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Tables of Lebedev, Mehler, and
Generalized Mehler Transforms

Fritz Oberhettinger

T.P. Higgins

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TABLES OF
LEBEDEV, MEHLER, AND GENERALIZED MEHLER TRANSFORMS

by

Professor Fritz Oberhettinger
Oregon State University

and

T. P. Higgins

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Introduction

Inversion formulas with kernels containing Bessel functions of purely imaginary order [1, vol. 2] and Legendre functions of complex index with the real part $-\frac{1}{2}$ (conical functions) [1, vol. 1] as variable have become prominent in recent times as methods in solving certain boundary value problems of the wave or heat conduction equation involving wedge or conically shaped boundaries. [3], [4], [6], [10], [11], [13].

These inversion formulas are:

A. Lebedev transform [7]

$$\begin{aligned} g(y) &= \int_0^{\infty} f(x) K_{ix}(y) dx, \\ (1) \quad f(x) &= 2\pi^{-2} x \sinh(\pi x) \int_0^{\infty} y^{-1} K_{ix}(y) g(y) dy. \end{aligned}$$

$K_{ix}(y)$ is the modified Hankel function $[K_{ix}(y) = \int_0^{\infty} \exp(-y \cosh t) \cos(xt) dt]$.

B. Mehler transform [5], [8], [9]

$$\begin{aligned} g(y) &= \int_0^{\infty} f(x) P_{ix-\frac{1}{2}}(y) dx, \\ (2) \quad f(x) &= x \tanh(\pi x) \int_1^{\infty} P_{ix-\frac{1}{2}}(y) g(y) dy. \end{aligned}$$

C. Generalized Mehler transform [12]

$$(3) \quad g(y) = \int_0^{\infty} f(x) P_{ix-\frac{1}{2}}^k(y) dx,$$

$$f(x) = \pi^{-1} x \sinh(\pi x) \Gamma(\tfrac{1}{2} - k + ix) \Gamma(\tfrac{1}{2} - k - ix) \int_1^{\infty} g(y) P_{ix-\frac{1}{2}}^k(y) dy.$$

For the condition of validity of the above formulas see the quoted literature. From formula 26 [1, vol. 1, p. 129] (see also the list of notations at the end of this report),

$$(4) \quad P_{ix-\frac{1}{2}}^k(\cosh a) = (2\pi \sinh a)^{-\frac{1}{2}} \left\{ \exp(-iax) \frac{\Gamma(-ix)}{\Gamma(\frac{1}{2} - k - ix)} {}_2F_1\left[\frac{1}{2} + k, \frac{1}{2} - k; 1 + ix; -\frac{1}{2} \exp(-a) \operatorname{csch} a\right] + \exp(iax) \frac{\Gamma(ix)}{\Gamma(\frac{1}{2} - k + ix)} \cdot \right. \\ \left. \cdot {}_2F_1\left[\frac{1}{2} + k, \frac{1}{2} - k; 1 - ix; -\frac{1}{2} \exp(-a) \operatorname{csch} a\right] \right\}.$$

This function obviously is an even function of x and is real for real parameters and real x .

The special case $k = 0$ in (3) yields the case (2). Furthermore from (4)

$$(5) \quad \begin{aligned} P_{ix-\frac{1}{2}}^{\frac{1}{2}}(\cosh a) &= (2/\pi)^{\frac{1}{2}} (\sinh a)^{-\frac{1}{2}} \cos(ax), \\ P_{ix-\frac{1}{2}}^{-\frac{1}{2}}(\cosh a) &= (2/\pi)^{\frac{1}{2}} (\sinh a)^{-\frac{1}{2}} \sin(ax). \end{aligned}$$

Therefore from (3) for $k = \frac{1}{2}$ putting $y = \cosh a$,

$$(6) \quad \begin{aligned} (\sinh a)^{\frac{1}{2}} g(\cosh a) &= (2/\pi)^{\frac{1}{2}} \int_0^{\infty} f(x) \cos(xa) dx, \\ f(x) &= (2/\pi)^{\frac{1}{2}} \int_0^{\infty} g(\cosh a) (\sinh a)^{\frac{1}{2}} \cos(xa) da. \end{aligned}$$

For $k = -\frac{1}{2}$

$$(7) \quad (\sinh a)^{\frac{1}{2}} g(\cosh a) = (2/\pi)^{\frac{1}{2}} \int_0^{\infty} x^{-1} f(x) \sin(xa) dx,$$

$$x^{-1} f(x) = (2/\pi)^{\frac{1}{2}} \int_0^{\infty} g(\cosh a) (\sinh a)^{\frac{1}{2}} \sin(xa) da.$$

But these are the Fourier cosine and the Fourier sine transformation formulas which are therefore a special case of (3).

The behavior of the kernel functions in (1), (2), (3) for large positive values of x and fixed argument y is of great importance. One has [1, vol. 2, p. 88],

$$(8) \quad K_{ix}(y) \sim (2\pi/x)^{\frac{1}{2}} \exp(-\frac{\pi}{2}x) \sin[x \log(2x/y) - x + \frac{\pi}{4}]$$

for large positive x and fixed y . Furthermore, from (4),

$$(9) \quad P_{ix-\frac{1}{2}}^k(y) \sim (2\pi \sinh a)^{-\frac{1}{2}} x^{k-\frac{1}{2}} [\exp(-iax - i\frac{\pi}{2}k + i\frac{\pi}{4}) + \exp(iax + i\frac{\pi}{2}k - i\frac{\pi}{4})]$$

for large positive x and fixed $y = \cosh a$.

Of further importance are representations of the different types of waves in the form of an integral transform of the kind expressed in (1), (2), and (3). Such representations are:

Cylindrical wave

$$(10) \quad K_0[\beta(r^2 + r'^2 - 2rr' \cos \Phi)^{\frac{1}{2}}] = \frac{2}{\pi} \int_0^{\infty} K_{ix}(\beta r) K_{ix}(\beta r') \cosh[x(\pi - |\Phi|)] dx,$$

$$0 \leq \Phi \leq 2\pi.$$

Spherical wave

$$\begin{aligned}
 (11) \quad & (R^2 + R'^2 - 2RR' \cos \theta)^{-\frac{1}{2}} \exp[-\beta(R^2 + R'^2 - 2RR' \cos \theta)^{\frac{1}{2}}] \\
 &= \frac{2}{\pi} (RR')^{-\frac{1}{2}} \int_0^{\infty} x \tanh(\pi x) P_{ix-\frac{1}{2}}(-\cos \theta) K_{ix}(\beta R) K_{ix}(\beta R') dx, \\
 & \quad 0 \leq \theta \leq 2\pi
 \end{aligned}$$

Generalized spherical wave

$$\begin{aligned}
 (12) \quad & (R^2 + R'^2 - 2RR' \cos \theta)^{-\frac{1}{2}} K_{\alpha} [\beta(R^2 + R'^2 - 2RR' \cos \theta)^{\frac{1}{2}}] \\
 &= 2^{\frac{1}{2}} \pi^{-\frac{3}{2}} (RR')^{-\alpha} (\sin \theta)^{\frac{1}{2}-\alpha} \int_0^{\infty} x \sinh(\pi x) \Gamma(\alpha + ix) \Gamma(\alpha - ix) \cdot \\
 & \quad \cdot P_{ix-\frac{1}{2}}^{\frac{1}{2}-\alpha}(-\cos \theta) K_{ix}(\beta R) K_{ix}(\beta R') dx, \\
 & \quad \operatorname{Re} \alpha > -1, \quad 0 \leq \theta \leq 2\pi
 \end{aligned}$$

The following tables (A), (B), (C) represent a list of integral transforms of the type (1), (2), (3). Most of the results displayed here are new and have been taken from unpublished material of the authors.

Certain combinations of Bessel functions which occur on the r.h.s. of these tables can be replaced by other combinations such as:

$$\begin{aligned}
 (13) \quad & J_{\alpha}(x) \cos(\tfrac{1}{2} \pi \alpha) - Y_{\alpha}(x) \sin(\tfrac{1}{2} \pi \alpha) = \tfrac{1}{2} \sec(\tfrac{1}{2} \pi \alpha) [J_{\alpha}(x) + J_{-\alpha}(x)] \\
 & \quad = -\tfrac{1}{2} \csc(\tfrac{1}{2} \pi \alpha) [Y_{\alpha}(x) - Y_{-\alpha}(x)],
 \end{aligned}$$

$$\begin{aligned}
 (14) \quad & J_{\alpha}(x) \sin(\tfrac{1}{2} \pi \alpha) + Y_{\alpha}(x) \cos(\tfrac{1}{2} \pi \alpha) = \tfrac{1}{2} \csc(\tfrac{1}{2} \pi \alpha) [J_{\alpha}(x) - J_{-\alpha}(x)] \\
 & \quad = \tfrac{1}{2} \sec(\tfrac{1}{2} \pi \alpha) [Y_{\alpha}(x) + Y_{-\alpha}(x)],
 \end{aligned}$$

$$\begin{aligned}
 (15) \quad J_{\alpha}(x)Y_{-\alpha}(y) + J_{-\alpha}(y)Y_{\alpha}(x) &= \csc(\pi\alpha) [J_{\alpha}(x)J_{\alpha}(y) - J_{-\alpha}(x)J_{-\alpha}(y)] \\
 &= \csc(\pi\alpha) [Y_{-\alpha}(x)Y_{-\alpha}(y) - Y_{\alpha}(x)Y_{\alpha}(y)],
 \end{aligned}$$

$$\begin{aligned}
 (16) \quad J_{\alpha}(x)Y_{\alpha}(y) - Y_{\alpha}(x)J_{\alpha}(y) &= J_{-\alpha}(x)Y_{-\alpha}(x) - Y_{-\alpha}(x)J_{-\alpha}(y) \\
 &= \csc(\pi\alpha) [J_{\alpha}(y)J_{-\alpha}(x) - J_{\alpha}(x)J_{-\alpha}(y)],
 \end{aligned}$$

$$\begin{aligned}
 (17) \quad J_{\alpha}(x)Y_{-\alpha}(y) - J_{-\alpha}(x)Y_{\alpha}(y) &= \sin(\pi\alpha) [J_{\alpha}(x)J_{\alpha}(y) + Y_{\alpha}(x)Y_{\alpha}(y)] \\
 &= \sin(\pi\alpha) [J_{-\alpha}(x)J_{-\alpha}(y) + Y_{-\alpha}(x)Y_{-\alpha}(y)].
 \end{aligned}$$

Table A

Lebedev Transform

$$g(y) = \int_0^{\infty} f(x) K_{ix}(y) dx$$

$$f(x) = 2\pi^{-2} \sinh(\pi x) \int_0^{\infty} y^{-1} K_{ix}(y) g(y) dy$$

$f(x)$	$g(y) = \int_0^\infty f(x) K_{ix}(y) dx$
x^2	$\frac{1}{2}\pi y \exp(-y)$
x^{2n}	$\frac{1}{2}\pi (-1)^n \left[\frac{d^{2n}}{dz^{2n}} \exp(-y \cosh z) \right]_{z=0}$
$(a^2 + x^2)^{-1}$	$\frac{1}{2}\pi a^{-1} \int_0^\infty \exp(-y \cosh t - at) dt$
$(a^2 + x^2)^{-\frac{1}{2}}$	$\int_0^\infty \exp(-y \cosh t) K_0(at) dt$
$\exp(-ax)$	$a \int_0^\infty (a^2 + t^2)^{-1} \exp(-y \cosh t) dt$
$\cos(ax)$	$\frac{1}{2}\pi \exp(-y \cosh a)$
$x \sin(ax)$	$\frac{1}{2}\pi y \sinh a \exp(-y \cosh a)$
$\sin(ax) \sinh(bx)$	$\frac{1}{2}\pi \exp(-y \cos b \cosh a) \sin(y \sin b \sinh a)$
$\cos(ax) \cosh(bx)$	$\frac{1}{2}\pi \exp(-y \cos b \cosh a) \cos(y \sin b \sinh a)$
$\sinh(ax) \sinh(bx)$	$\frac{1}{2}\pi \exp(-y \cos a \cos b) \sinh(y \sin a \sin b)$ $a + b \leq \frac{1}{2}\pi$
$\cosh(ax) \cosh(bx)$	$\frac{1}{2}\pi \exp(-y \cos a \cos b) \cosh(y \sin a \sin b)$ $a + b \leq \frac{1}{2}\pi$

$f(x)$	$g(y) = \int_0^\infty f(x) K_{1x}(y) dx$
$\operatorname{sech}(\frac{1}{2}\pi x)$	$\frac{1}{2}\pi\{1 - y[K_0(y)L_{-1}(y) + L_0(y)K_1(y)]\}$
$\operatorname{sech}(\pi x)\cosh(ax)$	$\frac{1}{2}\pi \exp(y \cos a) \operatorname{Erfc}[(2y)^{\frac{1}{2}} \cos(\frac{1}{2}a)]$ $a \leq \frac{3\pi}{2}$
$\operatorname{sech}(\frac{1}{2}\pi x)\cosh(ax)$	$y \int_0^\infty (y^2 + t^2)^{-\frac{1}{2}} \exp(-t \cos a) K_1[(y^2 + t^2)^{\frac{1}{2}}] dt$ $a \leq \pi$
$\operatorname{csch}(\frac{1}{2}\pi x) \sinh(ax)$	$\sin a \int_0^\infty \exp(-t \cos a) K_0[(y^2 + t^2)^{\frac{1}{2}}] dt$ $a \leq \pi$
$\operatorname{csch}(\pi x) \sinh(ax)$	$\frac{1}{2} \sin a \int_0^\infty \exp(-t \cos a) K_0(y + t) dt$ $a \leq \frac{3\pi}{2}$
$\tanh(\pi x) \sinh(ax)$	$\frac{1}{2}\pi \exp(-y \cos a) \operatorname{Erf}[(2y)^{\frac{1}{2}}\sin(\frac{1}{2}a)]$ $a \leq \frac{1}{2}\pi$
$\operatorname{sech}(\pi x) \sinh(ax) \sinh(bx)$	$\frac{1}{4}\pi\{\exp[y \cos(a+b)]\operatorname{Erfc}[(2y)^{\frac{1}{2}}\cos(\frac{1}{2}a + \frac{1}{2}b)]$ $- \exp[y \cos(a-b)]\operatorname{Erfc}[(2y)^{\frac{1}{2}}\cos(\frac{1}{2}a - \frac{1}{2}b)]\}$ $a + b \leq \frac{3\pi}{2}$
$\operatorname{sech}(\pi x)\cosh(ax)\cosh(bx)$	$\frac{1}{4}\pi\{\exp[y \cos(a+b)]\operatorname{Erfc}[(2y)^{\frac{1}{2}}\cos(\frac{1}{2}a + \frac{1}{2}b)]$ $+ \exp[y \cos(a-b)]\operatorname{Erfc}[(2y)^{\frac{1}{2}}\cos(\frac{1}{2}a - \frac{1}{2}b)]\}$ $a + b \geq \frac{3\pi}{2}$
$x \tanh(\frac{1}{2}\pi x)$	$yK_0(y)$

$f(x)$	$g(y) = \int_0^\infty f(x) K_{1x}(y) dx$
$\operatorname{csch}(bx) \sinh(\pi x) \cosh(ax)$	$\frac{1}{2} \pi^2 b^{-1} \sum_{n=0}^{\infty} (-1)^n \epsilon_n I_{\frac{n}{b}}(y) \cos(n\pi a/b)$ $b - a \geq \frac{1}{2} \pi$
$\tanh(\pi x) \sinh(bx) \operatorname{csch}(ax)$	$\frac{1}{4} \pi \{ \exp[-y \cos(b+a)] \operatorname{Erf}[(2y)^{\frac{1}{2}} \sin(\frac{1}{2}b + \frac{1}{2}a)]$ $+ \exp[-y \cos(b-a)] \operatorname{Erf}[(2y)^{\frac{1}{2}} \sin(\frac{1}{2}b - \frac{1}{2}a)] \}$ $a + b \leq \frac{1}{2} \pi$
$x \sinh(\pi x) \Gamma(k + \frac{1}{2}ix) \Gamma(k - \frac{1}{2}ix)$	$2^{1-2k} \pi^2 y^{2k}$ $0 \leq \operatorname{Re} k \leq \frac{1}{4}$
$x \sinh(\pi x) \Gamma(k - \frac{1}{2}ix) \Gamma(k + \frac{1}{2}ix) \cdot$ $\cdot \Gamma(\frac{1}{2} - k - \frac{1}{2}ix) \Gamma(\frac{1}{2} - k + \frac{1}{2}ix)$	$2^{\frac{3}{2}} \pi^{\frac{5}{2}} y^{\frac{1}{2}} K_{\frac{1}{2}-2k}(y)$
$x \tanh(\frac{1}{2}\pi x) P_{\frac{1}{2}ix-\frac{1}{2}}(z)$	$y K_0[y(\frac{1}{2} + \frac{1}{2}z)^{\frac{1}{2}}]$
$x \tanh(\pi x) P_{ix-\frac{1}{2}}(z)$	$(\frac{1}{2}\pi y)^{\frac{1}{2}} \exp(-zy)$
$x \operatorname{sech}(\pi x) \tanh(\pi x) P_{ix-\frac{1}{2}}(z)$	$-(2\pi)^{-\frac{1}{2}} y^{\frac{1}{2}} \exp(zy) \operatorname{Ei}(-zy - y)$
$x \sinh(\pi x) \operatorname{sech}(2\pi x) P_{i2x-\frac{1}{2}}(z)$	$2^{-\frac{7}{4}} y^{\frac{1}{4}} \exp(\frac{1}{2}z^2 y - \frac{1}{2}y) D_{-\frac{3}{2}}[z(2y)^{\frac{1}{2}}]$
$x \sinh(\frac{1}{2}\pi x) [P_{ix-\frac{1}{2}}(z)]^2$	$\frac{1}{2} \pi y \{ J_0[\frac{1}{2}y(z^2 - 1)^{\frac{1}{2}}] \}^2$
$x \sinh(\frac{1}{2}\pi x) P_{\frac{1}{2}ix-\frac{1}{2}}^k(z)$	$\pi 2^{-\frac{5}{2}k-1} (1+z)^{\frac{1}{2}k} y^{1+k} J_{-k}[y(\frac{1}{2}z - \frac{1}{2})^{\frac{1}{2}}]$ $\operatorname{Re} k \leq 0$

$f(x)$	$g(y) = \int_0^\infty f(x) K_{1x}(y) dx$
$x \sinh(\frac{1}{2}\pi x) \Gamma(\frac{1}{2}-k+\frac{1}{2}ix) \Gamma(\frac{1}{2}-k-\frac{1}{2}ix) \cdot$ $\cdot P_{\frac{1}{2}ix-\frac{1}{2}}^k(z)$	$\pi 2^{\frac{3}{2}k} (z-1)^{-\frac{1}{2}k} y^{1-k} K_k[y(\frac{1}{2} + \frac{1}{2}z)^{\frac{1}{2}}]$ $\text{Re } k \leq \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) P_{ix-\frac{1}{2}}^k(z)$	$z^{-\frac{1}{2}} \pi^{\frac{3}{2}} (z^2 - 1)^{-\frac{1}{2}k} y^{\frac{1}{2}-k} \exp(-zy)$ $\text{Re } k \geq \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+2ix) \Gamma(\frac{1}{2}-k-2ix) \cdot$ $\cdot P_{12x-\frac{1}{2}}^k(z)$	$\pi 2^{\frac{3}{2}k} y^{\frac{1}{2}-k} \Gamma(\frac{3}{2}-k) (z^2-1)^{-\frac{1}{2}k} \exp(z^2 y - y) \cdot$ $\cdot D_{k-\frac{3}{2}}(2zy^{\frac{1}{2}}), \quad \text{Re } k \leq \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+\frac{1}{2}ix) \Gamma(\frac{1}{2}-k-\frac{1}{2}ix) \cdot$ $\cdot P_{\frac{1}{2}ix-\frac{1}{2}}^k(z)$	$2^{-\frac{1}{2}k} \pi^2 (1+z)^{-\frac{1}{2}k} y^{1-k} J_{-k}[y(\frac{1}{2}z - \frac{1}{2})^{\frac{1}{2}}]$ $0 \leq \text{Re } k \leq \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{4}-\frac{1}{2}k+\frac{1}{2}ix) \Gamma(\frac{1}{4}-\frac{1}{2}k-\frac{1}{2}ix) \cdot$ $\cdot P_{ix-\frac{1}{2}}^k(z)$	$2^{\frac{1}{2}+k} \pi^2 y^{\frac{1}{2}} J_{-k}[y(z^2 - 1)^{\frac{1}{2}}]$ $\text{Re } k \leq \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{3}{4}-\frac{1}{2}k+\frac{1}{2}ix) \Gamma(\frac{3}{4}-\frac{1}{2}k-\frac{1}{2}ix) \cdot$ $\cdot P_{ix-\frac{1}{2}}^k(z)$	$2^{k-\frac{1}{2}} \pi^2 y^{\frac{3}{2}} J_{-k}[y(z^2 - 1)^{\frac{1}{2}}]$ $\text{Re } k \leq \frac{3}{2}$
$x \sinh(\frac{1}{2}\pi x) P_{\frac{1}{2}ix-\frac{1}{2}}^k(z) P_{\frac{1}{2}ix-\frac{1}{2}}^{-k}(z)$	$\frac{1}{2} \pi y J_k[\frac{1}{2}y(z^2 - 1)^{\frac{1}{2}}] J_{-k}[\frac{1}{2}y(z^2 - 1)^{\frac{1}{2}}]$
$x \sinh(\pi x) \Gamma(k+\frac{1}{2}+\frac{1}{2}ix) \Gamma(k+\frac{1}{2}-\frac{1}{2}ix) \cdot$ $\cdot [P_{\frac{1}{2}ix-\frac{1}{2}}^{-k}(z)]^2$	$\pi^2 y \{J_k[\frac{1}{2}y(z^2 - 1)^{\frac{1}{2}}]\}^2$ $\text{Re } k \geq -\frac{1}{2}$

$f(x)$	$g(y) = \int_0^\infty f(x) K_{ix}(y) dx$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot [P_{ix-\frac{1}{2}}^k(z)]^2$	$\pi(\frac{1}{2}\pi/y)^{\frac{1}{2}} \exp(-z^2 y) I_{-k}[(z^2 - 1)y]$ $\text{Re } k \leq \frac{1}{2}$
$x \sinh(\frac{1}{2}\pi x) [Q_{\frac{1}{2}ix-\frac{1}{2}}(z) + Q_{-\frac{1}{2}ix-\frac{1}{2}}(z)]$	$-\frac{1}{2}\pi^2 y Y_0[y(\frac{1}{2}z - \frac{1}{2})^{\frac{1}{2}}]$
$J_{ix}(a) + J_{-ix}(a)$	$\pi J_0[(a^2 - y^2)^{\frac{1}{2}}] \quad , y < a$ $0 \quad , y > a$
$i \sin(ax) [J_{ix}(b) - J_{-ix}(b)]$	$\frac{1}{2}\pi J_0(z_1) - \frac{1}{2}\pi J_0(z_2) \quad , 2by \sinh a < b^2 - y^2$ $\frac{1}{2}\pi J_0(z_1) \quad , 2by \sinh a > b^2 - y^2$ $y < b$ $0 \quad , 2by \sinh a < y^2 - b^2$ $\frac{1}{2}\pi J_0(z_1) \quad , 2by \sinh a > y^2 - b^2$ $y > b$ $z_{1/2} = (b^2 - y^2 \pm 2by \sinh a)^{\frac{1}{2}}$
$\cos(ax) [J_{ix}(b) + J_{-ix}(b)]$	$\frac{1}{2}\pi J_0(z_1) + \frac{1}{2}\pi J_0(z_2) \quad , 2by \sinh a < b^2 - y^2$ $\frac{1}{2}\pi J_0(z_1) \quad , 2by \sinh a > b^2 - y^2$ $y < b$ $0 \quad , 2by \sinh a < y^2 - b^2$ $\frac{1}{2}\pi J_0(z_1) \quad , 2by \sinh a > y^2 - b^2$ $y > b$ $z_{1/2} = (b^2 - y^2 \pm 2by \sinh a)^{\frac{1}{2}}$

$f(x)$	$g(y) = \int_0^{\infty} f(x) K_{ix}(y) dx$
$x \left\{ \begin{array}{l} \sinh(\frac{1}{2}\pi x) [J_{ix}(a) + J_{-ix}(a)] \\ i \cosh(\frac{1}{2}\pi x) [Y_{ix}(a) - Y_{-ix}(a)] \end{array} \right\} K_{ix}(a)$	$\frac{1}{2}\pi \sin(\frac{1}{2}a^2/y)$
$x \left\{ \begin{array}{l} \sinh(\frac{1}{2}\pi x) [Y_{ix}(a) + Y_{-ix}(a)] \\ -i \cosh(\frac{1}{2}\pi x) [J_{ix}(a) - J_{-ix}(a)] \end{array} \right\} K_{ix}(a)$	$-\frac{1}{2}\pi \cos(\frac{1}{2}a^2/y)$
$[J_{i\frac{1}{2}x}(a)]^2 + [Y_{i\frac{1}{2}x}(a)]^2$	$2\pi^{-1} K_0 \{ \frac{1}{2} [y + (y^2 - 4a^2)^{\frac{1}{2}}] \} K_0 \{ \frac{1}{2} [y - (y^2 - 4a^2)^{\frac{1}{2}}] \}$
$-ix \operatorname{sech}(\pi x) [J_{ix}(a) Y_{-ix}(a) - J_{-ix}(a) \cdot Y_{ix}(a)]$	$\exp(-y + \frac{1}{2}a^2/y) \operatorname{Erfc}[a(2y)^{-\frac{1}{2}}]$
$x \sinh(\frac{1}{2}\pi x) \{ [J_{i\frac{1}{2}x}(a)]^2 + [Y_{i\frac{1}{2}x}(a)]^2 \}$	$2y(4a^2 + y^2)^{-\frac{1}{2}}$
$x \sinh(\frac{1}{2}\pi x) [J_{i\frac{1}{2}x}(a) J_{i\frac{1}{2}x}(b) + Y_{i\frac{1}{2}x}(a) Y_{i\frac{1}{2}x}(b)]$	$2y(y^2 + 4ab)^{-\frac{1}{2}} \cos\{ \frac{1}{2} [(a/b)^{\frac{1}{2}} - (b/a)^{\frac{1}{2}}] (y^2 + 4ab)^{\frac{1}{2}} \}$
$x \sinh(\frac{1}{2}\pi x) [J_{i\frac{1}{2}x}(a) Y_{i\frac{1}{2}x}(b) - J_{i\frac{1}{2}x}(b) Y_{i\frac{1}{2}x}(a)]$	$-2y(4ab + y^2)^{-\frac{1}{2}} \sin\{ \frac{1}{2} [(a/b)^{\frac{1}{2}} - (b/a)^{\frac{1}{2}}] (4ab + y^2)^{\frac{1}{2}} \}$
$x \sinh(\frac{1}{2}\pi x) [J_{i\frac{1}{2}x}(a) Y_{-i\frac{1}{2}x}(b) + J_{-i\frac{1}{2}x}(b) Y_{i\frac{1}{2}x}(a)]$	$-2y(4ab - y^2)^{-\frac{1}{2}} \cos\{ \frac{1}{2} [(a/b)^{\frac{1}{2}} + (b/a)^{\frac{1}{2}}] (4ab - y^2)^{\frac{1}{2}} \}$ $, y < 2(ab)^{\frac{1}{2}}$ 0 $, y > 2(ab)^{\frac{1}{2}}$
$x [J_{k+i\frac{1}{2}x}(a) Y_{k-i\frac{1}{2}x}(a) - Y_{k+i\frac{1}{2}x}(a) \cdot J_{k-i\frac{1}{2}x}(a)]$	$i 2^{2k+1} a^{2k} y (4a^2 + y^2)^{-\frac{1}{2}} [y + (y^2 + 4a^2)^{\frac{1}{2}}]^{-2k}$

$f(x)$	$g(y) = \int_0^\infty f(x) K_{ix}(y) dx$
$\cosh(ax) K_{ix}(b)$	$\frac{1}{2} \pi K_0[(b^2 + y^2 + 2by \cos a)^{\frac{1}{2}}], \quad a \leq \pi$
$x \sinh(\pi x) K_{ix}(a)$	$\frac{1}{8} a \pi (2y/\pi)^{-\frac{1}{2}} \exp[-y - a^2/(8y)]$
$x \tanh(\pi x) K_{ix}(a)$	$\frac{1}{2} \pi (ay)^{\frac{1}{2}} (a + y)^{-1} \exp(-a - y)$
$x \sinh(bx) K_{ix}(a)$	$\frac{1}{2} \pi a y \sin b(a^2 + y^2 + 2ay \cos b)^{-\frac{1}{2}} \cdot$ $\cdot K_1[(a^2 + b^2 + 2ay \cos b)^{\frac{1}{2}}], \quad b \leq \pi$
$x(c^2 + x^2)^{-1} \sinh(\pi x) K_{ix}(a)$	$\frac{1}{2} \pi^2 I_c(y) K_c(a), \quad y < a$ $\frac{1}{2} \pi^2 I_c(a) K_c(y), \quad y > a$
$x \sinh(\pi x) \Gamma(c + i\frac{1}{2}x) \Gamma(c - i\frac{1}{2}x) K_{ix}(a)$	$2^{1-2c} \pi^2 (ay/z)^{2c} K_{2c}(z)$ $\text{Re } c \geq 0$ $z = (y^2 + a^2)^{\frac{1}{2}}$
$x \sinh(2\pi x) \Gamma(c + ix) \Gamma(c - ix) K_{ix}(a)$	$2^c \pi^{\frac{5}{2}} [\Gamma(\frac{1}{2} - c)]^{-1} (a^{-1} - y^{-1}) K_c(y - a)$ $0 \leq \text{Re } c \leq \frac{1}{2}$
$x \sinh(\pi x) \Gamma(c + ix) \Gamma(c - ix) K_{ix}(a)$	$2^{c-1} \pi^{\frac{3}{2}} (a^{-1} + y^{-1})^{-c} \Gamma(\frac{1}{2} + c) K_c(y + a)$ $\text{Re } c \geq 0$
$[\Gamma(\frac{3}{4} + i\frac{1}{2}x) \Gamma(\frac{3}{4} - i\frac{1}{2}x)]^{-1} x \tanh(\pi x) K_{ix}(a)$	$\frac{1}{2} (\pi ay/z)^{\frac{1}{2}} \exp(-z)$ $z = (a^2 + y^2)^{\frac{1}{2}}$

$f(x)$	$g(y) = \int_0^\infty f(x) K_{ix}(y) dx$
$x \sinh(\pi x) P_{-\frac{1}{2}+ix}^k(z) K_{ix}(a)$	$2^{-k-2} \pi^{\frac{3}{2}} (z^2-1)^{\frac{1}{2}k+\frac{1}{2}} (z-\tau)^{-\frac{1}{2}k-\frac{1}{2}} J_{-k-\frac{1}{2}}[c(z-\tau)^{\frac{1}{2}}]$ $, z > \tau$ 0 $, z < \tau$ $c = (2ay)^{\frac{1}{2}}, \tau = \frac{1}{2}(a/y + y/a)$
$x \tanh(\pi x) P_{-\frac{1}{2}+ix}(z) K_{ix}(a)$	$\frac{1}{2} \pi (ay)^{\frac{1}{2}} (a^2+y^2+2azy)^{-\frac{1}{2}} \exp[-(a^2+y^2+2azy)^{\frac{1}{2}}]$
$x \sinh(\pi x) \Gamma(c+ix) \Gamma(c-ix) P_{-\frac{1}{2}+ix}^{\frac{1}{2}-c}(z) \cdot K_{ix}(a)$	$2^{-\frac{1}{2}} \pi^{\frac{3}{2}} (ay/b)^c (z^2-1)^{\frac{1}{2}c-\frac{1}{4}} K_c(b)$ $b = (y^2+a^2+2ayz)^{\frac{1}{2}}$
$x [I_{-\frac{1}{2}ix}(a) I_{-\frac{1}{2}ix}(b) - I_{\frac{1}{2}ix}(a) I_{\frac{1}{2}ix}(b)]$	$2iy(4ab-y^2)^{-\frac{1}{2}} \cosh\{\frac{1}{2}[(a/b)^{\frac{1}{2}} - (b/a)^{\frac{1}{2}}](4ab-y^2)^{\frac{1}{2}}\}$ $, y < 2(ab)^{\frac{1}{2}}$ 0 $, y > 2(ab)^{\frac{1}{2}}$
$x \sinh(\frac{1}{2}\pi x) [I_{\frac{1}{2}ix}(a) + I_{-\frac{1}{2}ix}(a)] K_{\frac{1}{2}ix}(b)$	$\pi y(4ab-y^2)^{-\frac{1}{2}} \exp\{\frac{1}{2}[(a/b)^{\frac{1}{2}} - (b/a)^{\frac{1}{2}}](4ab-y^2)^{\frac{1}{2}}\}$ $, y < 2(ab)^{\frac{1}{2}}$ $\pi y(y^2-4ab)^{-\frac{1}{2}} \sin\{\frac{1}{2}[(a/b)^{\frac{1}{2}} - (b/a)^{\frac{1}{2}}](y^2-4ab)^{\frac{1}{2}}\}$ $, y > 2(ab)^{\frac{1}{2}}$
$x \tanh(\pi x) [I_{ix}(a) + I_{-ix}(a)] K_{ix}(a)$	$-\frac{1}{2} i \pi \exp(-y - \frac{1}{2} a^2/y) \operatorname{Erf}[ia(2y)^{-\frac{1}{2}}]$
$I_{k-\frac{1}{2}ix}(a) K_{k+\frac{1}{2}ix}(a) + I_{k+\frac{1}{2}ix}(a) K_{k-\frac{1}{2}ix}(a)$	$\pi I_k\{\frac{1}{2}[(4a^2+y^2)^{\frac{1}{2}}-y]\} K_k\{\frac{1}{2}[(4a^2+y^2)^{\frac{1}{2}}+y]\}$

$f(x)$	$g(y) = \int_0^{\infty} f(x) K_{1x}(y) dx$
$\sinh(\pi x) [K_{\frac{1}{2}ix+\frac{1}{2}}(a)K_{\frac{1}{2}ix+\frac{1}{2}}(b) - K_{\frac{1}{2}ix-\frac{1}{2}}(a)K_{\frac{1}{2}ix-\frac{1}{2}}(b)]$	$0, y < 2(ab)^{\frac{1}{2}}$ $2i\pi^2(y^2-4ab)^{-\frac{1}{2}}\cos\{\frac{1}{2}[(b/a)^{\frac{1}{2}}-(a/b)^{\frac{1}{2}}](y^2-4ab)^{\frac{1}{2}}\}, y > 2(ab)^{\frac{1}{2}}$
$x \sinh(\pi x) [K_{\frac{1}{2}ix}(a)]^2$	$0, y < 2a$ $\pi^2 y(y^2-4a^2)^{-\frac{1}{2}}, y > 2a$
$x \sinh(\frac{1}{2}\pi x) K_{\frac{1}{2}ix}(a) K_{\frac{1}{2}ix}(b)$	$\frac{1}{2}\pi^2 y z^{-1} \exp[-\frac{1}{2}(ab)^{-\frac{1}{2}}(az+bz)]$ $z = (y^2 + 4ab)^{\frac{1}{2}}$
$x \sinh(\pi x) K_{\frac{1}{2}ix}(a) K_{\frac{1}{2}ix}(b)$	$0, y < 2(ab)^{\frac{1}{2}}$ $\pi^2 y(y^2-4ab)^{-\frac{1}{2}}\cos\{\frac{1}{2}[(a/b)^{\frac{1}{2}}-(b/a)^{\frac{1}{2}}](y^2-4ab)^{\frac{1}{2}}\}, y > 2(ab)^{\frac{1}{2}}$
$x \sinh(\pi x) K_{ix}(a) K_{ix}(b)$	$\frac{1}{4}\pi^2 \exp[-\frac{1}{2}y(\frac{a}{b} + \frac{b}{a} + \frac{ab}{y})]$
$x \tanh(\pi x) K_{ix}(a) K_{ix}(b)$	$\frac{1}{4}\pi^2 \exp[\frac{1}{2}(\frac{ay}{b} + \frac{by}{a} + \frac{ab}{y})] \cdot$ $\cdot \operatorname{Erfc}[2^{-\frac{1}{2}}(\frac{ay}{b} + \frac{by}{a} + \frac{ab}{y})]$
$x \sinh(\pi x) K_{\frac{1}{2}ix+c}(a) K_{\frac{1}{2}ix-c}(a)$	$0, y < 2a$ $2^{-2c-1} a^{-2c} \pi^2 z^{-1} y[(y+c)^{2c} + (y-z)^{2c}]$ $z = (y^2-4a^2)^{\frac{1}{2}}, y > 2a$

$f(x)$	$g(y) = \int_0^{\infty} f(x) K_{ix}(y) dx$
$x \sinh(\pi x) K_{\frac{1}{2}ix+ic}(a) K_{\frac{1}{2}ix-ic}(a)$	$0, y < 2a$ $\pi^2 y(y^2 - 4a^2)^{-\frac{1}{2}} \cos\{2c \log [\frac{1}{2}ya^{-1} + (\frac{1}{4}y^2a^{-2} - 1)^{\frac{1}{2}}]\}$ $, y > 2a$
$x \sinh(\frac{1}{2}\pi x) S_{0,ix}(a)$	$\frac{1}{2}\pi ay(a^2 + y^2)^{-1}$
$x \tanh(\pi x) S_{0,2ix}(a)$	$-\frac{1}{8}(2y/\pi)^{-\frac{1}{2}} a \exp[-y+a^2/(8y)] Ei[-a^2/(8y)]$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-\frac{1}{2}k-\frac{1}{2}ix) \Gamma(\frac{1}{2}-\frac{1}{2}k+\frac{1}{2}ix) \cdot$ $\cdot S_{k,ix}(a)$	$2^k a^{k+1} \pi y^{1-k} (a^2 + y^2)^{-1}$ $Re\ k \leq 1$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot S_{2k,2ix}(a)$	$\pi(2y/\pi)^{-\frac{1}{2}} 2^{2k-3} a \Gamma(1-k) \exp[-y+a^2/(8y)] \cdot$ $\cdot \Gamma[k, a^2/(8y)]$ $Re\ k \leq \frac{1}{2}$
$x \tanh(\pi x) [D_{-\frac{1}{2}+ix}(a) D_{-\frac{1}{2}-ix}(-a) +$ $+ D_{-\frac{1}{2}+ix}(-a) D_{-\frac{1}{2}-ix}(a)]$	$\pi y^{\frac{1}{2}} \cos[a(2y)^{\frac{1}{2}}]$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+i\frac{1}{2}x) \Gamma(\frac{1}{2}-k-i\frac{1}{2}x) \cdot$ $\cdot W_{k,\frac{1}{2}ix}(a)$	$(4a)^k \pi^2 y^{1-2k} \exp[-\frac{1}{2}a-y^2/(4a)]$ $Re\ k \leq \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) W_{k,ix}(2a)$	$\pi(\frac{1}{2}\pi)^{\frac{1}{2}} a \Gamma(1-k) y^{\frac{1}{2}-k} (a+y)^{k-1} \exp(-a-y)$ $Re\ k \leq \frac{1}{2}$

Table B

Mehler Transform

$$g(y) = \int_0^{\infty} f(x) P_{ix-\frac{1}{2}}(y) dx$$

$$f(x) = x \tanh(\pi x) \int_1^{\infty} P_{ix-\frac{1}{2}}(y) g(y) dy$$

As noted in the introduction, a Mehler transform pair can be obtained from any generalized Mehler transform by setting $k = 0$. In general, the transform pairs that can be so obtained have not been included in Table B.

$f(x)$	$g(y) = \int_0^\infty f(x) P_{ix-\frac{1}{2}}(y) dx$
$x^{-1} \tanh(\pi x)$	$2[y + (y^2 - 1)^{\frac{1}{2}}]^{-\frac{1}{2}} K\{[y + (y^2 - 1)^{\frac{1}{2}}]^{-1}\}$
$x \tanh(\pi x) (a^2 + x^2)^{-1}$	$Q_{a-\frac{1}{2}}(y)$
$\tanh(\pi x) \operatorname{sech}(\pi x) \sinh(ax)$	$2^{\frac{1}{2}} \pi^{-1} (y + \cosh a)^{-\frac{1}{2}} \arctan[(1 - \cosh a)^{\frac{1}{2}} (y + \cosh a)^{-\frac{1}{2}}]$
$\sin(ax) \tanh(\pi x)$	$(2 \cosh a - 2y)^{-\frac{1}{2}}, y < \cosh a$ $0, y > \cosh a$
$(\operatorname{sech} \pi x)^2 \cos(ax)$	$\pi^{-1} (\frac{1}{2} y - \frac{1}{2} \cosh a)^{-\frac{1}{2}} \arctan\left[\frac{(y - \cosh a)^{\frac{1}{2}}}{1 + \cosh a}\right], y > \cosh a$ $2^{-\frac{1}{2}} \pi^{-1} (\frac{1}{2} \cosh a - \frac{1}{2} y)^{-\frac{1}{2}} \cdot \log \left[\frac{(\cosh a + 1)^{\frac{1}{2}} + (\cosh a - y)^{\frac{1}{2}}}{(\cosh a + 1)^{\frac{1}{2}} - (\cosh a - y)^{\frac{1}{2}}} \right], y < \cosh a$
$\cosh(ax) [\operatorname{sech}(\pi x)]^2$	$2^{-\frac{1}{2}} (y - \cosh a)^{-\frac{1}{2}} - 2^{\frac{1}{2}} \pi^{-1} (y - \cosh a)^{\frac{1}{2}} \cdot \arctan[(1 + \cosh a)^{\frac{1}{2}} (y - \cosh a)^{-\frac{1}{2}}]$
$x \sinh(\pi x) \Gamma(\alpha - \frac{1}{2}x) \Gamma(\alpha + \frac{1}{2}x) \cdot \Gamma(\frac{1}{2} - \alpha - \frac{1}{2}x) \Gamma(\frac{1}{2} - \alpha + \frac{1}{2}x)$	$2\pi^2 (y^2 - 1)^{-\frac{1}{2}} \{ [y + (y^2 - 1)^{\frac{1}{2}}]^{\frac{1}{2} - 2\alpha} + [y - (y^2 - 1)^{\frac{1}{2}}]^{\frac{1}{2} - 2\alpha} \}$ $0 \leq \operatorname{Re} \alpha \leq \frac{1}{2}$

$f(x)$	$g(y) = \int_0^{\infty} f(x) P_{ix-\frac{1}{2}}(y) dx$
$x \tanh(\pi x) \Gamma(\alpha - \frac{1}{2}x) \Gamma(\alpha + \frac{1}{2}x) \cdot$ $\cdot \Gamma(\frac{1}{2} - \alpha + \frac{1}{2}x) \Gamma(\frac{1}{2} - \alpha - \frac{1}{2}x)$	$2\pi^2 \sec(2\pi\alpha) (y^2 - 1)^{-\frac{1}{2}} \cdot$ $\cdot \{ [y + (y^2 - 1)^{\frac{1}{2}}]^{\frac{1}{2} - 2\alpha} - [y - (y^2 - 1)^{\frac{1}{2}}]^{\frac{1}{2} - 2\alpha} \}$ $0 \leq \operatorname{Re} \alpha \leq \frac{1}{2}$
$[\psi(\frac{1}{2} + ix) + \psi(\frac{1}{2} - ix)] \cos(ax)$	$-2^{-\frac{1}{2}} \pi (\cosh a - y)^{-\frac{1}{2}}, y < \cosh a$ $(\frac{1}{2}y - \frac{1}{2} \cosh a)^{-\frac{1}{2}} [-\gamma - \log 4 + \frac{1}{2} \log(y^2 - 1) - \log(y - \cosh a)], y > \cosh a$
$x \tanh(\pi x) \Gamma(\frac{1}{2} - \alpha + ix) \Gamma(\frac{1}{2} - \alpha - ix) P_{-\frac{1}{2} + ix}^{\alpha}(z)$	$(z^2 - 1)^{-\frac{1}{2}\alpha} \Gamma(1 - \alpha) (z + y)^{\alpha - 1}$ $\operatorname{Re} \alpha \leq \frac{1}{2}$
$x \tanh(\pi x) [\operatorname{sech}(\pi x)]^2 P_{ix-\frac{1}{2}}(z)$	$\pi^{-2} (y - z)^{-1} \log(\frac{y+1}{z+1})$
$x \sinh(\frac{1}{2}\pi x) \operatorname{sech}(\pi x) P_{\frac{1}{2}ix-\frac{1}{2}}(z)$	$2^{-\frac{1}{2}} \pi^{-1} c^{\frac{1}{2}} \{ 2E[(\frac{1}{2} - \frac{1}{2}yc)^{\frac{1}{2}}] - K[(\frac{1}{2} - \frac{1}{2}yc)^{\frac{1}{2}}] \}$ $c = (y^2 + \frac{1}{2}z - \frac{1}{2})^{-\frac{1}{2}}$
$x \sinh(\frac{1}{2}\pi x) \operatorname{sech}(\pi x) P_{\frac{1}{2}ix-\frac{1}{2}}^{\alpha}(z)$	$2^{-\frac{3}{2}-\frac{1}{2}\alpha} (1+z)^{\frac{1}{2}\alpha} (y^2 + \frac{1}{2}z - \frac{1}{2})^{-\frac{3}{2}-\frac{1}{2}\alpha} \cdot$ $\cdot P_{\frac{1}{2}+\alpha}^{\alpha} [y(y^2 + \frac{1}{2}z - \frac{1}{2})^{-\frac{1}{2}}]$
$x \sinh(\frac{1}{2}\pi x) \operatorname{sech}(\pi x) \cdot$ $\Gamma(\frac{1}{2} - \alpha + \frac{1}{2}x) \Gamma(\frac{1}{2} - \alpha - \frac{1}{2}x) P_{\frac{1}{2}ix-\frac{1}{2}}^{\alpha}(z)$	$2^{\frac{3}{2}\alpha-1} \pi^{\frac{1}{2}} \Gamma(\frac{3}{2} - 2\alpha) (z-1)^{-\frac{1}{2}\alpha} (\frac{1}{2}z + \frac{1}{2})^{-\frac{1}{2}} \cdot$ $\begin{cases} (y^2 - \frac{z+1}{2})^{\frac{1}{2}\alpha-\frac{1}{2}} P_{\alpha-\frac{1}{2}}^{\alpha-1} [y(\frac{1}{2}z + \frac{1}{2})^{-\frac{1}{2}}], y > (\frac{1}{2}z + \frac{1}{2})^{\frac{1}{2}} \\ (\frac{z+1}{2} - y^2)^{\frac{1}{2}\alpha-\frac{1}{2}} P_{\alpha-\frac{1}{2}}^{\alpha-1} [y(\frac{1}{2}z + \frac{1}{2})^{-\frac{1}{2}}], y < (\frac{1}{2}z + \frac{1}{2})^{\frac{1}{2}} \end{cases}$ $\operatorname{Re} \alpha \leq \frac{1}{2}$

$f(x)$	$g(y) = \int_0^\infty f(x) P_{ix-\frac{1}{2}}(y) dx$
$x \tanh(\pi x) \Gamma(\frac{1}{2} - \alpha + \frac{i}{2}x) \Gamma(\frac{1}{2} - \alpha - \frac{i}{2}x) \cdot$ $\cdot P_{\frac{1}{2}ix-\frac{1}{2}}^\alpha(z)$	$2^{\frac{1}{2}-\frac{1}{2}\alpha} \pi^{\frac{1}{2}} \Gamma(\frac{3}{2} - 2\alpha) (z+1)^{-\frac{1}{2}\alpha} (y^2 + \frac{z-1}{2})^{\frac{1}{2}\alpha-\frac{3}{4}} \cdot$ $\cdot P_{\frac{1}{2}ix-\alpha}^\alpha[y(y^2 + \frac{z-1}{2})^{-\frac{1}{2}}]$ $\text{Re } \alpha \leq \frac{1}{2}$
$x \tanh(\pi x) \Gamma(\frac{1}{4} - \frac{\alpha}{2} + \frac{i}{2}x) \Gamma(\frac{1}{4} - \frac{\alpha}{2} - \frac{i}{2}x) \cdot$ $\cdot P_{ix-\frac{1}{2}}^\alpha(z)$	$2^{1+\alpha} \pi^{\frac{1}{2}} (y^2+z^2-1)^{-\frac{1}{2}} (z^2-1)^{-\frac{1}{2}\alpha} [y+(y^2+z^2-1)^{\frac{1}{2}}]^\alpha$ $\text{Re } \alpha \leq \frac{1}{2}$
$x \tanh(\pi x) \Gamma(\frac{3}{4} - \frac{\alpha}{2} + \frac{i}{2}x) \Gamma(\frac{3}{4} - \frac{\alpha}{2} - \frac{i}{2}x) \cdot$ $P_{ix-\frac{1}{2}}^\alpha(z)$	$2^\alpha \pi^{\frac{1}{2}} z (z^2-1)^{-\frac{1}{2}\alpha} (y + z^2 - 1)^{-\frac{3}{4}} \cdot$ $\cdot [y - \alpha(y^2 + z^2 - 1)^{\frac{1}{2}}][y + (y^2 + z^2 - 1)^{\frac{1}{2}}]^\alpha$ $\text{Re } \alpha \leq \frac{3}{2}$
$x \tanh(\pi x) [P_{-\frac{1}{2}+ix}^\alpha(a)]^2$	$\pi^{-1} (2a^2 - 1 - y)^{\frac{1}{2}} (y - 1)^{-\frac{1}{2}}, 1 < y < 2a^2 - 1$ $0, y > 2a^2 - 1$ $a > 1$
$x \sinh(\frac{1}{2}\pi x) \text{sech}(\pi x) P_{\frac{1}{2}ix-\frac{1}{2}}^\alpha(z) \cdot$ $\cdot P_{\frac{1}{2}ix-\frac{1}{2}}^{-\alpha}(z)$	$2^{-\frac{3}{2}} y^{-\frac{1}{2}} (z^2 - 1)^{\frac{1}{2}} c^{-1} \cdot$ $\cdot [(\alpha + \frac{1}{4}) P_{\frac{1}{4}}^\alpha(c/y) P_{\frac{1}{4}}^{-\alpha}(c/y) - (\alpha - \frac{1}{4}) \cdot$ $\cdot P_{\frac{1}{4}}^\alpha(c/y) P_{\frac{1}{4}}^{-\alpha}(c/y)]$ $c = (y^2 + z^2 - 1)^{\frac{1}{2}}$
$x \tanh(\pi x) \Gamma(\alpha + \frac{1}{2} + \frac{i}{2}x) \Gamma(\alpha + \frac{1}{2} - \frac{i}{2}x) \cdot$ $[P_{\frac{1}{2}ix-\frac{1}{2}}^{-\alpha}(z)]^2$	$2^{-\alpha-\frac{1}{2}} \pi^{\frac{1}{2}} \Gamma(2\alpha + \frac{3}{2}) (z^2 - 1)^{\frac{1}{2}} y^{-\frac{1}{2}} c^{-1} P_{\frac{1}{4}}^{-\alpha}(c/y) \cdot$ $\cdot P_{\frac{1}{4}}^{-\alpha}(c/y)$ $c = (y^2 + z^2 - 1)^{\frac{1}{2}}$

$f(x)$	$g(y) = \int_0^{\infty} f(x) P_{ix-\frac{1}{2}}(y) dx$
$x \tanh(\pi x) \Gamma(\frac{1}{2} - \alpha + ix) \Gamma(\frac{1}{2} - \alpha - ix) \cdot [P_{-\frac{1}{2}+ix}^{\alpha}(z)]^2$	$(z^2 - 1)^{-\alpha} (y + 1)^{-\frac{1}{2}} (y - 1 + 2z^2)^{-\frac{1}{2}} \cdot [y + z^2 + (y + 1)^{\frac{1}{2}} (y - 1 + 2z^2)^{\frac{1}{2}}]^{\alpha}$
$x \operatorname{sech}(\pi x) \sinh(\frac{\pi}{2}x) [Y_{ix}(a) + Y_{-ix}(a)]$	$(2a/\pi)^{\frac{1}{2}} \sin(ay - \frac{3}{4}\pi)$
$x \operatorname{sech}(\pi x) \sinh(\frac{\pi}{2}x) [J_{ix}(a) + J_{-ix}(a)]$	$(2a/\pi)^{\frac{1}{2}} \cos(ay - \frac{3}{4}\pi)$
$x \tanh(\pi x) \operatorname{sech}(\frac{\pi}{2}x) [J_{ix}(a) + J_{-ix}(a)]$	$2(a/\pi)^{\frac{1}{2}} [\sin(ay) - \cos(ay)]$
$x \tanh(\pi x) \operatorname{sech}(\frac{\pi}{2}x) [Y_{ix}(a) + Y_{-ix}(a)]$	$-2(a/\pi)^{\frac{1}{2}} [\sin(ay) + \cos(ay)]$
$x \tanh(\pi x) [J_{ix}(a) Y_{-ix}(b) + Y_{ix}(a) J_{-ix}(b)]$	$-2\pi^{-1} (ab)^{\frac{1}{2}} (a^2 + b^2 + 2aby)^{\frac{1}{2}} \sin[(a^2 + b^2 + 2aby)^{\frac{1}{2}}]$
$x \tanh(\pi x) [Y_{ix}(a) Y_{-ix}(b) - J_{ix}(a) J_{-ix}(b)]$	$2\pi^{-1} (ab)^{\frac{1}{2}} (a^2 + b^2 + 2aby)^{\frac{1}{2}} \cos[(a^2 + b^2 + 2aby)^{\frac{1}{2}}]$
$x \tanh(\pi x) \{[J_{ix}(a)]^2 + [Y_{ix}(a)]^2\}$	$2^{\frac{1}{2}} \pi^{-1} (y - 1)^{-\frac{1}{2}} \exp[-a(2y - 2)^{\frac{1}{2}}]$
$x \tanh(\pi x) \exp(-\pi x) [H_{ix}^{(1)}(a)]^2$	$-2^{\frac{1}{2}} \pi^{-1} (1 + y)^{-\frac{1}{2}} \exp[ia(2 + 2y)^{\frac{1}{2}}]$
$x \tanh(\pi x) [I_{ix}(a) + I_{-ix}(a)]$	$(2y - 2)^{-\frac{1}{2}} \sin[a(2y - 2)^{\frac{1}{2}}]$
$x \tanh(\pi x) K_{ix}(a)$	$(\frac{1}{2}a\pi)^{\frac{1}{2}} \exp(-ay)$

$f(x)$	$g(y) = \int_0^{\infty} f(x) P_{ix-\frac{1}{2}}(y) dx$
$x \tanh(\pi x) K_{i2x}(a)$	$\frac{1}{4} a K_0 \left[a \left(\frac{1}{2} + \frac{1}{2} y \right)^{\frac{1}{2}} \right]$
$x \operatorname{sech}(\pi x) \tanh(\pi x) K_{ix}(a)$	$-(2\pi)^{-\frac{1}{2}} a^{\frac{1}{2}} \exp(ay) \operatorname{Ei}(-ay - a)$
$x \sinh\left(\frac{1}{2}\pi x\right) \operatorname{sech}(\pi x) K_{i\frac{1}{2}x}(a)$	$\left(\frac{1}{2}\pi\right)^{\frac{1}{2}} a^{\frac{1}{2}} \exp(ay^2 - a) D_{-\frac{1}{2}}(2ye^{\frac{1}{2}})$
$x \sinh\left(\frac{1}{2}\pi x\right) \operatorname{sech}(\pi x) [J_{ix}(a) + J_{-ix}(a)] \cdot K_{ix}(a)$	$(2y)^{-\frac{1}{2}} \exp(-ay^{\frac{1}{2}}) \sin(ay^{\frac{1}{2}})$
$x \sinh\left(\frac{1}{2}\pi x\right) \operatorname{sech}(\pi x) \cdot [Y_{ix}(a) + Y_{-ix}(a)] K_{ix}(a)$	$-(2y)^{-\frac{1}{2}} \exp(-ay^{\frac{1}{2}}) \cos(ay^{\frac{1}{2}})$
$x \tanh(\pi x) [I_{ix}(a) K_{ix}(b) - K_{ix}(a) I_{ix}(b)]$	$(ab)^{\frac{1}{2}} (a^2 + b^2 - 2aby)^{-\frac{1}{2}} \cosh[(a^2 + b^2 - 2aby)^{\frac{1}{2}}]$ $y < \frac{1}{2} \frac{a}{b} + \frac{1}{2} \frac{b}{a}$ 0 otherwise
$x \sinh(\pi x) [K_{ix}(a)]^2$	$2^{-\frac{3}{2}} \pi (y - 1)^{-\frac{1}{2}} \cos[a(2y - 2)^{\frac{1}{2}}]$
$x \tanh(\pi x) K_{ix}(a) K_{ix}(b)$	$\frac{1}{2} \pi (ab)^{\frac{1}{2}} (a^2 + b^2 + 2aby)^{-\frac{1}{2}} \cdot \exp[-(a^2 + b^2 + 2aby)^{\frac{1}{2}}]$

$f(x)$	$g(y) = \int_0^{\infty} f(x) P_{ix-\frac{1}{2}}(y) dx$
$x \tanh(\pi x) K_{ix}(ae^{i\pi/4}) K_{ix}(ae^{-i\pi/4})$	$\pi 2^{-\frac{3}{2}} y^{-\frac{1}{2}} \exp[-a(2y)^{\frac{1}{2}}]$
$x \tanh(\pi x) \Gamma(\frac{1}{4} + \frac{i}{2}y) \Gamma(\frac{1}{4} - \frac{i}{2}y) S_{\frac{1}{2}, ix}(a)$	$4(a\pi)^{\frac{1}{2}} [\sin(ay) \text{Ci}(ay) - \cos(ay) \text{si}(ay)]$
$x \tanh(\pi x) \Gamma(\frac{1}{2} - \frac{1}{2}\alpha + ix) \cdot$ $\cdot \Gamma(\frac{1}{2} - \frac{1}{2}\alpha - ix) S_{\alpha, 2ix}(a)$	$\frac{1}{2}a[\Gamma(1 - \frac{1}{2}\alpha)]^2 S_{\alpha-1, 0}[a(\frac{1}{2} + \frac{1}{2}y)^{\frac{1}{2}}]$ $\text{Re } \alpha \leq 1$

Table C

Generalized Mehler Transform

$$g(y) = \int_0^{\infty} f(x) P_{ix-\frac{1}{2}}^k(y) dx$$

$$f(x) = \pi^{-1} \sinh(\pi x) \Gamma\left(\frac{1}{2} - k + ix\right) \Gamma\left(\frac{1}{2} - k - ix\right) \int_1^{\infty} g(y) P_{ix-\frac{1}{2}}^k(y) dy$$

$f(x)$	$g(y) = \int_0^{\infty} f(x) P_{ix-\frac{1}{2}}^k(y) dx$
a	$a(\frac{1}{2}\pi)^{\frac{1}{2}}[\Gamma(\frac{1}{2}-k)]^{-1}(y^2-1)^{\frac{1}{2}k}(y-1)^{-k-\frac{1}{2}}$ $\text{Re } k < \frac{1}{2}$
$\cos(ax)$	$(\frac{1}{2}\pi)^{\frac{1}{2}}[\Gamma(\frac{1}{2}-k)]^{-1}(y^2-1)^{\frac{1}{2}k}(y-\cosh a)^{-k-\frac{1}{2}}$ $y > \cosh a$ 0 $y < \cosh a$ $\text{Re } k < \frac{1}{2}$
$\cos(ax)\text{sech}(\pi x)$	$2^{-\frac{1}{2}-k}(1+y)^{\frac{1}{2}k}(y+\cosh a)^{-\frac{1}{2}-\frac{1}{2}k} \cdot$ $P_k^k[(1+\cosh a)^{\frac{1}{2}}(y+\cosh a)^{-\frac{1}{2}}]$
$\cos(ax)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)$	$(\frac{1}{2}\pi)^{\frac{1}{2}}\Gamma(\frac{1}{2}-k)(y^2-1)^{-\frac{1}{2}k}(y+\cosh a)^{k-\frac{1}{2}}$ $\text{Re } k \leq \frac{1}{2}$
$\cosh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)$	$(\frac{1}{2}\pi)^{\frac{1}{2}}\Gamma(\frac{1}{2}-k)(y^2-1)^{-\frac{1}{2}k}(y-1)^{k-\frac{1}{2}}$ $-\frac{1}{2} \leq \text{Re } k \leq \frac{1}{2}$
$\Gamma(\frac{1}{4}-\frac{k}{2}+\frac{i}{2}x)\Gamma(\frac{1}{4}-\frac{k}{2}-\frac{i}{2}x)\cos(ax)$	$\pi 2^{k+\frac{1}{2}}\Gamma(\frac{1}{2}-k)(y^2+\sinh^2 a)^{-\frac{1}{4}} \cdot$ $P_{-\frac{1}{2}}^k[\cosh a(y^2+\sinh^2 a)^{-\frac{1}{2}}]$
$\Gamma(\frac{3}{4}-\frac{k}{2}+\frac{i}{2}x)\Gamma(\frac{3}{4}-\frac{k}{2}-\frac{i}{2}x)\cos(ax)$	$\pi 2^{k-\frac{1}{2}}y\Gamma(\frac{3}{2}-k)(y^2+\sinh^2 a)^{-\frac{3}{4}} \cdot$ $P_{\frac{1}{2}}^k[\cosh a(y^2+\sinh^2 a)^{-\frac{1}{2}}]$

$f(x)$	$g(y) = \int_0^\infty f(x) P_{ix-\frac{1}{2}}^k(y) dx$
$x \sinh(\pi x) \Gamma(\alpha + ix) \Gamma(\alpha - ix)$	$\pi 2^{\alpha-\frac{1}{2}} \Gamma(\frac{1}{2} + \alpha) [\Gamma(\frac{1}{2} - k - \alpha)]^{-1} \cdot$ $\cdot (y-1)^{-\frac{1}{2}-k-\alpha} (y^2-1)^{\frac{1}{2}k}$ $\text{Re}(2\alpha + k - \frac{1}{2}) < 0$ $\text{Re } \alpha \geq 0$
$x \sinh(\pi x) \Gamma(\frac{1}{2} - k + ix) \Gamma(\frac{1}{2} - k - ix) \cdot$ $\cdot \Gamma(\alpha + \frac{k}{2} + \frac{i}{2}x) \Gamma(\alpha + \frac{k}{2} - \frac{i}{2}x)$	$\pi 2^{\frac{3}{2}-k-2\alpha} \Gamma(\frac{1}{2} + 2\alpha) y^{-\frac{1}{2}-2\alpha} (y^2-1)^{-\frac{1}{2}k}$ $\text{Re}(\alpha + \frac{k}{2}) \geq 0$ $\text{Re } k \leq \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2} - k + ix) \Gamma(\frac{1}{2} - k - ix) \cdot$ $\cdot \Gamma(\frac{3}{4} - \lambda + \frac{i}{2}x) \Gamma(\frac{3}{4} - \lambda - \frac{i}{2}x) [\Gamma(\frac{3}{4} - \frac{k}{2} + \frac{i}{2}x) \cdot$ $\cdot \Gamma(\frac{3}{4} - \frac{k}{2} - \frac{i}{2}x)]^{-1}$	$\pi 2^{1-k} [\Gamma(\lambda - \frac{k}{2})]^{-1} \Gamma(1 - \lambda - \frac{k}{2}) (y^2-1)^{\lambda-1}$ $\frac{1}{4} < \text{Re } \lambda \leq \frac{3}{4}$ $\text{Re } k \leq \frac{1}{2}$
$x \sinh(2\pi x) \Gamma(\alpha + ix) \Gamma(\alpha - ix) \cdot$ $\Gamma(\frac{1}{2} - k + ix) \Gamma(\frac{1}{2} - k - ix)$	$\pi 2^{\frac{1}{2}+\alpha} \Gamma(\frac{1}{2} + \alpha - k) [\Gamma(\frac{1}{2} - \alpha)]^{-1} \cdot$ $\cdot (y-1)^{k-\alpha-\frac{1}{2}} (y^2-1)^{-\frac{1}{2}k}$ $\text{Re}(2\alpha - k - \frac{1}{2}) < 0, \quad \text{Re } \alpha \geq 0$ $\text{Re } k \leq \frac{1}{2}$
$\Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) \tanh(\pi x) \sinh(ax)$	$-2^{k-1} \pi^{\frac{1}{2}} \Gamma(1-2k) (y-1)^{-\frac{1}{2}k} (y+1)^{-\frac{1}{4}} (y+\cos a)^{\frac{1}{2}k-\frac{1}{4}} \cdot$ $\cdot [P_{-k-\frac{1}{2}}^{k-\frac{1}{2}}(z) - P_{-k-\frac{1}{2}}^{k-\frac{1}{2}}(-z)]$ $z = (1 - \cos a)^{\frac{1}{2}} (1+y)^{-\frac{1}{2}}$ $\text{Re } k \leq \frac{1}{2}$

$f(x)$	$g(y) = \int_0^\infty f(x) P_{ix-\frac{1}{2}}^k(y) dx$
$\cos(ax) \operatorname{sech}(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix)$	$2^{\frac{1}{2}+k} \Gamma(1-2k)(y-1)^{-\frac{1}{2}k} (\cosh a - y)^{\frac{1}{2}k-\frac{1}{2}} \cdot$ $e^{-i\pi k} Q_{-k}^k [(\frac{1}{2} \cosh a - \frac{1}{2} y)^{-\frac{1}{2}} \cosh(\frac{1}{2} a)], y < \cosh a$ $2^k \Gamma(1-2k) \pi^{\frac{1}{2}} (y^2-1)^{-\frac{1}{4}} (y-1)^{\frac{1}{4}-\frac{1}{2}k} (y-\cosh a)^{\frac{1}{2}k-\frac{1}{4}} \cdot$ $P_{k-\frac{1}{2}}^k [(\frac{1}{2} + \frac{1}{2} y)^{-\frac{1}{2}} \cosh(\frac{1}{2} a)], y > \cosh a$ $\operatorname{Re} k < \frac{1}{2}$
$\cosh(ax) \operatorname{sech}(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix)$	$(\frac{1}{2}\pi)^{\frac{1}{2}} \Gamma(\frac{1}{2}-k)(y^2-1)^{-\frac{1}{2}k} \{(y-\cos a)^{k-\frac{1}{2}} +$ $2^{-k-\frac{1}{2}} \pi^{-\frac{1}{2}} \Gamma(1-k)[(y+1)(y-\cos a)]^{\frac{1}{2}k-\frac{1}{4}} \cdot$ $\cdot [P_{-k-\frac{1}{2}}^{k-\frac{1}{2}}(z) - P_{-k-\frac{1}{2}}^{k-\frac{1}{2}}(-z)]\}$ $z = (1 + \cos a)^{\frac{1}{2}}(1+y)^{-\frac{1}{2}}$ $\operatorname{Re} k \leq \frac{1}{2}$
$x \sinh(\pi x) \operatorname{sech}(2\pi x) P_{2ix-\frac{1}{2}}^k(z)$	$2^{-\frac{7}{2}-\frac{3}{2}k} (1+y)^{\frac{1}{2}k} (z^2 + \frac{1}{2}y - \frac{1}{2})^{-\frac{3}{4}-\frac{1}{2}k} \cdot$ $\cdot P_{\frac{1}{2}+k}^k [z(z^2 + \frac{1}{2}y - \frac{1}{2})^{-\frac{1}{2}}]$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot P_{-\frac{1}{2}+ix}^k(a)$	$\pi [\Gamma(k)]^{-1} (y^2-1)^{-\frac{1}{2}k} (a-y)^{k-1}, y > a > 1$ $0, 1 < y < a$ $0 < \operatorname{Re} k \leq \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-\alpha+ix) \Gamma(\frac{1}{2}-\alpha-ix) \cdot$ $\cdot \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) P_{-\frac{1}{2}+ix}^\alpha(z)$	$\pi \Gamma(1-\alpha-k)(z^2-1)^{-\frac{1}{2}\alpha} (y^2-1)^{-\frac{1}{2}k} \cdot$ $(z+y)^{k+\alpha-1}$ $\operatorname{Re}(\alpha, k) < \frac{1}{2}$

$f(x)$	$g(y) = \int_0^\infty f(x) P_{ix-\frac{1}{2}}^k(y) dx$
$x \tanh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) P_{-\frac{1}{2}+ix}^k(a)$	$\Gamma(1-k)(y^2-1)^{-\frac{1}{2}k}(y+a)^{k-1}$ $\text{Re } k \leq \frac{1}{2}$
$x \tanh(2\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) P_{2ix-\frac{1}{2}}^k(z)$	$2^{-\frac{3}{2}-\frac{1}{2}k} \pi^{\frac{1}{2}} \Gamma(\frac{3}{2}-2k)(z+1)^{-\frac{1}{2}k} (z^2 + \frac{y-1}{2})^{\frac{1}{2}k-\frac{3}{2}} \cdot$ $\cdot P_{\frac{1}{2}-k}^k[z(z^2 + \frac{y-1}{2})^{\frac{1}{2}}]$ $\text{Re } k \leq \frac{1}{2}$
$x \tanh(\pi x) \Gamma(\frac{1}{4}-\frac{k}{2}+\frac{i}{2}x) \Gamma(\frac{1}{4}-\frac{k}{2}-\frac{i}{2}x) \cdot$ $\cdot P_{ix-\frac{1}{2}}^k(z)$	$2^{1+k} \pi^{\frac{1}{2}} (y^2+z^2-1)^{-\frac{1}{2}k} (y^2-1)^{-\frac{1}{2}k} [z+(y^2+z^2-1)^{\frac{1}{2}}]^k$ $\text{Re } k \leq \frac{1}{2}$
$x \tanh(\pi x) \Gamma(\frac{3}{4}-\frac{k}{2}+\frac{i}{2}x) \Gamma(\frac{3}{4}-\frac{k}{2}-\frac{i}{2}x) \cdot$ $\cdot P_{ix-\frac{1}{2}}^k(z)$	$2^k \pi^{\frac{1}{2}} (y^2-1)^{-\frac{1}{2}k} (y^2+z^2-1)^{-\frac{3}{2}} y \cdot$ $\cdot [z-k(y^2+z^2-1)^{\frac{1}{2}}][z+(y^2+z^2-1)^{\frac{1}{2}}]^k$ $\text{Re } k \leq \frac{3}{4}$
$x \sinh(\pi x) \Gamma(\frac{1}{4}-\frac{\alpha}{2}+\frac{i}{2}x) \Gamma(\frac{1}{4}-\frac{\alpha}{2}-\frac{i}{2}x) \cdot$ $\cdot \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) P_{ix-\frac{1}{2}}^\alpha(z)$	$2^{1+\alpha} \pi^{\frac{3}{2}} (y^2-1)^{-\frac{1}{2}k} \Gamma(1-k-\alpha)(y^2+z^2-1)^{\frac{1}{2}k-\frac{1}{2}} \cdot$ $\cdot P_{-k}^\alpha[y(y^2+z^2-1)^{-\frac{1}{2}}]$ $\text{Re}(\alpha, k) < \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{3}{4}-\frac{\alpha}{2}+\frac{i}{2}x) \Gamma(\frac{3}{4}-\frac{\alpha}{2}-\frac{i}{2}x) \cdot$ $\cdot \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) P_{ix-\frac{1}{2}}^\alpha(z)$	$2^\alpha \pi^{\frac{3}{2}} \Gamma(2-\alpha-k) z (y^2-1)^{-\frac{1}{2}k} \cdot$ $\cdot (y^2+z^2-1)^{\frac{1}{2}k-1} P_{1-k}^\alpha[y(y^2+z^2-1)^{-\frac{1}{2}}]$ $\text{Re } \alpha < \frac{3}{2}$ $\text{Re } k < \frac{1}{2}$

$f(x)$	$g(y) = \int_0^\infty f(x) P_{ix-\frac{1}{2}}^k(y) dx$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-\alpha+\frac{i}{2}x) \Gamma(\frac{1}{2}-\alpha-\frac{i}{2}x) \cdot$ $\cdot \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) P_{i\frac{1}{2}x-\frac{1}{2}}^\alpha(z)$	$2^{\frac{1}{2}-\frac{1}{2}\alpha} \pi^{\frac{3}{2}} (1+z)^{-\frac{1}{2}\alpha} \Gamma(\frac{3}{2}-k-2\alpha) (y^2-1)^{-\frac{1}{2}k} \cdot$ $\cdot (y^2 + \frac{z-1}{2})^{\frac{1}{2}(\alpha+k-\frac{3}{2})} P_{\frac{1}{2}-\alpha-k}^\alpha [y(y^2 + \frac{z-1}{2})^{\frac{1}{2}}]$ $\text{Re}(\alpha, k) < \frac{1}{2}$
$x \sinh(\frac{1}{2}\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot P_{\frac{1}{2}ix-\frac{1}{2}}^\alpha(z)$	$2^{-\frac{3}{2}\alpha-\frac{1}{2}} \pi^{\frac{1}{2}} \Gamma(\frac{3}{2}-k) (z+1)^{\frac{1}{2}\alpha} (y^2-1)^{-\frac{1}{2}k} \cdot$ $(y^2 + \frac{z-1}{2})^{\frac{1}{2}(k-\alpha-\frac{3}{2})} P_{\alpha-k+\frac{1}{2}}^\alpha [y(y^2 + \frac{z-1}{2})^{\frac{1}{2}}]$ $\text{Re } k \leq \frac{1}{2}$
$x \sinh(\pi x) [\Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix)]^2 \cdot$ $\cdot [P_{-\frac{1}{2}+ix}^k(z)]^2$	$2^{-k} \pi^{\frac{1}{2}} \Gamma(\frac{1}{2}-k) (z^2-1)^{-k} (y-1)^{-\frac{1}{2}k} (y+1)^{\frac{1}{2}k-\frac{1}{2}} \cdot$ $\cdot (2z^2-1+y)^{k-\frac{1}{2}}$ $\text{Re } k < \frac{1}{2}$
$x \sinh(\pi x) \text{sech}(2\pi x) \cdot$ $\cdot \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) P_{i2x-\frac{1}{2}}^\alpha(z)$	$2^{\frac{3}{2}k-3} \pi^{\frac{1}{2}} \Gamma(\frac{3}{2}-2k) (y-1)^{-\frac{1}{2}k} (\frac{1}{2}y+\frac{1}{2})^{-\frac{1}{4}} \cdot$ $\begin{cases} (z^2 - \frac{y+1}{2})^{\frac{1}{2}k-\frac{1}{2}} P_{k-\frac{1}{2}}^{k-1} [z(\frac{1}{2}y+\frac{1}{2})^{-\frac{1}{2}}], & z > (\frac{1}{2}y+\frac{1}{2})^{\frac{1}{2}} \\ (\frac{y+1}{2} - z^2)^{\frac{1}{2}k-\frac{1}{2}} P_{k-\frac{1}{2}}^{k-1} [z(\frac{1}{2}y+\frac{1}{2})^{-\frac{1}{2}}], & z < (\frac{1}{2}y+\frac{1}{2})^{\frac{1}{2}} \end{cases}$ $\text{Re } k \leq \frac{1}{2}$
$x \sinh(\frac{1}{2}\pi x) \Gamma(\frac{1}{2}-\alpha+\frac{i}{2}x) \Gamma(\frac{1}{2}-\alpha-\frac{i}{2}x) \cdot$ $\cdot \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) P_{i\frac{1}{2}x-\frac{1}{2}}^\alpha(z)$	$2^{\frac{3}{2}\alpha} \pi \Gamma(\frac{3}{2}-k) \Gamma(\frac{3}{2}-k-2\alpha) (z-1)^{-\frac{1}{2}\alpha} (\frac{z+1}{2})^{-\frac{1}{4}} (y^2-1)^{-\frac{1}{2}k} \cdot$ $\begin{cases} (y^2 - \frac{z+1}{2})^{\frac{1}{2}(\alpha+k-1)} P_{\alpha-\frac{1}{2}}^{\alpha+k-1} [y(\frac{z+1}{2})^{-\frac{1}{2}}], & y > (\frac{z+1}{2})^{\frac{1}{2}} \\ (\frac{z+1}{2} - y^2)^{\frac{1}{2}(\alpha+k-1)} P_{\alpha-\frac{1}{2}}^{\alpha+k-1} [y(\frac{z+1}{2})^{-\frac{1}{2}}], & y < (\frac{z+1}{2})^{\frac{1}{2}} \end{cases}$ $\text{Re}(\alpha, k) < \frac{1}{2}$

$f(x)$	$g(y) = \int_0^\infty f(x) P_{ix-\frac{1}{2}}^k(y) dx$
$x \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot \{[J_{ix}(a)]^2 - [J_{-ix}(a)]^2\}$	$- i 2^{\frac{1}{2}+k} \pi a^{\frac{1}{2}-k} (y+1)^{-\frac{1}{4}} (y-1)^{-\frac{1}{4}} J_{\frac{1}{2}-k} [a(2y+2)^{\frac{1}{2}}]$ $\text{Re } k < \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot [J_{ix}(a) J_{-ix}(a) - Y_{ix}(a) Y_{-ix}(a)]$	$\pi^{\frac{1}{2}} 2^{\frac{1}{2}k+\frac{1}{4}} a^{\frac{1}{2}-k} (y+1)^{-\frac{1}{4}} (y-1)^{-\frac{1}{4}} Y_{\frac{1}{2}-k} [a(2y+2)^{\frac{1}{2}}]$ $-\frac{1}{2} \leq \text{Re } k \leq \frac{1}{2}$
$x \sinh(\pi x) K_{i2x}(a)$	$\pi 2^{-3-\frac{3}{2}k} (1+y)^{\frac{1}{2}k} a^{1+k} J_{-k} [(\frac{1}{2}ay - \frac{1}{2}a)^{\frac{1}{2}}]$ $\text{Re } k \leq 0$
$x \sinh(2\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) K_{i2x}(a)$	$2^{-2-\frac{1}{2}k} \pi^2 (1+y)^{-\frac{1}{2}k} a^{1-k} J_{-k} [(\frac{1}{2}ay - \frac{1}{2}a)^{\frac{1}{2}}]$ $\text{Re } k \leq \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) K_{i2x}(a)$	$2^{\frac{3}{2}k-2} \pi a^{1-k} (y-1)^{-\frac{1}{2}k} K_k [a(\frac{y+1}{2})^{\frac{1}{2}}]$ $\text{Re } k \leq \frac{1}{2}$
$x \tanh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) K_{ix}(a)$	$(\frac{1}{2}\pi a)^{\frac{1}{2}} \Gamma(1-k) (y-1)^{-\frac{1}{2}k} (y+1)^{\frac{1}{2}k} \cdot$ $\exp(ay) \Gamma(-k, ay+a)$ $\text{Re } k \leq \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{4} \frac{k}{2} + \frac{i}{2} x) \Gamma(\frac{1}{4} \frac{k}{2} - \frac{i}{2} x) K_{ix}(a)$	$2^{\frac{1}{2}+k} \pi^2 a^{\frac{1}{2}} J_{-k} [a(y^2-1)^{\frac{1}{2}}]$ $\text{Re } k \leq \frac{1}{2}$

$f(x)$	$g(y) = \int_0^\infty f(x) p_{ix-\frac{1}{2}}^k(y) dx$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) K_{ix}(a)$	$2^{-\frac{1}{2}} \pi^{\frac{3}{2}} a^{\frac{1}{2}-k} (y^2-1)^{-\frac{1}{2}k} \exp(-ay)$ $\text{Re } k \leq \frac{1}{2}$
$x \sinh(\frac{1}{2}\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) K_{i\frac{1}{2}x}(a)$	$\pi 2^{\frac{1}{2}-k} a^{\frac{1}{4}-\frac{1}{2}k} \Gamma(\frac{3}{2}-k) (y^2-1)^{-\frac{1}{2}k} \exp(ay^2-a) \cdot$ $D_{k-\frac{3}{2}}(2ya^{\frac{1}{2}})$ $\text{Re } k \leq \frac{1}{2}$
$x \sinh(\frac{1}{2}\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot [J_{ix}(a) + J_{-ix}(a)] K_{ix}(a)$	$i(\frac{1}{2}\pi)^{\frac{1}{2}} (y^2-1)^{-\frac{1}{2}k} (2y/a^2)^{\frac{1}{2}k-\frac{1}{4}} \cdot$ $\cdot \{ \exp(i\frac{\pi}{4}-i\frac{\pi}{4}k) K_{k-\frac{1}{2}}[a(2iy)^{\frac{1}{2}}] - \exp(i\frac{\pi}{4}k-i\frac{\pi}{4}) K_{k-\frac{1}{2}}[a(-2iy)^{\frac{1}{2}}] \}$ $\text{Re } k \leq \frac{1}{2}$
$x \sinh(\frac{1}{2}\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot [Y_{ix}(a) + Y_{-ix}(a)] K_{ix}(a)$	$-(\frac{1}{2}\pi)^{\frac{1}{2}} (y^2-1)^{-\frac{1}{2}k} (2y/a^2)^{\frac{1}{2}k-\frac{1}{4}} \cdot$ $\{ \exp(i\frac{\pi}{4}-i\frac{\pi}{4}k) K_{k-\frac{1}{2}}[a(2iy)^{\frac{1}{2}}] + \exp(i\frac{\pi}{4}k-i\frac{\pi}{4}) K_{k-\frac{1}{2}}[a(-2iy)^{\frac{1}{2}}] \}$ $\text{Re } k \leq \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k-ix) \Gamma(\frac{1}{2}-k+ix) \cdot$ $\cdot [K_{ix}(a)]^2$	$\pi^{\frac{3}{2}} 2^{-\frac{3}{4}+\frac{1}{2}k} (y+1)^{-\frac{1}{4}} (y-1)^{-\frac{1}{2}k} K_{\frac{1}{2}-k}[a(2y+2)^{\frac{1}{2}}]$ $\text{Re } k \leq \frac{1}{2}$
$x \sinh(\pi x) K_{ix}(a) K_{ix}(b)$	$0 \quad y < \tau$ $2^{-k-2} \pi^{\frac{3}{2}} c^{\frac{1}{2}+k} (y^2-1)^{\frac{1}{2}k} (y-\tau)^{-\frac{1}{2}k-\frac{1}{4}} J_{-k-\frac{1}{2}}[c(y-\tau)^{\frac{1}{2}}]$ $y > \tau$ $c = (2ab)^{\frac{1}{2}}$ $\tau = \frac{1}{2}(a/b+b/a)$

$f(x)$	$g(y) = \int_0^\infty f(x) P_{ix-\frac{1}{2}}^k(y) dx$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot [\Gamma(-k+ix) \Gamma(-k-ix)]^{-1} [K_{ix}(a)]^2$	$\pi^{\frac{3}{2}} 2^{-\frac{k}{2}-\frac{7}{4}} a^{\frac{k}{2}-\frac{1}{2}} y^{\frac{1}{2}-k} (y^2-1)^{\frac{1}{2}k} J_{-\frac{1}{2}-k}[a(2y-2)^{\frac{1}{2}}]$ $\text{Re } k \leq -\frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot K_{ix}(a) K_{ix}(b)$	$2^{-\frac{1}{2}} \pi^{\frac{3}{2}} (ab)^{\frac{1}{2}-k} (y^2-1)^{-\frac{1}{2}k} \cdot$ $(a^2+b^2+2aby)^{\frac{1}{2}k-\frac{1}{4}} K_{\frac{1}{2}-k}[(a^2+b^2+2aby)^{\frac{1}{2}}]$ $\text{Re } k \leq \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) W_{k,ix}(a)$	$\frac{1}{2} \pi a (y+1)^{\frac{1}{2}k} (y-1)^{-\frac{1}{2}k} \exp(-\frac{1}{2}ay)$ $\text{Re } k \leq \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) \cdot$ $W_{\frac{1}{2}k-\frac{1}{4}, \frac{1}{2}x}(a)$	$2^{1-k} \pi a^{\frac{3}{4}-\frac{1}{2}k} (y^2-1)^{-\frac{1}{2}k} \exp(\frac{1}{2}a-ay^2)$ $\text{Re } k \leq \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) W_{\frac{1}{2}k+\frac{1}{4}, \frac{1}{2}x}(a)$	$2^{1-k} \pi a^{\frac{5}{4}-\frac{1}{2}k} y (y^2-1)^{-\frac{1}{2}k} \exp(\frac{1}{2}a-ay^2)$ $\text{Re } k \leq \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot \Gamma(\frac{1}{4}+\frac{k}{2}+\frac{i}{2}x) \Gamma(\frac{1}{4}+\frac{k}{2}-\frac{i}{2}x) W_{\frac{1}{4}-\frac{k}{2}, \frac{i}{2}x}(a)$	$\pi 2^{1-k} a^{\frac{3}{4}-\frac{k}{2}} (y^2-1)^{-\frac{1}{2}k} \exp(ay^2-\frac{1}{2}a) \text{Erfc}(ya^{\frac{1}{2}})$ $\text{Re } k < \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot \Gamma(\frac{1}{2}-\alpha+\frac{i}{2}x) \Gamma(\frac{1}{2}-\alpha-\frac{i}{2}x) W_{\alpha, \frac{i}{2}x}(a)$	$\pi^{\frac{3}{2}} 2^{\alpha+\frac{5}{4}-\frac{k}{2}} a^{\frac{3}{4}-\frac{k}{2}} \Gamma(\frac{3}{2}-2\alpha-k) \cdot$ $(y^2-1)^{-\frac{1}{2}k} \exp(\frac{1}{2}ay^2-\frac{1}{2}a) D_{2\alpha+k-\frac{3}{2}}[y(2a)^{\frac{1}{2}}]$ $\text{Re}(i, k) < \frac{1}{2}$

$f(x)$	$g(y) = \int_0^\infty f(x) P_{ix-\frac{1}{2}}^k(y) dx$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot \Gamma(\frac{1}{4}+\frac{k}{2}+\frac{i}{2}x) \Gamma(\frac{1}{4}+\frac{k}{2}-\frac{i}{2}x) W_{k, \frac{1}{2}ix}(a)$	$\pi 2^{1-k} a^{\frac{3}{2}-k} (y^2-1)^{-\frac{1}{2}k} \exp(-\frac{1}{2}a^2+a^2y^2) \operatorname{Erfc}(ay)$ $-\frac{1}{2} < \operatorname{Re} k < \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-\alpha+ix) \Gamma(\frac{1}{2}-\alpha-ix) \cdot$ $\cdot \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) W_{\alpha, ix}(a)$	$\pi (\frac{1}{2}a)^{\frac{1}{2}-\frac{1}{2}k} \Gamma(1-\alpha) \Gamma(1-\alpha-k) (y-1)^{-\frac{1}{2}k} (y+1)^{-\frac{1}{2}k} \cdot$ $\cdot \exp(\frac{1}{4}ay - \frac{1}{4}a) W_{\alpha-\frac{k}{2}-\frac{1}{2}, -\frac{k}{2}}(\frac{1}{2}a+\frac{1}{2}ay)$ $\operatorname{Re}(\alpha, k) < \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-\frac{\alpha}{2}+\frac{i}{2}x) \Gamma(\frac{1}{2}-\frac{\alpha}{2}-\frac{i}{2}x) \cdot$ $\cdot \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) S_{\alpha, ix}(a)$	$2^{\frac{1}{2}+\alpha} \pi a^{\frac{3}{2}-1-k} \Gamma(\frac{3}{2}-\alpha-k) y^{\frac{1}{2}} (y^2-1)^{-\frac{1}{2}k} S_{\alpha+k-1, \frac{1}{2}}(ay)$ $\operatorname{Re} \alpha < 1$ $\operatorname{Re} k < \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot S_{k+\frac{1}{2}, ix}(a)$	$\pi a^{\frac{3}{2}} y K_k[a(y^2-1)^{\frac{1}{2}}]$ $\operatorname{Re} k \leq \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot S_{2k, 2ix}(a)$	$\pi 2^{\frac{1}{2}k-2} a^{k+1} (1+y)^{\frac{1}{2}k} K_k[a(\frac{1}{2}y-\frac{1}{2})^{\frac{1}{2}}]$ $\operatorname{Re} k \leq \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot S_{k-\frac{1}{2}, ix}(a)$	$\pi a^{\frac{1}{2}} K_k[a(y^2-1)^{\frac{1}{2}}]$ $\operatorname{Re} k \leq \frac{1}{2}$

List of Abbreviations, Symbols and Notations

ϵ_n = Neumann's numbers, $\epsilon_0 = 1$, $\epsilon_n = 2$, $n = 1, 2, 3, \dots$

$\binom{a}{b}$ = Binomial coefficient, $\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}$

γ = Euler's constant, $\gamma = 0.57721\dots$

1. Elementary functions

Trigonometric and inverse trigonometric functions:

$\sin x$, $\cos x$, $\tan x = \sin x / \cos x$, $\cot x = \cos x / \sin x$,

$\sec x = 1/\cos x$, $\csc x = 1/\sin x$, $\arcsin x$, $\arccos x$, $\arctan x$,

$\operatorname{arccot} x$

Hyperbolic functions:

$\sinh x = (e^x - e^{-x})/2$, $\cosh x = (e^x + e^{-x})/2$, $\tanh x = \sinh x / \cosh x$,

$\operatorname{ctnh} x = \cosh x / \sinh x$, $\operatorname{sech} x = 1/\cosh x$, $\operatorname{csch} x = 1/\sinh x$.

2. Orthogonal polynomials

Legendre polynomials:

$$P_n(x) = 2^{-n}(n!)^{-1} \frac{d^n}{dx^n} (x^2-1)^n = {}_2F_1(-n, n+1; 1; \frac{1-x}{2})$$

Gegenbauer's polynomials:

$$C_n^a(x) = [n!\Gamma(2a)]^{-1}\Gamma(2a+n) {}_2F_1(-n, 2a+n; a+1/2; \frac{1-x}{2})$$

Chebyscheff polynomials:

$$T_n(x) = \cos(n \arccos x) = {}_2F_1\left(-n, n; \frac{1}{2}; \frac{1-x}{2}\right) = \frac{n}{2} \lim_{a \rightarrow 0} \Gamma(a) C_n^a(x)$$

$$U_n(x) = (1-x^2)^{-\frac{1}{2}} \sin[(n+1) \arccos x] \\ = x(n+1) {}_2F_1\left(\frac{1-n}{2}, \frac{3+n}{2}; \frac{3}{2}; 1-x^2\right)$$

Jacobi polynomials:

$$P_n^{(\beta, \alpha)}(x) = [n! \Gamma(1+\beta)]^{-1} \Gamma(1+\beta+n) {}_2F_1\left(-n, n+\alpha+\beta+1; \beta+1; \frac{1-x}{2}\right)$$

Laguerre polynomials:

$$L_n^a(x) = (n!)^{-1} x^{-a} e^x \frac{d^n}{dx^n} (e^{-x} x^{n+a}) = [n! \Gamma(1+a)]^{-1} \Gamma(a+1+n) {}_1F_1(-n; 1+a; x)$$

$$L_n(x) = L_n^0(x)$$

Hermite polynomials:

$$He_n(x) = (-1)^n \exp(x^2/2) \frac{d^n}{dx^n} \exp(-x^2/2)$$

$$He_{2n}(x) = (-1)^n 2^{-n} (n!)^{-1} (2n)! {}_1F_1\left(-n; \frac{1}{2}; \frac{1}{2}x^2\right)$$

$$He_{2n+1}(x) = (-1)^n 2^{-n} (n!)^{-1} (2n+1)! {}_1F_1\left(-n; \frac{3}{2}; \frac{1}{2}x^2\right)$$

3. Gamma function and related functions

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \quad \operatorname{Re} z > 0$$

ψ -function:

$$\psi(z) = \frac{d}{dz} \log \Gamma(z)$$

Beta function:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

See also under incomplete gamma function.

4. Riemann's and Hurwitz's zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \operatorname{Re} s > 1$$

$$\zeta(s, v) = \sum_{n=0}^{\infty} (n + v)^{-s}, \quad \operatorname{Re} s > 1$$

5. Legendre functions (definition according to Hobson)

$$P_a^\beta(z) = [\Gamma(1-\beta)]^{-1} \left(\frac{z+1}{z-1}\right)^{\beta/2} {}_2F_1\left(-a, a+1; 1-\beta; \frac{1-z}{2}\right)$$

$$Q_a^\beta(z) = 2^{-a-1} [\Gamma(a+3/2)]^{-1} e^{i\pi\beta} \sqrt{\pi} \Gamma(a+\beta+1) z^{-a-\beta-1} (z^2-1)^{\beta/2} \\ \cdot {}_2F_1\left(\frac{a+\beta+1}{2}, \frac{a+\beta+2}{2}; a+3/2; z^{-2}\right)$$

z is a point in the complex z plane cut along the real axis from $-\infty$ to $+1$.

$$P_a^\beta(x) = [\Gamma(1-\beta)]^{-1} \left(\frac{1+x}{1-x}\right)^{\beta/2} {}_2F_1\left(-a, a+1; 1-\beta; \frac{1-x}{2}\right), \quad -1 < x < 1$$

$$Q_a^\beta(x) = \frac{1}{2} e^{-i\pi\beta} [e^{-i\pi\beta/2} Q_a^\beta(x+i0) + e^{i\pi\beta/2} Q_a^\beta(x-i0)], \quad -1 < x < 1$$

$$P_a(z) = P_a^0(z); \quad Q_a(z) = Q_a^0(z); \quad P_a(x) = P_a^0(x); \quad Q_a(x) = Q_a^0(x)$$

6. Bessel functions

$$J_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{\alpha+2n}}{n! \Gamma(\alpha+n+1)}$$

$$Y_{\alpha}(z) = \cot(\pi\alpha) J_{\alpha}(z) - \csc(\pi\alpha) J_{-\alpha}(z)$$

$$H_{\alpha}^{(1)}(z) = J_{\alpha}(z) + iY_{\alpha}(z); \quad H_{\alpha}^{(2)}(z) = J_{\alpha}(z) - iY_{\alpha}(z)$$

7. Modified Bessel functions

$$I_{\alpha}(z) = e^{-i\pi\alpha/2} J_{\alpha}(ze^{i\pi/2}) = \sum_{n=0}^{\infty} \frac{(z/2)^{\alpha+2n}}{n! \Gamma(\alpha+n+1)}$$

$$\begin{aligned} K_{\alpha}(z) &= \frac{1}{2} \pi \csc(\pi\alpha) [I_{-\alpha}(z) - I_{\alpha}(z)] \\ &= \frac{1}{2} i\pi e^{i\pi\alpha/2} H_{\alpha}^{(1)}(ze^{i\pi/2}) = -\frac{1}{2} i\pi e^{-i\pi\alpha/2} H_{\alpha}^{(2)}(ze^{-i\pi/2}) \end{aligned}$$

8. Anger-Weber functions

$$J_{\alpha}(z) = \pi^{-1} \int_0^{\pi} \cos(z \sin t - at) dt$$

$$E_{\alpha}(z) = -\pi^{-1} \int_0^{\pi} \sin(z \sin t - at) dt$$

$$J_n(z) = J_n(z), \quad n=0,1,2,\dots$$

$$J_{\frac{1}{2}}(z) = (\pi z/2)^{-\frac{1}{2}} \{ \cos z [C(z) - S(z)] + \sin z [S(z) + C(z)] \} = E_{-\frac{1}{2}}(z)$$

$$J_{-\frac{1}{2}}(z) = (\pi z/2)^{-\frac{1}{2}} \{ \cos z [C(z) + S(z)] - \sin z [C(z) - S(z)] \} = E_{\frac{1}{2}}(z)$$

9. Struve functions

$$H_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{\alpha+2n+1}}{\Gamma(n+3/2) \Gamma(\alpha+n+3/2)} = 2^{1-\alpha} \pi^{-\frac{1}{2}} [\Gamma(\alpha+1/2)]^{-1} s_{\alpha,\alpha}(z)$$

$$L_{\alpha}(z) = -ie^{-i\pi\alpha/2} H_{\alpha}(ze^{i\pi/2})$$

10. Lommel functions

$$s_{\alpha, \beta}(z) = [(\alpha - \beta + 1)(\alpha + \beta + 1)]^{-1} z^{\alpha+1} {}_1F_2\left(1; \frac{\alpha - \beta + 3}{2}, \frac{\alpha + \beta + 3}{2}; -z^2/4\right); \quad \alpha \pm \beta \neq -1, -2, -3, \dots$$

$$S_{\alpha, \beta}(z) = s_{\alpha, \beta}(z) + 2^{\alpha-1} \Gamma\left(\frac{\alpha - \beta + 1}{2}\right) \Gamma\left(\frac{\alpha + \beta + 1}{2}\right) [\sin\left(\frac{\pi\alpha - \pi\beta}{2}\right) J_{\alpha}(z) - \cos\left(\frac{\pi\alpha - \pi\beta}{2}\right) Y_{\alpha}(z)]$$

Special cases of Lommel's functions:

$$s_{\alpha, \alpha}(z) = \pi^{\frac{1}{2}} 2^{\alpha-1} \Gamma(\alpha + 1/2) H_{\alpha}(z)$$

$$S_{\alpha, \alpha}(z) = \pi^{\frac{1}{2}} 2^{\alpha-1} \Gamma(\alpha + 1/2) [H_{\alpha}(z) - Y_{\alpha}(z)]$$

$$s_{0, \beta}(z) = \frac{1}{2} \pi \csc(\pi\beta) [J_{\beta}(z) - J_{-\beta}(z)]$$

$$S_{0, \beta}(z) = \frac{\pi}{2} \csc(\pi\beta) [J_{\beta}(z) - J_{-\beta}(z) - J_{\beta}(z) + J_{-\beta}(z)]$$

$$s_{-1, \beta}(z) = -\frac{\pi}{2} \beta^{-1} \csc(\pi\beta) [J_{\beta}(z) + J_{-\beta}(z)]$$

$$S_{-1, \beta}(z) = \frac{\pi}{2} \beta^{-1} \csc(\pi\beta) [J_{\beta}(z) + J_{-\beta}(z) - J_{\beta}(z) - J_{-\beta}(z)]$$

$$s_{1, \beta}(z) = 1 + \beta^2 s_{-1, \beta}(z); \quad S_{1, \beta}(z) = 1 + \beta^2 S_{-1, \beta}(z)$$

$$S_{\frac{1}{2}, \frac{1}{2}}(z) = z^{-\frac{1}{2}}; \quad S_{\frac{3}{2}, \frac{1}{2}}(z) = z^{\frac{1}{2}}$$

$$S_{-\frac{1}{2}, \frac{1}{2}}(z) = z^{-\frac{1}{2}} [\sin z \operatorname{Ci}(z) - \cos z \operatorname{si}(z)]; \quad S_{-\frac{3}{2}, \frac{1}{2}}(z) = -z^{-\frac{1}{2}} [\sin z \operatorname{si}(z) + \cos z \operatorname{Ci}(z)]$$

$$\lim_{\alpha \rightarrow \beta} [\Gamma(\beta - \alpha)]^{-1} s_{\alpha-1, \beta}(z) = -2^{\beta-1} \Gamma(\beta) J_{\beta}(z)$$

Lommel functions of two variables:

$$U_{\alpha}(w, z) = \sum_{n=0}^{\infty} (-1)^n (w/z)^{\alpha+2n} J_{\alpha+2n}(z)$$

$$V_{\alpha}(w, z) = \cos(\frac{1}{2}w + \frac{1}{2}z^2/w + \frac{1}{2}\alpha\pi) + U_{2-\alpha}(w, z)$$

11. Gauss's hypergeometric function

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}, \quad |z| < 1$$

12. Generalized hypergeometric series

$${}_mF_n(a_1, a_2, \dots, a_m; b_1, b_2, \dots, b_n; z) = \frac{\Gamma(b_1) \cdots \Gamma(b_n)}{\Gamma(a_1) \cdots \Gamma(a_m)} \sum_{k=0}^{\infty} \frac{\Gamma(a_1+k) \cdots \Gamma(a_m+k)}{\Gamma(b_1+k) \cdots \Gamma(b_n+k)} \frac{z^k}{k!}$$

13. Confluent hypergeometric functions

$${}_1F_1(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)} \frac{z^n}{n!}$$

$${}_1F_1(a; a; z) = e^z, \quad {}_1F_1(a; 2a; 2z) = 2^{a-\frac{1}{2}} \Gamma(a + \frac{1}{2}) z^{\frac{1}{2}-a} e^z I_{a-\frac{1}{2}}(z)$$

$${}_1F_1(\frac{1}{2}; \frac{3}{2}; ix) = e^{ix} {}_1F_1(1; \frac{3}{2}; -ix) = (\frac{1}{2}\pi/x)^{\frac{1}{2}} [C(x) + iS(x)]$$

Whittaker's functions:

$$M_{\alpha, \beta}(z) = z^{\beta+\frac{1}{2}} e^{-\frac{1}{2}z} {}_1F_1(\beta-\alpha + \frac{1}{2}; 2\beta+1; z)$$

$$W_{\alpha, \beta}(z) = \frac{\Gamma(-2\beta)}{\Gamma(-\alpha-\beta+\frac{1}{2})} M_{\alpha, \beta}(z) + \frac{\Gamma(2\beta)}{\Gamma(\beta-\alpha+\frac{1}{2})} M_{\alpha, -\beta}(z)$$

Special cases of Whittaker's functions:

$$M_{0, \beta}(z) = \Gamma(1+\beta) 2^{2\beta} I_{\beta}(z/2) \sqrt{z}; \quad W_{0, \beta}(z) = (z/\pi)^{\frac{1}{2}} K_{\beta}(z/2)$$

$$M_{\alpha, 0}(z) = z^{\frac{1}{2}} e^{-\frac{1}{2}z} L_{\alpha-\frac{1}{2}}(z); \quad M_{\frac{1}{4}, \frac{1}{4}}(z) = -i \frac{1}{2} \pi^{\frac{1}{2}} z^{\frac{1}{4}} e^{-\frac{1}{2}z} \text{Erf}(iz^{\frac{1}{2}})$$

Parabolic cylinder function:

$$D_{\alpha}(z) = 2^{(\alpha+\frac{1}{2})/2} z^{-\frac{1}{2}} W_{(\alpha+\frac{1}{2})/2, \frac{1}{4}}(z^2/2)$$

$$D_n(z) = e^{-z^2/4} \text{He}_n(z), \quad n=0, 1, 2, \dots$$

$$D_{-1}(z) = (\pi/2)^{\frac{1}{2}} e^{z^2/4} \text{Erfc}(2^{-\frac{1}{2}}z)$$

$$D_{-\frac{1}{2}}(z) = (\frac{1}{2} z/\pi)^{\frac{1}{2}} K_{\frac{1}{4}}(z^2/4)$$

Error integrals:

$$\operatorname{Erf}(z) = 2\pi^{-\frac{1}{2}} \int_0^z e^{-t^2} dt = 2\pi^{-\frac{1}{2}} {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -z^2\right) = 2(\pi z)^{-\frac{1}{2}} e^{-z^2/2} M_{-\frac{1}{2}, \frac{1}{2}}(z^2)$$

$$\operatorname{Erfc}(z) = 1 - \operatorname{Erf}(z) = 2\pi^{-\frac{1}{2}} \int_z^\infty e^{-t^2} dt = (\pi z)^{-\frac{1}{2}} e^{-z^2/2} W_{-\frac{1}{2}, \frac{1}{2}}(z^2) = \pi^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}, z^2\right)$$

$$\operatorname{Erf}(x^{\frac{1}{2}} e^{i\pi/4}) = 2^{\frac{1}{2}} e^{i\pi/4} [C(x) - i S(x)]$$

$$\operatorname{Erfc}(x^{\frac{1}{2}} e^{i\pi/4}) = 1 - C(x) - S(x) - i[C(x) - S(x)]$$

Fresnel's integrals:

$$C(x) = (2\pi)^{-\frac{1}{2}} \int_0^x t^{-\frac{1}{2}} \cos t dt; \quad S(x) = (2\pi)^{-\frac{1}{2}} \int_0^x t^{-\frac{1}{2}} \sin t dt$$

Exponential integral:

$$-\operatorname{Ei}(-z) = \int_z^\infty t^{-1} e^{-t} dt = -\gamma - \log z - \sum_{n=1}^\infty \frac{(-z)^n}{n \cdot n!} = z^{-\frac{1}{2}} e^{-z/2} W_{-\frac{1}{2}, 0}(z) = \Gamma(0, z),$$

$$-\pi < \arg z < \pi$$

$$\begin{aligned} \overline{\operatorname{Ei}}(x) &= \frac{1}{2} [\operatorname{Ei}(x+i0) + \operatorname{Ei}(x-i0)] = -\text{P.V.} \int_{-x}^\infty t^{-1} e^{-t} dt \\ &= \gamma + \log x + \sum_{n=1}^\infty \frac{x^n}{n \cdot n!}, \quad x > 0 \end{aligned}$$

$$\operatorname{Ei}(-ix) = \operatorname{Ci}(x) - i \operatorname{Si}(x); \quad \overline{\operatorname{Ei}}(ix) = \operatorname{Ci}(x) + i\pi + i \operatorname{Si}(x)$$

Sine and cosine integral:

$$\begin{aligned} \operatorname{Si}(x) &= \int_0^x t^{-1} \sin t dt; \quad \operatorname{si}(x) = - \int_x^\infty t^{-1} \sin t dt = \operatorname{Si}(x) - \frac{\pi}{2} \\ &= \frac{1}{2} [\operatorname{Ei}(-ix) - \operatorname{Ei}(ix)] \end{aligned}$$

$$\operatorname{Ci}(x) = - \int_x^\infty t^{-1} \cos t dt = \frac{1}{2} [\operatorname{Ei}(-ix) + \operatorname{Ei}(ix)] = \gamma + \log x + \sum_{n=1}^\infty \frac{(-1)^n x^{2n}}{2n(2n)!}$$

Incomplete gamma function:

$$\gamma(a, z) = \int_0^z t^{a-1} e^{-t} dt = \frac{1}{a} z^a {}_1F_1(a; a+1; -z)$$

$$\Gamma(a, z) = \Gamma(a) - \gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt = z^{(a-1)/2} e^{-z/2} W_{(a-1)/2, a/2}(z)$$

$$\Gamma(\frac{1}{2}, z^2) = \pi^{\frac{1}{2}} \operatorname{Erfc}(z); \quad \Gamma(0, z) = -\operatorname{Ei}(-z); \quad \gamma(\frac{1}{2}, z^2) = \pi^{\frac{1}{2}} \operatorname{Erf}(z)$$

14. Elliptic integrals and theta functions

Complete elliptic integrals:

$$K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 t)^{-\frac{1}{2}} dt = \frac{\pi}{2} {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; k^2)$$

$$E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 t)^{\frac{1}{2}} dt = \frac{\pi}{2} {}_2F_1(-\frac{1}{2}, \frac{1}{2}; 1; k^2)$$

Theta functions:

$$\theta_1(v, t) = (-it)^{-\frac{1}{2}} \sum_{-\infty}^{\infty} (-1)^n \exp[-i\pi(v + n - \frac{1}{2})^2 t^{-1}]$$

$$\theta_2(v, t) = (-it)^{-\frac{1}{2}} \sum_{-\infty}^{\infty} (-1)^n \exp[-i\pi(v + n)^2 t^{-1}]$$

$$\theta_3(v, t) = (-it)^{-\frac{1}{2}} \sum_{-\infty}^{\infty} \exp[-i\pi(v + n)^2 t^{-1}]$$

$$\theta_4(v, t) = (-it)^{-\frac{1}{2}} \sum_{-\infty}^{\infty} \exp[-i\pi(v + n - \frac{1}{2})^2 t^{-1}]$$

Symbol	Name of the Function	Listed under
$C(x)$	Fresnel's integral	13
$Ci(x)$	Cosine integral	13
$C_n^a(x)$	Gegenbauer's polynomial	2
$D_a(x)$	Parabolic cylinder function	13
$E(k)$	Complete elliptic integral	14
$Ei(-x)$	Exponential integrals	13
$\overline{Ei}(x)$		
$Erf(z)$	Error integrals	13
$Erfc(z)$		
$E_a(z)$	Anger-Weber function	8
${}_mF_n$	Hypergeometric function	11, 12, 13
$He_n(x)$	Hermite's polynomial	2
$H_a^{(1,2)}(x)$	Hankel's functions	6
$H_a(z)$	Struve's function	9
$I_a(z)$	Modified Bessel function	7
$J_a(z)$	Bessel's function	6
$J_a(z)$	Anger-Weber function	8
$K(k)$	Complete elliptic integral	14
$K_a(z)$	Modified Hankel function	7
$L_n^a(x)$	Laguerre's polynomial	2
$L_a(z)$	Struve's function	9
$M_{a,\beta}(z)$	Whittaker's functions	13
$W_{a,\beta}(z)$		
$P_n(x)$	Legendre's polynomials	2
$P_n^{(\alpha,\beta)}(x)$	Jacobi's polynomials	2

Symbol	Name of the Function	Listed under
$P_a^\beta(z)$ $P_a^\beta(x)$	} Legendre functions	5
$Q_a^\beta(z)$ $Q_a^\beta(x)$	} Legendre functions	5
$S(x)$	Fresnel's integral	13
$si(x)$ $Si(x)$	} Sine integrals	13
$s_{a,\beta}(z)$ $S_{a,\beta}(z)$	} Lommel's function	10
$T_n(x)$ $U_n(x)$	} Chebycheff's polynomials	2
$U_a(w,z)$ $V_a(w,z)$	} Lommel's function of two variables	10
$W_{a,\beta}(z)$	Whittaker's function	13
$Y_a(z)$	Neumann's function	6
$B(x,y)$	Beta function	3
$\Gamma(z)$	Gamma function	3
$\Gamma(a,z)$ $\gamma(a,z)$	} Incomplete gamma functions	13
$\psi(z)$	Psi function	3
$\theta_a(v,t)$	Theta functions	14
$\zeta(s)$	Riemann's zeta function	4
$\zeta(s,v)$	Hurwitz's zeta function	4

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