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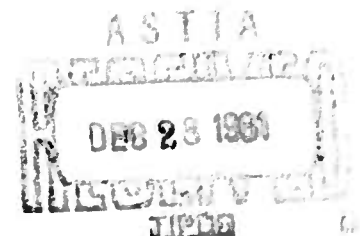
Procedures For Generating Normal
Random Variables, II

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Mathematics Research

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November 1961



PROCEDURES FOR GENERATING NORMAL RANDOM VARIABLES, II

by

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Mathematical Note No. 243

Mathematics Research Laboratory

BOEING SCIENTIFIC RESEARCH LABORATORIES

November 1961

1. Introduction

A method for generating a normal random variable in terms of uniform random variables will be described. The method is based on representing a density function as a mixture of simpler densities, as outlined in [1]. It is quite fast and requires little storage (60 constants). It is not quite as fast as the method of [2], but it is simpler, with less chance for prospective users being set adrift in a sea of details.

We will fashion the procedure with use on a computer with numbers having 35 binary digits (bits) in mind. The details may be modified for numbers of different lengths or for decimal computers.

The normal random variable x is generated in terms of independent uniform random variables $u_1, u_2, \dots$. Roughly, the procedure is this: we put $x = \frac{1}{2}u$ with frequency 35%; $x = \frac{1}{2} + \frac{1}{2}u$, 24%; $x = 1 + \frac{1}{2}u$, 13%; $x = 3/2 + \frac{1}{2}u$, 5%; and $x = 2 + \frac{1}{2}u$, with frequency about 1%. Thus with probability about .8 we put $x = a + \frac{1}{2}u$, the first 9 bits of u_1 being used to choose a from among $0, 1/2, 1, 3/2, 2$ and the remaining 26 bits of u_1 used to form $\frac{1}{2}u$.

Of the remaining 20% we use the modified rejection technique given in [3] with frequency 19% and the other 1% is devoted to the tail. The modified rejection technique is applied to one of regions 6-10 above the rectangles in Fig. 1, while rectangles 1-5 provide the representations $x = 0 + \frac{1}{2}u, \frac{1}{2} + \frac{1}{2}u$, etc. We use the polar representation of a random normal point to dispose of the tail.

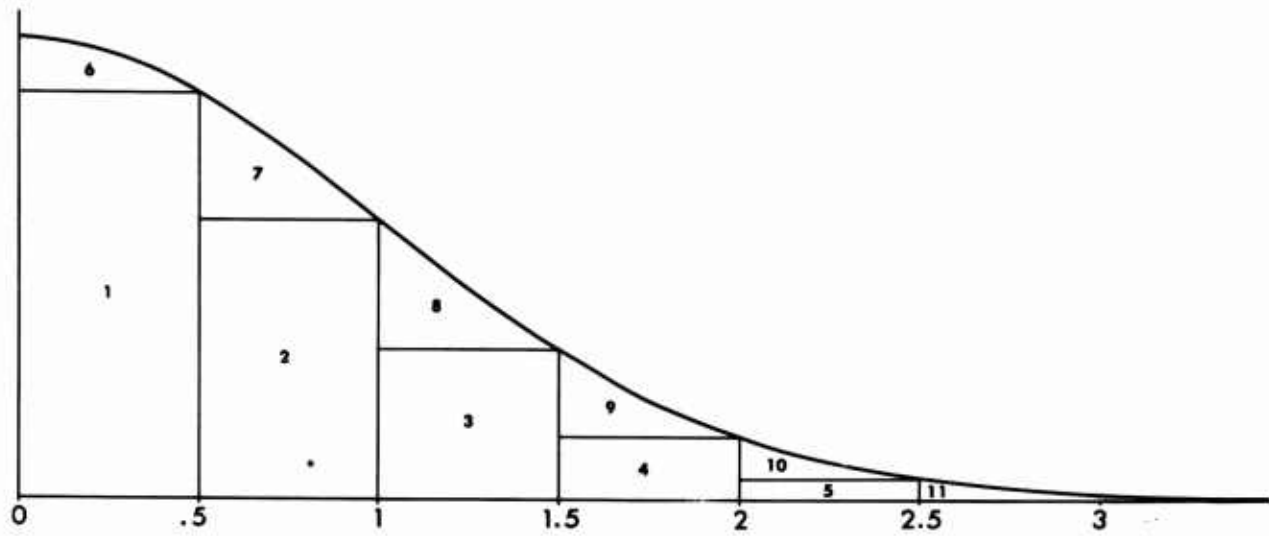


Fig. 1

The procedure has, in theory, unlimited accuracy; in practice, its accuracy is limited by the capabilities of the machine in single precision floating point.

2. The Rectangles

Let f be the absolute normal density,

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad x > 0.$$

We generate x with density f , attaching a random $\pm$ at some convenient point in the program. We write f as a mixture of 11 densities:

$$f(x) = \sum_{i=1}^{11} p_i g_i(x).$$

Fig. 1 shows how the terms in the mixture are stacked to form f .

The 5 rectangles provide the fast part of the program; $g_1, \dots, g_5$ being rectangles with base $\frac{1}{2}$, an x with such a density may be found by putting $x = a + \frac{1}{2}u$. For example, region 1 of Fig. 1 provides the representation $x = 0 + \frac{1}{2}u$, and this should be used with frequency equal to the area of region 1, $p_1 = .352065326764299$. But it is wasteful to test $u_1 < p_1$ in its full form, as this will require all 35 bits of u_1 ; much better is to use only, say, 9 bits of u_1 to test $u_1 < \hat{p}_1$ where $\hat{p}_1$ is the nearest we can get to p_1 using 9 bits, $\hat{p}_1 = 180/512 = .3515625$, then we may use the last 26 bits of u_1 to form $\frac{1}{2}u$ and put $x = \frac{1}{2}u$. Later, with probability $p_1 - \hat{p}_1$, we will put $x = \frac{1}{2}u_2$, using a completely new uniform number u_2 . We then will get by most of the time with a single uniform number u_1 . Each of the probabilities p_1, p_2, p_3, p_4, p_5

is broken up in this way; the numerical values appear in the steps of the program and in the summary of constants.

3. The "Nearly Linear" Method

We illustrate the nearly linear method with region 9 of Fig. 1. It represents the term $p_9 g_9$ in the mixture, so we must, with probability $p_9 = .034123172128170$, generate a random variable with density g_9 . This density is drawn in Fig. 2. Our method is to enclose the area under g in the trapezoid ADEF. The slope of CG and DE is -8 . We choose a point (z,y) uniformly from the trapezoid and set $x = z$ if $y < g(z)$, but choose a new point (z,y) if not. We test to see if the point (z,y) is in triangle ACG ($y < \text{const.} - 8z$), hoping to avoid testing $y < g(z)$. If that test is necessary, we take advantage of the exponential content of g , writing

$$g(z) = \frac{2}{p_9 \sqrt{2\pi}} [e^{-\frac{1}{2}z^2} - e^{-2}] = \frac{2e^{-9/8}}{p_9 \sqrt{2\pi}} [e^{-\frac{1}{2}(z^2 - 9/4)} - e^{-7/8}].$$

Then the test $y < g(z)$ hinges on whether this infinite sum is positive or negative:

$$ry - s + t - \frac{t^2}{2!} + \frac{t^3}{3!} - \frac{t^4}{4!} + \dots$$

with $r = p_9 \sqrt{\pi/2} e^{9/8}$, $s = 1 - e^{-7/8}$, $t = \frac{1}{2}(z + 3/2)(z - 3/2)$, $0 \leq t \leq 7/8$. This is an alternating series that converges rather quickly, and its sum is negative if and only if two successive partial sums are negative; positive if and only if two successive partial sums are positive. We should average only a few multiplications in order to

decide $y < g(z)$ with "unlimited" accuracy. Another advantage is that the structure of the test for g_9 is the same as that for the other g 's, only r , s , and t changing.

The point (z,y) in the trapezoid of Fig. 2 may be produced quite fast, putting

$$z = \frac{1}{2} \min(u_2, u_3) + 1.5,$$

$$y = d + 4 \max(u_2, u_3)$$

(choosing from triangle BDE) with probability $\frac{2}{2+d}$ and putting

$$z = \frac{1}{2} u_2,$$

$$y = du_3$$

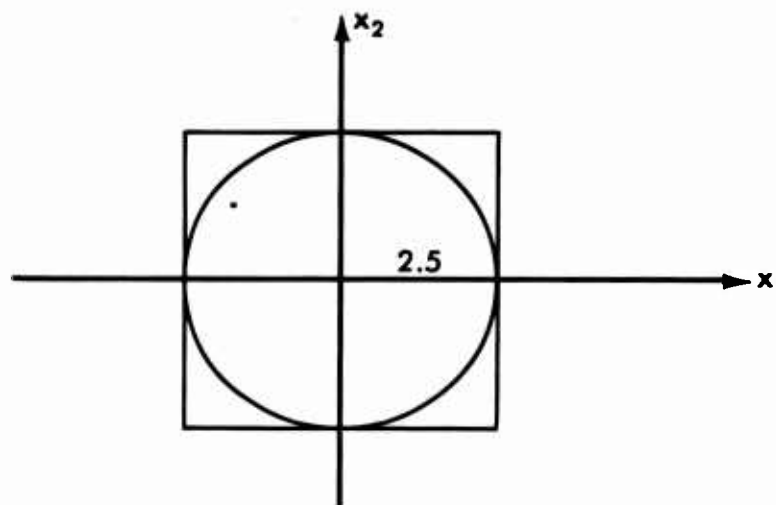
(choosing from ABEF) with probability $\frac{d}{2+d}$. If d is small we will avoid conventional multiplication most of the time, and by properly choosing d we will need only a few bits of u_1 to choose between the two methods for generating (z,y) , so that the remaining bits of u_1 may be used to form u_2 and u_3 .

For example, $p_9 = .034123172128170$. We let $\hat{p}_9 = 17/512 = .033203125$, then generate x with density g_9 with probability $\hat{p}_9$. In the course of the program, the test for this contingency is $\frac{460}{512} \leq u_1 < \frac{460 + 17}{512}$. If u_1 is in this range, then by making $\frac{2}{2+d} = \frac{14}{17}$, we may test $u_1 < \frac{460 + 14}{512}$, keeping the last 26 bits of u_1 available for forming u_2 and u_3 . This makes $d = 3/7 = .42857...$ whereas the smallest possible value for d is $d = .427...$. The loss is small, and the gain from not having to generate new u_2 and u_3 is great. We must generate x

from density g_9 with probability $p_9 - \hat{p}_9$ later on, using new u_2 and u_3 , but the average time loss for this compensating step is negligible.

4. The Tail

We need x conditioned by $|x| > 2.5$. Consider the plane of points (x_1, x_2) with x_1 and x_2 independent standard normal.



We want a point (x_1, x_2) outside the square, then one of the coordinates will serve as x . We use the polar representation (ρ, θ) to choose points outside the circle until we get one outside the square, then change to rectangular coordinates (x_1, x_2) .

The exterior of the circle has measure .044, the exterior of the square, .025. The ratio of the two is about .57 so that 57% of the time a point lying outside the circle will lie outside the square. If (x_1, x_2) lies outside the square, only once in 150 times will both $|x_1|$ and $|x_2|$ be > 2.5 , so it probably isn't worth the trouble to store the extra one

for future use, as this would mean that each time the tail was encountered in the program, the request for a possible stored value of x would almost always be denied.

It is elementary to verify that putting

$$x_1 = v_1 \left(\frac{25/4 + 2w}{v_1^2 + v_2^2} \right)^{\frac{1}{2}},$$

$$x_2 = v_2 \left(\frac{25/4 + 2w}{v_1^2 + v_2^2} \right)^{\frac{1}{2}},$$

with w exponential and v_1, v_2 uniform on $(-1,1)$ and conditioned by $v_1^2 + v_2^2 < 1$, will provide a point (x_1, x_2) outside the circle with the appropriate distribution.

5. Outline of a Program

We give a series of steps outlining a program based on the above procedure, assuming we have a subroutine which, when called for, will generate a new uniform number u in terms of the current u ; for example, by putting: $\text{new } u = 2^k u + u + 1 \pmod{2^{35}}$, as suggested by Rotenberg [5].

1. If $0 \leq u_1 < \frac{180}{512}$ put $x = \frac{1}{2}u$, with $\frac{1}{2}u$ formed from bits 10-35 of u_1 .
2. If $\frac{180}{512} \leq u_1 < \frac{303}{512}$ put $x = \frac{1}{2} + \frac{1}{2}u$, with $\frac{1}{2}u$ formed from bits 10-35 of u_1 .
3. If $\frac{303}{512} \leq u_1 < \frac{369}{512}$ put $x = 1 + \frac{1}{2}u$, with $\frac{1}{2}u$ formed from bits 10-35 of u_1 .
4. If $\frac{369}{512} \leq u_1 < \frac{396}{512}$ put $x = 1.5 + \frac{1}{2}u$, with $\frac{1}{2}u$ formed from bits 10-35 of u_1 .
5. If $\frac{396}{512} \leq u_1 < \frac{404}{512}$ put $x = 2.0 + \frac{1}{2}u$, with $\frac{1}{2}u$ formed from bits 10-35 of u_1 .
6. If $\frac{404}{512} \leq u_1 < \frac{433}{512}$ put $j = 7$, if $\frac{433}{512} \leq u_1 < \frac{460}{512}$ put $j = 8$,
 if $\frac{460}{512} \leq u_1 < \frac{477}{512}$ put $j = 9$, if $\frac{477}{512} \leq u_1 < \frac{492}{512}$ put $j = 6$, and
 if $\frac{492}{512} \leq u_1 < \frac{499}{512}$ put $j = 10$. Test $u_1 < q_j$, if yes, form
 $z = a_j + \frac{1}{2} \min(u_2, u_3)$ and $y = d_j + 4 \max(u_2, u_3)$; if no, form

$z = a_j + \frac{1}{2}u_2$, $y = d_j u_3$. In either case, u_2 is formed from bits 10-22 and u_3 from bits 23-35 of u_1 . [K] If $y < e_j - 8z$, put $x = z$; if not, let $t = \frac{1}{2}(z + a_j)(z - a_j)$ and form terms of the sequence

$$r_j y - s_j, r_j y - s_j + t, r_j y - s_j + t - \frac{t^2}{2!}, r_j y - s_j + t - \frac{t^2}{2!} + \frac{t^3}{3!}, \dots$$

until two successive terms have the same sign. If two successive terms are negative, put $x = z$, if two successive terms are positive, go to H.

7. If $\frac{499}{512} \leq u_1 < .980619788956091$ choose i from $1, 2, \dots, 5$ with probability $p_i - \hat{p}_i$, generate a uniform number u_2 and put $x = \frac{1}{2}(i-1) + \frac{1}{2}u_2$.

8. If $.980619788956091 \leq u_1 < .987580669348448$ choose j from among 6,7,8,9,10 with probability $p_j - \hat{p}_j$ and go to H.

9. If $.987580669348448 \leq u_1 < 1$, form

$$x_1 = v_1 \left(\frac{25/4 + 2w}{v_1^2 + v_2^2} \right)^{\frac{1}{2}},$$

$$x_2 = v_2 \left(\frac{25/4 + 2w}{v_1^2 + v_2^2} \right)^{\frac{1}{2}},$$

where w has the exponential distribution and v_1, v_2 are uniform on $(-1,1)$, conditioned by $v_1^2 + v_2^2 < 1$. Test $|x_1| > 2.5$; if yes, put $x = x_1$. If no, test $|x_2| > 2.5$; if yes, put $x = x_2$; if no, generate a new pair x_1, x_2 and try again.

H. Generate a new uniform number u . Test $u < b_j$, if yes, let

$z = a_j + \frac{1}{2} \min(u_2, u_3)$, $y = d_j + 4 \max(u_2, u_3)$; if no, let

$z = a_j + \frac{1}{2} u_2$, $y = d_j u_3$ where u_2 and u_3 are formed from the first and last parts of a new uniform number generated from u .

Now go to [K] of step 6.

6. Remarks

With probability $\cdot \frac{404}{512} = .79$ we will use the representation $x = a + \frac{1}{2}u$ of steps 1-5, and this is the principal reason for a fast program. However, even when, with frequency $\frac{95}{512}$ or about 19%, we require step 6, the time loss is not crushing, for most of the time we will get through step 6 without using a single conventional multiplication and will require only one uniform number u_1 .

It is in trying to avoid the use of more than one uniform number in step 6 that we make our only compromise with accuracy, for we use only 13 bits to represent numbers on an interval of length $\frac{1}{2}$. A program based on the outline above can be described as being as accurate as the single precision capacity of the machine, except that with probability $\frac{1}{5}$ the value of x may differ by about 2^{-14} from the value that should be delivered. Any nuts wanting accuracy greater than this can call up a new uniform number at that point, with a loss of perhaps 15 cycles, or a net loss of about $\frac{15}{5} = 3$ cycles in the average running time of the program.

Step 7 corrects for our inability to handle regions 1 to 5 using only 9 bits, and step 8 does the same for regions 6 to 10. Step 9 provides an x from the tail. Note that in step 9 a signed value of x is returned, whereas in the other steps a positive x is returned and a random $\pm$ must be assigned at some convenient point. The method for doing this is not critical in steps 7 or 8, but in the first 6 steps, particularly steps 1-5, the method should be chosen so that time is not wasted in this assignment.

Step 9 requires an exponential variable w . The ease with which w can be produced is probably more important than saving time here, since step 9 provides only one x in a hundred. If a logarithm subroutine is available, $w = -\ln u$ will serve, or the method in [4] may be used.

SUMMARY OF CONSTANTS

j	b_j	d_j	e_j	q_j	$\hat{p}_j$	a_j
6	12/15	1/2	3.0380	489/512	15/512=.029296875	0
7	28/29	1/14	7.8099	432/512	29/512=.056640625	1/2
8	25/27	4/25	11.930	458/512	27/512=.052734375	1
9	14/17	3/7	15.850	474/512	17/512=.033203125	3/2
10	5/7	4/5	19.780	497/512	7/512=.013671875	2

j	p_j	$p_j - \hat{p}_j$	r_j	s_j
6	.030859595783727	.001562720783727	.038676767667587	.117503097415405
7	.057793845069917	.001153220069917	.082078297231197	.312710721209028
8	.054178509659306	.001444134659306	.111952612794322	.464738571481000
9	.034123172128170	.000920047128170	.131731800427316	.583137980321492
10	.015552632751237	.001880757751237	.144029953116673	.675347532641650

i	p_i	$\hat{p}_i$	$p_i - \hat{p}_i$
1	.352065326764299	180/512=.3515625	.000502826764299
2	.241970724519143	123/512=.240234375	.001736349519143
3	.129517595665892	66/512=.12890625	.000611345665892
4	.053990966513188	27/512=.052734375	.001256591513188
5	.017528300493569	8/512=.015625	.001903300493569

REFERENCES

- [1] G. Marsaglia, "Expressing a Random Variable in Terms of Uniform Random Variables", Annals of Mathematical Statistics, vol. 32, no. 3, Sept. 1961, pp. 894-898.
- [2] G. Marsaglia and M. D. MacLaren, "A Program for Generating Normal Random Variables", to appear.
- [3] G. Marsaglia, "Remark on Generating a Random Variable Having a Nearly Linear Density Function", Boeing Scientific Research Laboratories Document DL-82-0107, May 1961.
- [4] G. Marsaglia, "Generating Exponential Random Variables", Annals of Mathematical Statistics, vol. 32, no. 3, Sept. 1961, pp. 899-900.
- [5] A. Rotenberg, "A New Pseudo-Random Generator", Journal of the Association for Computing Machinery, vol. 7, no. 1, Jan. 1960, pp. 75-77.