + \frac{2Z_3'}{\gamma} \right) + A_4 \left(Z_4 - \frac{2Z_4'}{\gamma} \right) + \frac{\delta^2}{9} \pi \omega^2 \gamma \ell \right] \left[A_0 e^{-\frac{G}{2} \frac{\gamma}{\delta}} + \frac{G}{2} \frac{\gamma}{\delta} \right] \\
Q_\phi &= -54 \frac{\rho}{\delta^2} \cot \gamma \left[A_1 \left(-\frac{2Z_1'}{\gamma^3} + \frac{Z_1}{\gamma^2} \right) + A_2 \left(-\frac{2Z_2'}{\gamma^3} - \frac{Z_2}{\gamma^2} \right) + A_3 \left(-\frac{2Z_3'}{\gamma^3} + \frac{Z_3}{\gamma^2} \right) + A_4 \left(-\frac{2Z_4'}{\gamma^3} - \frac{Z_4}{\gamma^2} \right) \right] \\
N_\theta &= 27 \frac{\rho}{\delta^2} \cot \gamma \left[A_1 \left(\frac{4Z_1'}{\gamma^3} - \frac{2Z_1}{\gamma^2} + \frac{Z_1'}{\gamma} \right) + A_2(\dots) + A_3(\dots) + A_4(\dots) \right] - \rho \ell \pi \omega \gamma \\
M_\phi' &= 54 \frac{\rho}{\delta^2} \cot \gamma \left[A_1 \left(-\frac{2Z_1'}{\gamma^3} + \frac{Z_1}{\gamma^2} \right) + A_2(\dots) + A_3(\dots) + A_4(\dots) \right] - \frac{\rho \ell \pi \omega \gamma}{2} \\
M_\theta &= -\frac{9}{2} \frac{\rho}{\delta} \cot \gamma \left[A_1 \left(\frac{Z_1'}{\gamma} + \frac{2Z_1}{\gamma^2} + \frac{4Z_1'}{\gamma^3} \right) + A_2(\dots) + A_3(\dots) + A_4(\dots) \right] - \frac{\rho \delta^2 \pi \omega^2 \gamma}{4} + \frac{G \delta^2 \omega^2 \ell}{2} \\
M_\phi &= -9 \frac{\rho}{\delta} \cot \gamma \left[A_1 \left(\frac{Z_1'}{\gamma} - \frac{Z_1}{\gamma^2} - \frac{2Z_1'}{\gamma^3} \right) + A_2(\dots) + A_3(\dots) + A_4(\dots) \right] - \frac{\rho \delta^2 \pi \omega^2 \gamma}{4} + \frac{G \delta^2 \omega^2 \ell}{2}
\end{aligned} \right\} \quad (3.3.2 - 3)$$

where

$$Z_1(\gamma) = \text{BER } \gamma$$

$$Z' = \frac{dZ}{d\gamma}$$

$$Z_2(\gamma) = -\text{BER } \gamma$$

$$Z_3(\gamma) = -\frac{2}{\pi} \text{KEI } \gamma$$

$$Z_4(\gamma) = -\frac{2}{\pi} \text{KER } \gamma$$

and

$$\gamma^2 = \frac{12 \cot \gamma}{\delta} \ell$$

A_0, A_1, A_2, A_3 , and A_4 are to be determined by the boundary conditions. A_0 will be determined by the load condition at the initial time, i.e., $t^0 = 0$. The continuity of ψ at $\ell = 0$ immediately eliminates A_3 and A_4 .

The edge conditions, then, determine the values of A_1 and A_2 .

IV. ANALYSIS OF HEMISPHERICAL SHELL

The analysis discussed in this chapter has greater applicability than the title suggests. It may be applied to any spherical shell having a maximum ϕ equals to or less than $\pi/2$.

The method used here is similar to that described in paragraph 3.1, i.e., the simplified strain-displacement relation method. However, it is not suggested that this is the only method, on the contrary, the other

two methods presented in Paragraphs 3.2 and 3.3 may be applied. However, the application will not be discussed to avoid repetition.

4.1 Governing Equations

The independent variable ϕ used in the equations of equilibrium (2.1 - 1 to - 3) could be retained. The values of ρ_1 and ρ_2 are the same and are equal to the radius of the sphere ρ_0 . The radius of parallel circles r may be replaced by $\rho_0 \sin \phi$. Hence the equations of equilibrium

$$\frac{\partial(N_\phi \rho_0 \sin \phi)}{\partial \phi} - N_\theta \rho_0 \cos \phi - Q_\phi \rho_0 \sin \phi = 0 \quad (4.1 - 1)$$

$$\sin \phi (N_\phi \rho_0) + N_\theta \rho_0 \sin \phi + \frac{\partial}{\partial \phi} (Q_\phi \rho_0 \sin \phi) + p \rho_0^2 \sin \phi = 0 \quad (4.1 - 2)$$

$$\frac{\partial(M_\phi \rho_0 \sin \phi)}{\rho_0 \partial \phi} - M_\theta \cos \phi - Q_\phi \rho_0 \sin \phi = 0 \quad (4.1 - 3)$$

To eliminate $Q_\phi \rho_0 \sin \phi$, (4.1 - 1) is differentiated with respect to ϕ and is added to equation (4.1 - 2). Thus

$$\frac{\partial^2 N_\phi}{\partial \phi^2} + \left(2 \frac{\partial N_\phi}{\partial \phi} - \frac{\partial N_\theta}{\partial \phi} \right) \cos \phi + 2 N_\theta + p \rho_0 = 0 \quad (4.1 - 4)$$

Subtraction of equation (4.1 - 1) from equation (4.1 - 3) yields

$$\frac{\partial M_\phi}{\partial \phi} - \rho_0 \frac{\partial N_\phi}{\partial \phi} + (M_\phi - M_\theta) \cos \phi + \rho_0 (N_\theta - N_\phi) \cos \phi = 0 \quad (4.1 - 5)$$

The two equations of equilibrium (4.1 - 4 and 4.1 - 5) are then used. Consequently, the expressions for N_θ , N_ϕ , M_θ and M_ϕ are the same as those of equations (3.1.1 - 5) through (3.1.1 - 8). Substitution of the latter equations into equations (4.1 - 4) and (4.1 - 5) yields two governing differential equations

$$\begin{aligned} & \frac{2\gamma_0}{K T_k} (1 - e^{-K T_k}) \left[\frac{\partial^3 (2\epsilon_{\phi\theta} + \epsilon_{\theta\theta})}{\partial^2 \phi \partial t} + 3 \cos \phi \frac{\partial^2 \epsilon_{\phi\theta}}{\partial \phi \partial t} + 2 \frac{\partial (2\epsilon_{\phi\theta} + \epsilon_{\phi\theta})}{\partial t} \right] + \\ & + 2 G_0 \left(1 - \frac{\beta T_k}{2} \right) \left[\frac{\partial^3 (2\epsilon_{\phi\theta} + \epsilon_{\theta\theta})}{\partial \phi^2} + 3 \cos \phi \frac{\partial^2 \epsilon_{\phi\theta}}{\partial \phi} + 2 (2\epsilon_{\phi\theta} + \epsilon_{\phi\theta}) \right] + \frac{p \rho_0}{\delta} = \\ & = 2 G_0 \alpha T_k (3 - 2 \beta T_k) + 12 \gamma_0 \alpha \frac{\partial T_k}{\partial t} \left[\frac{e^{-K T_k}}{K T_k} + \frac{1 - e^{-K T_k}}{(K T_k)^2} \right] \end{aligned} \quad (4.1 - 6)$$

and

$$\frac{\partial^2(2\epsilon_{\phi M} + \epsilon_{\phi M})}{\partial t \partial \phi} + \cot \phi \frac{\partial(\epsilon_{\phi M} - \epsilon_{\phi M})}{\partial t} + \tilde{F}_1 \left[\frac{\partial(2\epsilon_{\phi M} + \epsilon_{\phi M})}{\partial \phi} + \cot \phi (\epsilon_{\phi M} - \epsilon_{\phi M}) \right] = 0 \quad (4.1 - 7)$$

where

$$\tilde{F}_1 = \frac{G_0 K T_e \left[\beta T_e \left(\rho_0 + \frac{d}{6} \right) - 2 \rho_0 \right]}{2 \gamma_0 \left[\left(\rho_0 + \frac{d}{K T_e} \right) \left(e^{-\frac{K T_e}{2}} - 1 \right) + \frac{d}{2} \left(1 + e^{-\frac{K T_e}{2}} \right) \right]} \quad (4.1 - 7a)$$

4.2 Method of Solution

Introduction of new variables $U = 2\epsilon_{\phi M} + \epsilon_{\phi M}$ and $V = \epsilon_{\phi M}$ into the governing equations (4.1 - 6) and (4.1 - 7) yields

$$\begin{aligned} \frac{2 \gamma_0}{K T_e} \left(1 - e^{-\frac{K T_e}{2}} \right) \frac{\partial}{\partial t} \left[\frac{\partial^2 U}{\partial \phi^2} + 3 \cot \phi \frac{\partial U}{\partial \phi} + 2(2U - 3V) \right] + 2 G_0 \left(1 - \beta \frac{T_e}{2} \right) \left[\frac{\partial^2 U}{\partial \phi^2} + 3 \cot \phi \frac{\partial U}{\partial \phi} + 2(2U - 3V) \right] + \frac{P_0}{\delta} = \\ = 2 G_0 \alpha T_e (3 - 2 \beta T_e) + \frac{12 \gamma_0 \alpha}{K T_e} \frac{\partial T_e}{\partial t} \left(e^{-\frac{K T_e}{2}} + \frac{1 - e^{-\frac{K T_e}{2}}}{K T_e} \right) \end{aligned} \quad (4.2 - 1)$$

and

$$\frac{\partial}{\partial t} \left[\frac{\partial U}{\partial \phi} + \cot \phi (3V - U) \right] + \tilde{F}_1 \left[\frac{\partial U}{\partial \phi} + \cot \phi (3V - U) \right] = 0 \quad (4.2 - 2)$$

Using the method of the separation of variables in equation (4.2 - 2) one finds

$$\frac{\partial U}{\partial \phi} + \cot \phi (3V - U) = \left[\frac{\partial U}{\partial \phi} + \cot \phi (3V - U) \right]_{t=0} e^{-\int_0^t \tilde{F}_1 dt} \quad (4.2 - 3)$$

The value of $\left[\frac{\partial U}{\partial \phi} + \cot \phi (3V - U) \right]_{t=0}$ has to be evaluated from the boundary conditions at the initial time, i.e., the temperature profile, the pressure load, the properties of material and the stress and strain induced at $t = 0$. If it is an elastic shell under pressure load without thermal stress, the analytic solution of elastic hemispherical shell given in various mechanics textbooks can be used to evaluate the initial condition. Thus knowing

$$(\epsilon_{\phi M})_{t=0} = \frac{1}{E_i \delta} (N_\phi - \nu_i N_\theta)_{ELASTIC}$$

and

$$(\epsilon_{\theta M})_{t=0} = \frac{1}{E_i \delta} (N_\theta - \nu_i N_\phi)_{ELASTIC}$$

By the definition of U and V , one writes

$$\begin{aligned} \left[\frac{\partial U}{\partial \phi} + \cot \phi (3U - V) \right]_{t=0} &= \left[\frac{\partial U}{\partial \phi} + \cot \phi (3V - U) \right]_{EL} \\ &= \frac{1}{E_i \delta} \left\{ \left[(2 - \gamma_i) \frac{\partial N_i}{\partial \phi} + (1 - 2\gamma_i) \frac{\partial N_{\phi}}{\partial \phi} \right]_{EL} + (1 + \gamma_i) (N_{\phi} - N_{\theta})_{EL} \cot \phi \right\} \end{aligned} \quad (4.2 - 4)$$

E_i and γ_i denote values of E and γ at the initial time.

Equation (4.2 - 1) may be written

$$\begin{aligned} \frac{2\eta_0(1 - e^{-Kt_e})}{KT_e} \left\{ \frac{\partial}{\partial t} \left[\frac{\partial^2 U}{\partial \phi^2} + 3 \cot \phi \frac{\partial V}{\partial \phi} - (3V - U) \right] + \frac{\partial}{\partial t} (3U - 3V) \right\} + \\ + 2G_0(1 - \frac{\beta E}{2}) \left[\frac{\partial^2 U}{\partial \phi^2} + 3 \cot \phi \frac{\partial V}{\partial \phi} - (3V - U) + (3U - 3V) \right] + \frac{P_0}{\delta} - \\ - 2G_0 T_e (3 - 2\beta T_e) - \frac{12\eta_0 \alpha}{KT_e} \frac{\partial T_e}{\partial t} \left(e^{-Kt_e} + \frac{1 - e^{-Kt_e}}{KT_e} \right) = 0 \end{aligned} \quad (4.2 - 5)$$

By differentiating equation (4.2 - 2) with respect to ϕ and adding to the resulting equation the product of $\cot \phi$ and equation (4.2 - 2) one obtains

$$\frac{\partial}{\partial t} \left[\frac{\partial^2 U}{\partial \phi^2} + 3 \cot \phi \frac{\partial V}{\partial \phi} - (3V - U) \right] = -\tilde{F}_1 \left[\frac{\partial^2 U}{\partial \phi^2} + 3 \cot \phi \frac{\partial V}{\partial \phi} - (3V - U) \right] \quad (4.2 - 6)$$

Substituting equation (4.2 - 4) into equation (4.2 - 3) and then differentiating with respect to ϕ one finds that

$$\begin{aligned} \frac{\partial^2 U}{\partial \phi^2} + 3 \cot \phi \frac{\partial V}{\partial \phi} - (3V - U) &= \left[\frac{\partial^2 U}{\partial \phi^2} + 3 \cot \phi \frac{\partial V}{\partial \phi} - (3V - U) \right]_{t=0} e^{-\int_0^t \tilde{F}_1 dt} = \\ &= \frac{e^{-\int_0^t \tilde{F}_1 dt}}{E_i \delta} \left\{ \left(\frac{\partial}{\partial \phi} + \cot \phi \right) \left[\left[(2 - \gamma_i) \frac{\partial N_i}{\partial \phi} + (1 - 2\gamma_i) \frac{\partial N_{\phi}}{\partial \phi} \right]_{EL} + \cot \phi (1 + \gamma_i) (N_{\phi} - N_{\theta})_{EL} \right] \right\} \end{aligned} \quad (4.2 - 7)$$

Substitution of (4.2 - 6) and (4.2 - 7) into equation (4.2 - 5) yields

$$\frac{\partial(U - V)}{\partial t} + \tilde{F}_3 (U - V) + \tilde{F}_2 = 0 \quad (4.2 - 8)$$

where

$$\begin{aligned}\tilde{F}_2 = \frac{KT_k}{6\eta_0(1-e^{-\kappa_k})} & \left\{ \frac{\rho\beta}{\delta} - 2G\alpha T_k(3-2\beta T_k) - 12\eta_0\alpha \frac{\partial T_k}{\partial t} \left(\frac{e^{-\kappa_k}}{\kappa_k} + \frac{1-e^{-\kappa_k}}{(\kappa_k)^2} \right) + \right. \\ & \left. + \left[-\frac{2\eta_0(1-e^{-\kappa_k})}{\kappa_k} \tilde{F}_1 + 2G\alpha \left(1 - \frac{\beta T_k}{2}\right) \right] e^{-\int_0^t \tilde{F}_2 dt} \left(\frac{\partial \tilde{U}}{\partial \phi} + \cot \phi \right) \left[\frac{\partial \tilde{U}}{\partial \phi} + \cot \phi (3\tilde{V} - \tilde{U}) \right] \right\}_{t=t_0} \\ \tilde{F}_3 = \frac{G_0 \left(1 - \frac{\beta T_k}{2}\right) \kappa_k}{\eta_0(1-e^{-\kappa_k})}\end{aligned}\quad (4.2 - 9)$$

The function $\tilde{F}_2$ contains temperature and pressure, material properties, and the strain at the initial time, hence, is a function of time and ϕ .

The first order partial differential equation (4.2 - 8) is readily solvable by the method of separation of variables.

$$\tilde{U} - \tilde{V} = e^{-\int_0^t \tilde{F}_3 dt} \left\{ -\int_0^t \tilde{F}_2 e^{\int_0^t \tilde{F}_3 dt} dt + [\tilde{U} - \tilde{V}]_{t=t_0} \right\}$$

Employing the same techniques as used in the evaluation of $\left[\frac{\partial \tilde{U}}{\partial \phi} + \cot \phi (3\tilde{V} - \tilde{U}) \right]_{t=t_0}$ it can be shown that

$$[\tilde{U} - \tilde{V}]_{t=t_0} = \frac{1-\nu_i}{E_i \delta} (N_\phi + N_\theta)_{EL}$$

Thus

$$\tilde{V} = \tilde{U} - e^{-\int_0^t \tilde{F}_3 dt} \left\{ -\int_0^t \tilde{F}_2 e^{\int_0^t \tilde{F}_3 dt} dt + \frac{1-\nu_i}{E_i \delta} (N_\phi + N_\theta)_{EL} \right\} \quad (4.2 - 10)$$

Replacing equation (4.2 - 10) for the value of $\tilde{V}$ in equation (4.2 - 3), one writes that

$$\frac{\partial \tilde{U}}{\partial \phi} + 2 \cot \phi \tilde{U} = \tilde{F}_4 \quad (4.2 - 11)$$

where

$$\begin{aligned}\tilde{F}_4 = \frac{e^{-\int_0^t \tilde{F}_3 dt}}{E_i \delta} & \left\{ (2-\nu_i) \frac{\partial N_\phi}{\partial \phi} + (1-2\nu_i) \frac{\partial N_\theta}{\partial \phi} + \cot \phi (1+\nu_i) (N_\phi + N_\theta) \right\}_{EL} + \\ & + 3 \cot \phi e^{-\int_0^t \tilde{F}_3 dt} \left\{ -\int_0^t \tilde{F}_2 e^{\int_0^t \tilde{F}_3 dt} dt + \frac{1-\nu_i}{E_i \delta} (N_\phi + N_\theta)_{EL} \right\}\end{aligned}\quad (4.2 - 12)$$

Using the method of separation of variables on equation (4.2 - 11) and integrating one obtains

$$\tilde{U} = -\frac{1}{\sin^2 \phi} \int_{\phi_0}^{\phi} \sin^2 \phi \tilde{F}_4 d\phi + \frac{\sin^2 \phi_0}{\sin^2 \phi} [\tilde{U}]_{\phi=\phi_0}$$

$[\tilde{U}]_{\phi=\phi_0}$ is the value of $\tilde{U}$ at time t and at $\phi = \phi_0$ which is the edge of the shell, consequently it is determined by the edge conditions.

For example, if the edge is of fixed support, then $[\sigma]_{\phi=\phi_0} = 0$

With value of U established V can be evaluated from equation (4.2 - 10). The membrane forces and moments are subsequently evaluated by equations (3.1.1 - 5) through (3.1.1 - 8).

V. BENDING ANALYSIS

5.1 General Shell

To more accurately describe the effects of edge conditions the following derivation includes the changes in curvature of the middle surface.

5.1.1 Governing equations

Let $\epsilon_{\phi\phi} = \epsilon_1 - \xi \chi_\phi$ and $\epsilon_{\theta\theta} = \epsilon_2 - \xi \chi_\theta$ where ϵ_1 and ϵ_2 are the middle surface strains and χ_ϕ and χ_θ are the changes of curvature.

$$\epsilon_1 = \frac{1}{r_1} \left(\frac{\partial v}{\partial \phi} - w \right) \quad (5.1.1-1)$$

$$\epsilon_2 = \frac{1}{r_2} \left(v \cot \phi - w \right) \quad (5.1.1-2)$$

$$\chi_\phi = \frac{1}{r_1} \frac{\partial}{\partial \phi} \left[\frac{1}{r_1} \left(v + \frac{\partial w}{\partial \phi} \right) \right] \quad (5.1.1-3)$$

$$\chi_\theta = \frac{1}{r_1} \left(v + \frac{\partial w}{\partial \phi} \right) \frac{\cot \phi}{r_2} \quad (5.1.1-4)$$

For the Kelvin-Voigt body, the stress-strain relationship is

$$\sigma_{\phi\phi} = 2 \left\{ \eta_0 e^{-k \left(T_m - \frac{\xi}{\delta} T_t \right)} \frac{\partial (\dots)}{\partial t} + G_0 \left[1 - \beta \left(T_m - \frac{\xi}{\delta} T_t \right) \right] (\dots) \right\} \left\{ 2 \epsilon_{\phi\phi} + \epsilon_{\theta\theta} - 3 \alpha \left(T_m - \frac{\xi}{\delta} T_t \right) \right\}$$

Proceeding with the method outlined by Timoshenko²⁵ where

$$N_\phi = \int_{-\delta/2}^{\delta/2} \sigma_{\phi\phi} d\xi$$

$$M_\phi = \int_{-\delta/2}^{\delta/2} \sigma_{\phi\phi} \xi d\xi$$

then

$$\begin{aligned} N_\phi &= \Gamma(2\xi + \xi_2) - \Lambda(2\chi_\phi + \chi_\theta) - \Gamma(3\alpha T_m) + \Lambda(3\alpha \frac{T_c}{\delta}) \\ M_\phi &= \Lambda(2\xi + \xi_2) - \Pi(2\chi_\phi + \chi_\theta) - \Lambda(3\alpha T_m) + \Pi(3\alpha \frac{T_c}{\delta}) \end{aligned}$$

where

$$\begin{aligned} \Gamma(\dots) &= 4\eta_0 e^{-\frac{KT_m}{T_c}} \frac{\delta}{KT_c} \sinh \frac{KT_c}{2} \frac{\partial(\dots)}{\partial t} + 2G_0(1-\beta T_m) \delta(\dots) \\ \Lambda(\dots) &= 4\eta_0 e^{-\frac{KT_m}{T_c}} \left(\frac{\delta}{KT_c}\right)^2 \left[\frac{KT_c}{2} \cosh \frac{KT_c}{2} - \sinh \frac{KT_c}{2} \right] \frac{\partial(\dots)}{\partial t} + \frac{1}{6} G_0 \beta T_c \delta^2(\dots) \\ \Pi(\dots) &= 4\eta_0 e^{-\frac{KT_m}{T_c}} \left(\frac{\delta}{KT_c}\right)^3 \left\{ \left[\left(\frac{KT_c}{2}\right)^2 + 2 \right] \sinh \frac{KT_c}{2} - KT_c \cosh \frac{KT_c}{2} \right\} \frac{\partial(\dots)}{\partial t} + \frac{1}{6} G_0 (1-\beta T_m) \delta^3(\dots) \end{aligned}$$

Assuming T_m and T_c independent of ℓ then

$$3\Gamma\left(\frac{\partial \psi}{\partial \phi} - \omega\right) = f_1(2N_\phi - N_\theta) + \Lambda\left[\frac{\partial}{\partial \phi}\left(\frac{\psi}{f_1} + \frac{1}{f_1} \frac{\partial \omega}{\partial \phi}\right)\right] - \Gamma(3\alpha T_m) + \Lambda(3\alpha \frac{T_c}{\delta}) \quad (5.1.1-5)$$

$$3\Gamma(\psi \cot \phi - \omega) = f_2(2N_\theta - N_\phi) + \Lambda\left[\left(\frac{\psi}{f_1} + \frac{1}{f_1} \frac{\partial \omega}{\partial \phi}\right) \cot \phi\right] - \Gamma(3\alpha T_m) + \Lambda(3\alpha \frac{T_c}{\delta}) \quad (5.1.1-6)$$

Eliminating ω from equations (5.1.1-5) and (5.1.1-6) we have

$$3\Gamma\left(\frac{\partial \psi}{\partial \phi} - \psi \cot \phi\right) = f_1(2N_\phi - N_\theta) - f_2(2N_\theta - N_\phi) + 3\Lambda\left[\frac{\partial}{\partial \phi}\left(\frac{\psi}{f_1} + \frac{1}{f_1} \frac{\partial \omega}{\partial \phi}\right) - \cot \phi\left(\frac{\psi}{f_1} + \frac{1}{f_1} \frac{\partial \omega}{\partial \phi}\right)\right] \quad (5.1.1-7)$$

By differentiating equation (5.1.1-6) with respect to ϕ and using equation (5.1.1-7) to eliminate $\frac{\partial \omega}{\partial \phi}$ then

$$\begin{aligned} 3\Gamma\left(\psi + \frac{\partial \omega}{\partial \phi}\right) &= f_1 \cot \phi (2N_\phi - N_\theta) - f_2 \cot \phi (2N_\theta - N_\phi) - \\ &\quad - \frac{\partial}{\partial \phi} \left[f_2 (2N_\theta - N_\phi) \right] + 3\Lambda \left[\frac{\psi}{f_1} + \frac{1}{f_1} \frac{\partial \omega}{\partial \phi} \right] \end{aligned} \quad (5.1.1-8)$$

Let $\mathcal{V} = \frac{1}{f_1}(\psi + \frac{\partial \omega}{\partial \phi})$ and $\mathcal{U} = \phi f_2$ then, from the equations of equilibrium for $p = 0$,

$$\begin{aligned} N_\phi &= -\frac{1}{f_2} \mathcal{U} \cot \phi \\ N_\theta &= -\frac{1}{f_1} \frac{\partial \mathcal{U}}{\partial \phi} \end{aligned}$$

With the above, equation (5.1.1-8) becomes the first governing equation.

$$\frac{1}{2} \left[r(\nabla) - \frac{1}{r} \lambda(\nabla) \right] = \frac{r_2}{r^2} \frac{\partial^2 U}{\partial \phi^2} + \frac{1}{r} \left[\frac{d}{d\phi} \left(\frac{r_2}{r} \right) + \frac{r_2}{r} \cot \phi \right] \frac{\partial U}{\partial \phi} - \frac{1}{r} \left[\frac{r_1}{r_2} \cot^2 \phi + \frac{1}{2} \right] U \quad (5.1.1-9)$$

To obtain the second governing equation, substitute the expressions for M_ϕ and M_θ into the equation of equilibrium

$$\frac{\partial}{\partial \phi} (r_2 M_\phi \sin \phi) - r_1 \cos \phi M_\theta - r_1 \sin \phi U = 0$$

then

$$\begin{aligned} -2 \sin \phi \pi \left\{ \frac{r_2}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{1}{r} \left[\frac{r_2}{r} \cot \phi + \frac{d}{d\phi} \left(\frac{r_2}{r} \right) \right] \frac{\partial V}{\partial \phi} - \frac{1}{r} \left[\frac{r_1}{r_2} \cot^2 \phi + \frac{1}{2} \right] V \right\} + \\ + \frac{1}{r} \lambda \left\{ \frac{\partial}{\partial \phi} [r_2 \sin \phi (2\epsilon_1 + \epsilon_2)] - r_1 \cos \phi (2\epsilon_2 + \epsilon_1) \right\} = U \sin \phi \end{aligned} \quad (5.1.1-10)$$

To eliminate the middle surface strains from (5.1.1-10) consider the following equation of equilibrium

$$\frac{\partial}{\partial \phi} (r_2 N_\phi \sin \phi) - N_\theta r_1 \cos \phi - U \sin \phi = 0 \quad (5.1.1-11)$$

Substituting the expressions for N_ϕ and N_θ into (5.1.1-11) we have

$$\begin{aligned} \frac{1}{r} \lambda \left\{ \frac{\partial}{\partial \phi} [r_2 \sin \phi (2\epsilon_1 + \epsilon_2)] - r_1 \cos \phi (2\epsilon_2 + \epsilon_1) \right\} = 2 \sin \phi \lambda \left\{ \frac{1}{r} \left[\lambda \left(\frac{r_2}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \right. \right. \right. \\ \left. \left. \left. + \frac{1}{r} \left[\frac{d}{d\phi} \left(\frac{r_2}{r} \right) + \frac{r_2}{r} \cot \phi \right] \frac{\partial V}{\partial \phi} - \frac{1}{r} \left[\frac{r_1}{r_2} \cot^2 \phi + \frac{1}{2} \right] V \right] \right\} + \frac{1}{r} \lambda \left\{ \frac{1}{r} (U \sin \phi) \right\} \end{aligned} \quad (5.1.1-12)$$

With equations (5.1.1-10) and (5.1.1-11) we now have the second governing equation

$$\begin{aligned} -2\pi \left\{ \frac{r_2}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{1}{r} \left[\frac{d}{d\phi} \left(\frac{r_2}{r} \right) + \frac{r_2}{r} \cot \phi \right] \frac{\partial V}{\partial \phi} - \frac{1}{r} \left[\frac{r_1}{r_2} \cot^2 \phi + \frac{1}{2} \right] V \right\} + \\ + 2\lambda \left\{ \frac{1}{r} \left[\lambda \left(\frac{r_2}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{1}{r} \left[\frac{d}{d\phi} \left(\frac{r_2}{r} \right) + \frac{r_2}{r} \cot \phi \right] \frac{\partial V}{\partial \phi} - \frac{1}{r} \left[\frac{r_1}{r_2} \cot^2 \phi + \frac{1}{2} \right] V \right] \right\} = U - \frac{1}{r} \lambda \left\{ \frac{1}{r} (U) \right\} \end{aligned} \quad (5.1.1-13)$$

$$\text{Let } L(\dots) = \frac{f_2}{f_1^2} \frac{\partial^2(\dots)}{\partial \phi^2} + \frac{1}{f_1} \left[\frac{d}{d\phi} \left(\frac{f_2}{f_1} \right) + \frac{f_2}{f_1} \omega_T \phi \right] \frac{\partial(\dots)}{\partial \phi} - \frac{1}{f_1} \left[\frac{f_2}{f_1} \omega_T^2 \phi \right] (\dots)$$

then the governing equations become

$$-2\pi \left\{ L(V) + \frac{1}{2f_1} V \right\} + 2\lambda \left\{ \frac{1}{f_1} \left[\lambda \left(L(V) + \frac{1}{2f_1} V \right) \right] \right\} = U - \frac{1}{f_1} \lambda \left\{ \frac{1}{f_1} (U) \right\} \quad (5.1.1-13a)$$

$$L(U) - \frac{1}{2f_1} U = \frac{3}{2} \left[\Gamma(V) - \frac{1}{f_1} \lambda(V) \right] \quad (5.1.1-9a)$$

5.2 Conical Shell

For the conical shell

$$f_2 = \ell \tan \gamma$$

$$\frac{1}{f_1} = 0$$

$$f_1 \partial \phi = \partial \ell$$

then $L(\dots)$ becomes $L(\dots) = \tan \gamma \left[\ell \frac{\partial^2(\dots)}{\partial \ell^2} + \frac{\partial(\dots)}{\partial \ell} - \frac{1}{\ell}(\dots) \right]$
and the governing equations are

$$-2\pi \left\{ L(V) \right\} + 2\lambda \left\{ \frac{1}{f_1} \left[\lambda \left\{ L(V) \right\} \right] \right\} = U \tan \gamma \quad (5.2-1)$$

$$L(U) = \frac{3}{2} \Gamma(V) \quad (5.2-2)$$

where U is now $\phi \ell$.

Solving equations (5.2-1) and (5.2-2) for V then

$$-2\pi \left\{ L(V) \right\} + 2\lambda \left\{ \frac{1}{f_1} \left[\lambda \left\{ L(V) \right\} \right] \right\} = \frac{3}{2} \Gamma(V) \quad (5.2-3)$$

Assuming a product solution $V = \Phi(\ell) T(t)$ then

$$\frac{L(\Phi)}{\Phi} = \frac{\frac{3}{2} \Gamma(T)}{-2\pi(T) + 2\lambda \left\{ \frac{1}{f_1} \left[\lambda(T) \right] \right\}} = -\mu^4 \quad (5.2-4)$$

where the constant μ^4 must be $\mu^4 = 12(1-\nu^2)/\sigma^2$ for the solution to conform to the initial condition.

The solution as a function of ℓ is

$$\Phi(\ell) = C_1 \left[Z_1(\gamma) + \frac{2}{\gamma} Z_2'(\gamma) \right] + C_2 \left[Z_2(\gamma) - \frac{2}{\gamma} Z_1'(\gamma) \right]$$

where $\eta = 2\lambda\sqrt{l}$, $\lambda^4 = \mu^4 \cot^2 \gamma$, $z_1 = \text{BER}(\eta)$, $z_2 = -\text{BEI}(\eta)$, $z' = \frac{dz}{d\eta}$

The equation resulting from (5.2-3) describing the solution as a function of time is a first order differential-integral equation which, when differentiated, yields

$$\begin{aligned} & \left[\frac{3}{2} + A_1 - A_2 \right] \frac{dT}{dt^2} + \left\{ \left[(3 - A_2) A_3 - A_2 A_5 \right] (1 - \beta R t) e^{KRt} + 2 A_1 A_4 e^{KRt} - KR \left(\frac{3}{2} + A_1 - A_2 \right) \right\} \frac{dT}{dt} + \\ & + \left\{ \left[\frac{3}{2} A_3^2 - A_2 A_3 A_5 \right] (1 - \beta R t)^2 e^{2KRt} + \left[\beta R A_2 A_5 - \frac{3}{2} \beta R A_3 \right] (1 - \beta R t) e^{KRt} + \right. \\ & \left. + A_1 A_4^2 e^{2KRt} \right\} T = 0 \end{aligned} \quad (5.2 - 5)$$

where

$$\begin{aligned} T_m &= R t \\ A_1 &= 2\mu^4 \left(\frac{d}{K T_e} \right)^2 \left[\frac{K T_e}{2} \coth \frac{K T_e}{2} - 1 \right]^2 \\ A_2 &= 2\mu^4 \left(\frac{d}{K T_e} \right)^2 \left[\left(\frac{K T_e}{2} \right)^2 + 2 - K T_e \coth \frac{K T_e}{2} \right] \\ A_3 &= \frac{G_0 K T_e}{2 \eta_0 \sinh \frac{K T_e}{2}} \\ A_4 &= G_0 \beta T_e (K T_e)^2 / 24 \eta_0 \left[\frac{K T_e}{2} \cosh \frac{K T_e}{2} - \sinh \frac{K T_e}{2} \right] \\ A_5 &= G_0 (K T_e)^3 / 24 \eta_0 \left\{ \left[\left(\frac{K T_e}{2} \right)^2 + 2 \right] \sinh \frac{K T_e}{2} - K T_e \cosh \frac{K T_e}{2} \right\} \end{aligned}$$

By expressing the coefficients of T and its derivatives as Taylor series then T(t) may be expressed as a power series

$$T(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n + \dots$$

where

$$\begin{aligned}
 a_2 &= - \left(a_1 [B_2 - KR B_1] + a_0 [B_3 + \beta R B_4] \right) / 2 B_1 \\
 a_3 &= - \left(a_1 [B_2 KR - \beta R A_6] + 2 a_2 [B_2 - KR B_1] + a_0 [B_3 2KR - 2\beta R A_5 + \beta R B_4 KR - (\beta R)^2 B_4] + a_1 [B_3 + \beta R B_4] \right) / 3 \cdot 2 B_1 \\
 &\vdots \\
 a_{n+2} &= - \left\{ a_1 \left[B_2 \frac{(KR)^n}{n!} - \beta R A_6 \frac{(KR)^{n-1}}{(n-1)!} + \dots + (n+1) a_{n+1} [B_2 - KR B_1] + \right. \right. \\
 &\quad \left. \left. + a_0 \left[B_3 \frac{(2KR)^n}{n!} - 2\beta R B_5 \frac{(2KR)^{n-1}}{(n-1)!} + (\beta R)^2 B_5 \frac{(2KR)^{n-2}}{(n-2)!} + \beta R B_4 \frac{(KR)^n}{n!} - (\beta R)^2 B_4 \frac{(KR)^{n-1}}{(n-1)!} \right] + \right. \\
 &\quad \left. + \dots + a_n [B_3 - \beta R B_4] \right\} / n(n-1) B_1
 \end{aligned}$$

where

$$\begin{aligned}
 A_6 &= (3 - A_2) A_3 - A_2 A_5 \\
 B_1 &= \frac{3}{2} + A_1 - A_2 \\
 B_2 &= 2 A_1 A_4 + (3 - A_2) A_3 - A_2 A_5 \\
 B_3 &= \frac{3}{2} A_3^2 - A_2 A_3 A_5 + A_1 A_4^2 \\
 B_4 &= A_2 A_5 - \frac{3}{2} A_3 \\
 B_5 &= \frac{3}{2} A_3^2 - A_2 A_3 A_5
 \end{aligned}$$

The first two coefficients (a_0, a_1) are then determined from the particular boundary conditions.

5.2.1 Simplified Conical Shell

Since the programmed temperature rates in the test schedule produced a temperature difference across the thickness of less than one degree, the following analysis neglects the differential ($T_t = 0$).

The operators defined in section 5.1.1 become

$$\begin{aligned}
 \Gamma(\dots) &= 2 \left\{ \eta_0 e^{-K T_m} \frac{\partial(\dots)}{\partial t} + G_0 (1 - \beta T_m)(\dots) \right\} \delta \\
 \Lambda(\dots) &= 0 \\
 \Pi(\dots) &= 2 \left\{ \eta_0 e^{-K T_m} \frac{\partial(\dots)}{\partial t} + G_0 (1 - \beta T_m)(\dots) \right\} \delta^3 / 12
 \end{aligned}$$

Let $\Gamma(\dots) = 2 \left\{ \eta_0 e^{-\beta T_m} \frac{\partial(\dots)}{\partial t} + \zeta_0 (1 - \beta T_m)(\dots) \right\}$, then equation (5.2-3)
 becomes

$$\frac{\delta^3}{6} \Gamma[LL(V)] = -\frac{3}{2} \delta \Gamma(V) \quad (5.2.1-1)$$

then, with a product solution, $V = \Phi(\ell) T(t)$ we have

$$\Gamma(T) \left\{ LL(V) + \frac{9}{\delta^2} \Phi \right\} = 0 \quad (5.2.1-2)$$

where Φ is the same as that in section 5.2 and $T(t)$ must be determined from the boundary conditions.

5.2.2 Boundary Conditions

Consider the edge of the shell fixed against rotation and expansion, then at $\ell = \ell_0$

$$V = 0 \quad \text{FOR ALL } t \quad (5.2.2-1)$$

$$E_2 = \frac{\partial E_2}{\partial t} = 0 \quad \text{FOR ALL } t \quad (5.2.2-2)$$

Then

$$\overline{V}_{\ell=\ell_0} = 0 = \left[\overline{z}_{10} + \frac{2}{\gamma_0} \overline{z}'_{20} + A_0 \left(\overline{z}_{20} - \frac{2}{\gamma_0} \overline{z}'_{10} \right) \right] T(t)$$

so

$$A_0 = - \left(\overline{z}_{10} + \frac{2}{\gamma_0} \overline{z}'_{20} \right) / \left(\overline{z}_{20} - \frac{2}{\gamma_0} \overline{z}'_{10} \right)$$

Also

$$3\delta \Gamma(E_2) = 2N_\theta - N_\rho + 3\delta \Gamma(\alpha T_m) \quad (5.2.2-3)$$

where

$$N_\phi = 6\delta \cot \gamma \left[\frac{z_2}{\gamma^2} - \frac{2z_2'}{\gamma^3} + A_0 \left(-\frac{z_1}{\gamma^2} - \frac{2z_2'}{\gamma^3} \right) \right] \Gamma(T) - \rho l \tan \gamma / 2$$

$$N_\theta = 3\delta \cot \gamma \left[\frac{z_2'}{\gamma} - \frac{2z_2}{\gamma^2} + \frac{4z_2'}{\gamma^3} + A_0 \left(-\frac{z_1'}{\gamma} + \frac{2z_2}{\gamma^2} + \frac{4z_2'}{\gamma^3} \right) \right] \Gamma(T) - \rho l \tan \gamma$$

Applying equation 5.2.2-2 then

$$\Gamma(T) = \frac{\frac{3}{2} \rho l_0 \tan \gamma - 6\alpha \delta \left[\eta_0 e^{-kT_m} \frac{\partial T_m}{\partial t} + G_0 (1 - \beta T_m) T_m \right]}{6\delta \cot \gamma \left[\frac{z_{20}'}{\gamma_0} - \frac{3z_{20}}{\gamma_0^2} + \frac{6z_{20}'}{\gamma_0^3} + \left(-\frac{z_{10}}{\gamma_0} + \frac{2z_{20}}{\gamma_0^2} \right) \left(-\frac{z_{10}'}{\gamma_0} + \frac{3z_{20}'}{\gamma_0^2} + \frac{6z_{20}'}{\gamma_0^3} \right) \right]}$$

The inner surface stresses are then written as

$$\sigma_\phi = \frac{N_\phi}{\delta} + \frac{6}{\delta^2} M_\phi = 6 \cot \gamma \left[\frac{z_2}{\gamma^2} - \frac{2z_2'}{\gamma^3} + A_0 \left(-\frac{z_1}{\gamma^2} - \frac{2z_2'}{\gamma^3} \right) \right] \Gamma(T) - \rho l \tan \gamma / 2 -$$

$$- 6 \cot \gamma \left[\frac{z_1'}{\gamma} - \frac{z_1}{\gamma^2} - \frac{2z_2'}{\gamma^3} + A_0 \left(\frac{z_2'}{\gamma} - \frac{z_2}{\gamma^2} + \frac{2z_2'}{\gamma^3} \right) \right] \Gamma(T)$$

$$\sigma_\theta = \frac{N_\theta}{\delta} + \frac{6}{\delta^2} M_\theta = 3 \cot \gamma \left[\frac{z_2'}{\gamma} - \frac{2z_2}{\gamma^2} + \frac{4z_2'}{\gamma^3} + A_0 \left(-\frac{z_1'}{\gamma} + \frac{2z_2}{\gamma^2} + \frac{4z_2'}{\gamma^3} \right) \right] \Gamma(T) -$$

$$- \rho l \tan \gamma - 3 \cot \gamma \left[\frac{z_1'}{\gamma} + \frac{2z_2}{\gamma^2} + \frac{4z_2'}{\gamma^3} + A_0 \left(\frac{z_2'}{\gamma} + \frac{2z_2}{\gamma^2} - \frac{4z_2'}{\gamma^3} \right) \right] \Gamma(T)$$

(5.2.2-4)

Consider, now, the hinged expansion edge; that is, an edge free to rotate completely and to expand to some degree.

Then, at $l = l_0$

$$M_\phi = 0 \quad \text{FOR ALL } t \quad (5.2.2-5)$$

$$\epsilon_2 = \alpha_e T_e \quad (5.2.2-6)$$

where $\alpha_e T_e$ denotes the expansion of the mount. So with

$$M_\phi = -\frac{\delta^3}{12} \left\{ 4\lambda^2 \left[\frac{z_1'}{\gamma} - \frac{z_1}{\gamma^2} - \frac{2z_2'}{\gamma^3} + A_0 \left(\frac{z_2'}{\gamma} - \frac{z_2}{\gamma^2} + \frac{2z_2'}{\gamma^3} \right) \right] \right\} \Gamma(T)$$

then with (5.2.2-6)

$$A_0 = - \frac{\frac{z_{10}'}{\gamma_0} - \frac{z_{10}}{\gamma_0^2} - \frac{2z_{20}'}{\gamma_0^3}}{\frac{z_{20}'}{\gamma_0} - \frac{z_{20}}{\gamma_0^2} + \frac{2z_{20}'}{\gamma_0^3}}$$

Applying equations (5.2.2-3) and (5.2.2-6) then

$$r(T) = \frac{\frac{3}{2} \rho_0 \pi \nu \gamma - 6\delta \left[\gamma_0 e^{-\kappa T_m} \frac{\partial}{\partial t} (\alpha T_m - \alpha_e T_e) + G_0 (1 - \beta T_m) (\alpha T_m - \alpha_e T_e) \right]}{6\delta \cot \gamma \left[\frac{z'_{20}}{\gamma_0} - \frac{3z_{20}}{\gamma_0^2} + \frac{6z'_{20}}{\gamma_0^3} + A_0 \left(-\frac{z'_{20}}{\gamma_0} + \frac{3z_{20}}{\gamma_0^2} + \frac{6z'_{20}}{\gamma_0^3} \right) \right]}$$

5.3 Spherical Shell

Neglecting, again, the temperature differential, equations (5.1.1-9a) and (5.1.1-13a) become, for the spherical shell of radius a ,

$$L(V) - \frac{1}{2a} V = \frac{3}{2} \delta r'(T) \quad (5.3-1)$$

$$-\frac{d^3}{6} r \left\{ L(V) + \frac{1}{2a} V \right\} = V \quad (5.3-2)$$

where $L(\dots) = \frac{1}{a} \left[\frac{\partial^2(\dots)}{\partial \phi^2} + \cot \phi \frac{\partial(\dots)}{\partial \phi} - \cot^2 \phi (\dots) \right]$

The exact solution of equations (5.3-1) and (5.3-2) is a form of the hypergeometrical series whose convergence is unsatisfactory for $\sqrt{a/\delta} > 10$. The following solution will be an approximation developed by Timoshenko²⁵ for shells whose angle is not small.

Let $Q_1 = Q_\phi \sqrt{\sin \phi}$ and $V_1 = V \sqrt{\sin \phi}$ then the first derivatives in equations (5.3-1) and (5.3-2) disappear. Neglecting the quantity in comparison with its second derivative we have

$$\frac{\partial^2 Q_1}{\partial \phi^2} = \frac{3}{2} \delta r'(T) \quad (5.3-3)$$

$$\frac{\partial^2 V_1}{\partial \phi^2} = -\frac{6a^2}{\delta^2} Q_1 \quad (5.3-4)$$

the solution of which is

$$Q_\phi = e^{-\lambda \psi} \left[\sin(\xi - \psi) \right]^{-\frac{1}{2}} \sin(\lambda \psi + \gamma) T(t) \quad (5.3-5)$$

where $\psi = \xi - \phi$, ξ is the shell half angle (not small), $\lambda^2 = \frac{9a^2}{4\delta^2}$ and γ is an arbitrary constant.

Then

$$N_\phi = -e^{-\lambda\psi} [\sin(\xi-\psi)]^{-\frac{1}{2}} \cot(\xi-\psi) \sin(\lambda\psi+\gamma) T(t) - \frac{P_a}{2}$$

$$N_\theta = \frac{\lambda}{2\delta} e^{-\lambda\psi} [\sin(\xi-\psi)]^{-\frac{1}{2}} \left[2 \cos(\lambda\psi+\gamma) - \left\{ 2 - \frac{1}{\lambda} \cot(\xi-\psi) \right\} \sin(\lambda\psi+\gamma) \right] T(t) - \frac{P_a}{2}$$

$$M_\phi = \frac{a}{2\lambda} e^{-\lambda\psi} [\sin(\xi-\psi)]^{-\frac{1}{2}} \left[\cos(\lambda\psi+\gamma) + \sin(\lambda\psi+\gamma) \right] T(t)$$

$$M_\theta = \frac{a}{4\lambda} e^{-\lambda\psi} [\sin(\xi-\psi)]^{-\frac{1}{2}} \left[\left\{ 1 + \frac{3}{2} \cot(\xi-\psi) \right\} \cos(\lambda\psi+\gamma) + \sin(\lambda\psi+\gamma) \right] T(t)$$

Employing the fixed edge conditions on a hemisphere the inner surface stresses are then

$$\sigma_\phi = -\frac{e^{-\lambda\psi} \cot(\frac{\pi}{2}-\psi)}{\delta \sqrt{\sin(\frac{\pi}{2}-\psi)}} \sin(\lambda\psi+\frac{\pi}{2}) T(t) - \frac{P_a}{2\delta} + \frac{3a}{\delta^2 \lambda} \frac{e^{-\lambda\psi}}{\sqrt{\sin(\frac{\pi}{2}-\psi)}} \left[\cos(\lambda\psi+\frac{\pi}{2}) + \sin(\lambda\psi+\frac{\pi}{2}) \right] T(t)$$

$$\sigma_\theta = \frac{\lambda e^{-\lambda\psi}}{2\delta \sqrt{\sin(\frac{\pi}{2}-\psi)}} \left[2 \cos(\lambda\psi+\frac{\pi}{2}) - \left\{ 2 - \frac{1}{\lambda} \cot(\frac{\pi}{2}-\psi) \right\} \sin(\lambda\psi+\frac{\pi}{2}) \right] T(t) - \frac{P_a}{2\delta} + \frac{3a}{2\delta^2 \lambda} \frac{e^{-\lambda\psi}}{\sqrt{\sin(\frac{\pi}{2}-\psi)}} \left\{ \left[1 + \frac{3}{2} \cot(\frac{\pi}{2}-\psi) \right] \cos(\lambda\psi+\frac{\pi}{2}) + \sin(\lambda\psi+\frac{\pi}{2}) \right\} T(t)$$

where

$$T(t) = -\frac{1}{2\lambda} \left(\frac{3}{2} P_a - 6a\delta \left[\gamma_0 e^{-\kappa T_m} \frac{\partial T_m}{\partial t} + G_0 (1-\beta T_m) T_m \right] \right)$$

With the hinged expansion edge, the inner surface stresses are

$$\sigma_\phi = -\frac{e^{-\lambda\psi} \cot(\frac{\pi}{2}-\psi)}{\delta \sqrt{\sin(\frac{\pi}{2}-\psi)}} \sin(\lambda\psi-\frac{\pi}{4}) T(t) - \frac{P_a}{2\delta} + \frac{3a}{\delta^2 \lambda} \frac{e^{-\lambda\psi}}{\sqrt{\sin(\frac{\pi}{2}-\psi)}} \left[\cos(\lambda\psi-\frac{\pi}{4}) + \sin(\lambda\psi-\frac{\pi}{4}) \right] T(t)$$

$$\sigma_\theta = \frac{\lambda e^{-\lambda\psi}}{2\delta \sqrt{\sin(\frac{\pi}{2}-\psi)}} \left[2 \cos(\lambda\psi-\frac{\pi}{4}) - \left\{ 2 - \frac{1}{\lambda} \cot(\frac{\pi}{2}-\psi) \right\} \sin(\lambda\psi-\frac{\pi}{4}) \right] T(t) - \frac{P_a}{2\delta} + \frac{3a}{2\delta^2 \lambda} \frac{e^{-\lambda\psi}}{\sqrt{\sin(\frac{\pi}{2}-\psi)}} \left\{ \left[1 + \frac{3}{2} \cot(\frac{\pi}{2}-\psi) \right] \cos(\lambda\psi-\frac{\pi}{4}) + \sin(\lambda\psi-\frac{\pi}{4}) \right\} T(t)$$

where

$$T(t) = \frac{1}{2\sqrt{2}\lambda} \left[\frac{P_a}{2} - 6\delta \left\{ \gamma_0 e^{-\kappa T_m} \frac{\partial}{\partial t} (\alpha T_m - \alpha_e T_e) + G_0 (1-\beta T_m) (\alpha T_m - \alpha_e T_e) \right\} \right]$$

5.4 The Simplified Conical Shell as a Maxwell Body

For a Maxwell body

$$\frac{1}{G} \frac{\partial \sigma_x}{\partial t} + \frac{1}{\eta} \sigma_x = 2 \frac{\partial}{\partial t} [2\epsilon_x + \epsilon_y - 3\alpha T_m]$$

Proceeding as in section 5.1 and with

$$\Gamma(\dots) = \frac{1}{G} \frac{\partial(\dots)}{\partial t} + \frac{1}{\eta}(\dots)$$

then

$$N_\theta = 12 \delta \cot \gamma \left[\frac{\bar{z}_2}{\gamma^2} - \frac{2\bar{z}_1'}{\gamma^3} + A_0 \left(-\frac{\bar{z}_1}{\gamma^2} - \frac{\bar{z}_2'}{\gamma^3} \right) \right] \frac{1}{\Gamma} \left\{ \frac{dT}{dt} \right\} - \frac{Pl \tan \gamma}{2}$$

$$N_\theta = 6 \delta \cot \gamma \left[\frac{\bar{z}_2'}{\gamma} - \frac{2\bar{z}_2}{\gamma^2} + \frac{4\bar{z}_1'}{\gamma^3} + A_0 \left(-\frac{\bar{z}_1'}{\gamma} + \frac{2\bar{z}_1}{\gamma^2} + \frac{4\bar{z}_2'}{\gamma^3} \right) \right] \frac{1}{\Gamma} \left\{ \frac{dT}{dt} \right\} - Pl \tan \gamma$$

Applying the fixed edge conditions, then

$$A_0 = -\frac{\bar{z}_0 + \frac{2}{\gamma_0} \bar{z}_{20}'}{\bar{z}_0 - \frac{2}{\gamma_0} \bar{z}_{10}'}, \quad \frac{1}{\Gamma} \left\{ \frac{dT}{dt} \right\} = \frac{\frac{3}{2} Pl_0 - 6\delta \int_0^t \frac{G_0(1-\beta T_m) dt}{\gamma_0 e^{-KRt}} \int_0^t G_0(1-\beta T_m) \frac{dT_m}{dt} dt \int_0^t \frac{G_0(1-\beta T_m)}{\gamma_0 e^{-KRt}} dt}{12 \delta \cot \gamma \left[\frac{\bar{z}_{20}'}{\gamma_0} - \frac{3\bar{z}_{20}}{\gamma_0^2} + \frac{6\bar{z}_{10}'}{\gamma_0^3} + A_0 \left(-\frac{\bar{z}_{10}'}{\gamma_0} + \frac{3\bar{z}_{10}}{\gamma_0^2} + \frac{6\bar{z}_{20}'}{\gamma_0^3} \right) \right]}$$

For a constant temperature rate $T_m = Rt$. The integral then becomes

$$\int_0^t G_0(1-\beta Rt) R e^{\int_0^t \frac{G_0(1-\beta Rt)}{\gamma_0} e^{KRt} dt} dt$$

For $e^{KRt} < 0.1 \frac{\gamma_0}{G_0}$

the exponential is essentially unity so

$$\int_0^t G_0(1-\beta Rt) R e^{\int_0^t \frac{G_0(1-\beta Rt)}{\gamma_0} e^{KRt} dt} dt \approx G_0(1-\frac{\beta R}{2} t) Rt ; \left(t < \frac{\log 0.1 \frac{\gamma_0}{G_0}}{KR} \right)$$

After $t > \frac{\log 0.1 \frac{\gamma_0}{G_0}}{KR}$

the quantity $(1-\beta Rt)$ can be considered constant compared to the exponential so

$$\int_0^t G_0(1-\beta Rt) R e^{\int_0^t \frac{G_0(1-\beta Rt)}{\gamma_0} e^{KRt} dt} dt \approx G_0(1-\beta Rt) R \left[\bar{E}_i \left(\frac{G_0(1-\beta Rt)}{KR \gamma_0 e^{-KRt}} \right) - \bar{E}_i(0.1) \right]$$

where

$$t' = \frac{\log 0.1 \frac{\gamma_0}{G_0}}{KR}$$

and $\bar{E}_i(\dots)$

is defined as

$$\bar{E}_i(\log x) = \int_0^x \frac{dy}{\log y}$$

The time function is then

$$\frac{1}{K_2} \left[K_1 - 6\alpha \delta G_0 \left(1 - \frac{\beta R t}{2} \right) R t \right] \quad \text{for } t < t'$$

$$\frac{1}{K_2} \left[K_1 - 6\alpha \delta e^{-\frac{G_0(1-\beta R t')}{K R \eta_0 C - K R t'}} \left\{ G_0 \left(1 - \frac{\beta R t'}{2} \right) R t' + G_0 (1 - \beta R t) R \left[\bar{E}_i \left(\frac{G_0(1-\beta R t)}{K R \eta_0 C - K R t} \right) - \bar{E}_i(0.1) \right] \right\} \right] \quad \text{for } t > t'$$

VI. EXPERIMENTAL RESULTS AND VERIFICATION OF ANALYTICAL METHODS

6.1 Basic Approach to the Test Program

In the early stages of the current investigation, it was decided that, for experimental verification of theoretical methods of predicting the viscoelastic response of shells of revolution subjected to normal pressure and elevated temperature and thermal gradients, records of the strain as a function of time would be necessary. Accordingly, steps were taken to find appropriate means of making such elevated temperature measurements.

The objectives of the experimental investigation were as follows:

- (1) To obtain strain versus time curves for normal pressures and applied temperatures and thermal rates.
- (2) To determine the mode and progression of failure, and the load condition necessary to cause failure under load and transient temperatures.
- (3) To verify or disprove the method of analysis.

6.1.1 Test conditions

The experimental program for verification of a method of analysis does not necessarily duplicate any specific design conditions. It is required that the tests cover the parameters included in the method of analysis. These parameters should extend over a wide range of the values.

For convenience, a uniform normal pressure of 12 psi is chosen to be applied simultaneously with various combinations of temperature. A maximum test temperature of 500°F is chosen, based on a level of temperature sufficiently high to introduce material property variation and buckling.

The tests are designed to eliminate as many extraneous variables as possible. However, in the present study it is impossible to eliminate parameters such as edge constraints, and discontinuities of temperature due to mass. These variables are difficult to evaluate. In the correlation, it is not clear whether the method of analysis is at fault or the accuracy of the experimental data is questionable. The interrelationship of the above mentioned inseparable parameters and their influence on the overall stress distributions are not satisfactorily established.

6.1.2 Number of Tests

The number of tests is an important factor in the design and cost of an experimental investigation. Usually repetition of test conditions with several specimens is made to include the effect of test scatter.

A more economical approach is to test each specimen under different conditions and to compare the test results with theoretical results over the range of the variables. This approach permits checking over a large range of values with a minimum number of tests and test specimens.

The approach employed in this investigation is designed after the latter method; however, where certain doubts existed, the test conditions are repeated with another model.

6.1.3 Test Specimen

The size of the test specimen is another important consideration particularly in elevated temperature structural research. In such investigations, large specimens are required to eliminate certain testing difficulties; for example, temperature gradients are established by temperature difference along the length of a specimen. If the specimen were too small, temperature-sensing elements would be too closely spaced along the specimen with probable disturbance of the required gradient.

The design of the specimen is based on the following specifications:

- (1) Geometry - conical and hemispherical shells are used analytical and testing simplicity.
- (2) Radius - a base radius of 12 inches is chosen as a convenient specimen size.
- (3) Thickness - the thickness is based on a radius-to-thickness ratio corresponding to a convenient anticipated buckling stress.
- (4) Material - Aluminum is chosen so that material property variation with temperature, creep, and buckling occur within a convenient temperature and load range. This is of primary importance in the design of the test facility and the instrumentation.

6.2 Test Procedure

Prior to an actual test run, an unrestrained heating of the model provided data necessary for temperature compensation of the strain gages. The difference between these data and the actual test data compensated the effects of a varying coefficient of linear thermal expansion, coefficient of resistivity of the gages, and coefficients of linear thermal expansion of the gage cement and gage material. The compensated strain data is then, in effect, the total strain minus the linear thermal expansion.

With the strain-temperature data recorded and the model cooled, the shell is bolted to the steel mounting plate. The selected time-temperature function is set on the temperature programmer.

Instead of using an ice-water bath for the cold junction of each thermocouple, a separate thermocouple employing an ice-water bath provides the temperature of the recording thermocouple input to the recorder. This temperature, which is monitored on a precision millivolt potentiometer, is the effective cold junction temperature of the recording thermocouples. The strain gage bridges are balanced and zero points of all instrumentation and reference temperatures of thermocouples are noted. The oven is placed over the model and the power controller is connected to the lamps. The model is evacuated to conform to the prescribed external load, after which the recording instruments and the temperature programmer are started.

After completion of the transient portion of the test program, the temperature programmer is placed in a "hold" condition to maintain the steady-state temperature for the desired period of time. Usually the steady-state is maintained for five minutes except for collapse studies in which case thirty minutes are provided.

When the test is completed, the power controller is shut off, the model is allowed to cool and then unbolted for the next unrestrained run.

6.3 Analysis of Test Results

The experimental results are presented in Fig. 6.5-1 through 6.5-18 and included in Appendix C.

The low thermal resistance of models requires a low gain condition on the power controller to prevent a pulsing heat input. As a result of the low gain, the steady-state temperatures are 40°F to 60°F lower than that programmed in the initial tests.

The base temperatures are considerably lower than that of the rest of the shell due to heat conduction to the steel mounting plate. In an effort to remedy the low temperature at the base, the feedback thermocouple for Zone 3 was moved closer to the base. This produces a shell

temperature 40°F to 80°F higher than the program temperature, but the temperatures near the base are from 100°F to 150°F lower than the steady-state temperature. However, the resulting temperature gradient occupies only the lower three inches of the shell.

The data from the differential thermocouples are merely qualitative. Calculations show a thermal gradient across the shell thickness of 1°F for a programmed temperature rate of 20°F/sec whereas the data indicates 30°F. The large experimental values are due to a poor thermal bond between the insulated junction and the surface of the shell.

A characteristic feature of all tests is the relaxation of stress as steady-state temperature conditions are maintained. The reduction of compressive stress is due to the expansion of the steel mounting plate and a relaxation of the material. The relative contribution of each condition depends upon the temperature of the base. To separate the two effects requires insulation of the base. However, it is felt that the major factor in the reduction of compressive stress is that of material relaxation. This conclusion is supported by some of the zonal tests. In those tests where the base temperature was relatively low, say about 120°F, in which case yielding of the base should not be a problem, the compressive stresses along the model slant height decayed although the compressive stresses near the base increased with an increasing steady-state temperature.

As is expected the stresses are compressive throughout the shell with those in the circumferential direction being of the greater magnitude. Maximum values are observed at the base, and the stress decreases rapidly along the meridian for the lower section. The upper regions of the shells experience approximately the same stress magnitude. The meridional stresses are similar to the circumferential stresses but with less pronounced maxima and minima.

Collapse was encountered in a 1/16" thick conical shell during an applied heating rate of 20°F/sec (to 500°F) with a surface pressure equivalent to 22"Hg differential pressure. The same model had experienced the same program previously. The data are presented in Table C-11. Approximately at 80% of the transient period gage 4M showed an immediate change in strain. The change is believed to be caused by a slight buckle at the support. No further buckling occurred until the transient condition had been sustained for five minutes when all gages registered an immediate jump. Thereafter gages 3M and 3P showed increasing compression to five and one-half minutes when complete collapse occurred.

The collapsed cone has five non axisymmetric nodes with the wall in the region of the base almost flat against the support. After collapse the support for instrumentation wiring interfered with progression causing a tilt of the apex toward the more seriously buckled area. See Fig. 6.3-1.

6.4 Simplified Strain-Displacement Method

Compared in Fig. 7.1-4 and 7.1-5 are the experimental data for a 1/16" thick cone for 20°F/sec to 500°F with 11.5 psi pressure and the theoretical stress calculated from the equations presented in section 3.1 for a similar program. As seen from the figures the theoretical stress behavior is opposite to that of the experimental. This behavior is believed due to neglecting the change in curvature in the strain expressions thereby relegating the analysis to a non-bending case (membrane theory).

6.5 Bending Analysis Neglecting the Temperature Differential, T_r

Compared in Fig. 6.5-1 through Fig. 6.5-18 are the theoretical and experimental data for one test on a 1/32" thick cone, five test on a 1/16" cone, one test on a 1/8" cone, and two tests on a 1/16" thick hemisphere. All theoretical data are calculated for the hinged expansion edge boundary condition, i.e., the mounting plate is considered to assume the temperature of the edge of the model and the edge offers no resistance to rotation in the meridional direction. A characteristic of all theoretical data is a large stress value during the early time periods when the response should be nearly elastic. The time periods are 1/3, 1/2, and end of transient.

The Kelvin-Voigt body may be pictured as a spring and a dashpot in parallel with each element having the same displacement and the resulting force being the sum of that in each. Since the applied boundary condition of a restrained edge is essentially an applied strain rate, the large stresses are developed in the viscous member conforming to the boundary condition. At such time when the temperature is increased to cause an appreciable effect on the coefficient of viscosity (the coefficient varies inversely as the temperature) the viscous resistance becomes negligible and the response is one which is mainly elastic. From section 5.2 the thermal stress is governed by the term $\gamma_0 e^{-K T_m} \frac{\partial T_m}{\partial t} + G_0 (1 - \beta T_m) T_m$ so with $\gamma_0 \sim 10^{12}$ and $G_0 \sim 10^6$ the viscous stress is necessarily large for low temperatures.

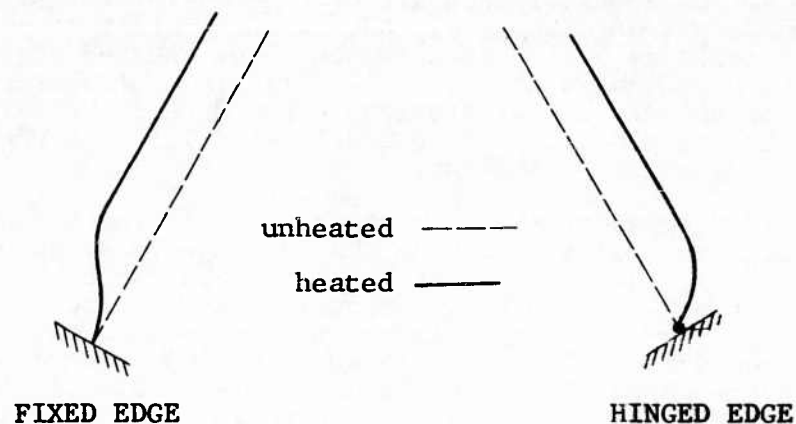
At higher temperatures the behavior of the theoretical curves are similar to that of the experimental, that is, an increasing compressive stress toward the edge increasing also with time.

Since the stresses are calculated at the inner surface they show the effect of the bending moment close to the edge. The theoretical bending moment accounts for the majority of the stress in the meridional direction. The moment changes rapidly with distance from the edge and depends considerably upon the boundary conditions.

The experimental boundary conditions cannot be assumed ideally hinged or fixed. Due to heat conducted from the edge to the mounting plate the edge will expand but not to the extent assumed in the expansion boundary. Also the edge may rotate but is not ideally hinged.

Actual edge conditions would require an analysis investigating a flanged edge.

The comparison in Fig. 7.1-4 and 7.1-5 shows the effect of boundary conditions on the inner surface stress. The two different sets of boundary conditions should bracket the actual situation. In the fixed edge, the meridian may not rotate and the edge may not expand. For the hinged expansion edge, the meridional moment at the base vanishes and the edge may expand with the steel plate ($\alpha_{\text{aluminum}} \sim 14 \times 10^{-6}$, $\alpha_{\text{steel}} \sim 7 \times 10^{-6}$). The fixed case produces tensile stress and the hinged case compressive stress near the edge on the inner surface of the shell in the meridional direction. The effect on the middle surface is shown in the following figure.

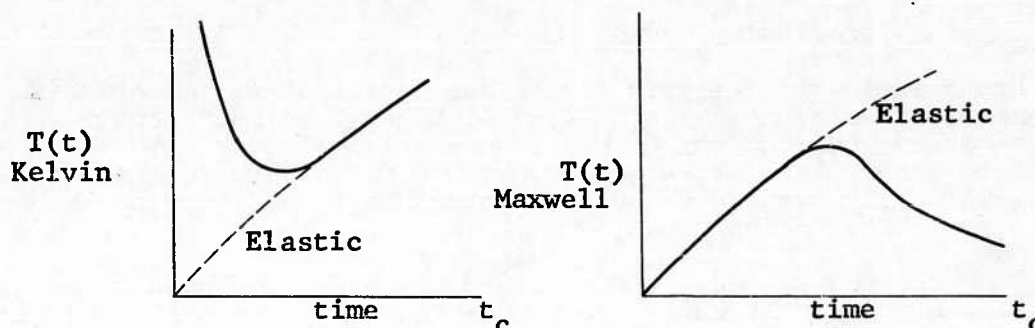


The circumferential stress depends mainly upon the degree of restraint at the edge and is proportional to the difference between the expansion of the shell and the expansion of the mount.

The experimental values compare more favorably with the theoretical stress for the hinged expansion edge. The maximum experimental circumferential stress is about 60% of that for the hinged edge while 40% of that for the fixed edge. The maximum experimental meridional stress is about 75% of that for the hinged edge while the fixed edge stress is of the same order of magnitude but of opposite sign. While the experimental stresses should fall between those of the two boundary conditions it is seen that the measured stress is still lower than either. The Kelvin-Voigt body does not consider the relaxation of stress at constant strain at high temperatures or the yielding of the material for high stress.

6.6 Comparison of the Maxwell and Kelvin-Voigt Bodies

In Fig. 6.6-1 and 6.6-2 the viscoelastic response of a 1/16" thick cone subjected to 20°F/sec to 500°F is compared for a Maxwell body and a Kelvin-Voigt body. The early response of the Kelvin-Voigt body is, again, of a highly viscous nature while the Maxwell body response is essentially elastic. At higher temperatures when the Kelvin-Voigt body becomes elastic the Maxwell response is fluidic due to the applied strain rate being absorbed in the viscous component. The theoretical stress is essentially the product of a function of slant length and a function of time. With the function of slant length being the same for both bodies the time function is presented in the following figure:



Moreover, at the steady state condition the Kelvin-Voigt body maintains a constant stress while the Maxwell body will decay exponentially.

Compared in Fig. 6.6-3 and 6.6-4 are the results of the temperature dependent properties analysis, the temperature independent properties analysis, and the temperature dependent properties elastic analysis. The property values for the temperature independent analysis were chosen at a temperature corresponding to 10 seconds. The response of the temperature independent analysis depends upon the choice of the coefficient of viscosity; if the temperature is low the response will be viscous and if the temperature is sufficiently high the response will be elastic.

The variation of the elastic solution from that of the viscoelastic solution at high temperatures is due to the choice of Poisson's ratio being 1/3 for the elastic analysis and 1/2 for the viscoelastic. Again the experimental stress is lower than any theoretical prediction which is believed due to the obscure boundary conditions and a yielding at the edge.

VII. SUMMARY AND CONCLUSIONS

The experimental results in some instances are in disagreement with the temperature dependent viscoelastic analysis. This is generally true for the early transient stage and the steady-state stress conditions. In an effort to explain these differences, a number of questionable elements of both the theoretical analysis and the experimental investigation were studied. These points are discussed in the following paragraphs, and when possible, recommendations to eliminate the individual effects are given.

7.1 Theoretical Analysis

7.1.1 Viscoelastic models

Employed in the analysis is the assumption of the material's response as that of a simple Kelvin-Voigt body. A much better approximation would be the development of a model consisting of a number of Kelvin-Voigt elements in series. The difficulties in the analysis during the early transient and the steady-state condition would benefit from a more descriptive model.

From the comparisons in Fig. 6.6-3 and 6.6-4 it appears that the elastic analysis would sufficiently describe the stress response in the models tested as compared to the viscoelastic analysis presented. However, neither analysis predicts the steady-state response.

7.1.2 Effect of material properties

In the present investigation, an empirical expression for η was established by employing some limited data established by ALCOA at higher temperature ranges for aluminum alloys. Because the viscosity/temperature relationship is based on such limited data, three values were used in the calculations. The effect of varying η is demonstrated in Fig. 7.1-1 and 7.1-2. Changes of an order of magnitude are employed. As seen from the figures, the only effect is to advance or retard the transition from a viscous response to an elastic response.

Although some error may exist in the choice of G and β , it is felt that both are minor and have little effect upon the results.

RECOMMENDATION: More reliable data on the dependance of thermal properties of alloys with temperature are required for accurate analyses. In particular, viscosity/temperature data on alloys below the melting point should be established.

7.1.3 Inaccuracies in the applied boundary conditions

Agreement of the boundary conditions employed in the analysis with those actually produced in a test program is difficult to achieve. Although such disagreement is present, it is felt that the effect is localized to the edge and apex (in the case of cones) regions as far as the stresses in the shell are concerned.

7.2 Experimental Study

Aside from the usual sources of errors encountered in experimental stress investigations and data reduction, the investigators feel that the following items are major possible contributors to the existing disagreement between theoretical and experimental results.

7.2.1 Inaccuracy in correcting strain data to eliminate temperature effects in strain gages

There is much to be done to perfect high temperature strain measuring techniques. The range of temperatures of interest is subject to a number of technical problems both in terms of model preparation and testing procedures for $T > 400^{\circ}\text{F}$. In fact, the experience gained in this investigation indicates that it requires six to eight months to develop techniques for such an investigation.

RECOMMENDATION: High temperature strain gages which require less complicated application procedures are needed. These strain gages should be temperature compensating. Since the completion of the present investigation, other high temperature strain gages have been brought to the attention of the investigators. It would be of interest to note whether these newer developments eliminate the difficulties of application, curing, and measuring as experienced in this investigation.

7.2.2 Effect of model support on temperature distribution

The steel plate which supported the oven and the model had considerable effect on the model temperature near the support. Although steps were taken to eliminate the plate as a heat sink, its effects are evident in the experimental results. The existence of such a heat sink is perhaps the major difficulty of the equipment. It created an undesirable thermal gradient along the shell length. Otherwise the equipment performed well. The equipment was not employed to the maximum capacity.

RECOMMENDATION: For more effective use of the present equipment design, a ceramic coating should be used at the shell-support connection.

7.2.3 Imperfection of the Model

In Appendix A.3, the effect of geometrical irregularities of the initial shape are discussed.

The 1/32" thick conical shells were more likely to contain initial imperfections. This probably accounts for the poorer correlation between analytical and experimental results. However for the thicker models, not only was better correlation achieved but in several cases there was good agreement of analytical and experimental results.

On the basis of these qualitative results, the following recommendation is made:

RECOMMENDATION: Study should be made of surfaces of revolution which have unsymmetrical initial imperfections.

7.3 General Comment

In the foregoing discussion, several important aspects of the investigation are outlined. In connection therewith, the investigators feel that the results are a step in the solution of the complicated viscoelastic shell problem. Nevertheless, in addition, several other important aspects of the analysis bear further investigation; these include:

1. Extension of the theory to include the effect of internal ring- and meridional-frames, and other surface constraints.
2. Extension of the method to include viscoelastic and/or elastic layered systems of two and three layers. The use of outer liners should be investigated.
3. Future investigations might also include both unsymmetrical body forces and temperature distributions.

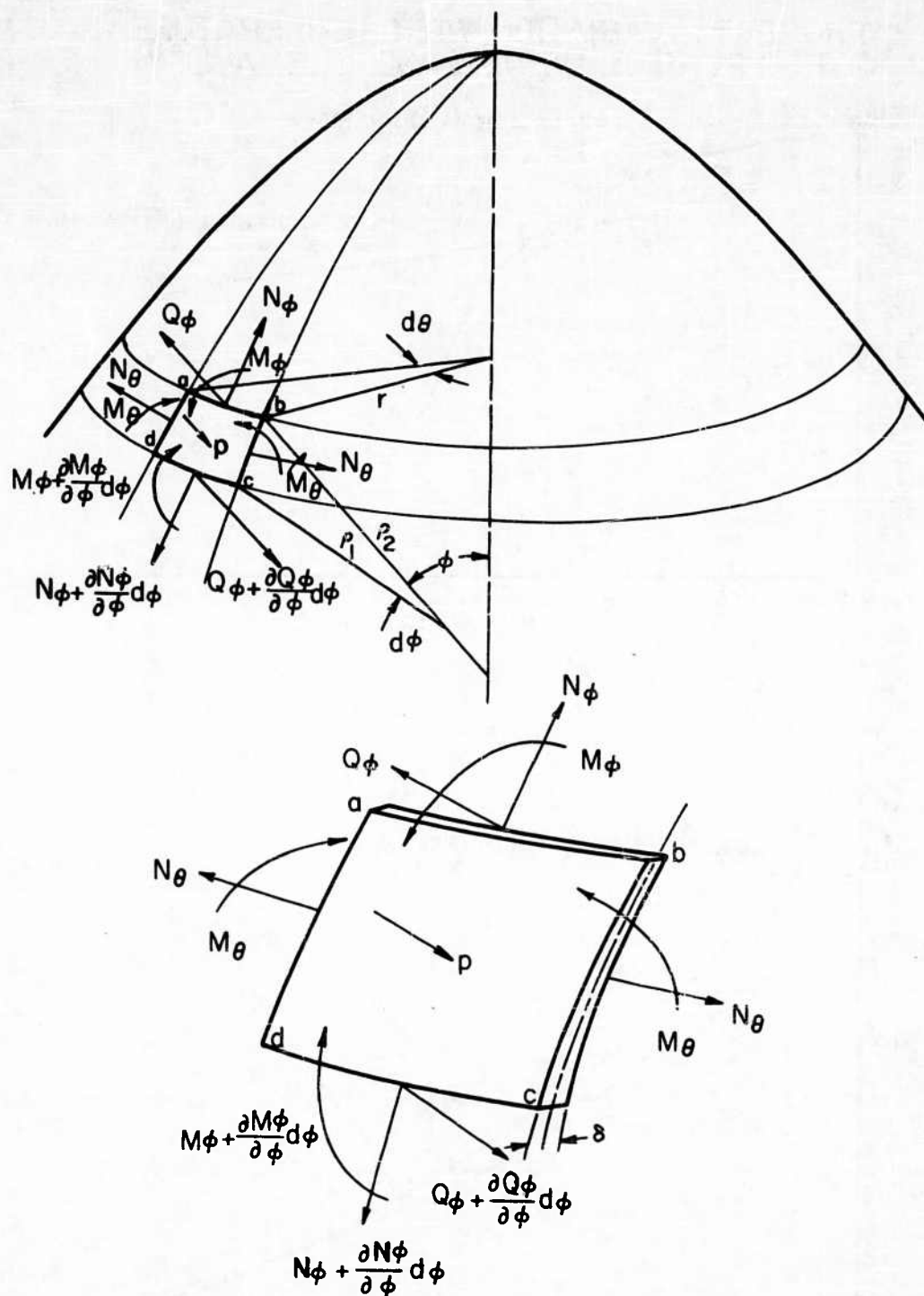
It is needless to say that each of these endeavors are entire subjects within themselves.

In connection with Item 2, it is felt that the present method is too complicated to be applied to layered systems. For this reason, the investigators studied the application of energy methods to the solution of viscoelastic shell problems.

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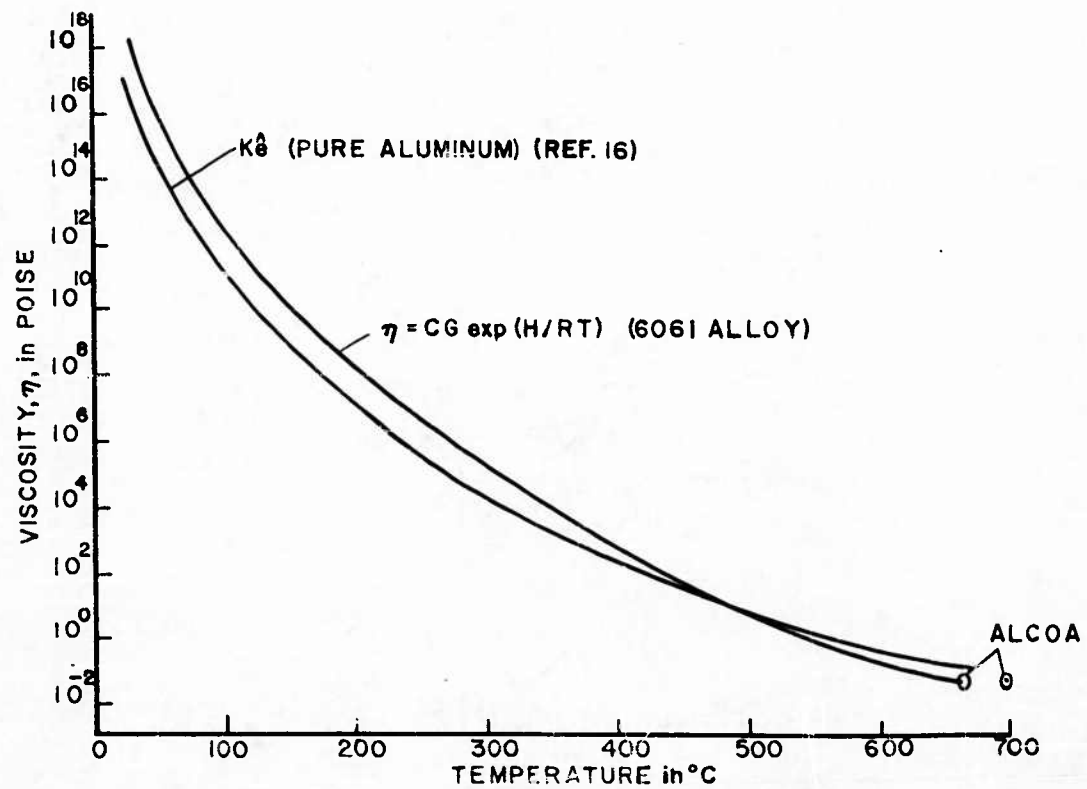
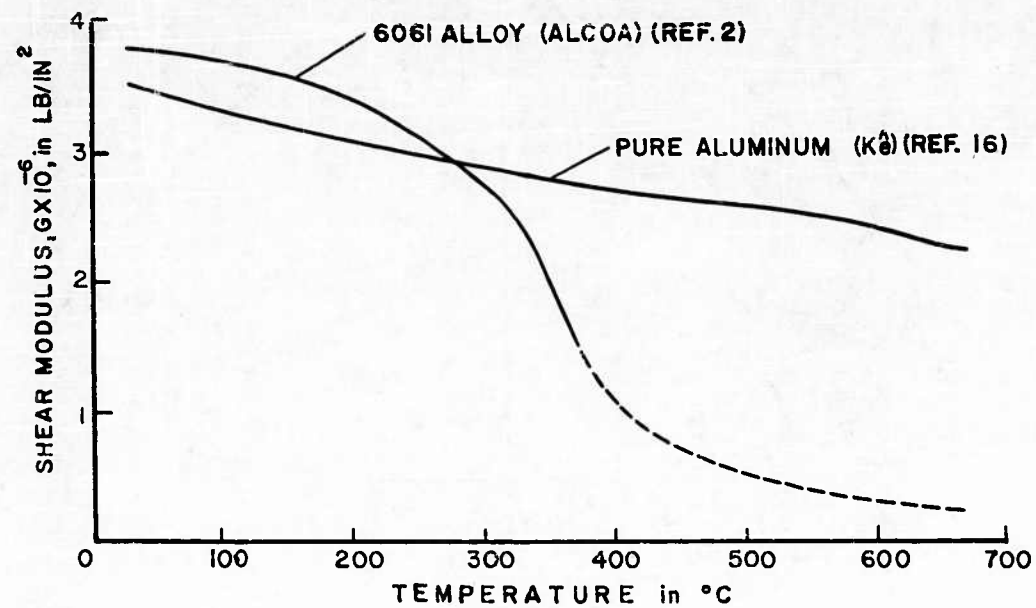
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SHELL MOMENTS AND FORCES
FIG. 2.1-1

Fig. 2.1.3-1
PROPERTY VARIATION WITH TEMPERATURE
FOR PURE ALUMINUM AND 6061 ALLOY



PROPERTY VARIATION WITH TEMPERATURE FOR
PURE ALUMINUM AND 6061 ALLOY

Table 2.1.3-1

T (°C)	G x 10 ⁻⁶ (psi)		(poise)	
	pure aluminum*	6061 alloy**	pure aluminum*	6061 alloy
25	3.48	3.80	2.2x10 ¹⁶	8.15x10 ¹⁷
100	3.34	3.78	2.1x10 ¹¹	4.22x10 ¹²
200	3.04	3.44	1.3x10 ⁷	1.45x10 ⁸
285	2.90	2.85	4.8x10 ⁴	3.66x10 ⁵
350	2.76	2.10	2.0x10 ³	9.30x10 ³
450	2.61	0.65#	43.0	52.9
550	2.47	0.35#	2.2	1.38
660	2.18	0.20#	0.18	0.06**
670	2.18	--	0.14	--
700	--	--	--	0.055**

* Estimated by Kê. Refer to Kê, Ting-sui: Experimental Evidence of the Viscous Behavior of Grain Boundaries in Metal, Physical Review, Vol. 71, pp. 533-546, 1947.

** ALCOA values.

Estimated from G-T curve.

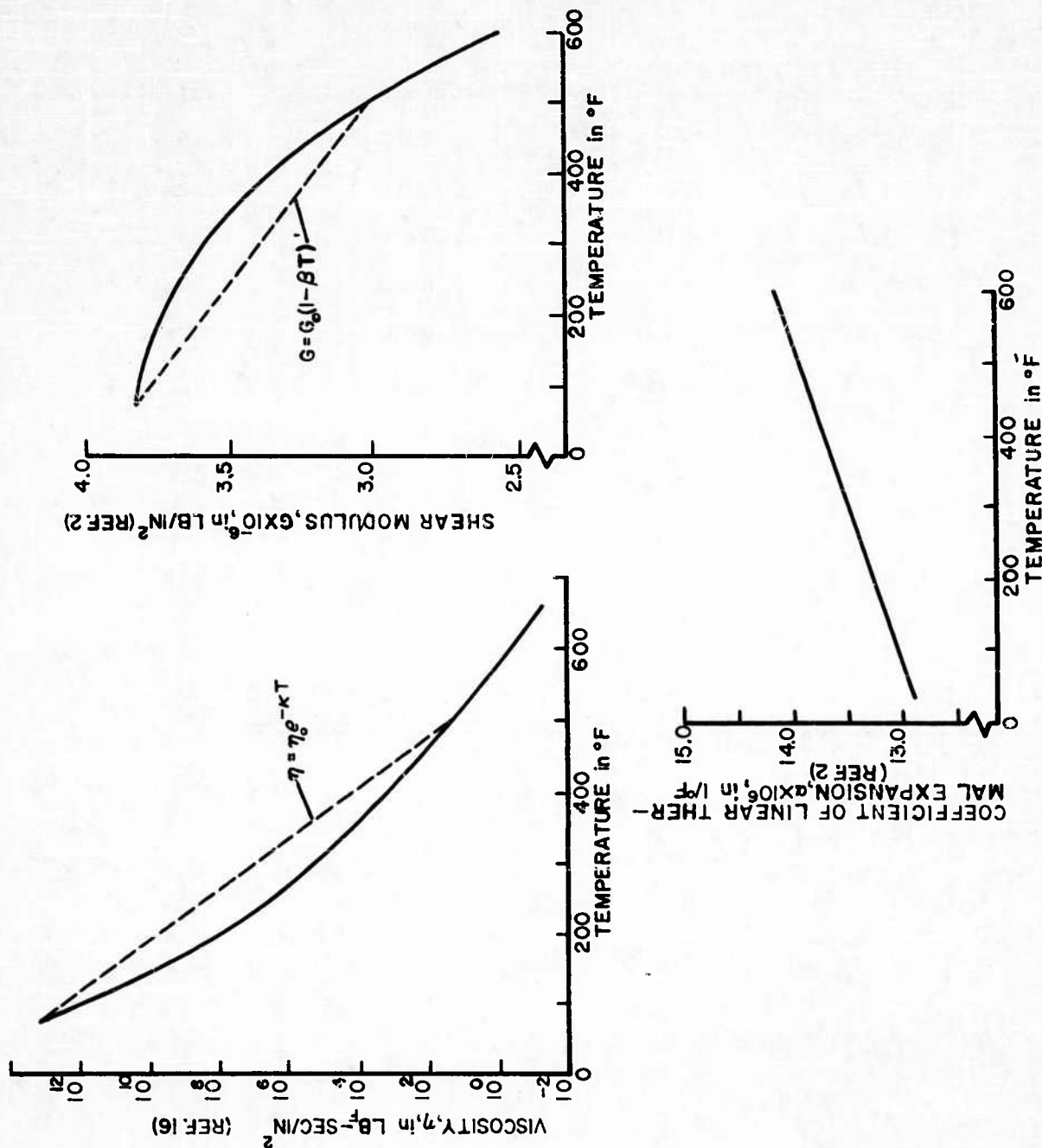
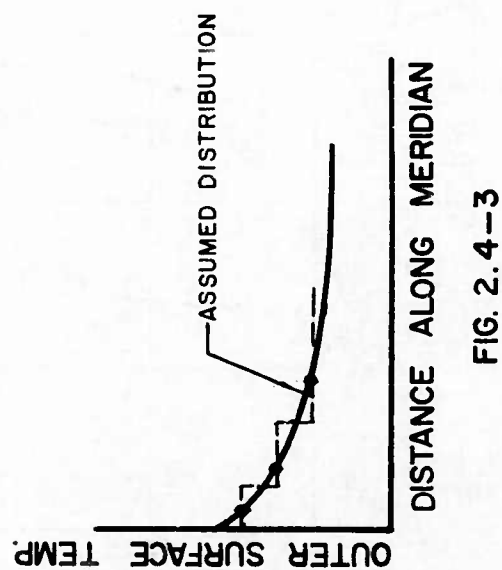
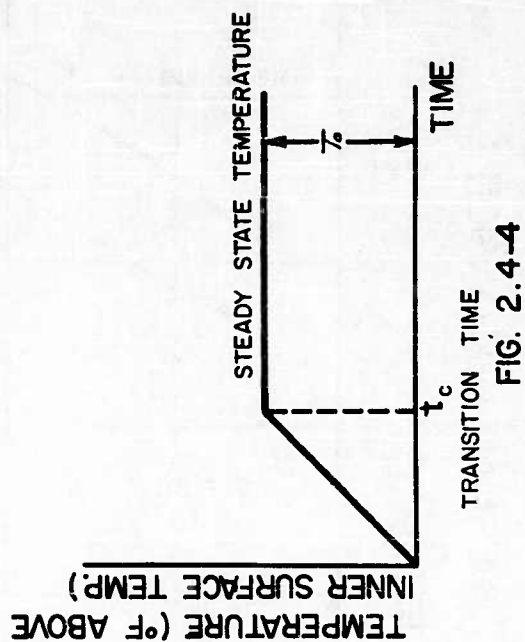
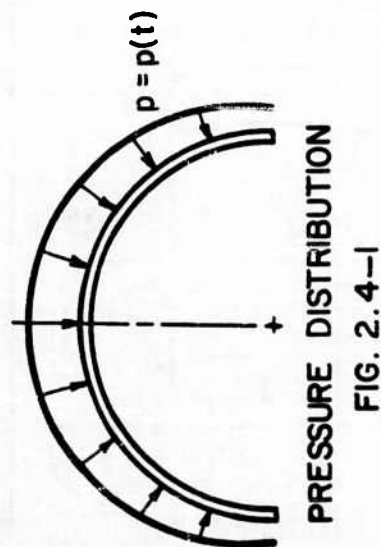
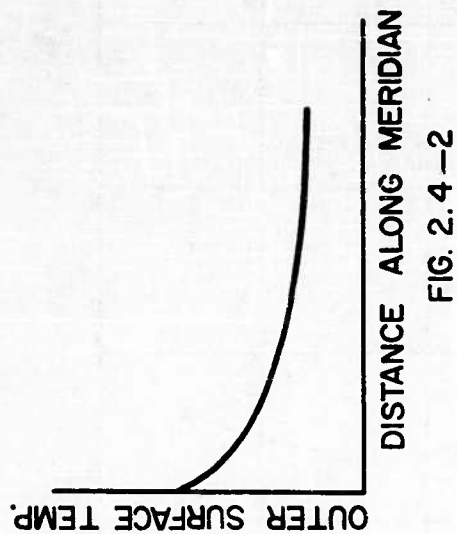


Fig. 2.1.3-2
 MATERIAL PROPERTY VARIATION WITH TEMPERATURE FOR 6061 ALLOY



MEAN COEFFICIENT OF THERMAL EXPANSION

(IN./IN./°F x 10⁶) REF. 4.

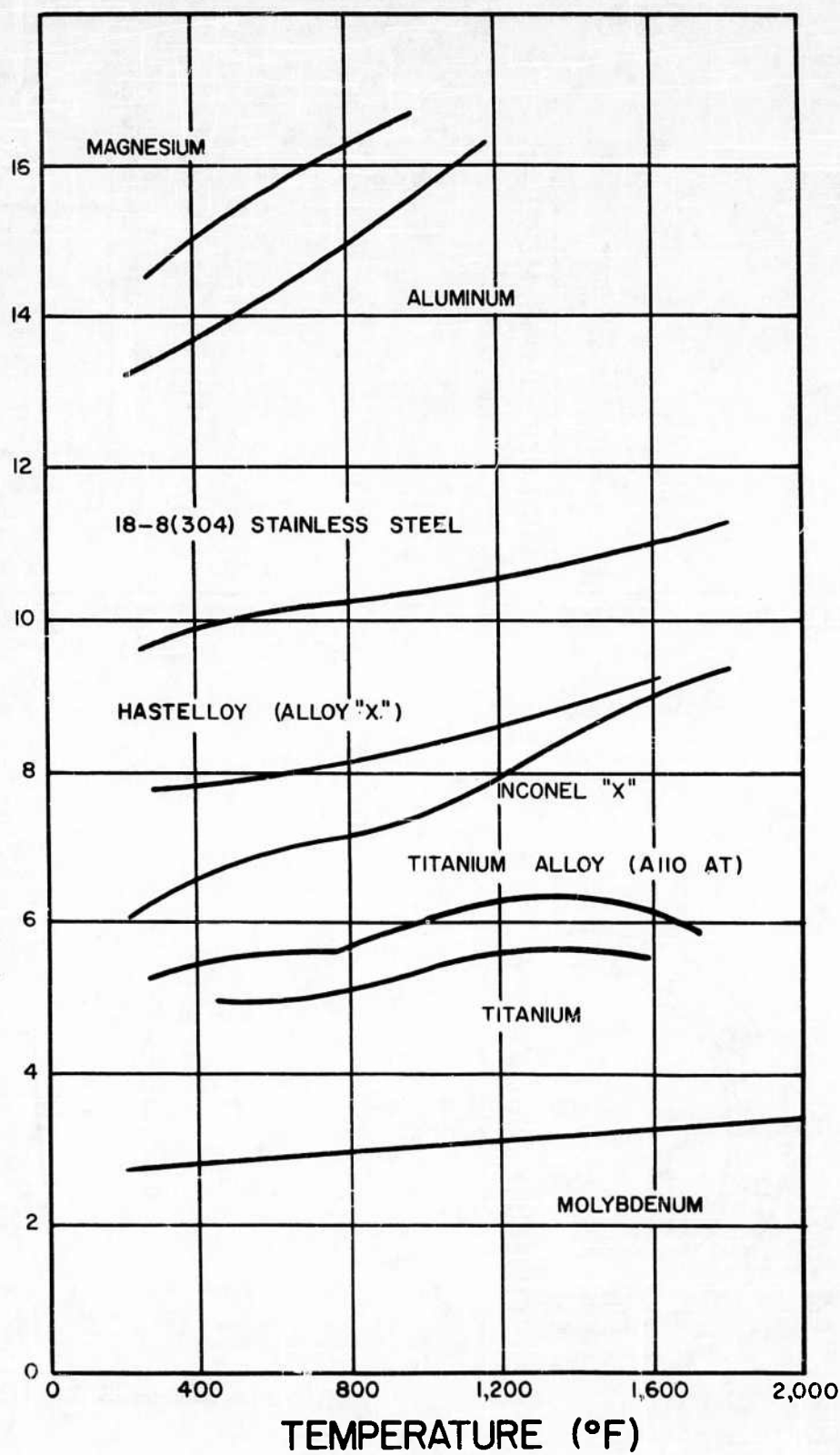


FIG. 2.6-1

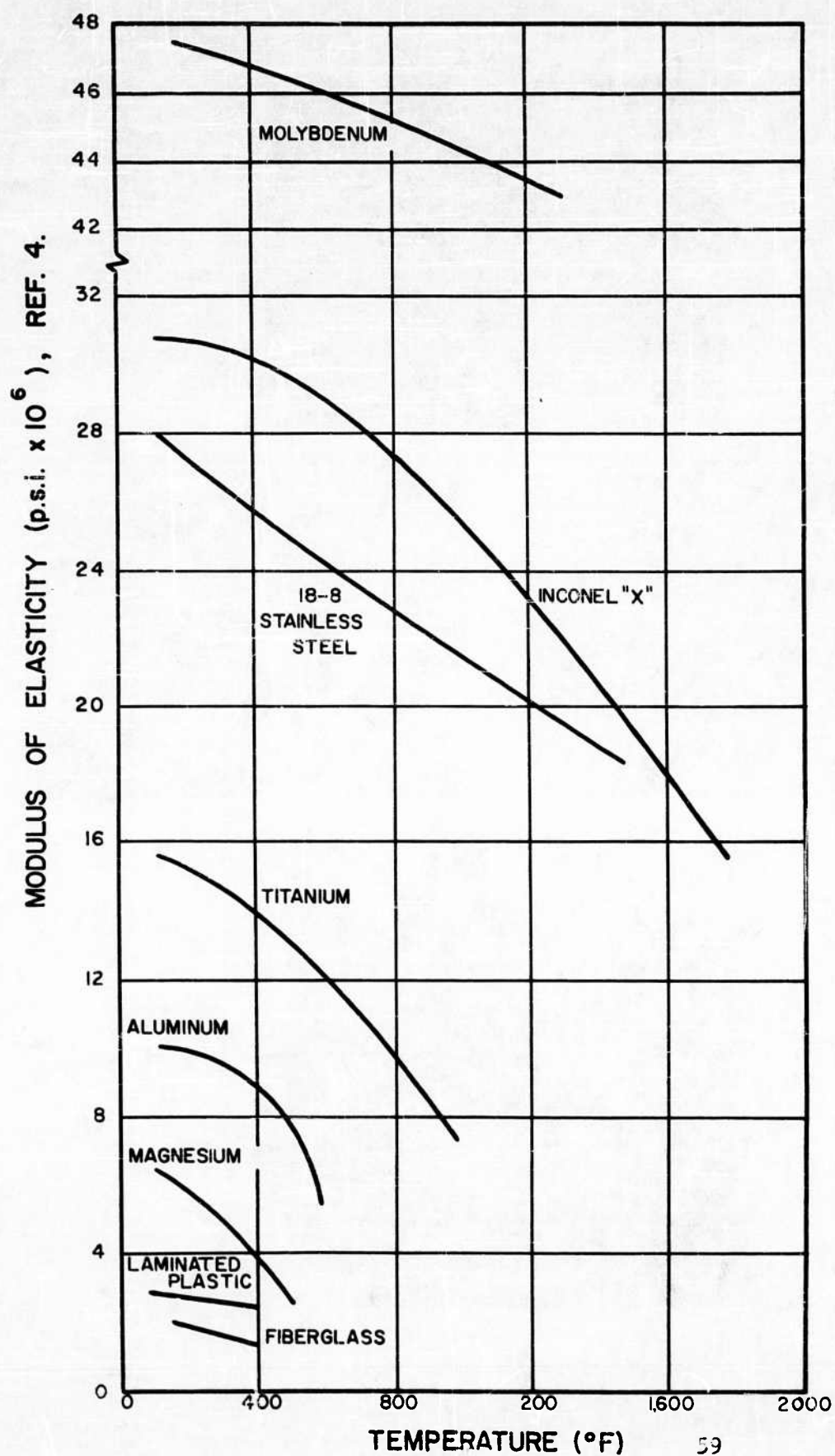


FIG. 2.6-2

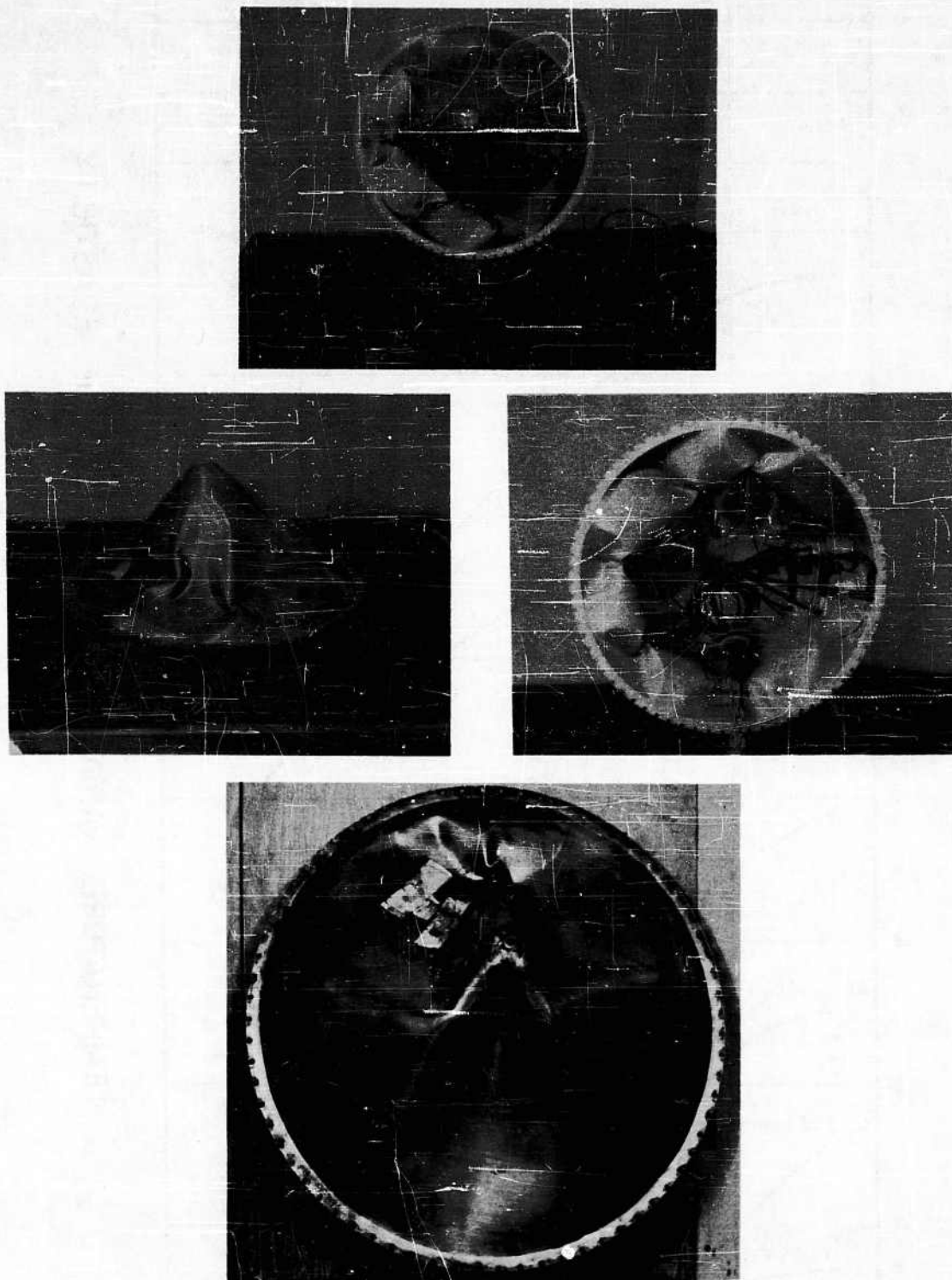
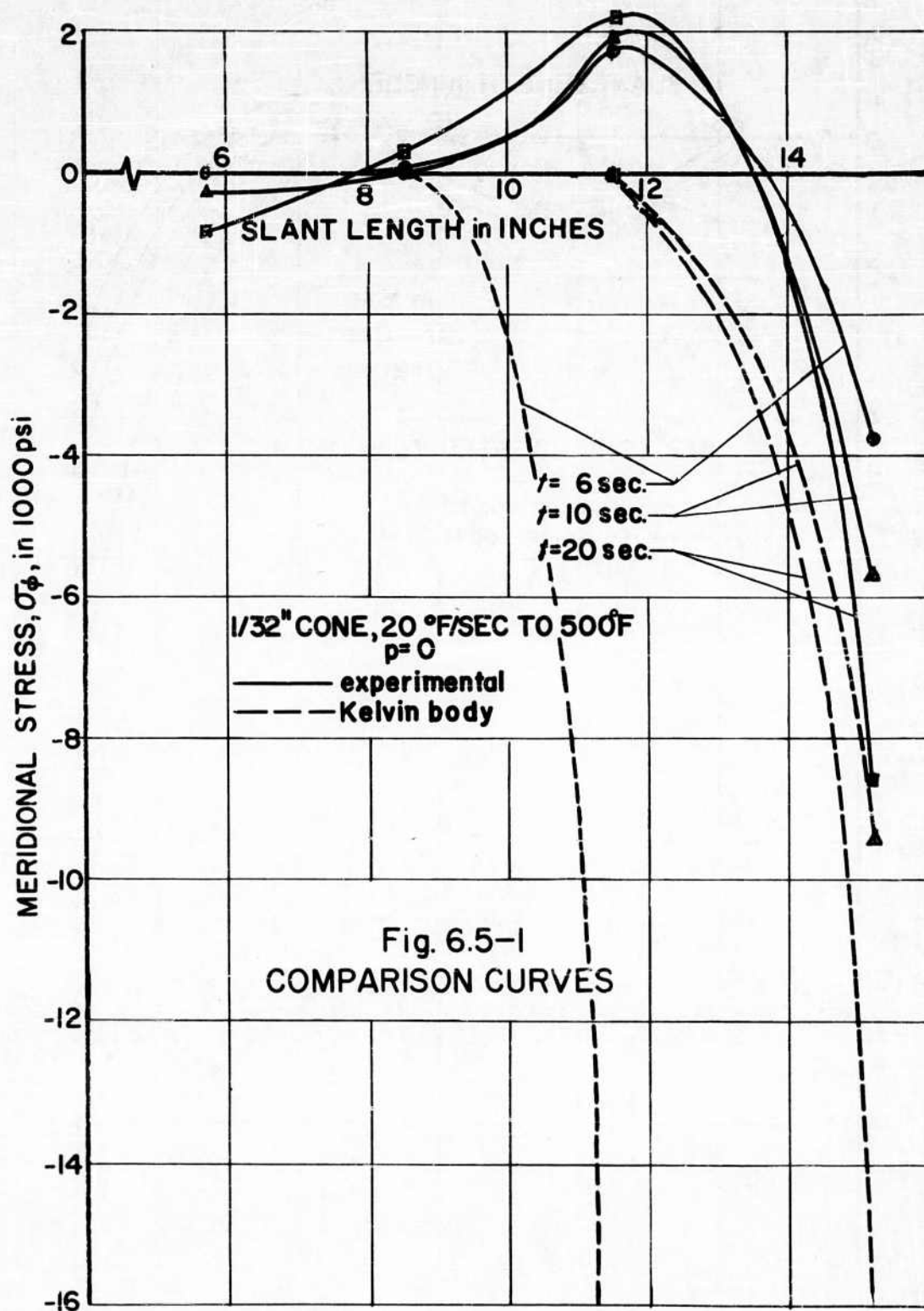
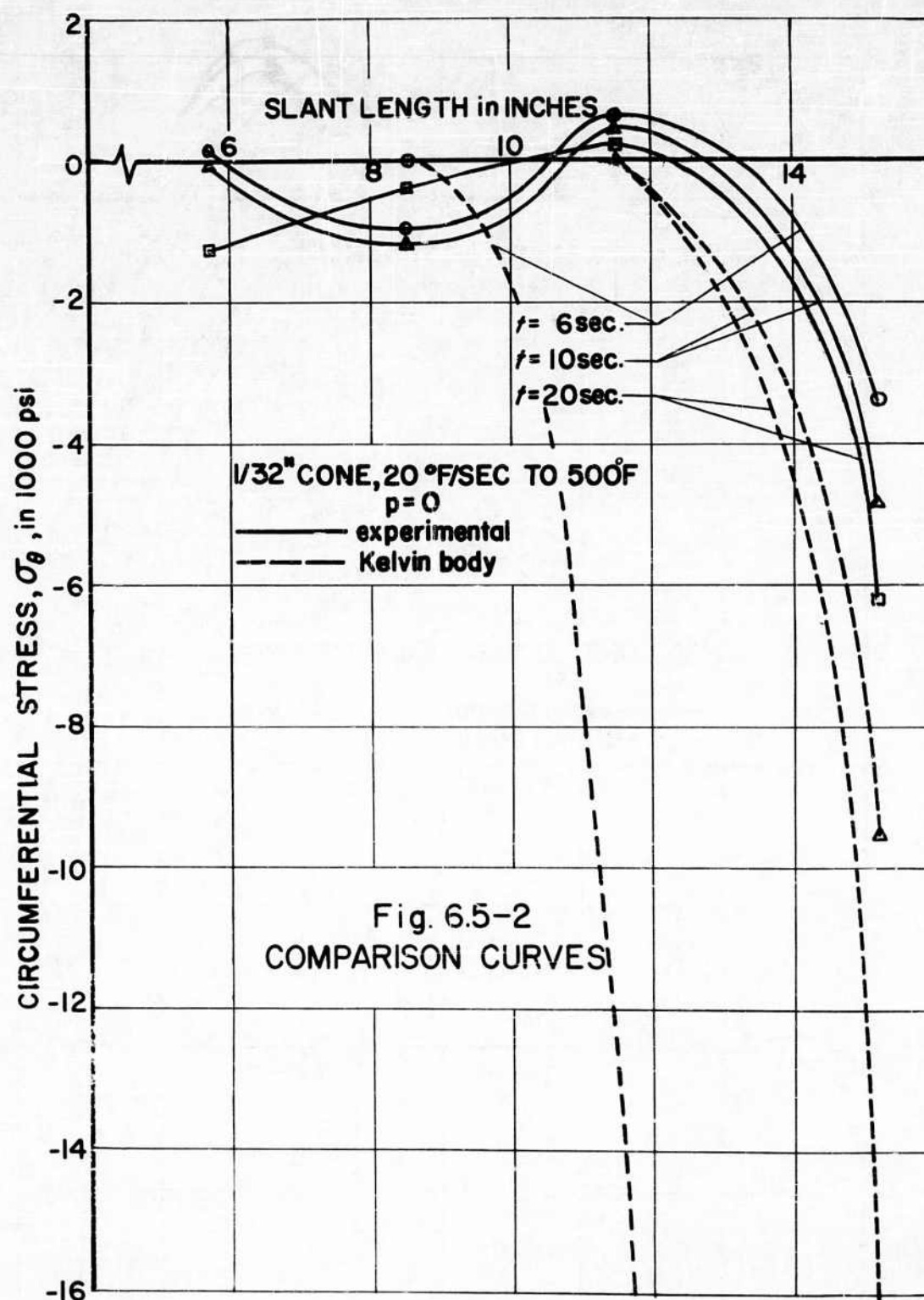
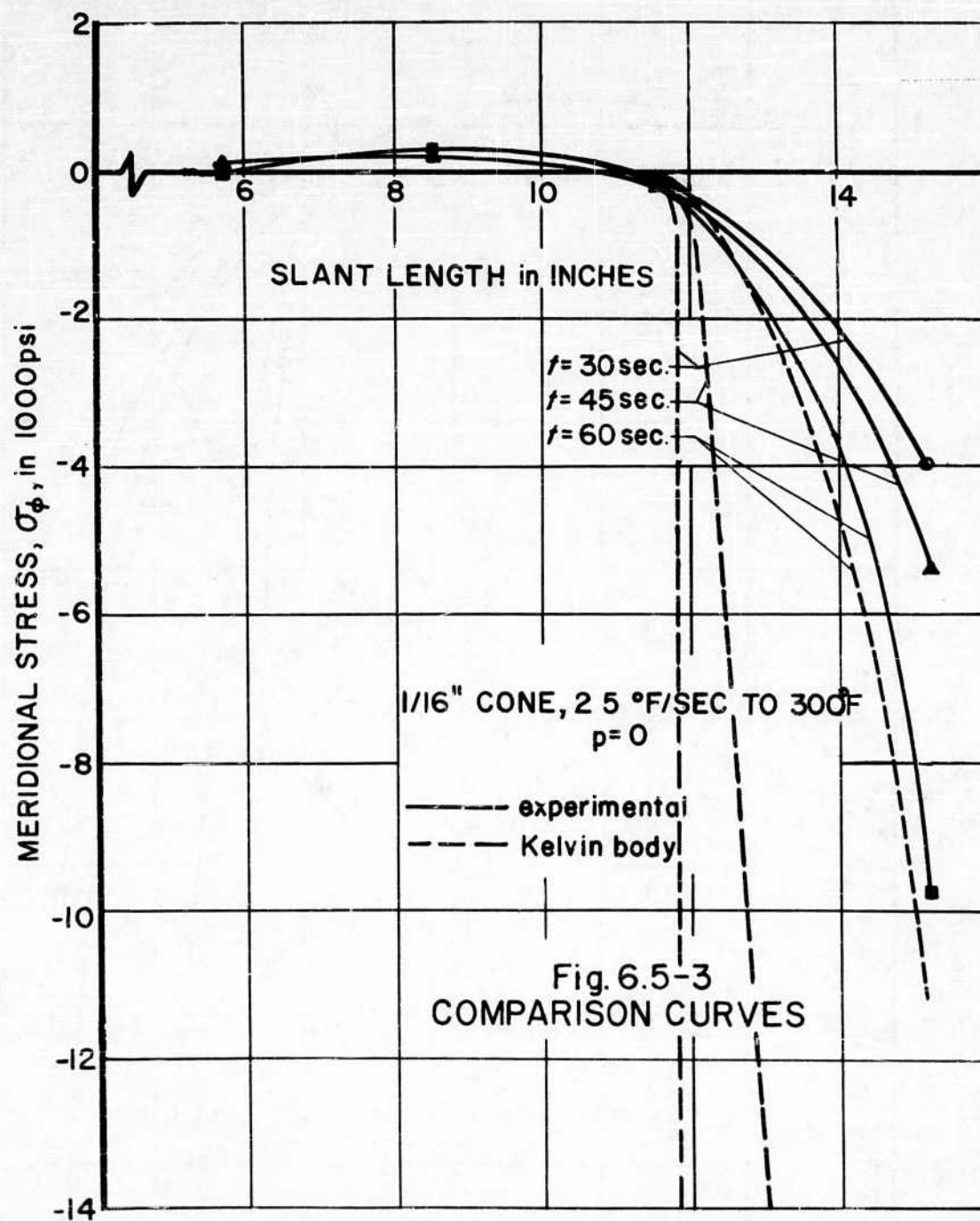
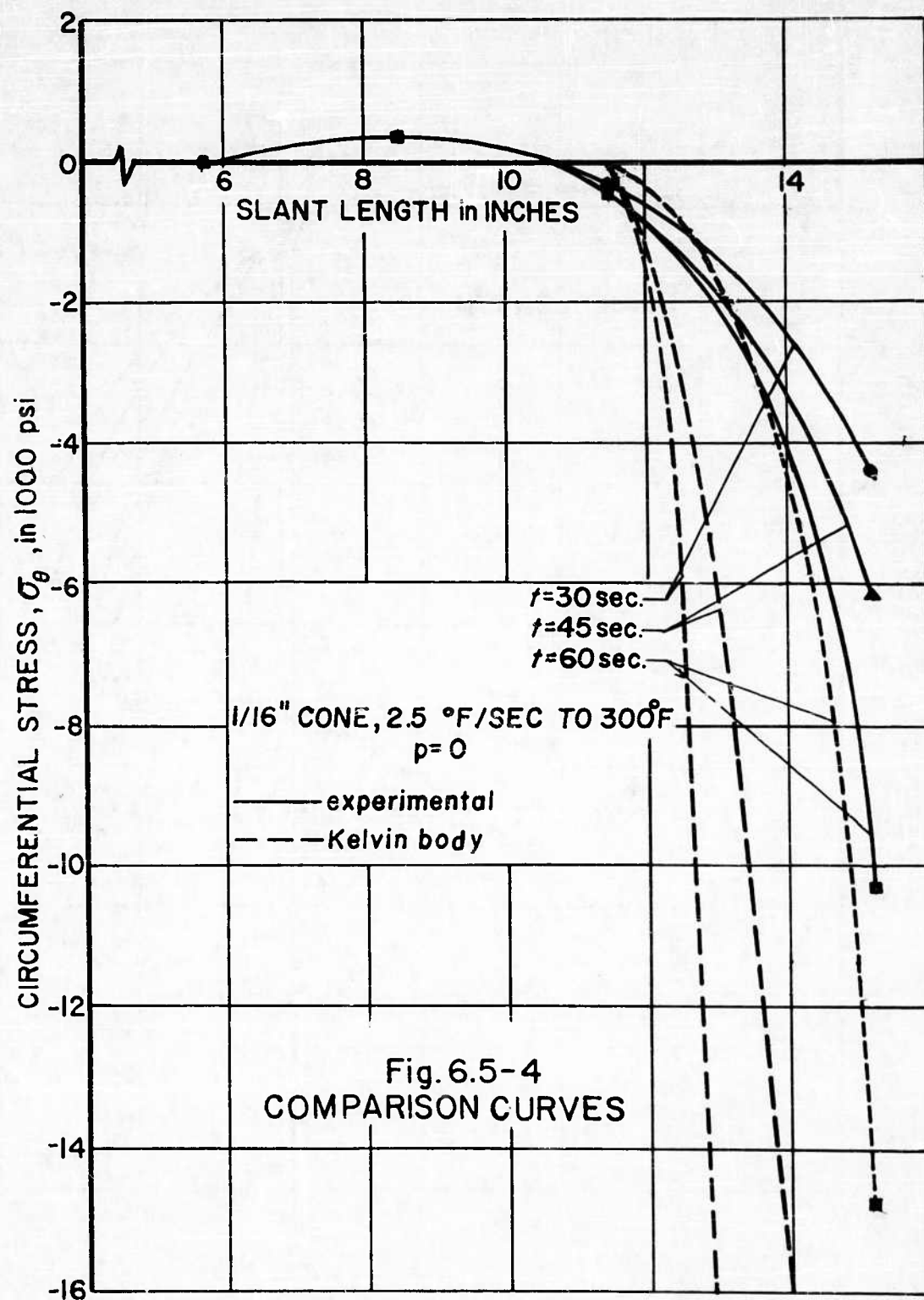


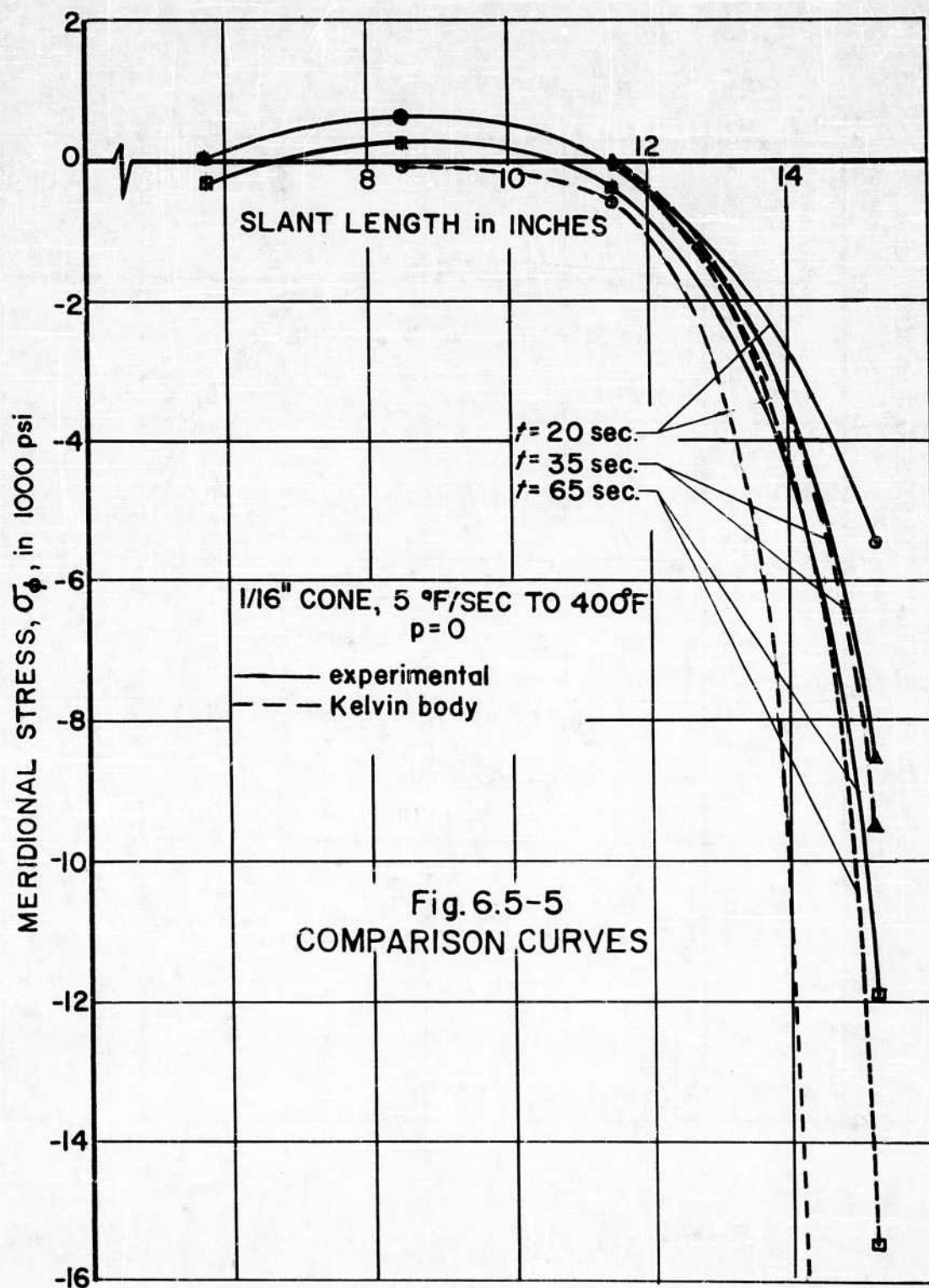
Fig. 6.3-1 Collapsed 1/16" Thick Conical Shells

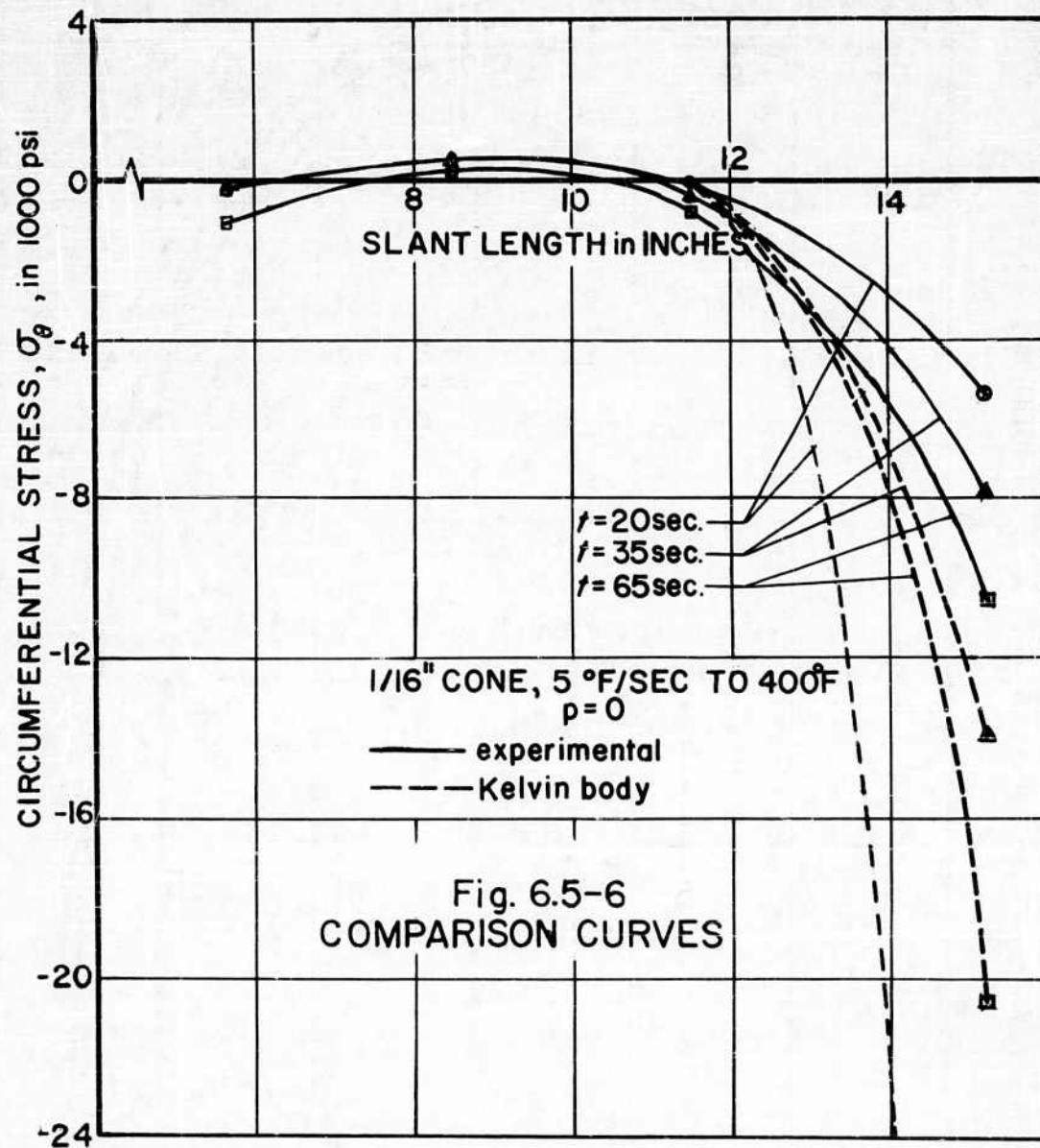


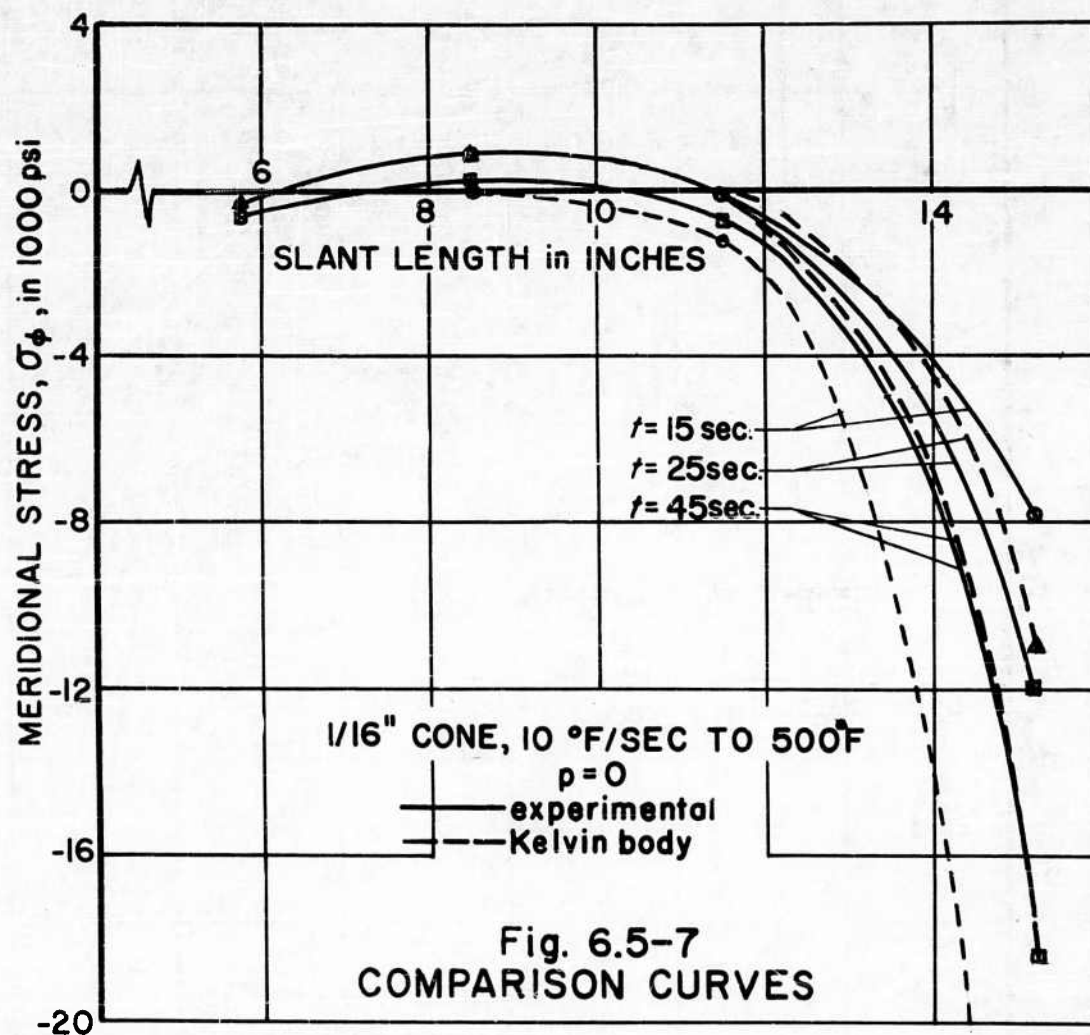


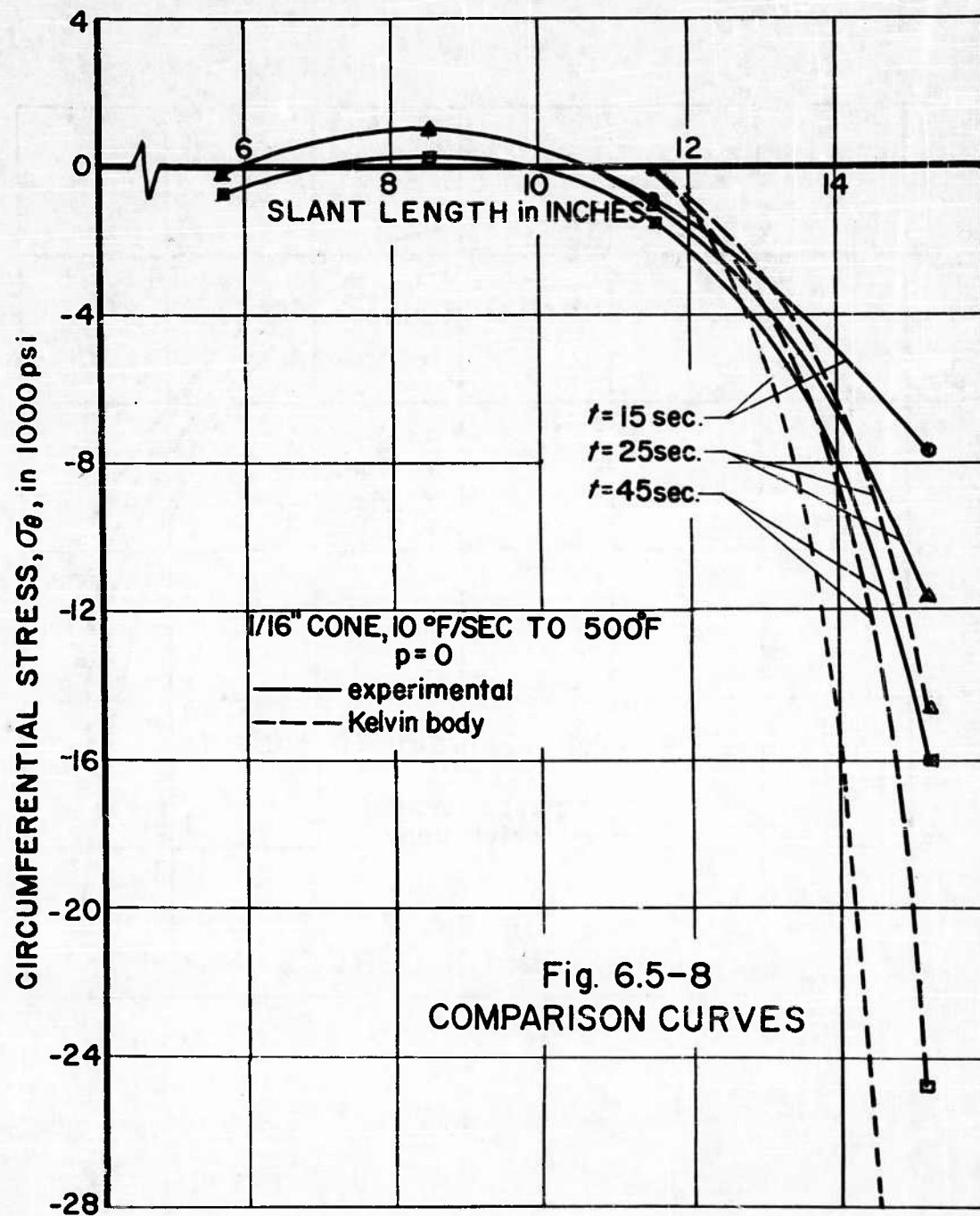


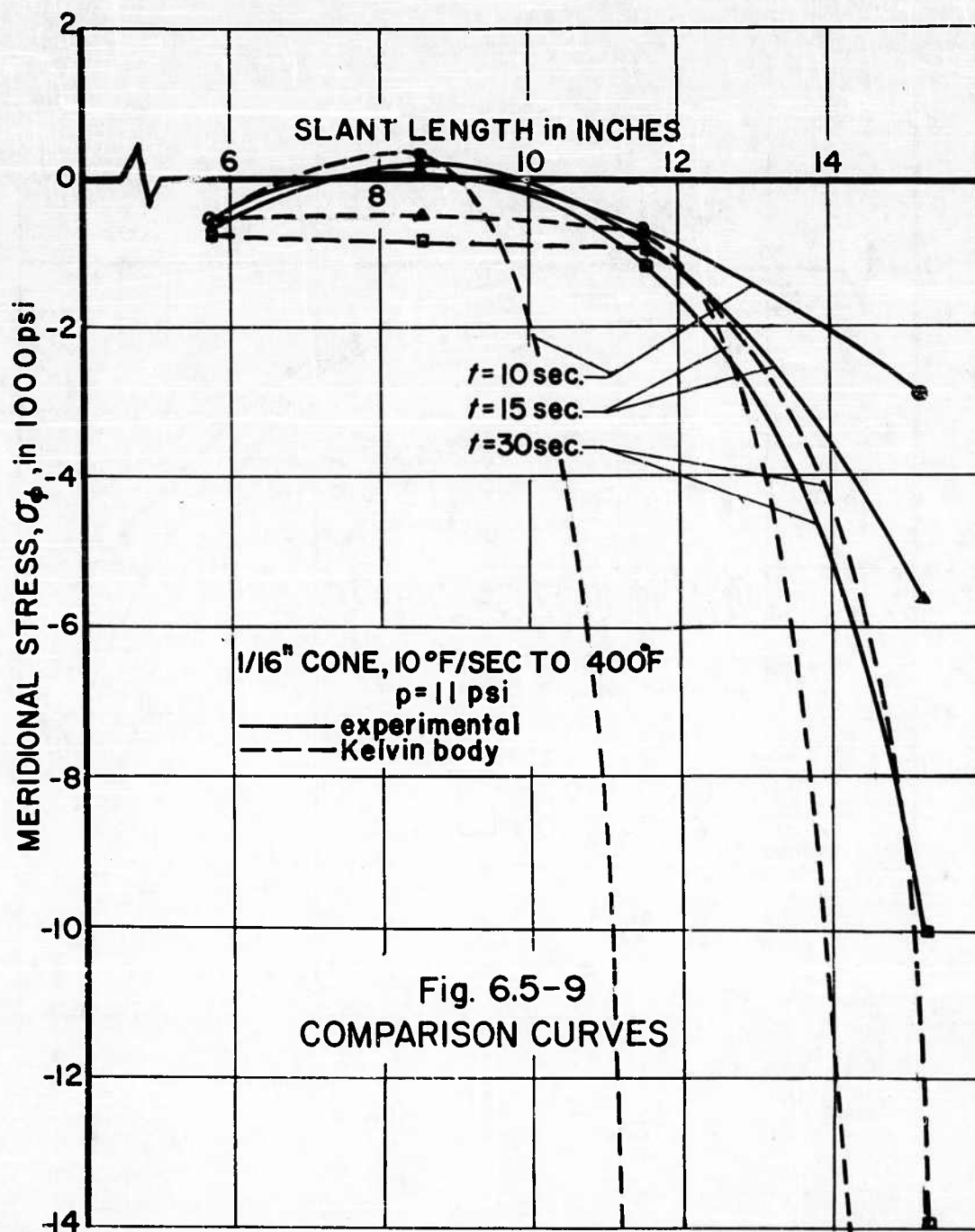


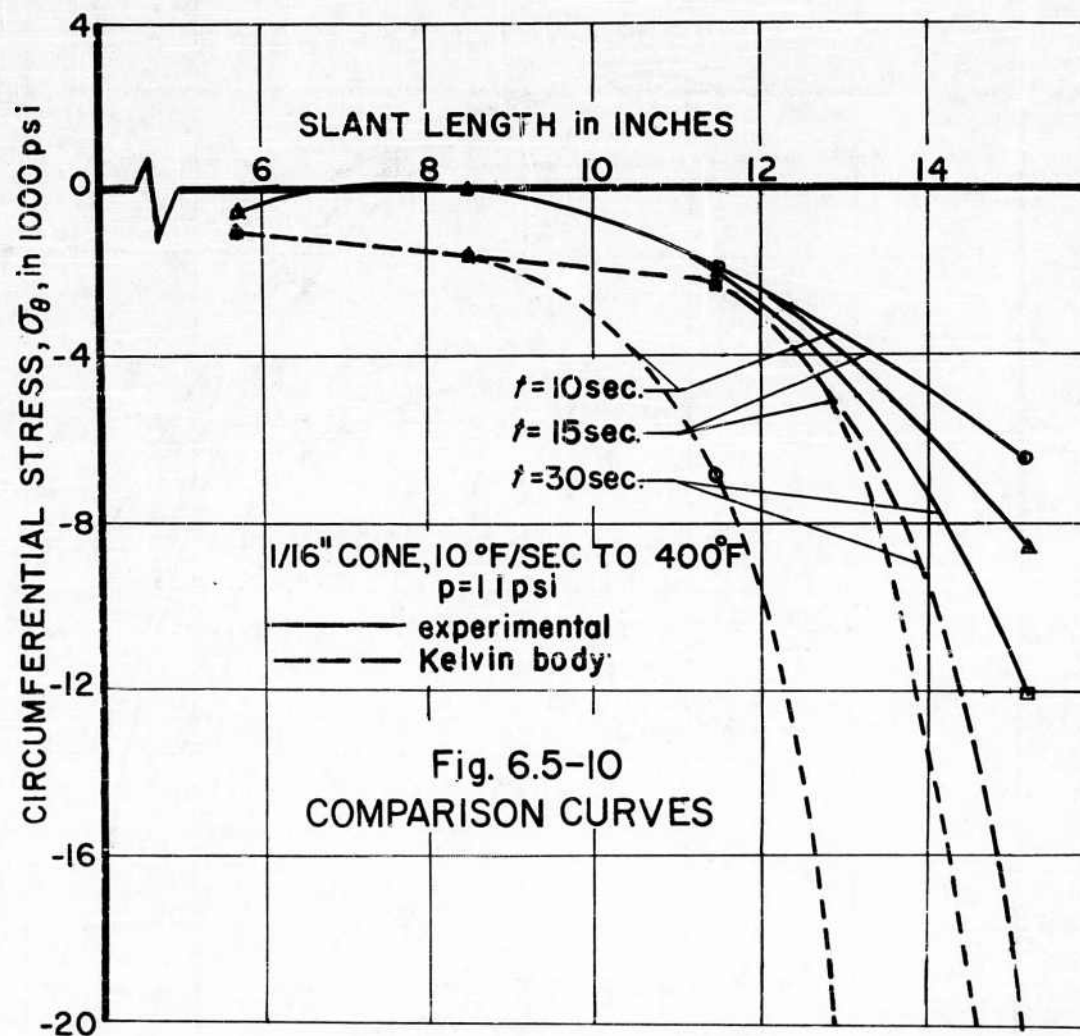


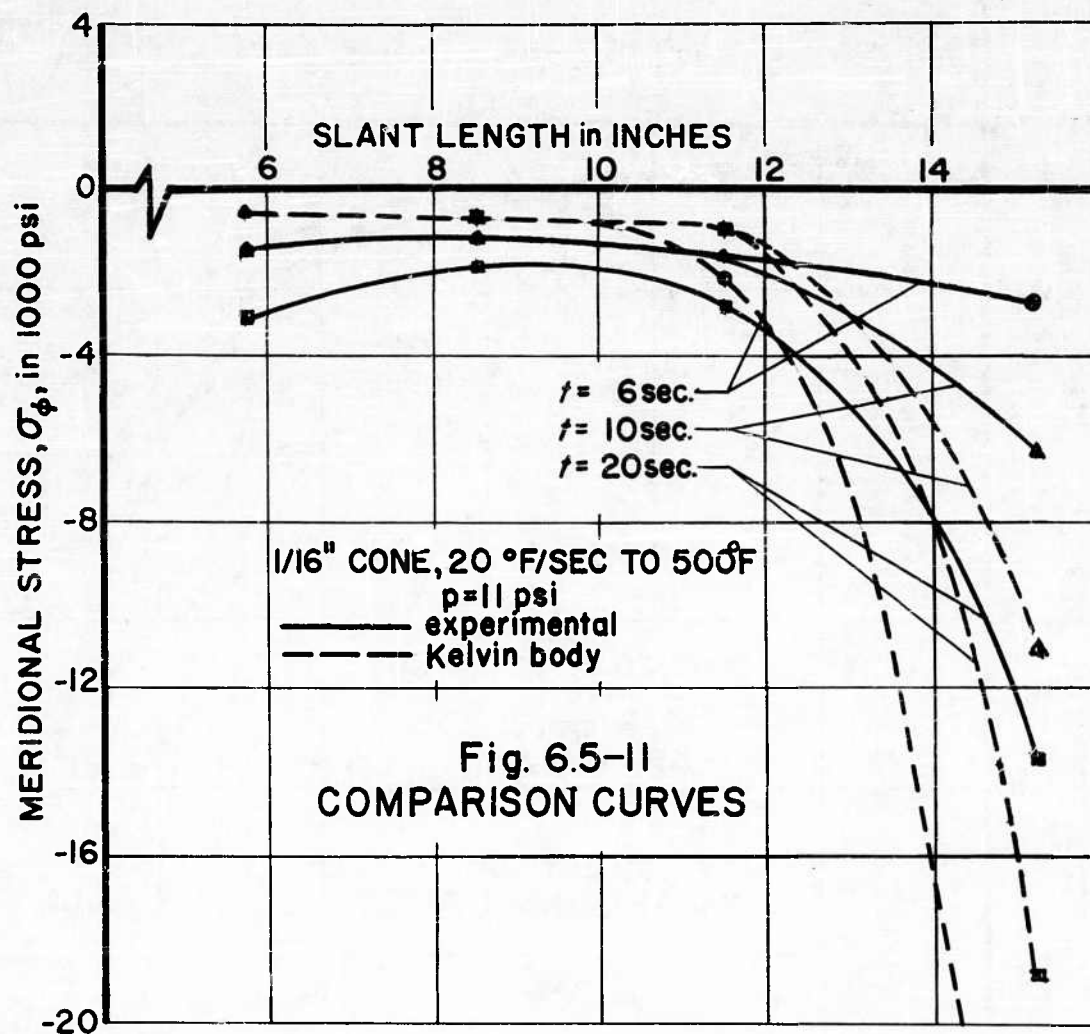


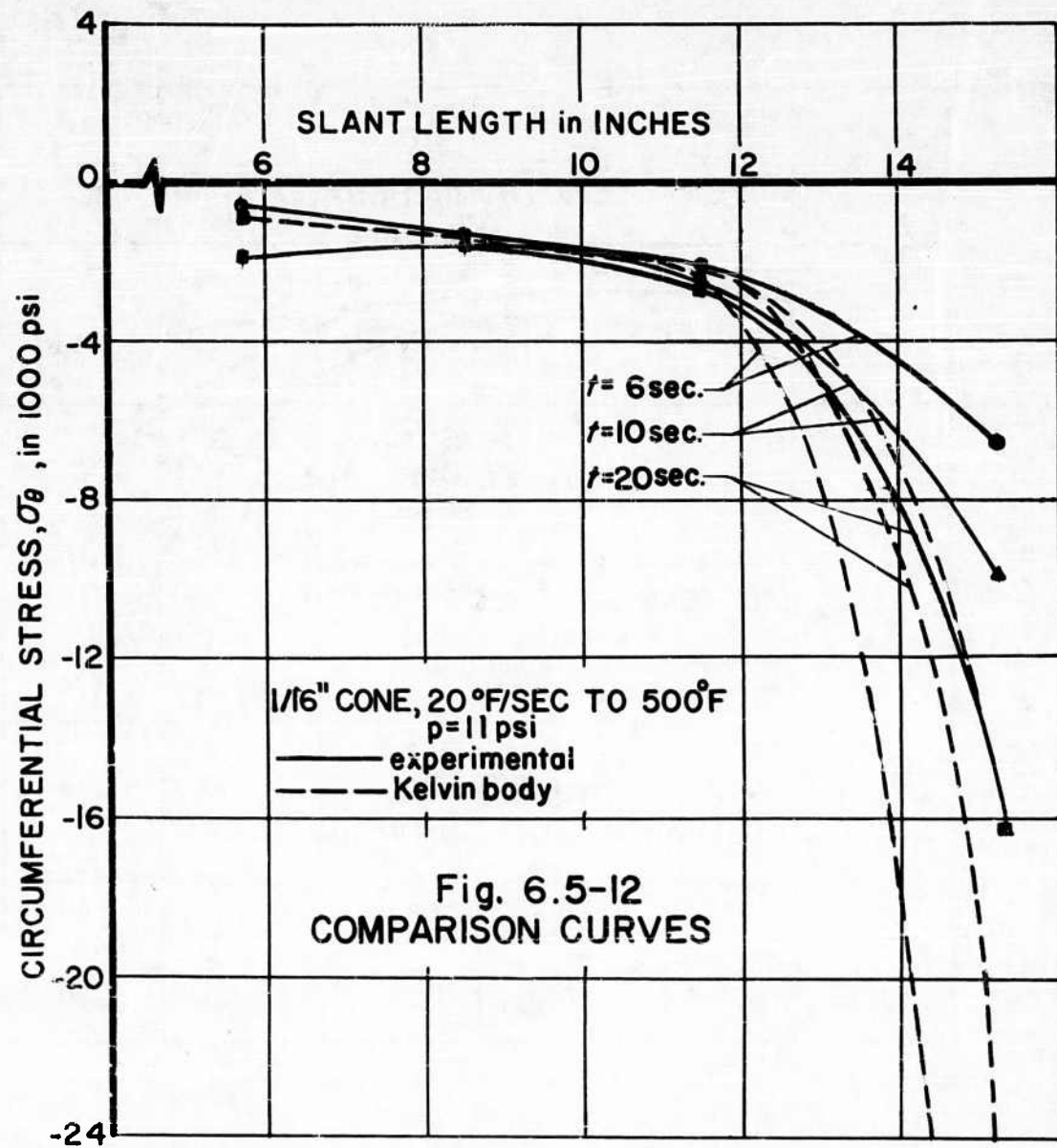


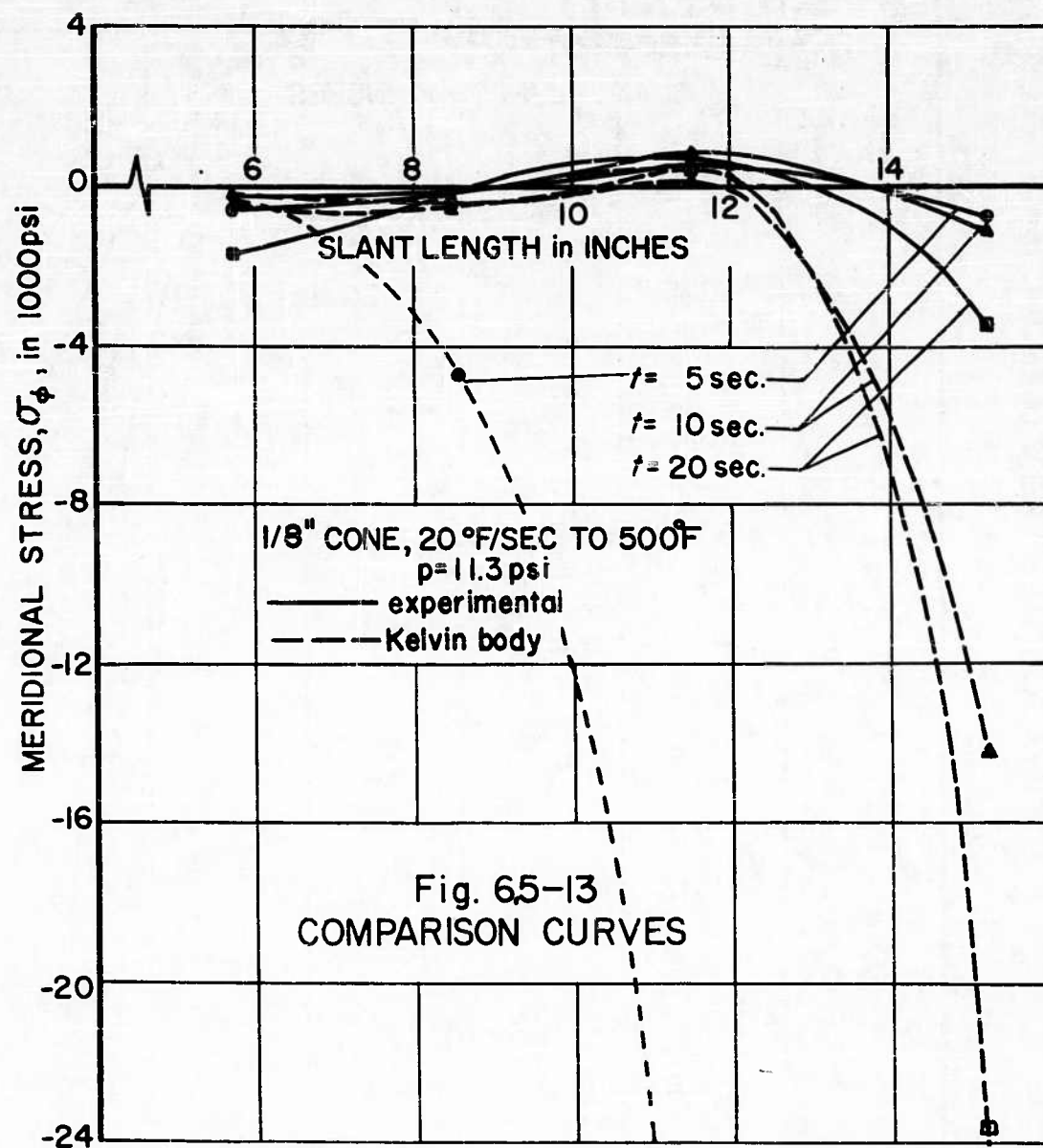


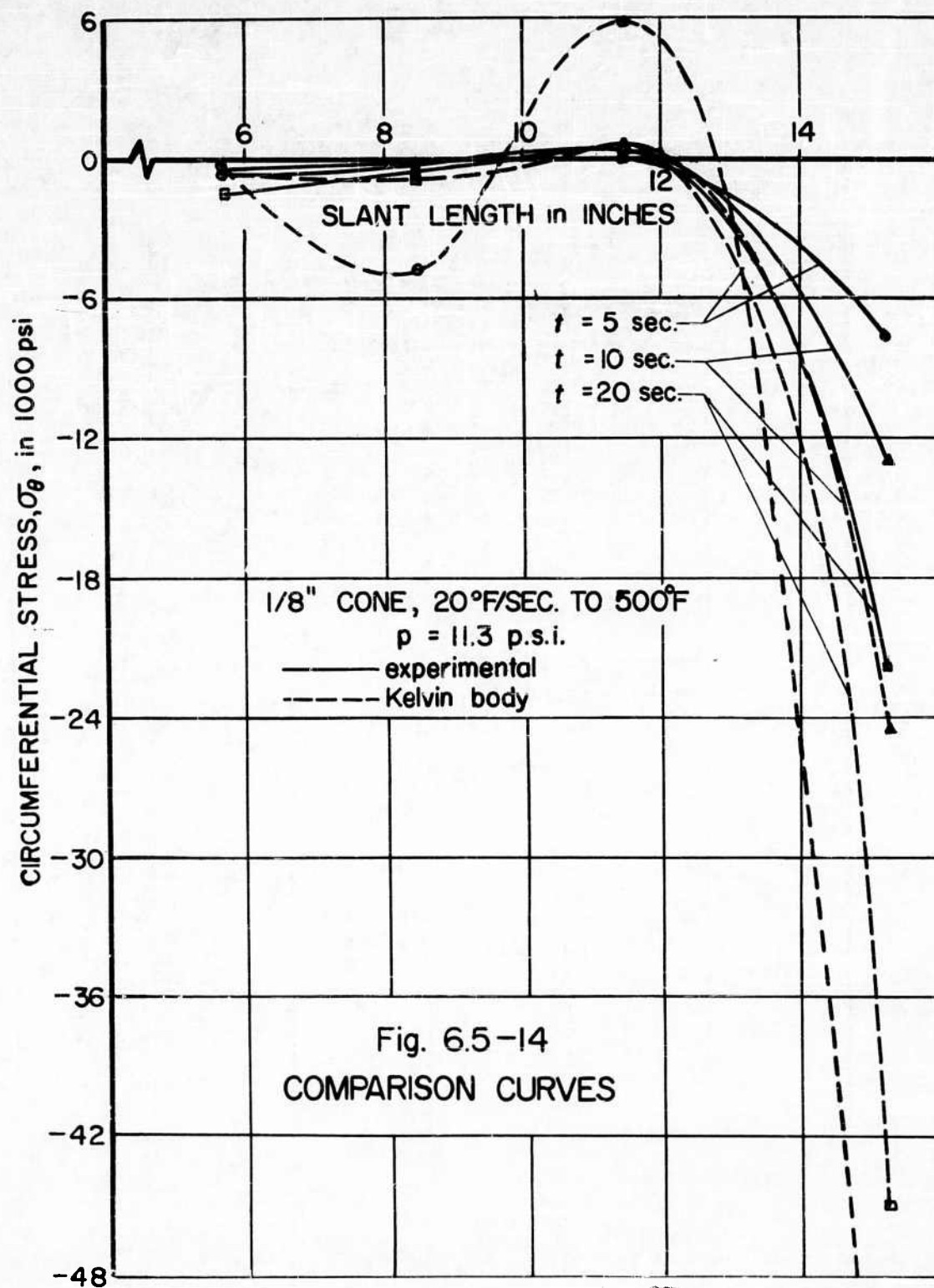


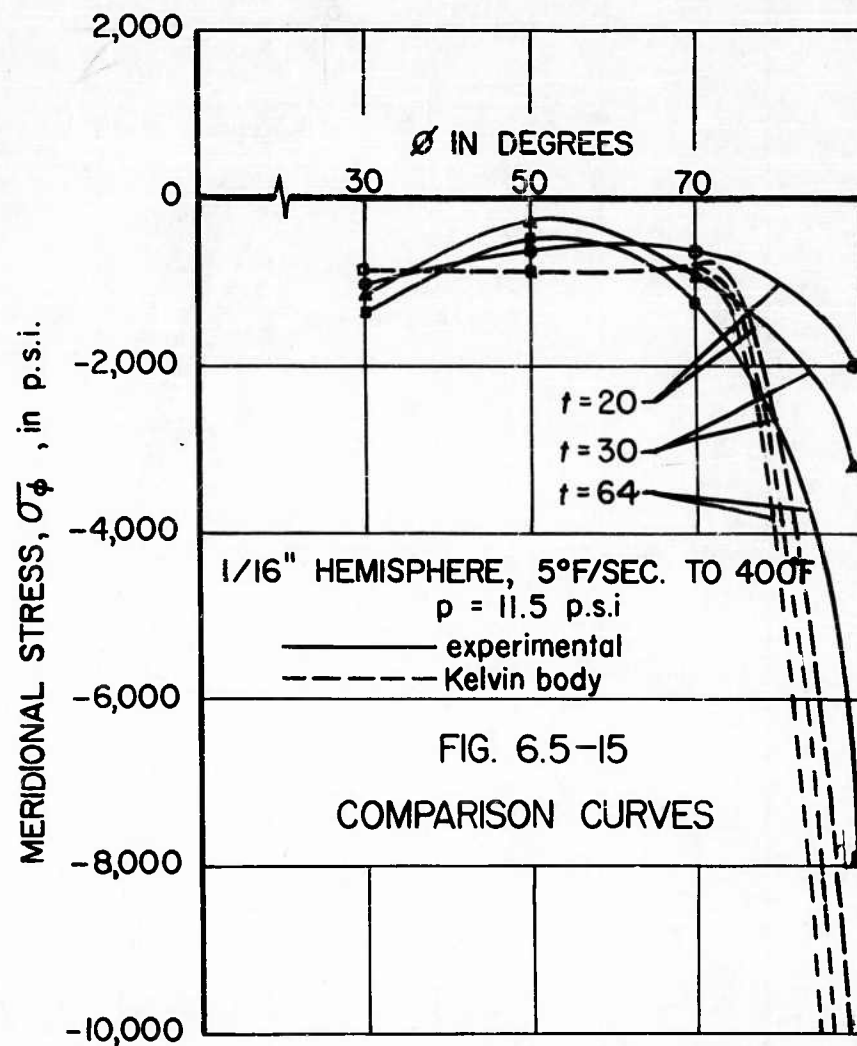


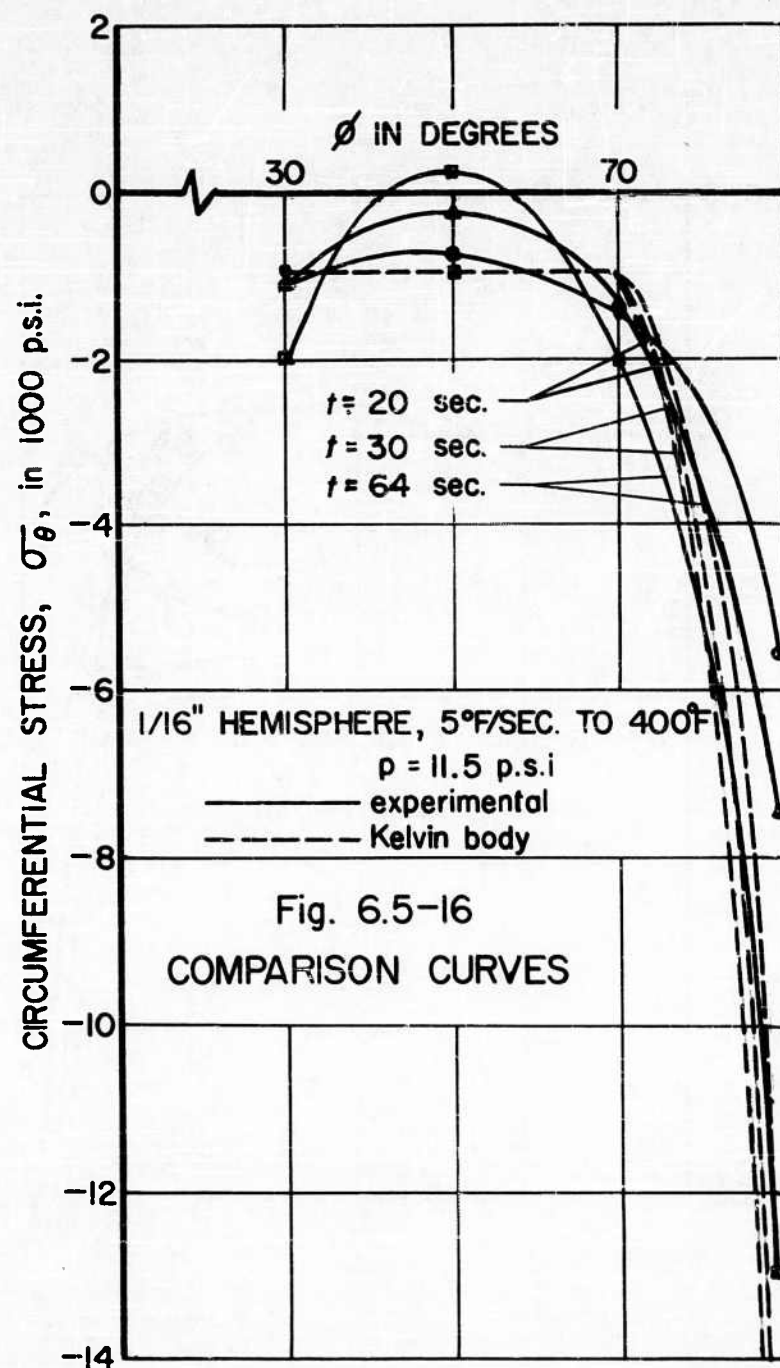


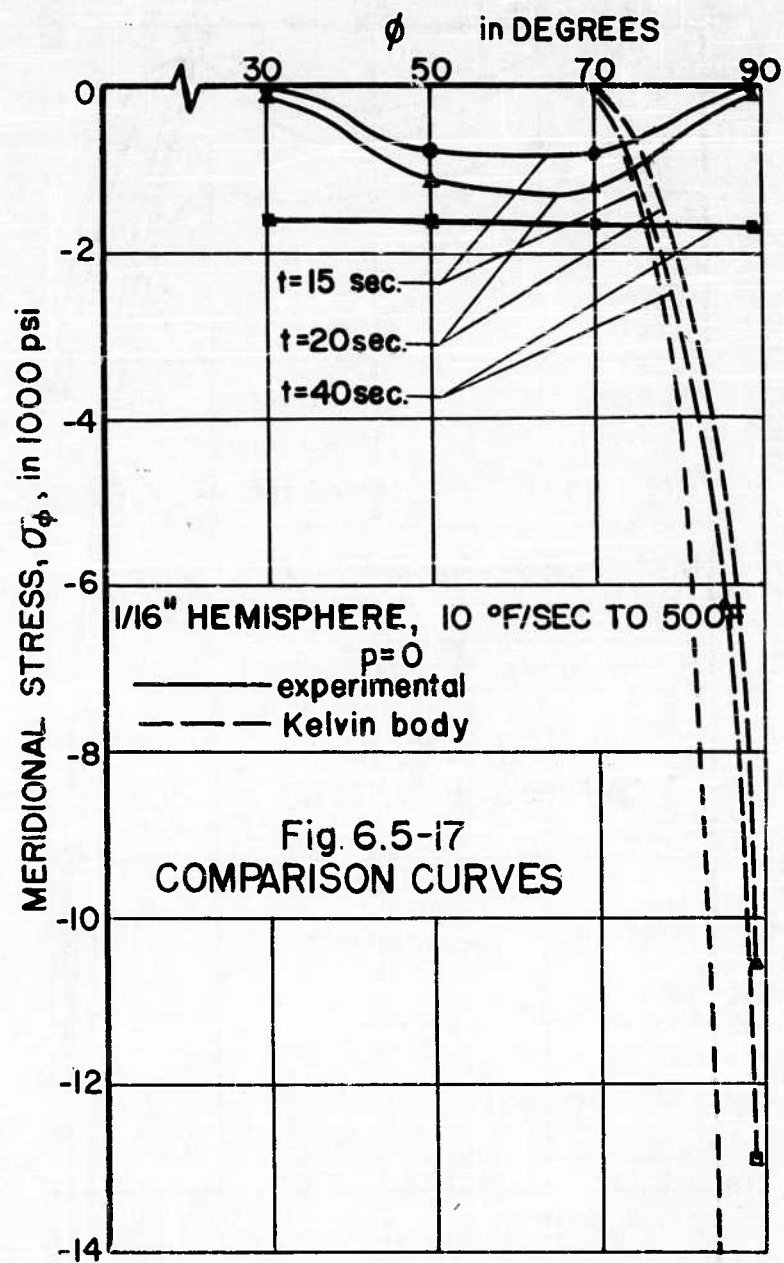


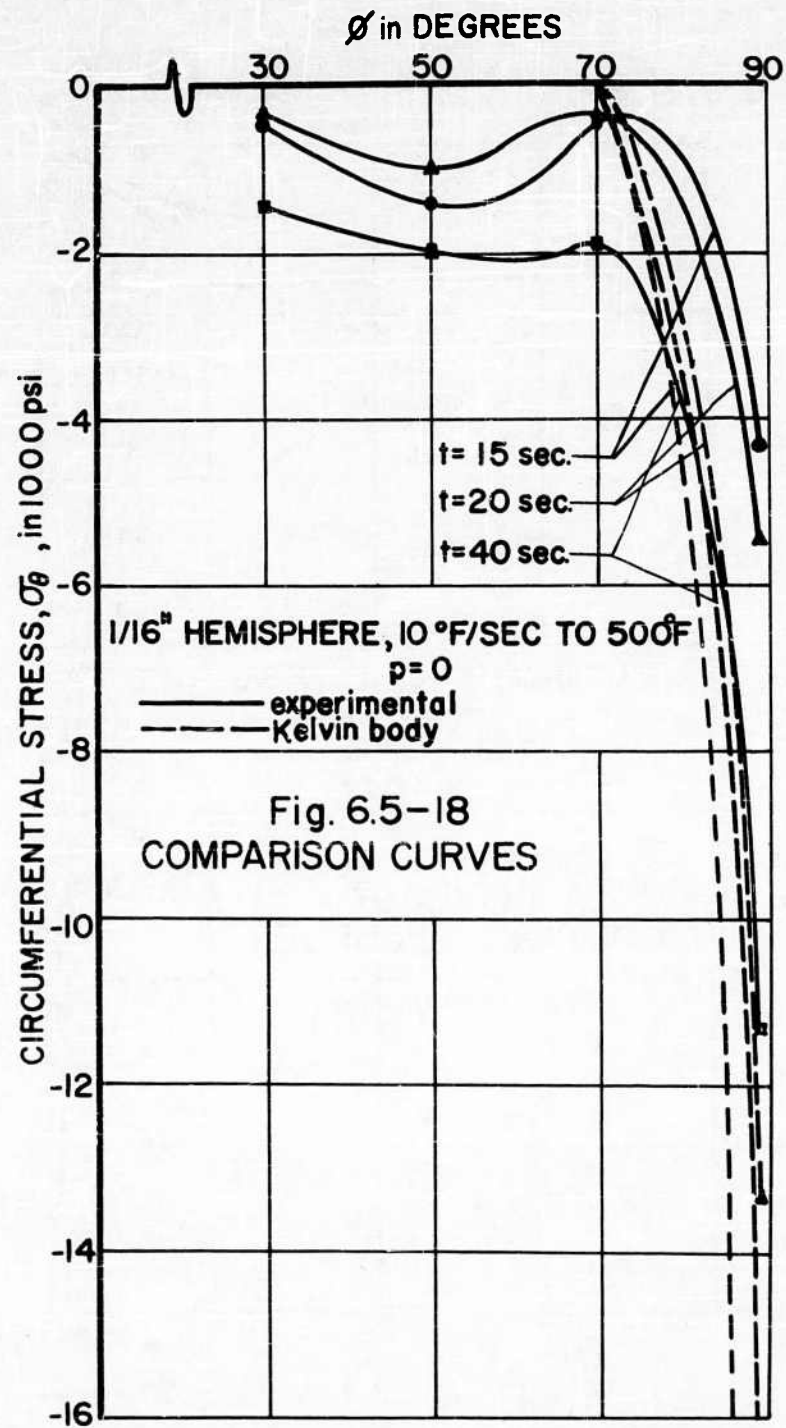


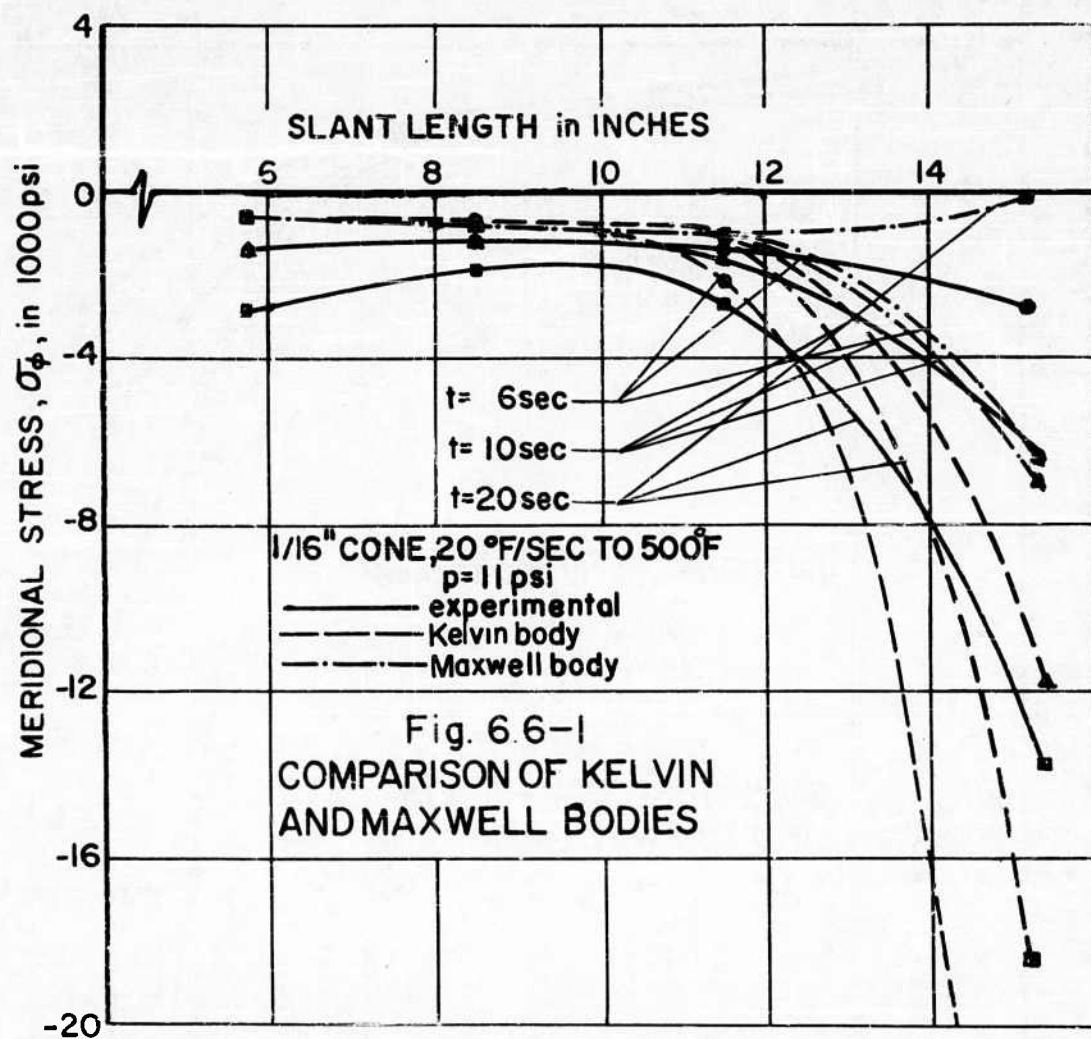


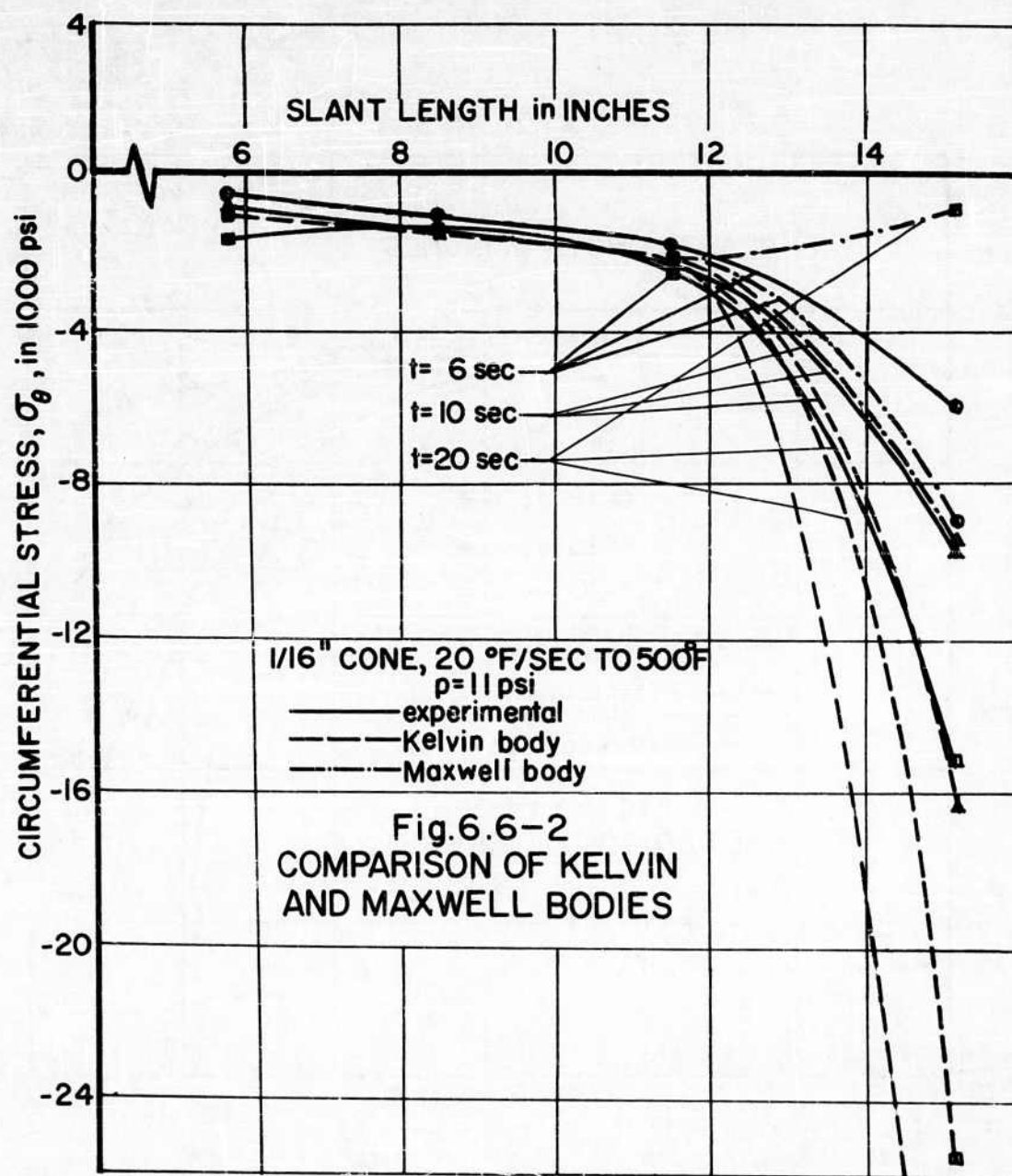


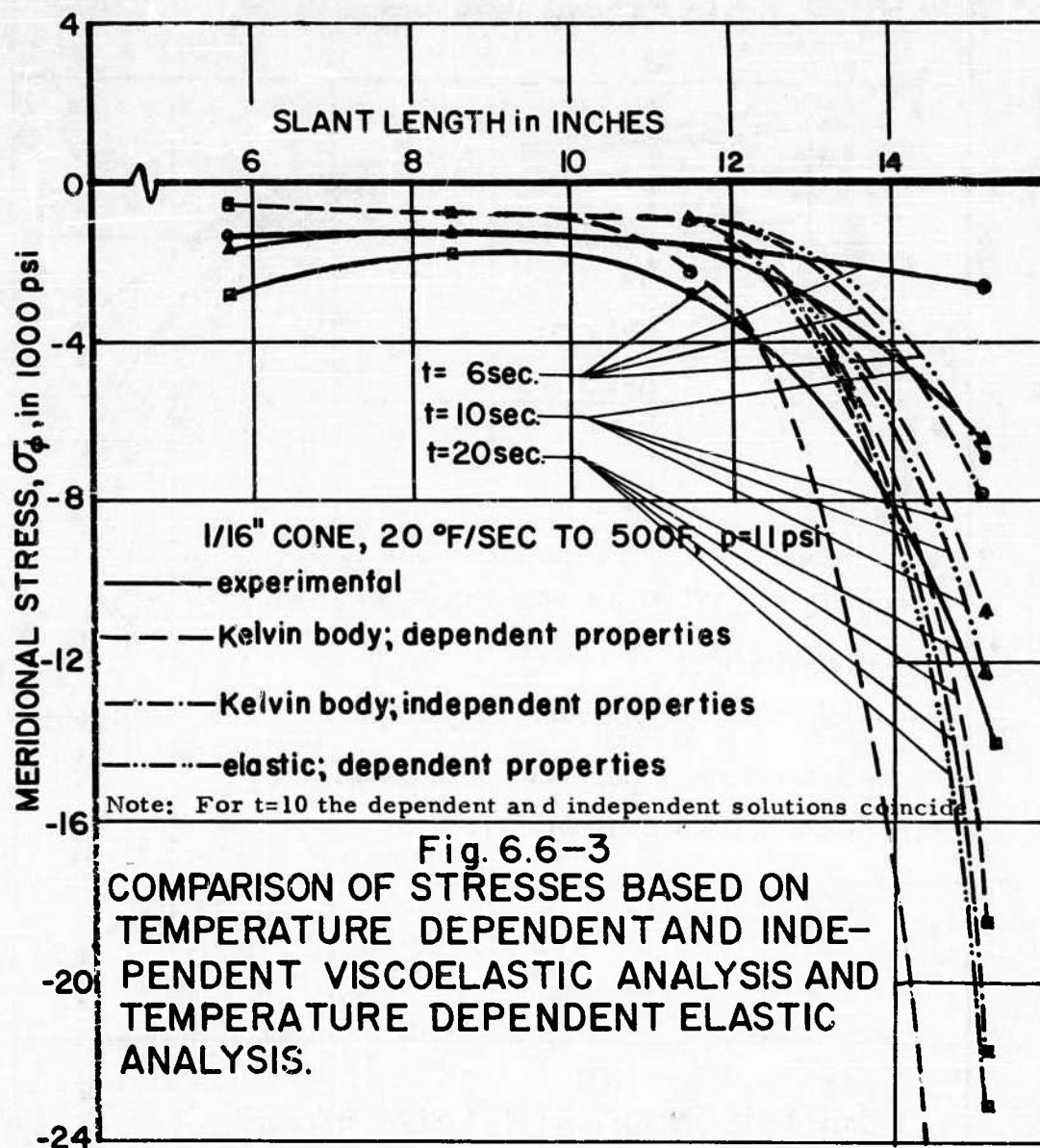


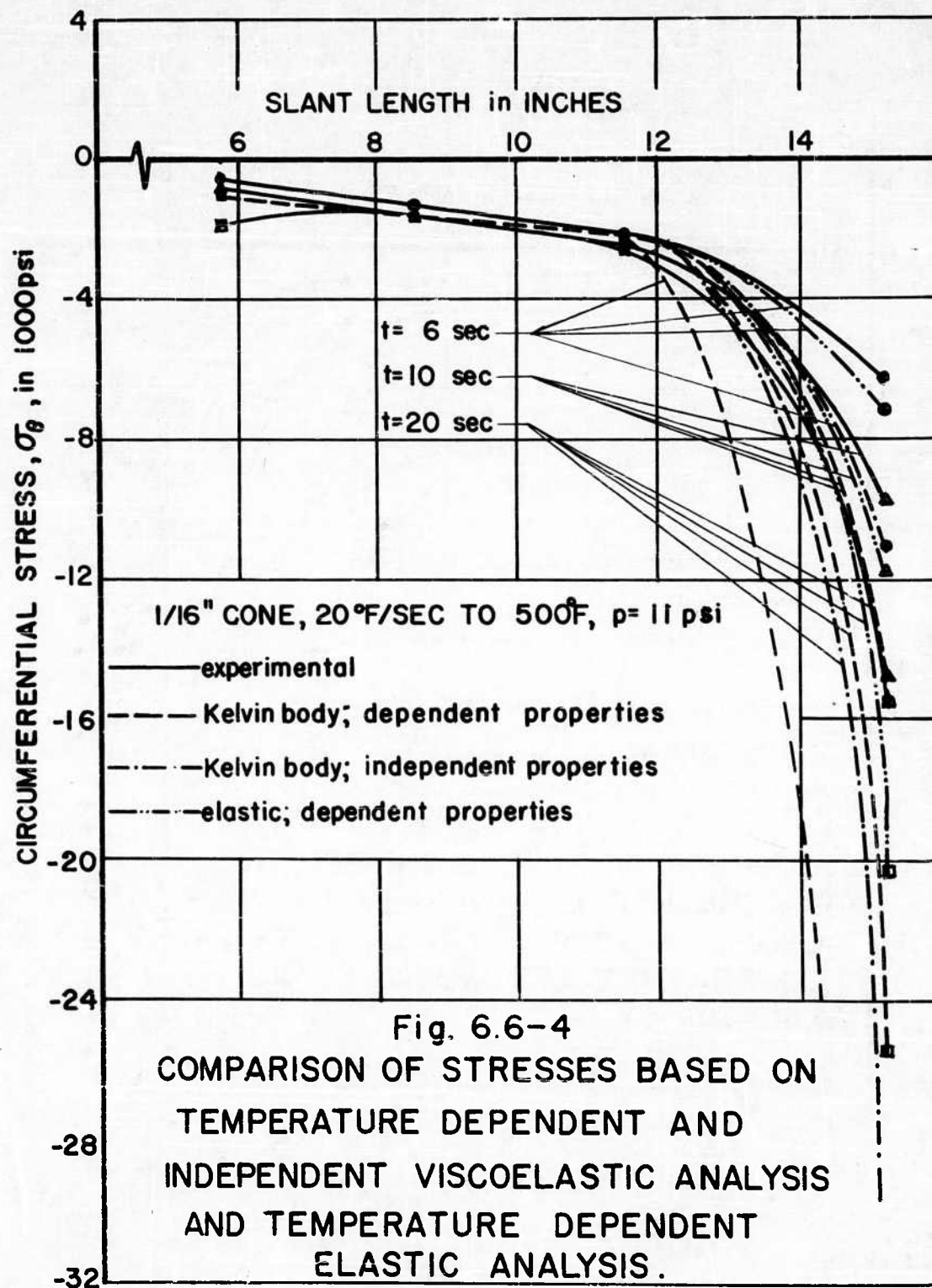


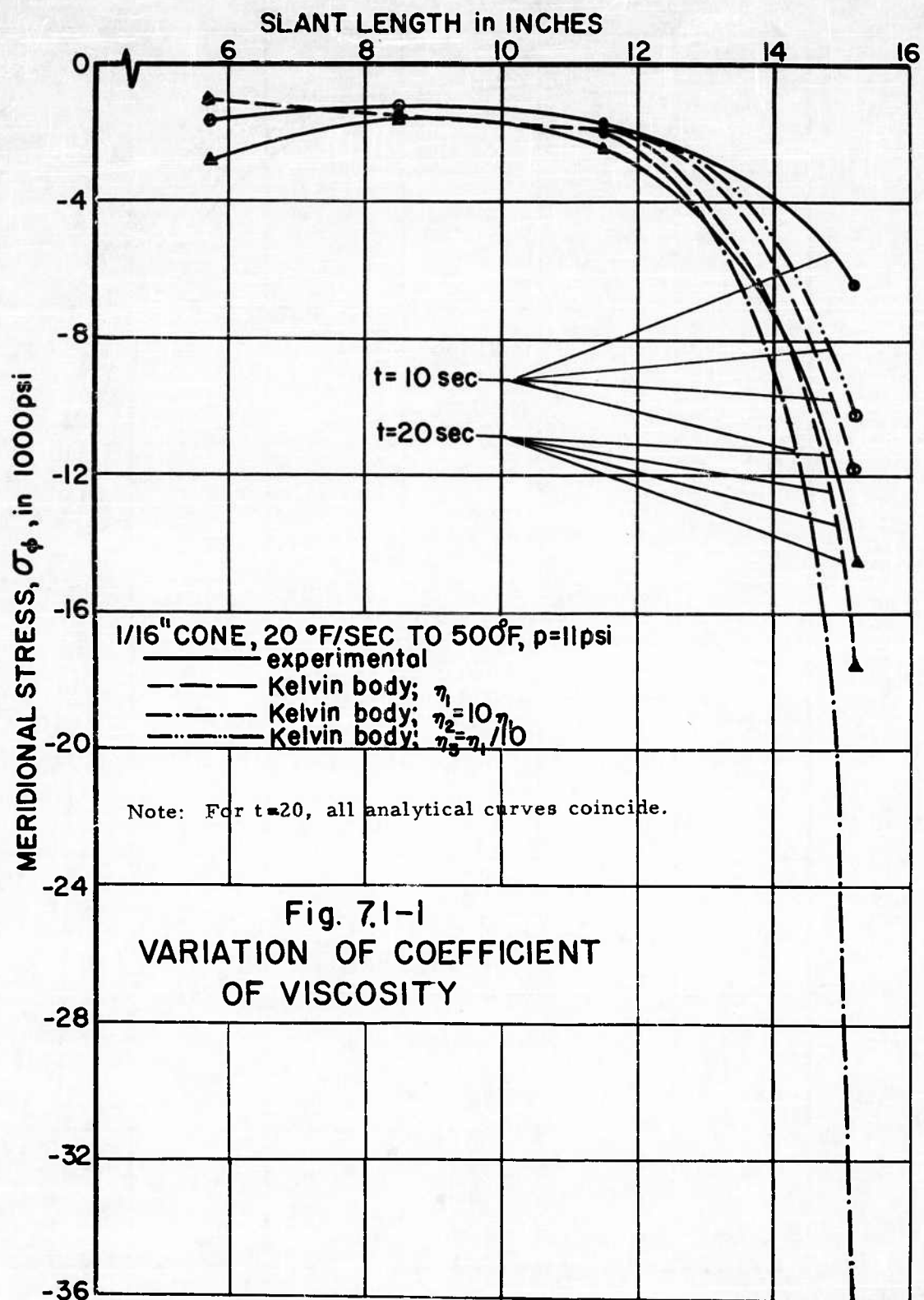


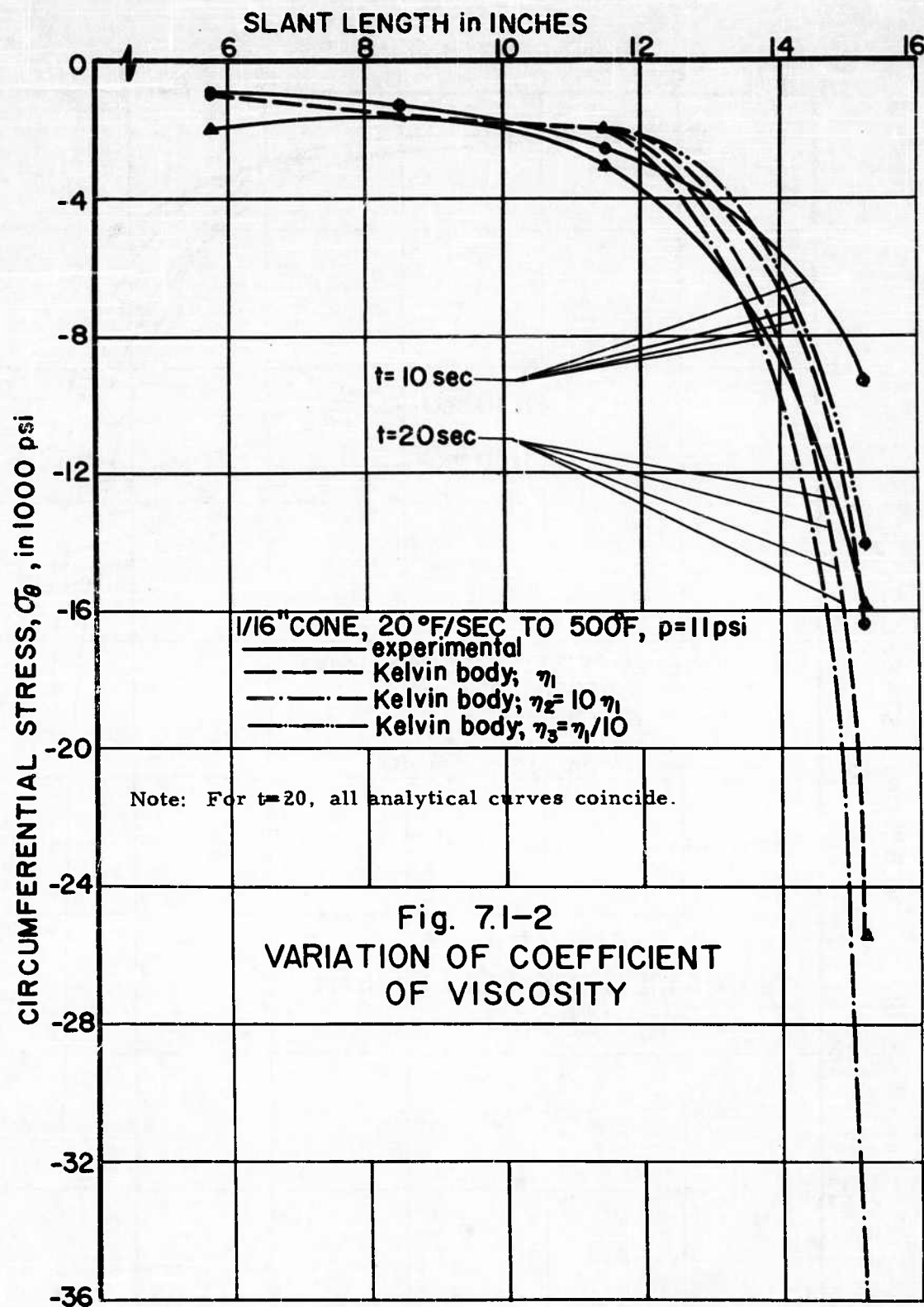


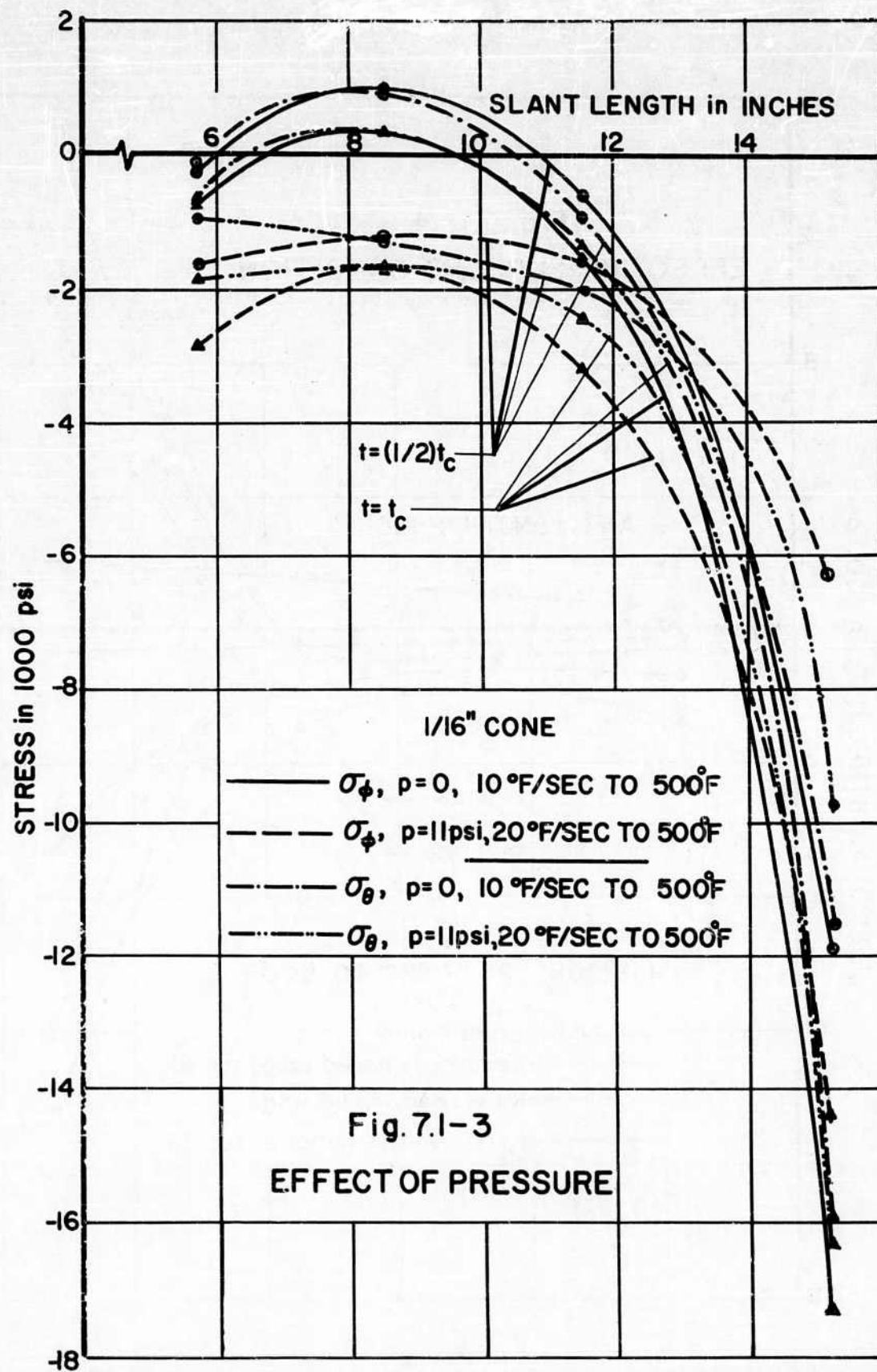


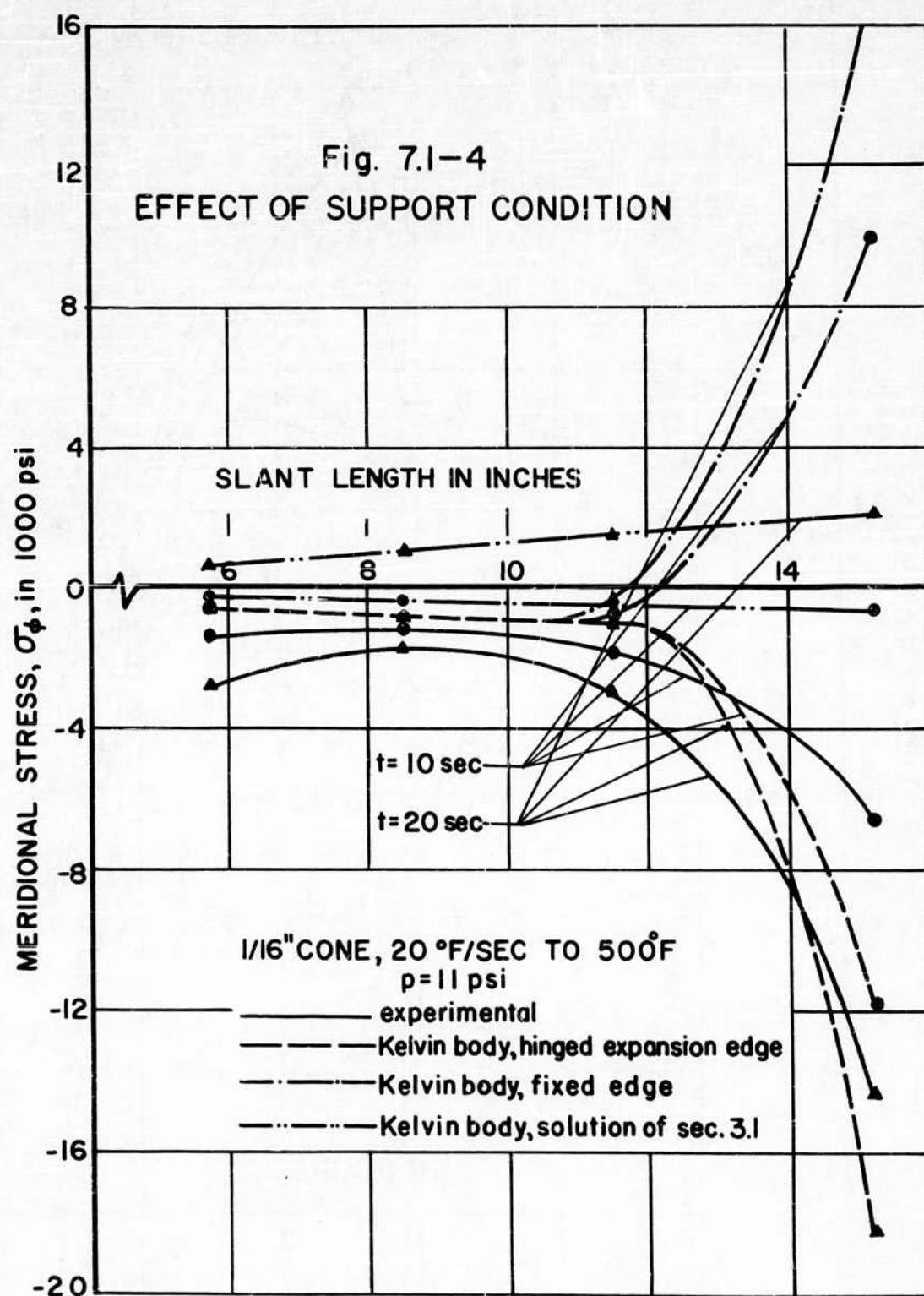


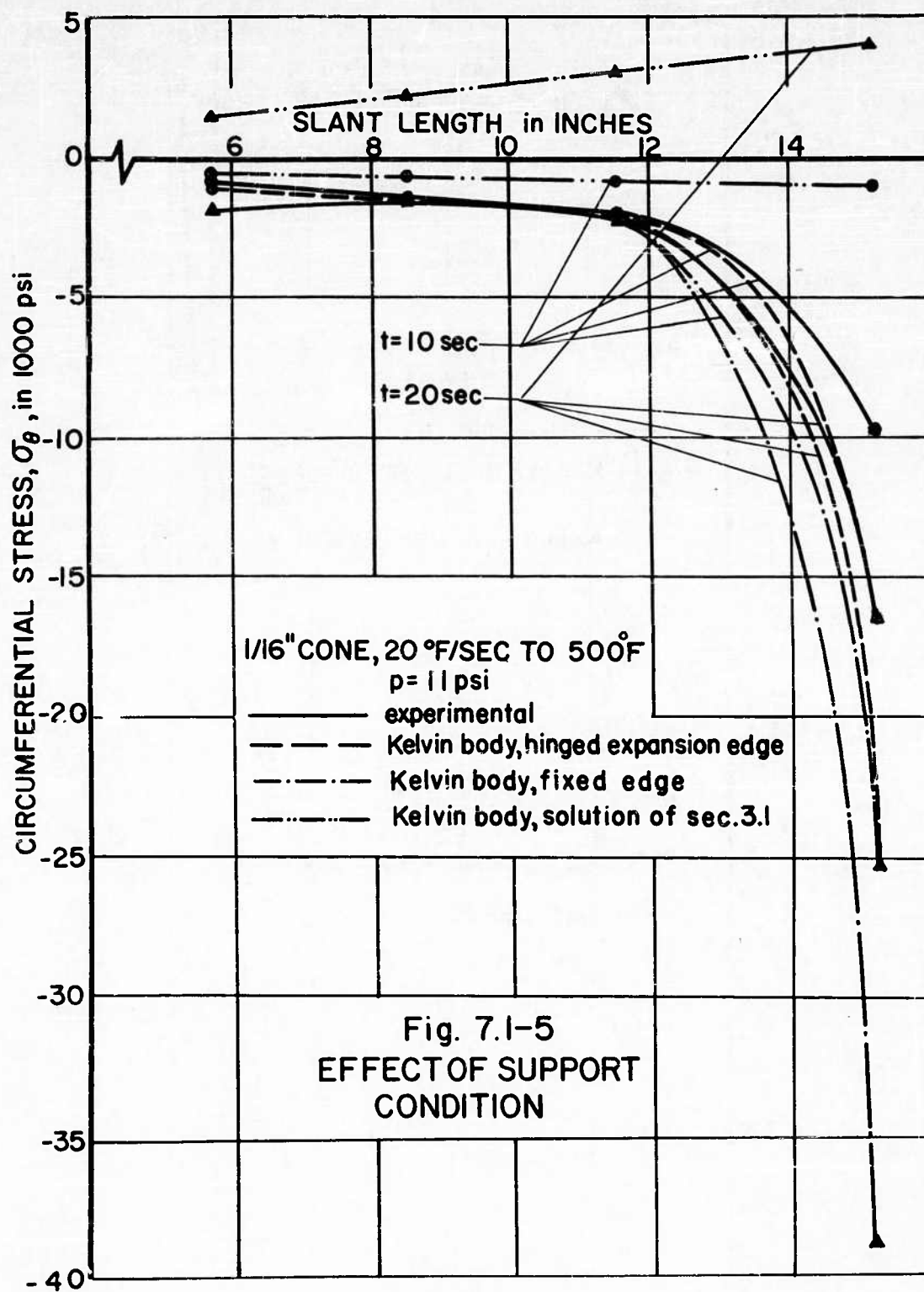


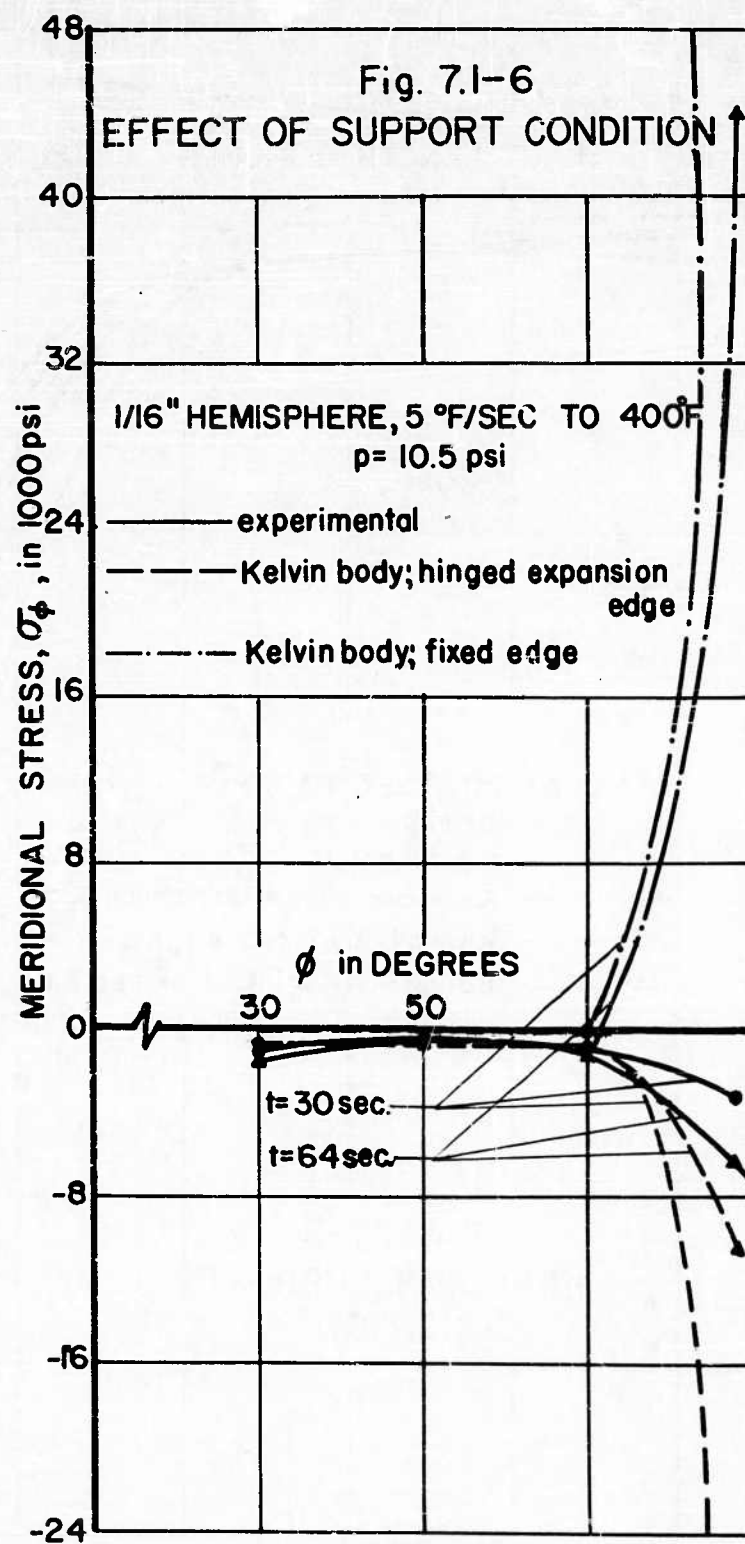


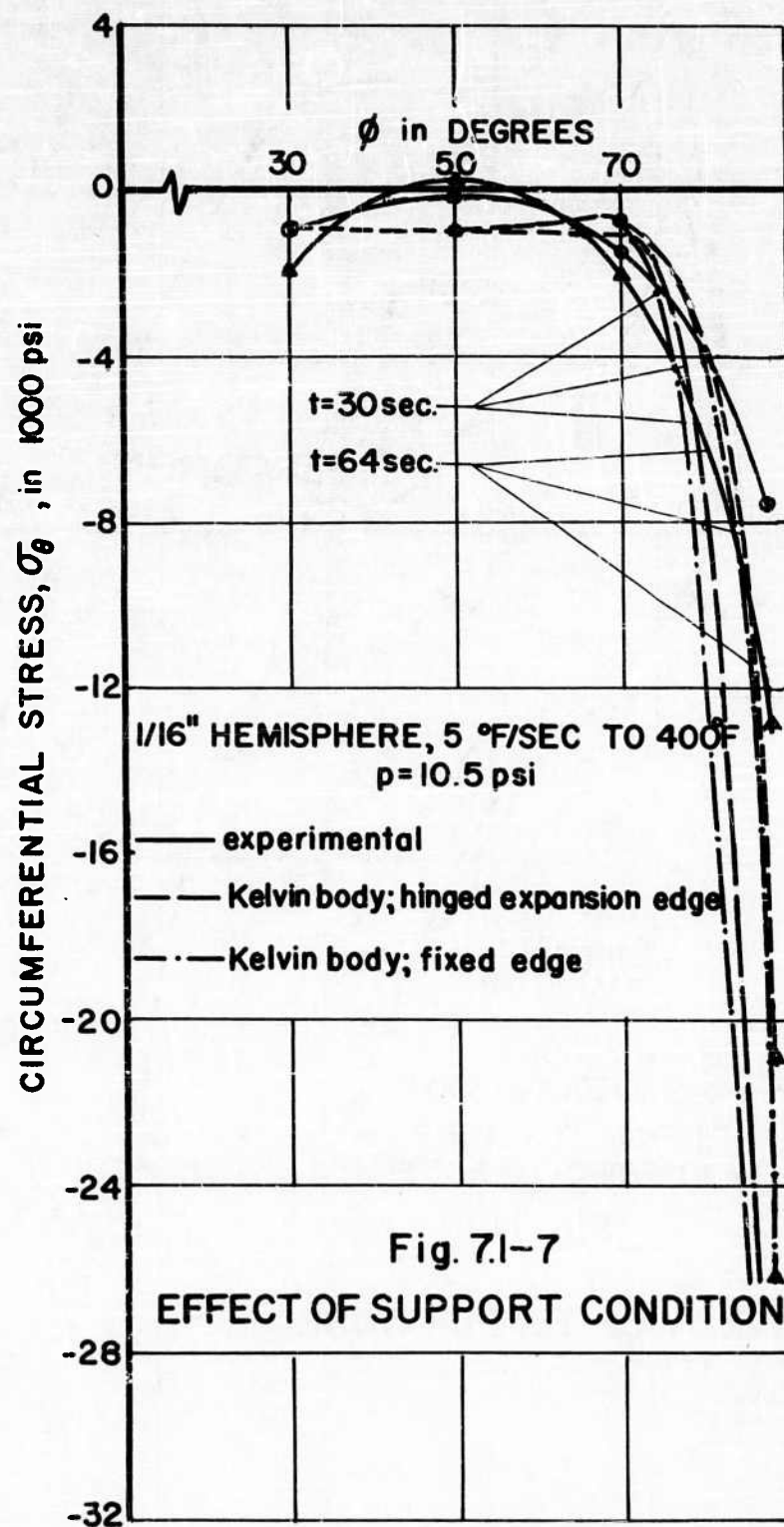


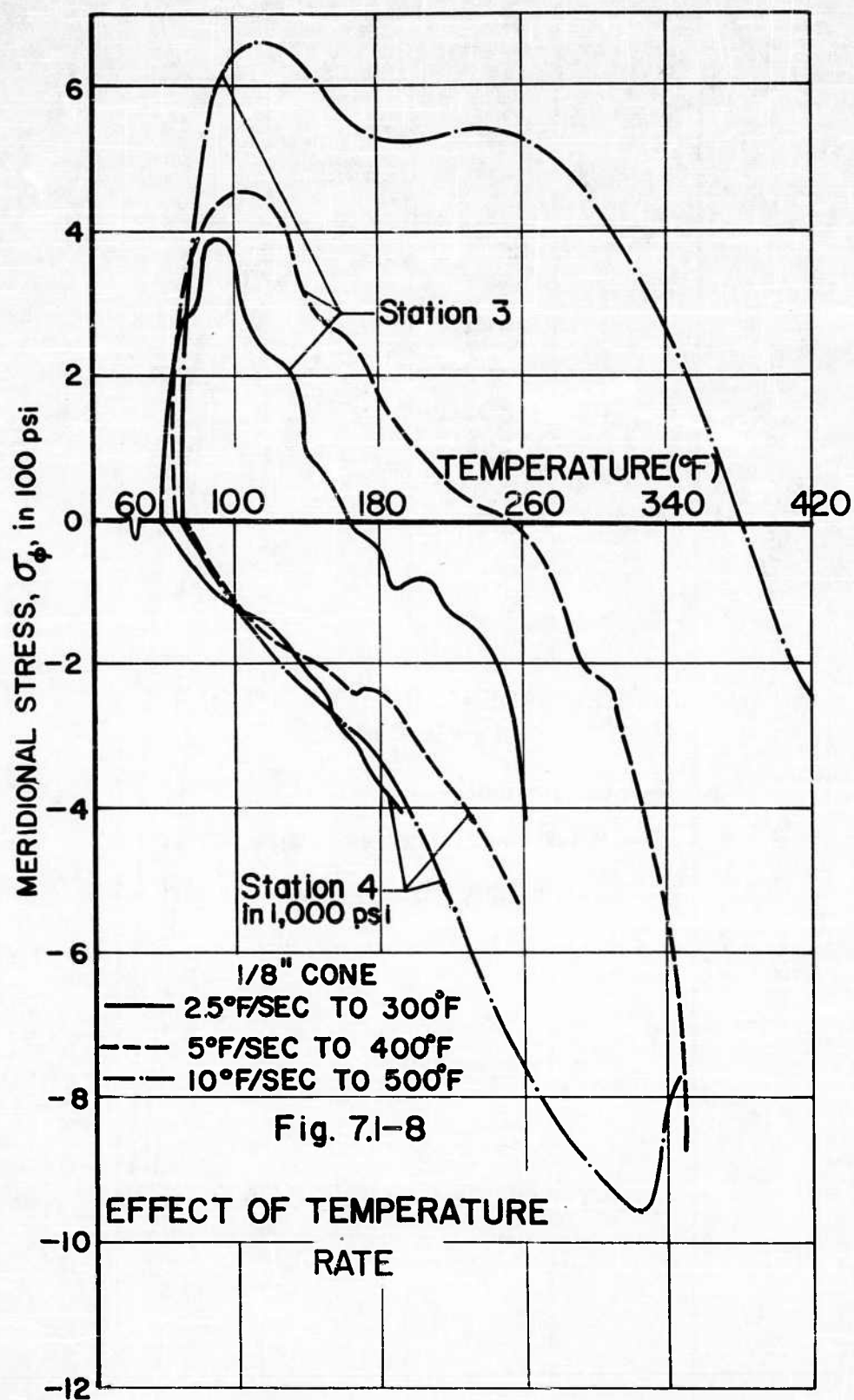












APPENDIX A
MATHEMATICAL DETAILS

Appendix A.1 Simplification of the Strain-displacement Relationship by the Order of Magnitude Consideration

The strain-displacement relationship for the general case are

$$\epsilon_{\phi\xi} = \frac{1}{r_1} \left(\frac{\partial v}{\partial \phi} - w \right) - \frac{1}{r_1} \frac{\partial}{\partial \phi} \left(\frac{v}{r_1} + \frac{\partial w}{r_1 \partial \phi} \right) \xi$$

$$\epsilon_{\theta\xi} = \frac{1}{r_2} (v \cot \phi - w) - \frac{\cot \phi}{r_2} \left(\frac{v}{r_1} + \frac{\partial w}{r_1 \partial \phi} \right) \xi$$

As applied to the conical shell

$$\begin{aligned} r_1 &= \infty \\ r_1 \partial \phi &= \partial l \\ r &= l \sin \gamma \\ r_2 &= l \tan \gamma \end{aligned}$$

The strain-displacement relation becomes

$$\epsilon_{\phi\xi} = \frac{\partial v}{\partial l} - \xi \frac{\partial^2 w}{\partial l^2}$$

$$\epsilon_{\theta\xi} = \frac{1}{l} (v - w \cot \gamma) - \xi \frac{\partial w}{l \partial l}$$

Introduction of the dimensionless variables

$$\begin{aligned} v' &= \frac{v}{\delta} \\ w' &= \frac{w}{\delta} \\ l' &= \frac{l}{l_0} \\ \xi' &= \frac{\xi}{\delta} \end{aligned}$$

gives for the strains

$$\epsilon_{\phi\zeta} = \frac{\delta}{l_0} \left(\frac{\partial v'}{\partial l'} \right) - \left(\frac{\delta}{l_0} \right)^2 \frac{\partial^2 w'}{\partial l'^2}$$

$$\epsilon_{\theta\zeta} = \frac{\delta}{l_0} \left(\frac{v' - w' \cos \gamma}{l'} \right) - \left(\frac{\delta}{l_0} \right)^2 \frac{\partial w'}{\partial l'}$$

Since for a thin shell

$$\frac{\delta}{l_0} < 1$$

and

$$\left(\frac{\delta}{l_0} \right)^2 \ll \left(\frac{\delta}{l_0} \right) < 1$$

it is obvious that the terms having the order of magnitude $(\delta/l_0)^2$ may be neglected. Therefore,

$$\epsilon_{\phi\zeta} \doteq \frac{\delta}{l_0} \left(\frac{\partial v'}{\partial l'} \right)$$

$$\epsilon_{\theta\zeta} \doteq \frac{\delta}{l_0} \left(\frac{v' - w' \cos \gamma}{l'} \right)$$

or

$$\epsilon_{\phi\zeta} \doteq \frac{\partial v}{\partial l}$$

$$\epsilon_{\theta\zeta} \doteq \frac{v - w \cos \gamma}{l}$$

Appendix A.2 Elastic Analysis of a Thin Conical Shell of Constant Thickness under Constant Pressure and Thermal Loads.

Equations of Equilibrium

$$\begin{aligned}\frac{d}{dl}(N_\phi l) &= N_\theta \\ \frac{d}{dl}(Q_\phi l) + N_\theta \cot \gamma + p l &= 0 \\ \frac{d}{dl}(M_\phi l) - M_\theta - Q_\phi l &= 0 \\ N_\phi + (Q_\phi + \frac{p l}{2}) \tan \gamma &= 0\end{aligned}$$

Temperature Profile Across the Shell Thickness

$$T = T_t \left(\frac{1}{2} - \frac{z}{\delta} \right)$$

Stress-Strain Relation

$$\begin{aligned}N_\phi = N_\theta &= \frac{E\delta}{1-\nu^2} \left[\frac{dr}{dl} + \frac{r-w \cot \gamma}{l} - (1+\nu) \alpha \frac{T_t}{2} \right] \\ N_\theta &= \frac{E\delta}{1-\nu^2} \left[\frac{r-w \cot \gamma}{l} + \nu \frac{dr}{dl} - (1+\nu) \alpha \frac{T_t}{2} \right] \\ M_\phi &= -\frac{E\delta^3}{12(1-\nu^2)} \left[\frac{d^2 w}{dl^2} + \frac{\nu}{l} \frac{dw}{dl} - (1+\nu) \alpha \frac{T_t}{\delta} \right] \\ M_\theta &= -\frac{E\delta^3}{12(1-\nu^2)} \left[\nu \frac{d^2 w}{dl^2} + \frac{1}{l} \frac{dw}{dl} - (1+\nu) \alpha \frac{T_t}{\delta} \right]\end{aligned}$$

Working Variables

$$\begin{aligned}W &= Q_\phi l \\ \chi &= \frac{dw}{dl}\end{aligned}$$

Governing Equations

$$l \frac{d^2 \chi}{dl^2} + \frac{d\chi}{dl} - \frac{\chi}{l} - (1+\nu) \frac{1}{\delta} \frac{d(\alpha T_c)}{dt} + \frac{12W(1-\nu^2)}{E\delta^3} = 0$$

$$l \frac{d^2 W}{dl^2} + \frac{dW}{dl} - \frac{W}{l} + \frac{3}{2}Pl - E\delta \chi \cos^2 \gamma - \frac{E\delta^2 l}{2 \tan \gamma} \frac{d(\alpha T_c)}{dl} = 0$$

Solutions

$$\chi = A_1 \left(Z_1 + \frac{2Z_1'}{\gamma} \right) + A_2 \left(Z_2 - \frac{2Z_2'}{\gamma} \right) + \frac{3}{2} \frac{Pl \tan \gamma}{E\delta}$$

$$N_\theta = 4E\delta \cos \gamma \left[A_1 \left(-\frac{2Z_1'}{\gamma^2} + \frac{Z_1}{\gamma^2} \right) + A_2 \left(-\frac{2Z_2'}{\gamma^2} - \frac{Z_2}{\gamma^2} \right) \right] - \frac{Pl \tan \gamma}{2}$$

$$N_\theta = 2E\delta \cos \gamma \left[A_1 \left(\frac{Z_1'}{\gamma} - \frac{2Z_2'}{\gamma^2} + \frac{4Z_1'}{\gamma^3} \right) + A_2 \left(-\frac{Z_2'}{\gamma} + \frac{2Z_2}{\gamma^2} + \frac{4Z_2'}{\gamma^3} \right) \right] - Pl \tan \gamma$$

$$M_\theta = -\frac{E\delta^2 \cos \gamma}{\sqrt{12(1-\nu^2)}} \left\{ A_1 \left[\gamma \frac{2Z_1'}{\gamma} + (1-\nu) \left(\frac{4Z_1}{\gamma^2} + \frac{8Z_1'}{\gamma^3} \right) \right] + A_2 \left[\gamma \frac{2Z_2'}{\gamma} + (1-\nu) \left(\frac{4Z_2}{\gamma^2} - \frac{8Z_2'}{\gamma^3} \right) \right] \right\} + \frac{E\delta^2 \alpha T_c}{12(1-\nu)} - \frac{Pl^2 \tan^2 \gamma}{8(1-\nu)}$$

$$M_\phi = -\frac{E\delta^2 \cos \gamma}{\sqrt{12(1-\nu^2)}} \left\{ A_1 \left[\frac{2Z_1'}{\gamma} (1-\nu) \left(\frac{4Z_1}{\gamma^2} + \frac{8Z_1'}{\gamma^3} \right) \right] + A_2 \left[\frac{2Z_2'}{\gamma} (1-\nu) \left(\frac{4Z_2}{\gamma^2} - \frac{8Z_2'}{\gamma^3} \right) \right] \right\} + \frac{E\delta^2 \alpha T_c}{12(1-\nu)} - \frac{Pl^2 \tan^2 \gamma}{8(1-\nu)}$$

where $Z_1 = Z_1(\gamma)$ and $Z_2 = Z_2(\gamma)$ are ber and -bei functions of γ respectively, and $\gamma^2 = \frac{4\cos \gamma}{\sqrt{12(1-\nu^2)}} l$. The values of A_1 and A_2 are to be determined by the edge conditions of the shell.

Appendix A.3 Geometrical Imperfections

In recent years, considerable discussions have been made to explain discrepancies between shell theories and experimental results. Many investigators feel that these discrepancies are due to the geometrical imperfections that exist in the models.

The present investigators made a preliminary study of the effect of geometrical imperfections on the behavior of spherical shells under normal pressure to obtain some understanding of their effects. The analysis is based on the axisymmetrical type imperfections.

As in the case of other investigations by Chen (6), Kaplan and Fung (15), Budiansky (5), it is found that axisymmetrical analysis could not account for the total difference between theory and experiment. The mode of collapse of the models in the present investigation gives evidence that the initial imperfections of the models were probably unsymmetrical. Experimental evidence obtained by Kaplan and Fung leads on to the conclusion that an investigation of the effect of geometrical imperfections should be based on unsymmetrical rather than axisymmetrical analysis.

However, for such an analysis to be meaningful, it must be an exhaustive study supported by a number of experimental tests. The study need not involve temperature. Such an investigation was not attempted as part of the present program.

Previous Investigations:

To the knowledge of the writers, only five studies have been made about the effect of initial imperfection on the behavior of thin shells. Four of the studies were concerned with spherical shells and one dealt with general thin shells. In chronological order the studies and their general results are as follows:

Mushtari presented the equilibrium equations of the middle surface of a thin shell which have irregularities of the order of the shell thickness. The relations were applied to thin shells subjected to normal pressure.

Klein used two parameters in connection with experimental data to provide a better correlation of analytical and experimental results. He attempted to show that by proper choice of the allowable collapse pressure, the effect of initial imperfections of the shell could be taken into account.

Gerard and Becker using an empirical approach based on an "unevenness factor" proposed that the

geometrical portion of the initial imperfections governs the average behavior of the spherical plates which the residual stress and other factors may contribute to the scatter in the experimental data.

Chen considering the axisymmetric buckling under uniform pressure of a shallow portion of a spherical shell, carried out a Rayleigh-Ritz solution of the variational equation equivalent to the governing equations. He concluded that the buckling pressures may be appreciably affected by the presence of imperfection of the middle surface and that its effect depend not only on the magnitude but also on the location of the maximum imperfection and the mode of the imperfection.

Budiansky, treating the same problem as Chen, based his analysis on the integral-equation formulation. Budiansky found that P_{cr} appears to approach 1 in an oscillatory fashion as λ (geometrical parameter) increases. Comparing the variation buckling pressures of initially perfect and imperfect clamped shallow spherical shells, he found that both followed the same trend.

Comparison of these investigations show very little agreement of the results. Within a certain range of λ , say $\lambda = 4$ to $\lambda = 5$, fairly good agreement has been observed. Beyond this narrow range, the correct theoretical variation of P_{cr} with λ is questionable.

Method of Analysis

To study the possible causes for discrepancies which exist between the present analytical and experimental results, a simplified analysis was made of thin spherical shells with axisymmetrical imperfections.

For simplification, the study is restricted to the hemispherical shells subjected to pressure only. Following a method proposed by Budiansky, we consider the buckling of such shells whose deviated surface may be represented by

$$z_0 = H \left[1 - \left(\frac{r}{a} \right)^2 - \epsilon c(r) \right]$$

where $e(r)$ is the shape of the imperfection, ϵ is the ratio of the downward initial displacement at the center of the shell to the rise H of the perfect shell.

For simplicity the shape of imperfection is taken as

$$c(r) = \left[1 - \left(\frac{r}{a} \right)^2 \right]^2$$

By the aid of the geometrical parameter λ defined by

$$\lambda^4 = 48(1-\nu^2) \left(\frac{H}{t}\right)^2$$

and the non-dimensional variables

$$\chi = \frac{\lambda r}{a}$$

$$\theta = \left(\frac{\lambda a}{2H}\right) \beta$$

$$\Phi = \left[\frac{12(1-\nu^2)a}{\lambda E t^3}\right]^{1/2} \psi$$

where ψ is the stress function, $\beta = (dw/dr)$ is the rotation of an element of the shell, q_0 is the classical buckling pressure of a complete spherical shell having the same radius of curvature $R = a^2/2H$ as the given shallow spherical shell.

The non-dimensional equilibrium and compatibility equations for an initially perfect spherical shell are

$$\begin{aligned} (\chi \theta') - \frac{\theta}{\chi} + \chi \Phi &= -2\rho\chi + \theta\Phi \\ (\chi \Phi') - \frac{\Phi}{\chi} - \chi \theta &= -\frac{1}{2}\theta^2 \end{aligned}$$

The prime denotes differentiation with respect to χ and ρ . A non-dimensional pressure parameter ρ is defined as

$$\rho = \frac{q}{q_0}$$

where

$$q_0 = \frac{2E}{\sqrt{3(1-\nu^2)}} \left(\frac{t}{R}\right)^2$$

The non-dimensional boundary conditions appropriate for a clamped shell

$$\theta(\lambda) = 0$$

$$\lambda \Phi'(\lambda) - \nu \Phi(\lambda) = 0$$

the first condition states that the rotation at $r = a$ is zero. The second boundary condition fulfills the requirement that the horizontal displacement at the shell edge must vanish.

For the initially imperfect shell, the basic equations take the form

$$(x\theta')' - \frac{\theta}{x} + x\Phi' = -2\rho x + \theta\Phi + \epsilon h\Phi$$

$$(x\Phi')' - \frac{\Phi}{x} - x\theta = -\frac{1}{2}\theta^2 - \epsilon h\theta$$

$$\text{WHERE } h = \frac{\lambda^2}{2} \epsilon'$$

The boundary conditions, of course, are still valid.

The solution of these equations and the boundary conditions can be represented by two integral equations

$$\theta(x) = \gamma(x) + dA_1(x) + eA_2(x)$$

$$\Phi(x) = z(x) + dB_1(x) + eB_2(x) - 2\rho x$$

where

$$\gamma(x) = \int_0^1 G(x, \xi) r(\xi) d\xi + \int_0^1 H(x, \xi) s(\xi) d\xi$$

$$z(x) = -\int_0^1 H(x, \xi) r(\xi) d\xi + \int_0^1 G(x, \xi) s(\xi) d\xi$$

$$r = \theta\Phi + \epsilon h\Phi$$

$$s = -\frac{1}{2}\theta^2 - \epsilon h\theta$$

$$e = -\gamma(\lambda)$$

$$d = 2\rho\lambda(1-\nu) - (1-\nu)z(\lambda) - \frac{1}{\lambda} \int_0^1 (\lambda^2 \gamma + \lambda s) dx$$

$$G(x, \xi) = \text{BEI}' x \text{KER}' \xi + \text{BEI}' x \text{KEI}' \xi \quad (0 \leq x \leq \xi)$$

$$= \text{BEI}' \xi \text{KER}' x - \text{BEI}' \xi \text{KEI}' x \quad (0 \leq \xi \leq x)$$

$$H(x, \xi) = \text{BER}' x \text{KER}' \xi - \text{BEI}' x \text{KEI}' \xi \quad (0 \leq x \leq \xi)$$

$$= \text{BER}' \xi \text{KER}' x - \text{BEI}' \xi \text{KEI}' x \quad (0 \leq \xi \leq x)$$

$$A_1(x) = -\frac{1}{D} (\text{BER}' \lambda \text{BEI}' x - \text{BEI}' \lambda \text{BER}' x)$$

$$B_1(x) = -\frac{1}{D} (\text{BER}' \lambda \text{BER}' x + \text{BEI}' \lambda \text{BEI}' x)$$

$$A_2(x) = -(1+\nu)B_1(x) - \frac{1}{D} (\text{BER}' \lambda \text{BEI}' x - \text{BEI}' \lambda \text{BER}' x)$$

$$B_2(x) = (1+\nu)A_1(x) + \frac{1}{D} (\text{BER}' \lambda \text{BER}' x + \text{BEI}' \lambda \text{BEI}' x)$$

$$D = (1+\nu) [(\text{BER}' \lambda)^2 + (\text{BEI}' \lambda)^2] - \lambda (\text{BER}' \lambda \text{BEI}' \lambda - \text{BEI}' \lambda \text{BER}' \lambda)$$

The solution of the basic equations is obtained by iterative procedures in terms of matrix operations with the integrations being performed numerically on the basis of Simpson's rule. For the particular application under discussion, the following data are selected for comparison of the buckling stress of a perfect spherical shell with that of a geometrically imperfect spherical shell:

$$\begin{aligned}\lambda &= 12 \\ \epsilon &= 0.05 \\ \nu &= 0.33 \\ \Delta x &= 0.25\end{aligned}$$

where Δx is the invariant spacing of stations in the numerical integration.

The results of the calculation may be stated as follows for the critical pressure

$$q_{cr} = 0.75 q_0$$

where q_{cr} is the local maximum pressure at which buckling occurs, q_0 is the buckling pressure of a complete sphere without bending.

For a perfect spherical shell, one finds that

$$q_{cr} = 0.96 q_0$$

In terms of stresses, one can write that

$$\begin{aligned}\sigma_{cr} &= \frac{0.96 E}{\sqrt{3(1-\nu^2)}} \left(\frac{t}{R} \right) && \text{Perfect shell} \\ &= \frac{0.75 E}{\sqrt{3(1-\nu^2)}} \left(\frac{t}{R} \right) && \text{Imperfect shell}\end{aligned}$$

This means that the predicted buckling stress band on an "imperfect shell" analysis is approximately 80% of that given by the "perfect shell". This reduction is not sufficient to bring the analytical results within the range of available experimental results for elastic hemispherical shells.

However, a reduction of the analytical stresses obtained in the present viscoelastic analysis by a similar magnitude gives a more favorable comparison of analytical and experimental results for the hemispherical shell.

ADDITIONAL REFERENCES FOR APPENDIX A.3

(not stated in main grouping)

Mushtari, H. M. (in Russian), "On the Elastic Equilibrium of a Thin Shell with Initial Irregularities in the Form of the Middle Surface", Prikladnaia Matematika i Mekhanika, Vol. 15, pp. 743-750, 1951.

Klein, B., "Parameters for Predicting the Initial and Final Collapse Pressures of Uniformly Loaded Spherical Shells", Journal of the Aeronautical Sciences, Vol. 22, No. 1, pp. 69-70, 1955.

Gerard, G. and Becker, H., "Handbook of Structural Stability, Part III - Buckling of Curved Plates of Shells", U. S. N. A. C. A. TN 3783, pp. 59-65, August 1957.

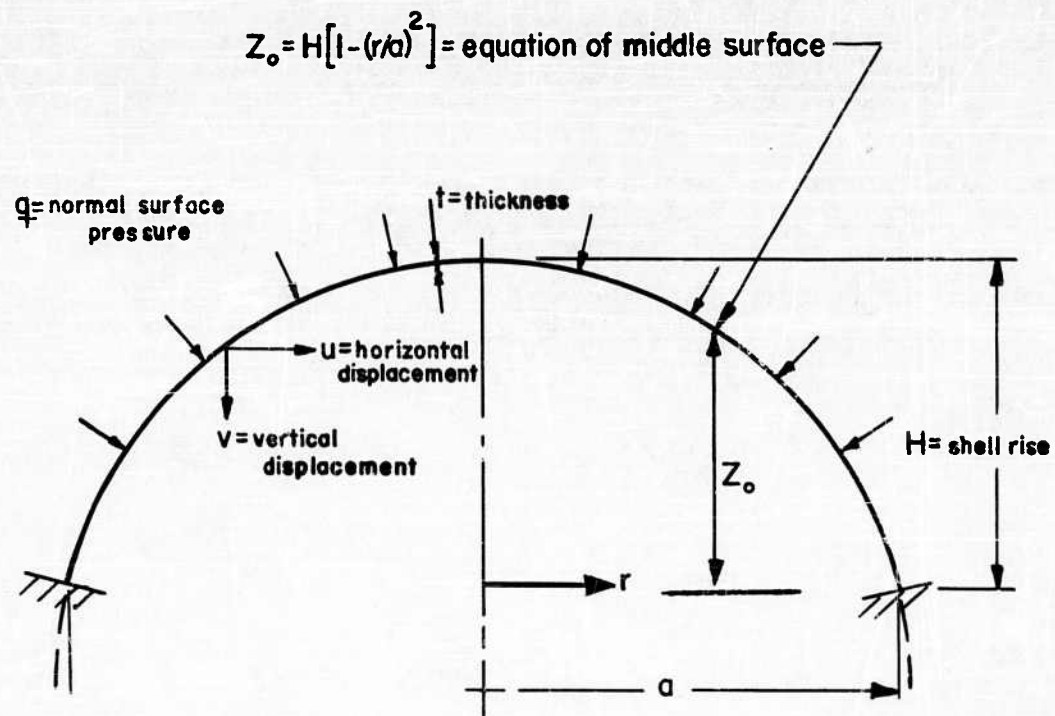


Fig. A.3-1
SHELL GEOMETRY

APPENDIX B
EXPERIMENTAL INVESTIGATION

B.1 Experimental Facility

(a) Heating System

The functional arrangement of the experimental facility is represented by the block diagram shown in Fig. B-1.

The radiant heating oven consists of a polished aluminum reflector with attached heating lamps. The heating lamps are the GE T3/CL quartz type. Thirty-two 2000 T3/CL lamps form the heating unit for Zone I; Zones II and III each consist of fifty-six 1000 T3/CL lamps (See Fig. B-2 for heating zones).

The maximum power dissipation of the entire unit is 478 KVA which can be increased by replacing the lamps of Zones II and III with 2000 T3/CL lamps. The capacity of the unit corresponds to 100 BTU/sq. ft/sec. The heat input to the model is less, depending upon a number of parameters including distance from model, model configuration, model surface, etc. The efficiency is estimated to be 50-85%.

The oven is equipped with polished aluminum heat shields to maintain zonal temperature control. Two sets are provided; one set for conical shells and the other for hemispherical shells.

The model SPG6266W three-phase temperature controller provides zonal control of temperature in the oven by proportioning the a-c heater power through ignitrons controlled by a closed-loop thermocouple feedback circuit. Temperatures may be automatically maintained to within 0.5% by employing precision potentiometric circuitry held in continuous calibration by a standard cell. The load power may also be manually varied from zero to full power by a front panel dial. The temperature controller is used to program temperature as a function of time by switching to the remote temperature programmer. The temperature controller can proportion a continuous load power of 390 KVA for the three phases with water-cooled ignitrons. Each load phase is controlled independently, thereby providing three separate zones.

The controller is a complete three-phase ignitron powered control unit combined with three d-c potentiometric circuits and high gain a-c amplifiers. Accurate specimen temperature control is accomplished by comparing a low level d-c potentiometer milli-voltage with the thermal electromotive force from the specimen thermocouple. Any positive error between these voltages is chopped and a-c amplified to a relatively high voltage level. The amplified a-c output voltage is converted back to d-c in a diode modulator circuit and used to control the firing angle of the ignitrons. The ignitrons, in turn, proportion the power to the oven to restore the thermocouple to the set point temperature and to reduce the low level error to within the proportional band of control. Compensation for ambient temperature changes at the thermocouple cold junction is automatic.

(b) Temperature Programmer

The Research, Inc., Data-Trak function generator (FGE5035) provides an output proportional to the ordinate of the chosen temperature-time function. It is a curve follower in that two conducting lines bordering the desired curve generate a high electrostatic potential field so a null-seeking probe can be servo-driven to follow the zero potential line.

The temperature-time graph is fastened to each of three drums such that the abscissa (time) is the rotational position of the drum. The probe is mounted between two parallel slide bars beneath the drum. As the servo positions the probe, a flexible cable from the probe carriage rotates a precision potentiometer to provide an output voltage proportional to the translational position of the probe.

Since there is no contact between the conducting lines or the probe the system does not rely upon sensing current or a magnetic flux.

One can also regulate the drum cycle from 10 to 300 seconds providing a flexible time scale. In addition, by placing the generator in a "hold" condition a given situation can be maintained for as long as desired.

(c) Vacuum-Pressure System

The vacuum system is automatic with a capability of permitting a pressure difference across the shell to vary within $\pm 1/8$ inch of mercury. The mercury column is positioned by two pins, $1/4$ inch apart, through the relay and the vacuum pump. The tank acts as a reservoir and a check valve while the relief valve allows the system to be opened to the atmosphere at will.

B.2 Instrumentation

Modified HT-600 strain gages, manufactured by the High Temperature Instruments Corporation are used. The gage is modified in that in addition to the HT-600 alloy, which has a negative coefficient of resistivity with temperature, an alloy with a positive coefficient is employed, thereby reducing the apparent strain. The gages are applied in a meridional and parallel direction at four stations along a meridian using Allen PBX cement (See Fig B-2). Obtaining a good bond between either the pre-coat and the aluminum surface or the second coat and the pre-coat was not always successful. As available testing time became short this dictated operation with less than full strain gage instrumentation. Each gage used a three wire connection to the bridge thereby negating effects of change in lead wire resistance. The signal is picked off a one gage bridge, recorded, and then corrected for temperature effects. Since the apparent strain versus temperature curves varied widely for different gages, each gage was provided with individual curves. Also, the nature of the curves changed with each temperature cycle thereby necessitating strain-temperature data not more than one cycle distant from the test run.

To calibrate the strain gage circuits, a decade resistance is shunted around the active gage and the resistance varied while the galvanometer deflection is recorded. Knowing the gage factor, one can interpret the resistance change as a strain and the deflection then calibrated in terms of micro-inches/inch/inch of deflection.

The temperature recording circuit is simply a measurement of the current induced by the potential difference between the hot and cold junctions. On later models differential thermocouples (Fig B-3) provided data on the temperature difference across the shell wall. One junction is a spot welded to the inner surface while the other, after being led through a small hole in the wall a distance from the measuring area, was cemented opposite to the previous junction. Since one junction had to be electrically insulated from the other, yet in good thermal contact with the outer surface, the mechanical bond of the copper oxide cement used often failed, providing only qualitative data on the temperature differential. An aluminum foil-backed tape shielded the outer junction from direct radiation while also sealing the hole in the shell wall. The calibration procedure is as follows: the hot and cold junctions are brought to the same temperature; a voltage is induced across the thermocouple resistance and the galvanometer by a precision millivolt potentiometer; with the aid of standard potential difference-temperature charts for the particular thermocouple, the circuit is calibrated for $^{\circ}\text{F}$ versus deflection.

All the instrumentation wiring in the model was supported by a brass center post bolted to the apex of the model. Brass rod branches from the center post supported leads to the individual gages.

A porcelain filter disk provided a means of passing the wiring through a pressure differential at high temperatures. The individual leads passed through the holes in the disk with the PBX cement acting as a potting agent. The disk was then bolted to the steel mounting plate atop a quad ring providing a pressure seal.

B.3 Test Specimens

The test specimens are shown in Figs B-4 and B-5. The specimen is attached to a 5/8 inch thick steel plate. A silicone rubber "Quad" ring is inserted around the specimen support to provide an adequate vacuum seal.

The specimens are coated with soot to increase the absorptivity and to provide a common surface for all tests.

B.4 Material Properties

Although the model manufacturer and Alcoa assured the investigators that soaking the models for a period of thirty minutes at 650 $^{\circ}\text{F}$ anneals the material and removes any residual stresses (produced in spinning), a series of supplementary tests were conducted to check certain important mechanical properties of the material.

Models were tested before and after soaking at room temperature under the action of a concentrated load. It was felt that this would

produce the desired biaxial stress condition without complicating the test program.

The tests were limited to the determination of the modulus of elasticity and of Poisson's Ratio. Because of the considerable scatter of test data, the investigators feel that these data are valueless to the investigation and are omitted.

The mechanical properties involved in the analysis are: shear modulus of elasticity, Poisson's ratio, coefficient of viscosity, and the coefficient of linear thermal expansion.

(a) Shear Modulus of Elasticity

The following approximate values for shear modulus are given:

Temperature (°F)	75	212	300	400	500	600	700
$G \times 10^{-6}$ psi	3.8	3.7	3.6	3.4	3.0	2.6	1.5

(b) Poisson's Ratio

At elevated temperatures, Poisson's ratio is a multivaried function of elastic strain, plastic strain, creep strain (time dependent), and anisotropy (varies with strain). Poisson's ratio, therefore, is no longer a unique property. Tests conducted by the investigators show a range of values from 0.33 to 0.45. At elevated temperatures, it seems that the results are highly sensitive to the specimen geometry and test procedure, rather than a strong dependence on the properties of the material.

An attempt to eliminate such variables is useless without an exhaustive number of well controlled tests.

(c) Coefficient of Viscosity

No attempt was made to establish experimentally the coefficient of viscosity of the subject material. To the knowledge of the writers, no data exist of the variation of viscosity with temperature below the melting point for aluminum alloys. Although some attempts have been made to establish such a viscosity-temperature curve for pure aluminum, the use of the results for the present application is highly questionable.

It is agreed, however, that the viscosity-temperature curve for pure metal may be quite different from that of alloys. The alloy may show:

1. Different rate of decrease of viscosity with increase of temperature.

2. Different rate at which the slope of the viscosity-temperature curve increases as the liquid state is approached.

3. A greater coefficient of viscosity.

Based on tests conducted above the melting points (6), the results of pure aluminum and aluminum alloys indicate the following:

1. In all cases, the viscosity-temperature curve of the low per cent alloys (99% aluminum) is similar to that of the pure metal.

2. For higher per cent alloys (98% or less aluminum), the presence of the alloy had a characteristic effect.

Since the coefficient of viscosity is one of the principal parameters in the temperature-dependent viscoelastic analysis, an attempt is made to estimate the viscosity-temperature curve for the material of the present investigation. An expression of the form

$$\eta = CG \exp(H/RT)$$

Where C is a constant, G is modulus of rigidity (dynes/cm²), H is heat of activation (calories/mole), R is the gas constant, and T is the absolute temperature (°K), is used to represent the curve. This functional relationship is a typical form of the variation of viscosity with temperature for some polycrystalline solids and liquids.

In the present investigation, the empirical expression was taken as:

$$\eta = 1.819 \times 10^{-20} G \exp (18000/T)$$

the above expression is based on established values by Alcoa (2) at the temperature range for aluminum alloys.

The graph of this expression is shown in Fig. 2.1.3-1 together with the results obtained by K&E (16) for pure aluminum.

(d) Coefficient of Thermal Expansion

No attempt was made to investigate the coefficient of thermal expansion. The values used in the analysis are based on Reference 2 and communication with Alcoa.

Fig. 2.1.3-2 shows the variation of these properties with temperature for 6061 aluminum alloy.

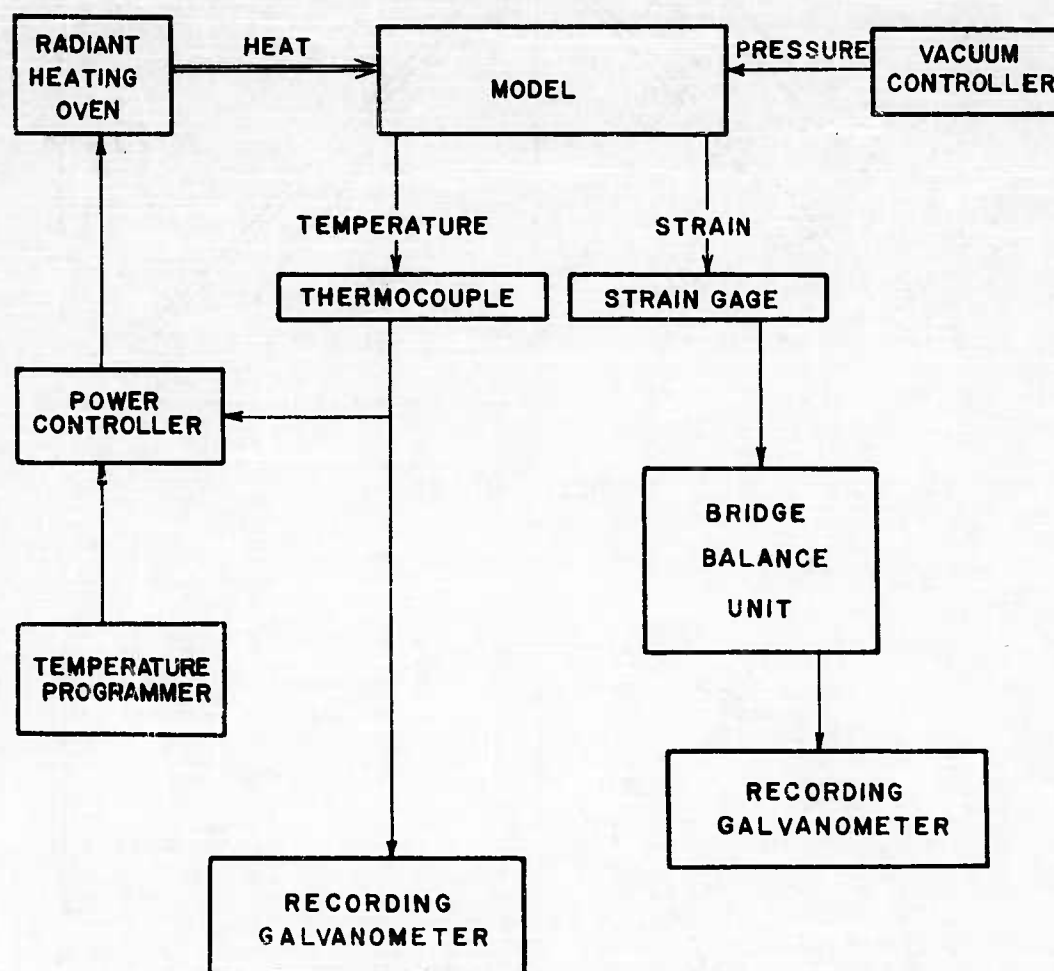
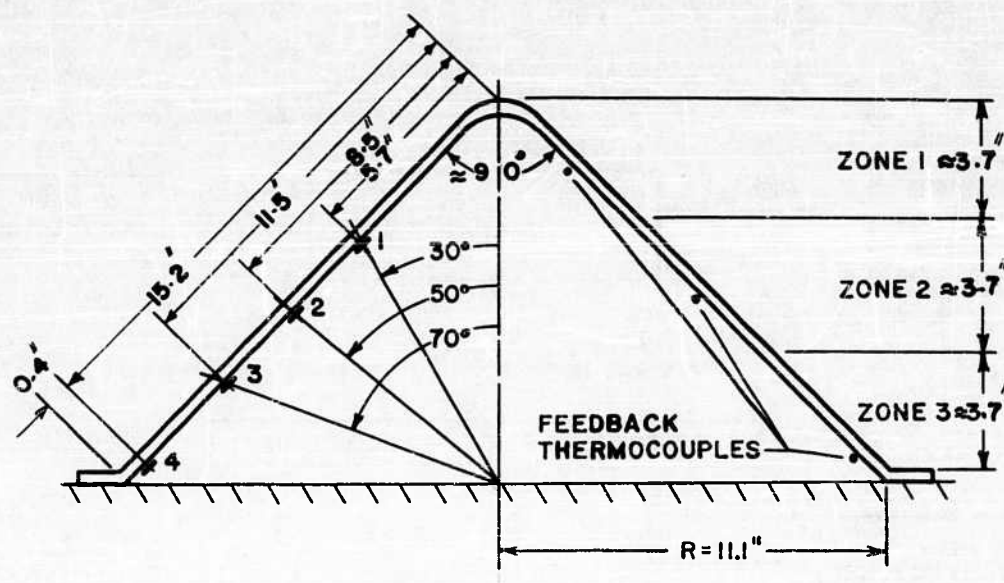
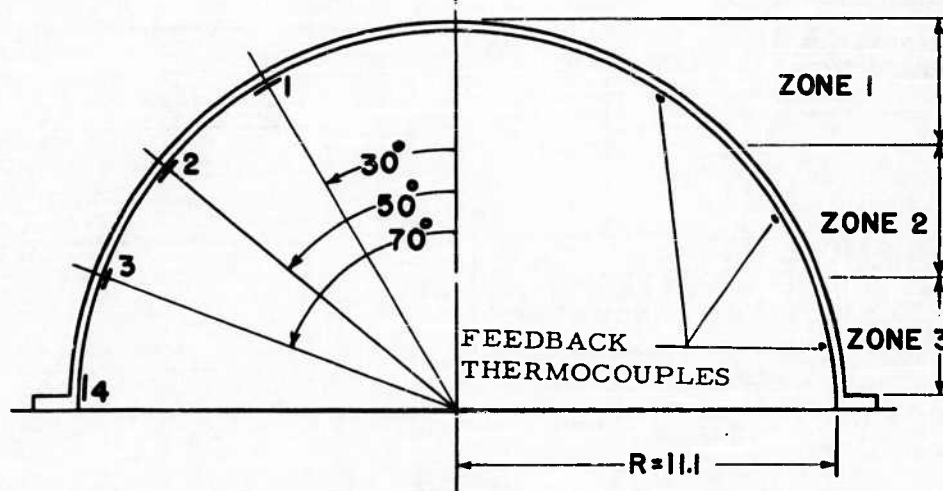


Fig B-I
TESTING FACILITY BLOCK DIAGRAM



CONICAL MODEL



HEMISPHERICAL MODEL

FIG. B-2A
INSTRUMENTATION

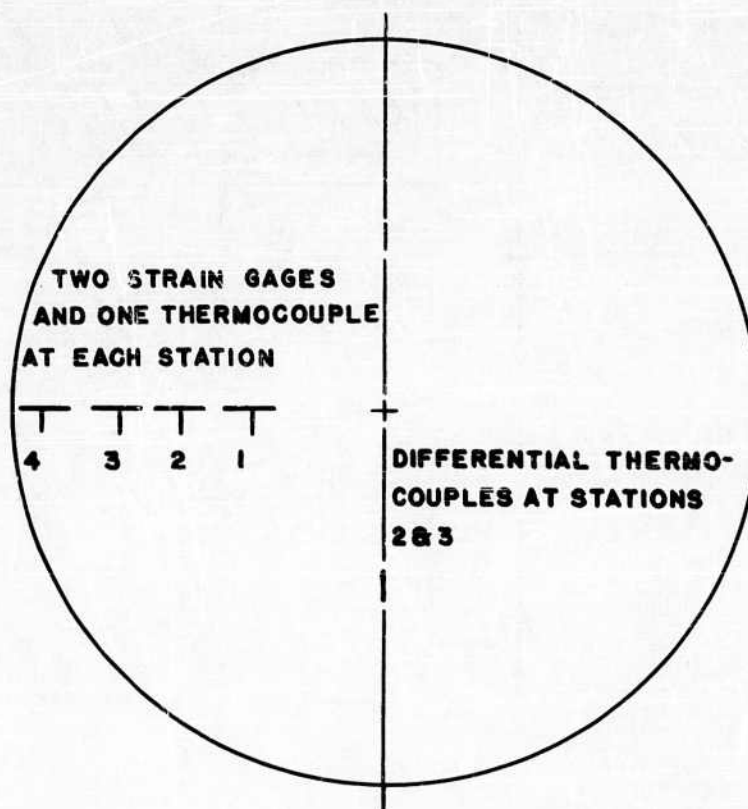


Fig. B-2 B
INSTRUMENTATION

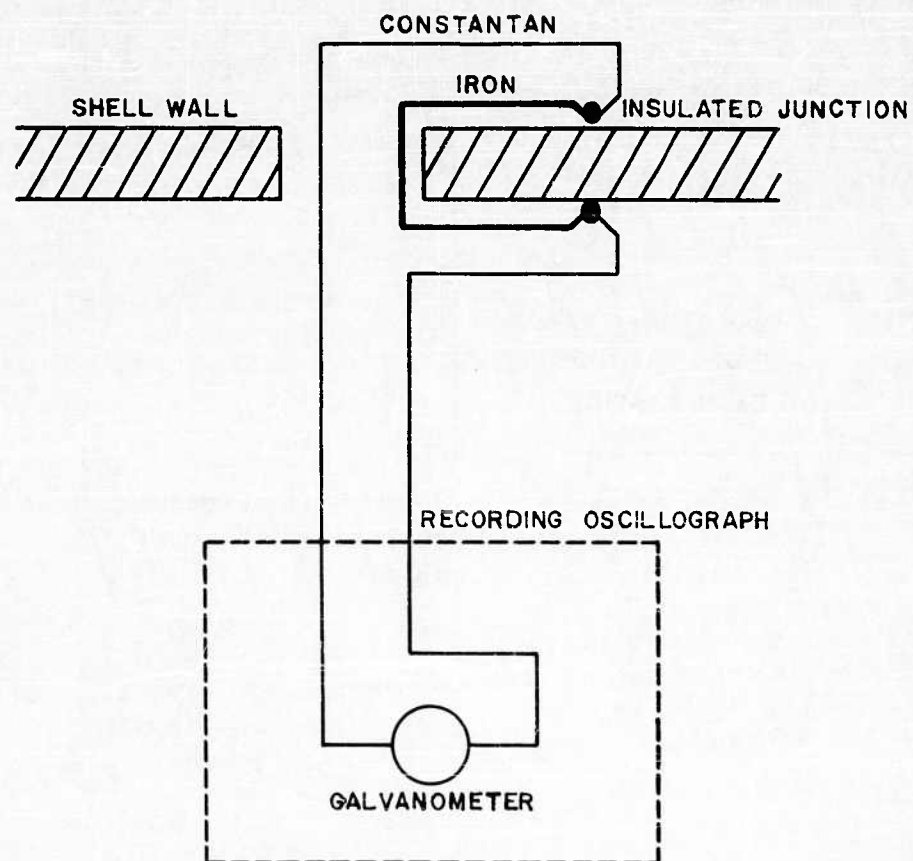


Fig. B-3
DIFFERENTIAL THERMOCOUPLE CIRCUIT



Fig. B-4 Conical Shell Model



Fig. B-5 Hemispherical Shell Model



Fig. B-6 Instrumented Conical Shell Model

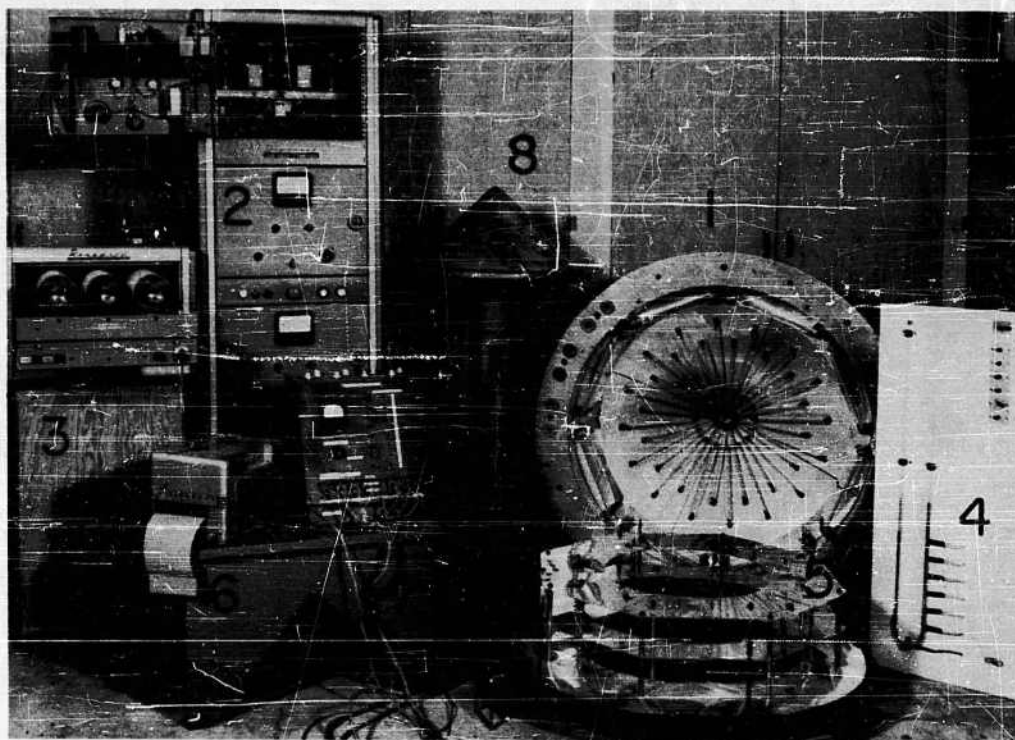


Fig. B-7 Experimental Facilities

LEGEND 1. Radiant Heating Oven, 2. Power Controller,
3. Temperature Programmer, 4. Vacuum Controller, 5. Zonal
Heat Shields, 6. Oscillograph, 7. Bridge Balance Unit,
8. Model.

APPENDIX C
EXPERIMENTAL DATA

TABLE C-1 TEST PROGRAM

MODEL	STEADY STATE TEMPERATURE	RATE ($^{\circ}\text{F}/\text{sec}$)	PRESSURE
1/32" Cones	400 $^{\circ}\text{F}$	5	None
	400 $^{\circ}\text{F}$	10	None
	500 $^{\circ}\text{F}$	20	None
	(1) 400 (2) 300 (3) 200	(1) 10 (2) 5 (3) 2.5	None
1/16" Cones	300 $^{\circ}\text{F}$	2.5	None
	400 $^{\circ}\text{F}$	5	None
	500 $^{\circ}\text{F}$	10	None
	400 $^{\circ}\text{F}$	10	22" Hg
	500 $^{\circ}\text{F}$ *	20	22" Hg
	(1) 500 (2) 400 (3) 300	(1) 50 (2) 40 (3) 30	22" Hg
1/8" Cones	300 $^{\circ}\text{F}$	2.5	None
	400 $^{\circ}\text{F}$	5	None
	500 $^{\circ}\text{F}$	10	None
	400 $^{\circ}\text{F}$	10	24" Hg
	500 $^{\circ}\text{F}$	20	23" Hg
	(1) 400 (2) 300 (3) 200	(1) 10 (2) 5 (3) 2.5	24" Hg
	(1) 400 (2) 300 (3) 200	(1) 20 (2) 10 (3) 5	23" Hg
	(1) 500 (2) 400 (3) 300	(1) 10 (2) 5 (3) 2.5	23" Hg
	(1) 500 (2) 400 (3) 300	(1) 20 (2) 10 (3) 5	23" Hg
	(1) 500 (2) 400 (3) 300	(1) 50 (2) 40 (3) 30	23" Hg
1/16" Hemispheres	400	5	None
	400	5	21" Hg
	500	10	None
	(1) 400 (2) 300 (3) 200	(1) 10 (2) 5 (3) 2.5	21" Hg
	(1) 500 (2) 400 (3) 300	(1) 10 (2) 5 (3) 2.5	21" Hg
	(1) 500 (2) 400 (3) 300	(1) 20 (2) 10 (3) 5	21" Hg
	(1) 500 (2) 400 (3) 300	(1) 50 (2) 40 (3) 30	18" Hg
	(1) 500 (2) 400 (3) 300	(1) 50 (2) 40 (3) 30	18" Hg

* Repeated

TABLE C-2

Temperatures and Stresses in 1/32" Thick Cone for a Program of 5 °F/sec. to 400 °F.

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	73	0	0	73	0	0	73	0	0	73	0	0
5	89	-10	20	80	284	91	76	324	162	78	284	274
10	117	+10	90	106	80	202	100	768	585	108	1120	1300
15	147	-30	100	127	71	273	125	697	465	125	1760	1890
20	177	-50	40	155	50	240	155	660	340	143	2410	2310
25	200	-79	40	178	70	287	179	640	370	158	2880	2560
30	225	-108	29	201	-70	396	207	831	495	177	3160	2740
35	251	-193	125	228	-19	403	236	754	353	193	3690	3085
40	276	-261	125	250	-68	438	267	912	446	210	3870	3120
45	301	-315	230	278	-87	445	295	874	317	224	4250	3240
50	324	-378	265	301	-210	563	323	928	104	242	4420	3240
55	350	-409	251	326	-122	567	351	874	102	258	4680	3280
60	373	-524	248	352	-102	400	381	933	82	274	4950	3280
90	391	-445	246	407	36	108	438	149	201	274	4400	2625
120	397	-234	9	424	124	70	448	191	452	277	4265	2490
150	400	126	72	430	264	229	448	339	670	282	4200	2380
180	400	0	135	430	396	387	448	583	940	286	4175	2285
210	400	99	288	430	510	502	446	766	1140	286	4250	2275
240	400	108	315	430	581	563	446	783	1190	290	4100	2040
270	400	108	333	430	660	642	442	1045	1280	292	4000	2020
300	400	162	378	430	748	722	442	1140	1500	295	4000	1890
330	400	225	441	430	792	774	442	1140	1500	298	3980	1840
360	400	225	441	430	845	827	442	1200	1520	300	3840	1700
390	400	270	459	430	854	854	442	1260	1700	300	3840	1700
420	400	270	459	430	862	871	442	1260	1700	300	3840	1700

TABLE C - 3

Temperatures and Stresses in 1/32" Thick Cone for a Program of 10 °F/sec. to 400 °F.

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	90	0	0	91	0	0	91	0	0	91	0	0
5	112	40	0	100	320	200	97	1400	1000	108	190	30
10	166	120	40	142	492	210	136	2650	1680	156	1630	2410
15	220	-305	108	186	80	458	186	2517	995	198	2653	3792
20	275	-444	-193	235	78	539	238	2560	980	232	4018	5086
25	326	-510	-425	285	96	538	294	2800	980	270	5170	6200
30	375	-524	-552	330	141	432	347	2740	567	301	6200	6952
35	326	-364	-546	360	-259	370	384	1146	-127	294	5683	6520
40	375	-245	-364	371	0	92	398	864	-135	288	5340	6130
45	389	-73	-164	380	37	27	410	823	-54	288	5070	5880
50	388	9	90	387	146	63	418	640	-142	286	4990	5760
55	388	55	18	392	208	136	423	522	-177	286	4920	5640
60	388	145	46	396	235	235	428	610	-35	286	4920	5540
90	391	428	228	414	347	356	432	580	35	288	4920	5540
120	393	610	320	424	549	610	442	802	540	290	4830	5540
150	393	720	410	424	673	788	442	872	750	290	4800	5400
180	393	837	491	424	761	823	442	942	977	298	4800	5400
210	393	900	555	424	903	1044	442	977	1080	300	4800	5400
240	391	992	646	424	1044	1200	438	970	1150	304	4560	5210
270	391	992	646	424	1090	1270	438	1000	1260	308	4550	5180

TABLE C - 4

Temperatures and Stresses in 1/32" Thick Cone for a Program of 200 F/sec. to 5000 F.

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	66	0	0	66	0	0	66	0	0	66	0	0
2	66	0	0	66	0	0	66	0	0	66	0	0
4	99	81	121	74	131	-464	70	1116	639	84	-873	-560
6	146	50	150	102	121	-858	94	1654	913	133	-1650	-2050
8	190	-20	200	131	10	-970	124	1740	636	168	-3580	-3370
10	235	-127	127	167	-20	-1140	161	1870	620	200	-3600	-3930
12	276	-270	-97	202	-20	-1140	200	1980	535	250	-5000	-4740
14	321	-455	-275	243	225	-980	242	2050	333	262	-6270	-5710
16	362	-594	-417	275	290	-878	278	2180	241	289	-7080	-6200
18	404	-657	-603	314	390	-693	320	2230	-104	317	-7640	-6560
20	445	-617	-852	350	409	-595	358	2550	110	347	-7840	-6300
22	484	-768	-1085	387	455	-400	396	2880	154	369	-8200	-6210
24	512	-940	-1250	420	62	-542	435	2100	-240	383	-8220	-6100
26	515	-867	-1190	437	385	-507	460	342	-1026	371	-8000	-5640
28	510	-745	-1150	447	-597	-631	476	-420	-1400	359	-7300	-5510
30	510	-704	-1030	455	-602	-507	487	-769	-1560	353	-7200	-5100
35	509	-526	-842	470	-304	-460	510	-	-	347	-6600	-5100
40	506	-300	-672	482	-234	-150	522	-	-	343	-6620	-5125
45	506	-194	-518	490	-215	-83	535	-	-	341	-6350	-4950
50	506	-113	-445	498	-238	-156	542	-	-	341	-6350	-4950
55	506	-57	-390	503	-230	-148	547	-	-	341	-6350	-4950
60	506	-16	-332	510	-186	-24	553	-	-	341	-6340	-4900
180	512	729	356	540	465	883	563	-1330	-444	341	-5880	-3920
300	515	826	526	540	907	1450	565	-1060	-37	363	-5200	-3210
420	515	1050	575	545	1050	1590	565	-791	200	386	-4940	-2940
540	515	1070	535	551	1140	1730	565	-1010	150	392	-4800	-2730
660	515	1134	591	551	1220	1875	565	-1010	150	400	-4530	-2550

TABLE C -5

Temperatures and Stresses in 1/32" Thick Cone for a Zonal Program:
 Zone 1, 10 °F/sec. to 400 °F; Zone 2, 5 °F/sec. to 300 °F; Zone 3, 2.5 °F/sec. to 200 °F.

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	93	-	0	93	0	0	93	0	0	93	0	0
5	104	-	50	93	225	125	93	250	220	93	50	10
10	142	-	161	93	313	131	100	869	788	101	272	0
15	182	-	200	121	333	90	109	1020	768	103	111	141
20	219	-	424	142	260	161	119	919	686	105	121	191
25	258	-	601	160	190	190	126	690	430	108	60	252
30	298	-	830	180	100	209	136	670	420	113	151	222
35	317	-	893	200	-	198	152	970	620	126	432	774
40	328	-	695	217	-	285	168	970	680	134	543	985
45	335	-	639	238	-	225	184	650	480	139	532	1075
50	335	-	564	242	-	49	190	446	377	136	442	954
55	335	-	564	245	-	68	196	544	435	136	600	1000
60	335	-	545	245	39	127	200	653	653	134	482	924
90	333	-	253	248	205	283	204	713	752	134	382	754
120	330	-	183	250	331	409	206	683	831	134	382	754
150	330	-	132	250	341	458	206	683	831	134	382	754
180	328	-	85	250	390	526	206	683	831	134	382	754
210	326	-	65	250	409	565	206	713	920	134	382	754
240	325	-	56	250	478	614	206	732	980	138	231	704
270	325	-	9	250	517	653	206	732	980	138	231	704
300	325	-	9	250	565	663	206	881	1100	138	221	653
330	325	-	47	250	585	720	206	881	1100	138	221	653
360	325	-	47	250	585	720	206	881	1100	138	221	653

TABLE C - 6
Temperatures and Stresses in 1/16" Thick Cone for a Program of 2.5 °F/sec. to 300°F

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	69	0	0	67	0	0	67	0	0	67	0	0
5	75	51	71	69	172	142	69	223	274	89	19	15
10	85	112	30	83	203	274	83	263	354	102	1840	-1630
15	99	91	10	96	263	283	98	323	364	115	2500	-2460
20	112	151	81	111	323	364	114	293	354	126	2950	-3140
25	125	91	10	126	271	331	128	280	400	138	3515	-3900
30	139	81	30	140	279	337	144	228	307	150	3920	-4435
35	151	121	81	152	278	337	161	208	337	161	4380	-5040
40	164	80	40	166	226	293	176	224	331	172	4930	-5720
45	177	112	80	181	156	293	193	239	398	183	5360	-6190
50	191	69	40	195	175	331	208	251	367	194	5860	-6760
55	203	59	59	208	321	347	224	297	374	205	6280	-7230
60	215	109	59	222	297	307	239	171	307	216	6780	-7781
65	230	20	88	234	171	307	253	47	113	227	7390	-8260
70	241	98	10	248	131	216	269	225	291	237	7970	-8760
75	252	49	49	262	225	291	283	310	319	248	8360	-9130
80	266	125	212	276	375	413	299	310	319	258	8950	-9600
85	279	38	125	289	469	450	314	497	535	265	9450	-10700
90	287	125	163	302	563	610	327	535	535	265	9720	-10280
120	304	286	267	325	784	888	344	722	694	260	8900	-9590
150	313	352	361	336	822	954	346	103	103	260	8650	-9250
180	320	492	540	341	954	954	346	112	112	260	8370	-8940
210	324	595	652	343	973	973	344	140	140	260	8260	-8830
240	328	680	775	343	879	879	342	168	168	260	8030	-8580
270	330	737	832	343	879	879	340	9	9	258	8030	-8540
300	330	737	832	343	879	879	338	47	47	258	7900	-8370
330	331	784	888	343	563	535	337	103	103	256	7900	-8330
360	331	822	954	342	638	610	336	112	112	256	7850	-8270
390	331	851	954	342	685	675	335	140	140	255	7770	-8160
420	331	879	973	342	722	694	334	168	168	255	7700	-8050

TABLE C - 7

Temperatures and Stresses in 1/16" Thick Cone for a Program of 5 °F/sec. to 400 °F

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	80	0	0	80	0	0	80		0	80	0	0
5	94	20	223	89	102	284	92		- 70	103	- 1310	- 1310
10	119	0	141	117	788	616	124		- 151	133	- 3150	- 3160
15	144	80	180	147	752	591	151		- 280	153	- 4230	- 4200
20	170	60	140	172	680	580	180		- 50	173	- 5450	- 5220
25	193	- 20	30	199	663	525	209		- 405	193	- 6450	- 6130
30	219	69	158	225	600	520	216		- 382	216	- 7490	- 6960
35	244	78	215	249	673	497	236		- 350	236	- 8580	- 7900
40	269	49	116	275	704	463	255		- 470	255	- 10400	- 8640
45	293	77	115	303	696	496	267		- 482	277	- 10100	- 9254
50	318	133	209	328	650	490	355		- 567	297	- 10800	- 9800
55	343	- 28	103	353	577	409	385		- 582	315	- 11500	- 10200
60	368	- 221	626	378	487	367	414		- 543	333	- 12060	- 10750
65	381	- 311	1043	398	270	216	433		- 616	339	- 11900	- 10600
70	385	- 328	1201	408	171	45	440		- 513	335	- 11300	- 10200
75	388	- 346	1265	412	117	27	447		- 441	331	- 11000	- 9900
80	393	- 382	1356	417	106	9	450		- 441	331	- 10800	- 9770
85	395	- 364	1430	424	159	44	454		- 435	331	- 10800	- 9770
90	407	- 362	1475	426	180	18	456		- 413	331	- 10600	- 9600
120	415	- 261	1460	443	174	70	461		- 215	331	- 10200	- 9320
150	420	62	1370	449	294	173	461		- 68	331	- 10000	- 9090
180	423	9	1415	452	536	389	461		85	331	- 9770	- 8850
210	426	185	1340	452	666	450	458		188	331	- 9570	- 8670
240	426	273	1260	452	822	554	454		250	331	- 9500	- 8530
270	428	273	1180	452	960	666	452		325	334	- 9320	- 8360
300	430	273	1180	452	1020	683	451		328	334	- 9220	- 8230
330	432	273	1180	452	1100	770	450		380	334	- 9200	- 8175
360	435	343	1117	452	1140	780	447		458	335	- 9000	- 7900
390		370	1110	451	1160	790	446		485	335	- 8970	- 7900
420		378	1065	451	1160	790	446		485	335	- 8970	- 7900

TABLE C - 8

Temperatures and Stresses in 1/16" Thick Cone for a Program of 10 °F/sec. to 500°F

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	66	0	0	66	0	0	66	0	0	66	0	0
5	76	71	223	70	528	426	73	528	426	83	1560	- 1070
10	121	- 353	- 181	115	626	757	123	626	757	142	5310	- 4960
15	175	- 289	- 70	166	860	1140	178	860	1140	188	7800	- 7650
20	224	- 315	- 216	220	955	1014	233	955	1014	233	10090	- 9900
25	274	- 251	- 106	273	1030	1050	293	1030	1050	276	1190	- 11570
30	325	- 480	- 453	327	888	936	354	888	936	320	13880	- 13280
35	378	- 660	- 761	380	576	714	414	576	714	360	15780	- 14860
40	427	- 628	- 584	433	510	580	471	510	580	400	16690	- 15380
45	470	- 726	- 701	482	300	359	528	300	359	436	17310	- 15890
50	482	- 670	- 770	504	- 300	- 138	554	- 300	- 138	431	16040	- 14930
55	487	- 616	- 757	515	- 536	- 264	563	- 536	- 264	420	15050	- 14190
60	491	- 589	- 805	525	- 537	- 284	571	- 537	- 284	420	14840	- 14040
90	507	- 235	- 575	556	- 574	- 250	591	- 574	- 250	418	13790	- 13300
120	517	128	- 272	568	- 377	- 52	588	- 377	- 52	418	13200	- 12830
150	523	329	- 88	573	- 88	146	583	- 88	146	420	12820	- 12470
180	528	569	245	575	146	350	580	146	350	420	12400	- 12000
210	529	717	355	575	300	438	576	300	438	420	12250	- 11800
240	530	812	473	575	467	569	574	467	569	422	12100	- 11660
270	530	906	527	575	518	628	574	518	628	422	11970	- 11500
300	533	905	562	575	620	708	572	620	708	425	11800	- 11320
330	535	811	546	575	723	781	570	723	781	425	11770	- 11200
360	538	1045	694	575	737	825	570	737	825	426	11750	- 11150
390	542	1046	760	575	759	832	568	759	832	426	11550	- 10930
420	542	1085	775	575	854	883	565	854	883			

TABLE C - 9
 " Temperatures and Stresses in 1/16 Thick Cone for a Program of 10 °F/sec. to 400 °F
 with 22 " Hg Differential Pressure.

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	67	-264	-528	66	-508	-1066	65		-903	66	3880	-1180
5	93	-374	-405	80	102	-254	76		-1210	92	1230	-3070
10	141	-551	-521	129	171	71	124		-2210	149	-2800	-6300
15	191	-645	-556	178	259	220	176		-2315	196	-5560	-8590
20	240	-666	-530	228	294	294	226		-2350	237	-7440	-9050
25	289	-586	-336	278	318	309	284		-2180	276	-8890	-11130
30	338	-770	-573	326	142	161	336		-2130	316	-10000	-12000
35	377	-1110	-1120	371	-294	-138	389		-2250	347	-10900	-12660
40	384	-1130	-1301	388	-855	-710	414		-2260	338	-9570	-11660
45	390	-1100	-1280	395	-1095	-996	425		-2240	332	-8820	-11100
50	393	-1150	-1440	404	-1200	-1070	436		-2260	330	-8460	-10800
55	395	-1070	-1390	409	-1226	-1100	443		-2180	330	-8180	-10610
60	397	-1090	-1460	414	-1282	-1080	446		-2240	327	-7914	-10440
90	410	-850	-1230	437	-1320	-1190	458		-2150	325	-7350	-10050
120	416	-694	-1100	450	-1100	-1040	458		-2050	325	-7144	-9860
150	422	-573	-890	454	-886	-920	456		-1950	325	-6871	-9640
180	426	-336	-540	455	-714	-774	453		-1860	325	-6730	-9440
210	430	-246	-396	455	-568	-671	453		-1800	326	-6650	-9280
240	430	-194	-334	455	-464	-610	450		-1750	326	-6530	-9240
270	430	-132	-282	455	-344	-482	446		-1770	329	-6320	-9052
300	430	-123	-246	455	-301	-525	446		-1770	329	-6251	-9000
330	433	-144	-141	455	-250	-464	446		-1700	329	-6170	-8840
360	435	-35	-97	455	-163	-396	443		-1700	329	-6080	-8770
390	438	9	44	455	-163	-396	443		-1650	329	-5940	-8550
420	440	35	44	455	-163	-396	443		-1650	329	-5940	-8550

TABLE C - 10

Temperatures and Stresses in 1/16" Thick Cone for a Program of
20 °F/sec. to 500°F with 22" Hg Differential Pressure

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	81	- 524	- 524	80	- 661	- 1090	82		- 741	85	1090	125
2	124	- 675	- 68	107	- 1026	- 1070	97		- 1400	108	2340	- 891
4	166	- 1013	- 280	140	- 1230	- 1380	140		- 1600	150	- 678	- 4150
6	205	- 1320	- 514	179	- 1210	- 1320	174		- 1940	183	- 2576	- 6350
8	246	- 1390	- 585	217	- 1225	- 1290	214		- 2050	216	- 4540	- 8300
10	286	- 1600	- 885	253	- 1200	- 1260	253		- 2020	243	- 6460	- 9790
12	330	- 2020	- 1340	293	- 1290	- 1345	297		- 1930	277	- 8140	- 11160
14	368	- 2570	- 1855	330	- 1440	- 1480	334		- 2130	307	- 9750	- 12300
16	407	- 2660	- 2050	364	- 1640	- 1570	372		- 2300	332	- 11680	- 13730
18	444	- 2610	- 1820	395	- 1780	- 1690	410		- 2330	360	- 13300	- 15000
20	480	- 2830	- 1870	430	- 1630	- 1630	450		- 2300	390	- 14400	- 15900
25	490	- 2620	- 2245	462	- 1910	- 1940	495		- 2580	384	- 12400	- 14400
30	490	- 2360	- 2090	470	- 1890	- 2090	515		- 2620	378	- 11300	- 13650
35	490	- 2250	- 2040	478	- 1890	- 2090	523		- 2590	375	- 10760	- 13000
40	490	- 2120	- 1980	483	- 1850	- 2030	530		- 2630	375	- 10470	- 13000
45	490	- 1980	- 1900	487	- 1870	- 2040	536		- 2550	375	- 10300	- 13000
50	490	- 1825	- 1825	493	- 1860	- 2000	540		- 2570	375	- 10300	- 13000
55	490	- 1750	- 1790	495	- 1860	- 2000	544		- 2620	375	- 10300	- 13000
60	490	- 1600	- 1640	500	- 1840	- 1970	545		- 2620	375	- 10000	- 12800
90	490	- 1100	- 1260	512	- 1600	- 1800	545		- 2460	375	- 9790	- 12700
120	490	- 807	- 940	515	- 1270	- 1510	542		- 2420	380	- 9600	- 12600
150	490	- 553	- 697	515	- 970	- 1260	540		- 2320	380	- 9600	- 12600
180	490	- 276	- 310	515	- 564	- 960	540		- 2320	380	- 9600	- 12600
210	490	- 66	- 55	515	- 426	- 852	536		- 2290	383	- 9320	- 12350
240	490	66	110	515	- 266	- 680	536		- 2290	385	- 9440	- 12400
270	490	144	200	515	- 75	- 618	533		- 2310	385	- 9320	- 12400
300	490	254	332	515	- 53	- 575	530		- 2260	385	- 9260	- 12300
600	490	553	697	510	160	- 437	530		- 2690	406	- 7600	- 10900
900	490	686	818	510	266	- 383	525		- 3220	412	- 6670	- 10200
1140	490	752	918	510	372	- 330	520		- 2120	417	- 6000	- 9600

TABLE C - 11

Temperatures and Stresses in 1/16" Thick Cone for a Program
of 20 °F/sec. to 500°F with 22" Hg. Differential Pressure, with collapse

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	84	- 524	- 456	84	-	-	84	- 1070	- 1780	84	2300	- 1000
5	194	-1090	- 179	167	752	935	171	- 1240	- 2020	180	2970	- 6640
10	294	-1730	- 853	258	316	294	269	- 1300	- 2170	262	7700	- 10230
16.4	416	-2740	- 1860	372	90	440	396	- 1500	- 2400	356	12140	- 12978
20	486	-3260	- 2410	440	291	31	474	- 1840	- 2920	410	11300	- 13300
25	490	-3000	- 2930	472	49	522	513	- 1900	- 2950	395	11900	- 12020
30	490	-2830	- 2880	480	725	1200	531	- 2080	- 3050	386	9200	- 11440
35	490	-2700	- 2830	490	693	1310	543	- 2200	- 3130	383	8760	- 11100
40	490	-2640	- 2820	497	686	1430	552	- 2370	- 3230	380	8530	- 11000
45	490	-2530	- 2770	500	735	1490	557	- 2470	- 3340	380	8360	- 10940
50	490	-2460	- 2780	505	745	1530	563	- 2570	- 3350	380	8190	- 10800
55	490	-2360	- 2750	510	754	1550	565	- 2500	- 3330	380	8090	- 10750
60	490	-2320	- 2700	512	820	1600	565	- 2430	- 3370	380	8010	- 10700
120	492	-1670	- 2200	525	392	1390	565	- 1820	- 2880	386	7620	- 10550
180	492	-1220	- 1720	527	125	1070	565	- 1350	- 2670	400	7750	- 10700
240	492	-1030	- 1500	525	384	910	560	- 1220	- 2690	400	7150	- 10080
300	492	-987	- 1330	522	634	740	550	- 990	- 2570	400	7420	- 10080
332.5	490	-930	- 724	522	295	920	550	- 1250	- 3070	400	6710	- 9670
332.5	490	-959	- 526	522	384	670	550	- 1230	- 2950	400	6700	- 9620
360	490	-1580	- 733	515	36	693	535	- 2290	- 4070	393	6320	- 9530
390	487	-2850	- 1650	515	567	1290	533	- 3770	- 6000	390	6180	- 9660
420	487	-3730	- 2250	510	990	1930	530	- 5900	- 9360	390	6040	- 9700

TABLE C - 12

Temperatures and Stresses in 1/16" Thick Cone for a Zonal Program with 22" Hg Differential Pressure
 Zone 1, 50 °F/sec. to 500 °F; Zone 2, 40 °F/sec. to 400 °F; Zone 3, 30 °F/sec. to 300 °F.

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	72	325	274	72	690	842	72	487	436	72	2550	1350
1	73	264	91	72	487	750	72	740	1420	75	2590	1167
2	130	61	1421	89	670	477	80	640	1110	90	1680	2260
3	169	540	610	117	438	122	103	1090	1580	120	20	4370
4	219	1560	108	153	456	50	137	1550	2030	155	1780	5880
5	257	2050	115	184	278	260	165	1580	2150	177	3500	7000
6	303	1920	296	214	158	325	194	1640	2140	198	4820	7840
7	346	2340	850	248	68	520	226	1820	2300	225	6530	8930
8	391	2750	1203	280	212	595	258	1940	2400	248	7700	9680
9	433	3300	1725	313	142	360	290	2230	2750	264	8500	10290
10	457	4025	2870	345	914	765	312	2530	3080	262	8250	9950
11	455	3900	3290	360	1380	1050	320	2300	2980	254	7780	9390
12	453	3600	3230	364	1380	1000	323	2000	2800	246	7460	9050
13	450	3340	3030	365	1290	932	325	1800	2600	241	7050	8700
14	449	3180	2900	365	1140	830	325	1650	2380	240	6800	8440
15	449	3070	2820	367	1040	800	327	1580	2330	240	6770	8440
20	448	2840	2670	370	892	708	340	1420	2140	230	6130	7950
25	447	2770	2640	372	874	780	350	1400	2120	229	5840	7750
30	447	2790	2670	374	782	754	354	1400	2120	227	5620	7540
60	440	2480	2570	381	773	745	363	1420	1980	230	5400	7390
90	440	2270	2380	380	613	613	360	1270	1830	230	5180	7320
120	440	2000	2060	380	393	460	360	1120	1740	230	5180	7320
150	436	1850	1940	380	192	285	357	1020	1650	232	5180	7320
180	435	1700	1820	380	20	175	357	930	1550	233	5180	7320
210	435	1620	1630	378	120	55	355	865	1490	235	5160	7270
240	435	1520	1560	378	247	35	355	865	1490	235	5160	7270
270	435	1470	1510	378	330	73	355	810	1420	235	5160	7270
300	435	1430	1430	378	366	120	355	770	1390	235	5160	7270
330	435	1420	1450	378	412	137	352	745	1310	235	5160	7270
360	435	1380	1320	378	457	156	352	680	1300	235	5160	7270

TABLE C - 13
Temperatures and Stresses in 1/8" Thick Cone for a Program of 2.5° F/sec. to 300° F.

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	60	0	0	58	0	0	57	0	0	58	0	0
5	78	211	70	74	244	81	66	255	233	62	222	377
10	91	189	24	87	322	111	78	300	455	74	616	1309
15	101	422	22	99	433	144	88	389	488	82	867	2031
20	115	440	33	110	484	99	103	297	352	91	1064	2453
25	126	462	0	123	484	99	115	231	110	98	1243	3058
30	136	451	11	134	429	88	127	220	198	106	1342	3685
35	148	352	0	146	418	77	138	99	242	113	1375	4020
40	160	294	32	157	360	66	150	55	209	120	1507	4697
45	172	196	32	170	349	120	162	0	218	129	1837	5313
50	183	131	55	184	283	98	175	33	153	135	2090	5819
55	194	174	32	195	305	101	187	98	131	143	2309	6314
60	207	108	54	206	292	97	199	76	108	152	2596	6897
65	218	151	43	219	322	108	211	119	858	161	2907	7458
70	230	173	43	230	301	97	226	151	11	168	3212	8019
75	241	194	32	241	301	118	236	162	43	175	3586	8613
80	253	213	21	255	352	96	249	256	96	184	3850	9262
85	261	245	32	263	406	96	256	363	161	187	4026	9427
90	262	266	21	266	309	64	259	299	128	183	3894	3111
95	264	244	53	268	331	64	260	395	202	181	3795	8965
100	265	193	64	270	331	64	259	438	224	179	3696	8668
105	267	171	64	270	331	64	259	502	245	178	3616	8450
110	268	161	86	272	320	64	259	502	245	178	3637	8565
115	270	139	86	273	320	64	259	470	245	179	3692	8602
120	270	139	107	274	309	43	260	502	245	179	3670	8527
125	270	128	139	274	309	43	260	502	245	179	3659	8450
300	265	86	171	270	256	234	260	341	299	193	4026	9170

TABLE C - 14
Temperatures and Stresses in 1/8" Thick Cone for a Program of 5° F/sec. to 400° F.

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	66	0	0	66	0	0	66	0	0	66	0	0
5	66	0	0	66	0	0	66	0	0	66	0	0
10	80	45	57	77	57	23	69	296	285	68	68	205
15	105	136	23	102	194	68	93	445	536	92.5	934	2340
20	129	205	23	126	160	57	120	433	433	109	1500	4000
25	153	262	0	150	147	45	143	285	376	130	1850	5415
30	175	342	46	173	11	11	168	228	285	148	2030	6500
35	198	136	0	196	124	34	192	123	213	165	2400	7700
40	222	148	34	219	113	11	216	56	179	180	2400	8160
45	244	330	160	242	79	23	239	0	22	193	2840	9800
50	265	422	216	264	121	22	264	33	44	208	3400	11000
55	288	479	296	290	165	99	288	173	259	224	3900	12100
60	311	558	365	313	270	140	312	267	353	239	4600	13200
65	334	695	490	335	353	193	333	466	530	252	5130	12900
70	348	718	536	352	382	212	345	693	777	256	5200	14100
75	349	661	467	354	378	200	349	777	777	250	4800	13400
80	349	615	410	356	346	158	349	872	840	243	4600	13000
85	351	535	330	357	346	168	347	872	840	240	4570	12600
90	353	478	240	358	336	136	349	892	819	242	4540	12800
95	353	410	194	360	304	126	350	913	798	242	4500	12700
100	353	353	125	362	262	115	350	892	735	242	4500	12700
105	353	273	84	362	273	136	350	871	672	242	4500	12660
110	353	252	31	362	220	115	351	871	672	243	4560	12800
115	353	190	10	363	220	115	351	871	672	245	4520	12840
120	354	178	42	364	220	115	351	735	567	245	4520	12840

TABLE C - 15
 " 1/8 Thick Cone for a Program of 10 °F/sec. to 500 °F.

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	68	-	0	68	-	0	68	0	0	-	0	0
5	116	91	125	110	307	63	107	661	752	112	1680	4000
10	163	45	170	155	227	37	157	554	600	155	2780	7540
15	207	123	78	199	178	76	202	526	605	189	3850	10120
20	253	264	132	242	302	11	251	540	583	224	5780	12980
25	301	507	380	289	359	109	301	432	389	258	7620	15800
30	349	714	588	335	393	141	351	210	126	293	8870	18200
35	397	989	836	383	463	181	399	133	122	322	9600	20200
40	440	1250	1240	425	451	107	446	406	386	352	9830	22000
45	449	1230	1170	438	443	208	457	908	821	339	9340	21600
50	450	1140	1090	442	411	185	458	1170	985	326	8740	20500
55	453	940	900	444	313	147	458	1275	1010	324	8300	19550
60	452	873	873	447	313	147	460	1275	1060	320	8140	19100
90	455	300	310	454	171	6	466	1290	965	322	8000	19000
120	456	30	116	454	132	170	466	954	625	326	7910	18950
150	456	78	87	454	359	296	466	742	382	333	7770	18800
180	456	165	145	454	501	419	466	456	138	339	7750	18100
210	456	223	243	454	634	463	456	297	85	339	7920	19430
240	456	272	252	454	691	478	456	74	274	345	7760	19200
270	456	291	310	452	624	539	466	42	285	345	7760	19200
300	456	340	330	452	624	539	466	0	410	350	7760	19200
330	456	310	320	452	624	615	463	0	410	350	7760	19200
360	456	370	398	452	851	644	463	52	483	358	7700	19000
390	456	370	398	452	860	644	463	242	578	358	7630	19000
420	456	320	380	452	860	660	463	242	578	360	7650	19000
450	456	320	380	452	908	660	463	284	620	362	7650	19000
480	456	320	380	452	908	660	463	357	640	362	7700	19000

TABLE C - 16

Temperatures and Stress in 1/8" Thick Cone for a Program of 10° F/sec. to 400°F with 24" Hg Differential Pressure

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	82	-	240	83	-	536	82	-	810	82	3900	22
5	108	342	240	104	467	479	103	182	0	105	3160	3110
10	155	407	283	151	350	373	153	237	23	149	2700	6000
15	199	463	350	196	314	325	199	403	67	185	2170	8280
20	244	717	560	241	418	451	247	781	143	218	421	10600
25	288	935	737	286	518	670	296	734	22	249	638	12200
30	333	1560	1130	333	710	900	339	392	297	278	-	13900
35	357	1650	1300	360	742	909	366	73	572	281	-	14000
40	359	1430	1220	362	742	900	368	312	760	268	430	13100
45	359	1350	1140	364	752	961	367	593	967	257	792	12300
50	361	1220	1060	366	752	961	367	750	1020	254	891	12000
55	361	1220	1060	367	697	936	368	790	1060	253	1030	11850
60	361	1150	1000	368	676	926	368	853	1080	253	1030	11850
100	365	763	920	377	487	829	370	760	1000	256	1090	11800
140	363	491	606	377	280	725	370	572	780	260	1140	11750
180	365	470	533	377	124	622	370	406	676	267	1080	12000
220	365	440	523	374	104	550	368	312	562	272	1230	12000

TABLE C - 17
Temperatures and Stresses in 1/8" Thick Cone for a Program
of 20 °F/sec. to 500 °F with 23" Hg Differential Pressure

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4			
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress		
		M	C		M	C		M	C		M	C	
0	86	-	213	85	-	213	86	-	425	85	2806	-	213
2.5	122	-	223	109	121	121	111	182	182	128	2328	-	3306
5	169	-	310	160	200	200	165	490	490	175	718	-	7475
10	257	-	601	248	205	205	255	953	369	258	1028	-	12842
15	346	-	1221	340	28	28	348	1042	391	324	3225	-	16880
20	436	-	1658	430	229	229	435	711	272	387	3604	-	20575
25	455	-	1875	458	377	377	407	187	451	373	1983	-	19219
30	455	-	1823	458	291	291	467	510	638	350	1107	-	17856
35	456	-	1737	460	317	317	467	850	850	334	639	-	17052
40	456	-	1686	462	295	295	469	969	893	332	535	-	16845
100	461	-	1221	476	84	84	477	739	655	340	37	-	16565
480	462	-	812	475	749	749	475	8	34	360	603	-	16620
660	463	-	778	476	926	926	473	235	286	372	1316	-	16201
840	464	-	718	478	985	985	474	344	344	378	1598	-	15846
1020	464	-	658	476	1069	1069	472	294	504	384	2202	-	15051
1200	464	-	650	476	1145	1145	475	227	487	388	2475	-	14578
1380	462	-	616	478	1162	1162	476	168	512	392	2603	-	14424
1560	462	-	581	478	1162	1162	475	118	563	396	2812	-	14041
1740	463	-	596	478	1204	1204	474	109	596	397	2876	-	13940

TABLE C - 18

Temperatures and Stresses in 1/8" Thick Cone for a Zonal Program with 24" Hg Differential Pressure
 Zone 1, 10 °F/sec. to 400°F; Zone 2, 5 °F/sec. to 300°F; Zone 3, 2.5 °F/sec. to 200°F.

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	79	296	250	77	467	535	77	661	912	77	3200	1140
5	124	433	433	97	570	627	85	445	640	83	2200	445
10	166	531	554	126	588	655	96	445	801	84	2140	638
15	204	706	773	149	723	791	106	456	775	86	2190	798
20	247	1043	1090	175	870	802	118	445	801	90	2280	946
25	285	1380	1340	199	830	830	140	486	848	104	2280	1960
30	315	1850	1690	219	921	866	155	520	880	110	2350	2280
35	337	2130	1850	241	1160	1030	165	520	860	112	2500	2220
40	353	2140	1780	266	1450	1220	179	610	960	115	2630	2270
45	356	2100	1730	274	1600	1290	184	784	1110	115	2710	2280
50	354	2070	1720	274	1550	1280	188	795	1165	115	2820	2250
55	350	2000	1660	273	1490	1250	190	851	1180	115	2840	2230
60	350	1940	1640	274	1450	1240	191	828	1100	115	2860	2220
90	346	1600	1370	275	1260	1120	200	896	1140	121	2940	2500
120	346	1490	1250	275	1160	1080	205	840	1120	125	2930	2700
150	346	1340	1123	275	1060	1050	207	830	1075	128	2950	2840
180	346	1270	1040	275	992	1000	207	817	1040	130	3000	2890
210	346	1220	960	275	937	981	207	817	1040	132	2950	3090
240	346	1220	960	277	905	937	209	638	940	134	2980	3170
270	346	1200	890	277	872	926	209	638	940	136	3000	3170
300	346	1160	820	278	818	915	211	616	874	138	3080	3210
330	346	1120	810	278	818	915	211	616	874	140	3080	3210

TABLE C-19

Temperatures and Stresses in 1/8" Thick Cone for a Zonal Program with 23" Hg Differential Pressure
 Zone 1, 20 °F/sec. to 400 °F; Zone 2, 10 °F/sec. to 300 °F; Zone 3, 5 °F/sec. to 200 °F.

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	70	-	580	71	-	787	72	-	980	70	3520	182
5	117	-	400	97	-	775	78	-	501	74	3430	167
10	191	-	750	147	-	814	105	-	433	91	2950	1650
15	262	-	1220	192	-	918	136	-	440	108	2630	2720
20	323	-	1830	240	-	1090	164	-	463	122	2520	3530
25	351	-	2000	273	-	1210	188	-	644	128	2500	3920
30	349	-	1920	273	-	1300	193	-	795	124	2740	3630
35	349	-	1920	273	-	1300	198	-	885	122	2870	3470
40	347	-	1840	273	-	1300	201	-	907	122	2990	3300
45	345	-	1820	273	-	1290	201	-	918	122	3030	3240
50	348	-	1820	276	-	1230	205	-	896	122	3100	3210
55	346	-	1750	276	-	1200	206	-	907	122	3110	3250
60	346	-	1700	275	-	1170	206	-	930	122	3160	3330
90	347	-	1520	276	-	1110	209	-	874	126	3110	3470
120	349	-	1370	279	-	1050	212	-	806	126	3160	3500
150	349	-	1280	279	-	970	212	-	762	129	3160	3500
180	349	-	1200	280	-	915	212	-	717	129	3140	3550
210	349	-	1140	280	-	905	212	-	717	129	3120	3590
240	349	-	1130	280	-	828	212	-	650	132	3120	3590
270	349	-	1130	280	-	828	212	-	650	134	3240	3570

TABLE C - 20

Temperatures and Stresses in 1/8" Thick Cone for a Zonal Program with 23" Hg Differential Pressure
 Zone 1, 10 °F/sec. to 500°F; Zone 2, 5 °F/sec. to 400°F; Zone 3, 2.5 °F/sec. to 300°F.

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	66	- 262	- 559	63	- 353	- 707	64	- 479	- 969	64	3550	- 194
5	92	- 560	- 650	76	- 456	- 707	69	- 285	- 752	66	3480	- 342
10	133	- 638	- 707	103	- 581	- 752	78	- 171	- 592	70	3430	- 1050
15	174	- 859	- 994	129	- 661	- 893	92	- 34	- 388	79	3240	- 1710
20	213	- 1050	- 1100	151	- 667	- 893	110	- 23	- 422	87	3020	- 1880
25	251	- 1420	- 1410	175	- 712	- 892	124	- 91	- 581	90	3000	- 2170
30	292	- 1800	- 1790	200	- 773	- 974	137	- 147	- 508	96	2920	- 2500
35	333	- 2120	- 2010	223	- 910	- 1090	151	- 214	- 508	103	2610	- 2700
40	371	- 2550	- 2410	246	- 990	- 1120	164	- 226	- 576	106	2990	- 2800
45	403	- 3090	- 2670	268	- 1140	- 1230	178	- 271	- 588	108	3060	- 2920
50	421	- 3240	- 2720	290	- 1320	- 1330	189	- 336	- 616	111	3090	- 3040
55	436	- 3114	- 2560	314	- 1570	- 1430	203	- 202	- 604	114	3100	- 3410
60	452	- 3100	- 2500	341	- 1810	- 1580	216	- 325	- 683	119	3030	- 3630
70	466	- 2860	- 2300	370	- 2240	- 1840	240	- 462	- 814	125	3120	- 4490
80	460	- 2700	- 1990	370	- 1980	- 1770	256	- 583	- 858	139	2970	- 5450
90	456	- 2430	- 1950	372	- 1790	- 1640	276	- 616	- 864	156	2730	- 5710
120	447	- 1940	- 1580	372	- 1425	- 1400	289	- 670	- 842	166	2750	- 5900
150	445	- 1760	- 1430	375	- 1270	- 1290	291	- 616	- 767	168	2770	- 5920
180	445	- 1680	- 1380	375	- 1110	- 1190	292	- 530	- 680	171	2810	- 5920
210	446	- 1610	- 1230	375	- 1030	- 1130	292	- 508	- 616	171	2810	- 5980
240	446	- 1520	- 1170	375	- 936	- 1040	292	- 486	- 550	173	2900	- 6000
270	446	- 1450	- 1100	375	- 915	- 1040	292	- 475	- 518	175	2830	- 6000
300	446	- 1410	- 1060	375	- 915	- 1040	292	- 388	- 432	177	2830	- 6010
330	446	- 1390	- 990	376	- 820	- 946	292	- 292	- 367	177	2800	- 6000
360	446	- 1390	- 990	376	- 820	- 946	293	- 280	- 335	179	2840	- 6000
390	446	- 1360	- 980	376	- 780	- 936	293	- 270	- 302	179	2840	- 6000
420	446	- 1360	- 980	376	- 707	- 832	293	- 270	- 302	179	2840	- 6000

TABLE C - 21

Temperatures and Stresses in 1/8" Thick Cone for a Zonal Program with 23" Differential Pressure
 Zone 1, 20 °F/sec. to 500°F; Zone 2, 10 °F/sec. to 400°F; Zone 3, 5 °F/sec. to 300°F.

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	84	-	- 490	87	-	- 730	86	-	- 1040	87	3690	46
5	94	-	- 353	87	-	- 730	86	-	- 1040	98	3690	46
10	168	-	- 239	127	-	- 604	101	-	- 798	97	3680	559
15	247	-	- 726	175	-	- 667	133	-	- 912	120	3420	- 2000
20	322	-	- 1370	221	-	- 810	160	-	- 718	131	3120	- 2820
25	391	-	- 2480	268	-	- 935	188	-	- 660	144	3070	- 3570
30	430	-	- 2880	311	-	- 1170	214	-	- 660	154	3020	- 4100
35	460	-	- 2910	359	-	- 1470	241	-	- 721	161	3020	- 4430
40	461	-	- 2780	368	-	- 1560	258	-	- 968	167	3070	- 4780
45	458	-	- 2740	368	-	- 1460	276	-	- 970	182	2860	- 5720
50	455	-	- 2560	368	-	- 1340	291	-	- 983	194	2690	- 6450
55	450	-	- 2490	368	-	- 1310	284	-	- 994	192	2840	- 6150
60	447	-	- 2360	368	-	- 1310	296	-	- 1015	188	2980	- 5940
90	441	-	- 1790	371	-	- 1150	299	-	- 1025	184	3160	- 5540
120	442	-	- 1700	374	-	- 1080	299	-	- 972	187	3170	- 5670
150	442	-	- 1540	374	-	- 946	299	-	- 907	188	3200	- 5660
180	442	-	- 1430	374	-	- 925	300	-	- 842	190	3240	- 5640
210	442	-	- 1290	374	-	- 832	300	-	- 767	192	3240	- 5640
240	444	-	- 1290	376	-	- 821	300	-	- 680	194	3280	- 5790
270	444	-	- 1270	376	-	- 800	301	-	- 680	197	3280	- 5790
300	444	-	- 1260	376	-	- 738	301	-	- 594	198	3280	- 5790
330	444	-	- 1090	376	-	- 738	301	-	- 594	198	3280	- 5790

TABLE C - 22

Temperatures and Stresses in 1/8" Thick Cone for a Zonal Program with 23" Hg Differential Pressure
 Zone 1, 50 °F/sec. to 500 °F; Zone 2, 40 °F/sec. to 400 °F; Zone 3, 30 °F/sec. to 300 °F.

TIME Sec.	STATION 1				STATION 2				STATION 3				STATION 4			
	Temp.		Stress		Temp.		Stress		Temp.		Stress		Temp.		Stress	
	M	C	M	C	M	C	M	C	M	C	M	C	M	C	M	C
0	62	- 285	- 536	- 672	61	- 342	- 672	- 490	60	- 912	- 490	- 912	60	3260	68	68
1	67	- 57	- 330	- 524	61	- 171	- 524	- 342	60	- 832	- 342	- 832	60	3240	- 23	- 23
2	102	- 68	- 114	- 388	78	- 217	- 388	- 364	68	- 57	- 364	- 57	72	3120	- 741	- 741
3	141	- 147	- 271	- 365	112	- 285	- 365	900	89	581	900	581	95	2540	- 2630	- 2630
4	178	- 283	- 260	- 388	144	- 296	- 388	1120	112	684	1120	684	125	1800	- 4830	- 4830
5	224	- 377	- 421	- 565	185	- 450	- 565	1190	149	667	1190	667	158	870	- 7180	- 7180
6	260	- 627	- 660	- 723	219	- 667	- 723	1360	174	880	1360	880	184	123	- 8840	- 8840
7	302	- 983	- 940	- 704	257	- 748	- 704	1490	208	930	1490	930	210	- 650	- 10800	- 10800
8	345	- 1180	- 1140	- 756	291	- 864	- 756	1590	237	1040	1590	1040	234	- 1220	- 12090	- 12090
9	383	- 1470	- 1420	- 933	327	- 1110	- 933	1560	264	902	1560	902	249	- 1800	- 13230	- 13230
10	404	- 2150	- 2000	- 1220	356	- 1290	- 1220	1170	285	462	1170	462	253	- 1850	- 13380	- 13380
11	411	- 2360	- 2160	- 1270	364	- 1340	- 1270	950	293	280	950	280	249	- 1630	- 13050	- 13050
12	414	- 2480	- 2240	- 1260	364	- 1300	- 1260	712	297	108	712	108	243	- 1310	- 12600	- 12600
13	415	- 2540	- 2260	- 1240	364	- 1250	- 1240	648	300	86	648	86	240	- 1130	- 12300	- 12300
14	417	- 2600	- 2280	- 1220	364	- 1200	- 1220	530	301	43	530	43	234	- 877	- 11800	- 11800
15	419	- 2660	- 2350	- 1210	363	- 1140	- 1210	356	302	65	356	65	232	- 655	- 11520	- 11520
20	426	- 2810	- 2380	- 1190	362	- 1010	- 1190	0	307	238	0	238	216	112	- 10340	- 10340
25	430	- 2840	- 2470	- 1220	363	- 1020	- 1220	- 262	312	432	- 262	432	205	570	- 9520	- 9520
30	432	- 2820	- 2400	- 1200	365	- 990	- 1200	- 432	312	518	- 432	518	197	896	- 8860	- 8860
35	436	- 2790	- 2380	- 1200	365	- 990	- 1200	- 540	312	583	- 540	583	192	1140	- 8490	- 8490
40	436	- 2690	- 2300	- 1220	368	- 1020	- 1220	- 583	310	659	- 583	659	188	1380	- 8030	- 8030
45	436	- 2620	- 2200	- 1220	368	- 1020	- 1220	- 616	307	680	- 616	680	184	1370	- 7860	- 7860
50	439	- 2610	- 2200	- 1220	368	- 1020	- 1220	- 583	306	670	- 583	670	181	1410	- 7540	- 7540
55	439	- 2570	- 2160	- 1230	368	- 1040	- 1230	- 605	304	734	- 605	734	177	1460	- 7500	- 7500
60	439	- 2510	- 2120	- 1230	370	- 1040	- 1230	- 680	302	756	- 680	756	177	1560	- 7320	- 7320
70	439	- 2440	- 2090	- 1230	372	- 1040	- 1230	- 659	299	745	- 659	745	175	1670	- 7120	- 7120
80	442	- 2320	- 2000	- 1220	373	- 1050	- 1220	- 605	297	650	- 605	650	172	1790	- 6900	- 6900
90	442	- 2270	- 1910	- 1220	373	- 1020	- 1220	- 530	297	594	- 530	594	176	1710	- 7100	- 7100
120	445	- 2100	- 1740	- 1140	379	- 894	- 1140	- 464	298	486	- 464	486	177	1790	- 7070	- 7070
150	445	- 1930	- 1530	- 1050	379	- 822	- 1050	- 432	298	475	- 432	475	177	1860	- 7050	- 7050
180	446	- 1870	- 1480	- 960	380	- 742	- 960	- 334	299	356	- 334	356	180	1970	- 7060	- 7060
300	448	- 1790	- 1370	- 782	379	- 463	- 782	- 162	300	97	- 162	97	187	2170	- 7020	- 7020
420	447	- 1700	- 1235	- 690	380	- 412	- 690	- 77	302	97	- 77	97	191	2230	- 7040	- 7040
540	448	- 1620	- 1150	- 680	381	- 381	- 680	- 43	300	54	- 43	54	192	2420	- 6800	- 6800
720	447	- 1600	- 1090	- 660	381	- 330	- 660	54	300	43	54	43	196	2430	- 6760	- 6760
900	448	- 1510	- 1040	- 607	381	- 268	- 607	97	300	77	97	77	199	2530	- 6570	- 6570
1080	449	- 1500	- 1000	- 577	383	- 258	- 577	140	300	118	140	118	201	2530	- 6480	- 6480

TABLE C - 23
Temperatures and Stresses in 1/16" Thick Hemisphere for a Program 5 °F/sec. to 400°F

TIME Sec.	STATION 1			STATION 2			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C
0	77	0	0	77	0	0	77	0	0
5	98	-57	-182	105	-890	-786	97	114	-935
10	128	-125	-535	135	-1090	-1410	117	-1240	-2400
15	159	-55	-296	166	-780	-915	132	-2590	-3820
20	192	-34	-168	200	-619	-583	151	-3600	-5130
25	223	-178	-300	230	-664	-697	170	-4860	-6770
30	254	-132	-209	260	-836	-891	187	-7500	-8700
35	284	-206	-293	288	-1075	-1060	206	-7000	-9300
40	312	-226	-333	314	-1280	-1180	222	-8000	-10700
45	339	-180	-340	341	-1310	-1250	237	-9110	-12000
50	367	-10	-239	366	-1270	-1200	256	-10000	-13200
55	396	-62	-374	393	-1270	-1150	273	-10800	-15400
60	418	-183	-523	418	-1400	-1200	290	-	-
90	427	-110	-10	437	-673	-307	297	-	-
120	427	231	372	442	-226	295	305	-12200	-15900
150	427	473	594	442	70	670	307	-12300	-15900
180	427	583	684	442	393	1050	310	-12100	-15800
210	427	724	825	440	531	1240	314	-12200	-15700
240	425	775	885	440	747	1460	317	-12300	-15700
270	425	815	895	435	786	1307	320	-12400	-15700
300	422	796	885	433	934	1460	324	-12300	-15600
330	422	826	895	430	1090	1590	325	-12400	-15600
360	422	876	915	430	1170	1660	325	-12500	-15700
390	422	876	915	430	1220	1700	327	-12600	-15500
420	422	896	920	430	1290	1820	330	-12300	-15200
450	420	925	925	427	1350	1960	330	-12400	-15300
480	420	915	876	427	1410	1990	330	-12400	-15200
510	417	965	896	425	1500	2050	334	-12300	-15000
540	417	985	905	425	1570	2070	335	-12350	-15000

TABLE C - 24

Temperatures and Stresses in 1/16" Thick Hemisphere for a Program of 5 °F/sec. to 400 °F
With 21" Hg Differential Pressure.

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	74	-707	-627	74	-445	-741	74	-604		74	125	-1130
5	82	-707	-627	100	-1265	-1390	107	-1070		96	-68	-2620
10	106	-958	-1050	126	-1360	-1820	132	-1182		107	-673	-3440
15	135	-978	-1125	151	-893	-1210	164	-1290		123	-1230	-4430
20	159	-1040	-1130	177	-655	-655	192	-1460		138	-1930	-5600
25	186	-1080	-1035	201	-403	-358	222	-1530		154	-2475	-6280
30	213	-1165	-1110	230	-278	-155	252	-1560		170	-3280	-7490
35	235	-1365	-1440	255	-363	-110	283	-1530		187	-3930	-8550
40	258	-1390	-1595	280	-356	43	313	-1700		202	-4590	-9300
45	281	-1620	-1840	307	-524	64	342	-1890		218	-5240	-10260
50	304	-1560	-1930	331	-752	0	372	-2110		235	-5870	-11180
55	326	-1530	-2040	357	-588	189	400	-2350		250	-6420	-12020
60	348	-1400	-2030	380	-525	216	428	-2580		264	-7120	-13100
90	375	-738	-1730	401	-173	337	459	-2460		277	-7900	-13600
120	384	-300	-1430	406	-337	163	464	-2350		284	-8000	-13700
150	384	-155	-1290	408	-296	173	464	-2060		290	-8100	-13800
180	384	-124	-1110	408	-153	224	461	-1970		294	-8190	-14000
210	382	-10	-999	408	-20	357	460	-1930		300	-8200	-14000
240	378	52	-874	406	112	428	460	-1740		302	-8300	-14000
270	378	83	-853	405	255	622	460	-1640		304	-8300	-14000
300	378	177	-749	405	398	785	455	-1715		306	-8500	-14200
330	378	198	-676	405	449	826	455	-1617		308	-8570	-14400
360	378	198	-676	405	510	847	450	-1670		310	-8560	-14300
390	378	239	-655	405	510	847	450	-1670		314	-8500	-14300
420	378	239	-655	405	510	847	450	-1670		314	-8600	-14300

TABLE C - 25
Temperature and Stresses in 1/16" Thick Hemisphere for a Program of 10 °F/sec. to 500 °F

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	78	0	0	78	0	0	78	0	0	78	0	0
5	126	250	148	132	-114	-273	118	171	-946	115	1320	-946
10	187	134	-146	195	-380	-930	175	-56	-2490	148	520	-2490
15	245	-33	-451	254	-759	-1500	232	-330	-4330	181	45	-4330
20	305	-162	-356	314	-1060	-1940	294	-326	-5420	217	-121	-5420
25	364	-530	-707	375	-1310	-2270	356	-567	-7170	250	-682	-7170
30	424	-726	-1024	430	-1550	-2380	416	-805	-8430	283	-810	-8430
35	480	-912	-1090	486	-1750	-2360	475	-922	-9880	317	-1100	-9880
40	529	-1330	-1380	538	-1820	-2320	531	-958	-10700	350	-1050	-10700
45	550	-1610	-1460	555	-1580	-1930	565	-945	-11370	357	-1650	-11370
50	550	-1560	-1220	560	-1030	-1440	575	-743	-11360	358	-1600	-11360
55	545	-1380	-1000	560	-755	-1150	580	-720	-11200	360	-1400	-11200
60	540	-1290	-835	560	-466	-907	585	-673	-11200	360	-1400	-11200
90	535	-633	-61	560	483	-16	585	-238	-10830	368	-938	-10830
120	531	-184	396	555	1000	493	580	119	-10800	368	-887	-10800
150	527	18	580	550	1530	1100	575	371	-10740	370	-683	-10740
180	522	286	884	547	1810	1440	570	455	-10200	374	-235	-10200
210	520	366	930	542	2030	1680	565	669	-10100	375	-204	-10100
240	516	515	1060	536	2360	1970	560	831	-9980	377	-153	-9980
270	512	560	1070	535	2500	2070	555	981	-9735	380	133	-9735
300	512	570	1030	530	2670	2200	553	1026	-9530	380	205	-9530
330	510	562	970	527	2730	2250	550	1080	-9120	380	340	-9120
360	505	695	1060	527	2760	2330	546	1200	-9000	380	380	-9000
390	505	723	1070	525	2790	2430	546	1260	-9000	380	380	-9000

TABLE C - 26

Temperatures and Stresses in 1/16" Thick Hemisphere for a Zonal Program with 21" Hg Differential Pressure
 Zone 1, 10 °F/sec. to 400°F; Zone 2, 5 °F/sec. to 300°F; Zone 3, 2.5 °F/sec. to 200°F.

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	94	-454	-545	87	-387	-650	86	-	-501	87	250	-980
5	143	-567	-386	125	11	-320	108		216	98	1880	250
10	192	-549	-515	155	-124	-316	125		376	106	2280	11
15	240	-575	-520	186	-157	-393	141		306	111	1675	-125
20	289	-759	-542	217	-257	-549	162		465	122	2245	296
25	332	-842	-634	246	-189	-696	181		452	130	1820	-409
30	357	-628	-533	267	-324	-702	198		439	135	1670	-648
35	362	-544	-512	287	-454	-540	219		333	143	1590	-851
40	360	-408	-460	297	-454	-324	236		222	151	1616	-1017
45	360	-335	-376	297	-302	-86	245		276	153	1330	-1290
50	359	-303	-356	297	-194	-22	248		331	153	1280	-1300
55	359	-200	-303	297	-194	-22	248		331	153	1280	-1300
60	358	-146	-251	298	-194	-22	250		331	155	1280	-1300
90	357	-31	-94	298	-194	-22	252		442	158	1410	-1330
120	357	115	52	299	-184	22	252		552	160	1590	-1200
150	357	220	115	299	-162	86	252		690	160	1510	-1320
180	357	303	167	299	-76	259	250		684	161	1450	-1380
210	357	439	261	299	-22	346	248		762	163	1680	-1100
240	357	481	313	298	22	367	248		806	164	1660	-1160
270	357	492	366	298	32	400	248		861	165	1620	-1160
300	357	533	429	296	130	486	248		883	166	1770	-1020
330	357	575	470	296	162	497	246		927	167	1720	-1040
360	357	628	492	296	216	550	246		883	169	1780	-842
390	357	628	492	296	227	605	246		883	169	1780	-842
420	357	628	492	296	227	605	246		938	169	1780	-842

TABLE C - 27

Temperatures and Stresses in 1/16" Thick Hemisphere for a Zonal Program with 21 Hg Differential Pressure.
 Zone 1, 10 °F/sec. to 500 °F; Zone 2, 5 °F/sec. to 400 °F; Zone 3, 2.5 °F/sec. to 300 °F.

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	71	- 587	- 452	69	- 330	- 627	71	- 456	- 1015	70	- 137	- 1015
5	144	- 418	- 23	92	- 160	- 513	79	- 855	- 1450	75	- 285	- 1450
10	159	- 1000	- 881	123	- 433	- 798	97	- 490	- 2140	83	- 912	- 2140
15	205	- 1390	- 1140	153	- 542	- 791	115	- 353	- 1980	90	- 650	- 1980
20	249	- 1350	- 1130	183	- 698	- 900	134	- 307	- 1760	99	- 216	- 1760
25	291	- 1310	- 1100	209	- 840	- 1280	153	- 192	- 2040	106	- 524	- 2040
30	340	- 1380	- 1070	240	- 873	- 1410	170	- 34	- 1860	113	- 365	- 1860
35	385	- 1285	- 1275	270	- 1070	- 1660	191	- 168	- 2020	122	- 320	- 2020
40	427	- 1290	- 1390	298	- 1275	- 1640	209	- 168	- 2270	128	- 513	- 2270
45	446	- 1307	- 1445	317	- 1470	- 1780	226	- 211	- 2330	136	- 531	- 2330
50	445	- 1270	- 1400	334	- 1470	- 1660	246	- 355	- 2640	142	- 655	- 2640
55	445	- 1120	- 1320	351	- 1600	- 1760	266	- 344	- 2830	149	- 632	- 2830
60	445	- 1040	- 1290	370	- 1670	- 1760	289	- 259	- 3000	157	- 600	- 3000
90	445	- 707	- 953	393	- 1210	- 1110	363	- 312	- 4730	204	- 1040	- 4730
120	446	- 570	- 796	400	- 1000	- 908	376	- 146	- 4800	212	- 1280	- 4800
150	448	- 432	- 580	403	- 806	- 724	376	- 104	- 5000	218	- 1290	- 5000
180	448	- 285	- 442	403	- 704	- 580	376	- 83	- 4920	222	- 1040	- 4920
210	448	- 245	- 344	403	- 571	- 510	374	- 208	- 5060	225	- 1060	- 5060
240	448	- 108	- 187	403	- 377	- 235	372	- 312	- 4910	227	- 910	- 4910
270	448	- 98	- 167	400	- 326	- 245	370	- 301	- 4770	230	- 700	- 4770
300	448	- 49	- 108	400	- 143	- 92	368	- 364	- 4800	230	- 810	- 4800
330	448	- 39	- 59	398	- 153	- 143	368	- 468	- 4860	230	- 977	- 4860
360	448	- 9	- 9	398	- 122	- 133	367	- 416	- 4710	231	- 977	- 4710
390	448	- 20	- 50	397	- 51	- 10	364	- 364	- 4750	233	- 890	- 4750
420	448	20	50	395	- 61	- 40	364	- 447		233	- 920	

TABLE C - 28

Temperatures and Stresses in 1/16" Thick Hemisphere for a Zonal Program with 21" Hg Differential Pressure:
 Zone 1, 20 °F/sec. to 500 °F; Zone 2, 10 °F/sec. to 400 °F; Zone 3, 5 °F/sec. 300 °F.

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	76	-616	-560	76	-327	-621	76		-502	76	80	-741
5	193	-1340	-896	153	-361	-723	116		-490	94	307	-1220
10	275	-1380	-1020	207	-560	-974	149		-584	110	376	-1560
15	364	-1510	-1260	265	-946	-1470	185		-600	125	308	-2030
20	437	-1540	-1430	316	-1740	-2040	220		-583	139	477	-1920
25	455	-1290	-1620	352	-1700	-2060	256		-575	152	260	-2340
30	455	-1063	-1420	386	-1870	-2110	296		-653	165	158	-2840
35	451	-934	-1350	391	-1590	-1640	322		-710	180	90	-3015
40	450	-806	-1270	394	-1120	-1020	341		-567	194	90	-3416
45	450	-784	-1210	395	-1120	-1020	358		-504	204	-571	-4030
50	450	-773	-1160	395	-989	-969	363		-494	205	-1110	-4450
55	450	-720	-1150	395	-918	-908	367		-525	206	-1220	-4440
60	450	-666	-1130	395	-877	-857	371		-416	209	-1160	-4300
90	450	-418	-890	400	-602	-612	375		-218	214	-1050	-4500
120	450	-183	-494	400	-480	-480	375		-31	219	-1060	-4550
150	450	-21	-322	400	-334	-334	375		135	223	-1160	-4540
180	450	75	-161	400	-204	-214	372		146	225	-1200	-4660
210	450	128	-97	398	-122	-102	370		156	228	-1065	-4640
240	450	215	0	398	10	10	368		291	230	-1170	-4600
270	450	257	54	398	102	81	368		116	231	-1270	-4630
300	450	344	160	396	194	163	367		468	233	-1020	-4650
330	450	344	160	395	184	143	365		562	235	-1050	-4400
360	450	376	172	395	200	179	364		593	236	-966	-4350
390	450	387	225	395	262	242	363		603	236	-966	-4350
420	450	408	236	395	284	305	363		624	236	-966	-4350

TABLE C - 29

Temperatures and Stresses in 1/16" Thick Hemisphere for a Zonal Program with 18" Hg Differential Pressure:
 Zone 1, 50 °F/sec. to 500°F; Zone 2, 40 °F/sec. to 400°F; Zone 3, 30 °F/sec. to 300°F.

TIME Sec.	STATION 1			STATION 2			STATION 3			STATION 4		
	Temp.	Stress		Temp.	Stress		Temp.	Stress		Temp.	Stress	
		M	C		M	C		M	C		M	C
0	70	- 467	- 422	68	- 308	- 560	70	- 553		70	- 102	- 809
1	126	- 456	- 342	123	114	- 524	84	- 640		80	- 250	- 1250
2	166	- 915	- 859	156	- 56	- 768	103	- 274		96	46	- 1500
3	207	- 1360	- 1310	190	- 293	- 1250	129	- 388		116	1700	- 992
4	251	- 1810	- 1810	227	- 178	- 966	161	- 350		134	2740	- 990
5	300	- 2170	- 2130	264	- 176	- 1100	191	- 34		151	3420	- 1090
6	340	- 2400	- 2360	299	- 140	- 1180	220	- 377		166	3670	- 1390
7	390	- 2490	- 2520	338	138	- 1070	252	- 375		183	3360	- 2280
8	433	- 2800	- 2780	374	11	- 1360	281	- 356		195	2480	- 3320
9	460	- 3100	- 3170	394	- 775	- 1800	310	- 756		195	1160	- 4420
10	467	- 2760	- 3050	400	- 1000	- 1430	327	- 856		194	314	- 4700
11	469	- 2620	- 2950	402	- 867	- 1300	333	- 795		193	- 280	- 4760
12	472	- 2400	- 2750	404	- 683	- 950	336	- 551		193	- 627	- 4550
13	472	- 2290	- 2690	406	- 581	- 867	337	- 223		193	- 874	- 4540
14	472	- 2100	- 2540	406	- 460	- 735	339	74		193	- 1050	- 4600
15	472	- 2070	- 2530	406	- 388	- 622	341	180		193	- 1230	- 4650
30	468	- 2020	- 2320	406	- 296	- 592	360	840		196	- 1380	- 4640
45	465	- 2020	- 2320	406	- 204	- 480	366	1070		199	- 850	- 4410
60	464	- 1720	- 1830	406	- 102	- 326	367	1200		201	- 493	- 4300
360	447	- 980	- 1070	387	1040	896	346	1750		219	544	- 3230
660	446	- 882	- 892	389	1150	1030	341	1800		228	355	- 3470
960	447	- 784	- 725	388	1180	1060	340	1760		234	- 297	- 3560
1260	448	- 696	- 637	389	1280	1150	340	1640		240	121	- 3310
1560	449	- 666	- 540	390	1350	1200	336	1700		242	- 121	- 3340
1860	447	- 588	- 343	390	1410	1290	336	1720		244	253	- 3000
2160	447	- 559	- 362	388	1420	1320	333	1750		247	143	- 2780
2400	447	- 500	- 314	388	1420	1320	333	1800		248	99	- 2800

APPENDIX D
ENERGY ANALYSIS

ENERGY APPROACH TO ANALYSIS OF LINEAR
VISCOELASTIC SHELLS OF REVOLUTION

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ABSTRACT

A variational equation based on energy principles has been developed for thin hemispherical shells. The behavior of the shell is assumed to be characterized by a viscoelastic material with temperature dependent properties.

The Ritz method is applied by assuming that the shell displacements are functions of the curvilinear shell coordinates and that their unknown coefficients are functions of time.

A system of six ordinary differential equations of order two is developed. These equations result from operations on the variational equation.

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LIST OF SYMBOLS

$a_{\alpha\beta}$	Metric tensor of surface coordinates
E	Modulus of elasticity
e_{ij}	Components of strain deviatoric tensor
F	Free energy
f	Energy density/unit volume
G	Shear modulus
g_{ij}	Metric tensor
K	Kinetic energy
S_{ij}	Components of stress deviatoric tensor
T	Temperature field of the shell
t	Time
u^i, u_i	Displacement components
W	Work integral
x^i	Rectangular coordinates of the shell
y^i	Curvilinear coordinates of the shell

GREEK LETTERS

α	Coefficient of thermal expansion
β_1, β_2	Coefficients (functions of time)
δ	First variation
ϵ_{ij}	Components of strain tensor
γ_1, γ_2	Coefficients (functions of time)
η	Coefficient of viscosity
ρ	Density of the material/unit volume
ρ_1, ρ_2	Coefficients (functions of time)
σ_{ij}	Components of stress tensor

SUBSCRIPTS AND SUPERSSCRIPTS

i, j	Refer to numbers 1, 2, 3
o	Refer to datum value, middle surface reference
α, β, γ	Refer to number 1, 2, 3
$\text{DOT}(\cdot)$	represents differentiations with respect to time.

D.1 Introduction

A number of investigations have been directed towards determining the thermoelastic behavior of shell structures. In recent years several investigators^{1,2,3,4,5,8,9} have used variational methods to treat problems of thermoelasticity and certain linear thermoviscoelastic problems.

This report extends some of these results toward determining the viscoelastic behavior of shells of revolution under uniform pressure and thermal gradients. Although certain restrictions are made concerning the coupling of heat conduction and the viscoelastic phenomena, the analysis has general applicability.

The use of generalized coordinates and variational calculus leads to concepts of generalized forces of the Lagrangian type which are applicable to mechanical (static and dynamic) and thermal (with material property variations) problems.

First, the general three-dimensional thermoviscoelasticity formulation of the variational equation is presented in curvilinear coordinates. The energy and work integrals, including the material properties, are expanded into powers of the normal coordinate. This permits the introduction of thermal effects on the material properties.

The general equations are specialized to thin shells. For the thin shell analysis, the shell thickness is allowed to approach zero and only terms of shell thickness of unit power are retained. An example of a hemispherical shell is given to illustrate the procedure of analysis.

D.2 Mathematical Formulation

D.2.1 Variational Equation

The transient response of a shell to mechanical and thermal loadings can be investigated by standard energy methods without the complications connected with the analytical solution of the partial differential equations of equilibrium.

If the shell is subjected to simultaneous mechanical and thermal loading, according to Hamilton's principle the energy formulation is given by the variational equation

$$\delta \int_{t_0}^{t_1} (K + F - W) dt \quad (D.2.1-1)$$

where

δ is the first variation and integration with respect to time performed between the limits t_0 and t_1

K is kinetic energy

F is free energy (potential energy in isothermal analysis)

W is work integral

The function $(K + F - W)$ is identical to the Lagrangian function. In the following paragraphs, the proper form of equation D.2.1-1 is developed for application to the viscoelastic analysis of shell structures.

D.2.1.1 Energy and Work Integrals

The coupling between thermoelasticity and dynamics is expressed by the inclusion of the acceleration term in equilibrium equation, thus forming the usual equations of motion

$$\sigma_{ij,j} = \rho \ddot{u}_i \quad (D.2.1-2)$$

where σ_{ij} and u_i are the unknown field variables. This leads to the Lagrangian formulation, expressed in terms of kinetic energy

$$K = \frac{1}{2} \int_V \rho (\dot{u}_i)^2 dV \quad (D.2.1-3)$$

where ρ is the density of the material and ρdV represents the mass of the element which independent of time t , u_i is the covariant displacement vector, and σ_{ij} is the stress tensor, the dot represents differentiation with respect to time t . It is assumed that the variation of the density with time is negligible.

The Helmholtz free energy expression to be used as the potential energy form in the variational formulation is

$$F = \int_V f dV \quad (D.2.1-4)$$

where f is the energy density per unit volume. Equation D.2.1-4 can be expressed in terms of the stress and strain tensors as follows

$$F = \frac{1}{2} \int_V \sigma_{ij} \epsilon_{ij} dV \quad (D.2.1-5)$$

Now let us introduce the stress-strain relations. The material is assumed to that of a Kelvin body with temperature dependent properties, in which case the following function for the stress tensor is chosen.

$$\left. \begin{aligned} S_{ij} &= 2G\epsilon_{ij} + 2\eta\dot{\epsilon}_{ij}^* \\ \sigma_{ii} &= 3K\epsilon_{ii} + 3p\dot{\epsilon}_{jj} \end{aligned} \right\} \quad (D.2.1-6)$$

where α is linear thermal expansion coefficient, η is coefficient of viscosity, G is shear modulus, K is bulk modulus. The strain-displacement relations are

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{AND} \quad \epsilon_{ii} = u_{i,i}$$

Substitution of the stress-strain and strain displacement relations into the free energy expression yields

$$F = \int_V \left[G \left\{ u_{i,j} u_{i,j} - 3\alpha^2 T^2 \left(1 - \frac{2(1+\nu)}{1-2\nu} \right) \right\} + \eta \left\{ u_{i,j} u_{i,j} - 3\alpha^2 T \dot{T} \left(1 - \frac{3p}{2\eta} \right) \right\} \right] dV \quad (D.2.1-7)$$

To include the effects of static or transient external loading, it is necessary to express the work done on the mass of material by the body forces. The work integral may be written as

$$W = \int_V F_i u^i dV \quad (D.2.1-8)$$

where F_i is the body force per unit mass acting on the material. Or, in terms of the displacement vector u^i and the unit pressure intensity, we find

$$W = \int_V p_i u^i dV \quad (D.2.1-9)$$

It is assumed that the only external force acting on the shell is that of external normal pressure, P_3 . Therefore the work may be written as a two-dimensional integral, where now P_3 is a function of the curvilinear coordinates y^1 and y^2 .

$$W = \int_S P_3 u^3 ds \quad (D.2.1-10)$$

Combining equations (D.2.1-4), (D.2.1-8) and (D.2.1-10) we may now write equation (D.2.1-1) as follows

$$\begin{aligned} \delta \int_{t_0}^{t_1} \left[\int_V \left(\frac{\rho}{2} (\dot{u}_i)^2 dV - \int_V \left[G \left\{ u_{i,j} u_{i,j} - 3\alpha^2 T^2 \left(1 - \frac{2(1+\nu)}{1-2\nu} \right) \right\} + \eta \left\{ u_{i,j} u_{i,j} - 3\alpha^2 T \dot{T} \left(1 - \frac{3p}{2\eta} \right) \right\} \right] dV + \right. \right. \\ \left. \left. + \int_S P_3 u^3 ds \right] dt = 0 \end{aligned} \quad (D.2.1-11)$$

D.2.2 Geometrical Relations in Curvilinear Coordinates

General formulation of the variational equation appropriate to shell theory expressed in curvilinear y^i coordinate system requires the characterization of the middle-surface geometry. Let y^i represent an orthogonal curvilinear coordinates which are single-valued functions of the Cartesian coordinates x^i

$$y^i = y^i(x^i) \quad (D.2.2-1)$$

We may represent any point on the mean-surface by $y^1 y^2$, with y^3 as the local normal as shown in Figure D.2.1.

The volume element, dV , in the y^i coordinate system is given by

$$dV = \sqrt{|g_{ij}|} dy^1 dy^2 dy^3; \quad g_{ij} = 0, \quad i \neq j \quad (D.2.2-2)$$

and the element of area, dS on the y^i surface is

$$dS = \sqrt{|a_{\alpha\beta}|} dy^1 dy^2 \quad (D.2.2-3)$$

where $\sqrt{|g_{ij}|}$ is the square root of determinant of the metric tensor. In equation (2.2-2) $g_{ij} = g_{ij}(y^1, y^2, y^3)$ while in (2.2-3) $a_{\alpha\beta} = a_{\alpha\beta}(y^1, y^2)$.

For specific examples $\sqrt{|a_{\alpha\beta}|}$ has taken the following forms:

Hemispherical Shells: $\sqrt{|a_{\alpha\beta}|} = r^2 \sin \gamma^2 \quad (D.2.2-4)$

Conical Shells: $= \gamma^2 \sin^2 \gamma \quad (D.2.2-5)$

(vertical height is unity) where γ is one-half the apex angle.

D.2.3 Deformation of a Shell and Its Middle Surface

The expressions for the components of displacement of an arbitrary point of the shell in terms of displacement of the corresponding point of the middle surface are

$$\left. \begin{aligned} u^1 &= u_0^1 + u_1^1 y^3 \\ u^2 &= u_0^2 + u_1^2 y^3 \\ u^3 &= u_0^3 \end{aligned} \right\} \quad (D.2.3-1)$$

where u_0^i and u_1^i are functions of y^1 and y^2 only. The strain displacement relations in curvilinear coordinates can be expressed as

$$\eta_{ij} = \frac{1}{2} (a_{in} \xi_{,j}^2 + a_{jn} \xi_{,i}^2) \quad (\text{D.2.3-2})$$

where

η_{ij} covariant components of strain tensor

$\xi_{,j}^2$ covariant differential coefficient of contravariant displacement vector ξ^2 with respect to the curvilinear coordinates y^1

a_{in} metric tensor

Making use of the assumption that the normals to middle surface before bending remain normal after bending, we get

$$\begin{aligned} \eta_{\alpha 3} \Big|_{y^3=0} &= \frac{1}{2} (a_{\alpha\delta} \xi_{,3}^{\delta} + a_{3\delta} \xi_{,\alpha}^{\delta}) \\ &= \frac{1}{2} \left[\xi_{\alpha,3} + \xi_{3,\alpha} \right] \Big|_{y^3=0} = 0 \end{aligned} \quad (\text{D.2.3-3})$$

After performing covariant differentiation, simplification, and substituting the following expression

$$\begin{aligned} u_{\alpha} &= u^{\alpha} = \xi_{,\alpha} \sqrt{a^{\alpha\alpha}} = \frac{\xi_{,\alpha}}{\sqrt{a_{\alpha\alpha}}} \\ \text{or} \quad \xi_{,\alpha} &= u^{\alpha} \sqrt{a_{\alpha\alpha}} \end{aligned} \quad (\text{D.2.3-4})$$

we obtain

$$u_{,\alpha}^{\alpha} = - \left(\sqrt{E_{\alpha\alpha}} \frac{\partial u_0^3}{\partial y^{\alpha}} + L_{\alpha}^{\alpha} u_0^{\alpha} \right) \quad (\text{D.2.3-5})$$

where

$$\begin{aligned} a_{\alpha\alpha} \Big|_{y^3=0} &= E_{\alpha\alpha}, \quad E^{\alpha\alpha} = \frac{1}{E_{\alpha\alpha}} \\ \frac{\partial a_{\alpha\alpha}}{\partial y^3} \Big|_{y^3=0} &= -2L_{\alpha\alpha} \end{aligned} \quad \begin{array}{l} (\text{D.2.3-6}) \\ \text{no summation on } \alpha \end{array}$$

and

$$L_{\alpha}^{\alpha} = E^{\alpha\alpha} L_{\alpha\alpha}$$

D.3 Variational Equation in Terms of the Displacement of the Middle Surface of the Shell.

D.3.1 Expression of General Displacements in Terms of the Displacements of the Middle Surface

Consider now the general displacements. Employing the results of paragraph D.2.3, we can write that

$$\begin{aligned}
 \dot{u}^i \dot{u}^i &= (\dot{u}^1)^2 + (\dot{u}^2)^2 + (\dot{u}^3)^2 \\
 &= (\dot{u}_0^1)^2 + (\dot{u}_0^2)^2 + (\dot{u}_0^3)^2 - 2\gamma^3 \left[2\dot{u}_0^1 \left\{ \sqrt{E''} \frac{\partial \dot{u}_0^3}{\partial \gamma^1} + L_1' \dot{u}_0^1 \right\} + \right. \\
 &\quad \left. + 2\dot{u}_0^2 \left\{ \sqrt{E^{22}} \frac{\partial \dot{u}_0^3}{\partial \gamma^2} + L_2^2 \dot{u}_0^2 \right\} \right] + \\
 &\quad + (\gamma^3)^2 \left\{ E'' \left(\frac{\partial \dot{u}_0^3}{\partial \gamma^1} \right)^2 + 2\sqrt{E''} \frac{\partial \dot{u}_0^3}{\partial \gamma^1} L_1' \dot{u}_0^1 + (L_1' \dot{u}_0^1)^2 \right\} + \\
 &\quad + (\gamma^3)^2 \left\{ E^{22} \left(\frac{\partial \dot{u}_0^3}{\partial \gamma^2} \right)^2 + 2\sqrt{E^{22}} \frac{\partial \dot{u}_0^3}{\partial \gamma^2} L_2^2 \dot{u}_0^2 + (L_2^2 \dot{u}_0^2)^2 \right\} \quad (D.3.1-1)
 \end{aligned}$$

$$u_{ij} u_{ij} = (u_{1,1})^2 + (u_{1,2})^2 + (u_{1,3})^2 + (u_{2,1})^2 + (u_{2,2})^2 + (u_{2,3})^2 + (u_{3,1})^2 + (u_{3,2})^2 + (u_{3,3})^2 \quad (D.3.1-2)$$

From equation (D.2.3-5) we have

$$\begin{aligned}
 u_\alpha &= u_0^\alpha - \gamma^3 \left(\sqrt{E^{\alpha\alpha}} \frac{\partial u_0^3}{\partial \gamma^\alpha} + L_\alpha^\alpha u_0^\alpha \right) \\
 u_3 &= u_0^3
 \end{aligned}$$

Differentiating and substituting above result we can express equation (D3.1-2) as follows:

$$\begin{aligned}
 u_{ij} u_{ij} &= \left(\frac{\partial u_0^1}{\partial \gamma^1} - \gamma^3 A \right)^2 + \left(\frac{\partial u_0^1}{\partial \gamma^2} - \gamma^3 B \right)^2 + (C)^2 + \left(\frac{\partial u_0^2}{\partial \gamma^1} - \gamma^3 D \right)^2 + \\
 &\quad + \left(\frac{\partial u_0^2}{\partial \gamma^2} - \gamma^3 F \right)^2 + (H)^2 + \left(\frac{\partial u_0^3}{\partial \gamma^1} \right)^2 + \left(\frac{\partial u_0^3}{\partial \gamma^2} \right)^2 \quad (D.3.1-3)
 \end{aligned}$$

where

$$\left. \begin{aligned} A &= \frac{\partial}{\partial \dot{\gamma}'} \left(\sqrt{E''} \frac{\partial u_0^3}{\partial \dot{\gamma}'} + L_1' u_0' \right) \\ B &= \frac{\partial}{\partial \dot{\gamma}^2} \left(\sqrt{E''} \frac{\partial u_0^3}{\partial \dot{\gamma}^2} + L_1' u_0' \right) \\ C &= \sqrt{E''} \frac{\partial u_0^3}{\partial \dot{\gamma}'} + L_1' u_0' \end{aligned} \right\} \quad (D.3.1-4)$$

$$\left. \begin{aligned} D &= \frac{\partial}{\partial \dot{\gamma}'} \left(\sqrt{E^{22}} \frac{\partial u_0^3}{\partial \dot{\gamma}^2} + L_2^2 u_0^2 \right) \\ F &= \frac{\partial}{\partial \dot{\gamma}^2} \left(\sqrt{E^{22}} \frac{\partial u_0^3}{\partial \dot{\gamma}^2} + L_2^2 u_0^2 \right) \\ H &= \sqrt{E^{22}} \frac{\partial u_0^3}{\partial \dot{\gamma}^2} + L_2^2 u_0^2 \end{aligned} \right\} \quad (D.3.1-5)$$

D.3.2 Variational Equation for a General Shell of Revolution

By making use of equations (D.2.1-11), (D.2.2-2), (D.3.1-1) and (D.3.1-3), we can write the variational equation for a general shell of revolution (shell of general geometry) in the following form:

$$\begin{aligned} & \int_{t_0}^{t_1} \left[\iiint_V \frac{\rho}{2} \sqrt{|g_{ij}|} \left\{ (\dot{u}_0')^2 + (\dot{u}_0^2)^2 + (\dot{u}_0^3)^2 + (\dot{\gamma}^3)^2 \left[(C)^2 + (H)^2 \right] - 2\dot{\gamma}^3 [C u_0' + H u_0^2] \right\} d\dot{\gamma}' d\dot{\gamma}^2 d\dot{\gamma}^3 + \right. \\ & + \iiint_V G \sqrt{|g_{ij}|} \left\{ \left(\frac{\partial u_0'}{\partial \dot{\gamma}'} - \dot{\gamma}^3 A \right)^2 + \left(\frac{\partial u_0'}{\partial \dot{\gamma}^2} - \dot{\gamma}^3 B \right)^2 + C^2 + \left(\frac{\partial u_0^2}{\partial \dot{\gamma}'} - \dot{\gamma}^3 D \right)^2 + \left(\frac{\partial u_0^2}{\partial \dot{\gamma}^2} - \dot{\gamma}^3 F \right)^2 + H^2 + \left(\frac{\partial u_0^3}{\partial \dot{\gamma}'} \right)^2 + \left(\frac{\partial u_0^3}{\partial \dot{\gamma}^2} \right)^2 \right\} d\dot{\gamma}' d\dot{\gamma}^2 d\dot{\gamma}^3 - \\ & - \iiint_V G \sqrt{|g_{ij}|} 3\alpha^2 T^2 \left\{ 1 - \frac{2(1+\nu)}{1-2\nu} \right\} d\dot{\gamma}' d\dot{\gamma}^2 d\dot{\gamma}^3 + \iiint_V \eta \sqrt{|g_{ij}|} \left\{ \left(\frac{\partial u_0'}{\partial \dot{\gamma}'} - \dot{\gamma}^3 A \right) \left(\frac{\partial \dot{u}_0'}{\partial \dot{\gamma}'} - \dot{\gamma}^3 \dot{A} \right) + \right. \\ & + \left(\frac{\partial u_0'}{\partial \dot{\gamma}^2} - \dot{\gamma}^3 B \right) \left(\frac{\partial \dot{u}_0'}{\partial \dot{\gamma}^2} - \dot{\gamma}^3 \dot{B} \right) + C \dot{C} + \left(\frac{\partial u_0^2}{\partial \dot{\gamma}'} - \dot{\gamma}^3 D \right) \left(\frac{\partial \dot{u}_0^2}{\partial \dot{\gamma}'} - \dot{\gamma}^3 \dot{D} \right) + \left(\frac{\partial u_0^2}{\partial \dot{\gamma}^2} - \dot{\gamma}^3 F \right) \left(\frac{\partial \dot{u}_0^2}{\partial \dot{\gamma}^2} - \dot{\gamma}^3 \dot{F} \right) + H \dot{H} + \\ & + \left. \frac{\partial u_0^3}{\partial \dot{\gamma}'} \frac{\partial \dot{u}_0^3}{\partial \dot{\gamma}'} + \frac{\partial u_0^3}{\partial \dot{\gamma}^2} \frac{\partial \dot{u}_0^3}{\partial \dot{\gamma}^2} \right\} d\dot{\gamma}' d\dot{\gamma}^2 d\dot{\gamma}^3 - \iiint_V \eta \sqrt{|g_{ij}|} 3\alpha T \dot{T} \left(1 - \frac{3\nu}{2} \right) d\dot{\gamma}' d\dot{\gamma}^2 d\dot{\gamma}^3 - \int P u_3 \sqrt{|g_{ij}|} d\dot{\gamma}' d\dot{\gamma}^2 \Big] dt = 0 \end{aligned} \quad (D.3.2-1)$$

D.3.3 Variational Equation for a Hemispherical Shell

Now we expand equation (D.3.2-1) for application to hemispherical shells in terms of middle surface deformations.

From Equations (D.2.3-1) and (D.2.3-5) we have that

$$\left. \begin{aligned} u^\alpha &= u_0^\alpha - \gamma^3 \left(\sqrt{E^{\alpha\alpha}} \frac{\partial u_0^3}{\partial y^\alpha} + E^{\alpha\alpha} L_{\alpha\alpha} u_0^\alpha \right) \\ u^3 &= u_0^3 \\ E_{\alpha\alpha} &= a_{\alpha\alpha} \Big|_{\gamma^3=0}, \quad -2L_{\alpha\alpha} = \frac{\partial a_{\alpha\alpha}}{\partial y^3} \Big|_{\gamma^3=0} \end{aligned} \right| \quad (D.3.3-1)$$

$$\left. \begin{aligned} E_{11} &= a_{11} \Big|_{\gamma^3=0} = r^2 \sin^2 \phi, \quad E^{11} = \frac{1}{E_{11}} = \frac{1}{r^2 \sin^2 \phi}, \quad L_{11} = -r \sin^2 \phi \\ E_{22} &= a_{22} \Big|_{\gamma^3=0} = r^2, \quad E^{22} = \frac{1}{E_{22}} = \frac{1}{r^2}, \quad L_{22} = -r \end{aligned} \right| \quad (D.3.3-2)$$

From the above two sets of equations we can write that

$$u^1 = u_0^1 + \frac{\gamma^3}{r} \left(u_0^1 - \frac{1}{\sin \phi} \frac{\partial u_0^3}{\partial \theta} \right) \quad (D.3.3-3)$$

$$u^2 = u_0^2 + \frac{\gamma^3}{r} \left(u_0^2 - \frac{\partial u_0^3}{\partial \phi} \right) \quad (D.3.3-4)$$

The functions A, B, C, D, F and H take the following forms:

$$\begin{aligned} C &= \frac{1}{r} \left(u_0^1 - \frac{1}{\sin \phi} \frac{\partial u_0^3}{\partial \theta} \right) \\ H &= \frac{1}{r} \left(u_0^2 - \frac{\partial u_0^3}{\partial \phi} \right) \\ A &= \frac{\partial C}{\partial \theta} = \frac{1}{r} \left(\frac{\partial u_0^1}{\partial \theta} - \frac{1}{\sin \phi} \frac{\partial^2 u_0^3}{\partial \theta^2} \right) \\ B &= \frac{\partial C}{\partial \phi} = \frac{1}{r} \left(\frac{\partial u_0^1}{\partial \phi} - \frac{1}{\sin \phi} \frac{\partial^2 u_0^3}{\partial \phi \partial \theta} + \frac{\partial u_0^3}{\partial \theta} \cot \phi \operatorname{cosec} \theta \right) \\ D &= \frac{\partial H}{\partial \theta} = \frac{1}{r} \left(\frac{\partial u_0^2}{\partial \theta} - \frac{\partial^2 u_0^3}{\partial \theta^2} \right) \\ F &= \frac{\partial H}{\partial \phi} = \frac{1}{r} \left(\frac{\partial u_0^2}{\partial \phi} - \frac{\partial^2 u_0^3}{\partial \phi^2} \right) \end{aligned}$$

Also for a hemispherical shell

$$\sqrt{a_{\theta\theta}} = \sqrt{\begin{vmatrix} r^2 \sin^2 \phi & 0 \\ 0 & r^2 \end{vmatrix}} = \sqrt{r^4 \sin^2 \phi} = r^2 \sin \phi$$

After putting these values in (D.3.2-1), simplifying the expression and integrating with respect to y^3 through the thickness of the shell, we get the following variational equation for a hemispherical shell.

Variational equations for the hemispherical shell

$$\begin{aligned} & \delta \int_{t_0}^{t_1} \left[\iint_S \frac{\rho}{2} \sin^2 \phi \left\{ h \left[(\dot{u}_\theta')^2 + (\dot{u}_\phi')^2 + (\dot{u}_\phi'')^2 \right] + \frac{h^3}{12r^2} \left[\left(\dot{u}_\theta' - \frac{1}{\sin \phi} \frac{\partial u_\phi^3}{\partial \theta} \right)^2 + \left(\dot{u}_\phi' - \frac{\partial u_\theta^3}{\partial \phi} \right)^2 \right] \right\} d\phi d\theta + \right. \\ & + \iint_S G r^2 \sin \phi \left\{ h \left[\left(\frac{\partial u_\theta'}{\partial \theta} \right)^2 + \left(\frac{\partial u_\phi'}{\partial \phi} \right)^2 + \left(\frac{\partial u_\phi''}{\partial \theta} \right)^2 + \left(\frac{\partial u_\theta''}{\partial \phi} \right)^2 + \left(\frac{\partial u_\phi^3}{\partial \theta} \right)^2 + \left(\frac{\partial u_\theta^3}{\partial \phi} \right)^2 \right] + \frac{h}{r^2} \left[\left(\dot{u}_\theta' - \frac{1}{\sin \phi} \frac{\partial u_\phi^3}{\partial \theta} \right)^2 + \left(\dot{u}_\phi' - \frac{\partial u_\theta^3}{\partial \phi} \right)^2 \right] + \right. \\ & + \frac{h^3}{12r^2} \left[\left(\frac{\partial u_\theta'}{\partial \theta} - \frac{1}{\sin \theta} \frac{\partial^2 u_\theta'}{\partial \theta^2} \right)^2 + \left(\frac{\partial u_\phi'}{\partial \phi} - \frac{1}{\sin \phi} \frac{\partial^2 u_\phi'}{\partial \phi^2} + \frac{\partial u_\theta'}{\partial \theta} \cot \phi \operatorname{cosec} \phi \right)^2 + \left(\frac{\partial u_\theta''}{\partial \theta} - \frac{\partial^2 u_\phi^3}{\partial \theta \partial \phi} \right)^2 + \left(\frac{\partial u_\phi''}{\partial \phi} - \frac{\partial^2 u_\theta^3}{\partial \phi \partial \theta} \right)^2 \right] \Big\} d\phi d\theta + \\ & + \iint_S \eta r^2 \sin \phi \left\{ h \left[\left(\frac{\partial u_\theta'}{\partial \theta} \frac{\partial \dot{u}_\theta'}{\partial \theta} + \frac{\partial u_\phi'}{\partial \phi} \frac{\partial \dot{u}_\phi'}{\partial \phi} + \frac{\partial u_\phi''}{\partial \theta} \frac{\partial \dot{u}_\phi''}{\partial \theta} + \frac{\partial u_\theta''}{\partial \phi} \frac{\partial \dot{u}_\theta''}{\partial \phi} + \frac{\partial u_\phi^3}{\partial \theta} \frac{\partial \dot{u}_\phi^3}{\partial \theta} + \frac{\partial u_\theta^3}{\partial \phi} \frac{\partial \dot{u}_\theta^3}{\partial \phi} \right] + \right. \\ & + \frac{h}{r^2} \left[\left(\dot{u}_\theta' - \frac{1}{\sin \phi} \frac{\partial u_\phi^3}{\partial \theta} \right) \left(\dot{u}_\theta' - \frac{1}{\sin \phi} \frac{\partial \dot{u}_\phi^3}{\partial \theta} \right) + \left(\dot{u}_\phi' - \frac{\partial u_\theta^3}{\partial \phi} \right) \left(\dot{u}_\phi' - \frac{\partial \dot{u}_\theta^3}{\partial \phi} \right) \right] + \frac{h^3}{12r^2} \left[\left(\frac{\partial u_\theta'}{\partial \theta} - \frac{1}{\sin \theta} \frac{\partial^2 u_\theta'}{\partial \theta^2} \right) \left(\frac{\partial \dot{u}_\theta'}{\partial \theta} - \frac{1}{\sin \theta} \frac{\partial^2 \dot{u}_\theta'}{\partial \theta^2} \right) + \right. \\ & - \frac{1}{\sin \phi} \frac{\partial^2 u_\phi^3}{\partial \theta^2} + \left(\frac{\partial u_\phi'}{\partial \phi} - \frac{1}{\sin \phi} \frac{\partial^2 u_\phi'}{\partial \theta \partial \phi} + \frac{\partial u_\theta'}{\partial \theta} \cot \phi \operatorname{cosec} \phi \right) \left(\frac{\partial \dot{u}_\phi'}{\partial \phi} - \frac{1}{\sin \phi} \frac{\partial^2 \dot{u}_\phi'}{\partial \theta \partial \phi} + \frac{\partial \dot{u}_\theta'}{\partial \theta} \cot \phi \operatorname{cosec} \phi \right) + \\ & + \left. \left(\frac{\partial u_\theta''}{\partial \theta} - \frac{\partial^2 u_\phi^3}{\partial \theta \partial \phi} \right) \left(\frac{\partial \dot{u}_\theta''}{\partial \theta} - \frac{\partial^2 \dot{u}_\phi^3}{\partial \theta \partial \phi} \right) + \left. \left(\frac{\partial u_\phi''}{\partial \phi} - \frac{\partial^2 u_\theta^3}{\partial \phi \partial \theta} \right) \left(\frac{\partial \dot{u}_\phi''}{\partial \phi} - \frac{\partial^2 \dot{u}_\theta^3}{\partial \phi \partial \theta} \right) \right] \right\} d\phi d\theta + \iint_S N_1 (G \alpha^2 T_s^2) r^2 \sin \phi d\phi d\theta + \\ & + \left. \iint_S N_2 (q \alpha^2 T_s \dot{T}_s) r^2 \sin \phi d\phi d\theta + \iint_S p u_\phi^3 r^2 \sin \phi d\phi d\theta \right] dt = 0 \end{aligned}$$

WHERE

$$N_1 = \frac{3}{4} h \left(1 + \frac{h}{3} \right) \left[1 - \frac{2(1+\nu)}{1-2\nu} \right], \quad N_2 = \frac{3}{4} h \left(1 + \frac{h}{3} \right) \left[1 - \frac{3\rho}{2\eta} \right]$$

$T_s = T_s(t)$ IS SURFACE TEMPERATURE

where

$$N_1 = \frac{3}{4} h \left(1 + \frac{h}{3}\right) \left[1 - \frac{2(1+\nu)}{1-2\nu}\right]$$

$$N_2 = \frac{3}{4} h \left(1 + \frac{h}{3}\right) \left[1 - \frac{3\rho}{2\gamma}\right]$$

$$T_s = T_s(t)$$

is surface temperature

D.4 Integration of the Variational Equation for Thin Hemispherical Shell

D.4.1 Boundary Conditions

Consider a hemispherical shell which is clamped at the edge, $\phi = \frac{\pi}{2}$ (See Fig. D.4.1). At $\phi = \frac{\pi}{2}$ and $0 \leq \theta \leq 2\pi$, the deformations u , v , w and the rotation ψ become

$$u = v = w = 0$$

$$\psi = -\frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} + \frac{v}{r} = 0$$

In case of a hemispherical shell we have that

$$u = u' = u_0' + \frac{\gamma^3}{r} \left(u_0' - \frac{1}{\sin \phi} \frac{\partial u_0^3}{\partial \theta} \right)$$

$$v = u^2 = u_0^2 + \frac{\gamma^3}{r} \left(u_0^2 - \frac{\partial u_0^3}{\partial \phi} \right)$$

$$w = u^3 = u_0^3 \quad \alpha_1 \sim \theta, \alpha_2 \sim \phi$$

and

$$A_2^2 = r, \phi \cdot r, \phi = r^2 \quad \text{or} \quad A_2 = r$$

Considering now the boundary conditions, we can write that

$$u' \Big|_{\phi=\frac{\pi}{2}} = 0 = u_0' \Big|_{\phi=\frac{\pi}{2}} + \frac{\gamma^3}{r} \left(u_0' - \frac{\partial u_0^3}{\partial \theta} \right) \Big|_{\phi=\frac{\pi}{2}} = \left[u_0' + \frac{\gamma^3}{r} \left(u_0' - \frac{\partial u_0^3}{\partial \theta} \right) \right] \Big|_{\phi=\frac{\pi}{2}} - \frac{1}{r} \frac{\partial u_0^3}{\partial \phi} + \frac{1}{r} \left\{ u_0^2 + \frac{\gamma^3}{r} \left(u_0^2 - \frac{\partial u_0^3}{\partial \phi} \right) \right\} \Big|_{\phi=\frac{\pi}{2}} = 0$$

$$u^2 \Big|_{\phi=\frac{\pi}{2}} = \left[u_0^2 + \frac{\gamma^3}{r} \left(u_0^2 - \frac{\partial u_0^3}{\partial \phi} \right) \right] \Big|_{\phi=\frac{\pi}{2}}, \quad u^3 \Big|_{\phi=\frac{\pi}{2}} = u_0^3 \Big|_{\phi=\frac{\pi}{2}}$$

which can be written as

$$u_0' + \frac{\gamma^2}{\kappa} \left(u_0' - \frac{\partial u_0^3}{\partial \phi} \right) = 0 \quad (D.4.1-1)$$

$$u_0^2 + \frac{\gamma^2}{\kappa} \left(u_0^2 - \frac{\partial u_0^3}{\partial \phi} \right) = 0 \quad (D.4.1-2)$$

$$u_0^3 = 0 \quad (D.4.1-3)$$

$$u_0^2 + \frac{\gamma^2}{\kappa} \left(u_0^2 - \frac{\partial u_0^3}{\partial \phi} \right) - \frac{\partial u_0^3}{\partial \phi} = 0 \quad (D.4.1-4)$$

From equations (D.4.1-2) and (D.4.1-4)

$$\frac{\partial u_0^3}{\partial \phi} = 0$$

Hence

$$u_0' + \frac{\gamma^2}{\kappa} u_0' = 0$$

$$\text{OR } u_0' \left(1 + \frac{\gamma^2}{\kappa} \right) = 0$$

Therefore

$$u_0' = 0$$

Similarly

$$u_0^2 = 0$$

And from equations (D.4.1-3)

$$u_0^3 = 0$$

Thus we have the following boundary conditions:

$$\left. \begin{aligned} u_0' = u_0^2 = u_0^3 = 0 \\ \frac{\partial u_0^3}{\partial \phi} = 0 \end{aligned} \right\} \text{ AT } \phi = \frac{\pi}{2} \text{ AND } 0 \leq \theta \leq 2\pi$$

Also the functions u_1, u_2, u_3 must be periodic with respect to θ with a period 2π .

D.4.2 Expression of Displacements in Terms of Functions ϕ and θ

It is necessary to represent u_0^1, u_0^2, u_0^3 in terms of functions of ϕ and θ whose coefficients are unknown functions of t . These functions must satisfy the boundary conditions and should be periodic with respect to θ with a period of 2π . These conditions are fulfilled by the following functions:

$$u_0^1 = \beta_1 \cos \phi + \beta_2 \cos \phi \sin^2 \theta$$

$$u_0^2 = \gamma_1 \cos \phi + \gamma_2 \cos \phi \sin^2 \theta$$

$$u_0^3 = \left(\frac{\pi}{2} - \phi \right) \left(\rho_1 \cos \phi + \rho_2 \cos \phi \sin^2 \theta \right)$$

These functions or their derivatives are substituted in the variational equation for a thin hemispherical shell and integration is performed with respect to ϕ and θ over the given boundaries.

D.4.3 Integration of the Variational Equation for Thin Hemispherical Shell

For the hemispherical shell the integration has been carried out over the limits of $0 \leq \phi \leq \frac{\pi}{2}$ and $0 \leq \theta \leq \frac{\pi}{2}$. But as we must take the limits for as $0 \leq \theta \leq 2\pi$ i.e. go around the circle, a factor of 4 will be common in all integrations. Since it is a common factor, it will not affect our results. Some of the integrations have been carried out numerically. A singularity occurs at the apex and in one of the integrations an approximate value has been taken numerically a surface making an angle of $\phi = 0^\circ 30'$ with the axis. Thus the results of this analysis cannot be applied to the apex and a separate analysis will be needed from this region of the shell.

After having carried out the integration over the middle surface of the shell, we reduce the variational equation to functions of time only. This equation is shown as follows: (See next page).

Variational Equation for a Thin Hemispherical Shell

$$\begin{aligned}
 \int_{t_0}^{t_1} \left[\frac{\rho n^2 h}{2} \left\{ 0.524 \dot{\beta}_1^2 + 0.524 \dot{\beta}_1 \dot{\beta}_2 + 0.196 \dot{\beta}_2^2 + 0.524 \dot{\delta}_1^2 + 0.524 \dot{\delta}_1 \dot{\delta}_2 + 0.196 \dot{\delta}_2^2 + 0.478 \dot{\rho}_1^2 + 0.478 \dot{\rho}_1 \dot{\rho}_2 + 0.179 \dot{\rho}_2^2 \right\} + \right. \\
 + G n^2 h \left\{ 0.261 \dot{\beta}_2^2 + 1.047 \dot{\beta}_1^2 + 1.047 \dot{\beta}_1 \dot{\beta}_2 + 0.393 \dot{\beta}_2^2 + 0.261 \dot{\delta}_2^2 + 1.047 \dot{\delta}_1^2 + 1.047 \dot{\delta}_1 \dot{\delta}_2 + 0.393 \dot{\delta}_2^2 + \right. \\
 + 0.239 \dot{\rho}_2^2 + 0.445 \dot{\rho}_1^2 + 0.445 \dot{\rho}_1 \dot{\rho}_2 + 0.167 \dot{\rho}_2^2 \left. \right\} + G h \left\{ 0.524 \dot{\beta}_1^2 + 0.524 \dot{\beta}_1 \dot{\beta}_2 + 0.196 \dot{\beta}_2^2 + \right. \\
 + 6.837 \dot{\rho}_2^2 - 1.734 \dot{\beta}_1 \dot{\rho}_2 + 0.867 \dot{\beta}_2 \dot{\rho}_2 + 0.524 (\dot{\delta}_1^2 + 2 \dot{\delta}_1 \dot{\rho}_1 + \dot{\rho}_1^2) + 0.524 (\dot{\delta}_1^2 + \dot{\delta}_1 \dot{\rho}_1 + \dot{\rho}_1 \dot{\rho}_2 + \dot{\rho}_2 \dot{\delta}_2) + \\
 + 0.196 (\dot{\delta}_2^2 + 2 \dot{\delta}_2 \dot{\rho}_2 + \dot{\rho}_2^2) - (0.776 \dot{\rho}_1^2 + 0.776 \dot{\rho}_1 \dot{\rho}_2 + 0.291 \dot{\rho}_2^2) + 0.698 (\dot{\rho}_1^2 + \dot{\rho}_1 \dot{\rho}_2) + \\
 + 0.349 (\dot{\rho}_1 \dot{\delta}_2 + 2 \dot{\rho}_1 \dot{\rho}_2 + \dot{\rho}_2 \dot{\delta}_1) + 0.262 (\dot{\rho}_2^2 + \dot{\rho}_2 \dot{\delta}_2) \left. \right\} + \eta n^2 h \left\{ 0.261 \dot{\beta}_1 \dot{\beta}_2 + 1.047 \dot{\beta}_1 \dot{\beta}_1 + 0.524 \dot{\beta}_1 \dot{\beta}_2 + \right. \\
 + \dot{\beta}_1 \dot{\beta}_2) + 0.393 \dot{\beta}_2 \dot{\beta}_2 + 0.261 \dot{\delta}_2 \dot{\delta}_2 + 1.047 \dot{\delta}_1 \dot{\delta}_1 + 0.524 (\dot{\delta}_1 \dot{\delta}_2 + \dot{\delta}_1 \dot{\delta}_2) + 0.393 \dot{\delta}_2 \dot{\delta}_2 + 0.239 \dot{\rho}_2 \dot{\rho}_2 + \\
 + 0.445 \dot{\rho}_1 \dot{\rho}_1 + 0.263 (\dot{\rho}_1 \dot{\rho}_2 + \dot{\rho}_1 \dot{\rho}_2) + 0.167 \dot{\rho}_2 \dot{\rho}_2 \left. \right\} + \eta h \left\{ 0.524 \dot{\beta}_1 \dot{\beta}_1 + 0.262 (\dot{\beta}_1 \dot{\beta}_2 + \dot{\beta}_1 \dot{\beta}_2) + \right. \\
 + 0.196 \dot{\beta}_2 \dot{\beta}_2 + 6.837 \dot{\rho}_2 \dot{\rho}_2 - 0.867 (\dot{\beta}_1 \dot{\rho}_2 + \dot{\beta}_1 \dot{\rho}_2) + 0.434 (\dot{\beta}_2 \dot{\rho}_2 + \dot{\beta}_2 \dot{\rho}_2) + 0.524 (\dot{\delta}_1 \dot{\delta}_1 + \dot{\delta}_1 \dot{\rho}_1 + \dot{\rho}_1 \dot{\rho}_1) + \\
 + 0.262 (\dot{\delta}_1 \dot{\delta}_2 + \dot{\delta}_2 \dot{\delta}_1) + 0.262 (\dot{\delta}_1 \dot{\rho}_2 + \dot{\rho}_1 \dot{\rho}_2) + 0.262 (\dot{\rho}_1 \dot{\rho}_2 + \dot{\rho}_1 \dot{\rho}_2) + 0.262 (\dot{\rho}_1 \dot{\delta}_2 + \dot{\rho}_1 \dot{\delta}_2) + 0.196 (\dot{\delta}_2 \dot{\delta}_2 + \\
 + \dot{\delta}_2 \dot{\rho}_2) + 0.196 (\dot{\delta}_2 \dot{\rho}_2 + \dot{\rho}_2 \dot{\rho}_2) - 0.776 \dot{\rho}_1 \dot{\rho}_1 - 0.388 (\dot{\rho}_1 \dot{\rho}_2 + \dot{\rho}_1 \dot{\rho}_2) - 0.291 \dot{\rho}_2 \dot{\rho}_2 + 0.698 \dot{\rho}_1 \dot{\rho}_1 + \\
 + 0.349 (\dot{\rho}_1 \dot{\delta}_1 + \dot{\rho}_1 \dot{\rho}_1) + 0.176 (\dot{\rho}_1 \dot{\delta}_2 + \dot{\rho}_1 \dot{\rho}_2) + 0.349 (\dot{\rho}_1 \dot{\rho}_2 + \dot{\rho}_1 \dot{\rho}_2) + 0.175 (\dot{\rho}_2 \dot{\delta}_1 + \dot{\rho}_2 \dot{\rho}_1) + \\
 + 0.262 \dot{\rho}_2 \dot{\rho}_2 + 0.131 (\dot{\rho}_2 \dot{\delta}_2 + \dot{\rho}_2 \dot{\delta}_2) \left. \right\} + \\
 \left. + 1.571 n^2 N_1 G \alpha^2 T_s^2 + 1.571 n^2 N_2 \gamma \alpha^2 T_s \dot{T}_s + P n (1.854 \dot{\rho}_1 + 0.926 \dot{\rho}_2) \right] dt = 0
 \end{aligned}$$

(D.4.3-1)

$$\text{where } N_1 = \frac{3}{4} h \left(1 + \frac{h}{3} \right) \left[1 - \frac{2(1+\nu)}{1-2\nu} \right], \quad N_2 = \frac{3}{4} h \left(1 + \frac{h}{3} \right) \left[1 - \frac{3\rho}{2\eta} \right]$$

T_s is surface temperature and function of t only.

D.5. Development of Differential Equations for Thin Hemispherical Shell

D.5.1 Functional of the Variational Equation

The variational equation can be represented in the following form:

$$\delta \int_{t_0}^t F(\beta_1, \beta_2, \gamma_1, \dots, \dot{\beta}_1, \dot{\beta}_2, \dot{\gamma}_1, \dots, t) dt = 0 \quad (D.5.1-1)$$

The integrand F is known as functional. It consists of independent variable t and other known and unknown functions of t e.g. $\beta_1, \beta_2, \dot{\beta}_1, \eta, T$ etc. Expression for the functional F is shown as follows:

Functional F of the Variational Equation is

$$\begin{aligned} F = & \frac{\rho \dot{a}^2 h}{2} \left\{ 0.524 \dot{\beta}_1^2 + 0.524 \dot{\beta}_1 \dot{\beta}_2 + 0.196 \dot{\beta}_2^2 + 0.524 \dot{\gamma}_1^2 + 0.524 \dot{\gamma}_1 \dot{\gamma}_2 + 0.196 \dot{\gamma}_2^2 + 0.478 \dot{\rho}_1^2 + 0.478 \dot{\rho}_1 \dot{\rho}_2 + 0.179 \dot{\rho}_2^2 \right\} + \\ & + G \dot{a}^2 h \left\{ 0.261 \beta_2^2 + 1.047 \beta_1^2 + 1.047 \beta_1 \beta_2 + 0.393 \beta_2^2 + 0.261 \gamma_2^2 + 1.047 \gamma_1^2 + 1.047 \gamma_1 \gamma_2 + 0.393 \gamma_2^2 + 0.239 \rho_2^2 + 0.445 \rho_1^2 + \right. \\ & + 0.445 \rho_1 \rho_2 + 0.167 \rho_2^2 \left. \right\} + G h \left\{ 0.524 \rho_1^2 + 0.524 \rho_1 \rho_2 + 0.196 \rho_2^2 + 6.837 \rho_2^2 - 1.734 \beta_1 \rho_2 + 0.867 \beta_2 \rho_2 + 0.524 \gamma_1^2 + \right. \\ & + 1.048 \gamma_1 \rho_1 + 0.524 \rho_1^2 + 0.524 (\gamma_1 \gamma_2 + \gamma_1 \rho_2) + 0.524 (\rho_1 \rho_2 + \rho_1 \gamma_2) + 0.196 (\gamma_2^2 + 2 \gamma_2 \rho_1) + 0.196 \rho_2^2 - 0.776 \rho_1^2 - \\ & - 0.776 \rho_1 \rho_2 - 0.291 \rho_2^2 + 0.698 (\rho_1^2 + \rho_1 \rho_2) + 0.349 (\rho_1 \gamma_2 + 2 \rho_1 \rho_2 + \rho_2 \gamma_1) + 0.262 (\rho_2^2 + \rho_2 \gamma_2) \left. \right\} + \\ & + \eta \dot{a}^2 h \left\{ 0.261 \beta_2 \dot{\beta}_2 + 1.047 \beta_1 \dot{\beta}_1 + 0.524 (\beta_1 \dot{\beta}_2 + \dot{\beta}_1 \beta_2) + 0.393 \beta_2 \dot{\beta}_2 + 0.261 \gamma_2 \dot{\gamma}_2 + 1.047 \gamma_1 \dot{\gamma}_1 + 0.524 (\gamma_1 \dot{\gamma}_2 + \dot{\gamma}_1 \gamma_2) + \right. \\ & + 0.393 \gamma_2 \dot{\gamma}_2 + 0.239 \rho_2 \dot{\rho}_2 + 0.445 \rho_1 \dot{\rho}_1 + 0.223 (\rho_1 \dot{\rho}_2 + \dot{\rho}_2 \rho_1) + 0.167 \rho_2 \dot{\rho}_2 + \eta h \left\{ 0.524 \beta_1 \dot{\beta}_1 + 0.262 (\dot{\beta}_1 \beta_2 + \beta_1 \dot{\beta}_2) + \right. \\ & + 0.196 \beta_2 \dot{\beta}_2 + 6.837 \rho_2 \dot{\rho}_2 - 0.867 (\beta_1 \rho_2 + \dot{\beta}_1 \rho_2) + 0.434 (\beta_2 \dot{\rho}_2 + \dot{\beta}_2 \rho_2) + 0.524 \gamma_1 \dot{\gamma}_1 + 0.524 (\gamma_1 \dot{\rho}_1 + \dot{\gamma}_1 \rho_1) + \\ & + 0.524 \rho_1 \dot{\rho}_1 + 0.262 (\gamma_1 \dot{\gamma}_2 + \dot{\gamma}_1 \gamma_2 + \gamma_1 \dot{\rho}_2 + \dot{\gamma}_1 \rho_2) + 0.262 (\rho_1 \dot{\rho}_2 + \dot{\rho}_1 \rho_2 + \rho_1 \dot{\gamma}_2 + \dot{\rho}_1 \gamma_2) + 0.196 (\gamma_2 \dot{\gamma}_2 + \gamma_2 \dot{\rho}_2 + \dot{\gamma}_2 \rho_2) + \\ & + 0.196 \rho_2 \dot{\rho}_2 - 0.776 \rho_1 \dot{\rho}_1 - 0.388 (\rho_1 \dot{\rho}_2 + \dot{\rho}_1 \rho_2) - 0.291 \rho_2 \dot{\rho}_2 + 0.698 (\rho_1 \dot{\rho}_1) + 0.349 (\rho_1 \dot{\gamma}_1 + \dot{\rho}_1 \gamma_1) + \\ & + 0.349 (\rho_1 \dot{\rho}_2 + \dot{\rho}_2 \rho_1) + 0.175 (\rho_1 \dot{\gamma}_2 + \dot{\rho}_1 \gamma_2) + 0.175 (\gamma_1 \dot{\rho}_2 + \dot{\gamma}_1 \rho_2) + 0.262 \rho_2 \dot{\rho}_2 + 0.131 (\rho_2 \dot{\gamma}_2 + \dot{\rho}_2 \gamma_2) \left. \right\} + \\ & + 1.571 N_1 \dot{a}^2 G \dot{a}^2 T_5^2 + 1.571 N_2 \dot{a}^2 T_5 \dot{T}_5 + P_n (1.854 \rho_1 + 0.926 \rho_2) \end{aligned}$$

where $\beta_1, \beta_2, \gamma_1, \gamma_2, \rho_1, \rho_2, G, \eta, T_5$

are functions of time

D.5.2 Euler Lagrange Equations for the Thin Hemispherical Shell

We assume that value of functional F vanishes at both the time $t = t_0$ and $t = t_1$. It is on this assumption that we can make use of Euler-Lagrange equations for expression of the results in form of differential equations. The Euler-Lagrange equations for the functional F with unknowns $\beta_1, \beta_2, \gamma_1, \gamma_2, \rho_1, \rho_2$ and their time derivatives can be expressed in the following form.

$$\frac{d}{dt} \left(\frac{\partial F}{\partial \dot{\beta}_1} \right) - \frac{\partial F}{\partial \beta_1} = 0 \quad (D.5.2-1)$$

$$\frac{d}{dt} \left(\frac{\partial F}{\partial \dot{\beta}_2} \right) - \frac{\partial F}{\partial \beta_2} = 0 \quad (D.5.2-2)$$

$$\frac{d}{dt} \left(\frac{\partial F}{\partial \dot{\gamma}_1} \right) - \frac{\partial F}{\partial \gamma_1} = 0 \quad (D.5.2-3)$$

$$\frac{d}{dt} \left(\frac{\partial F}{\partial \dot{\gamma}_2} \right) - \frac{\partial F}{\partial \gamma_2} = 0 \quad (D.5.2-4)$$

$$\frac{d}{dt} \left(\frac{\partial F}{\partial \dot{\rho}_1} \right) - \frac{\partial F}{\partial \rho_1} = 0 \quad (D.5.2-5)$$

$$\frac{d}{dt} \left(\frac{\partial F}{\partial \dot{\rho}_2} \right) - \frac{\partial F}{\partial \rho_2} = 0 \quad (D.5.2-6)$$

The Euler-Lagrange Equations are

$$\rho_1 \ddot{\beta}_1 + 0.5 \rho_2 \ddot{\beta}_2 - (4n^2 + 2)G\beta_1 - (2n^2 + 1)G\beta_2 + 3.31G\rho_2 + (2n^2 + 1)\dot{\gamma}_1 + (n^2 + 0.5)\dot{\gamma}_2 - 1.655\dot{\rho}_2 = 0 \quad (D.5.2-7)$$

$$\rho_2 \ddot{\beta}_1 + 0.75 \rho_2 \ddot{\beta}_2 - (4n^2 + 2)G\beta_1 - (5n^2 + 1.5)G\beta_2 - 3.31G\rho_2 + (2n^2 + 1)\dot{\gamma}_1 + (2.5n^2 + 0.75)\dot{\gamma}_2 + 1.655\dot{\rho}_2 = 0 \quad (D.5.2-8)$$

$$\rho_1 \ddot{\gamma}_1 + 0.5 \rho_2 \ddot{\gamma}_2 - (4n^2 + 2)G\gamma_1 - (2n^2 + 1)G\gamma_2 - 3.34G\rho_1 - 1.666G\rho_2 + (2n^2 + 1)\dot{\beta}_1 + (n^2 + 0.5)\dot{\beta}_2 + 1.67\dot{\rho}_1 + 0.834\dot{\rho}_2 = 0 \quad (D.5.2-9)$$

$$\rho_2 \ddot{\gamma}_1 + 0.75 \rho_2 \ddot{\gamma}_2 - (4n^2 + 2)G\gamma_1 - (5n^2 + 1.5)G\gamma_2 - 2.0G\rho_1 - 2.5G\rho_2 + (2n^2 + 1)\dot{\beta}_1 + (2.5n^2 + 0.75)\dot{\beta}_2 + 1.67\dot{\rho}_1 + 1.24\dot{\rho}_2 = 0 \quad (D.5.2-10)$$

$$\rho_1 \ddot{h}_1 + 0.5 \rho_2 \ddot{h}_2 - (1.86n^2 + 1.86)Gh_1 - (0.73n^2 + 0.73)Gh_2 - 3.68Gh_1 - 1.83Gh_2 - 3.88\rho_1 + (0.93n^2 + 0.93)\dot{\gamma}_1 + (0.46n^2 + 0.46)\dot{\gamma}_2 + 1.83\dot{\beta}_1 + 0.92\dot{\beta}_2 = 0 \quad (D.5.2-11)$$

$$\rho_2 \ddot{h}_1 + 0.75 \rho_2 \ddot{h}_2 - (1.86n^2 + 4.79)Gh_1 - (3.40n^2 + 58.61)Gh_2 + 7.26Gh_1 - 3.63Gh_2 - 3.50Gh_1 - 2.74Gh_2 + (0.93n^2 + 0.93)\dot{\gamma}_1 + (1.70n^2 + 2.931)\dot{\gamma}_2 - 3.63\dot{\beta}_1 + 1.82\dot{\beta}_2 + 1.83\dot{\gamma}_1 + 1.37\dot{\gamma}_2 - 3.81\rho_1 = 0 \quad (D.5.2-12)$$

D.5.3 Discussion

The general variational equation for the investigation of the visco-elastic behavior of shells of revolution subjected to uniform pressure and thermal gradients are developed. The development is based on the assumption that the material behaves as a Kelvin body with temperature dependent properties. The equation is applied to a shell of general geometry.

By making use of the geometrical relations for a hemispherical shell, the general equation is specialized for application to thin hemispherical shells. This is accomplished by neglecting all higher order terms. The respective geometrical relations for conical shells are presented for a similar application.

The boundary conditions applied in the investigation are based on a clamped shell. The Ritz method is applied by assuming that the displacement functions which satisfy the boundary conditions and their respective coefficients are dependent on time. Other boundary conditions can be incorporated by an appropriate choice of the displacement functions.

Integration of the variational equation (now a function of the displacements and their respective functions) over the limits governed by the shell geometry, results in an equation of unknown functions of t .

From the variational equation a system of Euler Lagrange equation is developed. In the specific problem we have a system of six ordinary differential equations of order two. The coefficients of the unknown functions are various functions of time. The equations are assumed to be continuous and that their values vanish at $t = t_0$.

A solution of this system of equations, which determines the unknown functions, must be made numerically by digital computer. The solution is based on the time interval $t_0 \leq t \leq t_1$.

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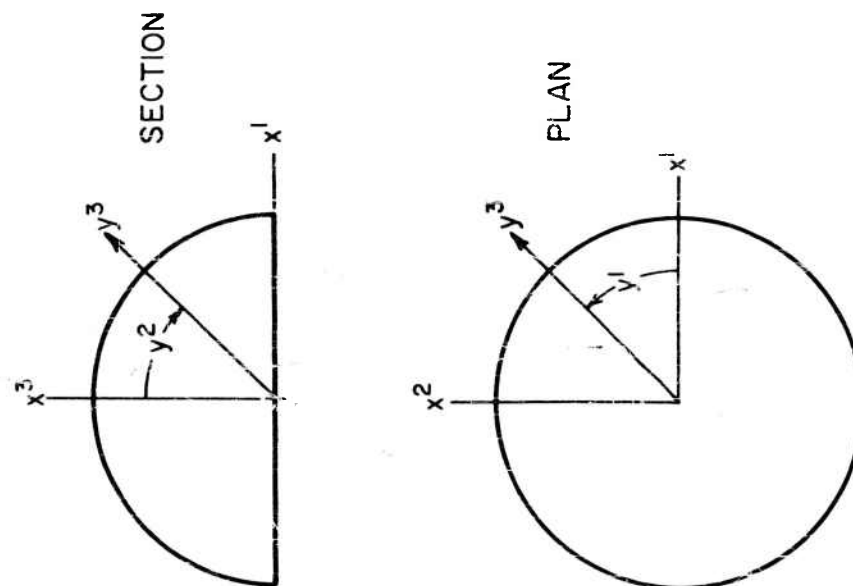


Fig. D-2.1

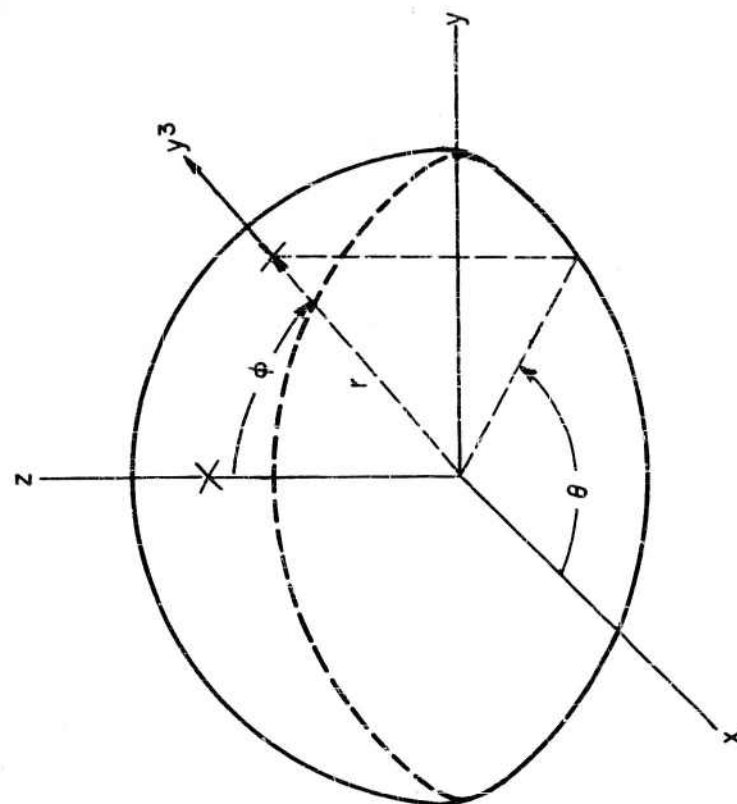


Fig. D-4.1

HEMISPHERICAL SHELL

University of Minnesota, Minneapolis, Minnesota. VISCOELASTIC BEHAVIOR OF SURFACES OF REVOLUTION UNDER COMBINED MECHANICAL AND THERMAL LOADS, by L. Albert Scipio II, See-foo Chien, James A. Moses, Mansa Singh. May 1961. 168 p. incl. illus. and tables. (Project 7063; Task 70524)(ARL TR 60-305) Unclassified Report This paper presents the results of an analytical and experimental study of thin conical and hemispherical shells subjected to combined mechanical and thermal loads. The analytical solutions are applicable for shells constructed from a linear viscoelastic material with temperature dependent properties. Experimental models were subjected to (over)	UNCLASSIFIED	University of Minnesota, Minneapolis, Minnesota. VISCOELASTIC BEHAVIOR OF SURFACES OF REVOLUTION UNDER COMBINED MECHANICAL AND THERMAL LOADS, by L. Albert Scipio II, See-foo Chien, James A. Moses, Mansa Singh. May 1961. 168 p. incl. illus. and tables. (Project 7063; Task 70524)(ARL TR 60-305) Unclassified Report This paper presents the results of an analytical and experimental study of thin conical and hemispherical shells subjected to combined mechanical and thermal loads. The analytical solutions are applicable for shells constructed from a linear viscoelastic material with temperature dependent properties. Experimental models were subjected to (over)	UNCLASSIFIED
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