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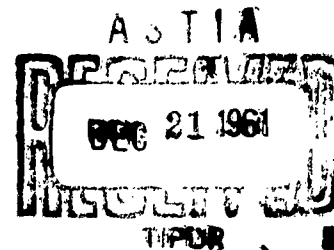
INTERPOLATION BY HARMONIC POLYNOMIALS

by

J.H. Curtiss

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Abstract:

Let $H_n(u; z)$ denote the harmonic polynomial of degree at most n found by interpolation in $2n+1$ points to a function u given on the boundary C of a region D of the complex z -plane. It is proved that (a) for any bounded D there always exist interpolation points on C so that H_n can be uniquely determined for each n , and (b) for a wide class of Jordan regions D and for boundary data u with a smooth first derivative on C the points of interpolation on C can be chosen so that $\lim_{n \rightarrow \infty} H_n(u; z)$ exists, $z \in \bar{Q} + D$, and gives the solution of the Dirichlet problem for u and D . Explicit formulas are derived for H_n in the case of interpolation on a circle and on an ellipse, and convergence is proved in these cases for arbitrary continuous boundary data. Various generalizations are indicated.

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Interpolation by Harmonic Polynomials*

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1. Introduction. Let U be a function on an open set D in the complex z -plane, $z = x + iy$, to the real numbers. The function U is said to be harmonic on D if it is continuous there together with its partial derivatives of the first two orders with respect to x and y , and if it satisfies Laplace's equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0.$$

A function is said to be harmonic at a point if it is harmonic in some neighborhood of the point. In what follows, we generally shall take D to be a region, by which we mean a non-empty open connected subset of the plane.

Suppose now that on the boundary C of a region D a continuous function u from C to the real numbers is given. It is well known that if D satisfies certain mild restrictions — for example, if its complement is such that no component reduces to a point — then there exists a unique function U harmonic on D and continuous on $D + C$ which coincides with u on C . The construction of this function U (or in purely theoretical contexts, the proof of its existence) is the substance of the famous Dirichlet problem, or first boundary value problem of potential theory.** There are many applications,

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** An up-to-date introduction to the problem will be found in [1, Chapter V].

and generalizations of the two-dimensional formulation given here.

A good many years ago, J.L. Walsh [13] [14] raised a question as to whether it might not be possible to represent the solution of the Dirichlet problem as the limit of a sequence of harmonic polynomials found by interpolation to the boundary data u on C in suitably chosen points. Although a number of constructive methods of finding solutions of the Dirichlet problem are now known, the suggestion of Walsh might yield a particularly simple analytic approximation process, if only a reasonably general convergence theory for it could be provided. It would appear that this has never been done heretofore.* In the present paper, it is established that there exist convergent sequences of harmonic interpolation polynomials for a wide class of regions and boundary data, but the last word on the general subject has not been said by any means.

In the references given above, Walsh indicates how to obtain an affirmative answer to his question for the special case in which C is a circle. Recently [18] he has supplemented this with a solution for the ellipse, but it is one in which it is necessary to impose a smoothness condition on the boundary data u beyond continuity. Convergence is then achieved on the ellipse as well as interior to it. Practical considerations and experience with similar but easier problems in complex polynomial interpolation suggest that the primary goal of a general convergence theory in the present problems might well be to establish convergence inside the region under minimal hypotheses on u and C , without regard to convergence on the boundary, where the problem is more delicate and where in the applications the sought-for function is known anyhow.

A survey of the general problem and of the analogous one for complex polynomial interpolation has recently been published by the author [4], in which a necessary and sufficient condition for convergence based on linear operator theory is given. But the condition

* For instance, the gap in theory is noted by Krylov [17, p. 489]. He presents in [17] a formal construction of the Green's function by harmonic polynomial interpolation, and in a numerical illustration it seems to give good results.

gives little indication as to whether in a given instance there exists a convergent sequence of harmonic interpolation polynomials and as to how to construct it if so.

In the present paper we first (Section 2) examine the structure of harmonic interpolation polynomials. It is proved that for any bounded region there always exists a point set S on the boundary such that a harmonic polynomial of degree at most n which assumes $2n + 1$ preassigned values at $2n + 1$ of these points S will be uniquely determined for $n = 1, 2, \dots$. The existence question requires some attention because unlike in the case of complex polynomial interpolation there can be situations involving distinct points of interpolation in which the harmonic interpolation polynomial of appropriate degree is indeterminate. A study of the structure of the interpolation polynomial is important because the basic difficulty in the whole convergence problem for harmonic polynomial interpolation is the absence of any compact formulas like the Lagrange and the Cauchy-Hermite interpolation formulas which are available in the complex polynomial case.

In Section 3 the convergence properties of harmonic interpolation polynomials are related to other types of harmonic polynomial approximations. It is thereby proved (Theorem 3.4) that for any region with a reasonably smooth boundary C and for any boundary data u possessing a smooth first derivative it is possible to solve the Dirichlet problem in terms of the limit of a sequence of harmonic polynomials found by interpolation to u on the boundary. The convergence takes place in the closed region. Generalizations and situations in which restrictions on the boundary can be eliminated are indicated. The results of this section do give an affirmative answer to Walsh's original question for a wide class of regions and boundary values, but they do not provide very explicit instructions as to how to choose successful interpolation points in practice.

Finally in Section 4 we consider a special choice of interpolation points which is the natural analogue, for a general boundary curve, of points equally spaced on a circle. For the cases of the circle and the ellipse, explicit formulas are derived for the harmonic interpolation

polynomials. The derivation is based on the classical Faber polynomials. Convergence is proved for continuous boundary values. Although Walsh's results [18] anticipate these to some extent, the formula for the polynomial for the ellipse is new, as is the proof of convergence inside the ellipse with no smoothness conditions on the boundary data beyond continuity.

The methods are largely elementary, in the sense that they are based on material usually covered in good first courses in classical complex variable theory and linear algebra. It is only in the proof of Theorem 3.4 that references to sophisticated work on approximation theory and conformal mappings are needed.

This is a paper on theory, not on practice, but it does contain at least one indication as to how the theory might be put to work in the computation laboratory. The all-important issue in the practical applications will certainly be the correct choice of the interpolation points on the boundary C of the region. It is shown below in Section 3, Theorems 3.2, 3.3, and 3.4, that in many, if not all cases, the successful choice of the points for the interpolation polynomial of n -th degree will be a choice which maximizes the absolute value of a certain determinant which first appears in the display (2.3) below. (The polynomials $p_j(z)$ which appear in (2.3) are any conveniently chosen complex polynomials in z with degrees coinciding with their subscripts. . The choice $p_j(z) = z^j$ would be permissible.) If it seemed worthwhile to do so, then for a curve C of a given shape it would certainly be feasible to calculate the maximizing points for some of the lower values of n and put them in a library ready to use for any boundary data u which might be presented. If this is all that is done, then the interpolation polynomial of degree n would have to be determined for each new boundary function by solving a certain set of linear equations in $2n + 1$ unknowns for the coefficients of the polynomial. But in Section 2 a formula is derived for the interpolation polynomial of degree n which expresses it as a linear combination of certain harmonic polynomials, there denoted by $B_k(z)$, $k = 1, \dots, 2n + 1$, which do not depend on the boundary data. The coefficients in the linear combination are merely the values of the boundary data

at the chosen $2n+1$ points on C . It would be possible to calculate the harmonic polynomials $B_k(z)$ once and for all, after the points of interpolation on C have been chosen, and to put them in the library along with the interpolation points. One would then have an exceedingly simple and rapid method of finding an analytic approximation to the solution of any Dirichlet problem that might be presented for that particular region.

2. The structure of harmonic interpolation polynomials. A harmonic polynomial of degree n is an expression of the form

$$h(z) = \alpha_0 + \sum_{j=1}^n (\alpha_j r^j \cos j\theta + \beta_j r^j \sin j\theta),$$

$$z = r(\cos \theta + i \sin \theta),$$

where $\alpha_0, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ are real, α_n and β_n are not both zero, and i is the imaginary unit. This can be written as

$$(2.1) \quad h(z) = a_0 + \sum_{j=1}^n (a_j z^j + \bar{a}_j \bar{z}^j),$$

where $a_0 = \alpha_0$, $a_j = (\alpha_j - i\beta_j)/2$, $j = 1, \dots, n$, and the bar denotes conjugate complex. The right side of (2.1) is the real part of a polynomial in z of degree n with coefficients $a_0, 2a_1, \dots, 2a_n$. It is also the imaginary part of a polynomial in z of degree n with coefficients $ia_0, 2ia_1, \dots, 2ia_n$. Thus every harmonic polynomial is the real part of a polynomial in z and also the imaginary part of a polynomial in z . Conversely, the real and imaginary parts of any polynomial in z are harmonic polynomials.

The real part of a polynomial in $\bar{z}$ is at the same time the real part of a related polynomial in z . The coefficients of the latter polynomial are the complex conjugates of the coefficients of the former polynomial. From this it follows that the real part (and also the imaginary part) of any finite linear combination of the monomials $1, z, z^2, \dots, \bar{z}, \bar{z}^2, \dots$ with complex coefficients is also a harmonic polynomial.

Theorem 2.1 . If a finite linear combination of the monomials $1, z, z^2, \dots, \bar{z}, \bar{z}^2, \dots$ vanishes for all z on the boundary of a bounded region D , then the coefficients are all zero.

For the proof, suppose first that the linear combination is a harmonic polynomial $h(z)$, and let it be written out in the form (2.1). By the maximum principle for harmonic functions [1, p. 179] $h(z)$ must vanish identically on the region D . Therefore the complex polynomial in z of which $h(z)$ is the real part must reduce to zero or to a pure imaginary constant [1, p. 69]. This implies in turn that all the coefficients in (2.1) must be zeros.

More generally, let the linear combination be

$$L = \sum_{j=0}^n a_j z^j + \sum_{j=1}^n b_j \bar{z}^j,$$

where of course some of the coefficients may be zero and a_0 now is not necessarily real. Then*

$$(2.2) \quad \Re L = \Re \left[\sum_0^n a_j z^j + \sum_1^n \bar{b}_j z^j \right].$$

Since L and therefore $\Re L$ vanishes on the boundary of D , the polynomial in z in the square brackets in (2.2) reduces to a pure-imaginary constant, and $a_j + \bar{b}_j = 0$, $j = 1, \dots, n$. Therefore

$$L = i\alpha + \sum_1^n (a_j z^j - \bar{a}_j \bar{z}^j), \quad \alpha \text{ real.}$$

This expression is the imaginary part of the polynomial $i\alpha + \sum_1^n 2a_j z^j$, multiplied by i , so it is a harmonic polynomial multiplied by i , and since it vanishes on the boundary of D , all of its coefficients must be zero. This completes the proof of Theorem 2.1 .

If $p_1(z), p_2(z), \dots, p_n(z)$ are complex polynomials in z of respective degrees $1, 2, \dots, n$, then the expression $b_0 + \sum_{j=1}^n (b_j p_j(z) + \bar{b}_j \overline{p_j(z)})$ with b_0 real is also a harmonic polynomial of degree n . With a rearrangement of terms it can be put into the form of $h(z)$ above.

* The symbol $\Re$ means "real part of".

If such a polynomial assumes given values $u_1, u_2, \dots, u_{2n+1}$ respectively at points $z_1, z_2, \dots, z_{2n+1}$, then the coefficients $b_0, b_1, \dots, b_n, \bar{b}_1, \dots, \bar{b}_n$ are a solution of the $2n+1$ linear algebraic equations

$$b_0 + \sum_{j=1}^n [b_j p_j(z_h) + \bar{b}_j \overline{p_j(z_h)}] = u_h, \quad h = 1, 2, \dots, 2n+1.$$

We shall now examine the existence and uniqueness of the solutions of such sets of equations.

Given the system of linear algebraic equations

$$P(z_h) = c_0 + \sum_{j=1}^n [c_j p_j(z_h) + d_j \overline{p_j(z_h)}] = u_h, \\ h = 1, 2, \dots, 2n+1,$$

where the numbers u_h are regarded as given complex numbers and the letters c_j and d_j stand for unknowns, a sufficient condition for the existence of a unique solution is the non-singularity of the matrix

$$(2.3) \quad A = \begin{bmatrix} 1 & p_1(z_1) & \dots & p_n(z_1) & \overline{p_1(z_1)} & \dots & \overline{p_n(z_1)} \\ \vdots & & & \vdots & & & \vdots \\ 1 & p_1(z_{2n+1}) & \dots & p_n(z_{2n+1}) & \overline{p_1(z_{2n+1})} & \dots & \overline{p_n(z_{2n+1})} \end{bmatrix}$$

which is equivalent to the non-vanishing of the determinant $\det A$. Now A can be transformed by elementary column transformations into a certain matrix A_0 which is the specialization of A in which $p_j(z)$ is replaced by z^j . A necessary and sufficient condition for the singularity of A_0 , and therefore for that of A , is that there exists some linear combination of the monomials $1, z, \dots, z^n, \bar{z}, \bar{z}^n$, with coefficients not all zero, which vanishes in all of the points $z_1, \dots, z_{2n+1}$. The real part of such a linear combination is a harmonic polynomial of degree not greater than n . Conversely any harmonic polynomial of degree not greater than n is such a linear combination. The upshot of this is that the necessary and sufficient condition for the non-vanishing of $\det A$ is that the points $z_1, \dots, z_{2n+1}$ shall not all lie on an algebraic curve given by an equation of the type $H(z) = 0$, where H is a harmonic polynomial of degree n or less. We henceforth proceed under the assumption that this non-

singularity condition is fulfilled. In specific instances its validity must be checked. It is an implication of the condition that the points $z_1, \dots, z_{2n+1}$ must all be distinct, for if they were not, the rank of A would of course be less than $2n+1$ and the columns would be linearly independent. We call the condition n -s for short (" n -s" for "non-singularity").

Suppose that in the system of equations $P(z_h) = u_h$ discussed above, the numbers u_h are all real. Then $P(z_h) = \overline{P(z_h)}$, $h = 1, \dots, 2n+1$, which means that the coefficients c_j and d_j satisfy the equations $c_0 - \bar{c}_0 + \sum_{j=1}^n [(c_j - \bar{d}_j) p_j(z_h) + (d_j - \bar{c}_j) \overline{p_j(z_h)}] = 0$, $h = 1, \dots, 2n+1$. Condition n -s now implies that $c_0 = \bar{c}_0$, $c_j = \bar{d}_j$, $j = 1, \dots, n$, so $P(z)$ is necessarily of the form $B(z) = b_0 + \sum_{j=1}^n (b_j p_j(z) + \overline{b_j p_j(z)})$, b_0 real. We formalize this in a theorem:

Theorem 2.2 . Given complex polynomials $p_1(z), p_2(z), \dots, p_n(z)$ in z of respective degrees $1, 2, \dots, n$; real numbers $u_1, u_2, \dots, u_{2n+1}$; and complex numbers $z_1, z_2, \dots, z_{2n+1}$ satisfying the n -s condition; there exists a unique linear combination of the polynomials $p_j(z)$ of the form $c_0 + \sum_{j=1}^n (c_j p_j(z) + d_j \overline{p_j(z)})$ which assumes the value u_h in the point z_h , $h = 1, \dots, 2n+1$. Furthermore $d_j = \bar{c}_j$, so this is a harmonic polynomial $B(z)$ of degree at most n .

Theorem 2.3 . In Theorem 2.2 if the numbers $u_1, u_2, \dots, u_{2n+1}$ are the respective values assumed in the points $z_1, \dots, z_{2n+1}$ by a harmonic polynomial $h(z) = c_0 + \sum_{j=1}^n (c_j z^j + \bar{c}_j \bar{z}^j)$ of degree at most n , then $B(z) \equiv h(z)$.

The proof consists in first putting $B(z)$ in the form $a_0 + \sum_{j=1}^n (a_j z^j + \bar{a}_j \bar{z}^j)$ and then using the non-singularity of A_0 to show that all the coefficients of $B(z) - h(z)$ must vanish. The theorem implies that there is essentially one and only one harmonic interpolation polynomial $B(z)$ under the hypotheses of Theorem 2.2 regardless of the choice of the base polynomials p_j .

Consider now the matrix A in (2.3) and modify it by replacing the k -th row-vector with the row-vector $\hat{p} = (1, p_1(z), \dots, p_n(z), \overline{p_1(z)}, \dots, \overline{p_n(z)})$. Let $A_k(z)$ denote the new matrix so obtained. The quotient $B_k(z) = \det A_k(z) / \det A$ is clearly a linear combination of the elements of $\hat{p}$ which vanishes for $z = z_1, z_2, \dots, z_{k-1}, z_{k+1}, \dots, z_{2n+1}$ and equals one for $z = z_k$. Therefore by Theorems 2.2 and 2.3 it is the unique harmonic polynomial of degree at most n which behaves in this way. It follows that given any real numbers $u_1, u_2, \dots, u_{2n+1}$ and any set of n complex polynomials in z in which one of each degree from 1 to n is present, the unique harmonic polynomial of degree at most n which assumes these values in points $z_1, \dots, z_{2n+1}$ satisfying the n -s condition can be represented by the formula

$$(2.4) \quad H_n(z) = \sum_{k=1}^{2n+1} u_k B_k(z) = \sum_{k=1}^{2n+1} u_k \frac{\det A_k(z)}{\det A}.$$

If we write

$$B_k(z) = b_{ko} + \sum_{j=1}^n [b_{kj} p_j(z) + \overline{b_{kj}} \overline{p_j(z)}],$$

then b_{ko} is the cofactor of the first element in the k -th row of $A_k(z)$ (or A) divided by $\det A$, b_{kj} , $j = 1, 2, \dots, n$, is the cofactor of $p_j(z)$ in $A_k(z)$ divided by $\det A$, and $\overline{b_{kj}}$, $j = 1, 2, \dots, n$, is the cofactor of $\overline{p_j(z)}$ divided by $\det A$. These cofactors divided by $\det A$ form the k -th column of the matrix A^{-1} , so if we denote this column-vector by $[A^{-1}]_k$, we can write $B_k(z) = \hat{p} \cdot [A^{-1}]_k$. Thus we get the compact formula

$$H_n(z) = \sum_{k=1}^{2n+1} u_k \hat{p} \cdot [A^{-1}]_k = \hat{p} A^{-1} \hat{u},$$

where $\hat{u}$ is the column-vector with elements $u_1, u_2, \dots, u_{2n+1}$.

We conclude with a result which is basic to certain convergence theorems in the next section.

Theorem 2.4 . Given any bounded region D , for each n , $n = 1, 2, \dots$, there always exists at least one set of $2n+1$ points on the boundary of D which satisfies the n -s condition.

arc α it is always possible to pass a "curve" with equation $a_1 u_1(z) + \dots + a_n u_n(z) = 0$ through the points, then α must coincide with such a curve. In our case, if no set of $2n+1$ points satisfying the n -s condition were to exist on the boundary of D , then each such point set must lie on a locus $H(z) = 0$, where H is a harmonic polynomial of degree at most n with coefficients not all zero. But then according to the above result the boundary of D would have to coincide with such a curve, and this would be impossible by the maximum principle for harmonic functions.

3. Convergence theorems for sequences of harmonic interpolation polynomials. Let D be an arbitrary bounded region and let C be its boundary. Consider the following infinite sequence of point sets lying on C :

$$\begin{aligned} S_1 &= \{z_{11}, z_{12}, z_{13}\} \\ S_2 &= \{z_{21}, z_{22}, z_{23}, z_{24}, z_{25}\} \\ &\vdots \\ S_n &= \{z_{n1}, z_{n2}, \dots, z_{n,2n+1}\} \\ &\vdots \end{aligned}$$

We assume throughout this section that for each n , S_n satisfies the n -s condition of Section 2. Let $B_{nk}(z)$, $k = 1, \dots, 2n+1$, denote for each k the (unique) harmonic polynomial of degree at most n which vanishes at all points of S_n except the point z_{nk} , at which it equals one. For any function u given on C , we construct the harmonic polynomial of degree at most n found by interpolation to u at the points S_n ; written in the form (2.4) this polynomial is

$$(3.1) \quad H_n(u; z) = \sum_{k=1}^{2n+1} u(z_{nk}) B_{nk}(z).$$

The method used in this section to study the convergence properties of the sequence $H_1, H_2, \dots$, consists in referring them to the properties of other sorts of harmonic polynomial approximations to u . We recall that if h is any harmonic polynomial of degree not more than n , then by Theorem 2.3,

$$H_n(h; z) = \sum_{k=1}^{2n+1} h(z_{nk}) B_{nk}(z) \equiv h(z).$$

Suppose now that at least for a particular u it is possible to solve the Dirichlet Problem for u and D ; let U denote the solution. For any z in the finite plane for which U exists,

$$H_n(u; z) - U(z) = \sum_{k=1}^{2n+1} [u(z_{nk}) - h(z_{nk})] B_{nk}(z) + h(z) - U(z).$$

Now let $T_n(z) = \sum_{k=1}^{2n+1} |B_{nk}(z)|$. By the maximum principle for harmonic functions,

$$|h(z) - U(z)| \leq \sup_{z \in C} |u(z) - h(z)|, \quad z \in D + C.$$

This with (3.1) implies that

$$(3.2) \quad |H_n(u; z) - U(z)| \leq \left[\sup_{z \in C} |u(z) - h(z)| \right] [1 + T_n(z)], \quad z \in D + C.$$

We base our convergence theorems all on this inequality. In the language of linear operator theory, the function $T_n(z)$ is the total variation of the kernel of H_n , regarded as a linear operator^{*}; or again, it is the norm of the functional H_n on the space of functions u . Clearly it must play a key role in the convergence theory.

In what follows we shall have frequent occasion to make use of the "o" and "O" notation when comparing orders of magnitudes of functions defined on the positive real integers. The reader will doubtless recall that the notation $a(z, n) = o(b(n))$ means that $\lim_{n \rightarrow \infty} a(z, n)/b(n) = 0$. Also, $a(z, n) = O(b(n))$ means that a constant M exists which is independent of n such that $a(z, n) \leq M b(n)$ for all n sufficiently large. A uniformity condition with respect to z on a set Z means that the limit in the "o" case is uniform with respect to z and the M in the "O" case is independent of z for z on Z .

Theorem 3.1. Let C be a Jordan curve^{**}, let u be continuous on C , and let $T_n(z) = O(1)$ uniformly on the subset γ of $D + C$. Then $\lim_{n \rightarrow \infty} H_n(u; z) = U(z)$

* For a further development of this point of view, see [4]. The comparison technique in (3.2) is a familiar one in the theory of interpolation; for example see [12, Chapter XIV], [6, Chapter IV].

** A Jordan curve (or simple closed curve) is homeomorphic to a circle.

uniformly on γ . If $T_n(z)$ is unbounded at any point z_0 , then there exists a continuous function u for which $\{H_n(u; z_0)\}$ is unbounded.

The theorem was first announced in [4]. The convergence statement follows from (3.2) and a theorem of Walsh [17, p. 169] to the effect that given any continuous function u on a Jordan curve, there exists a sequence of harmonic polynomials $h_1(z), h_2(z), \dots$ of respective degrees not greater than n which converges to u uniformly on C . The last sentence follows from linear operator theory; see [4].

Corollary. If $|B_{nk}(z)| = O(1/n)$, $k = 1, \dots, 2n+1$, uniformly in k and uniformly for z on any closed subset of the Jordan region D , and if u is continuous on C , then $\lim_{n \rightarrow \infty} H_n(u; z)$ exists uniformly for z on any closed subset of D and provides there the solution of the Dirichlet Problem for u and D .

The hypothesis of course implies that $T_n(z)$ is uniformly bounded on any closed subset of D . We have stated the corollary formally because a study of various special cases, such as the ones discussed in Section 4 below, seems to indicate that on any sufficiently smooth Jordan curve there always exists a sequence of point sets $S_1, S_2, \dots$ such that the condition on B_{nk} in the hypothesis of the corollary is satisfied.

It is to be noted that under the hypotheses of Theorem 3.1, the degree of convergence to u on D of any comparison sequence $h_1(z), h_2(z), \dots$ is duly reproduced by $H_1(u; z), H_2(u; z), \dots$ on γ . For example, if a sequence $h_1(z), h_2(z), \dots$ exists such that $u(z) - h_n(z) = O(n^{-q})$, $q > 0$, uniformly on C , then $H_n(u; z) - U(z) = O(n^{-q})$ uniformly on γ . It is known [3], [19], that if C is analytic* and if u has a k -th derivative $u^{(k)}$ with respect to arc length on C which satisfies the following condition (called a Lipschitz condition of order α):

* For any Jordan curve C there exist parametric representations of the form $z = f_1(\theta) + i f_2(\theta)$, where f_1 and f_2 are real continuous functions of period 2π , and where any two solutions θ of the equation for a given z on C differ by an integral multiple of θ . A Jordan curve is said to be analytic if there exists such a parametric representation in which f_1 and f_2 are analytic functions of θ and $|f_1'(\theta)| + |f_2'(\theta)| \neq 0$.

$$|u^{(k)}(z_1) - u^{(k)}(z_2)| \leq \lambda |z_1 - z_2|^\alpha, \quad \lambda > 0,$$

$$0 < \alpha \leq 1, \quad \text{all } z_1 \text{ and } z_2 \text{ on } C,$$

then there exist harmonic polynomials exhibiting the degree of convergence $O(n^{-k-\alpha})$.

The problem of finding point sets S_n on a general Jordan curve such that $T_n(z)$ is uniformly bounded for z inside the curve remains open as this is being written, but it is easy to show that there exist sets S_n for which the sequence $\{T_n(z)\}$ increases no more rapidly than does $2n+1$ for z on $D+C$. We now do this.

Theorem 3.2 . Let Γ be any closed bounded point set with the property that for each n a subset of $2n+1$ points satisfying the n -s condition exists. Then there exists a sequence of subsets of Γ , $S_1, S_2, \dots$, each satisfying the n -s condition, such that when $T_n(z)$ is constructed for the n -th subset, $T_n(z) \leq 2n+1$, $n = 1, 2, \dots$, $z \in \Gamma$.

For the proof, consider the matrix A appearing in (2.3) with $z_1, \dots, z_{2n+1}$ on Γ . (The polynomials $p_1, \dots, p_n$ in A may be chosen arbitrarily in what follows, except that as usual their degrees must agree with their subscripts.) Now $|\det A|$ is a continuous function of the $2n+1$ independent variables $z_1, \dots, z_{2n+1}$, and its domain Γ is a compact set. Therefore the function has a maximum on this set. The maximum cannot be zero, because the n -s condition in the Lemma is equivalent to stating that $\det A \neq 0$ for at least one S_n . Let S_n^* be any one of the sets S_n which maximize $|\det A|$ and denote the corresponding matrix by A^* . Consider now the matrix $A_k^*(z)$ obtained from A^* by replacing the k -th row-vector with $(1, p_1(z), \dots, p_n(z), \overline{p_1(z)}, \dots, \overline{p_n(z)})$. Because of the maximizing property of S_n^* , it must be true that $|\det A_k^*(z)| \leq |\det A^*|$ for all z on Γ . Let $B_{nk}(z)$ in the definition of $T_n(z)$ be specialized to $(\det A_k^*(z))/(\det A^*)$. Then $|B_{nk}(z)| \leq 1$ for all z on Γ , and the inequality in the Theorem follows from this.

The reasoning above is reminiscent of studies of interpolation points with extremal properties which have been made by Fekete, Leja, Shen, and others in connection with complex polynomial interpolation; see [17, 56 7.7-7.8].

Obviously the points S_n^* satisfy the n -s condition and are distinct.

From Theorem 2.4 it can be seen that at least whenever Γ contains the boundary of a bounded region, the hypotheses of Theorem 3.2 on Γ will be fulfilled.

Theorem 3.3 . Let D be a bounded simply connected region for which the Dirichlet Problem can be solved, at least for a certain function u continuous of the boundary C of D . If there exist harmonic polynomials $h_1(z), h_2(z), \dots$ of respective degrees no greater than $1, 2, \dots$, such that $u(z) - h_n(z) = o(1/n)$ uniformly on C , then there exists a sequence of point sets $S_1, S_2, \dots$ on C such that $\lim_{n \rightarrow \infty} H_n(u; z)$ exists uniformly on $D + C$ and provides the solution of the Dirichlet Problem for u and D .

Theorem 2.4 guarantees that the boundary C in this theorem satisfies the hypotheses imposed on Γ in Theorem 3.2, so there exists a certain sequence of subsets of C , say $S_1^*, S_2^*, \dots$, such that $T_n(z) \leq 2n+1$, $z \in D+C$ when $T_n(z)$ is constructed for S_n^* . Substituting this inequality into (3.2), we obtain

$$(3.3) \quad |H_n(u; z) - U(z)| \leq \sup_{z \in C} |u(z) - h_n(z)| (2n + 2), \quad z \in D + C,$$

from which the conclusion of Theorem 3.3 follows at once.

The polynomials $H_n(u; z)$ here interpolate to the boundary data u in the special points S_n^* which maximize $|\det A|$. For a brief discussion of computational aspects we refer the reader to the last paragraph of the Introduction, Section 1 above.

Under what conditions on $D + C$ and on the boundary data u do harmonic polynomials $h_n(z)$ exhibiting the degree of convergence $o(1/n)$ exist? We have indicated above that if C is an analytic Jordan curve and if the first derivative of u on C satisfies a Lipschitz condition of order α , then the desired harmonic approximation is available. It is possible however to relax the requirement of analyticity. By reference

to some work of Sewell [10][11] we shall now prove that the Dirichlet Problem can be solved by harmonic polynomial interpolation for a rather wide class of Jordan regions when the boundary data belong to a Lipschitz class.

Let C be a Jordan curve and let K be the region of the extended plane exterior to C . We consider a conformal mapping of K onto a region in another complex plane, which we call the w -plane. There exists a function $z = \psi(w)$, analytic and univalent on $|w| > 1$, which maps $|w| > 1$ conformally onto K so that the point at infinity in the w -plane goes into the point at infinity in the z -plane. By the Osgood-Caratheodory Theorem [2, p. 86] this function can be extended continuously onto $|w| = 1$ so that it gives a topological mapping of $|w| = 1$ onto C .

Definition. The curve C is of Type W provided that $\psi'(w) \neq 0$ on $|w| = 1$ and ψ'' is continuous on $|w| = 1$.

The primes indicate differentiation. Here and in what follows, the derivative of a function f given on a Jordan curve C means the derivative in a one-dimensional sense with respect to z on C :

$$\left[\frac{df}{dz} \right]_{z=z_1} = f'(z_1) = \lim_{z \rightarrow z_1} \frac{f(z) - f(z_1)}{z - z_1}, \quad f^{(k)}(z_1) = \left[\frac{df^{(k-1)}}{dz} \right]_{z=z_1},$$

$$k = 2, 3, \dots, \quad z, z_1 \in C.$$

Theorem 3.4 . Let D be a region of which the boundary is a Jordan curve C of Type W. Let u and its first derivative u' be continuous on C and let u' satisfy a Lipschitz condition:

$$|u'(z_1) - u'(z_2)| \leq \lambda |z_1 - z_2|^\alpha,$$

$$\lambda > 0, \text{ all } z_1 \text{ and } z_2 \text{ on } C,$$

of some order α , $0 < \alpha < 1$. Then there exists a sequence of point sets $S_1, S_2, \dots$, on C such that $\lim_{n \rightarrow \infty} H_n(u; z)$ exists uniformly on $D + C$ and provides the solution of the Dirichlet Problem for u and D .

It is the plan of the proof to show that under the hypotheses of the theorem there exist harmonic polynomials $h_2(z), h_3(z), \dots$, of respective

degrees at most 2, 3, ... such that

$$(3.4) \quad u(z) - h_n(z) = O\left(\left(\frac{\log n}{n}\right)^{\alpha+1}\right),$$

uniformly for z on C . Now

$$\left(\frac{\log n}{n}\right)^{\alpha+1} = \frac{1}{n} \frac{(\log n)^{\alpha+1}}{n^{\alpha}} = o\left(\frac{1}{n}\right),$$

so if we can obtain (3.4), then the hypotheses of Theorem 3.3 will be satisfied. Theorem 3.4 will then be an immediate consequence of Theorem 3.3.

The theorem of Sewell [11] which we need is the following:

Let the point set Γ consist of a finite number of bounded closed regions bounded by the set of mutually exterior Jordan curves $C_1, C_2, \dots, C_m$ each of Type W. For each j , $j = 1, \dots, m$, let the function f_j be analytic interior to C_j , be continuous in the corresponding closed region, and possess on C_j a continuous k -th derivative which satisfies a Lipschitz condition with exponent α , $0 < \alpha \leq 1$. Then for each integer n , $n \geq 2$, there exists a complex polynomial $P_n(z)$ of degree at most n in z , such that

$$(3.5) \quad f(z) - P_n(z) = O\left(\left(\frac{\log n}{n}\right)^{\alpha+k}\right),$$

uniformly for z on Γ .

In the present application, we use the case in which $m = 1$, $k = 1$, $\alpha < 1$.

We need the following facts concerning C , which are implied by the definition of Type W:

The curve C has a tangent at each point. If $\tau(s)$ denotes the tangent angle with the positive real axis, expressed as a function of arc length s on C , then $\tau(s)$ satisfies a Lipschitz condition with exponent unity uniformly in s . (See [11] for further explanation.) Let $z = \phi(w)$ be analytic and univalent for $|w| < 1$ and map $|w| < 1$ conformally onto D ; further let $z = \phi(w)$ be extended continuously onto $|w| = 1$. Then $z = \phi(e^{i\theta})$, $0 \leq \theta < 2\pi$, is a parametric equation for C . Finally let $w = \chi(z)$ be the inverse of $z = \phi(w)$. By a

theorem of Kellogg [8][5, pp. 34-35] the Lipschitz condition on $\tau(s)$ implies that $\phi'(w)$ and $\chi'(z)$ exist, and that for each α , $0 < \alpha < 1$, $\phi'(w)$ satisfies a Lipschitz condition in w on $|w| = 1$ with exponent α , and $\chi'(z)$ satisfies a Lipschitz condition in z on C with exponent α . In particular, ϕ' is continuous on $|w| = 1$ and χ' is continuous on C . From the continuity of the derivatives it follows that positive constants a and b exist such that

$$(3.6) \quad \left| \frac{\phi(w_1) - \phi(w_2)}{w_1 - w_2} \right| < a, \quad \text{all } |w_1| = 1, |w_2| = 1,$$

$$(3.7) \quad \left| \frac{\chi(z_1) - \chi(z_2)}{z_1 - z_2} \right| < b, \quad \text{all } z_1, z_2 \in C.$$

(The existence of such bounds is easily proved by an indirect argument. For a detailed direct proof which goes into the structure of the bounds, see [19, pp. 385-386].)

We return now to the function u of Theorem 3.4. The mapping $z = \phi(w)$ carries $u(z)$, z on C , into $u(\phi(w)) = v(w)$, $|w| = 1$. Now

$$v'(w) = \frac{du}{dw} = \frac{du}{dz} \cdot \frac{dz}{dw} = u'(z) \phi'(w), \quad |w| = 1,$$

and

$$\begin{aligned} v'(w_1) - v'(w_2) &= (u'(z_1) - u'(z_2)) \phi'(w_1) + u'(z_2)(\phi'(w_1) - \phi'(w_2)), \\ z_1 &= \phi(w_1), \quad z_2 = \phi(w_2), \quad z_1, z_2 \in C. \end{aligned}$$

Let M_1 be an upper bound for $|\phi'(w)|$, $|w| = 1$, and let M_2 be an upper bound for the modulus of the continuous function u' on C . Also let λ_1 be the constant in the Lipschitz condition satisfied by $\phi'(w)$ on $|w| = 1$. Then we obtain:

$$\begin{aligned} (3.8) \quad |v'(w_1) - v'(w_2)| &\leq \lambda M_1 |z_1 - z_2|^\alpha + \lambda_1 M_2 |w_1 - w_2|^\alpha \\ &= \lambda M_1 \left| \frac{\phi(w_1) - \phi(w_2)}{w_1 - w_2} \right|^\alpha |w_1 - w_2|^\alpha + \lambda_1 M_2 |w_1 - w_2|^\alpha \\ &\leq (\lambda M_1 a^\alpha + \lambda_1 M_2) |w_1 - w_2|^\alpha, \quad 0 < \alpha < 1. \end{aligned}$$

(For the last inequality we used (3.6).) Thus $v'(w)$ satisfies a Lipschitz condition of order α on $|w| = 1$.

A theorem of Walsh, Sewell, and Elliott [19, p. 388], based on work of Privalov [20, vol. I, pp. 121-122] on conjugate functions, now guarantees the existence of a function g analytic in the complex variable w for $|w| < 1$, continuous on $|w| \leq 1$, and with the properties that its real part coincides on $|w| = 1$ with v and also that its derivative g' satisfies a Lipschitz condition of order α on $|w| = 1$. The mapping $w = \chi(z)$ carries $g(w)$ into $g(\chi(z)) = f(z)$ which is analytic for z on D and continuous on $D + C$. Also, $\Re f(z) = u(z)$, z on C . The derivative f' exists at each point of C , because

$$\begin{aligned} \lim_{z \rightarrow z_1} \frac{f(z) - f(z_1)}{z - z_1} &= \lim_{z \rightarrow z_1} \left[\frac{g(w) - g(w_1)}{w - w_1} \cdot \frac{\chi(z) - \chi(z_1)}{z - z_1} \right] \\ &= g'(w_1) \chi'(z_1), \end{aligned}$$

$$w = \chi(z), \quad w_1 = \chi(z_1), \quad z, z_1 \in C.$$

Finally, as in (3.8),

$$\begin{aligned} |f'(z_1) - f'(z_2)| &= |(g'(w_1) - g'(w_2)) \chi'(z_1) + g'(w_2)(\chi'(z_1) - \chi'(z_2))| \\ &\leq \lambda_2 M_3 |w_1 - w_2|^\alpha + \lambda_4 M_4 |z_1 - z_2|^\alpha \\ &\leq (\lambda_2 M_3 b^\alpha + \lambda_4 M_4) |z_1 - z_2|^\alpha, \end{aligned}$$

$$w_1 = \chi(z_1), \quad w_2 = \chi(z_2), \quad |w_1| = |w_2| = 1,$$

where λ_2 and λ_4 are the Lipschitz constants of g' and χ' respectively, M_3 and M_4 are upper bounds for $|\chi'|$ and $|g'|$ respectively, and b is the bound which appears in (3.7).

This function f therefore satisfies the condition of Sewell's theorem with $m = 1$, $k = 1$, $\alpha < 1$. Therefore there exists a sequence of complex polynomials in z which satisfy (3.5). Let $h_n(z)$ in (3.4) be $\Re P_n(z)$ in (3.5). We have, since $u = \Re f$,

$$|u(z) - h_n(z)| \leq |f(z) - P_n(z)|,$$

and the existence of the required harmonic polynomial approximation to u follows from this inequality. This concludes the proof of Theorem 3.4.

The generality of the point set Γ in Sewell's result and in our Theorem 3.2 suggests that Theorems 3.3 and 3.4 remain true if D is replaced by a point set consisting of a finite number of mutually exterior bounded regions, $D_1, D_2, \dots, D_m$, with functions $u_1, u_2, \dots, u_m$ given respectively on their boundaries. (In the case of Theorem 3.4, the region D_k would be bounded by curves of Type W.) This is in fact true. The limit of the sequence of harmonic interpolation polynomials then simultaneously provides the solution of the Dirichlet Problems for D_1 and u_1 with z on D_1 , D_2 and u_2 with z on D_2 , and so on.

The result of Sewell used to establish (3.4) and thereby Theorem 3.4 seems to be nearly the best possible, in the sense that under the stated conditions on the approximated function f , the conditions on the curve C are nearly minimal for achieving the exhibited degree of approximation.* However, the hypotheses of Theorems 3.3 and 3.4 are strong enough to guarantee convergence on the boundary C as well as on D , and it was pointed out in the Introduction that because of this, in a sense these theorems overshoot the goal of the general convergence theory. We here amplify the remarks in the Introduction concerning the extra difficulties on the boundary.

Consider the case in which C is the unit circle. A glance at the first formula in Section 2 above will reveal that in that case, harmonic polynomial interpolation on C reduces to interpolation with trigonometric sums. In this case it is known [20, vol. II, p. 37], [6, p. 120] that a sequence of sets $S_1^*, S_2^*, \dots$ exists such that $T_n(z) = O(\log n)$, $|z| = 1$, which is much more favorable to convergence than the $O(n)$ given to us by Theorem 3.2. The points in S_n^* are equally spaced on $|z| = 1$. But to obtain convergence, additional smoothness conditions are needed for the

* Mergelyan [9, p. 84], using hypotheses on C which differ slightly from those of Sewell, arrives at a slightly weaker degree of convergence, and shows that his result is the best possible for the class of functions f and class of curves which he considers.

boundary values beyond mere continuity, for it is also known [20, vol. II, pp. 44 ff.] that if the point $z = 1$ belongs to each S_n^* , then there exists a function u continuous on $|z| = 1$ such that $\{H_n(u; z)\}$ diverges for all $z \neq 1$, where $H_n(u; z)$ is found by interpolation to u in S_n^* . For such a function u the Fourier series may converge uniformly [20, vol. II, p. 47], although (speaking qualitatively) the partial sums of the Fourier series generally do not give a particularly good trigonometric polynomial approximation to a merely continuous function. We shall show in the next section that for any continuous u , the above sequence $\{H_n(u; z)\}$ does converge on $|z| < 1$, no matter how badly it behaves on $|z| = 1$.

In defense of Theorems 3.3 and 3.4 it might be said that there are occasions in practice when it is of interest to know that an approximation to the solution of the Dirichlet Problem can be trusted on the boundary.

In the discussion immediately above it was brought out that even with the smoothest conceivable C (a circle) some smoothness conditions are still needed on u beyond continuity to guarantee convergence on the boundary. It is of some theoretical interest to note that if we are willing to consider the smoothest conceivable u , then C can be an entirely arbitrary Jordan curve and convergence will take place on C . We now express this result formally. The mapping function ψ which appears in the statement is the one used above to define curves of Type W.

Theorem 3.5 . Let D be a region bounded by an arbitrary Jordan curve C . Let the function U be harmonic on the closed set $D + C$. Let C_R denote the image in the z -plane of the circle $|w| = R > 1$ under the conformal map given by $z = \psi(w)$. Then (a) there exists a largest value of R , say $\rho \leq \infty$, such that U with its possible harmonic extensions is single-valued and harmonic at every point interior to C_ρ ; (b) there exists a sequence of point sets $S_1, S_2, \dots$ on C such that for any R , $1 < R < \rho$,

$$\limsup_{n \rightarrow \infty} \left[\max_{z \in D+C} |U(z) - H_n(U; z)|^{1/n} \right] \leq 1/R,$$

where H_n is the harmonic polynomial of degree at most n found by interpolation to U in the points S_n .

In other terms, the degree of convergence of the sequence of harmonic interpolation polynomials on $D + C$ is $O(1/R^n)$, for any R , $1 < R < \rho$.

Statement (a) in the conclusion of the theorem is a consequence of the monotonic character of the level curves C_R . For details the reader is referred to [17, chap. IV].

For statement (b), we make use of a theorem of Walsh [14][16] which states that there exists a sequence of harmonic polynomials $h_1, h_2, \dots, h_n, \dots$, of respective degrees at most $1, 2, \dots, n, \dots$, such that

$$(3.9) \quad |U(z) - h_n(z)| \leq \frac{M}{R^n}$$

for z on C and for any R , $0 < R < \rho$, where M is independent of z .

We let S_n be the point set S_n^* referred to in Theorem 3.2. Substituting into (3.2) we obtain

$$|H_n(U; z) - U(z)| \leq \frac{(2n + 2)M}{R^n}, \quad z \in D + C.$$

Taking the n -th root of both sides and then passing to the limit, we obtain the conclusion of Theorem 3.5, since $\lim_{n \rightarrow \infty} [(2n + 2)M]^{1/n} = 1$.

Walsh's harmonic approximation result embodied in (3.9) is valid when $D + C$ is replaced by any closed bounded point set Γ whose complement is connected and regular in the sense that it possesses a Green's function with pole at infinity. For details see [16] and [17, chap. 4]. Theorem 3.5 can be generalized correspondingly, provided that a hypothesis is inserted to the effect that for each n , Γ has a subset S_n satisfying the n -s condition.

By using interpolation techniques, Shen [17, pp. 173-174] constructed a proof of the existence of sequences of complex polynomials converging maximally (the terminology is that of Walsh) to a given complex function f analytic on a general closed bounded point set whose complement is connected and regular. Shen's polynomials were defined by interpolation to f in points with extremal properties similar to the extremal properties considered in Theorem 3.2 above. It would be of interest to see if with the aid of Theorem 3.2 an analogous existence proof could be given for harmonic polynomials and harmonic functions. Theorem 3.5 and the other theorems of this section of course are not existence theorems in the basic sense since they presuppose the existence of some sort of harmonic polynomial approximations.

4. Harmonic polynomial interpolation in transforms of the roots of unity. We now consider the following infinite sequence of point sets lying on a Jordan curve C in the z -plane:

$$\begin{aligned} S_1 &= \{ \psi(w_3^1), \psi(w_3^2), \psi(1) \}, \\ S_2 &= \{ \psi(w_5^1), \psi(w_5^2), \psi(w_5^3), \psi(w_5^4), \psi(1) \}, \\ &\vdots \\ S_n &= \{ \psi(w_{2n+1}^1), \psi(w_{2n+1}^2), \dots, \psi(w_{2n+1}^{2n}), \psi(1) \} \\ &\vdots \end{aligned}$$

where $w_{2n+1} = e^{2\pi i/(2n+1)}$ and $z = \psi(w)$ is the mapping function which was introduced in Section 3 to define curves of Type W. (That is, ψ is analytic and univalent on $|w| > 1$, continuous on $|w| \geq 1$, and maps $|w| > 1$ onto the region exterior to C so that the points at infinity correspond.) It is known that this sequence of interpolation points is of fundamental importance in complex polynomial interpolation to boundary values (see [4]), so it is natural to study it in connection with harmonic polynomial interpolation.

Let u be a continuous function on C . We choose an infinite sequence of base polynomials $p_1, p_2, \dots, p_n, \dots$ of respective degrees $1, 2, \dots$; we assume that S_{2n+1} satisfies the n -s condition of Section 2, so we set up the harmonic interpolation polynomial in the form (2.4) or (3.1):

$$(4.1) \quad H_n(u; z) = \sum_{k=1}^{2n+1} u[\psi(w_{2n+1}^k)] B_{nk}(z),$$

where

$$\begin{aligned} (4.2) \quad B_{nk}(z) &= b_{ko}^{(n)} + \sum_{j=1}^n [b_{kj}^{(n)} p_j(z) + \overline{b_{kj}^{(n)}} \overline{p_j(z)}] \\ &= b_{ko}^{(n)} + \Re 2 \sum_{j=1}^n b_{kj}^{(n)} p_j(z). \end{aligned}$$

Function B_{nk} is the unique harmonic polynomial of degree at most n which vanishes at all points of S_n except $\psi(w_{2n+1}^k)$, where it equals unity.

The above representations of H_n and B_{nk} are not quite as general as might be, in that it would be theoretically possible to choose a new

set of base polynomials p_j each time n is changed. We might then use the notation $p_{n1}, p_{n2}, \dots, p_{nn}$ for the n -th set. This generality is not needed in what follows.

We shall now calculate $H_n(u; z)$ explicitly for the special cases in which C is a circle and C is an ellipse.

For the circle $|z| = R$, the mapping function becomes $z = \psi(w) = R w$. We choose as the base polynomials $p_1 = z/R$, $p_2 = z^2/R^2$, $\dots$, $p_n = z^n/R^n$, $\dots$. The coefficients of $B_{nk}(z)$ are the solution of the following system of linear equations in the unknowns $b_{kj}^{(n)} = b_j$, $\bar{b}_{kj}^{(n)} = \bar{b}_j$:

$$(4.3) \quad b_0 + \sum_{j=1}^n b_j w_{2n+1}^{hj} + \sum_{j=1}^n \bar{b}_j \bar{w}_{2n+1}^{hj} = \delta_{hk},$$

$$h = 1, 2, \dots, 2n+1,$$

where $\delta_{hk} = 0$, $h \neq k$, $\delta_{kk} = 1$. The existence of a unique solution to (4.3) will be a by-product of our method of finding the solution. We shall need the following easily proved facts about the $(2n+1)$ -th roots of unity, w_{2n+1}^h , $h = 1, 2, \dots, 2n+1$:

$$(a) \quad w_{2n+1}^{hj} \bar{w}_{2n+1}^{hj} = 1, \quad \text{for any integers } j \text{ and } h.$$

$$(b) \quad \sum_{h=1}^{2n+1} w_{2n+1}^{hj} = \sum_{h=1}^{2n+1} \bar{w}_{2n+1}^{hj} = \begin{cases} 0, & j \not\equiv 0 \pmod{2n+1} \\ 2n+1, & j \equiv 0 \pmod{2n+1}. \end{cases}$$

We now transform the system (4.3) into another much simpler system as follows. First we add all the equations together. Duly referring to (b) above, we find that the sum equation is merely the equation $(2n+1)b_0 = 1$. Then for each j , $j = 1, 2, \dots, n$ we multiply the first equation by $\bar{w}_{2n+1}^j$, the second by $(\bar{w}_{2n+1}^j)^2$, $\dots$, the $(2n+1)$ -th by $(\bar{w}_{2n+1}^j)^{2n+1}$ and add the resulting equations. We find by referring to (a) and (b) that the sum equation reduces to $(2n+1)b_j = \bar{w}_{2n+1}^{jk}$. The same procedure with w_{2n+1} instead of $\bar{w}_{2n+1}$ yields $(2n+1)\bar{b}_j = w_{2n+1}^{jk}$.

In matrix language, this process simply amounts to replacing the linear system (4.3), which has a matrix we shall here denote by W , by a new system of linear equations in which the matrix is $W^* W$ and the right side is the k -th column of W^* , where W^* is the conjugate transpose of W . The matrix product $W^* W$ is a scalar matrix with diagonal elements all equal to $2n+1$. The matrix $(2n+1)^{-1/2} W$ is unitary. The absolute value of the determinant of W may be calculated as follows:

$$\det W^* W = \overline{\det W} \det W = |\det W|^2 = (2n+1)^{2n+1},$$

$$(4.4) \quad |\det W| = (2n+1)^{(2n+1)/2} = |\det W^*|.$$

This result is of some interest in connection with the proofs of Theorems 3.2 and 3.3, in which the determinant of the matrix for the analogous general case is maximized.*

The particular consequence of the above discussion needed here is that W is non-singular, so (4.3) has a unique solution and it is moreover the same as the solution to the new system having the matrix $W^* W$.

The solution of (4.3) is now obvious. Substituting into (4.2) we obtain

$$B_{nk}(z) = \frac{1 + \sum_{j=1}^n \left[\left(\frac{z \bar{w}_{2n+1}^k}{R} \right)^j + \left(\frac{\bar{z} w_{2n+1}^k}{R} \right)^j \right]}{2n+1}$$

By summing the geometric series, this can be written as

$$(4.5) \quad B_{nk}(z) = \frac{1}{2n+1} \left\{ \frac{R^2 - |z|^2}{|R w_{2n+1}^k - z|^2} - R \left[\left(\frac{z \bar{w}_{2n+1}^k}{R} \right)^{n+1} \frac{2R}{R - z \bar{w}_{2n+1}^k} \right] \right\}$$

* Suppose that in (4.3) we replace w_{2n+1}^h , $h = 1, \dots, 2n+1$ by any set of $2n+1$ points on the unit circle. The Hadamard determinant inequality yields the middle member of (4.4) as an upper bound for $|\det W|$ no matter how these points are chosen. Therefore the choice of points as the roots of unity has maximized $|\det W|$.

If z lies on some closed circular disk $|z| \leq R' < R$, the first term in braces is positive but less than unity and the second term approaches zero with n at a geometric rate of convergence. Therefore $B_{nk}(z) = O(1/(2n+1)) = O(1/n)$ uniformly in k , $k = 1, \dots, n$, and uniformly for $|z| \leq R'$. It follows from the Corollary of Theorem 3.1 that with (4.5) substituted into (4.1), $\lim_{n \rightarrow \infty} H_n(u; z)$ exists uniformly on $|z| \leq R' < R$, and provides there a solution of the Dirichlet problem for u and the region $|z| < R$.

It is easy to bypass Theorem 3.1 in the present case in demonstrating the convergence. We write $H_n(u; z)$ in the form

$$(4.6) \quad H_n(u; z) = \frac{1}{2\pi} \sum_{k=1}^{2n+1} u(R w_{2n+1}^k) \frac{R^2 - |z|^2}{|R w_{2n+1}^k - z|^2} \frac{2\pi}{2n+1} \\ - \sum_{k=1}^{2n+1} u(R w_{2n+1}^k) R \left[\left(\frac{z \bar{w}_{2n+1}^k}{R} \right)^{n+1} \frac{2R}{R - z \bar{w}_{2n+1}^k} \right]$$

The first summation term is for $|z| \neq R$ a Riemann sum approximating the Poisson integral [1, p. 180]

$$U(z) = \frac{1}{2\pi} \int_0^{2\pi} u(R e^{i\theta}) \frac{R^2 - |z|^2}{|R e^{i\theta} - z|^2} d\theta,$$

which represents the solution of the Dirichlet Problem for u and the disk $|z| < R$. It is an easy exercise in analysis to show that the second summation term approaches zero for $|z| < R$, so again we have $\lim_{n \rightarrow \infty} H_n(u; z) = U(z)$, $|z| < R$.

This argument establishes convergence on $|z| < R$ when u is merely Riemann integrable and not necessarily continuous.

On the circle $|z| = R$ itself, a calculation shows that

$$(4.7) \quad B_{nk}(R e^{i\theta}) = \frac{\sin(n + \frac{1}{2})(\theta - \frac{2\pi k}{2n+1})}{(2n+1) \sin \frac{1}{2}(\theta - \frac{2\pi k}{2n+1})},$$

and when this is substituted into $H_n(u; z)$, we obtain the trigonometric

sum of degree at most n found by interpolation to $u(R e^{i\theta})$ in the points $\theta_k = 2\pi k/(2n+1)$, $k = 1, \dots, 2n+1$. The identification of $H_n(u; z)$ on the circle with a trigonometric sum was previously mentioned in Section 3. As stated in Section 3, it is known [6, p. 120] that

$$T_n(R e^{i\theta}) = \sum_{k=1}^n |B_{nk}(R e^{i\theta})| = O(\log n).$$

From (3.2) and the facts given in the discussion following Theorem 3.1 it follows that if u satisfies a Lipschitz condition of order α , $0 < \alpha \leq 1$ on $|z| = R$, then $H_n(z) - U(z) = O(\log n/n^\alpha)$ uniformly on $|z| \leq R$, so convergence takes place uniformly on the closed disk.*

We turn to the case in which C is an ellipse, which we shall call E . If we let $z = x + iy$, the familiar equation $(x^2/a^2) + (y^2/b^2) = 1$, $a > b > 0$, may come first to mind, but here it is more convenient to define positive constants R and c respectively by $R = \sqrt{(a+b)/(a-b)}$, $c = \sqrt{a^2 - b^2}$ and write the equation of E in the form

$$(4.8) \quad \frac{x^2}{c^2(R + \frac{1}{R})^2} + \frac{y^2}{c^2(R - \frac{1}{R})^2} = 1, \quad c > 0, \quad R > 1.$$

The mapping function ψ for E is given by [1, pp. 76-77]

$$(4.9) \quad z = c(Rw + \frac{1}{Rw}), \quad |w| \geq 1.$$

This mapping function is analytic and univalent for $|w| > 1/R$ and gives a conformal map of this region onto the region in the z -plane which consists of the entire plane minus a cut on the real axis from $-c$ to $+c$. In this map, the exterior of any circle $|w| = \rho > 1/R$ is mapped onto the exterior of a certain ellipse E_ρ in the z -plane having an equation like (4.8), but with R replaced by ρR . It is then easily seen

* The explicit formula (4.5) may have been first published only in 1960 [4], but both it and the convergence results noted above were certainly known to Walsh in the late 1920's. He gave the convergence theorem to the author as a student exercise in 1932, and in [13] and [14] he pointed out the relation between harmonic polynomial interpolation on the circle and trigonometric interpolation.

that if $\rho < 1$, then E_ρ must lie inside E , and that given any point inside E , for some ρ sufficiently near to 1 this point also lies inside E_ρ .

Henceforth in the discussion we restrict w to the region $|w| > 1/R$. In this region the transformation (4.9) carries $Rw + 1/(Rw)$ into the polynomial z/c and $(Rw)^2 + 1/(Rw)^2$ into the polynomial $(z/c)^2 - 2$. We shall abbreviate a statement of this type by saying, for example, that $(Rw)^2 + 1/(Rw)^2$ "is" a polynomial in z . Now if it is true that $(Rw)^m + 1/(Rw)^m$ is a polynomial in z of degree m for $m = N - 1$ and $m = N$, then

$$(Rw)^{N+1} + \frac{1}{(Rw)^{N+1}} = [(Rw)^N + \frac{1}{(Rw)^N}](Rw + \frac{1}{Rw}) - [(Rw)^{N-1} + \frac{1}{(Rw)^{N-1}}]$$

is a polynomial in z of degree $N + 1$. The hypothesis is true for $N = 2$, so we have established that $p_m = (Rw)^m + 1/(Rw)^m$ is a polynomial of degree m in z for $m = 1, 2, \dots$.

We take the polynomials p_m as our base polynomials in constructing (4.1) and (4.2) for the ellipse. The points in S_n are of course now the points $c[Rw_{2n+1}^k + 1/(Rw_{2n+1}^k)]$, $k = 1, 2, \dots, 2n+1$.

The coefficients $b_j, \bar{b}_j$ of $B_{nk}(z)$ are the solution of the system of linear equations

$$(4.10) \quad b_0 + \sum_{j=1}^n b_j \left[(R w_{2n+1}^h)^j + \frac{1}{(R w_{2n+1}^h)^j} \right] + \sum_{j=1}^n \bar{b}_j \left[(R \bar{w}_{2n+1}^h)^j + \frac{1}{(R \bar{w}_{2n+1}^h)^j} \right] = \delta_{hk},$$

$$h = 1, 2, \dots, 2n+1.$$

To solve the system, we premultiply the matrix of the system and also the vector on the right hand side by the matrix W^* introduced above in the circle case. This time we arrive at the following system of linear equations:

$$(2n+1)b_0 = 1$$

$$(2n+1)[R^j b_j + \frac{1}{R^j} \bar{b}_j] = \bar{\omega}^j, \quad j = 1, 2, \dots, n$$

$$(2n+1)[\frac{1}{R^j} b_j + R^j \bar{b}_j] = \omega^j, \quad j = 1, 2, \dots, n,$$

where $\omega = w_{2n+1}^k$. Direct calculation shows that the determinant of the transformed system has the value $(2n+1)^{2n+1} \prod_{j=1}^n (R^{2j} - R^{-2j})$. If A denotes the matrix of the original system (4.10), then $\det W^* \det A$ has this value, so by (4.4),

$$|\det A| = (2n+1)^{(2n+1)/2} \prod_{j=1}^n (R^{2j} - \frac{1}{R^{2j}}) \neq 0.$$

The value of the determinant is of some interest in connection with finding interpolation points with extremal properties in connection with Theorems 3.2 and 3.3.

The discussion of the determinant of course implies that (4.10) has a unique solution which can be found from the transformed system. Accordingly we find that the solution is in part

$$b_0 = \frac{1}{2n+1}$$

$$b_j = \frac{(R\bar{\omega})^j - \frac{1}{(R\bar{\omega})^j}}{(2n+1)(R^{2j} - \frac{1}{R^{2j}})}, \quad j = 1, \dots, n,$$

and the complex conjugates of the numbers b_j complete the solution.

Substituting into (4.2), we obtain

$$(4.11) \quad (2n+1) B_{nk}(z) = 1 + R \sum_{j=1}^n \frac{(R\bar{\omega})^j - \frac{1}{(R\bar{\omega})^j}}{R^{2j} - \frac{1}{R^{2j}}} \left[(Rw)^j + \frac{1}{(Rw)^j} \right],$$

where as above $\omega = w_{2n+1}^k$, and $|w| > 1/R$.

To study the order of magnitude of $B_{nk}(z)$ as n becomes infinite, we let $w = \rho e^{i\theta}$, $\rho > 1/R$, and rewrite (4.11) in the form

$$(4.12) \quad (2n+1) B_{nk}(z) = 1 + R \sum_{j=1}^n \frac{\bar{\omega}^j e^{ij\theta} (R^{2j} \rho^j - \frac{1}{R^{2j} \rho^j}) + \omega^j e^{ij\theta} (\frac{1}{\rho^j} - \rho^j)}{R^{2j} - \frac{1}{R^{2j}}}.$$

From this it follows that

$$(4.13) \quad |B_{nk}(z)| \leq \frac{1 + 2 \sum_{j=1}^n \frac{|R^{2j} \rho^j - \frac{1}{R^{2j} \rho^j}| + |\frac{1}{\rho^j} - \rho^j|}{R^{2j} - \frac{1}{R^{2j}}}}{2n + 1}$$

Now let z lie on the locus E_ρ which is the map of $|w| = \rho$, where $(1/R) < \rho < 1$. It is possible to remove the absolute value indications in (4.13), and when this is done the right side becomes

$$\frac{1 + 2 \sum_{j=1}^n \frac{(R^j - \frac{1}{R^j}) \left[(R\rho)^j + \frac{1}{(R\rho)^j} \right]}{R^{2j} - \frac{1}{R^{2j}}}}{2n + 1}$$

$$\leq \frac{1 + 2 \sum_{j=1}^n \left[\frac{(R\rho)^j + \frac{1}{(R\rho)^j}}{R^j + \frac{1}{R^j}} \right]}{2n + 1}$$

$$\leq \frac{1 + 2 \sum_{j=1}^n \frac{(R\rho)^j + \frac{1}{(R\rho)^j}}{R^j}}{2n + 1}$$

$$\leq \frac{1 + 2 \sum_{j=1}^n \left(\rho^j + \frac{1}{R^j} \right)}{2n + 1}$$

The two geometric progressions in the numerator of the last member converge, so we have shown that $B_{nk}(z) = O(1/(2n+1)) = O(1/n)$ uniformly for z on E_ρ . But since B_{nk} is a harmonic polynomial, it follows by the maximum principle that $B_{nk}(z) = O(1/n)$ uniformly for z on and

interior to E_ρ . It will be recalled that E_ρ can be adjusted so as to contain any closed point set interior to E . Therefore according to the Corollary of Theorem 3.1, when $H_n(u; z)$ in (4.1) is constructed with $B_{nk}(z)$ as given by (4.11) or (4.12), $\lim_{n \rightarrow \infty} H_n(u; z)$ exists uniformly on any closed point set of the interior of the ellipse (4.8) and provides there the solution of the Dirichlet problem for the boundary data u .

Going back to (4.12), it can be seen that if $\rho = 1$, so that z lies on the ellipse E , then B_{nk} reduces to

$$B_{nk}(z) = \frac{1 + 2 \sum_{j=1}^n \cos j(\theta - \frac{2\pi k}{2n+1})}{2n+1}$$

$$= \frac{\sin(n + \frac{1}{2})(\theta - \frac{2\pi k}{2n+1})}{(2n+1) \sin \frac{1}{2}(\theta - \frac{2\pi k}{2n+1})}$$

This is the same as the formula (4.7) which we obtained in the circle case, and once again a Lipschitz condition on u will insure convergence in the closed region under consideration.

The identification of $H_n(u; z)$ for z on the ellipse with trigonometric interpolation was accomplished by Walsh [18] who then pointed out the convergence implication under suitable smoothness conditions on u . The present discussion eliminates any requirement on u beyond mere continuity for convergence interior to the ellipse.

For convenience in reference we summarize the above results formally.

Theorem 4.1 . Let C be either the circle $|z| = R$ or the ellipse

$$\frac{x^2}{c^2(R + \frac{1}{R})^2} + \frac{y^2}{c^2(R - \frac{1}{R})^2} = 1, \quad c > 0, \quad R > 1.$$

Let the function u be continuous on C . Let $H_n(u; z)$ be the harmonic polynomial of degree at most n which coincides with u in the points $z_{nk} = \psi(w_{2n+1}^k)$, $k = 1, 2, \dots, 2n+1$, where w_{2n+1} is a $(2n+1)$ -th root of unity and ψ gives the conformal map of $|w| > 1$ onto the exterior of C so that the points at infinity correspond. The polynomial

H_n exists and is uniquely determined. For the circle $H_n(u; z)$ is given by (4.6) for z not on C , and for the ellipse,

$$(4.14) \quad H_n(u; z) = \frac{\sum_{k=1}^{2n+1} u(z_{nk})}{1 + R^2 \sum_{j=1}^n \frac{(R\bar{\omega})^j - \frac{1}{(R\bar{\omega})^j}}{R^{2j} - \frac{1}{R^{2j}}} [(Rw)^j + \frac{1}{(Rw)^j}]},$$

$$\omega = w_{2n+1}^k, \quad z = c(Rw + \frac{1}{Rw}), \quad \frac{1}{R} < |w|.$$

In either case, $\lim_{n \rightarrow \infty} H_n(u; z)$ exists uniformly on any closed subset of the interior of C and provides there the solution of the Dirichlet problem for the data u . Also in either case, for z on C and with $z = \psi(e^{i\theta})$, $H_n(u; z)$ reduces to the trigonometric sum of order at most n which interpolates to $u(\psi(e^{i\theta}))$ in the points $\theta_k = 2\pi k/(2n+1)$, $k = 1, \dots, 2n+1$. If u satisfies a Lipschitz condition of order α , then $\lim_{n \rightarrow \infty} H_n(u; z)$ converges uniformly on and inside C .

It might be of interest to devise a proof of convergence in the ellipse case from (4.14) directly in such a way that the continuity condition on u could be relaxed to an integrability condition of some sort.

We conclude with a technical note looking toward the generalization of the above theorem to more or less arbitrary Jordan curves C . In constructing H_n for the ellipse, we could just as well have used $p_j^* = c^j((Rw)^j + 1/(Rw)^j)$, $j = 1, \dots, n$, as our base polynomials in z . After cancelling a c^j which would have appeared in the numerator and denominator of each term in the summation in (4.11), the formulas (4.11), (4.12), and (4.13) would have looked just the same as they do now. These polynomials p_j^* are the so-called Faber polynomials* for the region exterior to the ellipse E . In the case of the circle $|z| = R$, the Faber

* See [15, pp. 32-33] for a quick summary and a number of primary references. There appears to be a slight misprint in [15] in the recursion formula at the bottom of page 32, where the term a_n should be $(n+1)a_n$.

polynomials are $z, z^2, \dots, z^n, \dots$, which are equivalent for present purposes to the base polynomials which we used. Our method for finding $H_n(u; z)$ in each case thus consisted in expressing $B_{nk}(z)$ as the real part of a linear combination of the first n Faber polynomials, and then adjusting the coefficients of the linear combination so that the real part would vanish at $2n$ of the interpolation points and equal unity at the remaining point. The orthogonality properties of the values of the monomials $1, w, w^2, \dots, \bar{w}, \bar{w}^2, \dots$ in the roots of unity were used in the adjustment process.

In the general case, the mapping function $z = \psi(w)$ can be chosen so as to have a Laurent development

$$(4.15) \quad z = rw + a_0 + \frac{a_1}{w} + \frac{a_2}{w^2} + \dots, \quad r > 0, \quad |w| \geq 1,$$

where r is the transfinite diameter or capacity of the curve C . The Faber polynomials $P_n(z)$ for C are defined by stating that $P_n(z)$ must be of degree n , must have unity for the coefficient of z^n , and must be such that the Laurent development of P_n as a function of w is of the form

$$P_n(z) = (rw)^n + \frac{g_{n1}}{w} + \frac{g_{n2}}{w^2} + \dots, \quad |w| > 1.$$

In the case of the ellipse, all of the coefficients g_{nj} vanish except g_{nn} , and this paved the way to obtaining a simple explicit formula for the required linear combination of $1, P_1(z), P_2(z), \dots, P_n(z), \bar{P}_1(z), \bar{P}_2(z), \dots, \bar{P}_n(z)$. In more general cases one cannot hope to obtain manageable explicit formulas in terms of the coefficients in (4.15), but it may be possible to establish the asymptotic properties of $B_{nk}(z)$ by using known asymptotic properties of the polynomials P_n [15, p. 33] together with the orthogonality of the sequence $\{w^n\}$ on the unit circle.

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