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Effect of the Initial Acceleration
Upon the Wave Resistance of Ship Models

by

John V. Wehausen

Under Contract Number N-onr-222(30)



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Berkeley, California
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Hamburg**

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Introduction*

The wave resistance of a ship which is accelerating is primarily of interest in connection with towing-tank experiments. No matter how quickly the final desired speed is attained by the model, there remains always some question as to how long the influence of the initial acceleration persists and what form this takes. This question has been discussed by Havelock [1949 a,b] in two papers dealing with the wave resistance of circular cylinders started impulsively from rest. Further discussion of this case together with some general theoretical derivations may be found in an expository paper by Maruo [1957].

The purpose of the present report is to carry further this initial work of Havelock and Maruo in the way described below. An expression for the wave resistance of a thin ship in non-uniform motion was derived some years ago by Sretenskii [1939]. One may find the result together with derivations in Lunde [1951]. Havelock's work will be extended here in two directions: instead of a submerged circular cylinder we shall treat a ship form, albeit thin; instead of an impulsive start we shall assume a continuous rise from an initial velocity 0 at $t=0$ to a final velocity c_0 at $t=t_0$. As in Havelock's treatment, we obtain an asymptotic expression for large t . Such results are, however, of practical interest, for it is precisely the behavior for larger values of t which is of

*This work was carried out partly at the University of California with support from the Office of Naval Research and partly at the Institut für Schiffbau of the University of Hamburg during tenure of a Fulbright lectureship.

interest in tank experiments. However, in order to complete the asymptotic results somewhat, the series expansion for small t is also included. The region, of indeterminate extent, lying between the regions where these two series expansions are sufficiently accurate must be investigated numerically. The present investigation has an advantage over a purely numerical one in that much general information can be found without a special choice of ship form. Furthermore, it provides valuable qualitative information which would facilitate more extensive numerical calculations.

In obtaining his asymptotic expansions, Havelock made use of the method of stationary phase. In this report a different method, based upon the theory of Fourier integrals, is used. Although presumably all results can be obtained equally well by the method of stationary phase, the method used here seems to the author to be more straightforward in conception.

Since this report consists essentially of a working out of this asymptotic expansion, it is not possible to relegate mathematical details to an appendix. However, in the last section the various results are gathered together and some graphs are included which display the results of a numerical calculation for a ship of simple form.

Mathematical preliminaries

In this section we shall gather together various mathematical theorems which will be useful in the later developments. They are stated mostly without proof.

Theorem 1.* If $q(x)$ is integrable and of bounded variation in an interval containing x_0 , then

$$(1) \quad \lim_{t \rightarrow \infty} \int_a^b q(\xi) \frac{\sin t(\xi - x_0)}{\xi - x_0} d\xi = \begin{cases} \frac{\pi}{2} [q(x_0-0) + q(x_0+0)], & a < x_0 < b \\ \frac{\pi}{2} q(a+0), & x_0 = a \\ \frac{\pi}{2} q(b-0), & x_0 = b \\ 0, & x_0 < a, x_0 > b \end{cases}$$

Either a and/or b may be $-\infty$ or $+\infty$, respectively, if $q(x)/x$ has one of the following properties:

- a) $q(x)/x$ is absolutely integrable;
- b) $q(x)/x$ converges eventually monotonically to zero in either direction;
- c) $q(x)/x$ is of the form $h(x) \sin(px+q)$ where $h(x)$ eventually converges monotonically to zero as $x \rightarrow \pm \infty$

If $\sin t(\xi - x_0)$ is replaced by $\cos t(\xi - x_0)$ in (1), the integral may be interpreted as a Cauchy principal value. One may then prove that

$$(1a) \quad \lim_{t \rightarrow \infty} \int_a^b q(\xi) \frac{\cos t(\xi - x_0)}{\xi - x_0} d\xi = 0$$

Theorem 2.** Let $q(x)$ be N times continuously differentiable in the interval $a \leq x \leq b$. Then

*See Bochner [1932, Chap. I]

**See Erdélyi [1956, p. 47]

$$\int_a^b e^{it\xi} q(\xi) d\xi = B_N(t) - A_N(t) + o(t^{-N}),$$

$$(2) \quad A_N(t) = \sum_{n=0}^{N-1} i^{n-1} q^{(n)}(a) t^{-n-1} e^{ita},$$

$$B_N(t) = \sum_{n=0}^{N-1} i^{n-1} q^{(n)}(b) t^{-n-1} e^{itb}.$$

Either a or b may be $-\infty$ or $+\infty$, respectively, if $q^{(n)}(x) \rightarrow 0$ as $t \rightarrow -\infty$ or $+\infty$, respectively, for $0 \leq n \leq N-1$, and if $q^{(n)}(x)$ is integrable over the interval (a, b) .

Theorem 3.* Let $q(x)$ be N times continuously differentiable in the interval $a \leq x \leq t$ and let $0 < \lambda \leq 1$, $0 < \mu \leq 1$. Then

$$\int_a^b e^{it\xi} (\xi-a)^{\lambda-1} (b-\xi)^{\mu-1} q(\xi) d\xi = B_N(t) - A_N(t) + o(t^{-N}),$$

$$(3) \quad A_N(t) = \sum_{n=0}^{N-1} \frac{\Gamma(n+\lambda)}{n!} e^{\pi i(n+\lambda-2)/2} t^{-n-\lambda} e^{ita} \frac{d^n}{da^n} [(b-a)^{\mu-1} q(a)],$$

$$B_N(t) = \sum_{n=0}^{N-1} \frac{\Gamma(n+\mu)}{n!} e^{\pi i(n-\mu)/2} t^{-n-\mu} e^{itb} \frac{d^n}{db^n} [(b-a)^{\lambda-1} q(b)].$$

A special case of Theorem 3 is the following.

Theorem 4. Let $q(x)$ be N times continuously differentiable in $a \leq x \leq b$, let $q^{(n)}(b) = 0$, $0 \leq n \leq N-1$, and let $0 < \lambda < 1$.

* See Erdélyi [1956, p. 49]

Then

$$\int_a^b e^{it\xi} (\xi-a)^{\lambda-1} q(\xi) d\xi = -A_N(t) + O(t^{-N}),$$

(4)

$$A_N(t) = \sum_{n=0}^{N-1} \frac{\Gamma(n+\lambda)}{n!} e^{\pi i(n+\lambda-2)/2} q^{(n)}(a) t^{-n-\lambda} e^{iat}.$$

One may let $b = \infty$ provided that $q^{(n)}(x) \rightarrow 0$ as $x \rightarrow \infty$ for $0 \leq n \leq N-1$ and $(x-a)^{\lambda-1} q^{(N)}(x)$ is integrable over (a, ∞) .

We shall have occasion later on to treat integrals in which singularities of the form occurring in Theorem 1 as well as that in Theorem 3 appear. Consequently, let us consider the following integral:

$$(5) \quad G(t) = \int_a^b \frac{\sin t(\xi-x_0)}{\xi-x_0} (b-\xi)^{\mu-1} (\xi-a)^{\lambda-1} q(\xi) d\xi, \quad a < x_0 < b.$$

Theorem 1 can be applied directly to give

$$(6) \quad \lim_{t \rightarrow \infty} G(t) = \pi (b-x_0)^{\mu-1} (x_0-a)^{\lambda-1} q(x_0).$$

However, we shall proceed somewhat differently. Let $\epsilon > 0$ be such that $a < x_0 - \epsilon < x_0 < x_0 + \epsilon < b$, and divide $G(t)$ into separate integrals as follows:

$$\begin{aligned}
 G(t) &= \int_a^{x_0-\epsilon} \sin t (\xi - x_0) (\xi - a)^{\lambda-1} \frac{(b-\xi)^{\mu-1} q(\xi)}{\xi - x_0} d\xi \\
 &+ \int_{x_0-\epsilon}^{x_0+\epsilon} \sin t (\xi - x_0) \frac{(b-\xi)^{\mu-1} (\xi - a)^{\lambda-1} q(\xi) - (b-x_0)^{\mu-1} (x_0 - a)^{\lambda-1} q(x_0)}{\xi - x_0} d\xi \\
 &+ (b-x_0)^{\mu-1} (x_0 - a)^{\lambda-1} q(x_0) \int_{x_0-\epsilon}^{x_0+\epsilon} \frac{\sin t (\xi - x_0)}{\xi - x_0} d\xi \\
 (7) \quad &+ \int_{x_0+\epsilon}^b \sin t (\xi - x_0) (b-\xi)^{\mu-1} \frac{(\xi - a)^{\lambda-1} q(\xi)}{\xi - x_0} d\xi.
 \end{aligned}$$

To the first and last integrals we can immediately apply Theorem 3. The third integral can be written

$$\int_{-\epsilon t}^{\epsilon t} \frac{\sin \tau}{\tau} d\tau = 2 \int_0^{\epsilon t} \frac{\sin \tau}{\tau} d\tau = 2 \text{Si}(\epsilon t).$$

The following asymptotic development is well known:

$$(8) \quad \text{Si} t = \frac{\pi}{2} - \cos t \left(\frac{1}{t} - \frac{2!}{t^3} + \frac{4!}{t^5} - \dots \right) - \sin t \left(\frac{1}{t^2} - \frac{3!}{t^4} + \frac{5!}{t^6} - \dots \right)$$

and can be used here. This leaves only the second integral to discuss. For the moment let us set

$$f(\xi) = (b-\xi)^{\mu-1} (\xi - a)^{\lambda-1} q(\xi).$$

Consider the function

$$(9) \quad h(\xi) = \frac{f(\xi) - f(x_0)}{\xi - x_0}.$$

If f is continuous, h is continuous with the possible exception $\xi = x_0$, where it is not defined. However, if $f'(x_0)$ exists and we define

$$(10) \quad h(x_0) = f'(x_0)$$

then h is continuous there also. As a matter of fact, if f is $N+1$ times continuously differentiable at x_0 , then h is N times continuously differentiable if one defines

$$(11) \quad h^{(N)}(x_0) = \frac{1}{N+1} f^{(N+1)}(x_0)$$

This follows easily from the extended mean-value theorem:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \dots + \frac{1}{N!} f^{(N)}(x_0)(x-x_0)^N + \frac{1}{(N+1)!} f^{(N+1)}(\bar{x})(x-x_0)^{N+1}$$

or, after forming $h(x)$,

$$(12) \quad h(x) = f'(x_0) + \frac{1}{2} f''(x_0)(x-x_0) + \dots + \frac{1}{N!} f^{(N)}(x_0)(x-x_0)^{N-1} + \frac{1}{(N+1)!} f^{(N+1)}(\bar{x})(x-x_0)^N,$$

where $x_0 < \bar{x} < x$ or $x < \bar{x} < x_0$ and $\bar{x} = \bar{x}(x, x_0)$. The result follows immediately. It is evident that if $f(\xi)$ has $N+1$ derivatives in $(x_0 - \varepsilon, x_0 + \varepsilon)$, then also $f'(\xi)$ does, and hence that $h(\xi)$ has N derivatives. Consequently, we may apply Theorem 2 (or Theorem 3 which includes it) to the second integral above.

The function $G(t)$ then has the following asymptotic expansion:

$$\begin{aligned}
G(t) = & - \sum_{n=0}^{N-1} \frac{\Gamma(n+\lambda)}{n!} t^{-n-\lambda} \sin[(a-x_0)t + \frac{1}{2}(n+\lambda-2)\pi] \frac{d^n}{d\xi^n} \left[\frac{(b-\xi)^{\mu-1} q(\xi)}{\xi-x_0} \right]_{\xi=a} \\
& + \sum_{n=0}^{N-1} t^{-n-1} \sin(-\varepsilon t + \frac{n-1}{2}\pi) \frac{d^n}{d\xi^n} \left[\frac{(b-\xi)^{\mu-1} (\xi-a)^{\lambda-1} q(\xi)}{\xi-x_0} \right]_{\xi=x_0-\varepsilon} \\
& - \sum_{n=0}^{N-1} t^{-n-1} \sin(-\varepsilon t + \frac{n-1}{2}\pi) \frac{d^n}{d\xi^n} \left[\frac{(b-\xi)^{\mu-1} (\xi-a)^{\lambda-1} q(\xi) - (b-x_0)^{\mu-1} (x_0-a)^{\lambda-1} q(x_0)}{\xi-x_0} \right]_{\xi=x_0-\varepsilon} \\
& + \sum_{n=0}^{N-1} t^{-n-1} \sin(\varepsilon t + \frac{n-1}{2}\pi) \frac{d^n}{d\xi^n} \left[\frac{(b-\xi)^{\mu-1} (\xi-a)^{\lambda-1} q(\xi) - (b-x_0)^{\mu-1} (x_0-a)^{\lambda-1} q(x_0)}{\xi-x_0} \right]_{\xi=x_0+\varepsilon} \\
(13) \quad & + (b-x_0)^{\mu-1} (x_0-a)^{\lambda-1} q(x_0) \left\{ \pi - 2 \cos \varepsilon t \left[\frac{1}{\varepsilon t} - \frac{2!}{(\varepsilon t)^3} + \dots \right] - 2 \sin \varepsilon t \left[\frac{1}{(\varepsilon t)^2} - \frac{3!}{(\varepsilon t)^4} + \dots \right] \right\} \\
& - \sum_{n=0}^{N-1} t^{-n-1} \sin(\varepsilon t + \frac{n-1}{2}\pi) \frac{d^n}{d\xi^n} \left[\frac{(b-\xi)^{\mu-1} (\xi-a)^{\lambda-1} q(\xi)}{\xi-x_0} \right]_{\xi=x_0+\varepsilon} \\
& + \sum_{n=0}^{N-1} \frac{\Gamma(n+\mu)}{n!} t^{-n-\mu} \sin[(b-x_0)t + \frac{1}{2}(n+\mu)\pi] \frac{d^n}{d\xi^n} \left[\frac{(\xi-a)^{\lambda-1} q(\xi)}{\xi-x_0} \right]_{\xi=b} + O\left(\frac{1}{t^N}\right),
\end{aligned}$$

where the fifth sum is to be terminated with the term $(\varepsilon t)^N$. Upon comparison of the second and third and the fourth and sixth sums it is evident that they cancel except for the following part:

$$\begin{aligned}
& \sum_{n=0}^{N-1} t^{-n-1} (b-x_0)^{\mu-1} (x_0-a)^{\lambda-1} q(x_0) \left\{ \sin(-\varepsilon t + \frac{n-1}{2}\pi) \frac{d^n}{d\xi^n} \left[\frac{(b-\xi)^{\mu-1} (\xi-x_0)^{\lambda-1}}{\xi-x_0} \right]_{\xi=x_0-\varepsilon} - \sin(\varepsilon t + \frac{n-1}{2}\pi) \frac{d^n}{d\xi^n} \left[\frac{(b-\xi)^{\mu-1} (\xi-x_0)^{\lambda-1}}{\xi-x_0} \right]_{\xi=x_0+\varepsilon} \right\} \\
& = 2(b-x_0)^{\mu-1} (x_0-a)^{\lambda-1} q(x_0) \sum_{n=0}^{N-1} (-1)^n (\varepsilon t)^{-n-1} n! \begin{cases} (-1)^{\frac{n}{2}} \cos \varepsilon t, & n \text{ even} \\ (-1)^{\frac{n+1}{2}} \sin \varepsilon t, & n \text{ odd} \end{cases} \\
& = 2(b-x_0)^{\mu-1} (x_0-a)^{\lambda-1} q(x_0) \left\{ \cos \varepsilon t \left[\frac{1}{\varepsilon t} - \frac{2!}{(\varepsilon t)^3} + \dots \right] + \sin \varepsilon t \left[\frac{1}{(\varepsilon t)^2} - \frac{3!}{(\varepsilon t)^4} + \dots \right] \right\}.
\end{aligned}$$

Comparison of this with the fifth sum above shows that these just cancel except for the part

$$\pi (b-x_0)^{\mu-1} (x_0-a)^{\lambda-1} q(x_0)$$

It is evident then that the asymptotic expansion for $G(t)$ simplifies to the following:

$$G(t) = \pi (b-x_0)^{\mu-1} (x_0-a)^{\lambda-1} q(x_0)$$

$$(14) \quad - \sum_{n=0}^{N-1} \frac{\Gamma(n+\lambda)}{n!} t^{-n-\lambda} \sin[(a-x_0)t + \frac{1}{2}(n+\lambda-2)\pi] \frac{d^n}{d\xi^n} \left[\frac{(b-\xi)^{\mu-1} q(\xi)}{\xi-x_0} \right]_{\xi=a}$$

$$+ \sum_{n=0}^{N-1} \frac{\Gamma(n+\mu)}{n!} t^{-n-\mu} \sin[(b-x_0)t + \frac{1}{2}(n+\mu)\pi] \frac{d^n}{d\xi^n} \left[\frac{(\xi-a)^{\lambda-1} q(\xi)}{\xi-x_0} \right]_{\xi=b} + O(t^{-N}).$$

Here $q(x)$ must be N times continuously differentiable in $a \leq x \leq b$ and $N+1$ times differentiable in x_0 .

In case the singularity x_0 is at one of the limits of integration, the asymptotic development must be altered somewhat. The details of the derivation will not be repeated, but only the final result given for the case that x_0 is the upper limit.

The result follows:

$$\begin{aligned}
 G(t) &\equiv \int_a^{x_0} \frac{\sin t(\xi - x_0)}{\xi - x_0} (\xi - a)^{\lambda-1} q(\xi) d\xi \\
 (14a) \quad &= \frac{1}{2} \pi (x_0 - a)^{\lambda-1} q(x_0) \\
 &\quad - \sum_{n=0}^{N-1} \frac{\Gamma(n+\lambda)}{n!} t^{-n-\lambda} \sin[(a-x_0)t + \frac{1}{2}(n+\lambda-2)\pi] \frac{d^n}{d\xi^n} \left[\frac{q(\xi)}{\xi - x_0} \right]_{\xi=a} \\
 &\quad - \sum_{k=0}^{E[\frac{1}{2}(N-1)]} \frac{(-1)^k}{2k+1} t^{-2k-1} \frac{d^{2k+1}}{d\xi^{2k+1}} [(\xi - a)^{\lambda-1} q(\xi)]_{\xi=x_0} + O(t^{-N}).
 \end{aligned}$$

The function $q(x)$ must satisfy the same requirements as for (14).

Formulation of the Problem

In the following we shall take the z -axis downwards, the x -axis towards the bow and the y -axis toward the starboard. This is a right-handed system. The (x,y) -plane is taken in the undisturbed water surface, the (x,z) -plane through the ship's plane of symmetry and the (y,z) -plane through the midship section. The surface of the ship will be described by the function

$$(15) \quad \eta = \pm f(x, z).$$

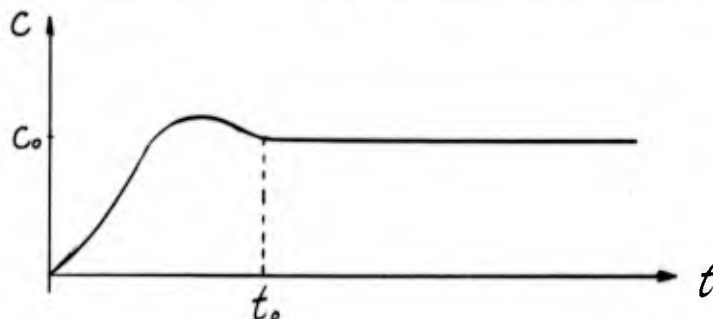
It will be assumed that the ship and water are in a state of rest at $t = 0$ and that thereafter the ship moves with velocity $c(t)$.

The wave-resistance integral. If one makes use of the same approximation as is used in deriving the classical Michell's integral, one may derive [see Lunde, 1951, pp. 40 ff.] the following expression for the wave resistance:

$$(16) \quad R = \frac{\rho c^3(t)}{\pi} \iint_S dx dz \iint_S d\xi d\zeta f_x(x, z) f_x(\xi, \zeta) \left\{ \frac{1}{\sqrt{(x-\xi)^2 + (z-\zeta)^2}} - \frac{1}{\sqrt{(x-\xi)^2 + (z+\zeta)^2}} \right\} \\ + \frac{\rho g}{\pi^2} \iint dx dz \iint d\xi d\zeta f_x(x, z) f_x(\xi, \zeta) \times \\ \int_0^\infty dk k e^{-k(z+\zeta)} \int_{-\pi}^\pi d\theta \int_0^t d\tau c(\tau) \cos[\sqrt{gk}(t-\tau)] \exp\left\{ ik\left[x-\xi + \int_\tau^t c(\tau') d\tau'\right] \cos\theta \right\}.$$

In the following we are going to restrict the form of $c(t)$.

We shall suppose that $C(+0) = 0$ and that $C(t) = C_0$, a constant, for $t \geq t_0 > 0$. Hence $C(t)$ may appear about as follows.



It will be convenient to decompose the integral with respect to t in the light of the assumed form for $C(t)$. We write it as follows:

$$\begin{aligned}
 & \int_0^{t_0} + \int_{t_0}^t C(\tau) \cos[\sqrt{g}K(t-\tau)] \exp\left\{ik\left[x-\xi + \int_{\tau}^t C(\tau') d\tau'\right] \cos\theta\right\} d\tau \\
 (17) \quad & = \int_0^{t_0} C(\tau) \cos[\sqrt{g}K(t-\tau)] \exp\left\{ik\left[x-\xi + \int_{\tau}^t C(\tau') d\tau'\right] \cos\theta\right\} d\tau \\
 & \quad + \int_0^{t-t_0} C(\tau+t_0) \cos[\sqrt{g}K(t-t_0-\tau)] \exp\left\{ik\left[x-\xi + \int_{\tau}^{t-t_0} C(\tau'+t_0) d\tau'\right] \cos\theta\right\} d\tau.
 \end{aligned}$$

If $t > t_0$, no great restriction inasmuch as we shall be dealing chiefly with the asymptotic expansion as $t \rightarrow \infty$, one has in the second integral above $C(\tau+t_0) = C(\tau'+t_0) = C_0$, and this integral simplifies to

$$(18) \quad C_0 \int_0^{t-t_0} \cos[\sqrt{g}K(t-t_0-\tau)] \exp\left\{ik\left[x-\xi + C_0(t-t_0-\tau) \cos\theta\right]\right\} d\tau.$$

If the final velocity had been attained impulsively, so that $t_0 = 0$, then the first integral would not occur at all and the second one would become

$$\begin{aligned}
 (19) \quad & C_0 \int_0^t \cos[\sqrt{gk}(t-\tau)] \exp\{ik[x-\xi + C_0(t-\tau)] \cos \theta\} d\tau \\
 & = C_0 \int_0^t \cos(\sqrt{gk} \tau) \exp\{ik[x-\xi + C_0 \tau] \cos \theta\} d\tau.
 \end{aligned}$$

Thus the results of an investigation of this last case can be carried over directly to the second integral in the more general case above.

We shall proceed as follows for the case of large $t > t_0$. First the asymptotic behavior of (19) will be investigated, then that of the first integral in (17). We note that the first integral in (16) doesn't enter into the behavior of R for $t > t_0$, since $C'(t)$ vanishes in this range. For values of $t < t_0$ this term is, of course, very important. It represents what is frequently called an added-mass force.

Wave resistance for an impulsive start

If the final velocity C_0 is attained immediately, then (16) takes the following simplified form after changing the order of integration:

$$\begin{aligned}
 (20) \quad R^*(t) &= \frac{\pi^2}{2\rho g C_0} R = \frac{1}{2} \int_0^\infty k dk \int_{-\pi}^\pi d\theta \int_0^t d\tau \cos\sqrt{gk} \tau e^{ikC_0\tau \cos\theta} \\
 &\quad \iint dxdz \iint d\xi d\eta f_x(x, z) f_x(\xi, \tau) e^{-k(\tau+\xi)} e^{ik(x-\xi)\cos\theta} \\
 &= \int_0^\infty k dk \int_0^{\pi/2} d\theta [P^2 + Q^2] \left\{ \frac{\sin[(KC_0 \cos\theta - \sqrt{gk})t]}{KC_0 \cos\theta - \sqrt{gk}} + \frac{\sin[(KC_0 \cos\theta + \sqrt{gk})t]}{KC_0 \cos\theta + \sqrt{gk}} \right\} \\
 &= \int_0^\infty k dk \int_0^1 \frac{d\lambda}{\sqrt{1-\lambda^2}} [P^2 + Q^2] \left\{ \frac{\sin(\lambda - \sqrt{\frac{v}{k}})KC_0 t}{(\lambda - \sqrt{\frac{v}{k}})KC_0} + \frac{\sin(\lambda + \sqrt{\frac{v}{k}})KC_0 t}{(\lambda + \sqrt{\frac{v}{k}})KC_0} \right\},
 \end{aligned}$$

where

$$(21) \quad P(\lambda, k) + iQ(\lambda, k) = \iint_S f_x(x, z) e^{-kz} e^{ik\lambda x} dx dz$$

and

$$v = g/C_0^2.$$

The various intermediate steps needed to derive the second equation in (20) may be found in Lunde [1951, p. 44]. The last is obtained by the substitution $\lambda = \cos\theta$.

The last integral in (20) will now be treated as a sum of two integrals corresponding to the two terms in curly brackets.

In the first integral we make the substitution

$$(22) \quad \alpha = \frac{1}{\lambda}, \quad \mu = k\lambda - \sqrt{vk} = \lambda \left(\sqrt{k} - \frac{\sqrt{v}}{2\lambda} \right)^2 - \frac{v}{4\lambda}$$

Then, solving for k one finds

$$\sqrt{k} = \frac{\sqrt{v}}{2\lambda} \pm \frac{1}{\sqrt{\lambda}} \sqrt{\mu + \frac{v}{4\lambda}}$$

or

$$k = \frac{v}{4\lambda^2} + \frac{1}{\lambda} \left[\left(\mu + \frac{v}{4\lambda} \right) \pm \sqrt{\frac{v}{\lambda} \left(\mu + \frac{v}{4\lambda} \right)} \right].$$

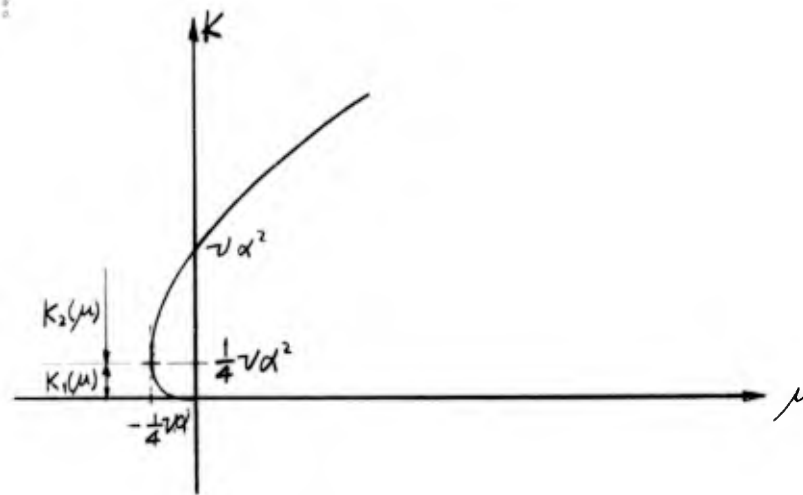
We define then

$$k_1(\mu) = \frac{1}{4} v \alpha^2 + \alpha \left[\mu + \frac{1}{4} v \alpha - \sqrt{v \alpha \left(\mu + \frac{1}{4} v \alpha \right)} \right]$$

(23)

$$k_2(\mu) = \frac{1}{4} v \alpha + \alpha \left[\mu + \frac{1}{4} v \alpha + \sqrt{v \alpha \left(\mu + \frac{1}{4} v \alpha \right)} \right].$$

The two functions are shown schematically on the following graph:



The corresponding derivatives are

$$\frac{d\mu}{dk} = \lambda - \frac{1}{2} \sqrt{\frac{v}{k}} = \frac{2\lambda\sqrt{k} - \sqrt{v}}{2\sqrt{k}} = \pm \sqrt{\frac{\lambda}{k}} \sqrt{\mu + \frac{v}{4\lambda}}$$

or

$$(24) \quad k_1'(\mu) = \frac{-\sqrt{k_1 \alpha}}{\sqrt{\mu + \frac{1}{4}v\alpha}}, \quad k_2'(\mu) = \frac{\sqrt{k_2 \alpha}}{\sqrt{\mu + \frac{1}{4}v\alpha}}.$$

In the second integral we let

$$(25) \quad \alpha = \frac{1}{\lambda}, \quad \mu = k\lambda + \sqrt{vk} = \lambda \left(\sqrt{k} + \frac{\sqrt{v}}{2\lambda} \right)^2 - \frac{v}{4\lambda}.$$

There is only one usable solution for k :

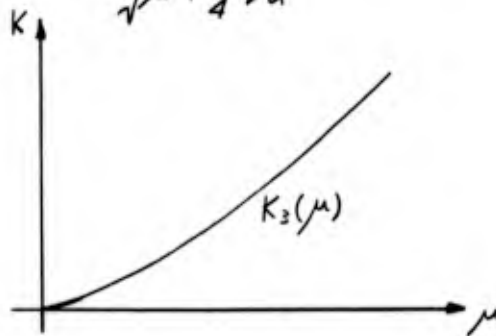
$$\sqrt{k_3} = -\frac{\sqrt{v}}{2\lambda} + \frac{1}{\sqrt{\lambda}} \sqrt{\mu + \frac{v}{4\lambda}}$$

or

$$(26) \quad k_3(\mu) = \frac{1}{4}v\alpha^2 + \alpha \left[\left(\mu + \frac{1}{4}v\alpha \right) - \sqrt{v\alpha \left(\mu + \frac{1}{4}v\alpha \right)} \right].$$

The derivative is

$$(27) \quad k_3'(\mu) = \frac{\sqrt{k_3 \alpha}}{\sqrt{\mu + \frac{1}{4}v\alpha}}$$



After making these two changes of variable, one obtains the following expression for R^* :

$$(28) \quad R^*(t) = \int_1^\infty d\alpha \frac{1}{\sqrt{\alpha(\alpha^2-1)}} \left\{ \int_{-\frac{1}{4}v\alpha}^0 \frac{k_1^{3/2}(\mu)}{\sqrt{\mu + \frac{1}{4}v\alpha}} \left[P^2\left(\frac{1}{\alpha}, k_1(\mu)\right) + Q^2 \right] \frac{\sin \mu c_0 t}{\mu c_0} d\mu \right. \\ \left. + \int_{-\frac{1}{4}v\alpha}^\infty \frac{k_2^{3/2}(\mu)}{\sqrt{\mu + \frac{1}{4}v\alpha}} \left[P^2\left(\frac{1}{\alpha}, k_2(\mu)\right) + Q^2 \right] \frac{\sin \mu c_0 t}{\mu c_0} d\mu \right. \\ \left. + \int_0^\infty \frac{k_3^{3/2}(\mu)}{\sqrt{\mu + \frac{1}{4}v\alpha}} \left[P^2\left(\frac{1}{\alpha}, k_3(\mu)\right) + Q^2 \right] \frac{\sin \mu c_0 t}{\mu c_0} d\mu \right\}.$$

It is to these integrals that we shall apply the theorems 1-4 and formulas (14) and (14a).

It will be convenient first to establish a few properties of $P+iQ$ and its derivatives. It follows immediately from (21) that

$$(29) \quad \begin{aligned} P_\lambda(\lambda, k) + i Q_\lambda(\lambda, k) &= ik \iint_S x f_x(x, z) e^{-kz} e^{ik\lambda x} dx dz \\ P_k(\lambda, k) + i Q_k(\lambda, k) &= \iint_S (-z + i\lambda x) f_x(x, z) e^{-kz} e^{ik\lambda x} dx dz. \end{aligned}$$

It is evident that

$$(30) \quad \lim_{k \rightarrow \infty} P + iQ = \lim_{k \rightarrow \infty} P_\lambda + iQ_\lambda = \lim_{k \rightarrow \infty} P_k + iQ_k = 0$$

and also, from a well-known theorem on Fourier coefficients, that

$$(31) \quad \lim_{\lambda \rightarrow \infty} P + iQ = \lim_{\lambda \rightarrow \infty} P_\lambda + iQ_\lambda = \lim_{\lambda \rightarrow \infty} P_k + iQ_k = 0.$$

It is also evident from an integration by parts with respect to x that

$$(32) \quad P(\lambda, 0) + iQ(\lambda, 0) = 0.$$

Formal application of formulas (14) or (14a) to the integrals inside the curly brackets in (28) yields the

expression below. A sufficient number of terms have been kept so that the remainder is $O((ct)^{-2})$.

$$\begin{aligned}
 (33) \quad R^*(t) = & \int_1^\infty d\alpha \frac{1}{\sqrt{\alpha(\alpha^2-1)}} \left\{ \frac{1}{2} \pi \frac{K_1^{3/2}(0)}{\sqrt{\frac{1}{4}v\alpha}} [P^2(\frac{1}{\alpha}, K_1(0)) + Q^2] \frac{1}{C_0} \right. \\
 & - \sqrt{\pi} (ct)^{-\frac{1}{2}} \sin(-\frac{1}{4}v\alpha ct - \frac{3}{4}\pi) K_1^{3/2}(-\frac{1}{4}v\alpha) \cdot \frac{-4}{v\alpha} [P^2(\frac{1}{\alpha}, K_1(-\frac{1}{4}v\alpha)) + Q^2] \frac{1}{C_0} \\
 & - \frac{\sqrt{\pi}}{2} (ct)^{-\frac{3}{2}} \sin(-\frac{1}{4}v\alpha ct - \frac{1}{4}\pi) \frac{\partial}{\partial \mu} \left[\frac{K_1^{3/2}(\mu)}{\mu} \{P^2(\frac{1}{\alpha}, K_1(\mu)) + Q^2\} \right] \cdot \frac{1}{C_0} \\
 & \qquad \qquad \qquad \mu = -\frac{1}{4}v\alpha \\
 & - (ct)^{-1} \frac{\partial}{\partial \mu} \left[\frac{K_1^{3/2}(\mu)}{\sqrt{\mu + \frac{1}{4}v\alpha}} \{P^2(\frac{1}{\alpha}, K_1(\mu)) + Q^2\} \right]_{\mu=0} \cdot \frac{1}{C_0} \\
 & + \pi \frac{K_2^{3/2}(0)}{\sqrt{\frac{1}{4}v\alpha}} [P^2(\frac{1}{\alpha}, K_2(0)) + Q^2] \frac{1}{C_0} \\
 & - \sqrt{\pi} (ct)^{-\frac{1}{2}} \sin(-\frac{1}{4}v\alpha ct - \frac{3}{4}\pi) K_2^{3/2}(-\frac{1}{4}v\alpha) \cdot \frac{-4}{v\alpha} [P^2(\frac{1}{\alpha}, K_2(-\frac{1}{4}v\alpha)) + Q^2] \frac{1}{C_0} \\
 & - \frac{\sqrt{\pi}}{2} (ct)^{-\frac{3}{2}} \sin(-\frac{1}{4}v\alpha ct - \frac{1}{4}\pi) \frac{\partial}{\partial \mu} \left[\frac{K_2^{3/2}(\mu)}{\mu} \{P^2(\frac{1}{\alpha}, K_2(-\frac{1}{4}v\alpha)) + Q^2\} \right] \frac{1}{C_0} \\
 & \qquad \qquad \qquad \mu = -\frac{1}{4}v\alpha \\
 & + \frac{\pi}{2} \frac{K_3^{3/2}(0)}{\sqrt{\frac{1}{4}v\alpha}} [P^2(\frac{1}{\alpha}, K_3(0)) + Q^2] \frac{1}{C_0} \\
 & - (ct)^{-1} \left[\frac{K_3^{3/2}(\mu)}{C_0 \mu \sqrt{\mu + \frac{1}{4}v\alpha}} \{P^2(\frac{1}{\alpha}, K_3(\mu)) + Q^2\} \right] \cos \mu ct \Big|_0^\infty \\
 & - (ct)^{-2} \frac{\partial}{\partial \mu} \left[\frac{K_3^{3/2}(\mu)}{C_0 \mu \sqrt{\mu + \frac{1}{4}v\alpha}} \{P^2(\frac{1}{\alpha}, K_3(\mu)) + Q^2\} \right] \sin \mu ct \Big|_0^\infty \\
 & \left. + O\left(\frac{1}{(ct)^2}\right) \right\}.
 \end{aligned}$$

Because of (32) and the fact that

$$K_1(0) = K_1'(0) = K_3(0) = K_3'(0) = 0$$

the first, fourth and eighth summands vanish. The ninth and tenth summands vanish at the upper limit, as follows easily from the form of K_3 and (30). (An analogous reasoning implies the vanishing of a possible contribution from the upper limit of the K_2 terms, and hence allows the applicability of the extended form of Theorem 4.) For the lower limit we need more precise information about the behavior of $K_3(\mu)$ near $\mu=0$. Since

$$K_3''(\mu) = \frac{1}{2} \frac{1}{\mu + \frac{1}{4}\nu\alpha} \left\{ \alpha - \frac{\sqrt{K_3}\alpha}{\sqrt{\mu + \frac{1}{4}\nu\alpha}} \right\},$$

one finds

$$K_3''(0) = \frac{8}{\nu}$$

and

$$K_3(\mu) = \frac{8}{\nu} \mu^2 + \dots$$

Hence the ninth and tenth summands vanish also at the lower limit. The fifth summand can be evaluated immediately since $K_2(0) = \nu\alpha^2$. Since

$$K_1(-\frac{1}{4}\nu\alpha) = K_2(-\frac{1}{4}\nu\alpha) = \frac{1}{4}\nu\alpha^2,$$

the second and sixth terms are the same and combine to give

$$-\sqrt{\pi} \frac{\nu\alpha^2}{c_0} (\nu c_0 t)^{-\frac{1}{2}} \left[P^2 \left(\frac{1}{\alpha} \cdot \frac{1}{4}\nu\alpha^2 \right) + Q^2 \right] \sin \left(\frac{1}{4}\nu c_0 t \alpha + \frac{3}{4}\pi \right).$$

In order to find the third and seventh terms, we must

evaluate the indicated derivatives:

$$\begin{aligned} & \frac{\partial}{\partial \mu} \left[\frac{K_i^{3/2}(\mu)}{\mu} \left\{ P^2\left(\frac{1}{\alpha}, K_i(\mu) + Q^2\right) \right\} \right] \\ &= -\frac{1}{\mu^2} K_i^{3/2}(\mu) \left\{ P^2\left(\frac{1}{\alpha}, K_i(\mu) + Q^2\right) \right\} \\ &+ (-1)^i \frac{K_i(\mu) \sqrt{\alpha}}{\mu \sqrt{\mu + \frac{1}{4} v \alpha}} \left[\frac{3}{2} \{ P^2 + Q^2 \} + 2 \{ P P_K + Q Q_K \} K_i(\mu) \right]. \end{aligned}$$

There is evidently a singularity in the second term at the point in question, namely $\mu = -\frac{1}{4} v \alpha$. We shall avoid this difficulty by interpreting the two limits as

$$\lim_{\varepsilon \rightarrow 0} \frac{\partial}{\partial \mu} \left[\quad \right]_{\mu = -\frac{1}{4} v \alpha + \varepsilon}$$

and combining them before taking the limit with respect to ε . It is then evident that only the first term contributes, and the third and seventh summands yield

$$-2\sqrt{\pi} \frac{v\alpha}{c_0} (vc_0 t)^{-3/2} \left[P^2\left(\frac{1}{\alpha}, \frac{1}{4} v \alpha^2\right) + Q^2 \right] \sin\left(\frac{1}{4} v c_0 t \alpha + \frac{1}{4} \pi\right).$$

After substituting in the integral (33), we obtain the following sum of three integrals with respect to α :

$$\begin{aligned} (34) \quad R^*(t) &= \frac{2\pi v}{c_0} \int_1^\infty \frac{\alpha^2}{\sqrt{\alpha^2 - 1}} \left[P^2\left(\frac{1}{\alpha}, v \alpha^2\right) + Q^2 \right] d\alpha \\ &- \sqrt{\pi} \frac{v}{c_0} (vc_0 t)^{-1/2} \int_1^\infty \frac{\alpha^{3/2}}{\sqrt{\alpha^2 - 1}} \left[P^2\left(\frac{1}{\alpha}, \frac{1}{4} v \alpha^2\right) + Q^2 \right] \sin\left(\frac{1}{4} v \alpha c_0 t + \frac{3}{4} \pi\right) d\alpha \\ &- 2\sqrt{\pi} \frac{v}{c_0} (vc_0 t)^{-3/2} \int_1^\infty \frac{\alpha^{1/2}}{\sqrt{\alpha^2 - 1}} \left[P^2\left(\frac{1}{\alpha}, \frac{1}{4} v \alpha^2\right) + Q^2 \right] \sin\left(\frac{1}{4} v \alpha c_0 t + \frac{1}{4} \pi\right) d\alpha \\ &+ O\left(\frac{1}{(vc_0 t)^2}\right). \end{aligned}$$

The first integral above is just Michell's Integral for the steady-state wave resistance, and need not concern us further. We denote it by $R^*(\infty)$. The remaining two integrals are still of a form to which we may apply Theorem 4. One obtains the following:

$$(35) \quad R^*(t) = R^*(\infty)$$

$$\begin{aligned} & -\sqrt{\pi} \frac{v}{c_0} (vc_0 t)^{1/2} \left\{ -\sqrt{\pi} \left(\frac{1}{4}vc_0 t\right)^{-1/2} \frac{1}{\sqrt{2}} [P^2(1, \frac{1}{4}v) + Q^2] \sin\left(\frac{1}{4}vc_0 t + \frac{3}{4}\pi - \frac{3}{4}\pi\right) \right. \\ & \quad \left. - \frac{1}{2} \sqrt{\pi} \left(\frac{1}{4}vc_0 t\right)^{-3/2} \frac{d}{d\alpha} \left[\frac{\alpha^{3/2}}{\sqrt{\alpha+1}} \left\{ P^2\left(\frac{1}{\alpha}, \frac{1}{4}v\alpha\right) + Q^2 \right\} \right]_{\alpha=1} \sin\left(\frac{1}{4}vc_0 t + \frac{3}{4}\pi - \frac{1}{4}\pi\right) \right. \\ & \quad \left. + O\left(\frac{1}{(vc_0 t)^2}\right) \right\} \\ & - 2\sqrt{\pi} \frac{v}{c_0} (vc_0 t)^{-3/2} \left\{ -\sqrt{\pi} \left(\frac{1}{4}vc_0 t\right)^{-1/2} \frac{1}{\sqrt{2}} [P^2(1, \frac{1}{4}v) + Q^2] \sin\left(\frac{1}{4}vc_0 t + \frac{1}{4}\pi - \frac{3}{4}\pi\right) \right. \\ & \quad \left. + O\left(\frac{1}{vc_0 t}\right) \right\} \\ & + O\left(\frac{1}{(vc_0 t)^2}\right); \end{aligned}$$

$$(36) \quad R^*(t) = R^*(\infty)$$

$$\begin{aligned} & + \pi\sqrt{2} \frac{v}{c_0} [P^2(1, \frac{1}{4}v) + Q^2] (vc_0 t)^{-1} \sin \frac{1}{4}vc_0 t \\ & + \pi\sqrt{2} \frac{v}{c_0} \left\{ \frac{1}{2} [P^2(1, \frac{1}{2}v) + Q^2] + 2v [PP_k + QQ_k] - 4 [PR_k + QQ_k] \right\} (vc_0 t)^{-2} \cos \frac{1}{4}vc_0 t \\ & + O\left(\frac{1}{(vc_0 t)^2}\right). \end{aligned}$$

This is the asymptotic expression for the integral (20) for an impulsive start. In view of the behavior of the asymptotic expressions in (35) it seems reasonable to suppose that one can replace $O\left(\frac{1}{(\nu c_0 t)^2}\right)$ by $o\left(\frac{1}{(\nu c_0 t)^2}\right)$. However, this has not been proved. One may note that the two oscillating terms in the asymptotic expression are much easier to compute than the limiting value $R^*(\infty)$.

Wave Resistance for a Gradual Start

We return now to the asymptotic evaluation of integral (16) for large values of t , in particular, for $t > t_0$. Because of the splitting of the time integral shown in (13), possible because of the special form of $C(\tau)$, we may also treat the integral (16) in two parts. The second part, corresponding to the second integral in (16), need not be further discussed because its asymptotic expansion can be taken over directly from that of (20), namely (36), by replacing t by $t - t_0$. We thus need to discuss only the integral corresponding to the first integral in (17), namely

$$(37) \quad \frac{PQ}{\pi^2} \iint dx dz \iint d\xi d\zeta f_x(x, \xi) f_x(\xi, \zeta) \int_0^\infty dk k e^{-k(\xi+\zeta)} \int_{-\pi}^{\pi} d\theta \int_{\tau}^{t_0} d\tau C(\tau) \cos \sqrt{gK}(t-\tau) \exp[ik(x-\xi)\cos\theta] \exp[ik\cos\theta] C(\tau) d\tau.$$

A change of the order of integration and slight further manipulation yield the following integral:

$$(38) \quad \frac{4PQ}{\pi^2} \int_0^\infty dk k \int_0^{\pi/2} d\theta [P^2 + Q^2] \int_0^{t_0} d\tau C(\tau) \cos \sqrt{gK}(t-\tau) \cos[k \cos\theta] \int_{\tau}^t C(\tau') d\tau' \\ = \frac{4PQ}{\pi^2} \int_0^\infty dk k \int_0^1 d\lambda \frac{P^2 + Q^2}{\sqrt{1-\lambda^2}} \int_0^{t_0} d\tau C(\tau) \cos \sqrt{gK}(t-\tau) \cos[k\lambda] \int_{\tau}^t C(\tau') d\tau',$$

where, as usual, $P(\lambda, k) + iQ(\lambda, k)$ is given by (21).

We shall write

$$\int_{\tau}^t C(\tau') d\tau' = \int_{\tau}^{t_0} C(\tau') d\tau' + C_0(t-t_0), \quad \sqrt{gK}(t-\tau) = \sqrt{gK}(t-t_0) + \sqrt{gK}(t_0-\tau).$$

The above integral may then be successively modified as

follows:

$$\begin{aligned}
 & \frac{2PQ}{\pi^2} \int_0^\infty dk k \int_0^1 d\lambda \frac{P^2+Q^2}{\sqrt{1-\lambda^2}} \int_0^{t_0} d\tau C(\tau) \left\{ \cos[\sqrt{qk+C_0k\lambda}(t-t_0)+\sqrt{qk}(t_0-\tau)+k\lambda] \int_\tau^{t_0} C(\tau') d\tau' \right. \\
 & \qquad \qquad \qquad \left. + \cos[\sqrt{qk-C_0k\lambda}(t-t_0)+\sqrt{qk}(t_0-\tau)-k\lambda] \int_\tau^{t_0} C(\tau') d\tau' \right\} \\
 & = \frac{2PQ}{\pi^2} \int_0^\infty dk k \int_0^1 d\lambda \frac{P^2+Q^2}{\sqrt{1-\lambda^2}} \left\{ \cos[\sqrt{k\nu}+k\lambda] C_0(t-t_0) \int_0^{t_0} d\tau C(\tau) \cos[\sqrt{qk}(t_0-\tau)+k\lambda] \int_\tau^{t_0} C(\tau') d\tau' \right. \\
 & \qquad \qquad \qquad - \sin[\sqrt{k\nu}+k\lambda] C_0(t-t_0) \int_0^{t_0} d\tau C(\tau) \sin[\sqrt{qk}(t_0-\tau)+k\lambda] \int_\tau^{t_0} C(\tau') d\tau' \\
 & \qquad \qquad \qquad + \cos[\sqrt{k\nu}-k\lambda] C_0(t-t_0) \int_0^{t_0} d\tau C(\tau) \cos[\sqrt{qk}(t_0-\tau)-k\lambda] \int_\tau^{t_0} C(\tau') d\tau' \\
 & \qquad \qquad \qquad \left. - \sin[\sqrt{k\nu}-k\lambda] C_0(t-t_0) \int_0^{t_0} d\tau C(\tau) \sin[\sqrt{qk}(t_0-\tau)-k\lambda] \int_\tau^{t_0} C(\tau') d\tau' \right\} \\
 & = \frac{8}{\pi^2} P C_0^2 \int_0^\infty dk k \int_0^1 d\lambda \frac{P^2+Q^2}{\sqrt{1-\lambda^2}} \left\{ C^+(\lambda, k) \cos[\sqrt{k\nu}+k\lambda] C_0(t-t_0) \right. \\
 (39) \qquad \qquad \qquad & \qquad \qquad \qquad - S^+(\lambda, k) \sin[\sqrt{k\nu}+k\lambda] C_0(t-t_0) \\
 & \qquad \qquad \qquad + \bar{C}(\lambda, k) \cos[\sqrt{k\nu}-k\lambda] C_0(t-t_0) \\
 & \qquad \qquad \qquad \left. - \bar{S}(\lambda, k) \sin[\sqrt{k\nu}-k\lambda] C_0(t-t_0) \right\},
 \end{aligned}$$

where

$$(40) \quad C^\pm(\lambda, k) + iS^\pm(\lambda, k) = \frac{1}{4} \nu \int_0^{t_0} d\tau C(\tau) \exp i[\sqrt{qk}(t_0-\tau) \pm k\lambda] \int_\tau^{t_0} C(\tau') d\tau'.$$

The functions C^\pm and S^\pm are dimensionless, and are independent of the hull form.

We shall now make use once again of the change of variables (22) in the last two terms of (39) and of (25)

in the first two terms. The integral (39) then takes the following form:

$$\begin{aligned}
 (41) \quad & \frac{8}{\pi^2} \rho C_0^2 \int_1^\infty d\alpha \frac{1}{\sqrt{\alpha(\alpha^2-1)}} \left\{ \int_0^\infty d\mu \frac{K_3^{3/2}(\mu)}{\sqrt{\mu + \frac{1}{4}\nu\alpha}} [P^2(\frac{1}{\alpha}, K_3(\mu)) + Q^2] \cdot [C^+(\frac{1}{\alpha}, K_3) \cos \mu C_0(t-t_0) \right. \\
 & \qquad \qquad \qquad \left. - S^+(\frac{1}{\alpha}, K_3) \sin \mu C_0(t-t_0)] \right. \\
 & + \int_{-\frac{1}{4}\nu\alpha}^0 d\mu \frac{K_1^{3/2}(\mu)}{\sqrt{\mu + \frac{1}{4}\nu\alpha}} [P^2(\frac{1}{\alpha}, K_1) + Q^2] \cdot [C^-(\frac{1}{\alpha}, K_1) \cos \mu C_0(t-t_0) \\
 & \qquad \qquad \qquad \left. + S^-(\frac{1}{\alpha}, K_1) \sin \mu C_0(t-t_0)] \right. \\
 & + \int_{-\frac{1}{4}\nu\alpha}^\infty d\mu \frac{K_2^{3/2}(\mu)}{\sqrt{\mu + \frac{1}{4}\nu\alpha}} [P^2(\frac{1}{\alpha}, K_2) + Q^2] \cdot [C^-(\frac{1}{\alpha}, K_2) \cos \mu C_0(t-t_0) \\
 & \qquad \qquad \qquad \left. + S^-(\frac{1}{\alpha}, K_2) \sin \mu C_0(t-t_0)] \right\}.
 \end{aligned}$$

We may now apply Theorems 2, 3 and 4 to the various integrals with respect to μ . Although a number of the terms will vanish, we shall include them all formally. Sufficient terms are kept so that the remainder is $\mathcal{O}((t-t_0)^2)$.

$$\begin{aligned}
 & \frac{8}{\pi^2} \rho C_0^2 \int_1^\infty d\alpha \frac{1}{\sqrt{\alpha(\alpha^2-1)}} \left\{ \frac{1}{C_0(t-t_0) \sqrt{\frac{1}{4}\nu\alpha}} \frac{K_3^{3/2}(0)}{\sqrt{\frac{1}{4}\nu\alpha}} [P^2(\frac{1}{\alpha}, K_3(0)) + Q^2] S^+(\frac{1}{\alpha}, K_3(0)) \right. \\
 & \qquad \qquad \qquad - \frac{1}{C_0^2(t-t_0)^2} \frac{\partial}{\partial \mu} \left[\frac{K_3^{3/2}(\mu)}{\sqrt{\mu + \frac{1}{4}\nu\alpha}} [P^2(\frac{1}{\alpha}, K_3(\mu)) + Q^2] C^+(\frac{1}{\alpha}, K_3) \right]_{\mu=0} \\
 & \qquad \qquad \qquad - \frac{1}{C_0(t-t_0) \sqrt{\frac{1}{4}\nu\alpha}} \frac{K_1^{3/2}(0)}{\sqrt{\frac{1}{4}\nu\alpha}} [P^2(\frac{1}{\alpha}, K_1(0)) + Q^2] S^-(\frac{1}{\alpha}, K_1(0)) \\
 & \qquad \qquad \qquad + \frac{1}{C_0^2(t-t_0)^2} \frac{\partial}{\partial \mu} \left[\frac{K_1^{3/2}(\mu)}{\sqrt{\mu + \frac{1}{4}\nu\alpha}} [P^2(\frac{1}{\alpha}, K_1(\mu)) + Q^2] C^-(\frac{1}{\alpha}, K_1) \right]_{\mu=0}
 \end{aligned}$$

$$\begin{aligned}
(42) \quad & -\frac{\sqrt{\pi}}{\sqrt{C_0(t-t_0)}} K_1^{3/2}(-\frac{1}{4}\nu\alpha) [P^2(\frac{1}{\alpha}, K_1) + Q^2] \left\{ \bar{C}(\frac{1}{\alpha}, K_1) \cos[-\frac{1}{4}\nu\alpha C_0(t-t_0) - \frac{3}{4}\pi] \right. \\
& \left. + \bar{S}(\frac{1}{\alpha}, K_1) \sin[-\frac{1}{4}\nu\alpha C_0(t-t_0) - \frac{3}{4}\pi] \right\} \\
& - \frac{1}{2} \frac{\sqrt{\pi}}{[C_0(t-t_0)]^{3/2}} \left\{ \frac{\partial}{\partial \mu} [K_1^{3/2} (P^2 + Q^2) \bar{C}]_{\mu = -\frac{1}{4}\nu\alpha} \cdot \cos[-\frac{1}{4}\nu\alpha C_0(t-t_0) - \frac{1}{4}\pi] \right. \\
& \left. + \frac{\partial}{\partial \mu} [K_1^{3/2} (P^2 + Q^2) \bar{S}]_{\mu = -\frac{1}{4}\nu\alpha} \cdot \sin[-\frac{1}{4}\nu\alpha C_0(t-t_0) - \frac{1}{4}\pi] \right\} \\
& - \frac{\sqrt{\pi}}{\sqrt{C_0(t-t_0)}} K_2^{3/2}(-\frac{1}{4}\nu\alpha) [P^2(\frac{1}{\alpha}, K_2) + Q^2] \left\{ \bar{C}(\frac{1}{\alpha}, K_2) \cos[-\frac{1}{4}\nu\alpha C_0(t-t_0) - \frac{3}{4}\pi] \right. \\
& \left. + \bar{S}(\frac{1}{\alpha}, K_2) \sin[-\frac{1}{4}\nu\alpha C_0(t-t_0) - \frac{3}{4}\pi] \right\} \\
& - \frac{1}{2} \frac{\sqrt{\pi}}{[C_0(t-t_0)]^{3/2}} \left\{ \frac{\partial}{\partial \mu} [K_2^{3/2} (P^2 + Q^2) \bar{C}]_{\mu = -\frac{1}{4}\nu\alpha} \cdot \cos[-\frac{1}{4}\nu\alpha C_0(t-t_0) - \frac{1}{4}\pi] \right. \\
& \left. + \frac{\partial}{\partial \mu} [K_2^{3/2} (P^2 + Q^2) \bar{S}]_{\mu = -\frac{1}{4}\nu\alpha} \cdot \sin[-\frac{1}{4}\nu\alpha C_0(t-t_0) - \frac{1}{4}\pi] \right\} + O\left(\frac{1}{C_0(t-t_0)}\right).
\end{aligned}$$

In order to evaluate the first four terms we recall

that $k_1(0) = k_1'(0) = k_3(0) = k_3'(0) = P(\lambda, 0) = Q(\lambda, 0) = 0$. One may easily confirm that

$$S^+(\lambda, 0) = S^-(\lambda, 0) = 0, \quad C^+(\lambda, 0) = C^-(\lambda, 0) = \frac{1}{4}\nu \int_0^t C(\tau) d\tau$$

It is evident that the first and third terms in (42) vanish.

In order to see that this is also the case for the second and fourth terms, we must show that $\frac{\partial}{\partial \mu} C^\pm(\lambda, k(\mu))$ does not spoil the behavior of the other terms. For $j = 1$ or 2

$$\begin{aligned}
(43) \quad & \frac{\partial}{\partial \mu} [C^\pm(\lambda, k_j(\mu)) + iS^\pm(\lambda, k_j(\mu))] = [C_k^\pm + iS_k^\pm] \cdot k_j'(\mu) \\
& = \frac{1}{4}\nu \int_0^t d\tau C(\tau) i \exp i[\sqrt{q}K_j(t_0 - \tau) \pm k_j\lambda] \int_\tau^{t_0} C(\tau') d\tau' \cdot [\sqrt{q}(t_0 - \tau) \pm \lambda] \int_\tau^{t_0} C(\tau') d\tau' \cdot \frac{(-i)k_j\alpha}{\sqrt{\mu + \frac{1}{2}\nu\alpha}} \\
& = \frac{(-i)k_j\nu\alpha}{\sqrt{\mu + \frac{1}{2}\nu\alpha}} \int_0^{t_0} d\tau C(\tau) [\sqrt{q}(t_0 - \tau) \pm \lambda k_j] \int_\tau^{t_0} C(\tau') d\tau' i \exp[\sqrt{q}K_j(t_0 - \tau) \pm k_j\lambda] \int_\tau^{t_0} C(\tau') d\tau'.
\end{aligned}$$

For $j=3$ the same sign holds as for $j=2$. It is evident that

$$\frac{\partial}{\partial \mu} C^{\pm}(\lambda, k_j(\mu)) \Big|_{\mu=0} = 0 \quad \text{for } j=1 \text{ or } 3$$

and hence that also the second and fourth terms in (42) vanish. The fifth and seventh terms are equal and may be combined:

$$-\frac{\sqrt{\pi}}{4} \frac{\nu^{3/2} \alpha^3}{\sqrt{C_0(t-t_0)}} \left[P^2\left(\frac{1}{\alpha}, \frac{1}{4}\nu\alpha^2\right) + Q^2 \right] \left\{ \bar{C}\left(\frac{1}{\alpha}, \frac{1}{4}\nu\alpha^2\right) \cos\left[\frac{1}{4}\nu\alpha C_0(t-t_0) + \frac{3}{4}\pi\right] - \bar{S}\left(\frac{1}{\alpha}, \frac{1}{4}\nu\alpha^2\right) \sin\left[\frac{1}{4}\nu\alpha C_0(t-t_0) + \frac{3}{4}\pi\right] \right\}.$$

The sixth and eighth terms both appear to have singularities at $\mu = -\frac{1}{4}\nu\alpha$, entering from the behavior of k_i' there. As in the case of the impulsive start, we shall interpret these two terms together as

$$\lim_{\varepsilon \rightarrow 0} \frac{\partial}{\partial \mu} \left[\quad \right]_{\mu = -\frac{1}{4}\nu\alpha + \varepsilon}.$$

The contribution of these two terms then vanishes.

We are now left with the following:

$$(44) \quad -\frac{2}{\pi^{3/2}} \frac{\rho C_0^2 \nu^{3/2}}{\sqrt{C_0(t-t_0)}} \int_1^{\infty} d\alpha \frac{\alpha^{5/2}}{\sqrt{\alpha^2-1}} \left[P^2\left(\frac{1}{\alpha}, \frac{1}{4}\nu\alpha^2\right) + Q^2 \right] \left\{ \bar{C}\left(\frac{1}{\alpha}, \frac{1}{4}\nu\alpha^2\right) \cos\left[\frac{1}{4}\nu C_0(t-t_0)\alpha + \frac{3}{4}\pi\right] - \bar{S}\left(\frac{1}{\alpha}, \frac{1}{4}\nu\alpha^2\right) \sin\left[\frac{1}{4}\nu C_0(t-t_0)\alpha + \frac{3}{4}\pi\right] \right\} + O\left(\frac{1}{C_0^2(t-t_0)^2}\right).$$

We may once more apply Theorem 4 in its extended form to this integral and obtain the following expression:

$$-\frac{2}{\pi^2} \frac{\rho c_0^2 v^{3/2}}{\sqrt{c_0(t-t_0)}} \left\{ -\sqrt{\pi} \frac{1}{\sqrt{\frac{1}{4} v c_0(t-t_0)}} \cdot \frac{1}{\sqrt{2}} [P^2(1, \frac{1}{4}v) + Q^2] \left\{ \bar{C}(1, \frac{1}{4}v) \cos[\frac{1}{4} v c_0(t-t_0) + \frac{3}{4}\pi - \frac{3}{4}\pi] \right. \right. \\ \left. \left. - \bar{S}(1, \frac{1}{4}v) \sin[\frac{1}{4} v c_0(t-t_0) + \frac{3}{4}\pi - \frac{3}{4}\pi] \right\} \right.$$

$$-\frac{1}{2} \sqrt{\pi} \frac{1}{[\frac{1}{4} v c_0(t-t_0)]^{3/2}} \left[\frac{\partial}{\partial \alpha} \left[\frac{\alpha^{5/2}}{\sqrt{\alpha+1}} [P^2(\frac{1}{2}, \frac{1}{4}v\alpha^2) + Q^2] \bar{C}(\frac{1}{2}, \frac{1}{4}v\alpha^2) \right] \right]_{\alpha=1} \times \\ \cdot \cos[\frac{1}{4} v c_0(t-t_0) + \frac{3}{4}\pi - \frac{1}{4}\pi]$$

$$-\frac{\partial}{\partial \alpha} \left[\frac{\alpha^{5/2}}{\sqrt{\alpha+1}} [P^2(\frac{1}{2}, \frac{1}{4}v\alpha^2) + Q^2] \bar{S}(\frac{1}{2}, \frac{1}{4}v\alpha^2) \right]_{\alpha=1} \sin[\frac{1}{4} v c_0(t-t_0) + \frac{3}{4}\pi - \frac{1}{4}\pi]$$

$$+ O\left(\frac{1}{v^2 c_0^2 (t-t_0)^2}\right) \left. \right\} + O\left(\frac{1}{v^2 c_0^2 (t-t_0)^2}\right)$$

$$= \frac{2\sqrt{2}}{\pi} \frac{\rho q}{c_0^2} [P^2(1, \frac{1}{4}v) + Q^2] \frac{1}{v c_0(t-t_0)} \left\{ \bar{C}(1, \frac{1}{4}v) \cos \frac{1}{4} v c_0(t-t_0) - \bar{S}(1, \frac{1}{4}v) \sin \frac{1}{4} v c_0(t-t_0) \right\}$$

$$-\frac{4\sqrt{2}}{\pi} \frac{\rho q}{c_0^2} \frac{1}{[v c_0(t-t_0)]^{3/2}} \left\{ \left[\frac{q}{4} (P^2+Q^2) - 2(PR_\lambda + QQ_\lambda) + v(PP_k + QQ_k) \right] \bar{C} \right. \\ \left. + (P^2+Q^2)(-\bar{C}_\lambda + \frac{1}{2} v \bar{C}_k) \right\} \sin \frac{1}{4} v c_0(t-t_0)$$

(45)

$$+ \left\{ \left[\frac{q}{4} (P^2+Q^2) - 2(PR_\lambda + QQ_\lambda) + v(PP_k + QQ_k) \right] \bar{S} \right. \\ \left. + (P^2+Q^2)(-\bar{S}_\lambda + \frac{1}{2} v \bar{S}_k) \right\} \cos \frac{1}{4} v c_0(t-t_0)$$

$$+ O\left(\frac{1}{v^2 c_0^2 (t-t_0)^2}\right).$$

This is as far as we shall carry the expansion of this integral.

We may now combine (45) with (36), properly modified, in order to obtain an asymptotic expansion for (16). We shall keep only terms through order $(t-t_0)^{-1}$ for the sake of perspicuity. However, one could, of course, also write down also the term of next higher order.

$$\begin{aligned}
 R(t) = & R(\infty) + \\
 & \frac{2\sqrt{2}}{\pi} \rho g v \frac{1}{v c_0 (t-t_0)} [P^2(1, \frac{1}{4}v) + Q^2(1, \frac{1}{4}v)] \left\{ [1 - \tilde{S}(1, \frac{1}{4}v)] \sin \frac{1}{4} v c_0 (t-t_0) \right. \\
 (46) \quad & \left. + \tilde{C}(1, \frac{1}{4}v) \cos \frac{1}{4} v c_0 (t-t_0) \right\} \\
 & + O\left(\frac{1}{v^2 c_0^2 (t-t_0)^2}\right).
 \end{aligned}$$

Since it is frequently convenient to have a formula in dimensionless forms, this is also given below. We shall use the half-length $l = \frac{1}{2}L$ in order to render the ship hull dimensionless, i.e. we introduce

$$(47) \quad \xi = \frac{x}{l}, \quad \zeta = \frac{z}{l}, \quad \tilde{f}\left(\frac{x}{l}, \frac{z}{l}\right) = \frac{1}{l} f(x, z), \quad \kappa = \kappa l, \quad v_0 = v l.$$

The ordinary Froude number $\mathcal{F} = c_0 / \sqrt{g l}$ is related to v_0 by $2v_0 = \mathcal{F}^{-2}$. Further, let

$$(48) \quad \tilde{P}(\lambda, \kappa) + i Q(\lambda, \kappa) = \iint \tilde{f}_{\xi}(\xi, \zeta) e^{-\kappa(\zeta - i\lambda\xi)} d\xi d\zeta,$$

$$\tilde{R} = R / \rho g L^3.$$

Then (46) may be rewritten as follows:

$$\begin{aligned}
 \tilde{R} &= \tilde{R}(\infty) + \\
 &+ \frac{\sqrt{2}}{4\pi} \frac{1}{v c_0 (t-t_0)} r_0 \left[\tilde{P}^2(1, \frac{1}{4} r_0) + \tilde{Q}^2(1, \frac{1}{4} r_0) \right] \left\{ [1 - \tilde{S}(1, \frac{1}{4} v)] \sin \frac{1}{4} v c_0 (t-t_0) \right. \\
 (49) \quad &\left. + \tilde{C}(1, \frac{1}{4} v) \cos \frac{1}{4} v c_0 (t-t_0) \right\} \\
 &+ O\left(\frac{1}{v^2 c_0^2 (t-t_0)^2}\right).
 \end{aligned}$$

It is also sometimes convenient to express the combination

$$v c_0 (t-t_0) = q(t-t_0)/c_0 \quad \text{as}$$

$$(50) \quad v c_0 (t-t_0) = 2 r_0 \frac{c_0 (t-t_0)}{L},$$

for $c_0 (t-t_0)/L$ gives the number of ship lengths travelled after reaching the terminal speed c_0 . Since the length L does not enter naturally into $\tilde{C} + i\tilde{S}$, and since v is not a dimensionless parameter, it is convenient to introduce another dimensionless variable, namely

$$(51) \quad \tau_0 = q t_0 / c_0$$

One may easily confirm that

$$\begin{aligned}
 \tilde{C}(1, \frac{1}{4} v) + i\tilde{S}(1, \frac{1}{4} v) &= \frac{1}{4} \frac{q}{c_0^2} \int_0^{t_0} d\tau C(\tau) \exp i \left[\frac{1}{2} \frac{q}{c_0} (t_0 - \tau) - \frac{1}{4} \frac{q}{c_0^2} \int_{\tau}^t C(\tau') d\tau' \right] \\
 (52) \quad &= \frac{1}{4} \tau_0 \int_0^1 d\bar{\tau} \bar{C}(\bar{\tau}) \exp i \frac{1}{2} \tau_0 \left[1 - \bar{\tau} - \frac{1}{2} \int_{\bar{\tau}}^1 \bar{C}(\bar{\tau}') d\bar{\tau}' \right],
 \end{aligned}$$

where

$$\bar{\tau} = \tau/t_0, \quad \bar{C}(\bar{\tau}) = \frac{1}{c_0} C(t_0 \bar{\tau}),$$

so that it is appropriate in (49) to write

$$S^-(\tau_0) \quad \text{and} \quad C^-(\tau_0).$$

The Initial Wave Resistance

The initial behavior of the second integral in (16) can also be determined through a power-series expansion. The computation of the first three derivatives is straightforward and will not be reproduced here. One finds

$$(53) \quad \frac{4\rho g}{\pi^2} \left[\frac{1}{2} c'(0)t^2 + \frac{1}{6} c''(0)t^3 \right] \int_0^\infty dk k \int_0^1 d\lambda \frac{P^2(\lambda, k) + Q^2(\lambda, k)}{\sqrt{1 - \lambda^2}} + O(t^4).$$

To this must be added the added-mass term, i.e. the first integral in (16).

A Numerical Computation

As an example we shall consider a hull of rectangular section with $H/l = \alpha$ and with parabolic waterlines. In dimensionless representation

$$(54) \quad \tilde{f}(\xi, \zeta) = \beta(\zeta^2 - 1), \quad \beta = b/l = B/L.$$

Then

$$(55) \quad \begin{aligned} \tilde{P}(1, \frac{1}{4}r_0) + i\tilde{Q} &= \int_1' d\xi \int_0^\alpha d\zeta 2\beta\zeta e^{i\frac{1}{4}r_0\xi} e^{-\frac{1}{4}r_0\zeta} \\ &= -i \frac{64\beta}{r_0^2} [1 - e^{-\frac{1}{4}r_0\alpha}] [\cos \frac{1}{4}r_0 - \frac{4}{r_0} \sin \frac{1}{4}r_0]. \end{aligned}$$

We also need some assumption about the initial part of the velocity function $C(t)$. We shall assume that

$$(56) \quad C(t) = \begin{cases} \frac{C_0}{t_0} t, & 0 \leq t \leq t_0 \\ C_0, & t \geq t_0 \end{cases}$$

Then, entering (52) with $\bar{C}(\bar{\tau}) = \bar{c}$, we find

$$(57) \quad \begin{aligned} \bar{C}(\tau_0) + i\bar{S}(\tau_0) &= \frac{1}{4}\tau_0 \int_0' d\bar{\tau} \bar{\tau} \exp i \frac{1}{8}\tau_0 [(\bar{\tau}-2)^2 - 1] \\ &= \frac{1}{i} e^{-i\frac{1}{8}\tau_0} \int_0' d\bar{\tau} \frac{i}{4}\tau_0 (\bar{\tau}-2) \exp [i\frac{1}{8}\tau_0 (\bar{\tau}-2)^2] \\ &\quad + \frac{1}{2}\tau_0 e^{-i\frac{1}{8}\tau_0} \int_0' d\bar{\tau} \exp [i\frac{1}{8}\tau_0 (\bar{\tau}-2)^2] \\ &= -i e^{-i\frac{1}{8}\tau_0} [e^{i\frac{1}{8}\tau_0} - e^{i\frac{1}{2}\tau_0}] + \sqrt{2}\tau_0 e^{-i\frac{1}{8}\tau_0} \int_{\frac{\sqrt{2}}{\sqrt{8}}}^{\frac{\sqrt{2}}{2}} d\sigma e^{i\sigma^2} \end{aligned}$$

$$= -\sin \frac{3}{8} \tau_0 - i \left(1 - \cos \frac{3}{8} \tau_0\right) \\ + \sqrt{\pi} \tau_0 \left[\cos \frac{1}{8} \tau_0 - i \sin \frac{1}{8} \tau_0 \right] \left[C\left(\frac{1}{2} \tau_0\right) - C\left(\frac{1}{8} \tau_0\right) + i \left(S\left(\frac{1}{2} \tau_0\right) - S\left(\frac{1}{8} \tau_0\right) \right) \right],$$

where $C(x)$ and $S(x)$ are the Fresnel integrals*

$$C(x) + iS(x) = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{x}} e^{i\sigma^2} d\sigma.$$

From (49) we may write

$$R(t) - R(\infty) = \frac{\sqrt{2}}{4\pi} \rho q L^3 r_0 [\tilde{P}^2 + \tilde{Q}^2] \sqrt{(1-S)^2 + (C)^2} \frac{1}{v c_0 (t-t_0)} \sin \left[\frac{1}{4} v c_0 (t-t_0) + \delta \right] + O(\quad) \\ (58) \quad = \frac{\sqrt{2}}{8\pi} \rho q L^3 [\tilde{P}^2 + \tilde{Q}^2] \sqrt{(1-S)^2 + (C)^2} \frac{1}{c_0 (t-t_0)/L} \sin \left[\frac{r_0}{2} \frac{c_0 (t-t_0)}{L} + \delta \right] + O(\quad) \\ = A(r_0, \tau_0) \frac{1}{c_0 (t-t_0)/L} \sin \left[\frac{r_0}{2} \frac{c_0 (t-t_0)}{L} + \delta \right] + O\left(\frac{1}{v^2 c_0^2 (t-t_0)^2}\right),$$

where

$$\cot \delta = \frac{1-S}{C}.$$

* These are the functions which are tabulated as $C(x)$ and $S(x)$ in Jahnke, Emde and Lösch, Tables of Higher Functions, 6th ed., McGraw-Hill, New York, Teubner, Stuttgart, 1960, pp. 34-35. The functions

$$C_F(u) + iS_F(u) = \int_0^u e^{i\frac{\pi}{2}t^2} dt$$

are also called Fresnel integrals and are more extensively tabulated than those used here. The functions are related by

$$C(x) + iS(x) = C_F\left(\sqrt{\frac{2}{\pi}}x\right) + iS_F\left(\sqrt{\frac{2}{\pi}}x\right).$$

Since

$$A(r_0, \tau_0) = \rho g L^3 \beta^2 M(r_0, \alpha) N(\tau_0),$$

$$\beta^2 M_0(r_0, \alpha) = \frac{\sqrt{2}}{8\pi} [\tilde{P}^2 + \tilde{Q}^2], \quad N(\tau_0) = \sqrt{(1-S)^2 + (C)^2},$$

it will be convenient to calculate separately $M(r_0, \alpha)$, $N(\tau_0)$ and $\delta(\tau_0)$. Figure 1 shows a graph of $M(r_0, 0.1)$ for the special choice of hull form (54). Figures 2 and 3 show the functions $N(\tau_0)$ and $\delta(\tau_0)$ for the special choice of acceleration (56).

In order to make the example more specific, one must further specify the hull form and the function $C(t)$. We shall assume that $L = 20$ ft., $\beta = 0.1$ and $2H/L = 0.1$. Three separate values of the Froude number were considered: $\mathcal{F} = 0.200, 0.250, 0.316$ or $V_0 = 12.5, 8, 5$, respectively. For a 20 foot model these correspond to $C_0 = 5.06, 6.32$ and 8.00 ft/sec respectively. According to information supplied by the towing tank (Skipsmodelltanken) in Trondheim these values of C_0 correspond to the following values of t_0 : $t_0 = 6.8, 7.4$ and 9.3 sec, respectively. These values yield in turn $\tau_0 = 43.5, 39.5$ and 37.2 , respectively. Since the carriage does not really have a constant acceleration up to the final value C_0 , one cannot, of course, take the specific values of τ_0 as indicating more than an approximate value. Fortunately, as reference to Figure 2 shows, the curve $N(\tau_0)$ is nearly flat in the region $30 < \tau_0 < 45$ and has approximately the value 0.2. This value

was used for $N(\tau_0)$ for all three cases. Values of δ were read from Figure 3. The computed oscillating part of the resistance is plotted in Figure 4 against $C_0(t-t_0)/L$. It is evident that the oscillatory behavior dies out only slowly and still has an amplitude between 0.05 and 0.10 lbs at 14 model lengths after attainment of the final speed. In order to show the amounts relative to the asymptotic values $R(\infty)$, the values of $R(\infty)$ were taken from Weinblum's Tables [1955]. They are $R(\infty) = 1.73, 3.50$ and 9.37 lbs, respectively. Figure 5 shows $R(\infty)/\rho g L^3 \beta^2$ plotted against τ_0 ; the scales are such that Figure 4 and Figure 1 may be directly compared to ascertain the relative importance of $M \cdot N$ and $R(\infty)/\rho g L^3 \beta^2$. Figure 6 shows the three resistance curves of Figure 5 in their proper places in a total wave resistance plot. The initial parts of the curves, i.e. equation (53), have not been plotted, for the assumed form of $C(t)$ in the region $0 < t < t_0$ is only a rather crude approximation to one which might occur with a real towing carriage. The effect of this discrepancy upon equation (52) will not be very large, for $C(t)$ enters always under an integral sign. The situation is, however, quite the reverse with equation (53) describing the initial wave resistance, where the derivatives $C'(0), C''(0)$, etc. enters. It is obvious that the assumed form of $C(t)$ for the initial region cannot provide useful information about this part of the resistance curve.

Some Conclusions

Although the numerical example has been worked out for a particularly simple hull shape and acceleration law, it is obvious, since the size of the computed oscillations are not negligible in comparison with the usual standards of accuracy attained in towing tests, that one may expect in a model towing test to experience this oscillation for a good part of the test run. Such oscillations have, in fact, been observed, and the data presently available show a very close agreement between predicted and observed frequency of oscillation. The theory does not indicate any way of avoiding the oscillations, but does show, as one might expect, that the usual procedure of taking the mean line through the oscillations is the correct one. It is possibly also of some comfort to know that these oscillations are naturally present and do not result from improper functioning or design of the equipment.

Finally, we recall a fundamental assumption, namely, that the velocity is prescribed and the force measured. For a constant-thrust towing apparatus in which the thrust is prescribed and the velocity measured one would not expect to observe an oscillation in velocity of the same proportional amplitude. The following computation is somewhat crude, but seems sufficiently convincing. We suppose that the oscillatory part of the velocity may be found approximately from

$$\frac{W}{g} \ddot{x} = A_0 \cos \omega t,$$

where W is the weight of the model plus attached apparatus. This equation neglects, of course, added-mass and damping forces and the fact that A is a function of t .

Then

$$\dot{x} = \frac{q}{W} A_0 \frac{1}{\omega} \sin \omega t = \frac{q}{W} A_0 \frac{4C_0}{q} \sin \omega t.$$

The ratio of the amplitude of the velocity oscillation to the mean velocity C_0 is $4A_0/W$. After consulting Figure 4, a liberal estimate for A_0 for the model considered earlier is $A_0 = 0.25$ lbs. For W we shall take 2500 lbs. Then $4A_0/W = 0.0004$, a negligible amount. Although a more precise estimate can be carried through, it is doubtful that it would change the conclusion.

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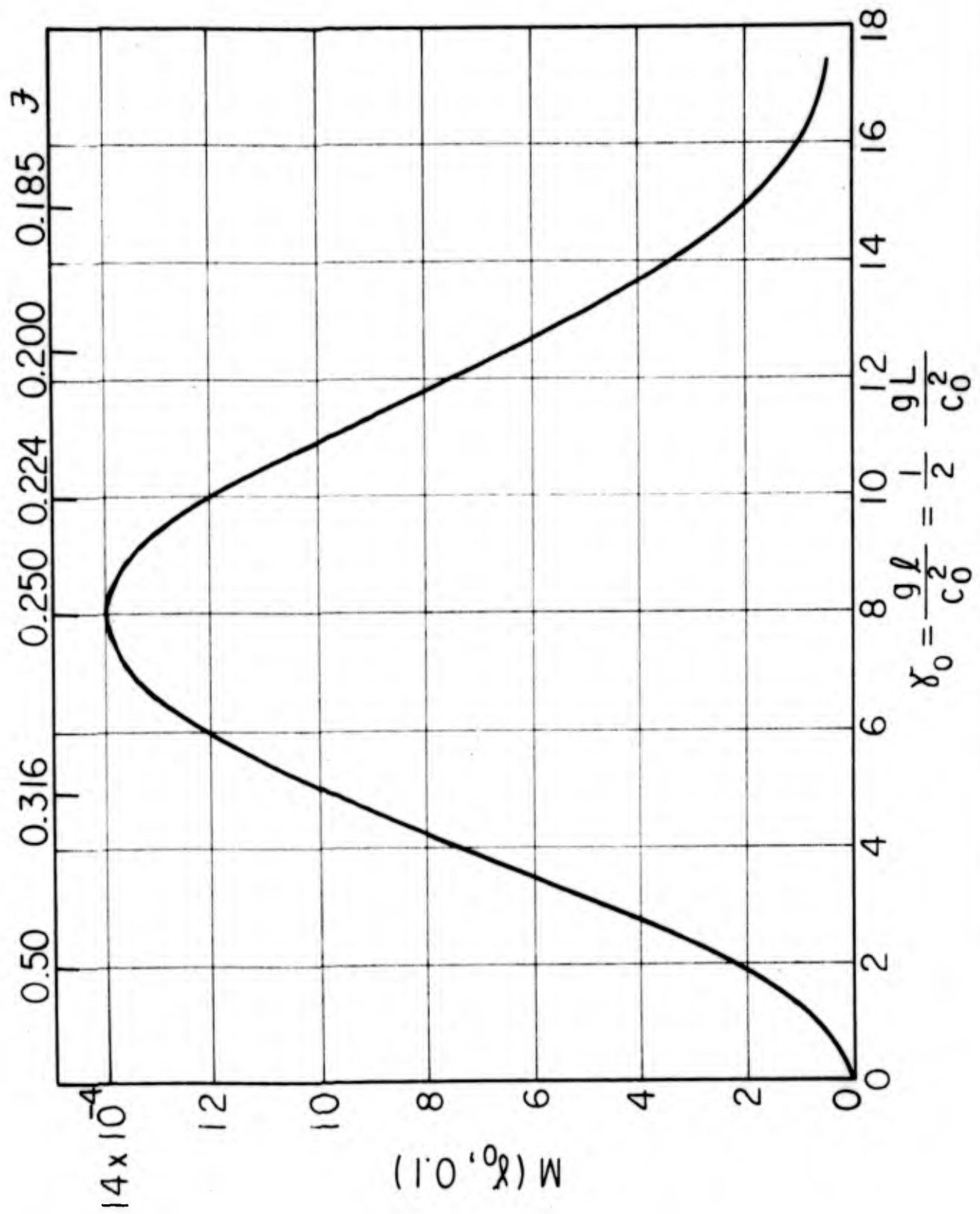


FIG. 1

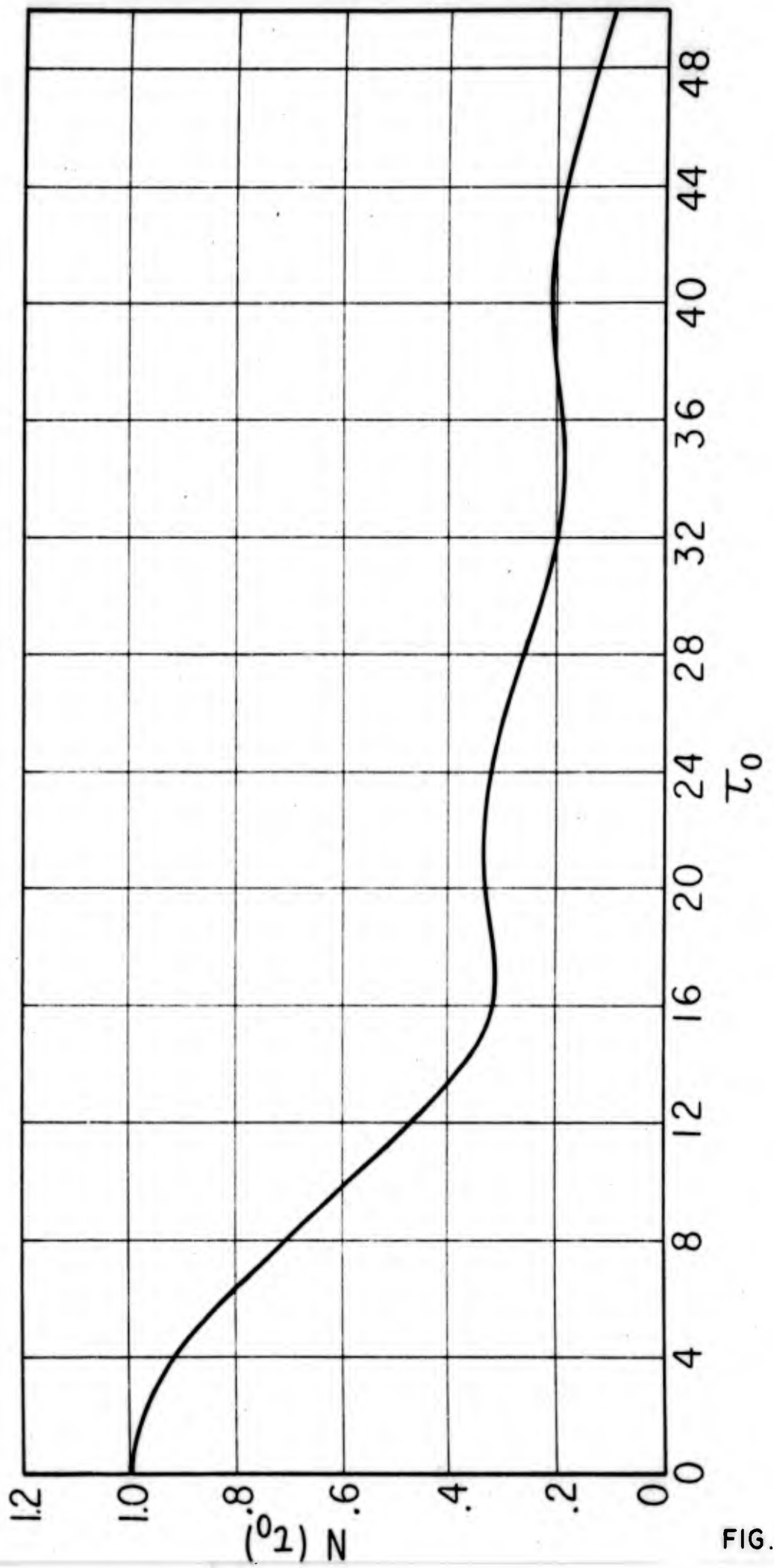


FIG. 2

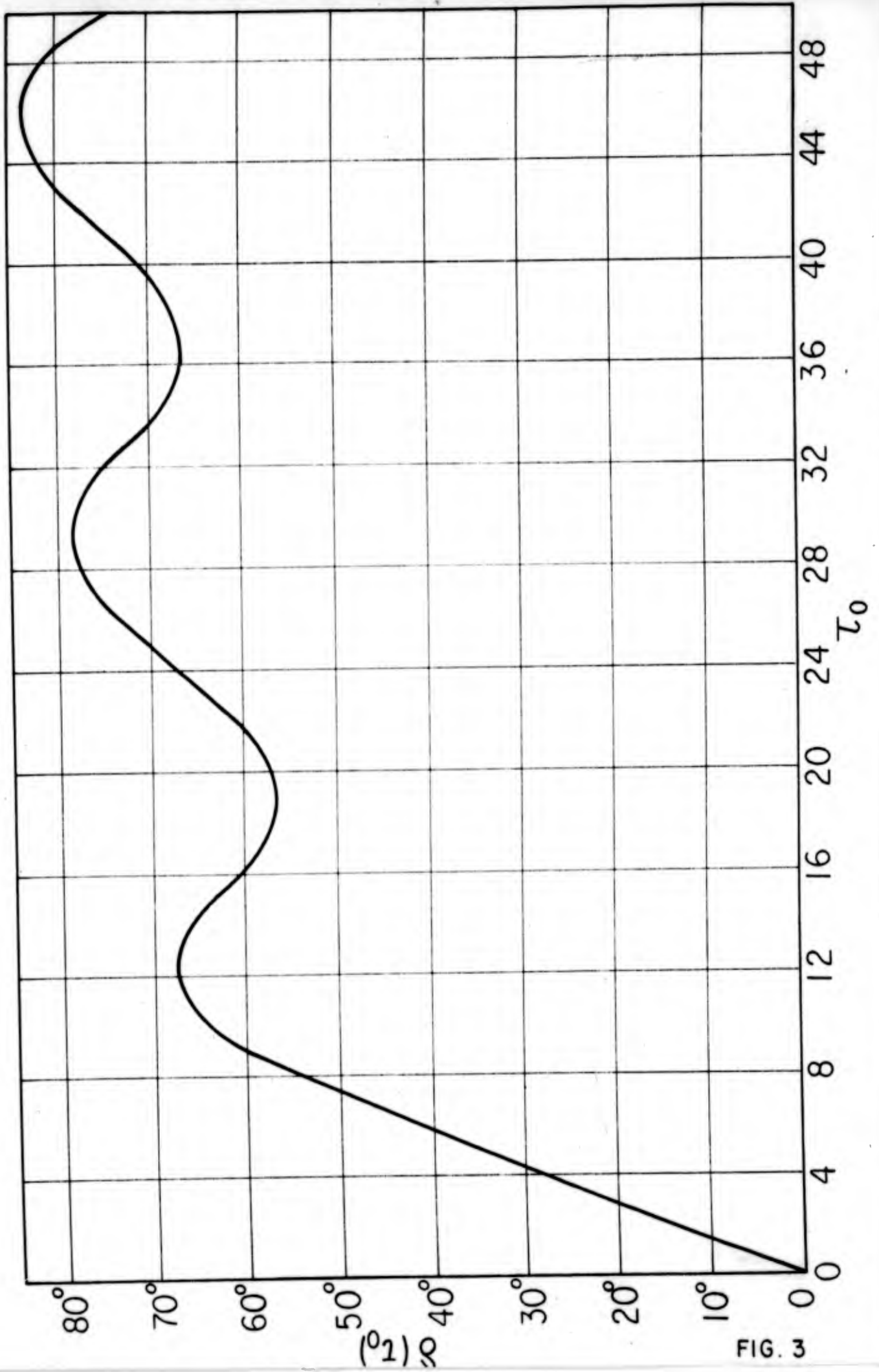


FIG. 3

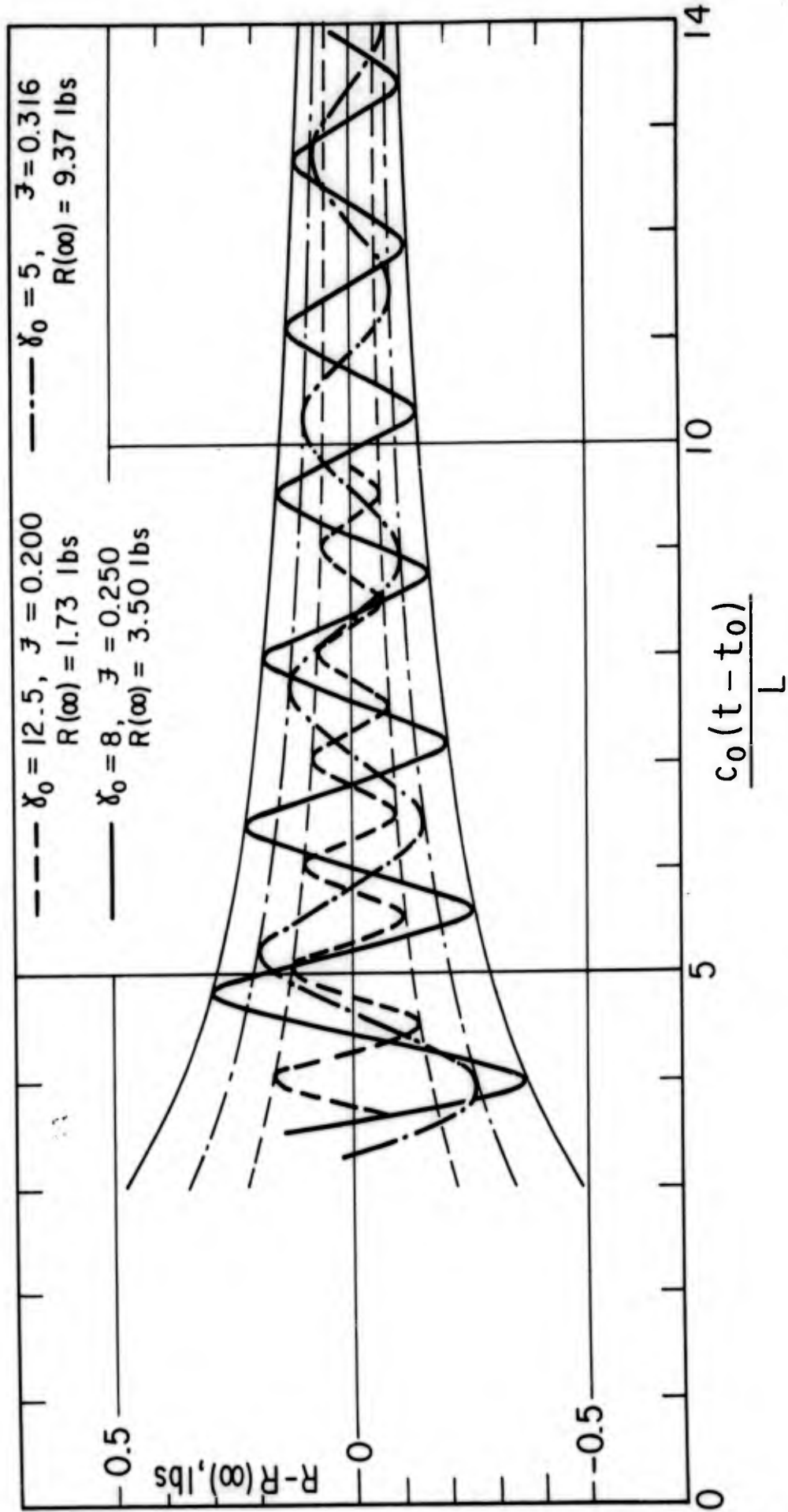


FIG. 4

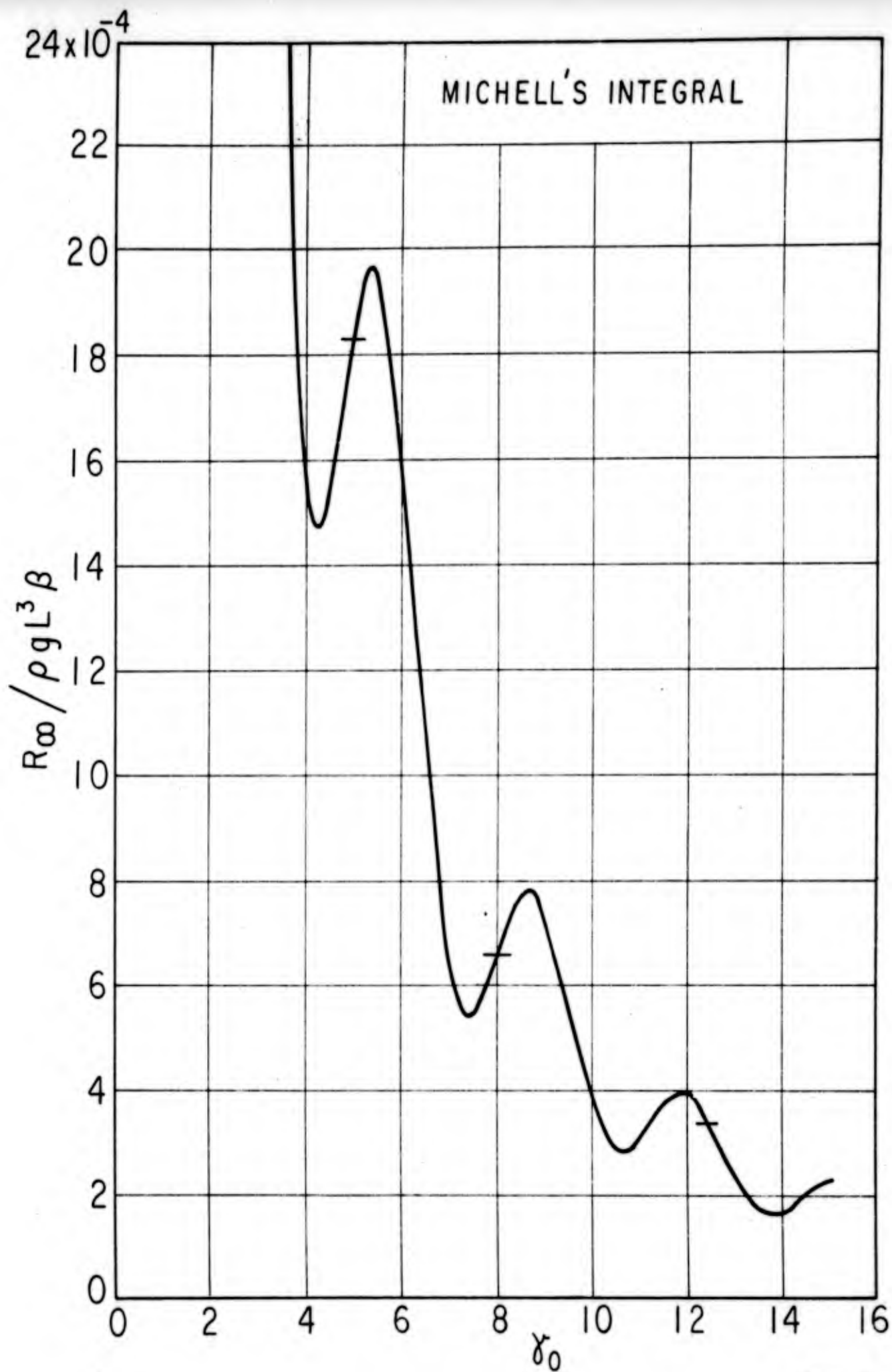


FIG. 5

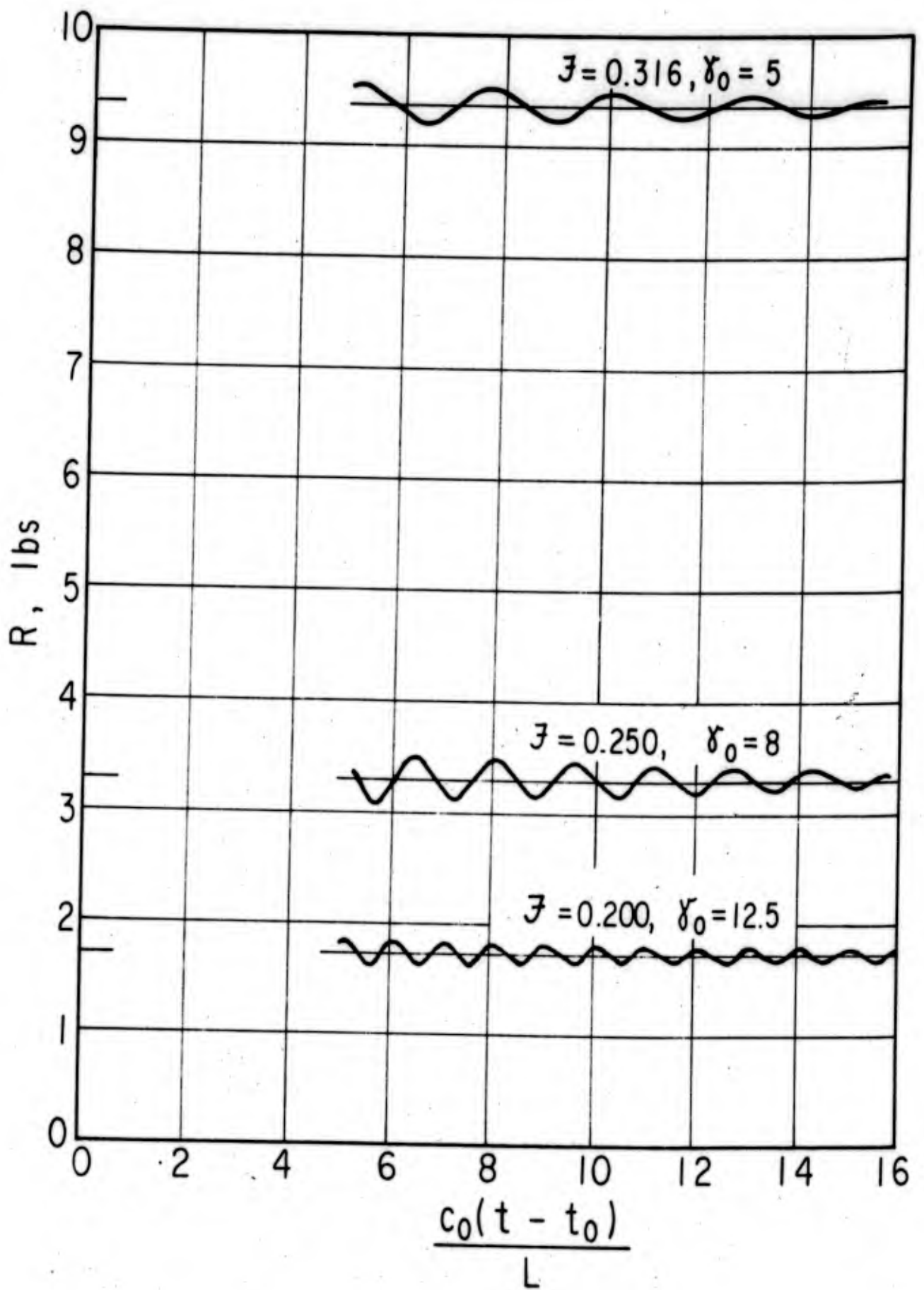


FIG. 6

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