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DATE: 16 October 1957

SUBJECT: Dynamics of a Rigid Bodied Missile
with Propellant Sloshing in Four Tanks

FROM: H. C. Thomson

- REFERENCES:
1. R-W Memo, "Motion of Fluid in a Cylindrical Tank", G. J. Gleghorn, 27 September 1955.
 2. GM 45.3-378, "Fluid Sloshing in Tanks of Arbitrary Shape", N. Trembath, 28 August 1957.
 3. GM-TM-146, "Equations of Motion of Missile with Sloshing," E. Heist, 8 February 1956.

INTRODUCTION

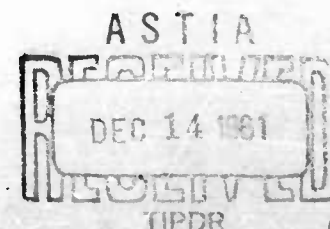
This memo summarizes the results obtained in an analysis of the dynamics of a rigid missile flying with fluid sloshing in four separate tanks, and represents an extension of the work described in Reference 3. The equations of motion are presented in a form suitable for use in programming an analog computer; and they are solved to yield the airframe transfer function for use in stability studies, etc.

DISCUSSION

Consider the missile diagrammed in Figure 1 with the sloshing propellants represented by the mass-spring analogy discussed in reference 1. The forces exerted on the airframe are due to aerodynamics, gravity, engine thrust and reaction forces of each of the first mode sloshing masses. To define the dynamics of the airframe in the pitch (or yaw) plane, fifteen equations are written: forces exerted on the airframe along and perpendicular to the velocity vector, torques exerted on the airframe about the missile (minus sloshing masses) c.g., forces exerted on each of the sloshing masses parallel to and perpendicular to the missile longitudinal axis:

$$m_0 \dot{V} = T \cos(\delta + \psi) - m_0 g \cos \gamma - D \cos \alpha_D + L \sin \alpha_D \\ - (F_{1N} + F_{2N} + F_{3N} + F_{4N}) \cos \psi + (F_1 + F_2 + F_3 + F_4) \sin \psi \quad (1)$$

$$m_0 V \dot{\gamma} = -T \sin(\delta + \psi) + m_0 g \sin \gamma - D \sin \alpha_D - L \cos \alpha_D \\ + (F_{1N} + F_{2N} + F_{3N} + F_{4N}) \sin \psi + (F_1 + F_2 + F_3 + F_4) \cos \psi \quad (2)$$



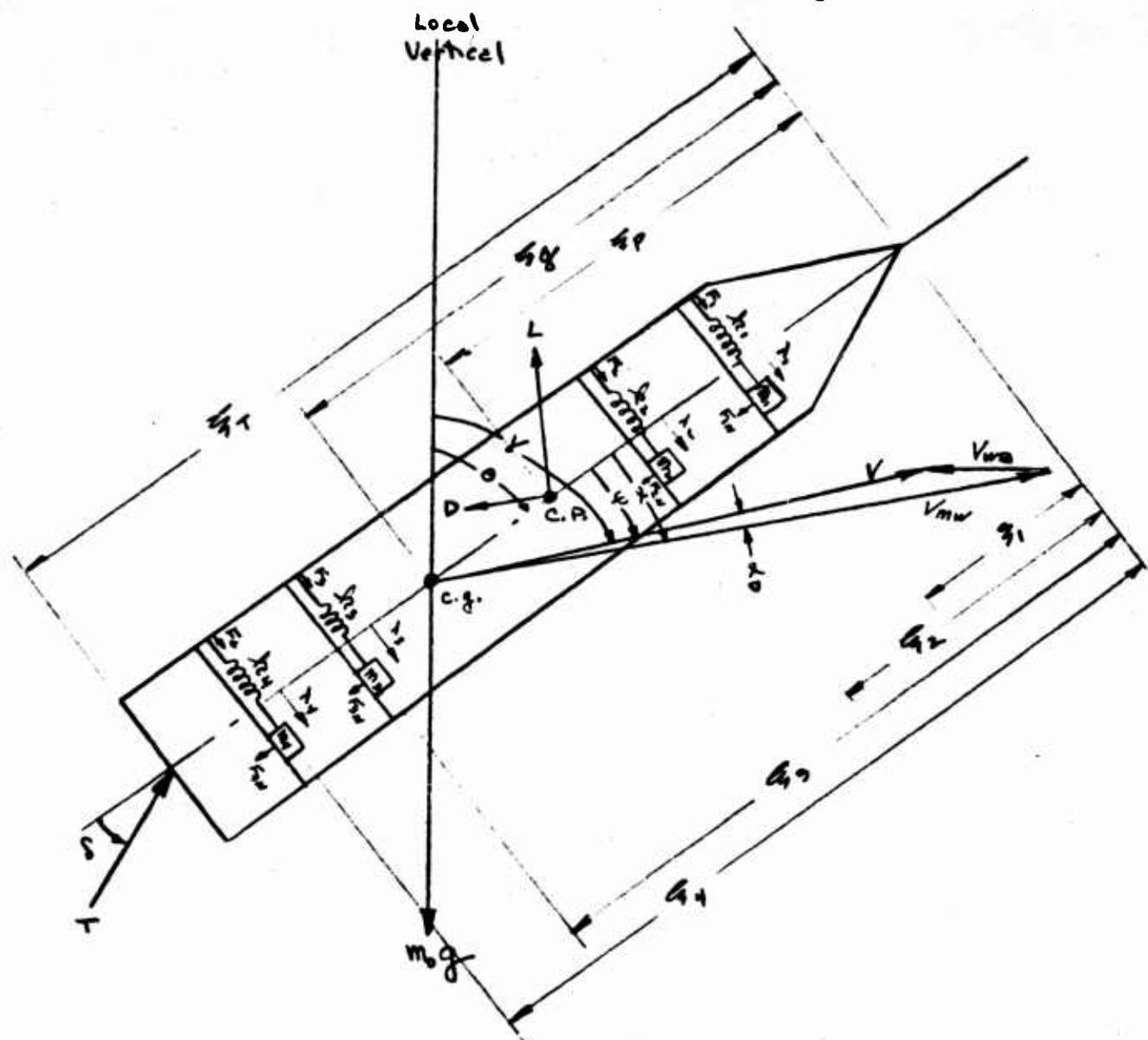


Figure 1
Missile Geometry

$$\begin{aligned} I_0 \ddot{\theta} = & T(\xi_T - \xi_g) \sin \theta - (\xi_g - \xi_p) (L \cos \alpha + D \sin \alpha) \\ & + F_{1N} \lambda_1 + F_{2N} \lambda_2 + F_{3N} \lambda_3 + F_{4N} \lambda_4 + F_1 (\xi_g - \xi_1) \\ & + F_2 (\xi_g - \xi_2) + F_3 (\xi_g - \xi_3) + F_4 (\xi_g - \xi_4) \end{aligned} \quad (3)$$

$$F_1 = m_1 [g \sin \theta - (\xi_g - \xi_1) \ddot{\theta} - \ddot{\lambda}_1 + \lambda_1 \dot{\theta}^2 - a_{cg \text{ lat}}] \quad (4)$$

$$F_1 = \omega_1^2 m_1 \lambda_1 \quad (5)$$

$$F_{1N} = m_1 [g \cos \theta - (\xi_g - \xi_1) \dot{\theta}^2 - 2 \dot{\lambda}_1 \dot{\theta} - \lambda_1 \ddot{\theta} + a_{cg \text{ long}}] \quad (6)$$

where $i = 1, 2, 3, 4$

and $a_{cg \text{ lat.}} = v \ddot{\gamma} \cos \psi + \dot{v} \sin \psi$

$a_{cg \text{ long.}} = -v \ddot{\gamma} \sin \psi + \dot{v} \cos \psi$

From these equations we may deduce the linearized equations of motion by making the usual small angle trigonometric approximations and by dropping non-linear terms.

$$\dot{v} = \frac{T-D}{M} - g \cos \gamma - \frac{L}{M} \alpha_D \quad (7)$$

$$\begin{aligned} v \ddot{\gamma} = & -\frac{T}{M} \theta - \frac{T-D}{M} \psi + g \sin \gamma - \frac{\mu_r}{\mu_c} \frac{T}{M} \alpha \\ & - \frac{m_1}{M} [(\xi_g - \xi_1) \ddot{\theta} + \ddot{\lambda}_1] - \frac{m_2}{M} [(\xi_g - \xi_2) \ddot{\theta} + \ddot{\lambda}_2] \\ & - \frac{m_3}{M} [(\xi_g - \xi_3) \ddot{\theta} + \ddot{\lambda}_3] - \frac{m_4}{M} [(\xi_g - \xi_4) \ddot{\theta} + \ddot{\lambda}_4] \end{aligned} \quad (8)$$

$$\begin{aligned} \ddot{\theta} = & \mu_c \theta - \mu_r \alpha + \frac{m_1 \lambda_1}{I_0} \left\{ \frac{T-D}{M} + \omega_1^2 (\xi_g - \xi_1) \right\} \\ & + \frac{m_2 \lambda_2}{I_0} \left\{ \frac{T-D}{M} + \omega_2^2 (\xi_g - \xi_2) \right\} + \frac{m_3 \lambda_3}{I_0} \left\{ \frac{T-D}{M} + \omega_3^2 (\xi_g - \xi_3) \right\} \\ & + \frac{m_4 \lambda_4}{I_0} \left\{ \frac{T-D}{M} + \omega_4^2 (\xi_g - \xi_4) \right\} \end{aligned} \quad (9)$$

$$\begin{aligned}
\lambda_1 \omega_1^2 = & -(\xi_g - \xi_1) \ddot{\theta} + \ddot{\lambda}_1 + \frac{T}{M} \delta + \frac{\mu_r}{\mu_c} \frac{T}{M} \alpha \\
& + \frac{m_1}{M} \left[(\xi_g - \xi_1) \ddot{\theta} + \ddot{\lambda}_1 \right] + \frac{m_2}{M} \left[(\xi_g - \xi_2) \ddot{\theta} + \ddot{\lambda}_2 \right] \\
& + \frac{m_3}{M} \left[(\xi_g - \xi_3) \ddot{\theta} + \ddot{\lambda}_3 \right] + \frac{m_4}{M} \left[(\xi_g - \xi_4) \ddot{\theta} + \ddot{\lambda}_4 \right]
\end{aligned} \tag{10}$$

$i = 1, 2, 3, 4$

where we define

$$\mu_c = \frac{T(\xi_T - \xi_g)}{I_o}$$

$$\mu_r = \frac{(\frac{L}{\alpha} + D)(\xi_g - \xi_p)}{I_o} = \frac{C_N \alpha A_q}{I_o} (\xi_g - \xi_p)$$

Equations 7 through 10 are those suitable for use in the programming of an analog computer.

In order to derive the engine angle to body angle transfer function in terms of the Laplace transform, the following procedure is used. Perturbation theory is employed to write the perturbed equations of motion from equations 7-10, considering $T, D, M, g, L, V, m_i, \xi_i, \xi_g, \mu_c, \mu_r, I_o$ as constants and all other quantities as variables. α_D is then set equal to zero (no wind) and one deduces that γ must also be zero.

Thus

$$\theta = -\alpha$$

Using this and transforming the equations, the following relations are arrived at:

$$\begin{aligned}
(s^2 - \mu_r) \theta(s) = & \mu_c \delta(s) + \frac{m_1 \lambda_1(s)}{I_o} \left\{ \frac{T-D}{M} + \omega_1^2 (\xi_g - \xi_1) \right\} \\
& + \frac{m_2 \lambda_2(s)}{I_o} \left\{ \frac{T-D}{M} + \omega_2^2 (\xi_g - \xi_2) \right\} \\
& + \frac{m_3 \lambda_3(s)}{I_o} \left\{ \frac{T-D}{M} + \omega_3^2 (\xi_g - \xi_3) \right\} \\
& + \frac{m_4 \lambda_4(s)}{I_o} \left\{ \frac{T-D}{M} + \omega_4^2 (\xi_g - \xi_4) \right\}
\end{aligned} \tag{11}$$

$$\begin{aligned}
 (s^2 + \omega_1^2) \lambda_1(s) &= \frac{T}{M} \delta(s) + \left[m_1 \lambda_1(s) + m_2 \lambda_2(s) + m_3 \lambda_3(s) + m_4 \lambda_4(s) \right] \frac{s^2}{M} \\
 &+ \left\{ \frac{m_1}{M} (\xi_g - \xi_1) + \frac{m_2}{M} (\xi_g - \xi_2) + \frac{m_3}{M} (\xi_g - \xi_3) + \frac{m_4}{M} (\xi_g - \xi_4) - (\xi_g - \xi_1) \right\} \times \\
 &s^2 - \frac{\mu_r}{\mu_c} \frac{T}{M} \theta(s)
 \end{aligned} \quad (12)$$

$i = 1, 2, 3, 4$

Expansion of (12) and solution of these along with (11) by determinants results in the following transfer function

$$\frac{\theta}{\delta}(s) = \frac{a_8 s^8 + a_6 s^6 + a_4 s^4 + a_2 s^2 + a_0}{b_{10} s^{10} + b_8 s^8 + b_6 s^6 + b_4 s^4 + b_2 s^2 + b_0} \quad (13)$$

where

$$a_8 = \mu_c (1 + c_1 + c_2 + c_3 + c_4)$$

$$\begin{aligned}
 a_6 = \mu_c & \left[\omega_1^2 (1 + c_2 + c_3 + c_4) + \omega_2^2 (1 + c_1 + c_3 + c_4) \right. \\
 & + \omega_3^2 (1 + c_1 + c_2 + c_4) + \omega_4^2 (1 + c_1 + c_2 + c_3) \left. \right] \\
 & - g_m (B_1 + B_2 + B_3 + B_4)
 \end{aligned}$$

$$\begin{aligned}
 a_4 = \mu_c & \left[\omega_1^2 \omega_2^2 (1 + c_3 + c_4) + \omega_1^2 \omega_3^2 (1 + c_2 + c_4) + \omega_1^2 \omega_4^2 (1 + c_2 + c_3) \right. \\
 & + \omega_2^2 \omega_3^2 (1 + c_1 + c_4) + \omega_2^2 \omega_4^2 (1 + c_1 + c_3) + \omega_3^2 \omega_4^2 (1 + c_1 + c_2) \left. \right] \\
 & - g_m \left[\omega_1^2 (B_2 + B_3 + B_4) + \omega_2^2 (B_1 + B_3 + B_4) \right. \\
 & + \omega_3^2 (B_1 + B_2 + B_4) + \omega_4^2 (B_1 + B_2 + B_3) \left. \right]
 \end{aligned}$$

$$\begin{aligned}
 a_2 = \mu_c & \left[\omega_1^2 \omega_2^2 \omega_3^2 (1 + c_4) + \omega_2^2 \omega_3^2 \omega_4^2 (1 + c_1) \right. \\
 & + \omega_1^2 \omega_2^2 \omega_4^2 (1 + c_3) + \omega_1^2 \omega_3^2 \omega_4^2 (1 + c_2) \left. \right] \\
 & - g_m \left[\omega_1^2 \omega_2^2 (B_3 + B_4) + \omega_1^2 \omega_3^2 (B_2 + B_4) + \omega_1^2 \omega_4^2 (B_2 + B_3) \right. \\
 & + \omega_2^2 \omega_3^2 (B_1 + B_4) + \omega_2^2 \omega_4^2 (B_1 + B_3) + \omega_3^2 \omega_4^2 (B_1 + B_2) \left. \right]
 \end{aligned}$$

$$a_0 = \mu_c \omega_1^2 \omega_2^2 \omega_3^2 \omega_4^2$$

$$-g_m \left[B_1 \omega_2^2 \omega_3^2 \omega_4^2 + B_2 \omega_1^2 \omega_3^2 \omega_4^2 \right. \\ \left. + B_3 \omega_1^2 \omega_2^2 \omega_4^2 + B_4 \omega_1^2 \omega_2^2 \omega_3^2 \right]$$

$$b_{10} = 1 + C_1 + C_2 + C_3 + C_4$$

$$b_8 = (A_1 - A_4) (C_1 B_4 - B_1 C_4) + (A_1 - A_2) (C_1 B_2 - B_1 C_2) \\ + (A_1 - A_3) (C_1 B_3 - B_1 C_3) + (A_2 - A_3) (C_2 B_3 - B_2 C_3) \\ + (A_2 - A_4) (C_2 B_4 - B_2 C_4) + (A_3 - A_4) (C_3 B_4 - B_3 C_4) \\ - A_1 B_1 - A_2 B_2 - A_3 B_3 - A_4 B_4 + (1 + C_2 + C_3 + C_4) \omega_1^2 \\ + (1 + C_1 + C_3 + C_4) \omega_2^2 + (1 + C_1 + C_2 + C_4) \omega_3^2 \\ + (1 + C_1 + C_2 + C_3) \omega_4^2 - \mu_r (1 + C_1 + C_2 + C_3 + C_4)$$

$$b_6 = (A_1 - A_4) (C_1 B_4 - B_1 C_4) (\omega_2^2 + \omega_3^2) + (A_1 - A_2) (C_1 B_2 - B_1 C_2) (\omega_3^2 + \omega_4^2) \\ + (A_1 - A_3) (C_1 B_3 - B_1 C_3) (\omega_2^2 + \omega_4^2) + (A_2 - A_3) (C_2 B_3 - B_2 C_3) (\omega_1^2 + \omega_4^2) \\ + (A_2 - A_4) (C_2 B_4 - B_2 C_4) (\omega_1^2 + \omega_3^2) + (A_3 - A_4) (C_3 B_4 - B_3 C_4) (\omega_1^2 + \omega_2^2) \\ - (A_1 B_1 + \mu_r C_1) (\omega_2^2 + \omega_3^2 + \omega_4^2) - (A_2 B_2 + \mu_r C_2) (\omega_1^2 + \omega_3^2 + \omega_4^2) \\ - (A_3 B_3 + \mu_r C_3) (\omega_1^2 + \omega_2^2 + \omega_4^2) - (A_4 B_4 + \mu_r C_4) (\omega_1^2 + \omega_2^2 + \omega_3^2) \\ + \omega_1^2 \omega_2^2 (1 + C_3 + C_4) + \omega_1^2 \omega_3^2 (1 + C_2 + C_4) + \omega_1^2 \omega_4^2 (1 + C_2 + C_3) \\ + \omega_2^2 \omega_3^2 (1 + C_1 + C_4) + \omega_2^2 \omega_4^2 (1 + C_1 + C_3) + \omega_3^2 \omega_4^2 (1 + C_1 + C_2) \\ - \mu_r (\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2) + \frac{\mu_r}{\mu_c} g_m (B_1 + B_2 + B_3 + B_4)$$

$$b_4 = \left[(A_1 - A_4) (C_1 B_4 - B_1 C_4) - A_1 B_1 - A_4 B_4 - \mu_r (1 + C_1 + C_4) \right] \omega_2^2 \omega_3^2 \\ + \left[(A_1 - A_2) (C_1 B_2 - B_1 C_2) - A_1 B_1 - A_2 B_2 - \mu_r (1 + C_1 + C_2) \right] \omega_3^2 \omega_4^2 \\ + \left[(A_1 - A_3) (C_1 B_3 - B_1 C_3) - A_1 B_1 - A_3 B_3 - \mu_r (1 + C_1 + C_3) \right] \omega_2^2 \omega_4^2$$

$$\begin{aligned}
& + \left[(A_2 - A_3)(C_2 B_3 - B_2 C_3) - A_2 B_2 - A_3 B_3 - \mu_r(1 + C_2 + C_3) \right] \omega_1^2 \omega_4^2 \\
& + \left[(A_2 - A_4)(C_2 B_4 - B_2 C_4) - A_2 B_2 - A_4 B_4 - \mu_r(1 + C_2 + C_4) \right] \omega_1^2 \omega_3^2 \\
& + \left[(A_3 - A_4)(C_3 B_4 - B_3 C_4) - A_3 B_3 - A_4 B_4 - \mu_r(1 + C_3 + C_4) \right] \omega_1^2 \omega_2^2 \\
& + (1 + C_1) \omega_2^2 \omega_3^2 \omega_4^2 + (1 + C_2) \omega_1^2 \omega_3^2 \omega_4^2 \\
& + (1 + C_3) \omega_1^2 \omega_2^2 \omega_4^2 + (1 + C_4) \omega_1^2 \omega_2^2 \omega_3^2 \\
& + \frac{\mu_r}{\mu_c} g_m \left[B_1 (\omega_2^2 + \omega_3^2 + \omega_4^2) + B_2 (\omega_1^2 + \omega_3^2 + \omega_4^2) \right. \\
& \quad \left. + B_3 (\omega_1^2 + \omega_2^2 + \omega_4^2) + B_4 (\omega_1^2 + \omega_2^2 + \omega_3^2) \right]
\end{aligned}$$

$$\begin{aligned}
b_2 = & \omega_1^2 \omega_2^2 \omega_3^2 \omega_4^2 - \left[A_1 B_1 + \mu_r(1 + C_1) \right] \omega_2^2 \omega_3^2 \omega_4^2 \\
& - \left[A_2 B_2 + \mu_r(1 + C_2) \right] \omega_1^2 \omega_3^2 \omega_4^2 - \left[A_3 B_3 + \mu_r(1 + C_3) \right] \omega_1^2 \omega_2^2 \omega_4^2 \\
& - \left[A_4 B_4 + \mu_r(1 + C_4) \right] \omega_1^2 \omega_2^2 \omega_3^2 + \frac{\mu_r}{\mu_c} g_m \left[(B_1 + B_4) \omega_2^2 \omega_3^2 \right. \\
& + (B_1 + B_3) \omega_2^2 \omega_4^2 + (B_1 + B_2) \omega_3^2 \omega_4^2 + (B_2 + B_4) \omega_1^2 \omega_3^2 \\
& + (B_2 + B_3) \omega_1^2 \omega_4^2 + (B_3 + B_4) \omega_1^2 \omega_2^2 \left. \right]
\end{aligned}$$

$$\begin{aligned}
b_0 = & -\mu_r \omega_1^2 \omega_2^2 \omega_3^2 \omega_4^2 \\
& + \frac{\mu_r}{\mu_c} g_m \left[B_1 \omega_2^2 \omega_3^2 \omega_4^2 + B_2 \omega_1^2 \omega_3^2 \omega_4^2 \right. \\
& \quad \left. + B_3 \omega_1^2 \omega_2^2 \omega_4^2 + B_4 \omega_1^2 \omega_2^2 \omega_3^2 \right]
\end{aligned}$$

$$A_1 = -\left(\frac{m_1}{M} - 1\right)(\xi_g - \xi_1) - \frac{m_2}{M}(\xi_g - \xi_2) - \frac{m_3}{M}(\xi_g - \xi_3) - \frac{m_4}{M}(\xi_g - \xi_4)$$

$$A_2 = -\frac{m_1}{M}(\xi_g - \xi_1) - \left(\frac{m_2}{M} - 1\right)(\xi_g - \xi_2) - \frac{m_3}{M}(\xi_g - \xi_3) - \frac{m_4}{M}(\xi_g - \xi_4)$$

$$A_3 = -\frac{m_1}{M}(\xi_g - \xi_1) - \frac{m_2}{M}(\xi_g - \xi_2) - \left(\frac{m_3}{M} - 1\right)(\xi_g - \xi_3) - \frac{m_4}{M}(\xi_g - \xi_4)$$

$$A_4 = -\frac{m_1}{M}(\xi_g - \xi_1) - \frac{m_2}{M}(\xi_g - \xi_2) - \frac{m_3}{M}(\xi_g - \xi_3) - \left(\frac{m_4}{M} - 1\right)(\xi_g - \xi_4)$$

$$B_1 = - \frac{m_1}{I_0} \left\{ \frac{T-D}{M} + \omega_1^2 (\xi_g - \xi_1) \right\}$$

$$B_2 = - \frac{m_2}{I_0} \left\{ \frac{T-D}{M} + \omega_2^2 (\xi_g - \xi_2) \right\}$$

$$B_3 = - \frac{m_3}{I_0} \left\{ \frac{T-D}{M} + \omega_3^2 (\xi_g - \xi_3) \right\}$$

$$B_4 = - \frac{m_4}{I_0} \left\{ \frac{T-D}{M} + \omega_4^2 (\xi_g - \xi_4) \right\}$$

$$C_1 = - \frac{m_1}{M}$$

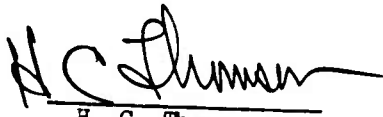
$$C_2 = - \frac{m_2}{M}$$

$$C_3 = - \frac{m_3}{M}$$

$$C_4 = - \frac{m_4}{M}$$

$$\xi_m = \frac{T}{M}$$

Note that equation (13) may be reduced to the two tank case by letting the sloshing masses and frequencies associated with those tanks be zero (for example, $m_3 = m_4 = \omega_3 = \omega_4 = 0$). Further, the effects of aerodynamics may be eliminated by letting $\mu_r = 0$.


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GLOSSARY OF SYMBOLS

A	Aerodynamic force reference area.
$a_{c.g.}$	Acceleration of center-of-gravity of airframe, less sloshing masses, with respect to earth reference.
c.g.	Center-of-gravity of missile airframe less sloshing masses.
c.p.	Center-of-pressure of aerodynamic forces acting on missile.
$c_{N_{\alpha}}$	Rate of change of normal force coefficient with respect to angle of attack.
D	Missile profile drag.
F	Force on missile wall due to extension or contraction of spring.
F_N	Normal force exerted on rail by sloshing mass.
g	Acceleration due to gravity.
I_o	Moment of inertia of missile less sloshing masses about axis perpendicular to center line and passing through center-of-gravity.
k	Spring constant ($\equiv \frac{F}{\Delta}$).
L	Lift force, component of total aerodynamic force which is perpendicular to local wind velocity.
M	Total mass of missile including sloshing masses.
m_o	mass of missile less sloshing masses.
m_1	Sloshing mass number 1.
m_2	Sloshing mass number 2.
m_3	Sloshing mass number 3.
m_4	Sloshing mass number 4.
q	Dynamic pressure ($\equiv \frac{\rho}{2} v^2$).
S	Differential operator $\frac{d}{dt}$.
T	Thrust.
V	Velocity of missile with respect to earth reference.
V_{WE}	Velocity of wind with respect to earth.
V_{MW}	Velocity of missile with respect to wind.
α	Missile angle of attack.
α_D	Angle between velocity of missile with respect to earth and velocity of missile with respect to wind.
γ	Missile trajectory angle with respect to vertical.
δ	Motor deflection Angle.
θ	Missile attitude angle with respect to vertical.
λ	Sloshing mass displacement with respect to missile centerline.
ξ	Station number measured from missile nose.
ψ	Angle between missile centerline and velocity of missile with respect to earth.
ω	Natural frequency of mass-spring combination ($\equiv \sqrt{k/m}$).

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