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THEORY AND APPLICATIONS OF CORRELATION TECHNIQUES

THESIS

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Paul Henry Hass - BS in EE

First Lieutenant, USAF

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Preface

This report is the results of my attempt to survey and present in an easily understandable manner the basic theory of correlation and its application to the analysis of communication systems. Any topic involving the use of statistical theory is inherently a mathematical dissertation. However, an attempt has been made to limit the report to the more important mathematical relations with emphasis placed upon physical interpretation.

The Summary of Linear System Relation, presented in Table I, page 33, is, I think, a relatively complete summary of the important relations for a linear system disturbed by random inputs. As a reference, it proved itself invaluable in the solution of the problems present in Sections IV and V.

The idea for this report was first suggested by Professor George W. Ogar, to whom I am deeply indebted. His patience and understanding, along with his many helpful suggestions is greatly appreciated.

And last but not least, a debt of gratitude is due my wife and children. May we once again renew our acquaintances.

Paul H. Hass

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Abstract

This report is a survey of the important principles of correlation theory, as applicable to the statistical analysis of communication systems. The underlying theory of correlation functions and their relations to the power-density-spectrum is presented. Linear systems, excited by random inputs, are analyzed and describing relations are developed from the correlation functions. Corresponding frequency domain relations in terms of the power-density spectrum are also developed. The mean-square-error as a design criterion is presented both from a general and a specific viewpoint. The specific approach is a problem which entails the excitation of a linear system by a message corrupted by noise. It is shown how the general development for the mean-square-error reduces to the specific problem presented.

THEORY AND APPLICATIONS OF CORRELATION TECHNIQUES

I. Introduction

The purpose of this report is to survey, present, and analyze the theory of correlation and its application to the statistical analysis of communication and control systems. The primary concern in any communication or control problem is the flow of messages or signals. In general, these messages or signals may take on a variety of different forms. Typical examples are; temperature changes, wind gusts on an aircraft, barometric changes, music, et cetera. In the past, conventional design procedures have been based on the response and characteristics that systems exhibited to certain predictable signals. Some examples of these predictable signals are the sinusoidal signal and step functions. It has often been said that any signal or message could be characterized by a sum of sinusoids or steps. However, is it possible to visualize what will be the effect in the output of a system due to this combination of inputs? Statistical design theory employing correlation techniques is an attempt to broaden the design to include consideration of general and random inputs.

The material presented in this report primarily concerns the underlying theory of correlation as applicable to the analysis of linear systems. The block diagram approach will be used in that linear systems under consideration will be characterized only by system functions. Little attention will be given to the internal configuration of any particular system.

The remainder of this report is divided into four major parts: (1) Correlation Functions, (2) Analysis of Linear Systems, (3) Problems and Examples, and (4) Mean-Square - Error Design Criterion. In the first part the correlation functions are introduced. Autocorrelation and cross-correlation are defined and their important properties are discussed. The second part involves the application of the information presented in part (1). Linear systems are discussed and describing relations are developed with the correlation functions. The third part is devoted to the solution of illustrative example. In part (4) the mean-square - error design criterion is presented. It is shown that the mean-square - error can be adequately described by correlation functions.

II. Correlation Functions

Autocorrelation Functions

Periodic Functions. The autocorrelation of a periodic function $f_1(t)$ is defined by the expression

$$\phi_{11}(\tau) = \frac{1}{T_1}\int_{-T_1/2}^{T_1/2} f_1(t) f_1(t+\tau)\, dt \tag{1}$$

where

$\phi_{11}(\tau)$ = the autocorrelation of $f_1(t)$

$T_1 = \frac{\omega_1}{2\pi}$, period of $f_1(t)$

and τ is a continuous displacement in the interval $(+\infty, -\infty)$, independent of t.

A close examination of the defining equation reveals that three distinct operations are involved: (1) displacement by an amount τ, (2) continuous multiplication, and (3) averaging by integration over a complete period.

An important property of the autocorrelation function is that its Fourier Transform is

$$\frac{F_1(n)\,\overline{F_1(n)}}{T_1} = \frac{|F_1(n)|^2}{T_1} \tag{2}$$

where $F_1(n)$ is the spectrum of $f_1(t)$

and $\overline{F_1(n)}$ is the complex conjugate of $F_1(n)$.

To establish this relation, the exponential Fourier expansion of $f_1(t)$ is

$$f_1(t) = \frac{1}{T_1}\sum_{n=-\infty}^{\infty} F_1(n)\epsilon^{j\omega_n t} \tag{3}$$

where

$$F_1(n) = a_n - jb_n = \int_{-T_1/2}^{T_1/2} f_1(t)\epsilon^{-j\omega_n t}\,dt \tag{4}$$

and

$$a_n = \int_{-T_1/2}^{T_1/2} f_1(t)\cos\omega_n t\,dt \qquad n=0,1,2,\cdots \tag{5}$$

$$b_n = \int_{-T_1/2}^{T_1/2} f_1(t)\sin\omega_n t\,dt \qquad n=1,2,\cdots \tag{6}$$

From equation (3) the expression for $f_1(t+\tau)$ is

$$f_1(t+\tau) = \frac{1}{T_1}\sum_{n=-\infty}^{\infty} F_1(n)\epsilon^{j\omega_n(t+\tau)} \tag{7}$$

Substituting the expressions for $f_1(t+\tau)$ into the defining equation for $\phi_{11}(\tau)$ gives

$$\phi_{11}(\tau) = \frac{1}{T_1}\int_{-T_1/2}^{T_1/2} f_1(t)dt\,\frac{1}{T_1}\sum_{n=-\infty}^{\infty} F_1(n)\epsilon^{j\omega_n(t+\tau)} \tag{8}$$

Interchanging the order of integration and summation

$$\phi_{11}(\tau) = \frac{1}{T_1}\sum_{n=-\infty}^{\infty} F_1(n)\epsilon^{j\omega_n \tau}\,\frac{1}{T_1}\int_{-T_1/2}^{T_1/2} f_1(t)\epsilon^{j\omega_n t}\,dt \tag{9}$$

The integral contained in the above equation is the complex conjugate of $F_1(n)$ as defined in equation (4). Therefore

$$\phi_{11}(\tau) = \frac{1}{T_1}\sum_{n=-\infty}^{\infty} \frac{F_1(n)\,\overline{F_1(n)}}{T_1}\,\epsilon^{j\omega_n \tau} \tag{10}$$

or

$$\phi_{11}(\tau) = \frac{1}{T_1}\sum_{n=-\infty}^{\infty} \frac{|F_1(n)|^2}{T_1}\,\epsilon^{j\omega_n \tau} \tag{11}$$

The value of the autocorrelation function at zero argument is

$$\phi_{11}(0) = \frac{1}{T_1}\int_{-T_1/2}^{T_1/2} f_1^2(t)\,dt \tag{12}$$

which is the mean-square-value of $f_1(t)$. From equation (11) and (12) it can be seen that the mean-square-value of the function $f_1(t)$ is equal to the sum, over the entire range of harmonic, of the square of the absolute value of its spectrum. If a load of 1 ohm is assumed, and $f_1(t)$ represents a voltage or current, then the mean power consumed is the sum of the power contributed by the harmonic into which $f_1(t)$ has been resolved. This is known as Parseval's Theorem for a periodic function.

Defining $\frac{|F_1(n)|^2}{T_1}$ as the power spectrum, and denote it by the symbol $\Phi_{11}(n)$ then from equation (11)

$$\phi_{11}(\tau) = \frac{1}{T_1} \sum_{n=-\infty}^{\infty} \Phi_{11}(n) \epsilon^{j\omega_n \tau} \tag{13}$$

and inversely

$$\Phi_{11}(n) = \int_{-T_1/2}^{T_1/2} \phi_{11}(\tau) \epsilon^{-j\omega_n \tau} d\tau \tag{14}$$

The above two equations form the autocorrelation theorem for a periodic function. That is, the autocorrelation function and the power spectrum are Fourier transforms of each other.

<u>Aperiodic</u> <u>Functions</u>. The basic tool in the analysis of an aperiodic function $f_1(t)$ is the Fourier Integral

$$f_1(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(\omega) \epsilon^{j\omega t} d\omega \tag{15}$$

and its inverse

$$F_1(\omega) = \int_{-\infty}^{\infty} f_1(t) \epsilon^{-j\omega t} dt \tag{16}$$

where $F_1(\omega)$ is the spectrum of $f_1(t)$. In general $F_1(\omega)$ may be complex. Whereas the spectrum of a periodic function was an amplitude spectrum, that for an aperiodic function is an amplitude-density spectrum. The restriction on the function $f_1(t)$ in order that it have a Fourier integral are that the integrals of both its squares and its absolute value have finite values, and that it posses a finite number of discontinuities in any finite interval (Ref. 11:239).

The defining equation for the autocorrelation function of an aperiodic signal is

$$\phi_{11}(\tau) = \int_{-\infty}^{\infty} f_1(t) f_1(t+\tau)\, dt \tag{17}$$

The main difference between the autocorrelation of a periodic function and that of an aperiodic function is that the latter does not include taking the mean value over a finite interval.

The square of the absolute value of the spectrum of an aperiodic function is the Fourier transform of the autocorrelation function. This is called the energy-density spectrum, $\Phi_{11}(\omega)$ and gives the distribution of signal energy with frequency. To establish the relation between $\phi_{11}(\tau)$ and $\Phi_{11}(\omega)$, the inverse Fourier transform of $f_1(t + \tau)$ is

$$f_1(t+\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(\omega)\, \epsilon^{j\omega(t+\tau)}\, d\omega \tag{18}$$

Substituting this expression into the defining equation for $\phi_{11}(\tau)$ gives

$$\phi_{11}(\tau) = \int_{-\infty}^{\infty} f_1(t)dt \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega)\epsilon^{j\omega(t+\tau)}d\omega \tag{19}$$

By interchanging the order of integration

$$\phi_{11}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega)\epsilon^{j\omega\tau}d\omega \int_{-\infty}^{\infty} f_1(t)\epsilon^{j\omega t}dt \tag{20}$$

In the above equation, the term $\int_{-\infty}^{\infty} f_1(t)\epsilon^{j\omega t}dt$ is the complex conjugate of $F_1(\omega)$. Therefore

$$\phi_{11}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F_1(\omega)|^2 \epsilon^{j\omega\tau} d\omega \tag{21}$$

or in terms of the energy-density spectrum

$$\phi_{11}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{11}(\omega)\, \epsilon^{j\omega\tau} d\omega \tag{22}$$

The inverse relation is

$$\Phi_{11}(\omega) = \int_{-\infty}^{\infty} \phi_{11}(\tau)\, \epsilon^{-j\omega\tau} d\tau \tag{23}$$

Equation (22) and (23) relate the autocorrelation function of an aperiodic function to its energy-density spectrum as Fourier transforms of each other. It is important to note that the physical interpretation given to $\Phi_{11}(\omega)$ is the energy-density spectrum of the aperiodic function and not the spectrum of $\phi_{11}(\tau)$.

The value of the autocorrelation function at zero argument is

$$\phi_{11}(0) = \int_{-\infty}^{\infty} f_1^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F_1(\omega)|^2\,d\omega \qquad (24)$$

This relation is often referred to as Parseval's equality.

Random Functions. A random process is an ensemble (or set) of time functions, such that the set can be described by certain probability distributions. If $f_1(t)$, $f_2(t)$, $f_3(t)$ ------ are functions comprising the ensemble, then the value of these function at the time t_1 must obey a certain probability distribution law. At any given time t_1 (see Figure 1) there must be a definite probability of any given member-function lying between specified values. A stationary random process is one in which the probability distribution functions do not change with time. Thermal noise voltage generated by the random motion of conduction electrons in a resistor would be a stationary random process, as long as the temperature remained constant.

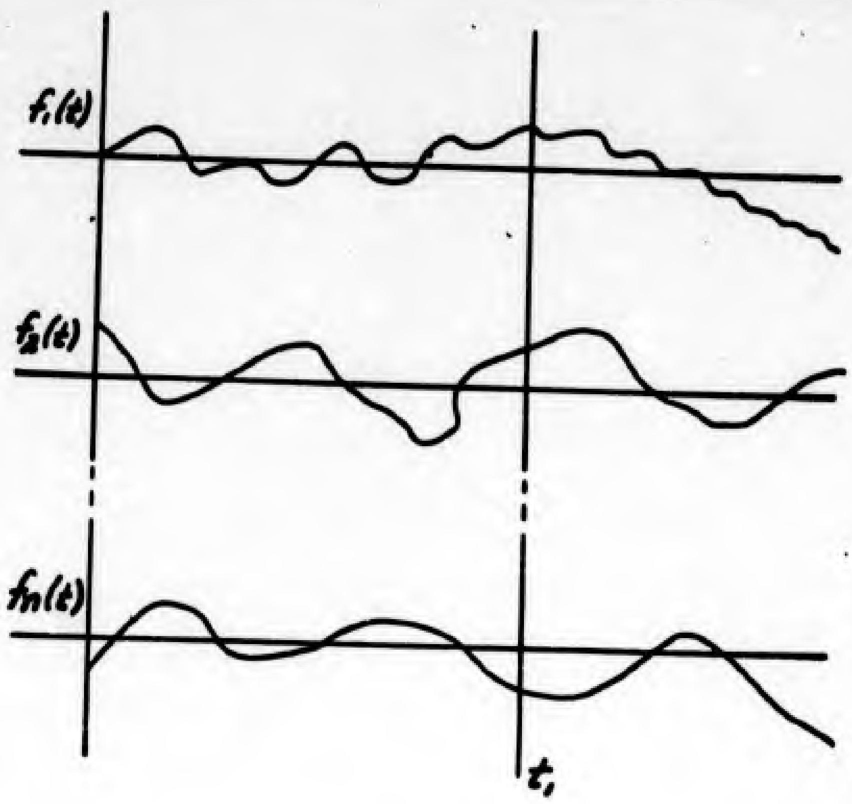

Figure 1

An Ensemble of Random Time Functions

Stationary random process are ergodic. This means that each record of the ensemble is statistically equivalent to every other record. In other words, if one member of a stationary random process is examined over a sufficiently long period, all important characteristics of the ensemble will be observed.

If $f_1(t)$ represents a member of a stationary random process, the autocorrelation of $f_1(t)$ is defined as

$$\phi_{11}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_1(t)\, f_1(t+\tau)\,dt \qquad (25)$$

The function $\phi_{11}(\tau)$ is assumed to exist for all values of the argument τ. From the previous discussion on random processes, the autocorrelation function is the same for each member of the ensemble. It is therefore a characteristic of the ensemble.

In general terms a random function or process is one in which the effect can not be related to the cause by a unique mathematical expression. The autocorrelation property of random functions is therefore an important tool in the analysis of communication and control systems. Through use of correlation functions an equivalent frequency domain expression for a random process can be found. Without the intermediate step of correlation this feature would not be possible, since random functions in general do not possess Fourier transforms.

The relation between the autocorrelation function, time domain representation of $f_1(t)$, and the power-density spectrum, frequency domain representation of $f_1(t)$, is

$$\phi_{11}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{11}(\omega)\epsilon^{j\omega\tau}\,d\omega \tag{26}$$

and

$$\Phi_{11}(\omega) = \int_{-\infty}^{\infty} \phi_{11}(\tau)\epsilon^{-j\omega\tau}\,d\tau \tag{27}$$

The above two relations are known as the Wiener-Khintchine Theorem. The physical significance of the power-density spectrum

$\Phi_{11}(\omega)$ can be seen by letting $\tau = 0$ in equation (25) and (27). From the defining equation for the autocorrelation function, equation (25)

$$\phi_{11}(0) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_1^2(t)\,dt \tag{28}$$

which is the mean-square-value of the function $f_1(t)$. From equation (27) the value of the autocorrelation function of zero argument is

$$\phi_{11}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{11}(\omega)\,d\omega \tag{29}$$

therefore

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_1^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{11}(\omega)\,d\omega \tag{30}$$

The above equation relates the mean-square of $f_1(t)$ to the power-density spectrum. If $f_1(t)$ represents a voltage across, or a current through a 1 ohm resistor, then the mean value of $f_1(t)$ is the mean power taken by the load. This mean power consumed is equal to the integral of $\Phi_{11}(\omega)$ with respect to the angular frequency ω over the entire range of frequencies. Consequently $\Phi_{11}(\omega)$ represents the power-density spectrum of $f_1(t)$ (Ref. 7:58). Since $\Phi_{11}(\omega)$ is defined as the Fourier transform of $\phi_{11}(\tau)$ which is an aperiodic function, it is a continuous spectrum of the autocorrelation function.

Properties of the Autocorrelation Function of a Random Process. Listed below are some of the more important properties of $\phi_{11}(\tau)$.

1) The autocorrelation function is an even function of τ.

$$\phi_{11}(\tau) = \phi_{11}(-\tau) \tag{31}$$

2) The value of the function at $\tau = 0$ is the mean-square value of $f_1(t)$.

$$\phi_{11}(0) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f_1^2(t)\,dt \tag{32}$$

3) If a random function contains a hidden periodic component, then the autocorrelation function contains components of the same period, however, any phase information that the periodic component may have had is lost.

4) The value of the autocorrelation function is a maximum at the origin.

$$\phi_{11}(0) > |\phi_{11}(\tau)| \qquad \tau \neq 0 \tag{33}$$

5) The value of $\phi_{11}(\tau)$ as $\tau \to \infty$ is zero if the function $f_1(t)$ contains no d.c. or periodic components.

6) A given autocorrelation function may correspond to

an infinite number of different time functions; however, any given time function has only one autocorrelation function.

<u>Alternate Expressions for the Relationship Between the Autocorrelation Function and its Associated Spectrum.</u> Since the autocorrelation function is an even function of τ, previous equations expressed in exponential form may be changed to trigonometric form when convenient. For the case of the periodic function

$$\phi_{11}(\tau) = \frac{1}{T_1} \sum_{n=-\infty}^{\infty} \Phi_{11}(n) \cos \omega_n \tau \tag{34}$$

and

$$\Phi_{11}(n) = \int_{-T_1/2}^{T_1/2} \phi_{11}(\tau) \cos \omega_n \tau \, d\tau \tag{35}$$

For the aperiodic and random functions

$$\phi_{11}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{11}(\omega) \cos \omega \tau \, d\omega \tag{36}$$

and

$$\Phi_{11}(\omega) = \int_{-\infty}^{\infty} \phi_{11}(\tau) \cos \omega \tau \, d\tau \tag{37}$$

Characteristics of the Power-Density Spectrum $\Phi_{11}(\omega)$ of a Random Function.[1] Listed below are some of the more important properties of the power-density spectrum of a random function.

1) $\Phi_{11}(\omega)$ is an even function of frequency

$$\Phi_{11}(\omega) = \Phi_{11}(-\omega) \tag{38}$$

2) $\Phi_{11}(\omega)$ measures the power-density spectra rather than the amplitude or phase spectra of the signal. This results in a loss of the relative phase information of the various frequency components.

3) A given power-density spectrum may represent an infinite number of time functions.

4) The power-density spectrum is a real and non-negative function for all values of frequency.

Cross-correlation Functions

Periodic Functions. The cross-correlation between two different periodic functions $f_1(t)$ and $f_2(t)$ is defined as

$$\phi_{12}(\tau) = \frac{1}{T_1}\int_{-T_1/2}^{T_1/2} f_1(t) f_2(t+\tau)\, dt \tag{39}$$

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[1]Truxal, J. C. Control System Synthesis, pp 442-443.

where $f_1(t)$ and $f_2(t)$ are of the same fundamental frequency. If the displacement is given instead to $f_1(t)$ then

$$\phi_{21}(\tau) = \frac{1}{T_1}\int_{-T_1/2}^{T_1/2} f_1(t+\tau) f_2(t)\, dt \tag{40}$$

The order of the subscript of ϕ indicates which function has been shifted by the amount τ. The two functions are related by

$$\phi_{12}(\tau) = \phi_{21}(-\tau) \tag{41}$$

The cross-power spectrum of the two periodic functions is

$$\Phi_{12}(n) = \frac{F_2(n)\overline{F_1(n)}}{T_1} \tag{42)a$$

and

$$\Phi_{21}(n) = \frac{F_1(n)\overline{F_2(n)}}{T_1} \tag{42)b$$

where $F_1(n)$ and $F_2(n)$ are the amplitude spectra of $f_1(t)$ and $f_2(t)$ respectively. Similar to the relations of the auto-correlation function, the cross-correlation function and the cross-power spectrum are Fourier transform of each other. Stated mathematically

$$\phi_{12}(\tau) = \frac{1}{T_1}\sum_{n=-\infty}^{\infty} \Phi_{12}(n)\,\epsilon^{j\omega_n\tau} \tag{43}$$

and

$$\Phi_{12}(n) = \int_{-T_1/2}^{T_1/2} \phi_{11}(\tau)\,\epsilon^{-j\omega_n \tau}\,d\tau \qquad (44)$$

Similar expressions exist for the relations between $\phi_{21}(\tau)$ and $\Phi_{21}(n)$. It is also true that

$$\Phi_{12}(n) = \overline{\Phi_{21}(n)} \qquad (45)$$

This can be readily seen from equations (42)a and (42)b.

Concerning the use of the term cross-power spectra. It is not always true that the transform of a cross-correlation function represents a power transfer. Where the operation performed on the input current and voltage of a system, then the cross-correlation function after transformation, would represent a power dissipation. However, if the operation was on the input and output for example, this would not correspond to a transfer of power through the system.

<u>Aperiodic and Random Functions</u>. Since the relations and expressions for the cross-correlation function of an aperiodic and random functions are similar in nature to those of the autocorrelation function, they will be listed here without discussion.

a) Aperiodic Functions.

Definition of the cross-correlation function:

$$\phi_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t) f_2(t+\tau)\, dt \tag{46}$$

and

$$\phi_{21}(\tau) = \int_{-\infty}^{\infty} f_1(t+\tau) f_2(t)\, dt \tag{47}$$

where

$$\phi_{12}(\tau) = \phi_{21}(-\tau) \tag{48}$$

Cross-energy-density spectrum.

$$\Phi_{12}(\omega) = \overline{F_1(\omega)}\, F_2(\omega) \tag{49}$$

and

$$\Phi_{21}(\omega) = F_1(\omega)\, \overline{F_2(\omega)} \tag{50}$$

where $F_1(\omega)$ and $F_2(\omega)$ are the amplitude-density spectra of $f_1(t)$ and $f_2(t)$ respectively and are determined by the following expressions

$$F_1(\omega) = \int_{-\infty}^{\infty} f_1(t)\, \epsilon^{-j\omega t}\, dt \tag{51}$$

and

$$F_2(\omega) = \int_{-\infty}^{\infty} f_2(t)\epsilon^{-j\omega t}dt \tag{52}$$

The cross-correlation function and cross-energy-density spectrum are Fourier transform of each other.

b) Random Functions. The defining equations for the cross-correlation of two stationary random functions $f_1(t)$ and $f_2(t)$ is

$$\phi_{12}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_1(t)f_2(t+\tau)\,dt \tag{53}$$

and

$$\phi_{21}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_1(t+\tau)f_2(t)\,dt \tag{54}$$

where $\phi_{12}(\tau) = \phi_{21}(-\tau)$. The reciprocal relations relating to the cross-power-density spectrum of $f_1(t)$ and $f_2(t)$ are

$$\phi_{12}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{12}(\omega)\epsilon^{j\omega\tau}d\omega \tag{55}$$

and

$$\Phi_{12}(\omega) = \int_{-\infty}^{\infty} \phi_{12}(\tau)\epsilon^{-j\omega\tau}d\tau \tag{56}$$

Properties of the Cross-correlation Function. Listed below are some of the more important properties of the cross-correlation function $\phi_{12}(\tau)$

1) $\phi_{12}(\tau)$ is not an even function of τ. In general, shifting of $f_2(t)$ ahead by an amount τ does not give the same result as a retardation (Ref. 13).

2) $\phi_{12}(\tau)$ does not necessarily posses a maximum at $\tau = 0$.

3) The cross-correlation of two "uncorrelated" stationary random processes is zero. By "uncorrelated" is meant that the two functions on which the cross-correlation is being performed are derived from independent sources.

4) $\phi_{12}(\tau \infty) = 0$. The cross-correlation functions tends to zero as the value of τ tends to infinity. This is true provided that either one or both of the two signals have no d.c. component, and, in addition, if the functions contain periodic components, those in one have no periods commensurable to the periods of the other. (Ref 7: 76)

III. Analysis of Linear Systems

Relation Between the Impulse Response and System Function

A linear system is characterized by its response to a unit impulse excitation. The response to a unit impulse is denoted by h(t). Since the impulse $\delta(t)$ is applied at t = o, h(t) is zero for $\tau < 0$ because the system can not respond before an excitation is applied. Another important property of a linear system is that if the unit impulse is delayed by an amount δ the impulse response is delayed by the same amount. (see figure 2).

The convolution integral mathematically relates the input and output in terms of the unit impulse response

$$f_o(t) = \int_{-\infty}^{\infty} h(v) f_i(t-v)\, dv \tag{57}$$

where

$f_o(t)$ is the system output

$h(v)$ is the impulse response

$fi(t-v)$ is the system input, delayed v sec.

A physical interpretation of the convolution integral is as follows: From figure 3 the input fi(v) is plotted as a function of v in part (a). In part (b) the input is folded about the

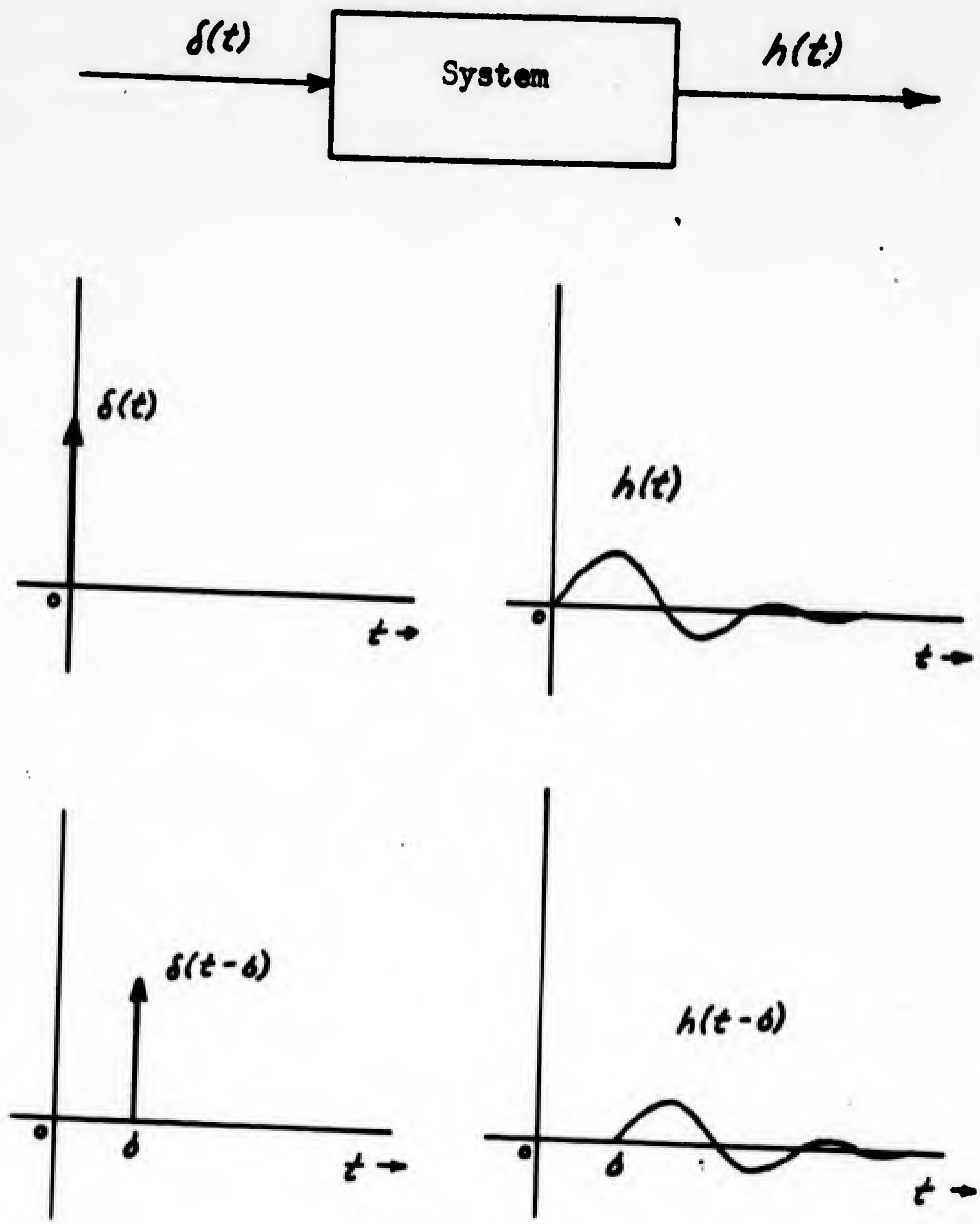

Figure 2

Relation Between Unit Impulse
Excitation and Unit Impulse Response
(Ref. 7:325)

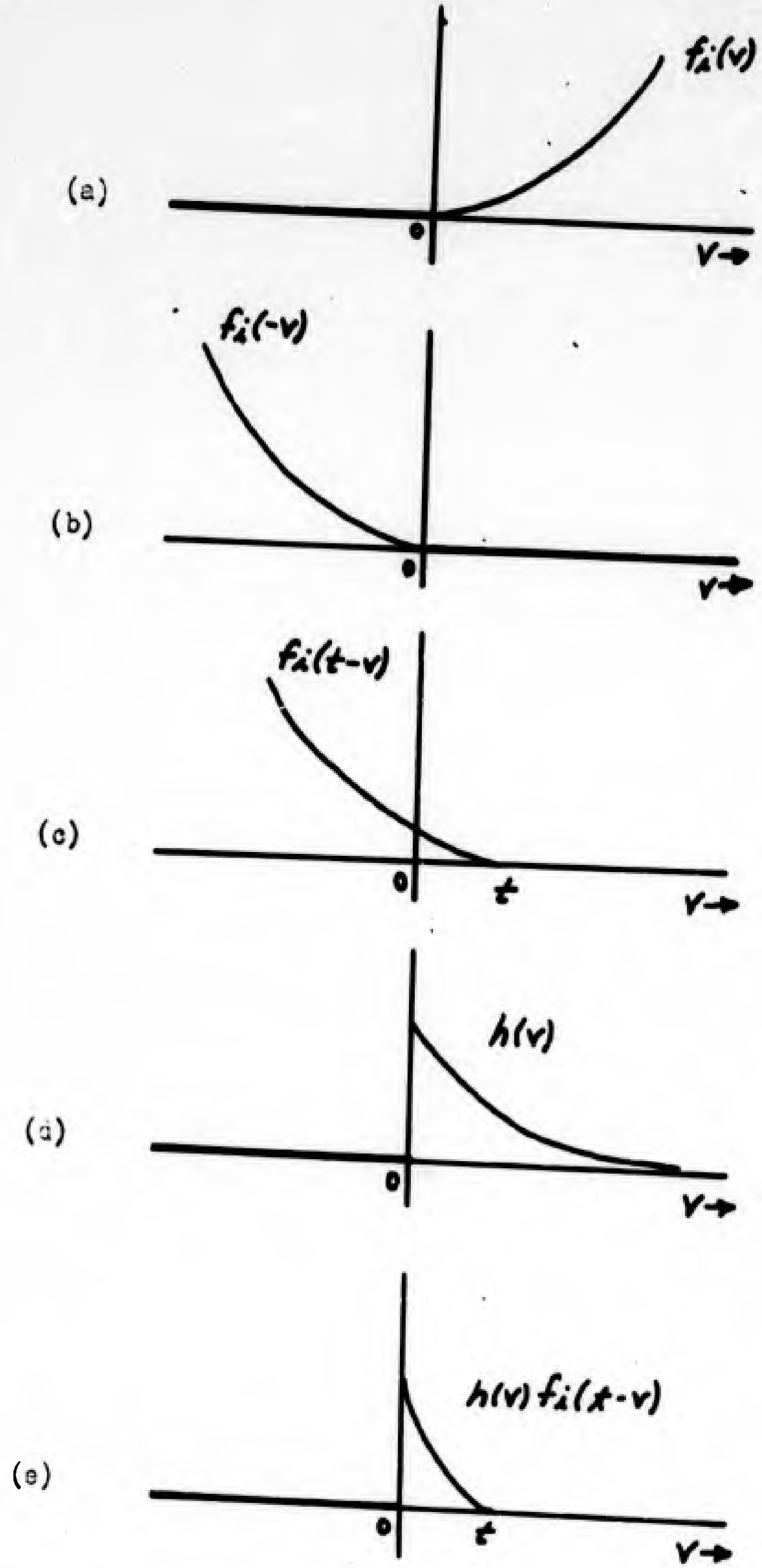

Figure 3

Integrand of the Convolution Integral

zero axis to form $f_i(-v)$. Part (c) shows the folded input advanced by a time t. The impulse response is plotted as a function of v in part (d). Part (e) is the product of parts (c) and (d) and corresponds to the integrand of the convolution integral. The area under the curve of part (e) corresponds to the output at the time t. Thus, h(v) can be considered a weighting function; the output at the time t is dependent upon the input at that same time and at all previous times. The convolution integral is a valid expression for periodic, aperiodic and random functions.

The system function $H(\omega)$ of a linear system is defined as

$$H(\omega) = \frac{E_o}{E_i} \tag{58}$$

where Eo and Ei are the complex amplitudes of the input and output respectively (Ref 7:328). The desired relations which related the impulse response and the system function are

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\,\epsilon^{j\omega t}\,d\omega \tag{59}$$

and

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\,\epsilon^{-j\omega t}\,dt \tag{60}$$

The two functions are Fourier transforms of each other.

The Output Autocorrelation Theorem

An expression of great importance in the analysis of linear systems with random inputs is that relating the input and output autocorrelation functions. To obtain this relation, the necessary step relating the two is the convolution integral. For the linear system shown in figure 4 the output autocorrelation function by definition is

$$\phi_{00}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_0(t)\, f_0(t+\tau)\, dt \qquad (61)$$

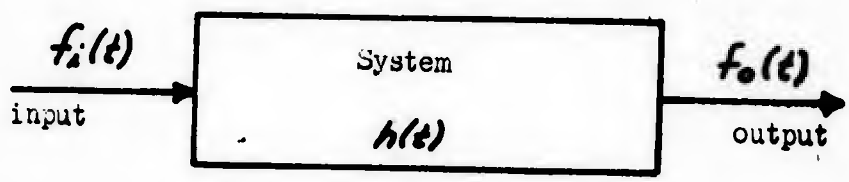

Figure 4
Linear System

Expressing the elements of the integrand in terms of the convolution integral gives

$$\phi_{00}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt \int_{-\infty}^{\infty} h(v)\, f_i(t-v)\, dv \int_{-\infty}^{\infty} h(\sigma)\, f_i(t+\tau-\sigma)\, d\sigma \qquad (62)$$

By interchanging the order of integration

$$\phi_{oo}(\tau) = \int_{-\infty}^{\infty} h(v)\,dv \int_{-\infty}^{\infty} h(\sigma)\,d\sigma \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_i(t-v)\,f_i(t+\tau-\sigma)\,dt \tag{63}$$

Since

$$\phi_{ii}(\tau+v-\sigma) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_i(t-v)\,f_i(t+\tau-\sigma)\,dt \tag{64}$$

The expression for the output autocorrelation becomes

$$\phi_{oo}(\tau) = \int_{-\infty}^{\infty} h(v)\,dv \int_{-\infty}^{\infty} h(\sigma)\,d\sigma\ \phi_{ii}(\tau+v-\sigma) \tag{65}$$

This is the desired time domain relation expressing the output in terms of the input autocorrelation and the systems's impulse response. The mean-square-value of the output may be found by setting $\tau = 0$ in the above equation.

Thus

$$\phi_{oo}(0) = \int_{-\infty}^{\infty} h(v)\,dv \int_{-\infty}^{\infty} h(\sigma)\,d\sigma\ \phi_{ii}(v-\sigma) \tag{66}$$

If the system under consideration is a filter, and the input $f_i(t)$ a noise or unwanted interference, the above equation provides an index as to the success of the filter in smoothing the noise (Ref 6:195).

An expression for the output power-density spectrum can be found by taking the Fourier transform of equation (65).

$$\Phi_{oo}(\omega) = \int_{-\infty}^{\infty} \epsilon^{-j\omega\tau} d\tau \int_{-\infty}^{\infty} h(v)\,dv \int_{-\infty}^{\infty} h(\delta)\,d\delta \; \phi_{ii}(\tau + v - \delta) \qquad (67)$$

Through a change of variables, x = $\tau + v - \delta$, the above becomes

$$\Phi_{oo}(\omega) = \int_{-\infty}^{\infty} \epsilon^{-j\omega(x+\delta-v)} dx \int_{-\infty}^{\infty} h(v)\,dv \int_{-\infty}^{\infty} h(\delta)\,d\delta \; \phi_{ii}(x) \qquad (68)$$

Interchanging the order of integration and rearranging terms gives

$$\Phi_{oo}(\omega) = \int_{-\infty}^{\infty} h(v)\epsilon^{j\omega v} dv \int_{-\infty}^{\infty} h(\delta)\epsilon^{-j\omega\delta} d\delta \int_{-\infty}^{\infty} \phi_{ii}(x)\epsilon^{-j\omega x} dx \qquad (69)$$

The first factor in the above equation is the conjugate of the system function, $\overline{H(\omega)}$; the second term is the system function $H(\omega)$; and the third term is the defining equation for the input power-density spectrum. Equation (69) may therefore be written as

$$\Phi_{oo}(\omega) = H(\omega)\,\overline{H(\omega)}\,\Phi_{ii}(\omega) \qquad (70)$$

or

$$\Phi_{oo}(\omega) = |H(\omega)|^2\,\Phi_{ii}(\omega) \qquad (71)$$

This is the desired frequency domain relation expressing the output power-density spectrum in terms of the system function and the input power-density spectrum. The significance of this equation is as follows; the convolution integral is a valid time domain expression for periodic, aperiodic, and random functions. Likewise the transform of the convolution integral results in valid expressions for periodic and aperiodic functions in the frequency domain. However, when random functions are involved it is not possible to transform the convolution integral; but through the intermediate step of correlation a corresponding expression in the frequency domain is possible.

The Input-Output Cross-correlation Theorem

Another important relation in the analysis of linear system may be derived by considering the cross-correlation between the input and output. The cross-correlation function is

$$\phi_{io}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_i(t) f_o(t+\tau)\, dt \tag{72}$$

Expressing the displaced output $f_o(t+\tau)$ in terms of the convolution integral yields

$$\phi_{io}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_i(t)\, dt \int_{-\infty}^{\infty} h(v) f_i(t+\tau-v)\, dv \tag{73}$$

or

$$\phi_{io}(\tau) = \int_{-\infty}^{\infty} h(v)\,dv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_i(t) f_i(t+\tau-v)\,dt \tag{74}$$

In terms of the input autocorrelation function the above equation becomes

$$\phi_{io}(\tau) = \int_{-\infty}^{\infty} h(v)\,\phi_{ii}(\tau-v)\,dv \tag{75}$$

Examination of the above equation reveals that the input-output cross-correlation function is the convolution of the input autocorrelation function and the system impulse response. Thus the cross-correlation function may be interpreted as the response the linear system exhibits when it is excited by the input autocorrelation.

The frequency domain expression of the input-output cross-correlation may be found by taking the Fourier transform of equation (75). Thus

$$\Phi_{io}(\omega) = \int_{-\infty}^{\infty} \phi_{io}(\tau)\,\epsilon^{-j\omega\tau}\,d\tau \tag{76}$$

or

$$\Phi_{io}(\omega) = \int_{-\infty}^{\infty} \epsilon^{-j\omega\tau}\,d\tau \int_{-\infty}^{\infty} h(v)\,\phi_{ii}(\tau-v)\,dv \tag{77}$$

Through a change of variables $x = \tau - v$ the above equation becomes

$$\Phi_{io}(\omega) = \int_{-\infty}^{\infty} h(v)\epsilon^{-j\omega v}\,dv \int_{-\infty}^{\infty} \phi_{ii}(x)\epsilon^{-j\omega x}\,dx \tag{78}$$

or

$$\Phi_{io}(\omega) = H(\omega)\,\Phi_{ii}(\omega) \tag{79}$$

The above equation shows that the input-output cross-power-density spectrum is equal to the product of the input power-density spectrum and the system function. This is an important relation in the analysis of linear systems. It is frequently used to determine the system function from measurements made on the input and output signals. The system function is the ratio of the cross-power-density spectrum to the input power-density spectrum. (Ref 10:118)

An expression for the output-input cross-correlation function may be obtained by starting with the expression previously determined for the input-output function and using the following property of the cross-correlation function

$$\phi_{io}(\tau) = \phi_{oi}(-\tau) \tag{80}$$

From equation (75), repeated here for convenience

$$\phi_{io}(\tau) = \int_{-\infty}^{\infty} h(v)\,\phi_{ii}(\tau - v)\,dv \tag{81}$$

then

$$\phi_{oi}(\tau) = \int_{-\infty}^{\infty} h(v)\, \phi_{ii}(-\tau - v)\, dv \tag{82}$$

Since $\phi_{ii}(-\tau-v)$ is an even function

$$\phi_{oi}(\tau) = \int_{-\infty}^{\infty} h(v)\, \phi_{ii}(\tau + v)\, dv \tag{83}$$

Therefore the output-input cross-correlation isthe convolution of the input autocorrelation and system impulse response, where time is running backwards, from positive value on through zero and on to minus values. This is due to the plus sign proceeding the variable of integration in equation (83). The output-input cross-power-density spectrum, found by taking the Fourier transform of equation (83) is

$$\Phi_{oi}(\omega) = \overline{H(\omega)}\, \Phi_{ii}(\omega) \tag{84}$$

This relation may also be derived directly from equation (79).

$$\Phi_{oi}(\omega) = \overline{\Phi_{io}(\omega)} = \overline{H_1(\omega)\, \Phi_{ii}(\omega)} \tag{85}$$

or since $\Phi_{ii}(\omega)$ is a purely real function

$$\Phi_{oi}(\omega) = \overline{H(\omega)}\, \Phi_{ii}(\omega) \tag{86}$$

Summary of Important Linear System Relations

Equations (71), (79), and (84) are the three fundamental relations describing a linear system in terms of its system function and associated power-density spectrums. From these three relations, it is possible to derive the expression listed in Table I. Table I provides a complete cross reference between the output, input-output, and output-input relations for a linear system. From the relations listed in Table I it can also be shown that the ratio of the output power-density spectrum to the input-output cross-power-density spectrum is equal to the ratio of the input power-density spectrum to the output-input cross-power-density spectrum.

$$\frac{\Phi_{oo}(\omega)}{\Phi_{io}(\omega)} = \frac{\Phi_{ii}(\omega)}{\Phi_{oi}(\omega)} \tag{87}$$

or

$$\Phi_{oo}(\omega)\,\Phi_{oi}(\omega) = \Phi_{ii}(\omega)\,\Phi_{io}(\omega) \tag{88}$$

Table I

Summary of Linear System Relations

I. Output Relations:

(a) $\Phi_{oo}(\omega) = |H_1(\omega)|^2 \Phi_{ii}(\omega)$

(b) $\Phi_{oo}(\omega) = H_1(\omega)\, \Phi_{oi}(\omega)$

(c) $\Phi_{oo}(\omega) = \overline{H_1(\omega)}\, \Phi_{io}(\omega)$

II. Input-Output Relations:

(a) $\Phi_{io}(\omega) = H_1(\omega)\, \Phi_{ii}(\omega)$

(b) $\Phi_{io}(\omega) = \dfrac{H_1(\omega)}{\overline{H_1(\omega)}}\, \Phi_{oi}(\omega)$

(c) $\Phi_{io}(\omega) = \dfrac{1}{\overline{H_1(\omega)}}\, \Phi_{oo}(\omega)$

III. Output-Input Relations:

(a) $\Phi_{oi}(\omega) = \overline{H_1(\omega)}\, \Phi_{ii}(\omega)$

(b) $\Phi_{oi}(\omega) = \dfrac{\overline{H_1(\omega)}}{H_1(\omega)}\, \Phi_{io}(\omega)$

(c) $\Phi_{oi}(\omega) = \dfrac{1}{H_1(\omega)}\, \Phi_{oo}(\omega)$

IV. Problems and Examples

This section is concerned with the application of the information presented in sections II and III. Sample problems are solved that illustrate the application of correlation functions and power-density spectrum.

Illustrative Example No. 1

The problem of detection of a periodic signal in the presence of noise illustrates many of the basic properties of correlation techniques. A message received consists of an additive mixture of signal and noise. It is assumed that the signal and noise are "uncorrelated", that is, they are both derived from independent sources, and that the noise has no d.c. component.

The received message fm(t) is given by

$$f_m(t) = s(t) + n(t) \tag{89}$$

where s(t) is the periodic signal component and n(t) is the random noise component. The autocorrelation of the message is

$$\phi_{mm}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_m(t)\, f_m(t+\tau)\, dt \tag{90}$$

or

$$\phi_{mm}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left[s(t) + n(t)\right]\left[s(t+\tau) + n(t+\tau)\right] dt \tag{91}$$

Expanding the integrand of the above

$$\phi_{mm}(\tau)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Big[s(t)s(t+\tau)+s(t)n(t+\tau)+n(t)s(t+\tau)+n(t)n(t+\tau)\Big]dt \tag{92}$$

By separating the above equation, the autocorrelation of the message becomes

$$\phi_{mm}(\tau)=\phi_{ss}(\tau)+\phi_{sn}(\tau)+\phi_{ns}(\tau)+\phi_{nn}(\tau) \tag{93}$$

where

$\phi_{ss}(\tau)$ = the autocorrelation of the signal

$\phi_{sn}(\tau)$ = the signal-noise cross-correlation

$\phi_{ns}(\tau)$ = the noise-signal cross-correlation

$\phi_{nn}(\tau)$ = the autocorrelation of the noise

Since the noise and signal components are "uncorrelated", the cross-correlation functions are zero and

$$\phi_{mm}(\tau)=\phi_{ss}(\tau)+\phi_{nn}(\tau) \tag{94}$$

Therefore the autocorrelation of the message consists of the sum of the autocorrelation of the signal and noise. If the noise component contains no d.c. component

$$\phi_{nn}(\pm\infty)=0 \tag{95}$$

and the value of $\phi_{mm}(\tau)$ for large values of τ becomes

$$\phi_{mm}(\tau) = \phi_{ss}(\tau) \tag{96}$$

This is illustrated in figure 5.

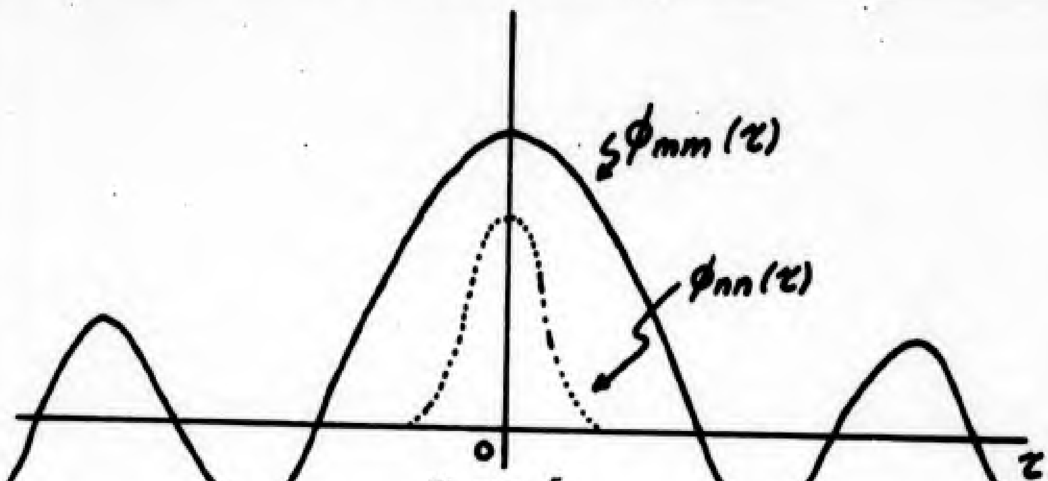

Figure 5

Autocorrelation Function of a
Periodic Signal and Noise

Therefore if the original signal was a sinusodial signal, the autocorrelation function of the message for large values of τ is a periodic function having the same fundamental frequency.

<u>Illustrative</u> <u>Example</u> <u>No.</u> <u>2</u>

For the system shown in figure 6 it is desired to find an expression relating the cross-power-density spectrum $\Phi_{jk}(\omega)$ in terms of the system functions and the input-output cross-power-density spectrum, $\Phi_{io}(\omega)$.

Figure 6

Cascaded Systems

Assuming linear systems the input-output cross-power-density spectrum is, from equation (79)

$$\Phi_{io}(\omega) = H_1(\omega)\, H_2(\omega)\, H_3(\omega)\, \Phi_{ii}(\omega) \tag{97}$$

This relates the input-output spectrum in terms of the input spectrum. The problem now is to find an expression relating $\Phi_{ii}(\omega)$ and $\Phi_{jk}(\omega)$. From the relations given in Table I

$$\Phi_{jj}(\omega) = H_1(\omega)\, \overline{H_1(\omega)}\, \Phi_{ii}(\omega) \tag{98}$$

and

$$\Phi_{jk}(\omega) = H_2(\omega)\, \Phi_{jj}(\omega) \tag{99}$$

Substituting (98) into (99) and solving for $\Phi_{ii}(\omega)$ yields

$$\Phi_{ii}(\omega) = \frac{1}{H_1(\omega)\,\overline{H_1(\omega)}\,H_2(\omega)}\,\Phi_{jk}(\omega) \tag{100}$$

Which is in turn substituted into (97) giving for $\Phi_{jk}(\omega)$.

$$\Phi_{jk}(\omega) = \frac{\overline{H_1(\omega)}}{H_3(\omega)}\,\Phi_{io}(\omega) \tag{101}$$

Examining the above, it is seen that the cross-power-density spectrum between the input and output of the second system in figure 6 is independent of $H_2(\omega)$. This could prove of tremendous importance in a system when the actual points for measuring $\Phi_{jk}(\omega)$ might be inaccesible.

<u>Illustrative Example No. 3</u>

For the feedback system shown in figure 7 it is desired to find an expression relating the output and input power-density spectrum in terms of the system functions.

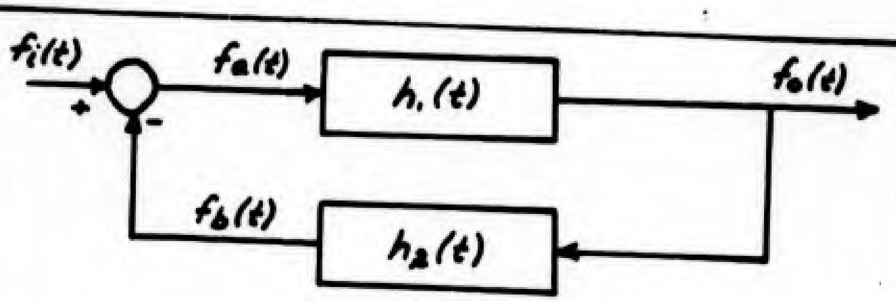

Figure 7

Feedback System

From figure 7

$$f_i(t) = f_a(t) + f_b(t) \tag{102}$$

The input autocorrelation function is

$$\phi_{ii}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_i(t) f_i(t+\tau)\, dt \tag{103}$$

Substituting the expression for fi(t) given in (102) into (103)

$$\phi_{ii}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left[f_a(t) + f_b(t)\right]\left[f_a(t+\tau) + f_b(t+\tau)\right] dt \tag{104}$$

Expanding the integrand of the above yields

$$\phi_{ii}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left[f_a(t) f_a(t+\tau) + f_a(t) f_b(t+\tau) + f_a(t+\tau) f_b(t) + f_b(t) f_b(t+\tau)\right] dt \tag{105}$$

or in terms of the autocorrelation functions

$$\phi_{ii}(\tau) = \phi_{aa}(\tau) + \phi_{ab}(\tau) + \phi_{ba}(\tau) + \phi_{bb}(\tau) \tag{106}$$

From the above equation, the frequency domain relation would be

$$\Phi_{ii}(\omega) = \Phi_{aa}(\omega) + \Phi_{ab}(\omega) + \Phi_{ba}(\omega) + \Phi_{bb}(\omega) \tag{107}$$

From figure 7

$$f_i(t) = f_a(t) + f_b(t) \tag{102}$$

The input autocorrelation function is

$$\phi_{ii}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_i(t) f_i(t+\tau)\, dt \tag{103}$$

Substituting the expression for fi(t) given in (102) into (103)

$$\phi_{ii}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left[f_a(t) + f_b(t)\right]\left[f_a(t+\tau) + f_b(t+\tau)\right] dt \tag{104}$$

Expanding the integrand of the above yields

$$\begin{aligned}\phi_{ii}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \Big[& f_a(t) f_a(t+\tau) + f_a(t) f_b(t+\tau) \\ &+ f_a(t+\tau) f_b(t) + f_b(t) f_b(t+\tau)\Big] dt\end{aligned} \tag{105}$$

or in terms of the autocorrelation functions

$$\phi_{ii}(\tau) = \phi_{aa}(\tau) + \phi_{ab}(\tau) + \phi_{ba}(\tau) + \phi_{bb}(\tau) \tag{106}$$

From the above equation, the frequency domain relation would be

$$\Phi_{ii}(\omega) = \Phi_{aa}(\omega) + \Phi_{ab}(\omega) + \Phi_{ba}(\omega) + \Phi_{bb}(\omega) \tag{107}$$

Referring to the relations given in Table I

$$\Phi_{aa}(\omega) = \frac{1}{H_1(\omega)\,\overline{H_1(\omega)}}\,\Phi_{oo}(\omega) \tag{108}$$

$$\Phi_{bb}(\omega) = H_2(\omega)\,\overline{H_2(\omega)}\,\Phi_{oo}(\omega) \tag{109}$$

$$\Phi_{ab}(\omega) = H_1(\omega)\,H_2(\omega)\,\Phi_{aa}(\omega) \tag{110}$$

and

$$\Phi_{ba}(\omega) = \overline{H_1(\omega)\,H_2(\omega)}\,\Phi_{aa}(\omega) \tag{111}$$

From equations (108) and (110) the expression for $\Phi_{ab}(\omega)$ in terms of the output power-density spectrum is

$$\Phi_{ab}(\omega) = \frac{H_2(\omega)}{\overline{H_1(\omega)}}\,\Phi_{oo}(\omega) \tag{112}$$

And from equations (108) and (111) the expression for $\Phi_{ba}(\omega)$ in terms of the output power-density spectrum is

$$\Phi_{ba}(\omega) = \frac{\overline{H_2(\omega)}}{H_1(\omega)}\,\Phi_{oo}(\omega) \tag{113}$$

Substituting equations (108), (109), (112), and (113) into (107) and simplifying yields

$$\Phi_{ii}(\omega) = \left[\frac{[1 + H_1(\omega)H_2(\omega)][1 + \overline{H_1(\omega)H_2(\omega)}]}{H_1(\omega)\overline{H_1(\omega)}}\right]\Phi_{oo}(\omega) \tag{114}$$

or

$$\frac{\Phi_{oo}(\omega)}{\Phi_{ii}(\omega)} = \left[\frac{H_1(\omega)}{1 + H_1(\omega)H_2(\omega)}\right]\left[\frac{\overline{H_1(\omega)}}{1 + \overline{H_1(\omega)H_2(\omega)}}\right] \tag{115}$$

which may be written as

$$\frac{\Phi_{oo}(\omega)}{\Phi_{ii}(\omega)} = \left|\frac{H_1(\omega)}{1 + H_1(\omega)H_2(\omega)}\right|^2 \tag{116}$$

This equation expresses the output power-density spectrum in terms of the input power-density spectrum and the system functions. This result can be compared qualitatively to the control ratio for a feedback system in servo theory. The control ratio is $\frac{C(s)}{R(s)}$ where C(s) is the Laplace transform of the input and R(s) is the Laplace transform of the output. This relation is given by the quotient of the transfer function G(s), to the quantity one plus the transfer function times the feedback transfer function H(s). However, when random signals are involved, the method of Laplace transforms is no longer valid; but through the use of correlation functions, an equally descriptive relation is available in terms of the power-density spectrum.

V. Mean-Square-Error Criterion

To formulate an expression for the mean-square-error consider the system shown in figure 8.

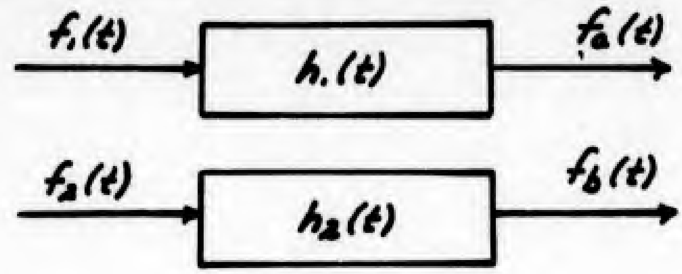

Figure 8

Signals in Two Systems

The cross-correlation between the output of the two systems is (Ref 7:349)

$$\phi_{ab}(\tau) = \int_{-\infty}^{\infty} h_1(v)\,dv \int_{-\infty}^{\infty} h_2(\sigma)\,d\sigma\ \phi_{12}(\tau+v-\sigma) \tag{117}$$

and the corresponding frequency domain relation is

$$\Phi_{ab}(\omega) = \overline{H_1(\omega)}\ H_2(\omega)\ \Phi_{12}(\omega) . \tag{118}$$

Define a new function $f_e(t)$ as the difference of the two outputs $f_a(t)$ and $f_b(t)$ as

$$f_e(t) = f_a(t) - f_b(t) \tag{119}$$

whose autocorrelation is by definition

$$\phi_{ee}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_e(t) f_e(t+\tau)\, dt \tag{120}$$

By substituting (119) into (120), expanding, and identifying correlation functions, the above equation becomes

$$\phi_{ee}(\tau) = \phi_{aa}(\tau) - \phi_{ab}(\tau) - \phi_{ba}(\tau) + \phi_{bb}(\tau) \tag{121}$$

Taking the Fourier transforms of (121) yields the power-density spectrum of the difference of the two signals as:

$$\Phi_{ee}(\omega) = \Phi_{aa}(\omega) - \Phi_{ab}(\omega) - \Phi_{ba}(\omega) + \Phi_{bb}(\omega) \tag{122}$$

Using the previously developed relations for the power-density spectrum, namely (see Table I)

$$\Phi_{aa}(\omega) = H_1(\omega)\, \overline{H_1(\omega)}\, \Phi_{11}(\omega) \tag{123}$$

$$\Phi_{bb}(\omega) = H_2(\omega)\, \overline{H_2(\omega)}\, \Phi_{22}(\omega) \tag{124}$$

and the results of equation (118), the expression for $\Phi_{ee}(\omega)$ becomes

$$\begin{aligned}\Phi_{ee}(\omega) = {} & H_1(\omega)\, \overline{H_1(\omega)}\, \Phi_{11}(\omega) - \overline{H_1(\omega)}\, H_2(\omega)\, \Phi_{12}(\omega) \\ & - H_1(\omega)\, \overline{H_2(\omega)}\, \Phi_{21}(\omega) + H_2(\omega)\, \overline{H_2(\omega)}\, \Phi_{22}(\omega)\end{aligned} \tag{125}$$

The importance and/or physical significance of this equation is as follows. If $f_a(t)$ represents the desired output of a physical system and $f_b(t)$ represents the actual output, then the difference between the two at any time t would be an instantaneous error e(t). Equation (125) relates the power-density spectrum of this instantaneous error to the system functions, input power-density spectra, and input cross-power-density spectra. The value of the autocorrelation function of e(t), equation (120), at $\tau = 0$ is the mean-square-value of e(t); called the mean-square-error ($\overline{e^2}$). Since $\phi_{ee}(\tau)$ is the Fourier transform of $\Phi_{ee}(\omega)$; or

$$\phi_{ee}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{ee}(\omega)\, \epsilon^{j\omega\tau}\, d\omega \tag{126}$$

an expression for the mean-square-error is

$$\phi_{ee}(0) = \overline{e^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{ee}(\omega)\, d\omega \tag{127}$$

Or substituting the value of $\Phi_{ee}(\omega)$ given by equation (125)

$$\overline{e^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[H_1(\omega)\overline{H_1(\omega)}\,\Phi_{11}(\omega) - \overline{H_1(\omega)}\,H_2(\omega)\,\Phi_{12}(\omega) - H_1(\omega)\overline{H_2(\omega)}\,\Phi_{21}(\omega) - H_2(\omega)\overline{H_2(\omega)}\,\Phi_{22}(\omega)\Big]d\omega \tag{128}$$

This expression relates the mean-square-error to the system function, input power-density spectra, and input cross-power-density spectra.

An example of the mean-square-error criterion, consider the system shown in figure 9. The input and output of system # 1 are the same. Therefore; its impulse response is an impulse or delta function applied at t = o.

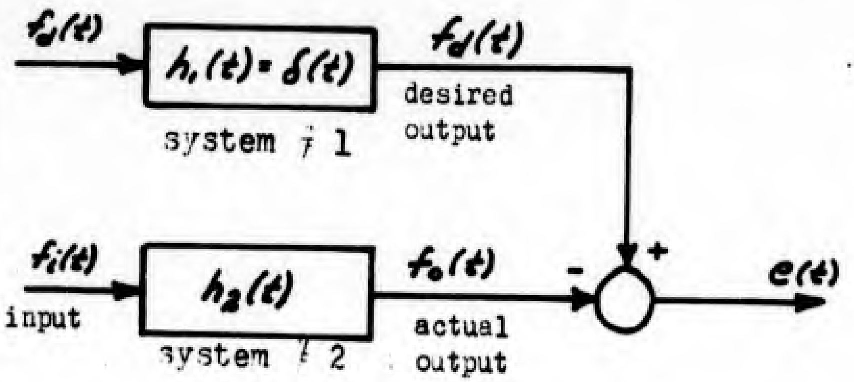

Figure 9

Error Between Output of Two Systems

(Ref. 10:121)

The instantaneous error is defined as the difference between the desired output and the actual output. Comparing figure 8 to figure 9, and using the expression for the mean-square-error given by equation (128), it can be seen that

$$\overline{e^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\Phi_{dd}(\omega) - H_2(\omega)\,\Phi_{di}(\omega) - \overline{H_2(\omega)}\,\Phi_{id}(\omega) + H_2(\omega)\,\overline{H_2(\omega)}\,\Phi_{ii}(\omega)\Big]d\omega \qquad (129)$$

Or since $\Phi_{id}(\omega)$ is the complex conjugate of $\Phi_{di}(\omega)$, the above equation may be expressed as

$$\overline{e^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\Phi_{dd}(\omega) - H_2(\omega)\,\overline{\Phi_{id}(\omega)} - \overline{H_2(\omega)}\,\Phi_{id}(\omega) + H_2(\omega)\,\overline{H_2(\omega)}\,\Phi_{ii}(\omega)\Big]d\omega \tag{130}$$

where

$\Phi_{ii}(\omega)$ = input power-density spectrum

$\Phi_{dd}(\omega)$ = desired output power-density spectrum

$\Phi_{id}(\omega)$ = input-desired output power-density spectrum.

Therefore, given the input, and specifying a desired output, the mean-square-error can be evaluated as a function of the system function $H_2(\omega)$.

As another example, consider the case where the input is an additive mixture of signal and noise

$$f_i(t) = s(t) + n(t) \tag{131}$$

The desired output in this case is exclusively related to the signal component of the input. The situation could be represented as shown in figure 10, where $h_d(t)$ is the impulse response of the desired system relating $s(t)$ and $f_d(t)$.

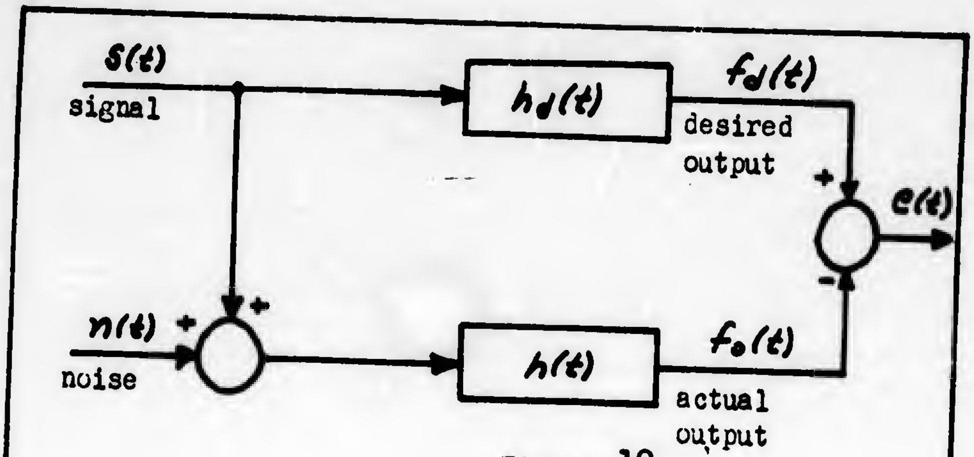

Figure 10

Signal And Noise Input

By comparing figure 10 to figure 6 the following equivalences can be established for the time signals

$$f_1(t) = S(t) \tag{132}$$

$$f_2(t) = S(t) + n(t) \tag{133}$$

$$f_a(t) = f_d(t) \tag{134}$$

$$f_b(t) = f_o(t) \tag{135}$$

An expression for the mean-square-error can be obtained from the following identies

$$\Phi_{11}(\omega) = \Phi_{ss}(\omega) \tag{136}$$

$$\Phi_{22}(\omega) = \Phi_{ss}(\omega) + \Phi_{nn}(\omega) \tag{137}$$

$$\Phi_{12}(\omega) = \Phi_{si}(\omega) = \Phi_{ss}(\omega) + \Phi_{sn}(\omega) \tag{138}$$

$$\Phi_{21}(\omega) = \Phi_{is}(\omega) = \Phi_{ss}(\omega) + \Phi_{ns}(\omega) \tag{139}$$

When these expressions are substituted into (128),

$$\overline{e^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[H_d(\omega)\overline{H_d(\omega)}\,\Phi_{ss}(\omega) - \overline{H_d(\omega)}\,H(\omega)\,\Phi_{si}(\omega)$$

$$- H_d(\omega)\,\overline{H(\omega)}\,\Phi_{is}(\omega) + H(\omega)\overline{H(\omega)}\big(\Phi_{ss}(\omega) + \Phi_{nn}(\omega)\big)\Big]d\omega \tag{140}$$

If the signal and noise are "uncorrelated" then the cross-correlation between the signal and noise would be zero, and the cross-power density spectra $\Phi_{is}(\omega)$ and $\Phi_{si}(\omega)$ would contain only a signal component, that is

$$\Phi_{si}(\omega) = \Phi_{is}(\omega) = \Phi_{ss}(\omega) \tag{141}$$

and the expression for the mean-square error would then be

$$\overline{e^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[|H(\omega)|^2\,\Phi_{nn}(\omega) + \Big\{|H(\omega)|^2 - \overline{H_d(\omega)}\,H(\omega) - H_d(\omega)\,\overline{H(\omega)} + |H_d(\omega)|^2\Big\}\,\Phi_{ss}(\omega)\Big]\,d\omega \tag{142}$$

or

$$\overline{e^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[|H(\omega)|^2\,\Phi_{nn}(\omega) + |H(\omega) - H_d(\omega)|^2\,\Phi_{ss}(\omega)\Big]\,d\omega \tag{143}$$

Under these conditions, the mean-square-error can be thought of as having two components. One corresponds to the input signal acting through the difference between the desired and actual system function, and the other component corresponds to the noise acting through the actual system function.

The above two examples have shown how the mean-square-error can be expressed as a function of the free parameters of the system. This is accomplished by first finding the power-density spect um of the instantaneous error, and relating this in turn to the mean-square-error through the use of equation (127).

A general procedure for the mean-square-error will now be developed considering the problems of filtering. In a filter the incoming signal is a message that has been corrupted by noise. It is desired to extract the message with a minimum of error in accordance with a chosen index. Therefore, either the system function $H(\omega)$ or the impulse response $h(t)$ of the filter must be determined that performs best. Once this is specified, the filter can then be constructed by well-known techniques.

In specifying the output, more of the filter is demanded than is usually physically realizable (Ref 7:358). There will be an error which will be the difference between the actual output and the specified or desired output

$$e(t) = f_d(t) - f_o(t) \tag{144}$$

where fd(t) is the desired output and fo(t) is the actual output. The mean-square-value of this instantaneous error is chosen as an index of success and is given by the expression (Ref 7:358)

$$\overline{e^2} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left[f_d(t) - f_o(t)\right]^2 dt \tag{145}$$

By relating the actual output to the input through the convolution integral it can be shown that

$$\overline{e^2} = \int_{-\infty}^{\infty} h(\tau)\,d\tau \int_{-\infty}^{\infty} h(\sigma)\,d\sigma\, \phi_{ii}(\tau-\sigma) - 2\int_{-\infty}^{\infty} h(\tau)\,d\tau\, \phi_{id}(\tau) + \phi_{dd}(0) \tag{146}$$

where

$h(t)$ = the impulse response of the filter

$\phi_{ii}(\tau)$ = the autocorrelation of the input

$\phi_{id}(\tau)$ = the input-desired output cross-correlation

$\phi_{dd}(0)$ = the mean-square value of the desired output

Therefore if the input of the filter is known, and the desired output is specified, the only unknown in the above equation is the impulse response of the filter. The problem is now one of selecting an impulse response that will minimize the mean-square-error. Through the use of calculus of variations it can be shown (Ref 7:364-7) that the condition for $\overline{e^2}$ to be a minimum is

$$\int_{-\infty}^{\infty} h(\sigma)\phi_{ii}(\tau-\sigma)\,d\sigma - \phi_{id}(\tau) = 0 \qquad \tau \geq 0 \tag{147}$$

This is known as the Weiner-Hopf equation. A physical interpretation is as follows. Rewriting the integral equation as

$$\phi_{id}(\tau) = \int_{-\infty}^{\infty} h(\sigma)\phi_{ii}(\tau-\sigma)\,d\sigma \qquad \tau \geq 0 \tag{148}$$

the right hand member is the convolution of the system impulse response and the input autocorrelation function. The solution of this equation involves finding an impulse response, which when convolved with the known input autocorrelation function will produce the specified input desired-output cross-correlation. The solution will have to be limited for $\tau \geq 0$. In other words, for the system shown in figure 11, it amounts to designing a system with an impulse response $h(\tau)$, such that when the system is excited by the input autocorrelation, an output $f(\tau)$ will be produced that is equal to the input-desired output cross-correlation.

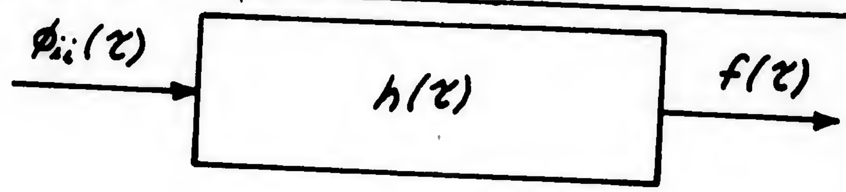

Figure 11
Interpretation of Integral Equation

It is interesting to note how the results of the general development agree with the results of the special problem considered in the beginning of the section. Referring to equation (146), and considering the case where the input is an additive mixture of "uncorrelated" signal and noise and the desired output is exclusively related to the signal component of the input alone, then

$$\phi_{ii}(\tau) = \phi_{ss}(\tau) + \phi_{nn}(\tau) \tag{149}$$

The corresponding expressions for $\phi_{id}(\tau)$ and $\phi_{dd}(0)$ would be (see Appendix A)

$$\phi_{id}(\tau) = \int_{-\infty}^{\infty} h_d(v)\,\phi_{ss}(\tau - v)\,dv \tag{150}$$

and

$$\phi_{dd}(0) = \int_{-\infty}^{\infty} h_d(v)\,dv \int_{-\infty}^{\infty} h(\delta)\,d\delta\,\phi_{ii}(v-\delta) \tag{151}$$

where $h_d(v)$ is the desired impulse response that will suppress the noise. Substituting equations (149), (150), and (151) into (146) yields

$$\begin{aligned} \overline{e^2} = & \int_{-\infty}^{\infty} h(\tau)\,d\tau \int_{-\infty}^{\infty} h(\delta)\,d\delta \left[\phi_{ss}(\tau-\delta) + \phi_{nn}(\tau-\delta)\right] \\ & -2\int_{-\infty}^{\infty} h(\tau)\,d\tau \int_{-\infty}^{\infty} h_d(v)\,dv\,\phi_{ss}(\tau - v) \\ & +\int_{-\infty}^{\infty} h_d(v)\,dv \int_{-\infty}^{\infty} h_d(\delta)\,d\delta\,\phi_{ss}(v-\delta) \end{aligned} \tag{152}$$

By rather simple but non-obvious manipulations (see Appendix B) the above equation becomes

$$\overline{e^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[H_d(\omega)\,\overline{H_d(\omega)}\,\Phi_{ss}(\omega) - \overline{H_d(\omega)}\,H(\omega)\,\Phi_{ss}(\omega) - H_d(\omega)\,\overline{H(\omega)}\,\Phi_{ss}(\omega) + H(\omega)\,\overline{H(\omega)}\big(\Phi_{ss}(\omega) + \Phi_{nn}(\omega)\big)\Big]d\omega \tag{153}$$

which is identical to equation (142). Therefore, it has been shown how the general development reduces to the specific problem examined earlier.

VI. Summary

Three main types of functions were discussed in this report; periodic, aperiodic, and random. The periodic function has a finite mean-square-value, the aperiodic has a finite integral-square-value, and the random function has a finite mean-square-value over the infinite time interval. The basic tool in the analysis of periodic functions was the Fourier series; and in the analysis of the aperiodic function it was the Fourier integral. However for random functions an intermediate step is necessary. This step is correlation, and is necessary because random functions do not possess Fourier transforms. Through the use of correlation, the power-density spectrum of a random function can be found. This spectrum is characteristic of all members of the ensemble, and is an important step in the analysis of random functions.

The response of linear systems disturbed by random inputs can be described by correlation functions. Corresponding relations in the frequency domain can be derived from these correlation functions. These frequency domain functions are called power-density spectra, and are related to the correlation functions by the Fourier integral. From the correlation functions and power-density spectrum functions, describing relations for a linear system can be developed. In the time domain these relations are between the impulse response and the correlation functions. In the frequency domain these relations are between the power-density spectrum and the system function. Specifically, these relations are as follows:

A. Time domain relations for a linear system disturbed by random inputs.

1. The output autocorrelation function can be expressed in terms of the input autocorrelation function and the impulse response of the system (Ref Equation 65).

2. The input-output cross-correlation function can be expressed as the convolution of the input autocorrelation function and the impulse response (Ref Equation 75).

3. The output-input cross-correlation function can be expressed as a special case of the convolution between the input autocorrelation function and the impulse response (Ref Equation 83).

B. Frequency domain relations for a linear system disturbed by random inputs.

Table I, page 39, gives a complete list of the frequency domain relations for a linear system. The input-output cross-power-density spectrum, input power-density spectrum, and the output-input cross-power-density spectrum are tablized each in terms of the other.

The detection of a periodic signal in the presence of noise is an important communication problem. Correlation theory and techniques have been of considerable importance in the solution of these problems. If it is assumed that the signal and noise are "uncorrelated", and that the noise has no d.c. component (these are not unrealistic assumptions), it was shown that the value of the autocorrelation function for large value of its argument is a periodic function of the same fundamental frequency as the signal.

If minimization of the mean-square-error is chosen as a design criterion for a communication system with random inputs, then this index is adequately described by correlation functions. For the problem of filtering, an impulse response function that optimizes the desired output can be expressed in terms of the input autocorrelation function and the input-desired output cross-correlation function. In the frequency domain the mean-square-error can be expressed as a function of the associated power-density spectra and the system function.

In conclusion, it can be stated that correlation theory provides a means whereby signals that are not periodic or aperiodic can be analyzed. It also provides an insight into the capabilities and limitations of linear systems in general.

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Appendix A

Mathematical Development of Equations (150) and (151)

From equation (145) the mean-square-error is

$$\overline{e^2} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left[f_d(t) - f_o(t)\right]^2 dt \tag{A-1}$$

where $f_o(t)$ is the actual output, and $f_d(t)$ is the desired output. For the problem under consideration the input is an additive mixture of signal and noise, or

$$f_i(t) = s(t) + n(t) \tag{A-2}$$

Relating the actual output to the input through the convolution integral gives

$$f_o(t) = \int_{-\infty}^{\infty} h(v)\left[s(t-v) + n(t-v)\right] dv \tag{A-3}$$

Also the relation between the desired output and the input would be

$$f_d(t) = \int_{-\infty}^{\infty} h_d(v)\left[s(t-v) + n(t-v)\right] dv \tag{A-4}$$

where $h_d(t)$ is the desired impulse response that will suppress the noise. Under these conditions, the convolution between $h_d(t)$ and $n(t)$ would be zero, and equation (A-4) may be written as

$$f_d(t) = \int_{-\infty}^{\infty} h_d(v)\, s(t-v)\, dv \tag{A-5}$$

Substituting equations (A-3) and (A-5) into (A-1) and squaring yields

$$\overline{e^2} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left[A^2 + B^2 + C^2 - 2AB - 2AC - 2BC\right] dt \qquad \text{(A-6)}$$

where

$$A = \int_{-\infty}^{\infty} h(v)\, s(t-v)\, dv \qquad \text{(A-7)}$$

$$B = \int_{-\infty}^{\infty} h(v)\, n(t-v)\, dv \qquad \text{(A-8)}$$

$$C = \int_{-\infty}^{\infty} h_d(v)\, s(t-v)\, dv \qquad \text{(A-9)}$$

Equation (A-6) for convenience may be written as

$$\overline{e^2} = I_1 + I_2 + I_3 - 2I_4 - 2I_5 - 2I_6 \qquad \text{(A-10)}$$

where

$$I_1 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A^2\, dt \qquad \text{(A-11)}$$

$$I_2 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} B^2\, dt \qquad \text{(A-12)}$$

$$I_3 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} C^2\, dt \qquad \text{(A-13)}$$

$$I_4 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} AB\, dt \qquad \text{(A-14)}$$

$$I_5 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} AC\,dt \tag{A-15}$$

and

$$I_6 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} BC\,dt \tag{A-16}$$

Consider now each of the terms of equation (A-10) separately.

$$I_1 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt \int_{-\infty}^{\infty} h(v)\,s(t-v)\,dv \int_{-\infty}^{\infty} h(\sigma)\,s(t-\sigma)\,d\sigma \tag{A-17}$$

where the variable σ is used in the second integral to avoid confusion with the first variable of integration. Interchanging the order of integration in I_1 gives

$$I_1 = \int_{-\infty}^{\infty} h(v)\,dv \int_{-\infty}^{\infty} h(\sigma)\,d\sigma \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} s(t-v)\,s(t-\sigma)\,dt \tag{A-18}$$

or since

$$\phi_{ss}(v-\sigma) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} s(t-v)\,s(t-\sigma)\,dt \quad \text{(A-19}$$

the expression for I_1 becomes

$$I_1 = \int_{-\infty}^{\infty} h(v)\,dv \int_{-\infty}^{\infty} h(\sigma)\,d\sigma\; \phi_{ss}(v-\sigma) \tag{A-20}$$

By identical means the relations for I_2, I_3, I_4, I_5, and I_6 would be

$$I_2 = \int_{-\infty}^{\infty} h(v)\,dv \int_{-\infty}^{\infty} h(\sigma)\,d\sigma\;\phi_{nn}(v-\sigma) \tag{A-21}$$

$$I_3 = \int_{-\infty}^{\infty} h_d(v)\,dv \int_{-\infty}^{\infty} h_d(\sigma)\,d\sigma\;\phi_{ss}(v-\sigma) \tag{A-22}$$

$$I_4 = \int_{-\infty}^{\infty} h(v)\,dv \int_{-\infty}^{\infty} h(\sigma)\,d\sigma\;\phi_{sn}(v-\sigma) \tag{A-23}$$

$$I_5 = \int_{-\infty}^{\infty} h_d(v)\,dv \int_{-\infty}^{\infty} h(\sigma)\,d\sigma\;\phi_{ss}(v-\sigma) \tag{A-24}$$

and

$$I_6 = \int_{-\infty}^{\infty} h(v)\,dv \int_{-\infty}^{\infty} h_d(\sigma)\,d\sigma\;\phi_{sn}(v-\sigma) \tag{A-25}$$

Since the signal and noise are "uncorrelated", their cross-correlation is zero, and the expression for I_4 and I_6 would be zero. When the expressions for I_1, I_2, I_3, and I_5 are substituted into equation (A-10) the mean-square-error becomes

$$\begin{aligned}\overline{e^2} = &\int_{-\infty}^{\infty} h(v)dv \int_{-\infty}^{\infty} h(\sigma)\,d\sigma\left[\phi_{ss}(v-\sigma) + \phi_{nn}(v-\sigma)\right] \\ &+ \int_{-\infty}^{\infty} h_d(v)\,dv \int_{-\infty}^{\infty} h_d(\sigma)d\sigma\;\phi_{ss}(v-\sigma) \\ &- 2\int_{-\infty}^{\infty} h_d(v)\,dv \int_{-\infty}^{\infty} h(\sigma)d\sigma\;\phi_{ss}(v-\sigma)\end{aligned} \tag{A-26}$$

Comparing equation (A-26) with the general equation for the mean-square-error, equation (146), reveals the following identies

$$\phi_{dd}(o) = \int_{-\infty}^{\infty} h_d(v)\,dv \int_{-\infty}^{\infty} h_d(\delta)\,\phi_{ss}(v-\delta) \tag{A-27}$$

and

$$\phi_{id}(\delta) = \int_{-\infty}^{\infty} h_d(v)\,\phi_{ss}(v-\delta)\,dv \tag{A-28}$$

Since $\phi_{ss}(v-\delta)$ is an even function, equation (A-28) may be written as

$$\phi_{id}(\delta) = \int_{-\infty}^{\infty} h_d(v)\,\phi_{ss}(\delta-v)\,dv \tag{A-29}$$

By a change of variables $\tau = \delta$, the above becomes

$$\phi_{id}(\tau) = \int_{-\infty}^{\infty} h_d(v)\,\phi_{ss}(\tau-v)\,dv \tag{A-30}$$

Equation (A-27) and (A-30) above are stated as equation (151) and (150) respectively in the body of the report.

Appendix B

Mathematical Development of Equation (153).

Starting with equation (152)

$$\overline{e^2} = \int_{-\infty}^{\infty} h(\tau)d\tau \int_{-\infty}^{\infty} h(\sigma)d\sigma \left[\phi_{ss}(\tau-\sigma) + \phi_{nn}(\tau-\sigma)\right] \tag{B-1}$$

$$-2\int_{-\infty}^{\infty} h(\tau)d\tau \int_{-\infty}^{\infty} h_d(v)dv \; \phi_{ss}(\tau-v)$$

$$+\int_{-\infty}^{\infty} h_d(v)dv \int_{-\infty}^{\infty} h_d(\sigma)d\sigma \; \phi_{ss}(v-\sigma)$$

or

$$\overline{e^2} = I_1 - 2I_2 + I_3 \tag{B-2}$$

where

$$I_1 = \int_{-\infty}^{\infty} h(\tau)d\tau \int_{-\infty}^{\infty} h(\sigma)d\sigma \left[\phi_{ss}(\tau-\sigma) + \phi_{nn}(\tau-\sigma)\right] \tag{B-3}$$

and

$$I_2 = \int_{-\infty}^{\infty} h(\tau)d\tau \int_{-\infty}^{\infty} h_d(v)dv \; \phi_{ss}(\tau-v) \tag{B-5}$$

and

$$I_3 = \int_{-\infty}^{\infty} h_d(\nu)\,d\nu \int_{-\infty}^{\infty} h_d(\delta)\,d\delta\, \phi_{ss}(\tau-\delta) \tag{B-5}$$

Examining each of the above integrals separately;

$$I_1 = \int_{-\infty}^{\infty} h(\tau)\,d\tau \int_{-\infty}^{\infty} h(\delta)\,d\delta\, \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\Phi_{ss}(\omega) + \Phi_{nn}(\omega)\Big]\epsilon^{j\omega(\tau-\delta)}\,d\omega \tag{B-6}$$

where the following substitution has been made

$$\phi_{ss}(\tau-\delta) + \phi_{nn}(\tau-\delta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\Phi_{ss}(\omega) + \Phi_{nn}(\omega)\Big]\epsilon^{j\omega(\tau-\delta)}\,d\omega \tag{B-7}$$

Equation (B-6) may be rearranged as follows

$$I_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\Phi_{ss}(\omega) + \Phi_{nn}(\omega)\Big]d\omega \int_{-\infty}^{\infty} h(\tau)\epsilon^{j\omega\tau}\,d\tau \int_{-\infty}^{\infty} h(\delta)\epsilon^{-j\omega\delta}\,d\delta \tag{B-8}$$

or since

$$\overline{H(\omega)} = \int_{-\infty}^{\infty} h(\tau)\epsilon^{j\omega\tau}\,d\omega \tag{B-9}$$

and

$$H(\omega) = \int_{-\infty}^{\infty} h(\delta)\, \epsilon^{-j\omega\delta}\, d\delta \tag{B-10}$$

the expression for I_1 becomes

$$I_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\Phi_{ss}(\omega) + \Phi_{nn}(\omega) \right] \overline{H(\omega)}\, H(\omega)\, d\omega \tag{B-11}$$

Now consider the expression for I_2 which may be written as

$$2I_2 = \int_{-\infty}^{\infty} h(\tau)\, d\tau \int_{-\infty}^{\infty} h_d(v)\, dv\, \phi_{ss}(\tau - v) + \int_{-\infty}^{\infty} h(\tau)\, d\tau \int_{-\infty}^{\infty} h_d(v)\, dv\, \phi_{ss}(\tau - v) \tag{B-12}$$

since $\phi_{ss}(\tau - v)$ is an even function

$$2I_2 = \int_{-\infty}^{\infty} h(\tau)\, d\tau \int_{-\infty}^{\infty} h_d(v)\, dv\, \phi_{ss}(\tau - v) + \int_{-\infty}^{\infty} h(\tau)\, d\tau \int_{-\infty}^{\infty} h_d(v)\, dv\, \phi_{ss}(-\tau + v) \tag{B-13}$$

By substituting the following expressions

$$\phi_{ss}(\tau - v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{ss}(\omega)\, \epsilon^{j\omega(\tau - v)}\, d\omega \tag{B-14}$$

and

$$\phi_{ss}(-\tau+v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{ss}(\omega)\,\epsilon^{j\omega(-\tau+v)}\,d\omega \tag{B-15}$$

into equation (B-13), and rearranging terms

$$2I_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{ss}(\omega)\,d\omega \int_{-\infty}^{\infty} h(\tau)\,\epsilon^{j\omega\tau}\,d\tau \int_{-\infty}^{\infty} h_d(v)\,\epsilon^{-j\omega v}\,dv$$
$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{ss}(\omega)\,d\omega \int_{-\infty}^{\infty} h(\tau)\,\epsilon^{-j\omega\tau}\,d\tau \int_{-\infty}^{\infty} h_d(v)\,\epsilon^{j\omega v}\,dv \tag{B-16}$$

or

$$2I_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{H(\omega)}\,H_d(\omega)\,\Phi_{ss}(\omega)\,d\omega$$
$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\,\overline{H_d(\omega)}\,\Phi_{ss}(\omega)\,d\omega \tag{B-17}$$

In a similar manner the expression for I_3 would be

$$I_3 = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_d(\omega)\,\overline{H_d(\omega)}\,\Phi_{ss}(\omega)\,d\omega \tag{B-18}$$

Substituting the expressions for I_1, I_2, and I_3 into equation (B-2) yields

$$\overline{e^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[H_d(\omega)\overline{H_d(\omega)}\,\Phi_{ss}(\omega) - \overline{H_d(\omega)}\,H(\omega)\,\Phi_{ss}(\omega) - H_d(\omega)\overline{H(\omega)}\,\Phi_{ss}(\omega) + H(\omega)\overline{H(\omega)}\big(\Phi_{ss}(\omega) + \Phi_{nn}(\omega)\big)\Big]d\omega \tag{B-18}$$

which is as stated in equation (153) of the report.

Vita

Paul H[redacted] Hass was born on [redacted], the son of Henry C[redacted] Hass, Jr. and Hazel [redacted] Hass. After completing his work in 1952 at the High School of [redacted], he enrolled at The Citadel, Charleston. In June of 1956 he was graduated with the Bachelor of Science Degree in Electrical Engineering. After receiving his comission as a Lieutenant in the United States Air Force, he entered active duty in September 1956. His military assignment prior to his coming to the Institute of Technology was as a Communication Officer at the Central GEEIA Region, Tinker AFB, Oklahoma.

Permanent address: [redacted]

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