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AN ISOPERIMETRIC EQUALITY; AND RELATED QUESTIONS

by

Robert Finn

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Introduction.

Let S be a piece of smooth two-dimensional surface, and let Γ be a closed curve which bounds a simple region \mathcal{G} on S . Let $\mathcal{L}(\Gamma)$ and $\mathcal{A}(\mathcal{G})$ be, respectively, the length and area of Γ and \mathcal{G} . A. Huber has proved in [1] the general inequality

$$(1) \quad \frac{\mathcal{L}^2(\Gamma)}{\mathcal{A}(\mathcal{G})} \geq 4\pi - 2\mu^+$$

where μ^+ is the surface integral of the Gaussian curvature K of S , evaluated over that part of \mathcal{G} where $K > 0$.

Equality holds in (1) if and only if $K \equiv 0$ in \mathcal{G} and Γ is a geodesic circle; thus the estimate is not a sharp one unless the Gaussian curvature vanishes identically on S .

One might suspect at first glance that (1) would be improved on replacing μ^+ by μ , the total curvature or curvatura integra of \mathcal{G} . The inequality obtained in this way would, however, be incorrect. For example, we may choose for Γ a unit circle on a plane, and then deform the plane interior to the concentric subcircle of radius $1/2$. This does not affect μ , but the deformation can always be arranged to make the left side of (1) as small as desired. Neither can one assert in any sense the existence, in general, of curves for which $\frac{\mathcal{L}^2}{\mathcal{A}} \leq 4\pi - 2\mu$, since $\mu = \mu^+$ on any surface of non-negative curvature.

Nevertheless, the quantity $4\pi - 2\mu$ does in some cases occur in a natural way in connection with a relation of the form (1).

To make the idea clear, consider a circular cone of vertex half-angle α , and a circle of radius r on the cone. We compute easily, $\mathcal{L}(r) = 2\pi r$, $\mathcal{A}(r) = \pi r^2 \csc \alpha$, $\mu(r) = \mu = 2\pi(1 - \sin \alpha)$, and hence $\frac{\mathcal{L}^2(r)}{\mathcal{A}(r)} = 4\pi - 2\mu$. This does not contradict Huber's theorem, as the surface is singular at the vertex. If the cone is smoothed out near this point, we obtain

$$(2) \quad \frac{\mathcal{L}^2(r)}{\mathcal{A}(r)} = 4\pi - 2\mu + o(1)$$

as $r \rightarrow \infty$. Since this estimate is obviously independent of how the smoothing is accomplished, we see that a result of this type can be expected to hold also for surfaces on which the curvature is of mixed sign.

We shall show in this paper that if "circles" on the surface are defined with respect to global conformal parameters on S , then an estimate of the form (2) is true for every complete simply-connected open surface on which the curvature is absolutely integrable. If certain smoothness hypotheses are satisfied by S at infinity, then these "circles" become, up to terms of negligible order, curves which are equidistant from a fixed point on S . In this sense, the image of a large circle in a plane on which S is represented conformally will be a "circle" on S . As corollaries of the method, asymptotic estimates for the lengths of the images of circles and of radial lines in such a reference plane are given, depending only on the curvature integral. Under some conditions, estimates above or below for the local stretching in the mapping can also be given. Also, new proofs are obtained, in the

case considered here, of theorems of Cohn-Vossen [2], Blanc and Fiala [3], and Huber [4].

The demonstrations have turned out to be considerably more intricate than I had at first anticipated, for the results can be obtained relatively easily when the curvature vanishes outside a compact subset of S . It seems, however, desirable to present the material in the generality imposed by the most natural geometrical assumptions.

It is hoped that the methods of this paper will find application also in other problems related to the geometry of two-dimensional surfaces. One possibility would be to study again some of the problems considered by Huber in [4] with a view to providing new and perhaps simpler demonstrations. We have, however, not carried out this program.

I should like to thank my colleagues, Professors T. Frankel and C. Loewner, for many stimulating conversations. I am indebted particularly to Professor P. Malliavin for a suggestion which has led to a considerable improvement of my original results.

1. Preliminary remarks; notation and definitions:

By an abstract surface S we shall mean an open Riemannian manifold whose metric is defined in terms of local parameters ξ, η by a positive definite quadratic form

$$(3) \quad ds^2 = E(\xi, \eta) d\xi^2 + 2F(\xi, \eta) d\xi d\eta + G(\xi, \eta) d\eta^2 .$$

Such a surface is in particular a Riemann surface with angle defined locally by the given metric. We shall assume that S is simply connected,

and in this case S can be mapped conformally, in the metric (3), either onto the unit disc $|z| < 1$ (hyperbolic case) or else onto the entire z -plane (parabolic case). We write

$$ds^2 = e^{2u(x,y)} (dx^2 + dy^2) = e^{2u(z)} |dz|^2 = \lambda^2(z) |dz|^2 .$$

Here we have used z , as we shall throughout this paper, to denote both the complex variable $x + iy$ and the pair of numbers (x, y) .

From a knowledge of E, F, G as functions of local parameters one can, by the Theorema Egregium, calculate the Gaussian curvature K at each point of S . In terms of the conformal parameters z , this relation takes the form

$$K = -e^{-2u} \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] = -e^{-2u} \Delta u$$

Thus, the curvatura integra, or surface integral of the Gaussian curvature, over some region \mathcal{G} on S , is expressed by

$$\mu(\mathcal{G}) = - \int \Delta u \, dx dy$$

the integration being taken over the corresponding region in the z -plane. This relation defines a mass distribution or measure $\mu(\mathcal{G})$ over S , such that the total mass $\mu(S) = \mu$ is the curvatura integra of S . We shall suppose that the positive and negative mass associated with S are individually finite, thus

$$T = \int_S |d\mu| = \int_z |\Delta u| \, dx dy < \infty .$$

We shall consider surfaces S which are complete in the sense of Hopf-Rinow [5], that is, every divergent path on S has infinite length. A path on S is said to be divergent if it is the topological image $p = p(t)$ of a half-open interval $0 \leq t < 1$, and if $p(t)$ lies outside any given compact set on S for all t sufficiently close to 1. Under our assumptions, we shall provide new proofs of the known facts, that $\mu \leq 2\pi$ and S is conformally parabolic.

In order to minimize technical complications, we assume throughout this paper that all entering functions are sufficiently regular to justify the operations which are performed. This will be guaranteed, for example, if $E, F,$ and G are in the Hölder class $C^{2+\alpha}$ with respect to suitable parameters. Most of our results are however independent of smoothness considerations, and hence will be true for all surfaces whose curvature is defined and integrable, and for which $E, F, G,$ can be approximated by smooth functions such that the mappings converge, and such that the integral curvature over an arbitrary open set converges in $L_1,$ to the given one. Such surfaces include, for example, polyhedra in which the curvature is concentrated at isolated singular points.

Throughout this paper the symbols A and C are used to denote constants, the value of which may change even within a given context. Thus, from $|z| < A(1 + |\xi|)$ we may conclude $|z| < A|\xi|$ for $|\xi| > 1$. The symbols ϵ and δ will be used in a similar sense, but will usually represent small quantities.

2. A theorem on conformal type:

The following result is contained in Huber [4, Theorem 15]. In the case required for this paper, it is possible to give a relatively simple

demonstration, which we present below.

Theorem 2.1: If an open simply-connected Riemann surface S admits a complete conformal metric $e^{u(z)}|dz|$, and if $T < \infty$, then S is parabolic.

Proof: If S were not parabolic, it could be represented conformally on the unit disc $\Sigma : |z| < 1$. Corresponding to the subcircle $\Sigma_r : |z| < r$ we would have the representation, valid whenever $|z| < r$,

$$u(z) = - \int_{\Sigma_r} g(z; \zeta) d\mu(\zeta) + h_r(z)$$

where $\mu(\zeta)$ is the associated measure, $g(z; \zeta)$ is the Green's function for the unit disc, and $h_r(z)$ is harmonic. Since by assumption,

$T = \int_{\Sigma} |d\mu(\zeta)| < \infty$, the integrals converge, uniformly in any Σ_{r_0} , as $r \rightarrow 1$. Since $u(z)$ does not depend on r , we conclude that

$h_r(z) \rightarrow h(z)$, again uniformly in any Σ_{r_0} , where $h(z)$ is harmonic.

Thus,

$$u(z) = - \int_{\Sigma} g(z; \zeta) d\mu(\zeta) + h(z) .$$

This relation contradicts the assumed completeness of the metric.

For consider ^{1/}the analytic function

$$w = F(z) = \int_0^z e^{h(z)+i\hat{h}(z)} dz$$

defined in Σ , where $\hat{h}(z)$ is a harmonic function conjugate to $h(z)$

in Σ . $F(z)$ maps the unit disc onto an unbranched Riemann surface over the w -plane, and takes $z = 0$ into the origin 0 in some sheet of the surface. Consider now all rays which originate at 0 in the w -plane. Not every such ray can be extended to infinity in the Riemann surface, for this would imply that the entire plane is a sheet of the surface, and the inverse mapping would determine a bounded analytic function over the plane. Hence at least one such ray γ_w meets the boundary at a finite point P . Denote by γ_z the inverse image of γ_w . Then,

$$\int_{\gamma_z} e^{u(z)} |dz| = \int_{\gamma_z} e^{v(z)} e^{h(z)} |dz| = \int_{\gamma_w} e^{v[z(w)]} |dw| ,$$

where we have set $v(z) = u(z) - h(z)$. We shall show that under the hypotheses, the last integral on the right is finite, thus establishing the existence on S of a divergent path of finite length. To do this, we study the function

$$v(z) = - \int_{\Sigma} g(z; \zeta) d\mu(\zeta) = - \int_{\Gamma} \mathcal{G}(w; \omega) dm(\omega) = v[z(w)] = V(w)$$

where $\mathcal{G}(w; \omega)$ is the Green's function for the image Γ of Σ , and $dm(\omega)$ is the corresponding mass distribution over Γ . We shall need two preliminary results, for which we postpone the demonstrations:

Lemma 2.1: There exists a finite constant C , depending only on Γ , on P , and on δ , (δ small), such that whenever w and ω have distances respectively less than $\delta/2$ and δ from P , then

$$|\mathcal{G}(w; \omega)| < \frac{1}{2\pi} \left[\log \frac{1}{r_{w\omega}} + C \right].$$

Lemma 2.2: There exists a finite constant C , depending only on Γ , on P , and on δ , such that whenever $|w - P| < \delta/4$ and $|\omega - P| > \delta$, then $|\mathcal{G}(w; \omega)| < C$.

To complete the proof, choose $\delta < \frac{1}{2}$ and so small that the total mass $\int_{\Sigma_\delta} |dm(\omega)|$ interior to the circle Σ_δ of radius δ about P is smaller than 2π . Denote the part of γ_w which lies interior to $\Sigma_{\delta/2}$ by γ , and let $\alpha(M)$ be the measure of that set E_M on γ , in which $|v(w)| > M$. We then have, by the above lemmas

$$M \alpha(M) < \int_{E_M} |v| |dw| < \frac{1}{2\pi} \int_{\Sigma_\delta} \left\{ \int_{E_M} (C + \log \frac{1}{r_{w\omega}}) |dw| \right\} |dm(\omega)| \\ + CT \alpha(M)$$

with, as before, $T = \int_{\Gamma} |dm(\omega)| = \int_{\Sigma} |d\mu(\zeta)|$. Now the integral in brackets on the right is clearly maximized when the set E_M is an interval containing ω at its mid point. Thus,

$$\int_{E_M} (C + \log \frac{1}{r_{w\omega}}) |dw| \leq (1 + C) \alpha(M) + \alpha(M) \log \frac{1}{2\alpha(M)}$$

so that, letting $\beta = \frac{1}{2\pi} \int_{\Sigma_\delta} |dm(\omega)|$, ($\beta < 1$), we have

$$M < C + \beta \log \frac{1}{\alpha(M)}$$

for some (new) constant C . Hence

$$\alpha(M) < C e^{-\frac{1}{\beta} M}$$

ad $M \rightarrow \infty$. Therefore, we may write

$$\int_{\gamma} e^{V(w)} |dw| = \int e^M |d\alpha(M)| \\ \leq C + \int_0^{\infty} e^{(1-\frac{1}{\beta})M} dM < \infty$$

and we conclude that the length of the path γ_2 is in the given metric finite.

It remains only to prove the Lemmas 2.1 and 2.2. We observe first that^{2/}

$$\mathcal{G}(w; \omega) = \frac{1}{2\pi} \log \frac{1}{r_{w\omega}} + \alpha_1(w; \omega) + \alpha_2(w; \omega)$$

where

$$\alpha_1 = \begin{cases} -\frac{1}{2\pi} \log \frac{1}{r} & \text{on } \gamma_1 \\ 0 & \text{on } \gamma_2, \end{cases} \quad \alpha_2 = \begin{cases} 0 & \text{on } \gamma_1 \\ -\frac{1}{2\pi} \log \frac{1}{r} & \text{on } \gamma_2, \end{cases}$$

γ_1 and γ_2 being the parts of the boundary of Π lying, respectively, interior and exterior to Σ_{δ} . Thus, since $\delta < 1/2$,

$$0 < \mathcal{G}(w; \omega) < \frac{1}{2\pi} \log \frac{1}{r_{w\omega}} + \alpha_2(w; \omega).$$

For fixed $w \in \Sigma_{\delta/2} \cap \Pi$, $\alpha_2(w; \omega)$ is harmonic and bounded in $\Sigma_{\delta} \cap \Pi$, $|\alpha_2(w; \omega)| < A < \infty$ in this region. By the mean value theorem, the change in boundary data for $\alpha_2(w; \omega)$ as w moves in $\Sigma_{\delta/2}$ does not exceed $\frac{\delta}{r_{Pw} - \delta/2}$, hence by the maximum principle for harmonic

functions, $|\alpha_2(w; \omega)| < A + 2$ in $\Sigma_\delta \cap \Pi$, uniformly for all w in $\Sigma_{\delta/2} \cap \Pi$. This proves Lemma 2.1.

From the above discussion we see that $|\alpha_2(w; \omega)| \leq A + 2$ if $w \in \Sigma_{\delta/2} \cap \Pi$, and ω lies on the boundary Γ_δ of Σ_δ . Thus $\mathcal{G}(w; \omega)$ is uniformly bounded on Γ_δ . Since $\mathcal{G}(w; \omega) = 0$ on the boundary of Π and, for the given choice of w , is harmonic in $(R - \Sigma_\delta) \cap \Pi$, (R being the whole space), Lemma 2.2 follows similarly from the maximum principle. This completes the proof of Theorem 2.1.

3. Estimates of length and area:

We derive first estimates of these quantities from below, which are valid for arbitrary surfaces. Let the surface S be represented conformally on a plane region which includes the disc $\Sigma_r: |z| \leq r$. Let Γ_r be the boundary of Σ_r and let $\mathcal{L}(r)$ and $\mathcal{A}(r)$ be, respectively, the length of the image γ_r of Γ_r on S and the area bounded by γ_r . Let $e^{u(z)}$ be the local length ratio in the mapping.

Theorem 3.1: Under the above hypotheses, there always holds

$$\mathcal{L}(r) \geq 2\pi e^{u_0} r \exp \left\{ -\frac{1}{2\pi} \int_0^r \frac{\mu(\Sigma_\rho)}{\rho} d\rho \right\}$$

$$\mathcal{A}(r) \geq 2\pi e^{2u_0} \int_0^r \rho \exp \left\{ -\frac{1}{\pi} \int_0^\rho \frac{\mu(\Sigma_\tau)}{\tau} d\tau \right\} d\rho$$

where $u_0 = u(0)$ and $\mu(\Sigma_\rho)$ is the curvatura integra over Σ_ρ .
Equality holds if and only if $u(z)$ is a function only of r .

Proof: Let μ be the measure associated with $u(z)$. From the relation $\Delta u = -Ke^{2u}$ we conclude

$$\int_{\Sigma_\rho} d\mu = \mu(\Sigma_\rho) = - \oint_{\Gamma_\rho} \frac{\partial u}{\partial n} ds = -\rho \frac{d}{d\rho} \oint_{\Gamma_\rho} u d\theta$$

where $\mu(\Sigma_\rho)$ is the curvatura integra. Hence

$$\begin{aligned} -\frac{1}{2\pi} \int_0^r \frac{\mu(\Sigma_\rho)}{\rho} d\rho &= \frac{1}{2\pi r} \oint_{\Gamma_r} u ds - u_0 \\ &= \frac{1}{2\pi r} \oint_{\Gamma_r} (u - u_0) ds = \frac{1}{2\pi r} \oint_{\Gamma_r} \log e^{(u-u_0)} ds \\ &\leq \log \frac{1}{2\pi r} \oint_{\Gamma_r} e^{(u-u_0)} ds \end{aligned}$$

from which the first inequality follows. \square

The second relation is proved similarly. In fact, we write

$$\begin{aligned} -\frac{1}{2\pi} \int_0^\rho \frac{\mu(\Sigma_\tau)}{\tau} d\tau &= \frac{1}{2\pi r} \oint_{\Gamma_r} (u - u_0) ds = \frac{1}{4\pi r} \oint_{\Gamma_r} \log e^{2(u-u_0)} ds \\ &\leq \frac{1}{2} \log \frac{1}{2\pi r} \oint_{\Gamma_r} e^{2(u-u_0)} ds . \end{aligned}$$

Thus,

$$e^{2u_0} \exp \left\{ -\frac{1}{\pi} \int_0^\rho \frac{\mu(\Sigma_\tau)}{\tau} d\tau \right\} \leq \frac{1}{2\pi r} \oint_{\Gamma_r} e^{2u} ds = \frac{1}{2\pi r} \frac{d}{dr} A(r)$$

from which the result follows on a further integration.

Corollary: Let S be simply connected and conformally parabolic, and suppose that $\mu(\Sigma_r) \leq 2\pi$ for all sufficiently large r. Then S has infinite area.

The proof is immediate from the second inequality of Theorem 3.1, for under the assumptions S can be mapped onto the entire finite plane.

Remark 1: If S is in addition complete, and if the curvature is absolutely integrable over S, then $\lim_{r \rightarrow \infty} \mu(\Sigma_r) \leq 2\pi$, cf., Huber [4, Theorem 10], also the corollary to Theorem 3.4 in this paper.

Remark 2: There exist complete surfaces S with finite area, which satisfy all the above hypotheses except the assumption $\mu(\Sigma_r) \leq 2\pi$ for all large r. An example is provided by the conformal metric

$$ds = \frac{|dz|}{(|z| + 1) \log(|z| + 2)} \text{ over the } z\text{-plane. See Huber [4, p. 61].}$$

If the surface S is assumed complete and of finite total curvature, the inequalities in Theorem 3.1 can, essentially, be replaced by equalities. This is the content of our next result.

Theorem 3.2: Let S be a complete, simply connected open surface, over which the curvature is absolutely integrable. Then S can be represented conformally on the whole z-plane, and in each such mapping the relation

$$(4) \quad \mathcal{L}(r) = 2\pi e^{u_0 + o(1)} r \exp \left\{ -\frac{1}{2\pi} \int_0^r \frac{\mu(\Sigma_\rho)}{\rho} d\rho \right\}$$

is asymptotically satisfied as $r \rightarrow \infty$. If in addition S has infinite

area (cf., the Corollary to Theorem 3.1), then

$$(5) \quad A(r) = 2\pi e^{2u_0 + o(1)} \int_0^r \rho \exp \left\{ -\frac{1}{\pi} \int_0^\rho \frac{\mu(\Sigma_\tau)}{\tau} d\tau \right\} d\rho$$

Theorem 3.2a: Let S be a complete, simply connected open surface for which^{4/} $T < \infty$ and $-\infty < \mu < 2\pi$, $\mu = \text{curvatura integra}$. Then S can be conformally represented on the whole z-plane and in each such mapping there holds asymptotically for large r,

$$(6) \quad A(r) = \frac{2\pi r^2 e^{2u_0 + o(1)}}{2\pi - \mu} \exp \left\{ -\frac{1}{\pi} \int_0^r \frac{\mu(\Sigma_\rho)}{\rho} d\rho \right\}.$$

Proof of Theorem 3.2: The first statement of the theorem follows from Theorem 2.1. In any representation on the z-plane, we then have for the logarithm of the local length ratio,

$$u(z) = u(0) - \frac{1}{2\pi} \int_{\Sigma_r} \log \left| 1 - \frac{z}{\zeta} \right| d\mu(\zeta) + h_r(z) - h_r(0)$$

for any r, where $h_r(z)$ is harmonic in Σ_r . The integral converges absolutely as $r \rightarrow \infty$, and we obtain as in the proof of Theorem 2.1,

$$u(z) = u(0) - \frac{1}{2\pi} \int \log \left| 1 - \frac{z}{\zeta} \right| d\mu(\zeta) + h(z) - h(0)$$

the integration being taken over the entire z-plane. We introduce

again the mapping

$$w = F(z) = \int_0^z e^{h(z)+i\hat{h}(z)} dz ,$$

and consider all rays on the unbranched Riemann surface defined by $F(z)$, starting at the image of the origin. If there should be one such ray which cannot be extended to infinity, the method of proof of Theorem 2.1 shows that the inverse image of this segment is a divergent path of finite length, thus contradicting the assumed completeness. We omit details.^{5/} Therefore all segments extend to infinity, so that one branch of the Riemann surface necessarily covers the w -plane. But the inverse image of $F(z)$ is by the monodromy theorem single valued. We conclude that $F(z) \equiv Az$ for a suitable constant A , and therefore $h(z) \equiv \text{const}$. Thus,

$$u(z) = u(0) - \frac{1}{2\pi} \int \log \left| 1 - \frac{z}{\xi} \right| d\mu(\xi)$$

for all z . We base the proof of the theorem on this representation.

Let $|z| = r$ and write

$$(7) \quad u(z) = u(0) + u_1(z) + u_2(z)$$

where

$$u_1(z) = \frac{-1}{2\pi} \int_{\Sigma_{r/2}} \log \left| \frac{z-\xi}{\xi} \right| d\mu(\xi)$$

$$u_2(z) = \frac{-1}{2\pi} \int_{\mathcal{E}_{r/2}} \log \left| \frac{z-\xi}{\xi} \right| d\mu(\xi)$$

\mathcal{E}_r being the exterior of Σ_r .

In $\Sigma_{r/2}$, we have $\log \left| \frac{z-\zeta}{\zeta} \right| = \log \left| \frac{z}{\zeta} \right| - \log \left| 1 - \frac{\zeta}{z} \right|^{-1}$,

and

$$\int_{\Sigma_{r/2}} \log \frac{1}{\left| 1 - \frac{\zeta}{z} \right|} |d\mu(\zeta)| \leq T \log \frac{1}{1-\eta} + \int_{\Sigma_{r/2} \setminus \Sigma_{\eta r}} \log \frac{1}{\left| 1 - \frac{\zeta}{z} \right|} |d\mu(\zeta)|$$

where

$$T = \int_{|\zeta| < \infty} |d\mu(\zeta)|, \quad \text{and} \quad 0 < \eta < 1/2.$$

In the last term on the right, the integrand is bounded, hence for a suitable constant A ,

$$\int_{\Sigma_{r/2}} \log \frac{1}{\left| 1 - \frac{\zeta}{z} \right|} |d\mu(\zeta)| \leq T \log \frac{1}{1-\eta} + A \int_{\mathcal{E}_{\eta r}} |d\mu(\zeta)|$$

If we choose $\eta = \eta(r)$ tending to zero but such that $\eta r \rightarrow \infty$, we see that

$$\int_{\Sigma_{\frac{1}{2}r}} \log \frac{1}{\left| 1 - \frac{\zeta}{z} \right|} |d\mu(\zeta)| = o(1)$$

as $r \rightarrow \infty$. Next, since $\left| \frac{z}{\zeta} \right|$ is bounded for $|\zeta| > \frac{1}{2}|z|$,

$$\begin{aligned} \int_{\Sigma_{\frac{1}{2}r}} \log \left| \frac{z}{\zeta} \right| d\mu(\zeta) &= \int_{\Sigma_r} \log \left| \frac{z}{\zeta} \right| d\mu(\zeta) + o(1) \\ &= \int_0^r \frac{\mu(\Sigma_\rho)}{\rho} d\rho + o(1) \end{aligned}$$

after an integration by parts. Thus

$$(8) \quad u_1(z) = \frac{-1}{2\pi} \int_0^r \frac{\mu(\Sigma_\rho)}{\rho} d\rho + o(1).$$

Consider now $\int_0^{2\pi} [e^{u_2(z)} - 1] d\theta$ for $|z| = r$ and $\theta = \arg z$.

Let $\alpha(M)$ be the measure of the set E_M of θ for which $|u_2(z)| > M$.

Then

$$(9) \quad M \alpha(M) < \int_{E_M} |u_2(z)| d\theta < \frac{1}{2\pi} \int_{C_{\frac{1}{2}r}} Q(\zeta) |d\mu(\zeta)|$$

where

$$Q(\zeta) = \left| \int_{E_M} \log \left| 1 - \frac{z}{\zeta} \right| d\theta \right|.$$

For $|\zeta| > \frac{1}{2}|z|$, $|1 - \frac{z}{\zeta}| < 3$, hence $\log |1 - \frac{z}{\zeta}|$ is bounded except at $z = \zeta$. It follows that $Q(\zeta)$ is maximized when E_M is an interval with its mid-point at $\arg \zeta$. For this configuration, we compute

$$Q(\zeta) < A(1 + |\log \alpha(M)|) \alpha(M)$$

for some constant A , and hence, from (9),

$$M < (1 + |\log \alpha(M)|) \epsilon(r)$$

where $\epsilon(r) \rightarrow 0$ as $r \rightarrow \infty$. We conclude

$$\alpha(M) < A e^{-\frac{1}{\epsilon(r)}M}$$

for a suitable A . But

$$\int_0^{2\pi} |e^{u_2(z)} - 1| d\theta \leq \int_{M>0} |e^M - 1| d\alpha(M) + \int_{M>0} |e^{-M} - 1| d\alpha(M)$$

from which there follows easily

$$(10) \quad \int_0^{2\pi} |e^{u_2(z)} - 1| d\theta = o(1)$$

as $r \rightarrow \infty$.

We may now write, for $|z| = r$, by (7), (8),

$$\begin{aligned} \mathcal{L}(r) &= r \int_0^{2\pi} e^{u(z)} d\theta = r e^{u_0} \int_0^{2\pi} e^{u_1(z)} e^{u_2(z)} d\theta \\ &= r e^{u_0+o(1)} \exp\left\{\frac{-1}{2\pi} \int_0^r \frac{\mu(\Sigma_\rho)}{\rho} d\rho\right\} \int_0^{2\pi} e^{u_2(z)} d\theta \\ &= 2\pi r e^{u_0+o(1)} \exp\left\{\frac{-1}{2\pi} \int_0^r \frac{\mu(\Sigma_\rho)}{\rho} d\rho\right\} \end{aligned}$$

by (10). This proves (4).

To prove (5), observe that

$$\frac{d\mathcal{A}(r)}{dr} = \int_{\Gamma_r} e^{2u(z)} ds.$$

By (7), (8), and (10), we find

$$(11) \quad \frac{dA(r)}{dr} = 2\pi e^{2u_0 + o(1)} r e^{-\frac{1}{\pi}} \int_0^r \frac{\mu(\Sigma_\rho)}{\rho} d\rho$$

from which (5) follows by an integration.

Proof of Theorem 3.2a: By Theorem 2.1, S can be conformally represented on the whole z -plane. By the corollary to Theorem 3.1, S has infinite area. Hence by Theorem 3.2,

$$A(r) = 2\pi e^{2u_0 + o(1)} \int_0^r \rho F(\rho) d\rho$$

where

$$(12) \quad F(\rho) = \exp \left\{ -\frac{1}{\pi} \int_0^\rho \frac{\mu(\Sigma_\tau)}{\tau} d\tau \right\}.$$

We have

$$(13) \quad \begin{aligned} \int_0^r \rho F(\rho) d\rho &= \frac{r^2}{2} F(r) + \frac{1}{2\pi} \int_0^r \rho^2 \frac{\mu(\Sigma_\rho)}{\rho} F(\rho) d\rho \\ &= \frac{r^2}{2} F(r) + \frac{\mu}{2\pi} \int_0^r \rho F(\rho) d\rho + \frac{1}{2\pi} \int_0^r [\mu(\Sigma_\rho) - \mu] \rho F(\rho) d\rho \end{aligned}$$

Again using the fact that S has infinite area, we obtain

$$\left[1 - \frac{\mu}{2\pi} + \epsilon(r) \right] \int_0^r \rho F(\rho) d\rho = \frac{r^2}{2} F(r)$$

where $\epsilon(r) \rightarrow 0$ as $r \rightarrow \infty$, and from this (6) follows.

From Theorems 3.2 and 3.2a we obtain immediately:

Theorem 3.3: Let S be a complete, simply connected open surface, for which $T < \infty$ and $-\infty < \mu < 2\pi$. Then S can be conformally represented on the whole z-plane, and asymptotically for large r, there holds

$$(14) \quad \frac{\mathcal{L}^2(r)}{A(r)} = 4\pi - 2\mu + o(1) .$$

We remark that it is easy to give examples of surfaces for which $\mu(\Sigma_\rho) \rightarrow \mu < 2\pi$, for which $T = \infty$, and for which (14) is incorrect.

Theorem 3.3a: Let S be a complete, simply connected open surface for which $\mu = 2\pi$. Then if S has infinite area, there holds

$$\frac{\mathcal{L}^2(r)}{A(r)} = 4\pi - 2\mu + o(1) .$$

Theorem 3.3b: Let S be a complete, simply connected open surface for which $\mu = 2\pi$. If exterior to a compact set on S the curvature is either non-positive or non-negative, then

$$\frac{\mathcal{L}^2(r)}{A(r)} = 4\pi - 2\mu + o(1) .$$

Proof of Theorem 3.3a: By Theorem 3.2,

$$\mathcal{L}^2(r) = 4\pi^2 e^{2u_0 + o(1)} r^2 F(r)$$

where $F(r)$ is defined by (12). By (13),

$$r^2 F(r) = \frac{1}{\pi} \int_0^r [\mu(\Sigma_\rho) - \mu] \rho F(\rho) d\rho .$$

By assumption $\mathcal{A}(r) = 2\pi e^{2u_0 + o(1)} \int_0^r \rho F(\rho) d\rho$ is unbounded. The result then follows from the fact that $\mu(\Sigma_\rho) \rightarrow \mu$.

Proof of Theorem 3.3b: If $\mu = 2\pi$ and $K \geq 0$ at infinity, one sees from Theorem 3.1 that $\mathcal{A} = \infty$, so that Theorem 3.3a applies. Suppose $K \leq 0$ at infinity. By Theorem 3.1,

$$\begin{aligned} \mathcal{A}(r) &\geq 2\pi e^{2u_0} \int_0^r \rho \exp\left\{-\frac{1}{\pi} \int_0^\rho \frac{\mu(\Sigma_\tau)}{\tau} d\tau\right\} d\rho \\ &= 2\pi e^{2u_0} \int_0^r \rho^{-1} H(\rho) d\rho \end{aligned}$$

defining $H(\rho)$. We compute

$$\frac{dH}{d\rho} = 2\rho \left[1 - \frac{\mu(\Sigma_\rho)}{2\pi}\right] \exp\left\{-\frac{1}{\pi} \int_0^r \frac{\mu(\Sigma_\rho)}{\rho} d\rho\right\}$$

and hence, under the assumptions, $\frac{dH}{d\rho} \leq 0$ for all sufficiently large ρ . We conclude $H \rightarrow 0$, since otherwise there would hold $\mathcal{A}(r) \rightarrow \infty$; hence $\mathcal{L}^2(r) \rightarrow 0$, qed,

We study now the image curves of the radial lines emanating from the origin in the z -plane on which the surface S is represented conformally. We shall give asymptotic estimates for the lengths $L(r)$ of these curves as the radial segments tend to infinity.

Theorem 3.4: Let S be a complete simply-connected open surface for which $T < \infty$. Then for any $\delta > 0$, there holds asymptotically for the length on S of the image of a radial segment of length r from the origin in the z-plane,

$$(15) \quad r^{1-\frac{\mu}{2\pi}-\delta} < L(r) < r^{1-\frac{\mu}{2\pi}+\delta}$$

Proof: We follow, essentially, the proof of Theorem 3.2. In the notation of that proof, we find

$$(16) \quad \begin{aligned} u_1(z) &= -\frac{1}{2\pi} \int_0^r \frac{\mu(\Sigma_\rho)}{\rho} d\rho + o(1) \\ &= -\frac{\mu}{2\pi} \log r + \int_1^r \frac{[\mu - \mu(\Sigma_\rho)]}{\rho} d\rho - \frac{1}{2\pi} \int_0^1 \frac{\mu(\Sigma_\rho)}{\rho} d\rho + o(1) \\ &= -\left(\frac{\mu}{2\pi} + o(1)\right) \log r . \end{aligned}$$

The length $L(r)$ of the image on S of a radial line is $L(r) = \int_0^r e^{u(z)} ds$, and we have, for large r_0 ,

$$(17) \quad e^{u_0} \int_{r_0}^r \rho^{\delta^-} e^{u_2(z)} d\rho \leq \int_{r_0}^r e^{u(z)} ds \leq e^{u_0} \int_{r_0}^r \rho^{\delta^+} e^{u_2(z)} d\rho$$

where δ^- and δ^+ are any two numbers such that $\delta^- < -\frac{\mu}{2\pi} < \delta^+$. For any δ , we have

$$(18) \quad \int_{r_0}^r \rho^\delta e^{u_2(z)} d\rho = \int_{r_0}^r \rho^\delta d\rho + \int_{r_0}^r \rho^\delta [e^{u_2(z)} - 1] d\rho$$

and if $r \leq 2r_0$ we may write

$$(19) \quad \left| \int_{r_0}^r \rho^\delta [e^{u_2(z)} - 1] d\rho \right| < C r^\delta \int_{r_0}^r |e^{u_2(z)} - 1| d\rho$$

Let E_M be the set on $[r_0, r]$ where $|u_2(z)| > M$ and let $\alpha(M)$ be its measure. Then

$$\alpha(M) \cdot M < \int_{E_M} |u_2(z)| ds < \int_{\mathcal{E}_{\frac{1}{2}r_0}} |d\mu_\delta| \int_{E_M} \left| \log \left| 1 - \frac{z}{\delta} \right| \right| ds_z .$$

The last integral is maximized for a ξ such that $\arg \xi = \arg z$. We have, for $\tau = \left| \frac{z}{\xi} \right|$ and E_M^* the image of E_M ,

$$\int_{E_M} \left| \log \left| 1 - \frac{z}{\xi} \right| \right| ds_z \leq |\xi| \int_{E_M^*} \left| \log |1 - \tau| \right| d\tau .$$

If $r \leq 2r_0$, then on the set E_M^* there holds always $\left| \frac{z}{\xi} \right| = \tau \leq 4$, and we obtain

$$\begin{aligned} \int_{E_M^*} \left| \log |1 - \tau| \right| d\tau &\leq A \alpha^*(M) \left[1 + \log \frac{1}{\alpha^*(M)} \right] \\ &= A \frac{1}{|\xi|} \alpha(M) \left[1 + \log \frac{|\xi|}{\alpha(M)} \right] . \end{aligned}$$

Thus,

$$\alpha(M) \cdot M < \epsilon(r) \cdot \alpha(M) \left[1 + \log \frac{2r_0}{\alpha(M)} \right]$$

where $\epsilon(r) \rightarrow 0$ as $r \rightarrow \infty$, from which

$$\alpha(M) < C \operatorname{re} \frac{1}{\epsilon(r)^M}$$

Hence, if $r \leq 2r_0$,

$$\left| \int_{r_0}^r [e^{u_2(z)} - 1] ds \right| \leq \left| \int_{M>0} [e^M - 1] d\alpha(M) \right| + \left| \int_{M>0} [e^{-M} - 1] d\alpha(M) \right|$$

$$\leq C \cdot \epsilon(r) \cdot r$$

for some constant C . Hence in this case, (19) becomes

$$(20) \quad \left| \int_{r_0}^r \rho^\delta [e^{u_2(z)} - 1] d\rho \right| < C \cdot \epsilon(r_0) \cdot r_0^{1+\delta}$$

Consider now an arbitrary $r > r_0$, and let n be the smallest integer for which $2^n r_0 > r$. Then $r < 2^n r_0 \leq 2r$, and we find from (20)

$$\left| \int_{r_0}^r \rho^\delta [e^{u_2(z)} - 1] d\rho \right| < C \cdot \epsilon(r_0) \cdot r_0^{1+\delta} [1 + 2^{1+\delta} + 2^{n(1+\delta)}]$$

$$= C \cdot \epsilon(r_0) \cdot r_0^{1+\delta} \frac{2^{(n+1)(1+\delta)} - 1}{2^{(1+\delta)} - 1}$$

$$(21) \quad < C \cdot \epsilon(r_0) \begin{cases} r^{1+\delta} & \text{if } \delta > -1 \\ r_0^{1+\delta} & \text{if } \delta < -1 \end{cases}$$

Using this inequality, the theorem follows easily from (17) and (18).

Remark: The example of the complete conformal metric $ds = \log(2+|z|)|dz|$ spread over the z -plane, shows that the constant δ cannot in general be removed from the exponent in (15). In this case $\mu = 0$ and $L(r) = r \log r + O(r)$.

Corollary: Under the hypotheses of Theorem 3.4, we have necessarily
 $\mu \leq 2\pi$.

For otherwise, $L(r)$ would by (15) be bounded on every radial segment, so that S could not be complete.

Theorem 3.5: Let S be as in Theorem 3.4, and let γ be a divergent path on S . Let γ_z be the inverse image of γ under a representation of S on the z -plane, and let $L_r(\gamma)$ be the length of that part of γ such that γ_z lies interior to a circle Σ_r of radius r about the origin. Then for any $\delta > 0$ there holds asymptotically

$$L_r(\gamma) \geq r^{1 - \frac{\mu}{2\pi} - \delta}$$

as $r \rightarrow \infty$.

Comparing this result with Theorem 3.4, we see that the images of the radial lines in the z -plane behave asymptotically as approximations to geodesics on S .

Proof: Setting $|\xi| = \rho$, we have

$$L_r(\gamma) = \int_{\gamma_z \cap \Sigma_r} e^{u(\xi)} ds \geq \int_{\rho \leq r} e^{u(\xi)} d\rho.$$

This inequality will be strengthened if we omit all arcs of γ_z on which values of ρ are repeated; that is, if the maximum values of ρ attained on γ_z for all arc lengths $s \leq s_0$ is ρ_0 , all arcs on γ_z for which $s > s_0$, $\rho < \rho_0$ are to be omitted in the integration. In this case the integration is monotonic in ρ and the estimates in the proof of Theorem 3.4 are easily seen to apply, so that for the length

$L_r(\gamma)$ of that part of γ_z for which $\rho < r$ we obtain $L_r(\gamma) \geq r^{1 - \frac{\mu}{2\pi} - \delta}$ by (15) for any $\delta > 0$, the stated result.

4. A geometrical assumption; sharpening of the above estimates:

In order to improve the estimate of Theorem 3.4, it is necessary to introduce a new assumption on the regularity of S at infinity. Such an assumption, if it is to be meaningful, must necessarily involve only quantities which can a-priori be determined in terms of the intrinsic geometry of the surface, and it should not depend on properties of the representations of S on the z -plane. The simplest such hypothesis which is available to us involves the rate of decay of the curvature as the point of evaluation moves to infinity on S . To make this concept precise, we select a fixed point P of S , and define the distance σ_Q from P to Q on S as the greatest lower bound of lengths of paths which join P to Q on S . We shall assume in this section^{6/} that there are fixed constants C and $\delta > 0$ such that uniformly for all Q on S , $|K| < C \sigma^{-2-\delta}$.

Under this hypothesis, we can sharpen our earlier estimates.

Theorem 4.1: Under the hypotheses of Theorem 3.4 and the additional hypothesis $|K| < C \sigma^{-2-\delta}$, there holds

$$L(r) = A r^{1 - \frac{\mu}{2\pi}} [1 + O(r^{-\epsilon})]$$

for some positive constants A and ϵ , whenever $\mu < 2\pi$.

Remark: The assumption $|K| < C \sigma^{-2-\delta}$ cannot be deleted, and even an assumption $|K| < C(\sigma \log \sigma)^{-2}$ is not sufficient. This can be seen from the example (which we have already considered in another context) of the conformal metric $ds = \log(2 + |z|)|dz|$ spread over the z -plane. For the surface S defined in this way, we have $T < \infty$, $\mu = 0$, $K = (r \log^2 r)^{-2} \sim (\sigma \log \sigma)^{-2}$, $L(r) = r \log r + O(r)$.

Proof of Theorem 4.1: As point P we choose the image of the origin in the z -plane (we are at liberty to choose an arbitrary P by adjusting the constant C) and we consider a radial segment from the origin of length r in the z -plane, defining a path \widehat{PQ} on S . Consider now a smooth path joining P to Q on S , whose length approximates the distance σ_Q to Q . Applying Theorem 3.5 to the inverse image of this path, we obtain, for given $\delta > 0$ and large r , $\sigma_Q + \epsilon \geq r^{1 - \frac{\mu}{2\pi} - \delta}$ for any $\epsilon > 0$. Hence

$$\sigma_Q \geq r^{1 - \frac{\mu}{2\pi} - \delta}$$

as $r \rightarrow \infty$.

By assumption, $|K(Q)| < \sigma_Q^{-2-\delta} < r^{\frac{\mu}{\pi} - 2 - \delta}$ (δ not the same in all contexts) as $r \rightarrow \infty$. Now

$$\mu = \int_S d\mu = \mu(\Sigma_r) + \int_{\mathcal{E}_r} K e^{2u} \rho d\rho d\theta$$

where \mathcal{E}_r is the exterior of Σ_r , and

$$\left| \int_{\mathcal{E}_r} K e^{2u} \rho d\rho d\theta \right| \leq \int_r^\infty \rho^{\frac{\mu}{\pi} - 1 - \delta} d\rho \int_0^\pi e^{2u(z)} d\theta .$$

The circuit integral on the right equals $\rho^{-1} \frac{dQ(\rho)}{d\rho}$. All the hypotheses of Theorem 3.2 are satisfied, so we may write from (11)

$$\begin{aligned} \int_0^\pi e^{2u(z)} d\theta &= 2\pi e^{2u_0 + o(1)} \frac{1}{e} \int_0^\rho \frac{\mu(\Sigma_\tau)}{\tau} d\tau \\ &= 2\pi e^{2u_0 + o(1)} \rho^{-\frac{\mu}{\pi} + o(1)} \end{aligned}$$

by (16). Thus,

$$\left| \int_{\mathcal{E}_r} K e^{2u(z)} \rho d\rho d\theta \right| \leq 2\pi e^{2u_0} \int_r^\infty \rho^{-1-\delta} d\rho = \frac{2\pi}{\delta} e^{2u_0} r^{-\delta}$$

for some $\delta > 0$. We have proved:

$$\mu(\Sigma_r) = \mu + O(r^{-\delta})$$

This estimate may be used to replace the estimate (16) by the more precise relation

$$u_1(z) = -\frac{\mu}{2\pi} \log r + A + O(r^{-\delta})$$

as may be seen by repeating the estimate of $u_1(z)$ in the proof of Theorem 3.2 for this case, and hence to replace (17) by

$$\int_{r_0}^r e^{u(z)} ds = e^A (1 + O(r^{-\delta})) \int_{r_0}^r \rho^{-\frac{\mu}{2\pi}} e^{u_2(z)} d\rho .$$

The relations (18) and (21) then yield

$$L(r) = e^A r^{1-\frac{\mu}{2\pi}} [1 + o(r^{-\epsilon})],$$

the stated result.

Similarly we may prove:

Theorem 4.2: Let S be an open, complete, simply connected surface
for which $T < \infty$, $\mu < 2\pi$, and let S be represented conformally on
the z-plane. Then (cf. Theorems 3.2, 3.3)

$$\mathcal{L}(r) = 2\pi e^{u_0} r^{1-\frac{\mu}{2\pi}} [1 + o(r^{-\epsilon})]$$

$$A(r) = \frac{2\pi^2}{2\pi - \mu} e^{2u_0} r^{2-\frac{\mu}{\pi}} [1 + o(r^{-\epsilon})],$$

$$\frac{\mathcal{L}^2(r)}{A(r)} = 4\pi - 2\mu + o(r^{-\epsilon})$$

for some constant $\epsilon > 0$.

We omit details.

5. Asymptotic estimates for the length ratio:

We denote the local length ratio in the representation of S on a plane by $\lambda(z) = e^{u(z)}$.

Theorem 5.1: Let S be an open, complete, simply-connected surface
with finite total curvature μ , and suppose the region in which $K > 0$
has compact support on S. Then in any conformal representation of S

on the z -plane, there is a constant C such that $\lambda(z) \leq C r^{-\frac{\mu}{2\pi}}$ as $r \rightarrow \infty$. If the region in which $K < 0$ is compact on S , then $\lambda(z) \geq C r^{-\frac{\mu}{2\pi}}$ for some C .

Proof: We use again the decomposition (7). Suppose that in a given representation of S , $K \leq 0$ outside the circle Σ_{r_0} . By (8), if $r > r_0$,

$$\begin{aligned} u_1(z) &= -\frac{1}{2\pi} \int_0^r \frac{\mu(\Sigma_\rho)}{\rho} d\rho + o(1) \\ &= -\frac{1}{2\pi} \int_0^{r_0} \frac{\mu(\Sigma_\rho)}{\rho} d\rho - \frac{1}{2\pi} \int_{r_0}^r \frac{\mu}{\rho} d\rho + \frac{1}{2\pi} \int_{r_0}^r \frac{[\mu - \mu(\Sigma_\rho)]}{\rho} d\rho + o(1). \end{aligned}$$

By assumption, the last integral on the right is non-positive, hence

$$\lambda_1(z) = e^{u_1(z)} \leq C r^{-\frac{\mu}{2\pi}}.$$

Consider now

$$u_2(z) = -\frac{1}{2\pi} \int_{\mathcal{E}_{r/2}} \log \left| \frac{z-\zeta}{\zeta} \right| d\mu(\zeta)$$

for $r > 2r_0$. We may neglect the integration over the region $|z-\zeta| < |\zeta|$, since in this region $d\mu(\zeta) \leq 0$. But if $|z-\zeta| > |\zeta|$ and $\zeta \in \mathcal{E}_{r/2}$, then $1 < \left| \frac{z-\zeta}{\zeta} \right| < 3$. Thus

$$\lambda_2(z) = e^{u_2(z)} \leq e^{o(1)}$$

which proves the first part of the result. The corresponding inequality, when $K \geq 0$ outside Σ_{r_0} , is proved similarly.

Corollary: Under the hypotheses of Theorem 5.1, if $-\infty < \mu < 2\pi$, then $L(r) \leq C r^{1-\frac{\mu}{2\pi}}$, $L(r) \geq C r^{1-\frac{\mu}{2\pi}}$ respectively, in the two cases considered. If $\mu = 2\pi$, then $L(r) \leq C \log r$, $L(r) \geq C \log r$, respectively.

Remark: If the curvature has compact support on S , then one sees easily that $\lambda = r^{-\frac{\mu}{2\pi}} [1 + O(r^{-1})]$. Estimates of this type cannot be expected, however, in a general case, even under assumptions of the type introduced in § 4. One may imagine, for example, a surface on which the curvature is concentrated at a sequence of points tending to infinity, the surface appearing in the neighborhood of each such point as the vertex of a cone. Such a surface can be constructed so that the curvature tends to zero at infinity as rapidly as desired, but λ will nevertheless be singular at each point of the sequence. In order to obtain asymptotic estimates of the above type on $\lambda(z)$ it would be necessary to introduce a new postulate on the local smoothness of the curvature on S .

FOOTNOTES

1. With reference to the ensuing discussion, cf. Osserman [6, Lemma 6].
2. Possible difficulties due to irregularity of the boundary can be avoided by a simple approximation procedure.
3. The last step is a consequence of a standard inequality between arithmetic and geometric means.
4. If $\mu > 2\pi$ then S cannot be complete, cf. the corollary to Theorem 3.4.
5. The function $\log \left| 1 - \frac{z}{\xi} \right|$, considered as function of (w, ω) , has all the properties of the Green's function which entered in the proof of Theorem 2.1 except that it is singular at the image ω_0 of $\xi = 0$. However, one sees easily from the maximum principle that for w near P , this function remains bounded on a fixed circle surrounding ω_0 . Hence the interior of this circle may be deleted from the image region and the proof of Theorem 2.1 repeated without further change.
6. This assumption assures a sufficient rate of decay so that the curvature is absolutely integrable on S . An assumption $|K| < C \sigma^{-2}$ would not suffice.

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