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SUCCESSIVE APPROXIMATION TECHNIQUES FOR TRAJECTORY OPTIMIZATION

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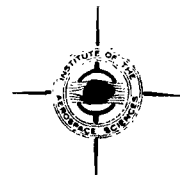
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Abstract

Three related successive approximation schemes for determination of optimal trajectories are developed with particular attention to treatment of inequality constraints on control variables. The first is a gradient method based upon a Euclidean metric with appropriate modification for handling of inequalities; the second employs a Min operation without use of a metric; and the third features a special integral-absolute value metric. The Pontryagin Principle is employed for construction of successive control function approximations. All three schemes employ an adjoint system for computation of influence functions and a 'penalty function' technique for handling of constraints on terminal values.

Illustrative calculations are presented for planar Earth-Mars transfer with rocket thrust variable between limits $T_1 \leq T \leq T_2$. The relative merits of the techniques are discussed from the viewpoint of digital computation.

Introduction

Recent work on successive approximation techniques for numerical solution of variational problems involving differential equations as subsidiary conditions has provided a clear indication of the practical usefulness of this class of methods when employed in conjunction with high speed digital computation^{1,2,3,4}. In the present paper we report some recent developments of techniques which are applicable to problems featuring inequality constraints imposed on the control variables.

Problem Formulation

The basic system of differential equations describing the process of interest is presumed given in first order form

$$\dot{x}_i = g_i(x_1, \dots, x_n, y_1, \dots, y_l, t) \quad (1)$$
$$i = 1, \dots, n$$

The variables x_i , $i = 1, \dots, n$, are state variables and the variables y_k , $k = 1, \dots, l$, are control variables. The functions g_i are assumed to possess continuous first partial derivatives with respect to their arguments.

Initial conditions $x_i(t_0) = \tilde{x}_{i_0}$, $i = 1, \dots, n$, are prescribed. In addition some of the terminal values $x_i(t_f) = \tilde{x}_{i_f}$ may also be prescribed. The terminal time t_f will usually not be fixed; thus its value is regarded as 'open' for optimization purposes.

We wish to minimize some function

$$P = P(x_{1_f}, \dots, x_{n_f}, t_f) \quad (2)$$

of the terminal values of the state variables and the terminal time. This is the general problem of Mayer, encompassing a large class of current problems in flight performance, control, and guidance. It has recently been observed by Hoelker⁵ that similarities in mathematical treatment of these problems provide a strong influence toward unification of future efforts.

'Penalty Function' Treatment Of Terminal Constraints

We may relax the requirement for meeting fixed terminal conditions on the x_i in favor of an approximation: the addition of terms to the function P of the form

$$P' = P + \frac{1}{2} \sum_{j=1}^m K_j (x_{j_f} - \tilde{x}_{j_f})^2, \quad (3)$$

$$K_j > 0$$

(The summation ranges only over those variables subject to specified terminal conditions.)

The notion underlying this approximation is that the solution of the problem $\text{Min } P'$ will, under appropriate conditions, tend toward the solution of the original problem as the magnitudes of the K_j are increased. This idea is due to Courant⁶. Its basis and application are discussed in some detail in Ref. 3. We may now deal with the problem of minimizing P' with terminal values of the x_i 'open.'

Adjoint System

As in earlier work^{1,3,7}, we employ an adjoint system of differential equations in variables λ_i obtained from the linearized version of (1)

$$\delta \dot{x}_i = \sum_{j=1}^n \frac{\partial g_j}{\partial x_j} \delta x_j + \sum_{k=1}^l \frac{\partial g_j}{\partial y_k} \delta y_k \quad (4)$$

$$i = 1, \dots, n$$

[The linearization is in the neighborhood of $y_k = \bar{y}_k(t)$, $x_i = \bar{x}_i(t)$, a solution of system (1).]

To obtain the adjoint system, one discards the control terms, transposes the matrix of coefficients, and changes the signs of the right members:

$$\dot{\lambda}_i = - \sum_{j=1}^n \frac{\partial g_j}{\partial x_i} \lambda_j, \quad i = 1, \dots, n \quad (5)$$

The partial derivatives $\frac{\partial g_j}{\partial x_i}$ are functions of t only, being evaluated along $y_k = \bar{y}_k(t)$, $x_i = \bar{x}_i(t)$.

The most significant property of adjoint systems is given by

$$\frac{d}{dt} \sum_{i=1}^n \lambda_i \delta x_i = \sum_{i=1}^n \sum_{k=1}^l \lambda_i \frac{\partial g_i}{\partial y_k} \delta y_k \quad (6)$$

which may be verified directly by differentiation and use of (4) and (5). Integrating both left and right members of (6) between definite limits we have

$$\left[\sum_{i=1}^n \lambda_i \delta x_i \right]_{t_0}^{t_f} = \int_{t_0}^{t_f} \sum_{i=1}^n \sum_{k=1}^l \lambda_i \frac{\partial g_i}{\partial y_k} \delta y_k dt \quad (7)$$

In terms of a function H^* defined by

$$H^*(x_1, \dots, x_n, \lambda_1, \dots, \lambda_n, y_1, \dots, y_l, t) = \sum_{i=1}^n \lambda_i g_i \quad (8)$$

this may be written:

$$\left[\sum_{i=1}^n \lambda_i \delta x_i \right]_{t_0}^{t_f} = \int_{t_0}^{t_f} \sum_{k=1}^l \frac{\partial H^*}{\partial y_k} \delta y_k dt \quad (9)$$

We shall need a version of (9) which incorporates higher order effects, in particular one for which linearization with respect to the control variables is avoided. Our development from the properties of adjoint systems follows that of Ref. 7. Working temporarily with finite increments $\Delta x_i = x_i - \bar{x}_i$, $\Delta y_k = y_k - \bar{y}_k$, we get from (1)

$$\begin{aligned} \dot{\Delta x}_1 = & g_1(\bar{x}_1 + \Delta x_1, \dots, \bar{x}_n + \Delta x_n, \bar{y}_1 + \Delta y_1, \dots, \bar{y}_l + \Delta y_l, t) \\ & - g_1(\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_l, t) \end{aligned} \quad (10)$$

From integration of

$$\frac{d}{dt} \sum_{i=1}^n \lambda_i \Delta x_i = \sum_{i=1}^n \dot{\lambda}_i \Delta x_i + \sum_{i=1}^n \lambda_i \dot{\Delta x}_i \quad (11)$$

we then obtain

$$\begin{aligned} \sum_{i=1}^n \lambda_i \Delta x_i \Big|_{t_0}^{t_f} &= \int_{t_0}^{t_f} \sum_{i=1}^n \dot{\lambda}_i \Delta x_i dt \\ &+ \int_{t_0}^{t_f} \sum_{i=1}^n \lambda_i [g_1(\bar{x}_1 + \Delta x_1, \dots, \bar{x}_n + \Delta x_n, \\ &\quad \bar{y}_1 + \Delta y_1, \dots, \bar{y}_l + \Delta y_l, t) \\ &\quad - g_1(\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_l, t)] dt \end{aligned} \quad (12)$$

If we employ an expansion of the functions g_i in a Taylor series in the Δx_i , including a remainder term

$$\begin{aligned} &g_1(\bar{x}_1 + \Delta x_1, \dots, \bar{x}_n + \Delta x_n, \bar{y}_1 + \Delta y_1, \dots, \bar{y}_l + \Delta y_l, t) \\ &= g_1(\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1 + \Delta y_1, \dots, \bar{y}_l + \Delta y_l, t) \\ &+ \sum_{j=1}^n \frac{\partial g_1}{\partial x_j} (\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1 + \Delta y_1, \dots, \bar{y}_l + \Delta y_l, t) \Delta x_j \\ &+ \frac{1}{2} \sum_{j=1}^n \sum_{s=1}^n \frac{\partial^2 g_1}{\partial x_j \partial x_s} (\bar{x}_1 + \xi_1 \Delta x_1, \dots, \bar{x}_n + \xi_n \Delta x_n, \\ &\quad \bar{y}_1 + \Delta y_1, \dots, \bar{y}_l + \Delta y_l, t) \Delta x_j \Delta x_s \end{aligned} \quad (13)$$

where $0 \leq \xi_q \leq 1$, $q = 1, \dots, n$, then
(12) becomes

$$\begin{aligned} \sum_{i=1}^n \lambda_i \Delta x_i \Big|_{t_0}^{t_f} &= \int_{t_0}^{t_f} \sum_{i=1}^n \lambda_i [g_i(\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1 + \Delta y_1, \dots, \bar{y}_\ell + \Delta y_\ell, t) \\ &\quad - g_i(\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_\ell, t)] dt \\ &\quad + \int_{t_0}^{t_f} \sum_{i=1}^n \sum_{j=1}^n \lambda_i \left[\frac{\partial g_i}{\partial x_j} (\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1 + \Delta y_1, \dots, \bar{y}_\ell + \Delta y_\ell, t) \right. \\ &\quad \left. - \frac{\partial g_i}{\partial x_j} (\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_\ell, t) \right] \Delta x_j dt \\ &\quad + \frac{1}{2} \int_{t_0}^{t_f} \sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^n \lambda_i \frac{\partial^2 g_i}{\partial x_j \partial x_s} (\bar{x}_1 + \xi_1 \Delta x_1, \dots, \bar{x}_n + \xi_n \Delta x_n, \\ &\quad \bar{y}_1 + \Delta y_1, \dots, \bar{y}_\ell + \Delta y_\ell, t) \Delta x_j \Delta x_s dt \end{aligned} \quad (14)$$

Note that the series development assumes the existence of the second partial derivatives of the g_i with respect to the state variables. In terms of the function H^* defined by (8), the expression (14) may be written:

$$\bar{\lambda}_1 \Delta x_1 \Big|_{t_0}^{t_f} = \int_{t_0}^{t_f} [H^*(\bar{y}_1 + \Delta y_1, \dots, \bar{y}_\ell + \Delta y_\ell, t) - H^*(\bar{y}_1, \dots, \bar{y}_\ell, t)] dt \quad (15)$$

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in which only the term of the right member dominant for small Δy_k is shown. Since the functions $\bar{x}_1(t)$, $\bar{\lambda}_1(t)$ may be regarded as known functions of t evaluated from (1) and (5) for the control functions $\bar{y}_k(t)$, the function H^* appearing in (15) may be considered a function of the Δy_k and t only.

Terminal Time Criterion

We employ as trajectory termination criterion the vanishing of the time de-

derivative of the function $P'(x_1, \dots, x_n, t)$:

$$\frac{d}{dt} P'(x_1, \dots, x_n, t) =$$

$$\frac{\partial P'}{\partial t} + \sum_{i=1}^n \frac{\partial P'}{\partial x_i} \dot{x}_i = 0 \quad (16)$$

This amounts to a one-dimensional search for the value of t_f which minimizes $P'(x_{1f}, \dots, x_{nf}, t_f)$, performed concurrently with numerical integration of the basic system (1). The terminal time t_f is determined by a zero of (16), $t > t_0$, for which $\frac{d}{dt} P'$ is negative.

Effect Of Control Variations On The Function P'

Beginning with some first approximation for the control functions $y_k = \bar{y}_k(t)$ and the corresponding solution of (1), $x_i = \bar{x}_i(t)$, we wish to obtain successive decrements in the performance function P' , eventually approaching a minimum. We assume that this first solution of (1) is terminated according to (16), and we designate the terminal time $t_f = \bar{t}_f$. If the control functions are altered, producing increments $\Delta x_i(t)$ in the state variables, a first order approximation to the increment in the terminal value of P' is given by

$$\Delta P' = \sum_{i=1}^n \frac{\partial P'}{\partial x_{if}} \Delta x_{if}(\bar{t}_f) + \left[\frac{\partial P'}{\partial t_f} + \sum_{i=1}^n \frac{\partial P'}{\partial x_{if}} \dot{x}_i(\bar{t}_f) \right] \Delta t_f \quad (17)$$

in which the partial derivatives of P' are evaluated for $x_{if} = \bar{x}_{if}$, $t_f = \bar{t}_f$. But by our choice of $\bar{t}_f$ determined by (16), the second member on the right of (17) vanishes.

Referring to (9) and (15) and noting that the $\Delta x_i(t_0)$ are zero as a consequence of fixed initial conditions on the x_i , we see that if boundary conditions

$$\bar{\lambda}_i(\bar{t}_f) = \frac{\partial P'}{\partial x_{if}} \quad (18)$$

are imposed upon the adjoint variables evaluated along $\bar{y}_k(t)$, $\bar{x}_i(t)$, then the following form of (15) is obtained:

$$\Delta P' = \int_{t_0}^{\bar{t}_f} [H^*(\bar{y}_1 + \Delta y_1, \dots, \bar{y}_l + \Delta y_l, t) - H^*(\bar{y}_1, \dots, \bar{y}_l, t)] dt \quad (19)$$

the higher order terms having been dropped.

Modified Gradient Method

Sketching first the basic gradient method for obtaining negative increments in P' , we employ (19) for construction of control variations $\Delta y_k(t)$. We wish to limit the 'step size' in some way as to guarantee the predominance of first order effects. A constraint on the 'distance' covered in the step as measured in the Euclidean sense

$$\int_{t_0}^{\bar{t}_f} \Delta y_k^2 dt = a_k^2, \quad k = 1, \dots, l \quad (20)$$

is the conventional means employed. If the constants a_k are taken sufficiently small, then (19) will be a good approximation to the actual change in P' and it is sensible to seek a minimum of (19) subject to (20).

For this purpose we apply the Pontryagin theory^{7,8,9}, the full advantage of this approach becoming clear as we later proceed with treatment of inequality constraints on the y_k . The integrals (19) and (20) are represented as the terminal values of variables $z_1(t)$, $\dots$, $z_{l+1}(t)$ satisfying the system:

$$\dot{z}_k = \Delta y_k^2, \quad z_k(t_0) = 0, \quad z_k(\bar{t}_f) = a_k^2 \quad k = 1, \dots, l \quad (21)$$

$$\dot{z}_{l+1} = H^*(\bar{y}_1 + \Delta y_1, \dots, \bar{y}_l + \Delta y_l, t) - H^*(\bar{y}_1, \dots, \bar{y}_l, t),$$

$$z_{l+1}(t_0) = 0 \quad (22)$$

Thus a statement in terms of an auxiliary Mayer problem is obtained: $\text{Min } z_{\ell+1}(\bar{t}_f)$.

Introducing adjoint variables $p_1(t)$, $\dots$, $p_{\ell+1}(t)$ and a function $\hat{H}$ defined by

$$\begin{aligned} \hat{H} = & \sum_{k=1}^{\ell} p_k \Delta y_k^2 \\ & + p_{\ell+1} [H^*(\bar{y}_1 + \Delta y_1, \dots, \bar{y}_\ell + \Delta y_\ell, t) \\ & - H^*(\bar{y}_1, \dots, \bar{y}_\ell, t)] \end{aligned} \quad (23)$$

we then write the conditions

$$\dot{p}_k = 0, \quad k = 1, \dots, \ell \quad (24) \quad \text{or}$$

$$\dot{p}_{\ell+1} = 0, \quad p_{\ell+1}(\bar{t}_f) = 1 \quad (25)$$

$$\begin{aligned} & \hat{H}(\Delta y_1, \dots, \Delta y_\ell, p_1, \dots, p_{\ell+1}, t) \\ & \geq \hat{H}(\hat{\Delta y}_1, \dots, \hat{\Delta y}_\ell, p_1, \dots, p_{\ell+1}, t) \end{aligned} \quad (26)$$

The $\Delta y_k = \hat{\Delta y}_k(t)$ minimizing $z_{\ell+1}(\bar{t}_f)$ are determined from (26), the Pontryagin principle, whose alternate form is

$$\text{Min}_{\Delta y_1, \dots, \Delta y_\ell} \hat{H} \quad (27)$$

The expressions (26) and (27) are equivalent to the necessary condition of Weierstrass, and represent a generalization thereof if inequality constraints on the Δy_k are operative, as will later be the case^{7,9}. In the absence of such constraints, the minimum of $\hat{H}$ sought in (27) must also be a stationary point, this following from the differentiability property of the functions g_i assumed earlier. The vanishing of the partial derivatives

$$\begin{aligned} \frac{\partial \hat{H}}{\partial \Delta y_k} = 2p_k \Delta y_k + \frac{\partial H^*}{\partial y_k} = 0, \\ k = 1, \dots, \ell \end{aligned} \quad (28) \quad 6$$

then determines the control increments as

$$\Delta y_k = - \frac{1}{2p_k} \frac{\partial H^*}{\partial y_k}, \quad k = 1, \dots, \ell \quad (29)$$

The constants $p_1, \dots, p_\ell$ must satisfy the requirements

$$\begin{aligned} z_k(\bar{t}_f) = & \int_{t_0}^{\bar{t}_f} \Delta y_k^2 dt = \\ & \frac{1}{4p_k^2} \int_{t_0}^{\bar{t}_f} \left(\frac{\partial H^*}{\partial y_k} \right)^2 dt = a_k^2 \end{aligned} \quad (30)$$

$$\begin{aligned} p_k = & \frac{\pm 1}{2|a_k|} \left[\int_{t_0}^{\bar{t}_f} \left(\frac{\partial H^*}{\partial y_k} \right)^2 dt \right]^{\frac{1}{2}}, \\ & k = 1, \dots, \ell \end{aligned} \quad (31)$$

Evidently the negative sign corresponds to a maximum of $z_{\ell+1}(\bar{t}_f) = \Delta P'$ and the positive sign to a minimum. Choosing the latter, we obtain

$$\Delta P' = - \sum_{k=1}^{\ell} |a_k| \left[\int_{t_0}^{\bar{t}_f} \left(\frac{\partial H^*}{\partial y_k} \right)^2 dt \right]^{\frac{1}{2}} \quad (32)$$

We have found the direction of 'steepest descent' in the sense of the Euclidean metric (20), and, according to (32), the increment $\Delta P'$ must be negative or zero. The functions $\frac{\partial H^*}{\partial y_k}(t)$ evaluated for $y_k = \bar{y}_k(t)$ assume the role of gradient components and the 'steepest descent' process is governed by

$$\begin{aligned} \Delta y_k = & b_k \frac{\partial H^*}{\partial y_k}, \quad b_k < 0, \\ & k = 1, \dots, \ell \end{aligned} \quad (33)$$

We now turn attention to means for handling inequality constraints on the control functions y_k . It is assumed that the inequalities can be converted to the form

$$\begin{aligned} y_{k1} &\leq y_k \leq y_{k2}, \\ k &= 1, \dots, l \end{aligned} \quad (34)$$

which is usually the case in applications. The development up through (26, 27) is identical for this case, except that the Min operation in (27) must be performed subject to the additional constraints (34):

$$\begin{aligned} \text{Min} \quad & \hat{H} \\ & \Delta y_1, \dots, \Delta y_l \\ & y_{k1} \leq y_k \leq y_{k2} \end{aligned} \quad (35)$$

An examination of the minimum problem (35) for vanishingly small step size $|a_k|$ leads to the formulas

$$\Delta y_k = b_k \frac{\partial H^*}{\partial y_k}, \quad b_k < 0 \quad (36)$$

if $\bar{y}_k$ is an interior point of the interval, i.e., for

$$y_{k1} < \bar{y}_k < y_{k2} \quad (37)$$

At the lower limit $\bar{y}_k = y_{k1}$,

$$\begin{aligned} \Delta y_k &= b_k \frac{\partial H^*}{\partial y_k} & \text{if } \frac{\partial H^*}{\partial y_k} \leq 0 \\ \Delta y_k &= 0 & \text{if } \frac{\partial H^*}{\partial y_k} \geq 0 \end{aligned} \quad \left. \vphantom{\begin{aligned} \Delta y_k &= b_k \frac{\partial H^*}{\partial y_k} \\ \Delta y_k &= 0 \end{aligned}} \right\} b_k < 0 \quad (38)$$

while at the upper limit $\bar{y}_k = y_{k2}$

$$\begin{aligned} \Delta y_k &= 0 & \text{if } \frac{\partial H^*}{\partial y_k} \leq 0 \\ \Delta y_k &= b_k \frac{\partial H^*}{\partial y_k} & \text{if } \frac{\partial H^*}{\partial y_k} \geq 0 \end{aligned} \quad \left. \vphantom{\begin{aligned} \Delta y_k &= 0 \\ \Delta y_k &= b_k \frac{\partial H^*}{\partial y_k} \end{aligned}} \right\} b_k < 0 \quad (39)$$

Thus we arrive at a modified gradient method.

In a descent process for which the steps are vanishingly small, essentially a continuous process, the governing equations (36), (38), (39) will succeed in holding the control variables y_k within the desired range $y_{k1} \leq y_k \leq y_{k2}$. There will be difficulty, however, if finite steps are taken, and to avoid this, it will be necessary to 'trim' the functions $\bar{y}_k + \Delta y_k$ obtained for finite b_k to conform to the inequality (34) before employing them in numerical integration of the basic system (1).

The general mode of operation for gradient calculations has been described in earlier publications^{1,3} for cases in which only a single control variable is employed. It consists of numerical solution of the basic system (1) for a number of values of the descent parameter b , the parameter being varied systematically in one-dimensional search for a minimum of P . In problems involving a number of control variables, an identical treatment may be given to each in turn; or alternatively, a more complicated multi-dimensional search versus the b_k may be implemented.

A Successive Approximation Scheme Employing the Min Operation

A shortcoming of a gradient process is that over intervals in which $\frac{\partial H^*}{\partial y_k}$ is small in magnitude, the corresponding changes in y_k will be small. After several steps the y_k may still be far from their optimal values over such 'insensitive' intervals owing to this feature of the gradient process. This, of course, stems from the rather arbitrary imposition of the Euclidean distance measure. From an engineering viewpoint this is unimportant if only the value of P is of inter-

est, as is often the case in flight performance work. It is inconvenient if a family of neighboring extremals are required, as, for example, in connection with a guidance study, because this requires that additional computations be performed to converge the control variable histories to within the desired accuracy.

It has been speculated that an appropriate procedure for treatment of this situation is transition to a scheme for systematic numerical solution of the Euler equations, and this appears plausible on first consideration. One finds, however, that the appropriate linear combinations of adjoint solutions do not yield a good approximation to the multiplier functions of the 'indirect' theory, and, in particular, the required initial values of the multipliers may be sufficiently in error to cause difficulty in an iterative adjustment process.

The successive approximation scheme of the present section was originally developed for refinement of control programs of near-minimal gradient solutions into close approximations to Euler solutions, i.e., to accelerate convergence in the later stages of the descent process. It appears, however, to possess merit as a primary computational scheme, as will be discussed later in connection with an example.

Instead of employing the somewhat arbitrary Euclidean metric (20), we choose an equally arbitrary alternative: discard the use of a metric, calculating new control variables y_k^* from the operation

$$\begin{aligned} \text{Min} \quad & H^*(y) \\ y_{k1} \leq y_k \leq y_{k2} \quad & (40) \\ k = 1, \dots, l \end{aligned}$$

In adopting the control $y_k = y_k^*(t)$ generated by the Min operation as our next approximation, we risk the violation of our linearizing assumptions, for this may represent a large step process. More conservatively, we may elect to replace the large step by an exploratory series of small ones, setting

$$y_k = \bar{y}_k(t) + \zeta[y_k^*(t) - \bar{y}_k(t)] \quad , \quad (41)$$

$$k = 1, \dots, l$$

and evaluating P' versus ζ , a one-dimensional search analogous to that performed versus step size (the b_k) in gradient computations.

There is, of course, a question of convergence with this scheme, hinging on whether or not the increment $\Delta P'$ given by (19) is negative for some ζ in the range $0 < \zeta < 1$ normally explored. Evidently a sufficient condition for $\Delta P' < 0$ for small ζ is

$$\begin{aligned} \lim_{\zeta \rightarrow 0} [H^*(\bar{y}_1 + \zeta(y_1^* - \bar{y}_1), \dots, \bar{y}_l + \zeta(y_l^* - \bar{y}_l), t) \\ - H^*(\bar{y}_1, \dots, \bar{y}_l, t)] = \sum_{k=1}^l \left[\frac{\partial H^*}{\partial y_k}(\bar{y}_1, \dots, \bar{y}_l, t) \right] (y_k^* - \bar{y}_k) < 0 \quad (42) \end{aligned}$$

which requires that the directional derivative of H^* be negative in the direction of the minimum point $y_k = y_k^*$. This requirement will be met globally, i.e. for all admissible starting points $y_k = \bar{y}_k$, if the function $H^*(y_1, \dots, y_l, t)$ possesses no stationary points or interior extrema in the region defined by (34), other than the minimum at $y_k = y_k^*$.

In the original conception of this technique as a refinement scheme, only a local version of the requirement discussed in the preceding paragraph requires consideration, and this requirement is automatically met if the gradient process has progressed sufficiently before transition to the Min H^* scheme. From the viewpoint of more general applications, the requirements on the function H^* discussed above constitute a limitation; yet the class of problems for which convergence is assured a priori is sufficiently large as to warrant considerable interest in the scheme.

A Successive Approximation Scheme Employing An Integral-Absolute Value Metric

A third method for computing optimum trajectories by successive approximations is obtained by substituting an integral-absolute value metric for the integral square metric of (20):

$$\int_0^{\bar{t}_f} |\Delta y_k| dt = a_k > 0, \quad k = 1, \dots, \ell$$

$$k = 1, \dots, \ell \quad (43)$$

Although apparently not limited to cases in which the optimal control is bang-bang, this method appears to offer the advantage in such cases that if the optimal control is bang-bang and the first approximation is taken to be bang-bang, all successive approximations will also be bang-bang.

By applying the Pontryagin theorem^{7,8,9} in a manner similar to that of (21) to (27), we obtain:

$$\dot{z}_k = |\Delta y_k|, \quad z_k(t_0) = 0,$$

$$z_k(\bar{t}_f) = a_k > 0, \quad k = 1, \dots, \ell \quad (44)$$

and

$$\begin{aligned} \dot{z}_{\ell+1} &= H^*(\bar{y}_1 + \Delta y_1, \dots, \bar{y}_\ell + \Delta y_\ell, t) \\ &- H^*(\bar{y}_1, \dots, \bar{y}_\ell, t), \quad z_{\ell+1}(t_0) = 0 \end{aligned} \quad (45)$$

From (19) the function we wish to minimize is

$$\Delta P' = z_{\ell+1}(\bar{t}_f) \quad (46)$$

Thus we have a new minimum problem to which we may apply the Pontryagin principle. A new function $\hat{H}$ is defined by

$$\begin{aligned} \hat{H} &= \sum_{k=1}^{\ell} p_k |\Delta y_k| + p_{\ell+1} [H^*(\bar{y}_1 + \Delta y_1, \dots, \\ &\bar{y}_\ell + \Delta y_\ell, t) - H^*(\bar{y}_1, \dots, \bar{y}_\ell, t)] \end{aligned} \quad (47)$$

where $p_1(t), \dots, p_{\ell+1}(t)$ are adjoint variables satisfying the system of differential equations

$$\dot{p}_k = 0, \quad k = 1, \dots, \ell \quad (48)$$

$$\dot{p}_{\ell+1} = 0, \quad p_{\ell+1}(\bar{t}_f) = 1$$

The Δy_k are now considered as new control variables with time-varying constraints which are compatible with the constraints on the control variables y_k . As in the previous cases, the constraints on y_k are assumed to be of the form

$$y_{k1} \leq y_k \leq y_{k2} \quad (49)$$

A necessary condition for $\Delta P'$ to be a minimum is that the Δy_k minimize $\hat{H}$ for all $t_0 \leq t \leq \bar{t}_f$ subject to (49). This is expressed as

$$\text{Min}_{\Delta y_1, \dots, \Delta y_\ell} \hat{H} \quad (50)$$

Convergence is assured by limiting the magnitude of the a_k defined in (43). However, we will see that this no longer requires that the change in the control variable be small at any particular t . From (48) and (49) we obtain

$$\begin{aligned} \hat{H} &= \sum_{k=1}^{\ell} p_k |\Delta y_k| + H^*(\bar{y}_1 + \Delta y_1, \dots, \\ &\bar{y}_\ell + \Delta y_\ell, t) - H^*(\bar{y}_1, \dots, \bar{y}_\ell, t) \end{aligned} \quad (51)$$

where p_k is a constant ($p_k = \rho_k$). Since the a_k were not previously specified, aside from the requirements that they be positive and small in magnitude, we may consider the ρ_k as constants to be later found by performing a search, evaluating $\Delta P'$ versus ρ_k .

The function $\hat{H}$ to be minimized given by (52) can be thought of geometrically as the superposition of a cone on the surface ΔH^* :

$$\begin{aligned} \Delta H^* &= H^*(\bar{y}_1 + \Delta y_1, \dots, \bar{y}_\ell + \Delta y_\ell, t) \\ &- H^*(\bar{y}_1, \dots, \bar{y}_\ell, t) \end{aligned} \quad (52)$$

Let us now consider the treatment of one control variable at a time. We rewrite (52) as

$$\hat{H} = \rho_j |\Delta y_j| + H^*(\bar{y}_j + \Delta y_j, \dots, \bar{y}_\ell, t) - H^*(\bar{y}_j, \dots, \bar{y}_\ell, t) \quad (53)$$

Examining (53), we see that the minimum of $\hat{H}$ with respect to Δy_j occurs either at a stationary point ($\partial H / \partial \Delta y_k = 0$), at the apex of the cone where $\Delta y_k = 0$, or on the boundaries determined by (49). For ρ_j very large, the minimum of $\hat{H}$ occurs at the apex of the cone for all time and $\Delta y_k = 0$ everywhere. As ρ_j is made smaller, a point is reached where the minimum changes from the apex of the cone to a boundary or an interior point over an infinitesimal interval of time. This value of ρ_j we will designate as the threshold value of ρ_j . If we continue to decrease ρ_j a small amount below its threshold value, Δy_k will take on non-zero values over finite intervals of time. It is to be noted that the Δy_k at any time are themselves not necessarily small. The final selection of ρ_j is determined by evaluating P' via (1) for several values of ρ_j below its threshold value and fitting a parabola to the points. The minimum of this parabola then determines the optimal ρ_j .

To illustrate the possible advantages of employing the integral-absolute value metric, let us consider a system linear in the y_k and for which the optimal control is bang-bang. For this case H^* can be written in the form

$$H^* = \phi_1(\bar{x}, \lambda, t) + y_k \phi_2(\bar{x}, \lambda, t) \quad (54)$$

giving for $\hat{H}$:

$$\hat{H} = \rho_k |\Delta y_k| + \phi_1(t) + \Delta y_k \phi_2(t) \quad (55)$$

If the first approximation to the control function is bang-bang, it is easily seen that all successive approximations are also bang-bang since this property propagates, due to (55).

For ρ_k very large (compared to $|\phi_2(t)|$), Δy_k is equal to zero for all t . The threshold value of ρ_k is ob-

tained when ρ_k is equal to the maximum value of $|\phi_2(t)|$ in a region where it is possible to change y_k subject to (55) and (49). If we decrease ρ_k a small amount beyond this threshold value, Δy_k will change by its maximum value for a limited period of time.

Low Thrust Example

For illustrative computations we choose the problem of planar low thrust transfer between circular orbits, which has been employed in earlier technique developments^{3,10}. The system of equations governing the motion is given by:

Radial Acceleration

$$\ddot{u} = g_1 = \frac{v^2}{R} - A_o \left(\frac{R_o}{R} \right)^2 + \frac{T}{m} \sin \theta \quad (56)$$

Circumferential Acceleration

$$\dot{v} = g_2 = -\frac{uv}{R} + \frac{T}{m} \cos \theta \quad (57)$$

Radial Velocity

$$\dot{R} = g_3 = u \quad (58)$$

Circumferential Angular Velocity

$$\dot{\psi} = g_4 = \frac{v}{R} \quad (59)$$

Propellant Expenditure

$$\dot{m} = g_5 = -\frac{T}{V_e} \quad (60)$$

The exhaust velocity V_e is taken constant and the thrust T variable between fixed limits:

$$T_1 \leq T \leq T_2 \quad (61)$$

If we assume a fully throttlable rocket, with a control parameter η ,

$$0 \leq \eta \leq 1 \quad (62)$$

then

$$T = T_2 \eta \quad (63)$$

The function H^* takes the form

$$\begin{aligned} H^* = & \lambda_1 \left(\frac{v^2}{R} - A_0 \left(\frac{R_0}{R} \right)^2 + \frac{T_2 \eta}{m} \sin \theta \right) \\ & + \lambda_2 \left(-\frac{uv}{R} + \frac{T_2 \eta}{m} \cos \theta \right) \\ & + \lambda_3 u + \lambda_4 \frac{v}{R} - \lambda_5 \frac{T_2 \eta}{v_e} \end{aligned} \quad (64)$$

and the adjoint variables satisfy the system of differential equations

$$\dot{\lambda}_i = -\frac{\partial H^*}{\partial x_i} \quad (65)$$

$i = 1, \dots, 5$

[Note that for an optimal trajectory the function H^* becomes the Hamiltonian function, i.e. as the successive approximation process converges, $H^* \rightarrow H$.]

The function P' is

$$\begin{aligned} P' = & P + \frac{1}{2} [K_1 (u_f - \tilde{u})^2 + K_2 (v_f - \tilde{v})^2 \\ & + K_3 (R_f - \tilde{R})^2 + K_4 (\psi_f - \tilde{\psi})^2 \\ & + K_5 (m_f - \tilde{m})^2] \end{aligned} \quad (66)$$

Where a terminal value of a variable x_i is unspecified, the corresponding K_i is taken zero. In our illustrative computations, we have chosen the problem of minimum time transfer, that is

$$P = t_f \quad (67)$$

We note that the function H^* , (64), is linear in η , a feature usually associated with bang-bang control. As recently pointed out by Lawden¹¹, however, there is a possibility that the collected coefficient of η in H^*

$$D = T_2 \left(\frac{\lambda_1}{m} \sin \theta + \frac{\lambda_2}{m} \cos \theta - \frac{\lambda_5}{v_e} \right) \quad (68)$$

may vanish over a finite interval of time. Such arcs satisfy the weak form, but not the strong form, of the Weierstrass condition; therefore fall in the gap between necessary and sufficient conditions for a minimum. The question of whether an arc $D = 0$ may or may not be minimizing is currently unresolved, and consequently we have no a priori assurance of a bang-bang throttle characteristic.

Some Computational Results

The three successive approximation schemes described earlier were mechanized for digital computation of planar transfers between the orbits of Earth and Mars, idealized as circular. A modified Adams numerical integration scheme was used with a fixed time interval of two days.

Having earlier obtained experience with the gradient method in a constant thrust version of this example¹⁰, we first performed the modification to incorporate the throttle variable η . The computer mechanization employed alternating descent cycles on the variables θ and η . After some experimentation to select penalty constants K_i , a family of transfers for various specified terminal values of the vehicle mass were computed (Figs. 1 and 2). Terminal values of radius and velocity components corresponding to the Mars orbit were specified. The terminal value of the heliocentric angle ψ was left open. The results indicated a bang-bang throttle characteristic, the transfers consisting of an initial full throttle period, a coasting period, and a final full throttle period.

One such transfer was adopted as a test specimen for experimentation with the three successive approximation techniques. The initial approximation consisted of circumferential thrust at full throttle in all three cases. Penalty constants were set at 'intermediate' values for the first sixty cycles, then increased by a factor of thirty. It was intended to divorce the effect of penalty constant manipulation, which has a pronounced effect on convergence, from the characteristics of the successive approximation methods in this way. This was not very successful because of strong interaction effects — the penalty constant adjustment technique should be 'tailored' to the method and to the problem under attack.

Successive approximations produced by the modified gradient method are shown in Figs. 3A and 3B. The decrease in the function P' versus number of descent cycles is illustrated in Fig. 3C.

Corresponding results for the $\text{Min } H^*$ scheme are shown in Figs. 4A, 4B, and 4C. Increments in the control variables θ and η were generated simultaneously during each cycle. Values of the interpolation parameter ζ corresponding to local minima of P' in the one-dimensional search process were found to be on the order of .0003 to .10, the smaller values arising in conjunction with large penalty constants.

In the first attempts at performing computations with the integral-absolute value metric, a difficulty was encountered: the procedure failed to continue to produce decreases in P' after six or seven cycles. This was traced to the fact that the search for a variation in throttle history is 'quantized', this arising from the finite number of numerical integration steps. Thus the 'smallest variation' in throttle history admitted by the integration procedure is $|\Delta\eta| = 1$ over one integration interval. If such a variation cannot produce a decrease in P' , the one-dimensional search fails. This difficulty was overcome by reducing the integration interval to one-tenth of that previously employed, i.e. from 2 days to .2 days. The increase in computing time, by a factor of approximately five, indicates a need for employment of a variable integration step feature in conjunction with this method.

The results of Figs. 5A, 5B, and 5C were obtained using alternate cycles of the integral-absolute value scheme on η , and the $\text{Min } H^*$ scheme on θ . As observed earlier, the bang-bang character of the throttle history is preserved throughout the process, this being a feature of the method.

Concluding Remarks

It is felt that the numerical results obtained are too limited to provide conclusions on the relative merits of the three methods. The differences in speed of convergence exhibited in Figs. 3, 4, and 5 are insignificant in comparison to improvements attainable by modest amounts of experimentation with penalty constant adjustment procedures.

Perhaps the most significant fact emerging from the experiments reported herein is that several successive approximation techniques can be successfully adapted to problems featuring inequality constraints on the control variables. The three methods examined are workable and possess the 'hammer and tongs' quality so desirable for engineering applications.

On the other hand, it is clear that continuing research in the class of successive approximation methods is likely to be fruitful, perhaps not only in the evolution of more efficient computational schemes, but also in contributing to the understanding of the various phenomena arising in variational problems of flight performance, control and guidance.

Acknowledgments

The modified gradient and $\text{Min } H^*$ techniques were developed under Contracts AF 29(600)-2671 and AF 29(600)-2733 with USAF Missile Development Center, monitored by AFOSR Directorate of Research Analysis, Holloman AFB, New Mexico. Ref. 3 presents derivations of these two schemes carried out by means other than that employed in the present paper.

The scheme employing the integral-absolute value metric was an offshoot of an investigation of the Pontryagin theory performed under Contract Nonr 3384(00) with ONR's Information Systems Branch⁷.

The low-thrust applications work was performed under NASA Contract NAS 8-1549 with the Aeroballistics Division of the Marshall Space Flight Center, Huntsville, Alabama.

Symbols

x_i	state variables
y_k	control variables
$(\dot{})$	derivative with respect to time
g_i	functional representation of basic system right members (Eq. 1)
t_0	initial time
t_f	terminal time
$(\sim)$	initial conditions

P	function of terminal values to be minimized	A_0	gravitational constant
P'	modified function including penalty terms (Eq. 3)	m	mass
K_j	positive weighting factors for penalty terms	θ	thrust direction angle
δx_i	state variable perturbations	ψ	heliocentric angle
δy_k	control variable perturbations	v_e	exhaust velocity
H^*	function defined by Eq. 8	η	throttle control variable
Δx_i	total change in state variable	$\Delta \eta$	increment in throttle control variable
Δy_i	total change in control variable	$()_f$	final values
ξ_q	$(0 \leq \xi_q \leq 1)$, constant	D	coefficient of η in H^*
$(\bar{})$	variable for nominal trajectory	References	
$\Delta P'$	total change in function P'		
z_k	system variable used for evaluating $\Delta P'$	1.	Kelley, H.J.; <u>Gradient Theory of Optimal Flight Paths</u> , ARS Semiannual Meeting, Los Angeles, Calif., May 9-12, 1960; ARS Journal, October 1960.
p_k	adjoint variables for z system	2.	Bryson, A.E., Carroll, F.J., Mikami, K. and Denham, W.F.; <u>Determination of the Lift or Drag Program that Minimizes Re-entry Heating with Acceleration or Range Constraints Using a Steepest Descent Computation Procedure</u> , IAS 29th Annual Meeting, New York, N.Y., January 23-25, 1961.
$\hat{H}$	function defined by Eqs. 23 and 47	3.	Kelley, H.J.; <u>Method of Gradients</u> , Chapter 6 of "Optimization Techniques", G. Leitmann, Ed., Academic Press, to appear.
$\hat{\Delta y}_k$	total change in y_k for gradient direction	4.	Bryson, A.E. and Denham, W.F.; <u>A Steepest-Ascent Method for Solving Optimum Programming Problems</u> , Report BR-1303, Raytheon Missile and Space Division, August 10, 1961.
a_k	positive constant	5.	Hoelker, R.F.; <u>The Evolution of Guidance Theory and Trajectory Analysis into a Single Scientific Discipline</u> , Meeting of the Institute of Navigation, Williamsburg, Va., June 1961.
b_k	$(0 > b_k)$, search parameter for gradient method	6.	Courant, R.; <u>Variational Methods for the Solution of Problems of Equilibriums and Vibrations</u> , Bulletin of the American Mathematical Society, Vol. 49, No. 1, January 1943.
y_{k1}	lower limit on control variable y_k	7.	Kopp, R.E.; <u>Pontryagin Maximum Principle</u> , Chapter 7 of "Optimization Techniques", G. Leitmann, Ed., Academic Press, to appear.
y_{k2}	upper limit on control variable y_k		
ζ	$(0 \leq \zeta \leq 1)$, searching parameter for Min H^* method		
y_k^*	control function for Min H^*		
ρ_k	search parameter for integral-absolute value metric method		
θ_1, θ_2	generic functions		
u	radial velocity		
v	circumferential velocity		
R	radial distance		
T	thrust		
R_0	initial radial distance		

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FIG. 1 PLANAR EARTH-MARS ORBIT TRANSFER PATHS
FOR MINIMUM TIME

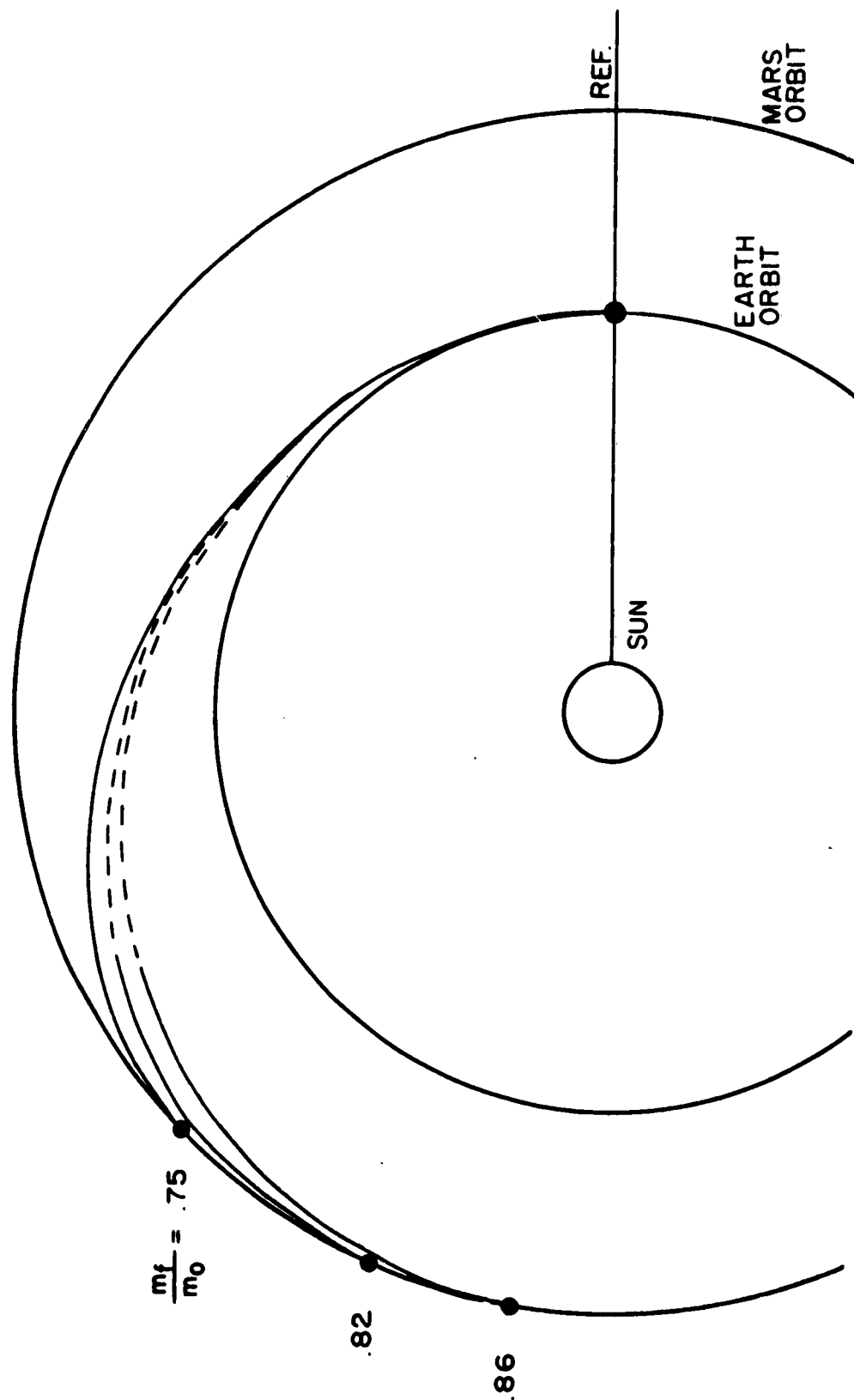


FIG. 2 MINIMUM TIME FOR PLANAR EARTH-
MARS ORBIT TRANSFER VERSUS FINAL MASS

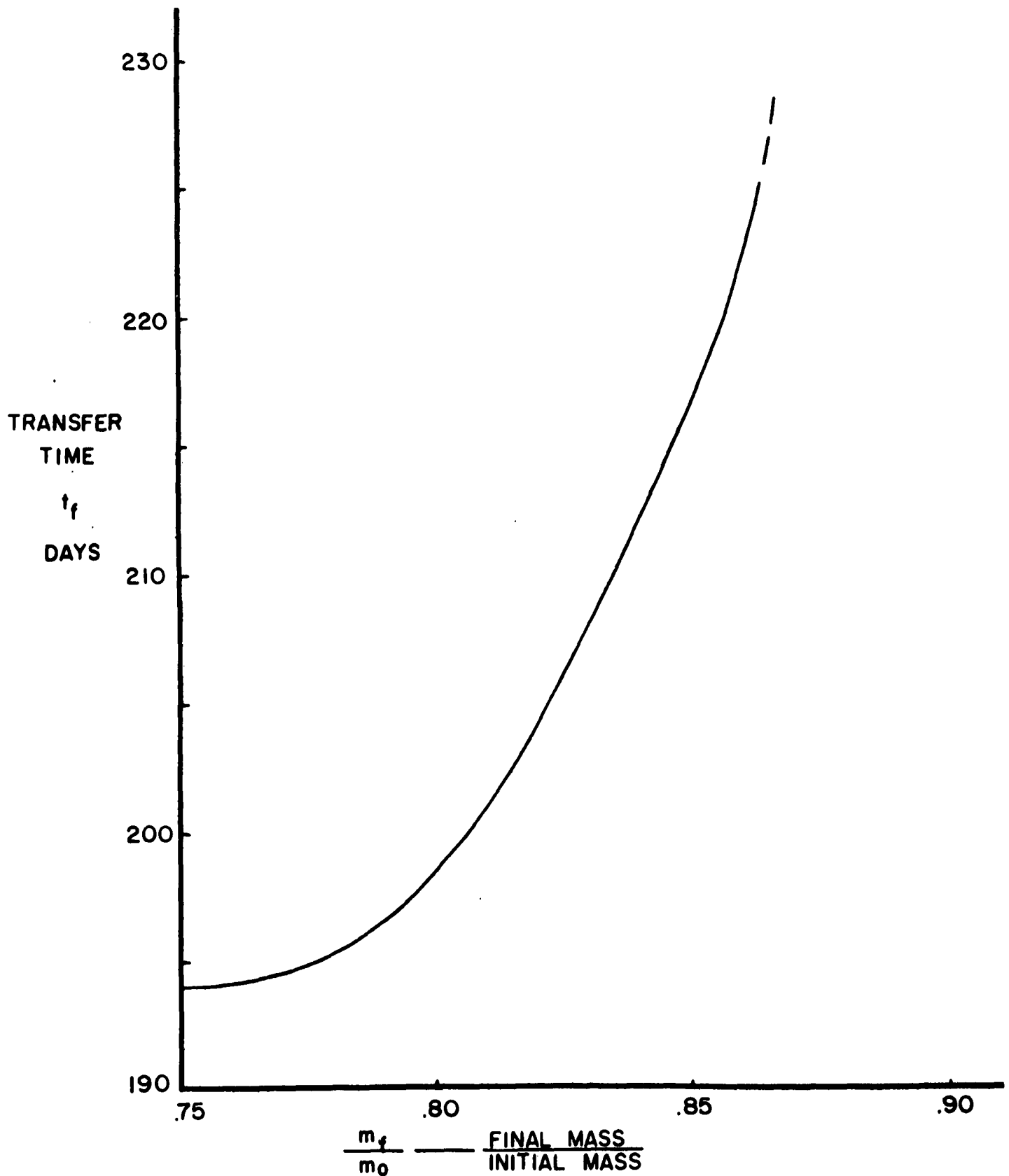
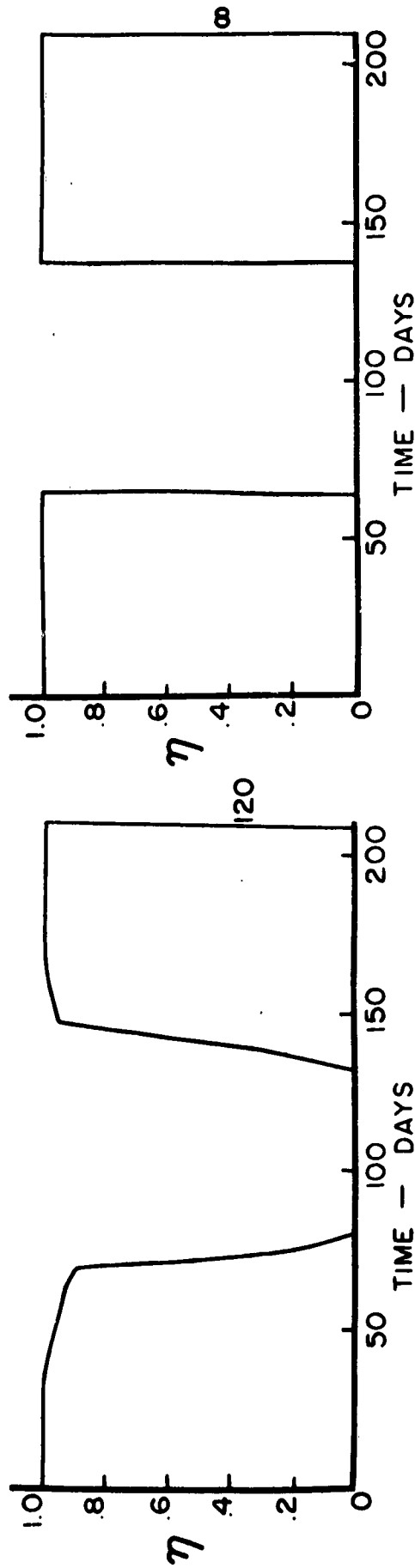
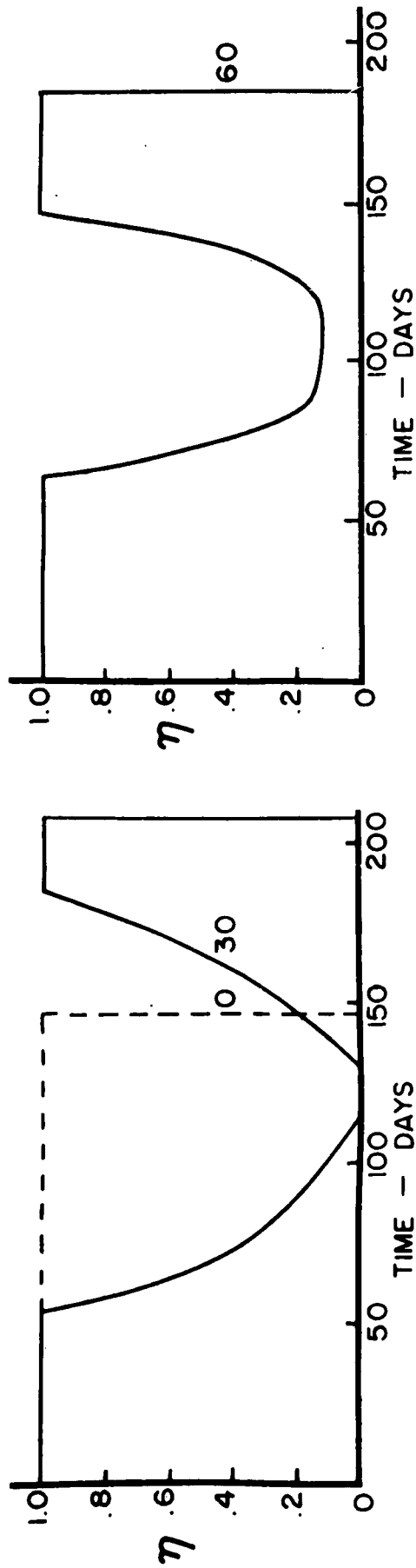
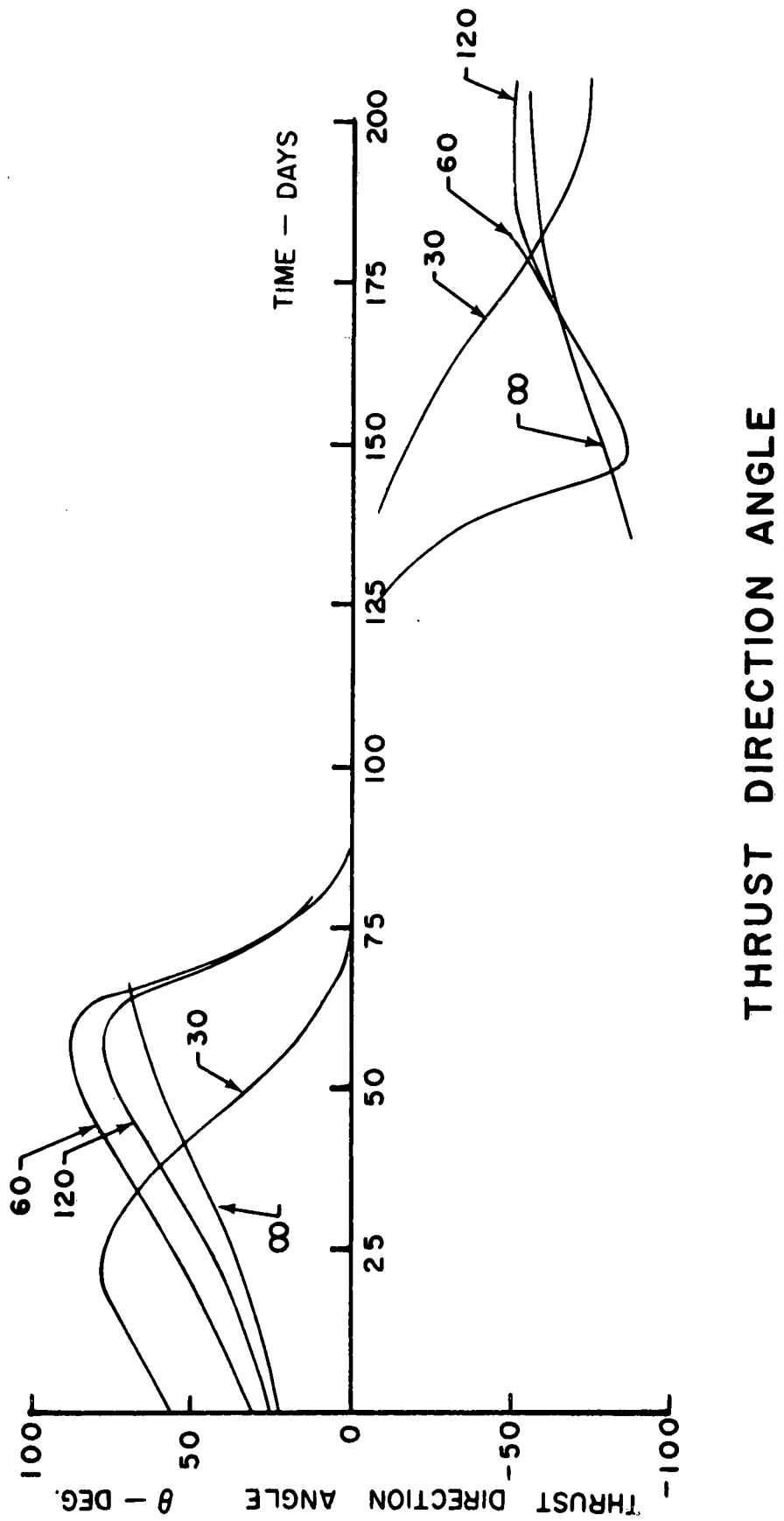


FIG. 3A SUCCESSIVE APPROXIMATIONS TO PLANAR EARTH-MARS
ORBIT TRANSFER VIA MODIFIED GRADIENT METHOD



THROTTLE CONTROL VARIABLE

FIG. 3B SUCCESSIVE APPROXIMATIONS TO PLANAR EARTH - MARS
ORBIT TRANSFER VIA MODIFIED GRADIENT METHOD



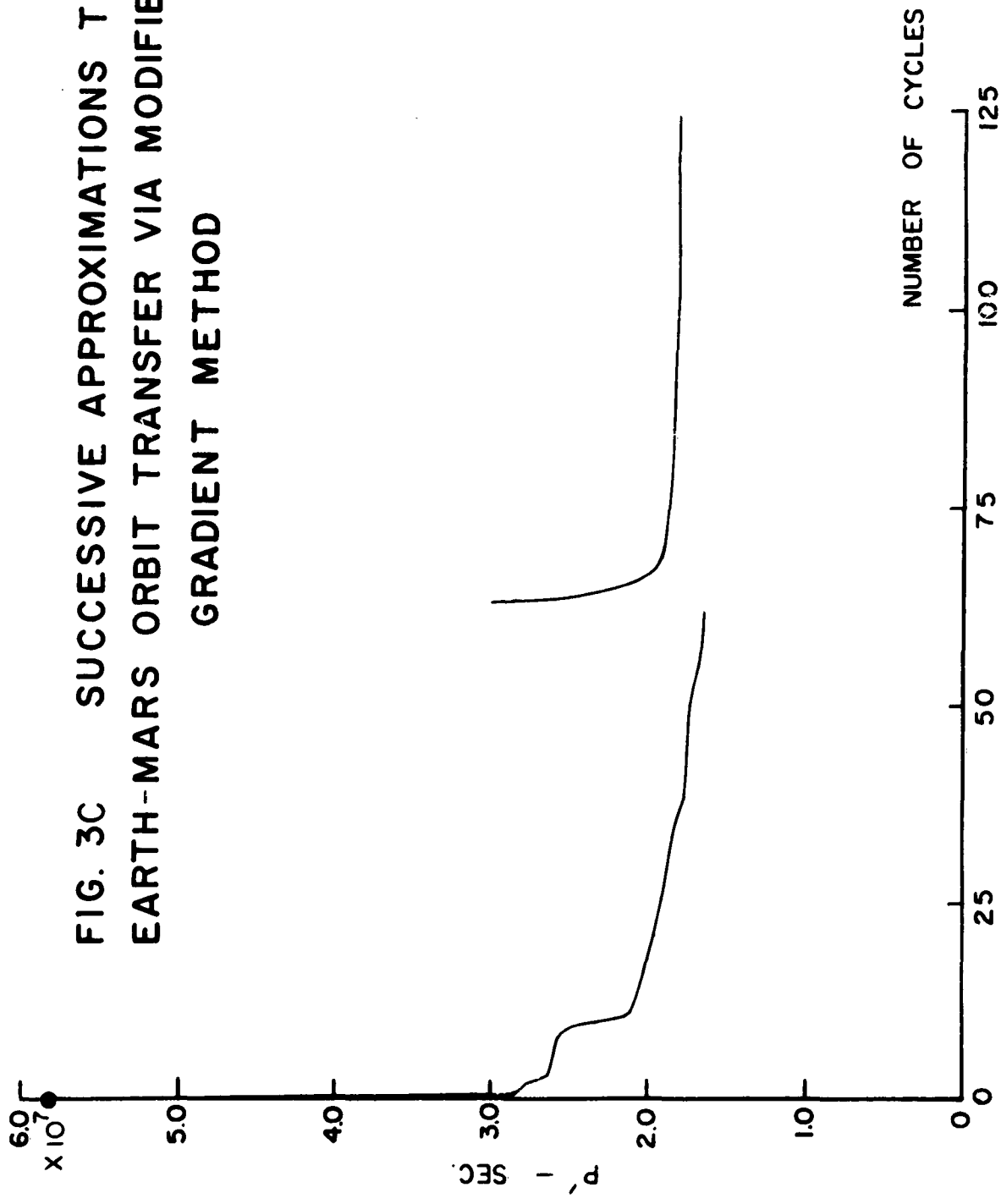


FIG. 3C SUCCESSIVE APPROXIMATIONS TO PLANAR
EARTH-MARS ORBIT TRANSFER VIA MODIFIED
GRADIENT METHOD

P' VERSUS NUMBER OF CYCLES

FIG. 4A SUCCESSIVE APPROXIMATION TO PLANAR EARTH-MARS
ORBIT TRANSFER VIA MIN H^* METHOD

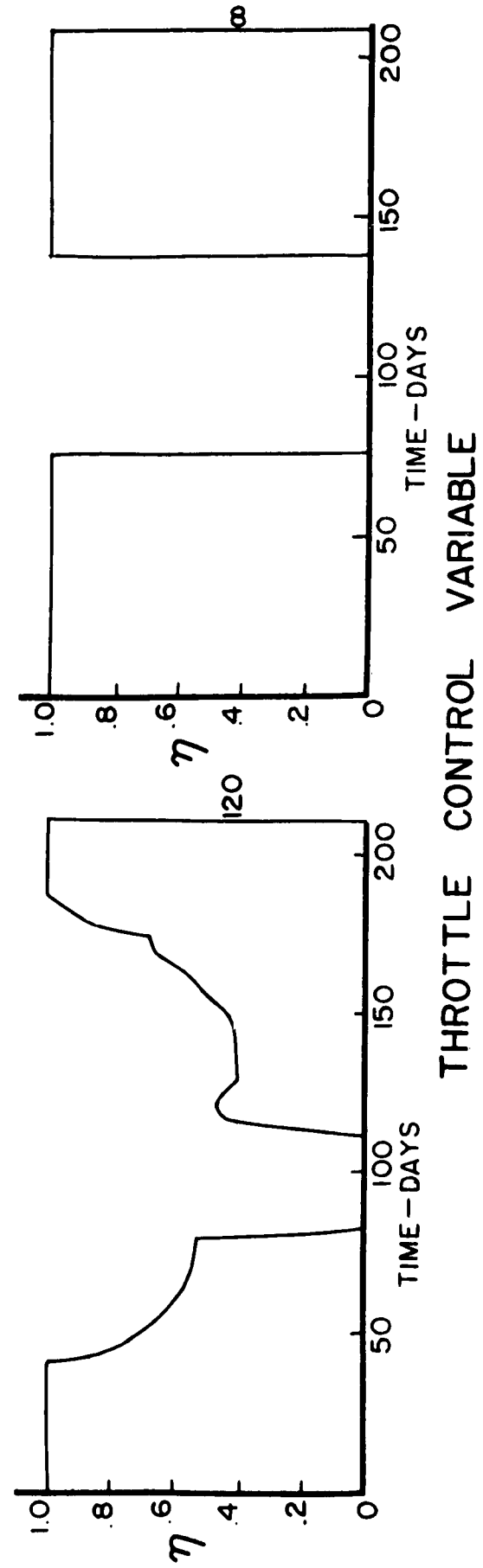
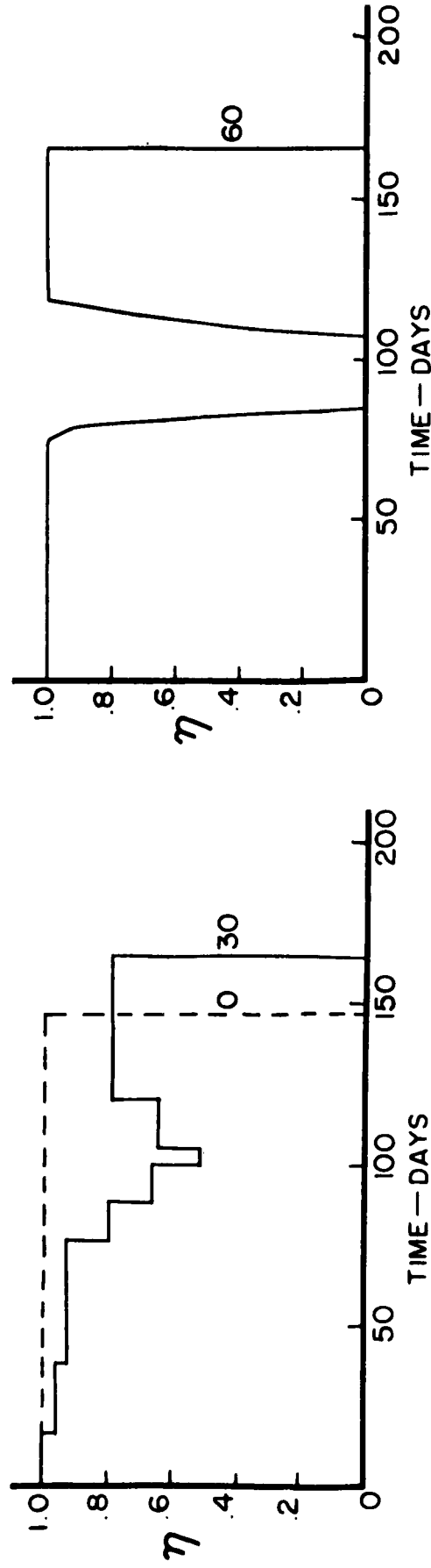
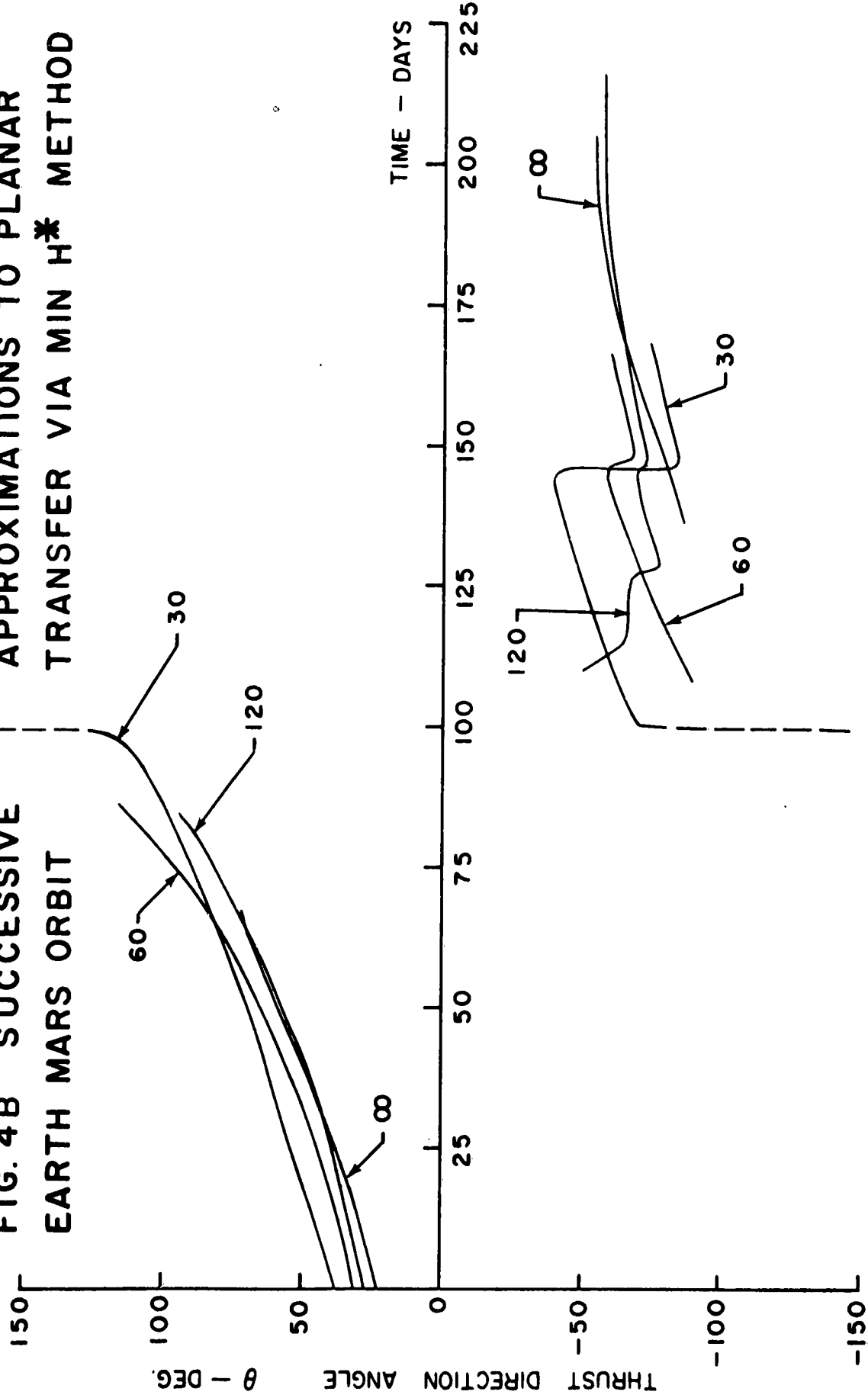


FIG. 4B SUCCESSIVE

EARTH MARS ORBIT

APPROXIMATIONS TO PLANAR

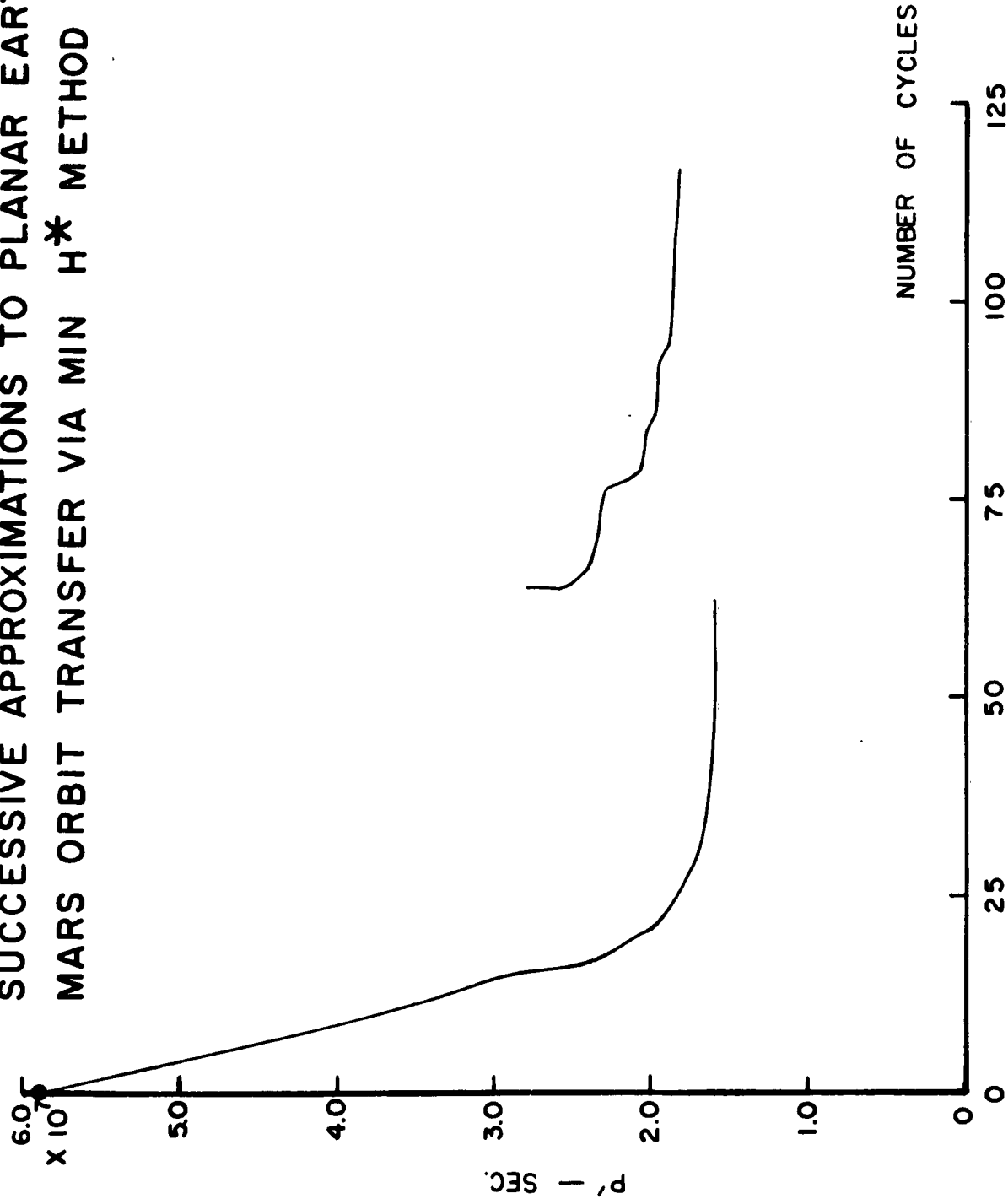
TRANSFER VIA MIN H^* METHOD



THRUST DIRECTION ANGLE

FIG. 4C

SUCCESSIVE APPROXIMATIONS TO PLANAR EARTH-
MARS ORBIT TRANSFER VIA MIN H^* METHOD



P' VERSUS NUMBER OF CYCLES

FIG. 5A SUCCESSIVE APPROXIMATIONS TO PLANAR EARTH-MARS
ORBIT TRANSFER VIA INTEGRAL - ABSOLUTE VALUE METHOD

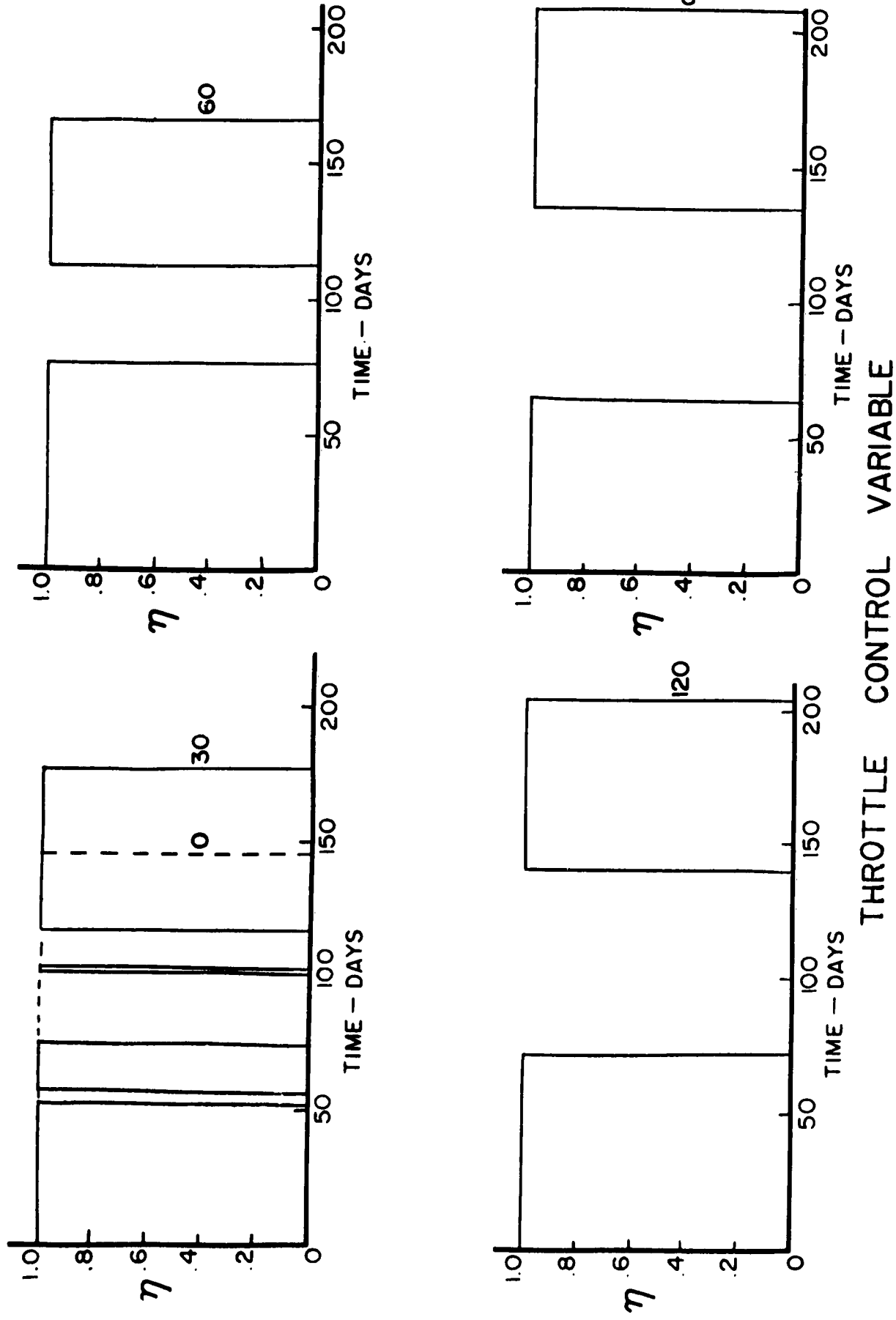


FIG. 5B SUCCESSIVE APPROXIMATIONS TO PLANAR EARTH-MARS
ORBIT TRANSFER VIA INTEGRAL-ABSOLUTE VALUE METHOD

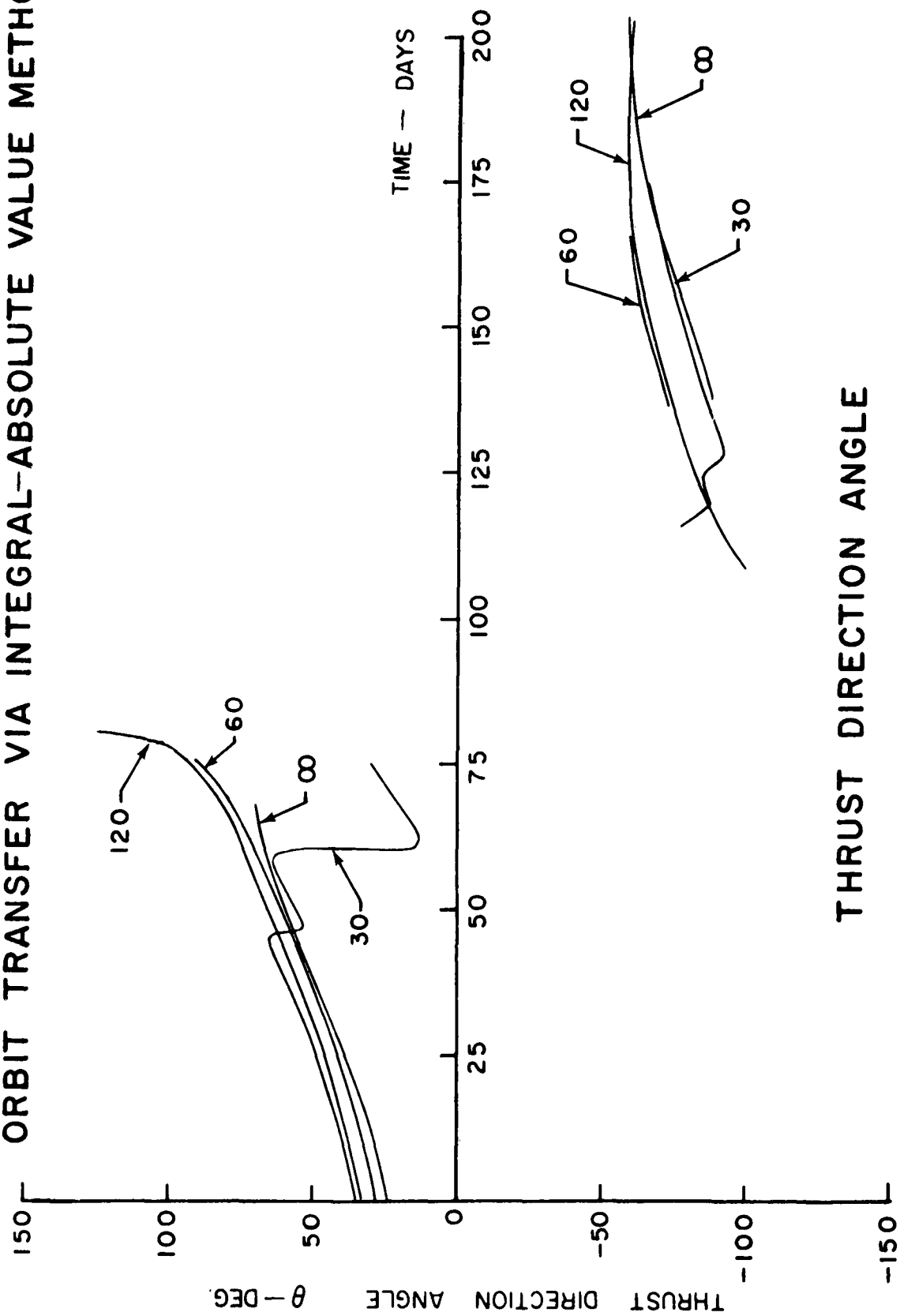
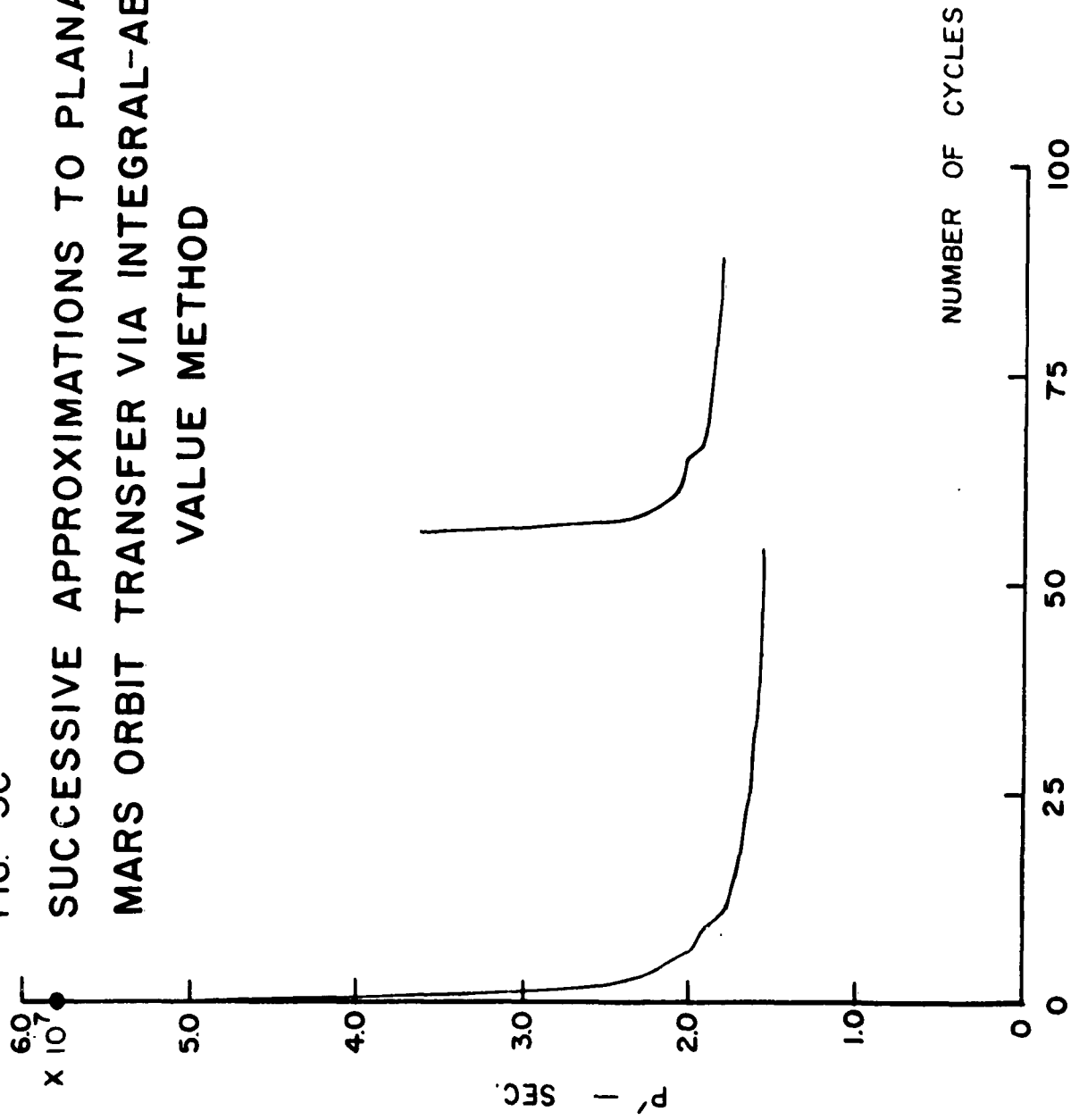


FIG. 5C
 SUCCESSIVE APPROXIMATIONS TO PLANAR EARTH-
 MARS ORBIT TRANSFER VIA INTEGRAL-ABSOLUTE
 VALUE METHOD



P' VERSUS NUMBER OF CYCLES

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