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VALUES OF LARGE GAMES, III: A CORPORATION WITH TWO LARGE STOCKHOLDERS

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PREFACE

This Memorandum is one of a continuing series of research articles by the author and his colleagues on the evaluation of economic and other games involving a large number of conflicting interests.

Titles of other contributions to the series will be found in the bibliography at the end of the Memorandum.

SUMMARY

This paper contains a numerical and graphical analysis of a class of games having two major players and a continuous infinity of minor players. For expository convenience, the games are interpreted in terms of the control by stockholders of a large corporation. This note provides an illustration of the theory of "oceanic games" presented in the preceding paper of this series (RM-2649).

CONTENTS

VALUES OF LARGE GAMES, III: A CORPORATION WITH TWO LARGE STOCKHOLDERS

1. INTRODUCTION

In a previous paper the value formulas were derived for certain voting games of the "oceanic" type: a few powerful major players pitted against a sea of infinitesimal minor players (see [2]). The present note contains a detailed numerical and graphical study of the two-major-player, constant-sum case.

We have interpreted this case as the game of "Corporation Control," but only for expository convenience; we do not anticipate any direct application of our findings to the theory or practice of corporation ownership. We do hope to demonstrate, however, in presenting such a thorough analysis of a relatively simple class of examples, that there is much variety and depth in the power relationships which the "value" concept can bring out when applied to an abstract political structure.

2. THE OCEANIC GAME

Consider a corporation with common stock held by two large interests and a great number of very small interests. Assume that control of corporation policy hinges entirely on a simple majority vote of the stock, and forget about proxies, the board of directors, marketability of shares, the wishes of management, and other such mundane matters. A formal model of the resulting, stripped-down political structure is provided by an oceanic game, designated symbolically by

$$(1) \qquad [1/2;\ w_1,\ w_2;\ \alpha] \ ,$$

where $\alpha = 1 - w_1 - w_2$ (see [2]). The positive numbers w_1, w_2, and α represent the relative voting strengths of the two major interests and the "ocean" of minor players, respectively. The "1/2" is the majority quota—i.e., the fraction of the total vote needed to win.

The range of the parameters is a right triangle in the (w_1, w_2)—plane; it can conveniently be split into four cases, as shown in Fig. 1. Region I, with its large ocean, represents the theoretically important "interior" case, discussed in [2] (p. 16ff.) and [3] (p. 8f. and 16). In Region II the ocean is smaller but still holds the "balance of power" between the strong major players. In the other regions a single player has dictatorial power, and the mathematically trivial games that result are of interest only as limiting cases.

In the following discussion we shall use the terms "value" and "power" almost interchangeably. This usage is in effect a definition of the latter term (see [4]). We trust that as the qualitative properties of the value emerge, they will retrospectively justify the choice of terminology.

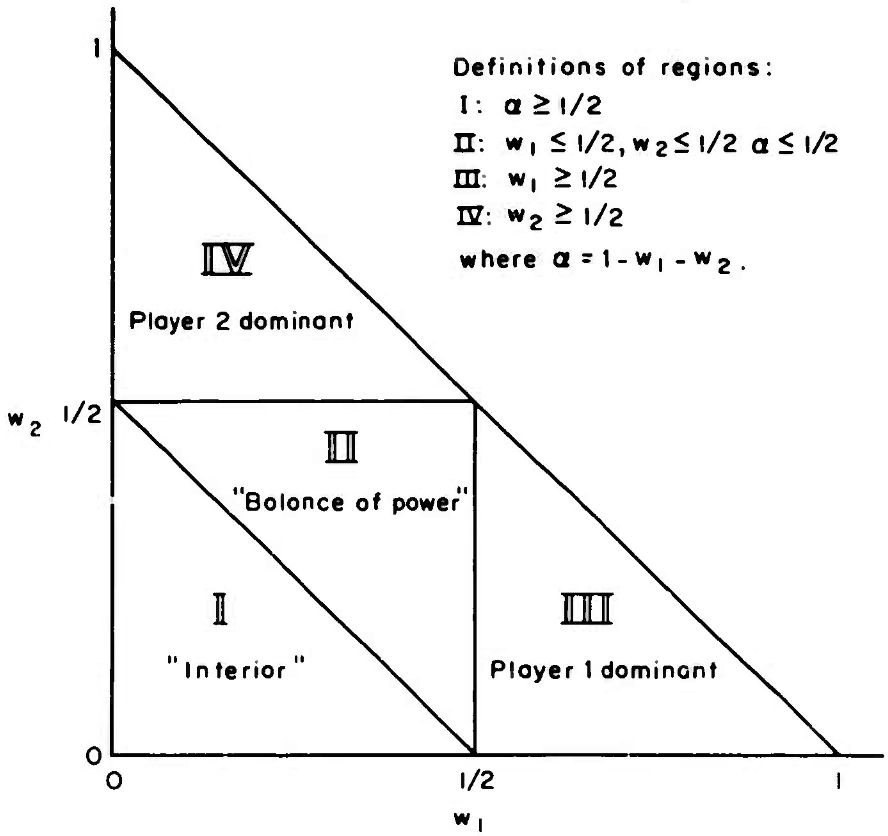

Fig. 1 — Range of parameters in the oceanic game $[1/2; w_1; w_2; \alpha]$

3. POWER INDICES AND POWER RATIOS

According to [2], the power index of the first major player in (1) is given by

$$\phi_1 = \begin{cases} \dfrac{w_1}{\alpha} - \dfrac{w_1 w_2}{\alpha^2} & \text{in Region I} \\[2ex] \dfrac{(1 - 2w_2)^2}{4\alpha^2} & \text{in Region II} \\[2ex] 1 & \text{in Region III} \\ 0 & \text{in Region IV} \end{cases}$$

The formula for the second player's power index is similar, with the obvious transpositions of w_1 and w_2, and III and IV. Finally, the total power Φ of the oceanic players is given by the relation

$$\phi_1 + \phi_2 + \Phi = 1. \tag{3}$$

We shall be interested not only in the power indices, as just defined, but also in the relative powers per share of stock, or <u>power ratios</u>, which are defined by

$$R_1 = \phi_1/w_1, \; R_2 = \phi_2/w_2, \; R_{oc} = \Phi/\alpha.$$

In Fig. 2 we have graphed the first player's power index and power ratio as functions of the game parameters. The corresponding figure for player 2 may be obtained by reflection in the line $w_1 = w_2$. Figure 3 gives similar information for the oceanic players. For example, if the first player holds 10% of the stock and the second player 40%, then their power

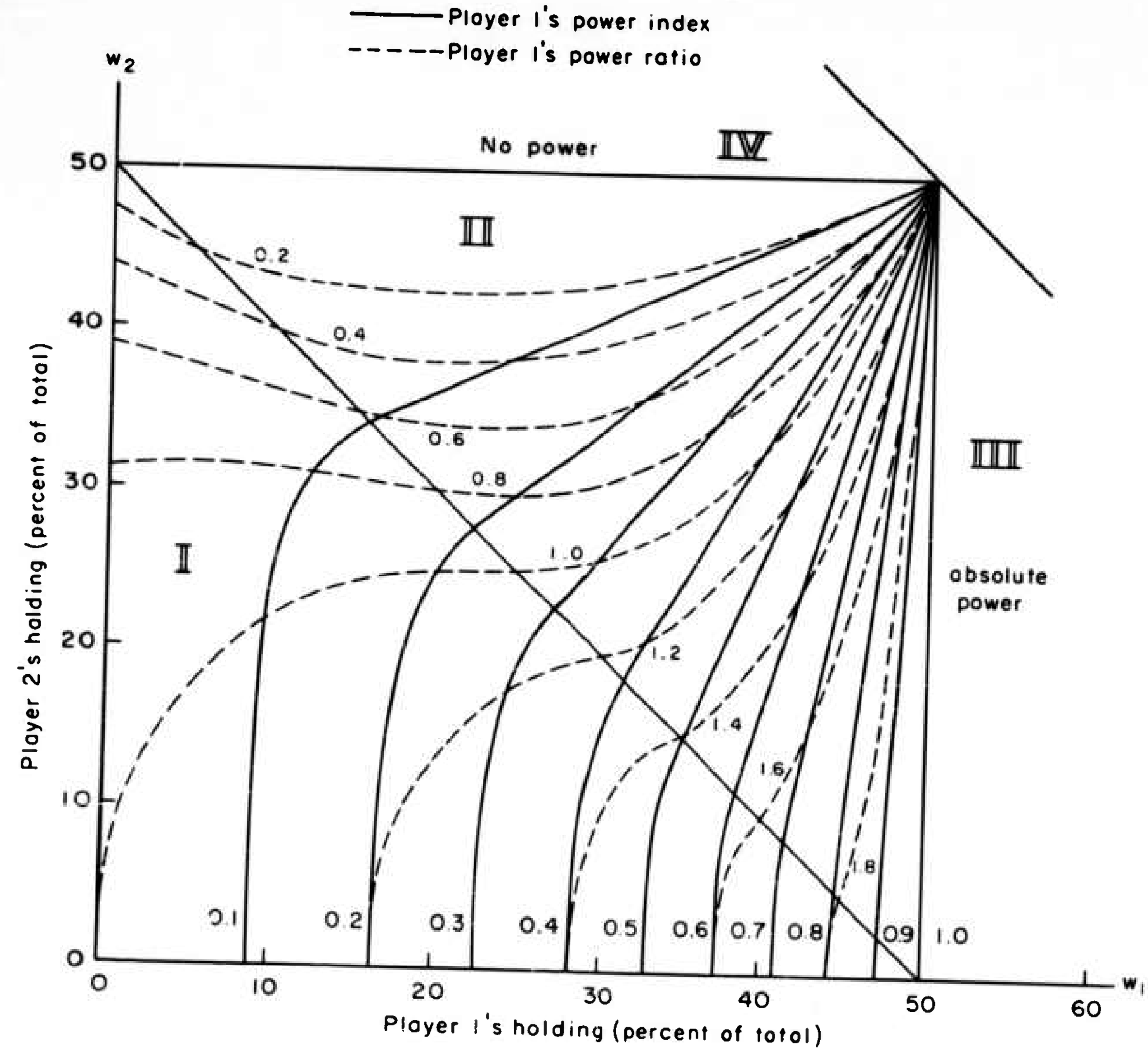

Fig. 2—Power of player 1 in the oceanic game

$$\left[\frac{1}{2}; w_1, w_2; 1-w_1-w_2\right]$$

indices are respectively 0.04 and 0.64, and their power ratios 0.4 and 1.6. The oceanic players have a combined power of 0.32 and a power ratio of 0.64.

Some comments on these graphs are in order. The functions plotted are all continuous across the case-boundaries, and along the edges of the large triangle with the exception of the point $w_1 = w_2 = 1/2$. The first derivatives are all continuous across the boundary between Regions I and II. The presence of oceanic players in the game appears to have a smoothing effect on the variations in power distribution; nothing remains of the discontinuous "fine structure" that is so conspicuous in finite-person games with many small players. (See [1], p. 7f. and [3], p. 7.)

When one of the major holdings, say w_2, is small, the power of the other player is insensitive to its magnitude, a fact made apparent by the vertical slope of the contours near the w_1-axis in Fig. 2. The same holds for the ocean's power ratio. This is intuitively plausible, since we would expect a small but finite block of shares to have virtually the same effect on the power distribution as an equivalent amount of "oceanic" stock. Indeed, if we let $w_2 \rightarrow 0$, then player 2 vanishes into the ocean without a ripple, to provide a perfectly continuous transition to the one-major-player case. (Compare [2], Theorem 4.)

Moderately powerful major players (as in the center of Region I) do somewhat better than the minor players on a

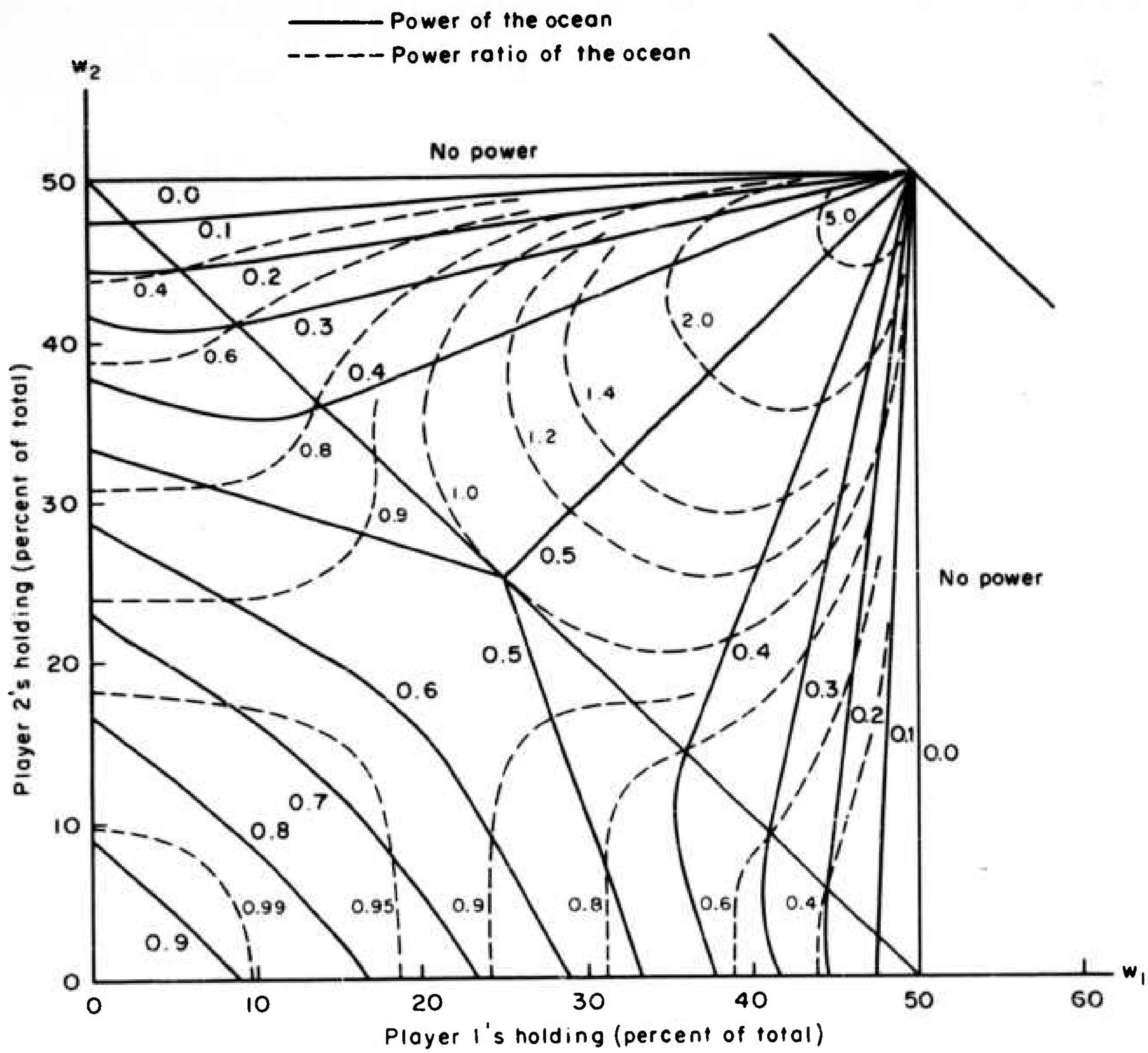

Fig. 3—Combined power of the oceanic players in the game

$$\left[\frac{1}{2}; w_1, w_2; 1-w_1-w_2\right]$$

power—per—share basis. As we cross the line into Region II, however, the ebbing tide of oceanic players becomes increasingly significant. The relative positions are shown in Fig. 4. The "balance of power" effect is strikingly illustrated as the two major players simultaneously approach 50% ownership, as in a fiercely contested proxy fight. The ocean actually retains half of the total power so long as the big players are evenly matched (and, we may imagine, are obliged to bid against each other for the support of the little players). The ocean's power ratio increases without limit in this situation.

4. A CROSS SECTION

In Table 1 we present in detail the fluctuations in power that would accompany the expansion of player 1's holding from nothing to more than half of the outstanding stock. Player 2's holding is kept constant at 30%. This particular cross section of the parameter space happens to cut through six of the eight regions shown in Fig. 4. The following comments are keyed to the table:

a. Player 2 has a commanding position.

b. Player 1's power ratio is slightly larger than the ocean's in this initial phase.

c. Almost all of player 1's gain has been at the expense of the ocean so far.

d. The ocean's power ratio surpasses that of player 1, which now starts to drop.

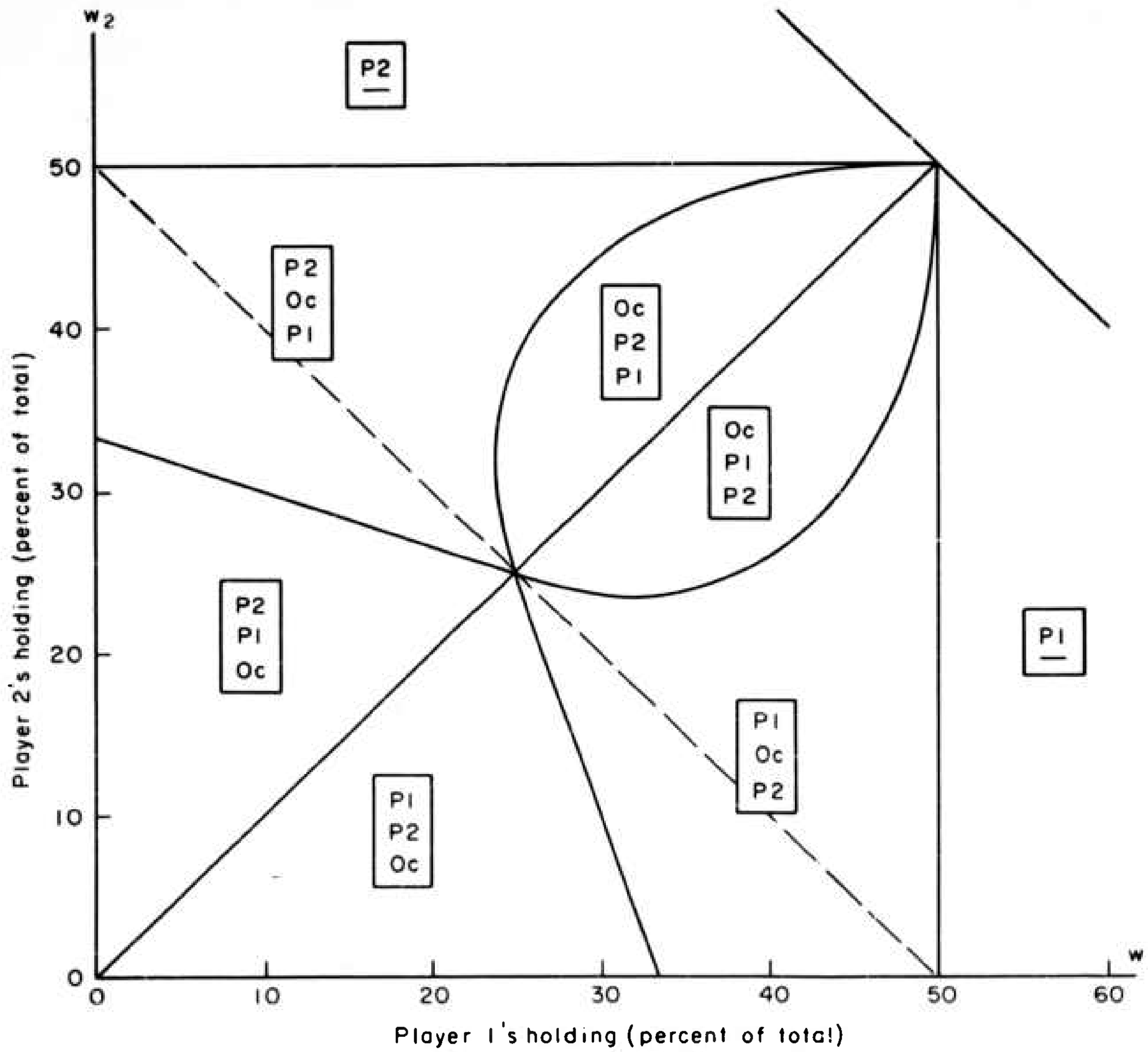

Fig. 4 — Ranking of players by power ratio

e. The oceanic players' total power starts to rise, despite their dwindling holdings. Player 2's power starts to slip significantly. Player 1's power ratio is lower now than ever before.

f. The game enters Region II. Note that player 2's shares are still "worth" half again as much as player 1's.

g. The major players for the first time have a combined power ratio less than 1.

h. This is the turning point for player 1's power ratio. Player 2 is losing ground rapidly as the ocean's total power continues to increase.

i. The symmetric point. The oceanic power index is at a local maximum.

j. Player 1's power ratio breaks out of the narrow interval in which it has been oscillating thus far.

k. Player 2 has gone into a sharp decline; he has less than half the power of player 1 even though the sizes of the two holdings are still comparable.

l. The oceanic power ratio reaches its maximum.

m. Player 1 sweeps towards absolute control.

n. Player 2's block of shares is "worth" much less than the smaller quantity of scattered oceanic shares.

TABLE 1

Shareholdings			Power indices			Power ratios			Notes
w_1	w_2	α	ϕ_1	ϕ_2	Φ	R_1	R_2	R_{oc}	*
0	30	70	.000	.429	.571	(.816)	1.429	.816	a
1	30	69	.008	.428	.563	.819	1.428	.816	b
2	30	68	.016	.428	.555	.822	1.427	.817	
3	30	67	.025	.428	.548	.824	1.426	.817	
4	30	66	.033	.427	.540	.826	1.423	.818	
5	30	65	.041	.426	.533	.828	1.420	.819	
6	30	64	.050	.425	.525	.830	1.416	.821	
7	30	63	.058	.423	.519	.831	1.411	.823	c
8	30	62	.067	.421	.512	.832	1.405	.826	
9	30	61	.075	.419	.506	.833	1.397	.829	
10	30	60	.083	.417	.500	.833	1.389	.833	d
11	30	59	.092	.414	.494	.833	1.379	.838	
12	30	58	.100	.410	.490	.832	1.367	.845	
13	30	57	.108	.406	.486	.831	1.354	.852	
14	30	56	.116	.402	.483	.829	1.339	.862	
15	30	55	.124	.397	.479	.826	1.322	.871	
16	30	54	.132	.391	.477	.823	1.303	.884	
17	30	53	.139	.384	.476	.819	1.282	.899	
18	30	52	.146	.377	.476	.814	1.257	.916	e
19	30	51	.153	.369	.478	.807	1.230	.936	
20	30	50	.160	.360	.480	.800	1.200	.960	f
21	30	49	.167	.350	.483	.793	1.168	.986	
22	30	48	.174	.340	.486	.789	1.134	1.013	g
23	30	47	.181	.330	.489	.787	1.100	1.040	
24	30	46	.189	.319	.492	.787	1.065	1.068	h
25	30	45	.198	.309	.494	.790	1.029	1.098	
26	30	44	.207	.298	.496	.795	.992	1.127	
27	30	43	.216	.286	.498	.801	.954	1.157	
28	30	42	.227	.274	.499	.810	.915	1.188	
29	30	41	.238	.262	.500	.821	.874	1.219	
30	30	40	.250	.250	.500	.833	.833	1.250	i
31	30	39	.263	.237	.500	.848	.791	1.281	
32	30	38	.277	.224	.499	.866	.748	1.312	j
33	30	37	.292	.211	.497	.886	.704	1.342	
34	30	36	.309	.198	.494	.908	.658	1.372	
35	30	35	.327	.184	.490	.933	.612	1.399	
36	30	34	.346	.170	.484	.961	.565	1.425	k
37	30	33	.367	.155	.478	.993	.517	1.447	
38	30	32	.391	.141	.469	1.028	.469	1.465	
39	30	31	.416	.126	.458	1.067	.420	1.477	
40	30	30	.444	.111	.444	1.111	.370	1.481	l
41	30	29	.476	.096	.428	1.160	.321	1.476	
42	30	28	.510	.082	.408	1.215	.272	1.458	
43	30	27	.549	.067	.384	1.276	.224	1.423	
44	30	26	.592	.053	.355	1.345	.178	1.365	
45	30	25	.640	.040	.320	1.422	.133	1.280	m
46	30	24	.694	.028	.278	1.510	.093	1.158	
47	30	23	.756	.017	.227	1.609	.057	.987	
48	30	22	.826	.008	.165	1.722	.028	.751	
49	30	21	.907	.002	.091	1.851	.008	.432	n
50	30	20	1.000	.000	.000	2.000	.000	.000	o
51	30	19	1.000	.000	.000	1.961	.000	.000	

*See text.

o. The game enters Region III. For maximum "efficiency" player 1 should stop when he has precisely half of the stock.

The reader should be cautioned against attaching too much significance to this little story. The value of a game can be justified as a measure of power, but only on a static basis. Once a game acquires a context, or a history, the delicate interplay of symmetry and ignorance on which the validity of the power index depends is lost.

5. SOME DYNAMIC CONSIDERATIONS

In spite of the warning we have just given, we would like to speculate for a bit on the possibility that the power, or the power-per-share ratio, may have an influence on the way that the players change their holdings over time. Of course, in practice this influence, if it exists, would have to be combined with all sorts of other tendencies, incentives, and mechanical constraints before meaningful dynamic predictions could be made. We are in no mood to enlarge our pleasantly uncluttered game model to include even the most important additional factors (e.g., money, time) that would be necessary, and so we confine our attention to a few simple exploratory remarks.

One hypothesis might be that shares tend to migrate to positions of increased power ratio. The resulting drift would then correspond roughly to the arrows shown in Fig. 5, based on Fig. 4:

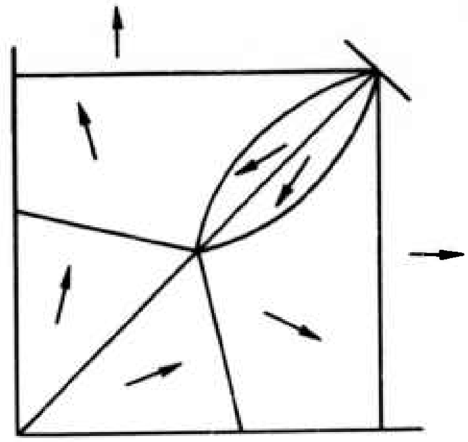

Fig. 5

We see at once, however, that this hypothesis implies motion in Regions III and IV — the "dictator" cases — which we would intuitively expect to be very stable. Powerless shares would gravitate into the portfolio of the winning player, despite his lack of incentive to acquire such shares, thereby diluting his holdings.

The flaw in the hypothesis. on reflection, seems to be the implicit assumption that shares, rather than players, are attracted by the prospect of increased power. To put the motivation where it belongs, we should assume instead that it is the marginal power to the player that governs the migration of shares. If a share happens to represent a smaller increment of power to its owner than to some other player, it is plausible to suppose, other things being equal, that a transfer might be arranged at a price agreeable to both parties.

To illustrate the general trend in shareholdings that would result from this principle, we have plotted in Fig. 6 the vector field $\vec{u} = (u_1, u_2)$ defined by

$$u_i = \frac{d\phi_i}{dw_i} - \frac{\Phi}{\alpha}, \qquad i = 1, 2. \tag{4}$$

This definition involves the assumption that transfers take place only into or out of the ocean — the open market — and not between the two major players directly. Note that "$+d\Phi/dw_i$" would not be correct for the second term of (4), since the ocean is not a single player.*

There are several noteworthy features of this vector field. One is that the larger of the two major holdings always tends to grow more rapidly than the other, which sometimes even tends to decrease. The ocean never tends to increase in size, but in the presence of two strong, evenly matched players ($w_1=w_2$ in Region II) it holds its own, since one of them tends to sell shares as fast as the other one buys them.

Another interesting point is the generally greater stability of Region I as compared with Region II, if we use the magnitude of $\vec{u}$ as a measure of stability. But the transition between the two regions is continuous. At the boundaries of Regions III and IV, on the other hand, there is a sharp discontinuity, indicated by the broken arrows. The magnitude of $\vec{u}$ drops abruptly to zero as soon as one player achieves absolute control.

*In drawing the arrows in Fig. 6 we have distorted the lengths for better legibility; the vector field actually shown has the form $c\vec{u}/(|\vec{u}|)^{1/2}$.

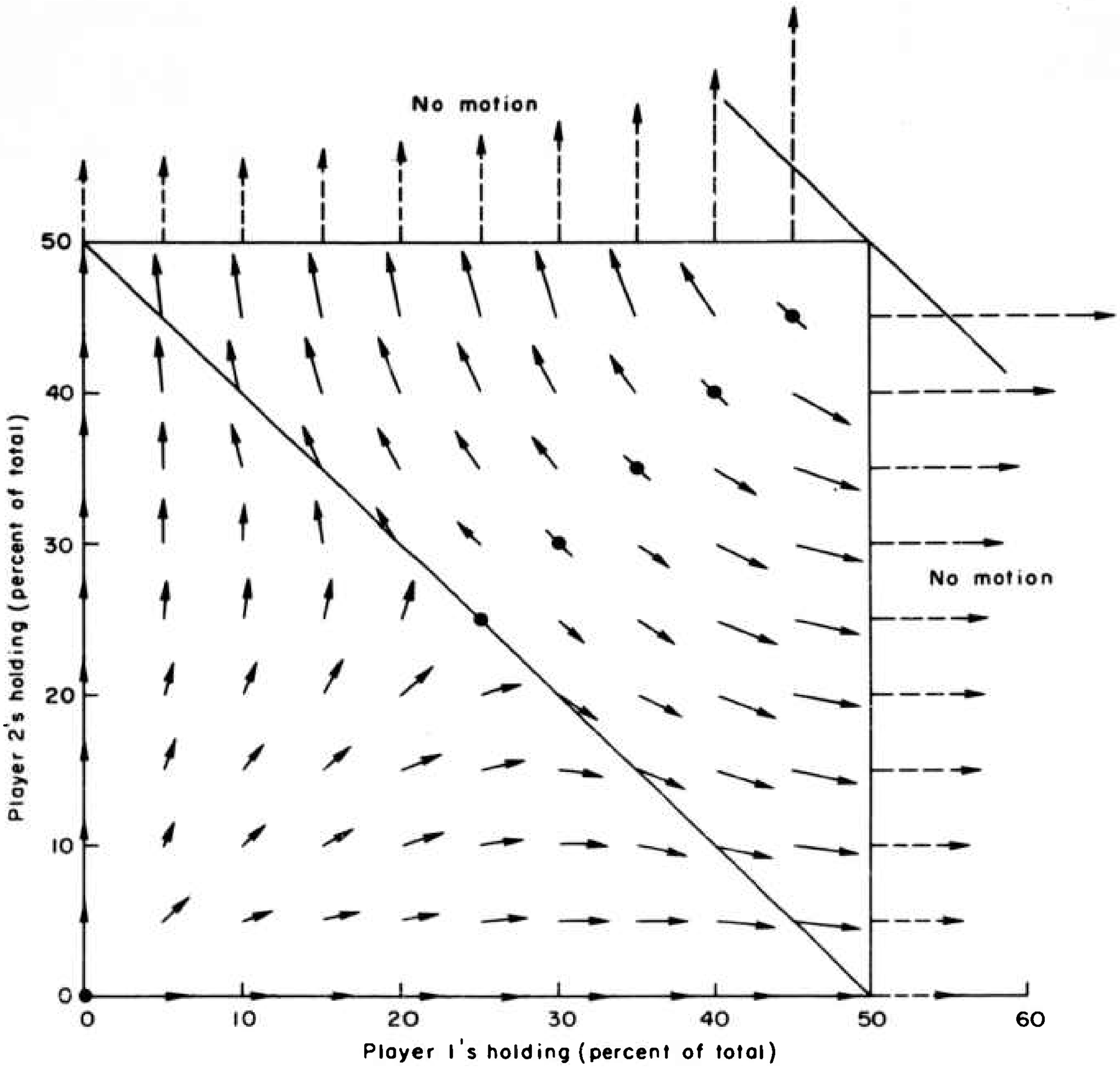

Fig. 6 — Migration of shareholdings induced by differences in marginal power per share

6. CRYSTALLIZATION

Another kind of dynamic evolution that might be investigated is the possible entry of a third major interest into the game via a process of "crystallization" out of the ocean. A theorem proved in [2] indicates that there is no first-order power incentive either favoring or inhibiting this possibility. Specifically, for w_3 close to zero, both the power ratio ϕ_3/w_3 and the marginal power per share $d\phi_3/dw_3$ are approximately equal to the oceanic ratio Φ/α.*

In other words, the situation in regard to a new major player is in equilibrium; whether it is stable or unstable depends on the second-order terms. Figure 7 shows how the sign of the difference,

$$\Delta = \phi_3\left[\tfrac{1}{2};\ w_1,\ w_2,\ \epsilon;\ \alpha - \epsilon\right] - \frac{\epsilon}{\alpha}\,\Phi\left[\tfrac{1}{2};\ w_1,\ w_2,\ 0;\ \alpha\right],$$

depends on the game parameters. In conformity with the theorem cited, Δ is of the order of magnitude of ϵ^2. A "+" sign means that conditions favor the entrance of a third major player; a "−" sign means the opposite. The dashed curves are approximate contours of equal Δ values. We see that in this case, too, the "balance of power" region is less stable than the "interior" region.

Setting $w_2 = 0$, we see that "crystallization" is favored in the one-major-player game only if that player has less than

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*See [2], Theorem 4. The case considered there is slightly different in that the added player's shares are not subtracted from the ocean, but the adjustment is easily made.

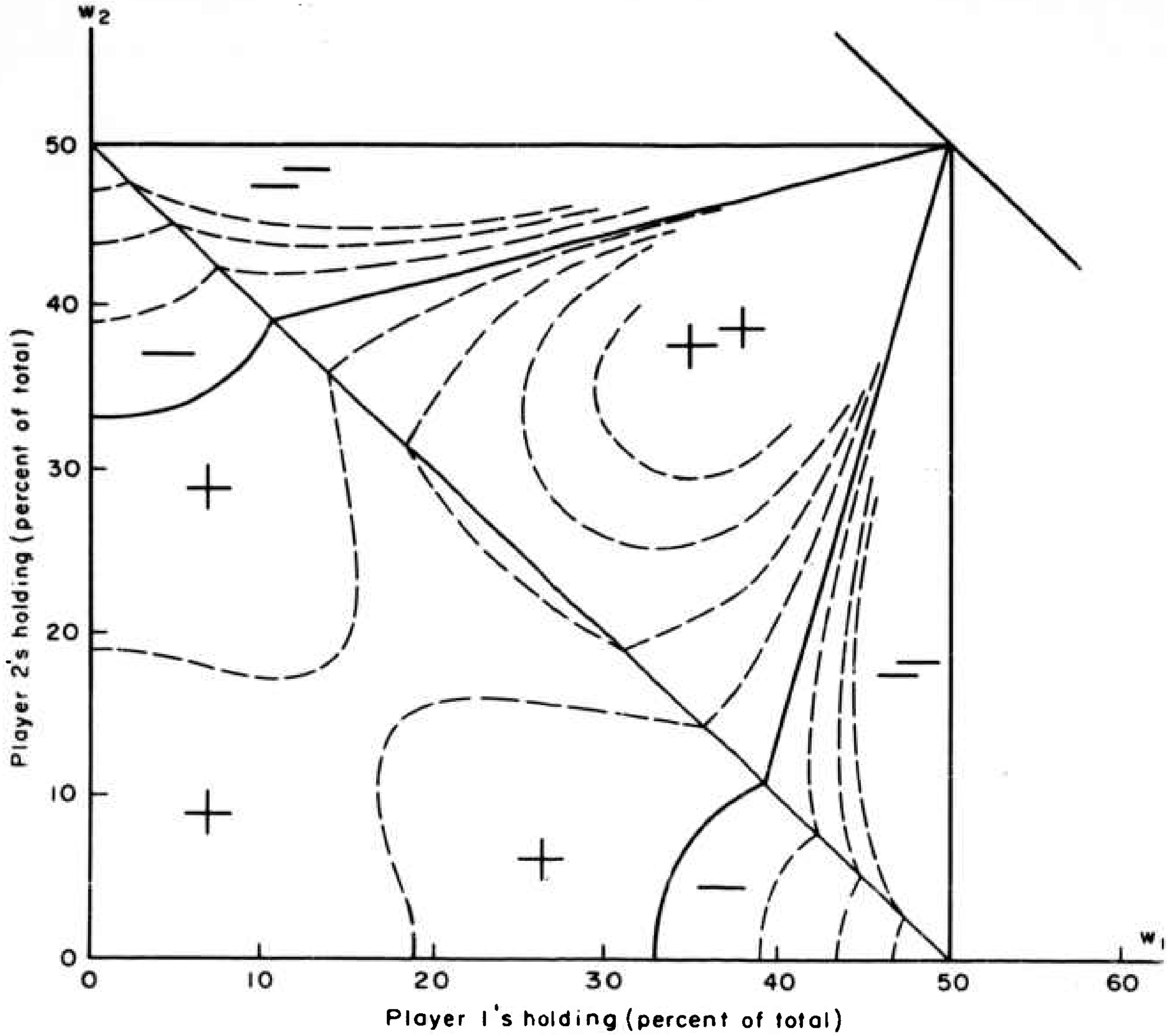

Fig. 7 — Prospects for the emergence of a third major player (+ = favorable, — = unfavorable)

1/3 of the stock. This fact can also be observed in Fig. 6 (less distinctly) by noting that the vector field in the neighborhood of the w_1-axis has a w_2-component that is positive for $w_1 < 1/3$ and negative for $w_1 > 1/3$. Again, in Table 1, the 'cyrstallizing' player starts off with a slight improvement over the oceanic ratio, since his major opponent holds only 3/10 of the stock.

The foregoing analysis may seem incomplete without a discussion of the tendency of a major interest to split up into two or more separate holdings, in pursuit of greater over-all power, or, inversely, to combine with (or sell out to) another major interest with the same end in view. We may remark that both of these maneuvers can be profitable under some circumstances. They have little to do with the 'oceanic' character of our present example, however, since they can equally well be studied in the context of ordinary finite games. Accordingly we shall not pursue the subject further at present.

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