WADD TECHNICAL REPORT 61-177

STRUCTURAL SAFETY UNDER CONDITIONS OF ULTIMATE LOAD FAILURE AND FATIGUE

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OCTOBER 1961

AERONAUTICAL SYSTEMS DIVISION

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Unclassified report

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OCTOBER 1961

DIRECTORATE OF MATERIALS AND PROCESSES
CONTRACT No. AF 33(616)-7042
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AERONAUTICAL SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
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FOREWORD

This report was prepared by the Department of Civil Engineering and Engineering Mechanics of Columbia University, under USAF Contract No. AF 33(616)-7042. The contract was initiated under Project No. 7351, "Metallic Materials", Task No. 73521, "Behavior of Metals". The work was administered under the direction of the Directorate of Materials and Processes, Deputy for Technology, Aeronautical Systems Division, with Mr. D. M. Forney, Jr. acting as project engineer.

This report covers the period of work 1 February 1960 to 31 November 1960.

The cooperation and continued interest of Mr. D. M. Forney, Jr. is gratefully acknowledged.

ABSTRACT

The purpose of this investigation is to analyze the concept of the safety of structures subject to operational loads that cause fatigue damage as well as to occasional excessive overloads that might produce ultimate load failure.

In Part I the relation between probability of failure and the reliability or the safety factor is discussed. Diagrams have been computed under the assumptions that the statistical variations of load and carrying capacity are expressed either by log-normal or by extremal distributions. The safety of multiple load-path structures, the probability of failure of simple structures under combined (primary and secondary) loads are also considered and the use of separate load factors for dead and live load is related to the concept of a single safety factor.

Part II deals mainly with the statistical properties of fatigue life distributions. Assuming a statistical-mechanical model for the fatigue mechanism, a new distribution of fatigue lives is derived. The concept of stress-interaction established in previous experimental research is used to reproduce the survivorship functions under random loading from the known survivorship functions associated with constant stress amplitude fatigue.

In Part III the risks of ultimate load and fatigue failures are combined and the reliability of aluminum specimens (AA 2024 Al) under both operational loads and occasional excessive overloads is investigated considering the interrelation with the risk-functions. The procedure is illustrated by a numerical example in which the truncated part of an exponential load spectrum is applied as operational (fatigue) loading while the rest of the spectrum produces the overloads.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:

W. J. TRAPP
Chief, Strength and Dynamics Branch
Metals and Ceramics Laboratory
Directorate of Materials and Processes

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a	=	$(\nu/\nu_0)^{\alpha}$ or $(\nu/\nu_0)^{\beta}$: appears in the approximated form of $F_\nu(\nu)$ associated with extremal distribution of R and S
$a', a'', \bar{a}$	=	constants
A	=	cross sectional area of circular column
A_1, A_N	=	initial and momentary resisting area of specimen
b	=	α/β or β/α : appears in the approximated form of $F_\nu(\nu)$ associated with extremal distribution of R and S
b', b'', c, c', C_j, d	=	constants
D	=	$R-S$ difference between resistance of structure and applied load
$f_X(x)$	=	probability density function of any quantity X
$f(q)(f(q))$	=	difference between design load S_q (specified resistance R_p) and mean $\bar{S}(\bar{R})$ of the distribution of $S(R)$ in terms of standard deviation $\sigma_S(\sigma_R)$.
$f(S_i), f^*(S_i)$	=	stress effect functions without and with interaction
$f_{S_1}(S)$	=	truncated distribution of load intensity
$f_{S_2}(S)$	=	distribution of load intensity representing the extremal part of load spectrum
$f(W,\theta,\phi)$	=	joint probability density function of W, θ and ϕ
$F_X(x)$	=	probability distribution function of any quantity X
$\bar{F}_X(x)$	=	$1 - F_X(x)$
$g(\Delta)$	=	history function
$g(W^*,\theta,\phi)$	=	joint probability density function of W^*, θ and ϕ
$G(\Delta)$	=	$\int_0^\Delta d\Delta/g(\Delta)$
$G(\Delta^*)=G^*$	=	critical value of $G(\Delta)$
$G_e(\Delta)$	=	function of Δ (analogous to $G(\Delta)$) associated with extremal distribution of fatigue life
h, h'	=	parameters of exponential distributions of load intensity
k', k	=	ratios of design load S_q and specified resistance R_p to mean $\bar{S}_d$ of the distribution of dead load S_d

ℓ	=	height of circular column
$L(N), L'(N)$	=	survivorship functions
$\overline{L(N_k)}$	=	mean value of $L(N_k)$ associated with k-th value of fatigue life N_k among a set of observations arranged in ascending order
$\mathcal{L}[f(t)]$	=	the Laplace transform of $f(t)$
m_i, m_i^*	=	means of stress effect function without and with interaction
m_{i0}, m_0	=	m_i/G^*
m_{i0}^*, m_0^*	=	m_i^*/G^*
$m_{i,max}$	=	value of m_i associated with maximum stress level $S_{i,max}$ in the spectrum
$\bar{m}_R$	=	quantity such that $N\bar{m}_R$ gives mean of the distribution of $G(\Delta)$ associated with random fatigue
$\bar{m}_{R0}$	=	$\bar{m}_R/G^*$
$\bar{m}_R'$	=	quantity analogous to $\bar{m}_R$ appearing in the distribution of $G_e(\Delta)$
N	=	number of load applications
N^*	=	reduced operational life in terms of number of load applications
N_0	=	minimum fatigue life
N'	=	number of load applications measured from the instant of the occurrence of the first failure in members of a redundant structure
N_i, N_i'	=	fatigue life at constant stress amplitude S_i without and with interaction
$\bar{N}_i, \check{N}_i$	=	mean and median of the distribution of fatigue life N_i
N_j	=	interval in terms of number of load applications between the (j-1)-th and j-th failures in members of a redundant structure
N_R	=	fatigue life associated with random fatigue
$\check{N}_R$	=	median of the distribution of N_R
p	=	probability of resistance R being smaller than specified value R_p
p_i	=	frequency ratio of cycles of stress amplitude S_i in spectrum

p_j = probability of failure of a member when $n-j+1$ members still exist in a redundant structure consisting of n nominally identical members

p_k = probability of failure of k-th member of a non-redundant structure

p_t = probability of the occurrence of load intensity belonging to extremal part of load spectrum

$P(\nu)$ = $F_\nu(\nu)$: probability distribution of ν

$P_f, P_f(N)$ = probability of failure

q = probability of load S being larger than design load S_q

$Q_j(\omega), Q(\omega)$ = characteristic functions

r_p = $R_p/\check{R}$ or $R_p/\tilde{R}$: ratio of specified resistance to central value of the distribution of resistance R.

$r(N), r'(N)$ = risk functions

R = resistance of structure

$\bar{R}, \check{R}$ = mean and median of the distribution of resistance R

$\tilde{R}$ = characteristic value of the extremal distribution of resistance R

R_1, R_N = initial and momentary static resistance of structure

$\bar{R}_N$ = mean of the distribution of momentary static resistance R_N

$R_p = R_{min}$ = specified resistance of structure

R_N^* = critical static resistance of structure

s_i = stress ratio obtained by dividing respective stress by ultimate strength S_u

$s_{i,min}$ = stress ratio associated with minimum stress level $S_{i,min}$

s_q = $S_q/\check{S}$ or $S_q/\tilde{S}$: ratio of design load to central value of the distribution of S

S = applied load acting on structure

$\bar{S}$, $\check{S}$ = mean and median of the distribution of load intensity S

$\tilde{S}$ = characteristic value of the extremal distribution of load intensity S

S^*	=	non-statistical applied load
S_{max}	=	design load or load intensity dividing load spectrum into operational and extremal part
S_{min}	=	minimum load intensity in load spectrum
S_d, S_ℓ	=	dead and live load
S_d^*, S_ℓ^*	=	quantities equal to design dead and design live load multiplied by associated load factors
$\bar{S}_d$, $\bar{S}_\ell$	=	means of the distributions of dead load S_d and live load S_ℓ
$S_{d,q}$, $S_{\ell,q}$	=	design dead and design live load associated with design load S_q
S_{i_k}	=	stress intensity S_i (of i-th stress level) appearing at k-th in the sequence of load applications
$S_{i,min}$, $S_{i,max}$	=	minimum and maximum stress level in load spectrum
S_u	=	ultimate strength
t	=	$\frac{1-Nm_{i0}}{\sqrt{N}\,\sigma_{i0}}$: reduced variable which transforms the proposed distribution of fatigue life N into normal distribution
t'	=	$\frac{N-N_0}{\bar{N}-N_0}$ or $\frac{N-N_0}{V-N_0}$: reduced variable in the distribution of fatigue life
T_R, T_R^*	=	return numbers
V_R, V_S, V_d, V_ℓ	=	coefficients of variation of the distributions of resistance R, of applied load S, of dead load S_d and of live load S_ℓ
$V(V_S, V_R)$	=	characteristic value of the extremal distribution of fatigue life (associated with constant amplitude and random fatigue)
$\vec{W}$, W	=	total load acting on circular column and its absolute value
$\vec{W}_p$, W_p	=	primary load acting on circular column and its absolute value
$\vec{W}_d$, W_d	=	dead load of circular column and its absolute value
$\vec{W}_\ell$	=	live load acting on circular column
W_{des}	=	design load for primary load
$W^*(\theta, \phi)$	=	critical load in the direction of (θ, ϕ) in circular column
Z	=	section modulus
α	=	scale parameter of the extremal distribution of resistance R

List of Symbols (continued)

α_i, α_R	= scale parameters of the extremal distributions of fatigue life associated with constant stress level and random fatigue
α_q	= load factor for dead load associated with design load S_q
β	= scale parameter of the extremal distribution of load intensity S
β_q	= load factor for live load associated with design load S_q
γ	= $\bar{S}_\ell/\bar{S}_d$: ratio of mean of the distribution of live load to mean of the distribution of dead load
$\Gamma(x)$	= $\int_0^\infty e^{-u} u^{x-1} du$: Gamma function
δ	= standard deviation of the distribution of the difference between log R and log S
δ_N	= standard deviation of the distribution of log $(N - N_0)$
δ_R, δ_S	= standard deviations of the distributions of log R and log S.
Δ	= accumulated fatigue damage
Δ^*	= critical fatigue damage
$\bar{\Delta}$	= $1 - \Delta$
ε	= $\sigma_0/\sqrt{m_0}$: parameter of the proposed distribution of fatigue life with reduced variable t'
$\vec{\zeta}, \vec{\zeta}_H, \vec{\zeta}_V$	= acceleration due to earthquake and its horizontal and vertical component
ζ_H, ζ_V	= magnitude of $\vec{\zeta}_H$ and $\vec{\zeta}_V$
$\zeta_{H_{des}}$	= design value for horizontal acceleration due to earthquake
η	= reciprocal of slope of log S - log m_0^* diagram
θ	= angle (Fig. 9)
κ, κ'	= constants
λ	= constant
μ_k, μ_k'	= k-th moment of a statistical variable
ν	= R/S : ratio of resistance of structure to applied load
$\bar{\nu}$	= "conventional" safety factor
ν_0	= $\check{R}/\check{S}$ or $\tilde{R}/\check{S}$: central safety factor

List of Symbols (continued)

ν', ν'_N	= R_1/S, R_N/S: ratios of initial and momentary static resistance to load intensity of extremal part of load spectrum
ξ	= reciprocal of slope of log S - log m_0 diagram
ρ	= radius of cross-section of circular column
σ	= standard deviation of the distribution of the difference between resistance and load
$\sigma_2, \sigma_3, \sigma_{23}$	= standard deviations of the distributions of stresses τ_2, τ_3 and their sum
σ_H, σ_V	= standard deviations of the distributions of ζ_H and ζ_V
σ_d, σ_ℓ	= standard deviations of the distributions of dead and live load
σ_i, σ_i^*	= standard deviations of stress effect function without and with interaction
σ_{i0}, σ_0	= σ_i/G^*
$\sigma_{i0}^*, \sigma_0^*$	= σ_i^*/G^*
$\sigma_{N_i}, \sigma_{N_R}$	= standard deviations of the distributions of fatigue lives N_i and N_R
σ_R, σ_S	= standard deviations of the distributions of resistance R and applied load S
$\bar{\sigma}_R$	= quantity such that $N\bar{\sigma}_R$ gives standard deviation of the distribution of $G(\Delta)$ associated with random fatigue
$\bar{\sigma}_{R0}$	= $\bar{\sigma}_R/G^*$
σ_{R_N}	= standard deviation of the distribution of momentary static resistance R_N
$\tau = \tau_E$ or τ_F	= stress at critical point E or F (Fig. 10) in circular column
τ_1, τ_2, τ_3	= stresses at bottom section of circular column due to primary load, due to vertical and horizontal acceleration caused by earthquake
τ_0	= material strength parameter
τ_{a0}, τ_a	= allowable stresses associated with primary load and combined load
τ_d	= W_d/A: stress due to dead load in circular column
τ_C^*, τ_T^*	= critical compressive and tensile stress
ϕ	= angle (Fig. 9)

$\phi_N(\Delta)$	=	monotonically decreasing function of Δ
$\Phi(t)$	=	$\int_{-\infty}^{t} e^{-u^2/2}\,du$: normal distribution function
$\bar{\Phi}(t)$	=	$1 - \Phi(t)$
$\overline{\bar{\Phi}(t_k)}$	=	mean of $\bar{\Phi}(t_k)$
ω_i, ω_i'	=	stress interaction factors associated with mean m_i and standard deviation σ_i of stress effect function $f(S_i)$.

I. INTRODUCTION

The probabilistic interpretation of the concept of structural safety and the statistical approach to the determination of safety factors has been developed in recent years in this country [1] as well as in several European countries [2]. This work has been essentially concerned with safety with respect to a single (ultimate) load application.

The increasing severity of the operational conditions of modern, dynamically loaded structures combined with the use of structural materials of higher "static" strength but reduced fatigue resistance has gradually changed the emphasis from ultimate load design to design for fatigue. The necessity has therefore arisen to extend the statistical approach for structural safety to the fatigue design of mechanical systems, or rather to develop a statistical procedure of safety analysis for the combination of ultimate load and fatigue for which such systems, whether they are structures, machine parts or mechanical parts of control systems, must be designed.

The considerable improvement and refinement in the methods of stress-analysis of mechanical systems made possible by the increased use of computers, is in no way matched by improvement in the safety analysis, which, in general, is still based on the use of more or less arbitrary safety factors that can neither be justified by rational argument nor related to a probability of failure. It is the purpose of this study to develop, on the basis of recent work concerned mainly with ultimate load failure [3], the principles of safety analysis for structures subject to fatigue under operational loads as well as to an occasional excessive overload.

1. Probability of Failure. Let S denote the statistical population of load intensities that can be expected to act on a structure, with probability density $f_S(S)$ and probability function $F_S(S)$, and R the carrying capacity (resistance) of the population of nominally identical structures, with probability density $f_R(R)$ and probability function $F_R(R)$. Then the probability of failure is

$$P_f = P(R<S) = P(R-S<0) = P(R/S<1) \tag{1.1}$$

provided $S > 0$. This probability is the stochastic limit of the proportion of structures out of the population of resistances $F_R(R)$, which will fail when a load selected at random from the population $F_S(S)$ is applied to a structure selected at random from the population $F_R(R)$.

The probability function $F_D(D)$ of the difference $D = (R-S)$ is evaluated as the marginal density of the joint distribution of D and R or of D and S. The probability of failure $P_f = F_D(0)$. Hence,

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$$P_f = \int_0^\infty F_R(S) f_S(S) dS = \int_0^\infty \bar{F}_S(R) f_R(R) dR \tag{1.2}$$

where $(S, R > 0)$, and $\bar{F}_S(R) = 1 - F_S(R)$.

Similarly the probability function $F_\nu(\nu)$ of the quotient $\nu = R/S$ is evaluated as the marginal density of the joint distribution of ν and S [4]:

$$F_\nu(\nu) = \int_0^\infty F_R(\nu S) f_S(S) dS \tag{1.3}$$

and therefore

$$P_f = F_\nu(1) = \int_0^\infty F_R(S) f_S(S) dS \tag{1.4}$$

an expression that is obviously identical with the second term of Eq. (1.2), since by definition $F_D(0) = F_\nu(1)$.

2. Probability of Survival: Reliability. The probability of failure P_f is not a measure of the safety of a structure subject to a random sequence of loads from the population $F_S(S)$ during its period of service. Such a measure is its probability of survival $L(N)$ under the N load applications constituting the sequence and is given by

$$L(N) = (1 - P_f)^N \tag{2.1}$$

provided P_f is independent of N. Physically this limitation defines conditions of failure under a single application of "ultimate" load such that the resistance of the structure is unaffected by any load preceding the "ultimate" load that causes failure.

Since $P_f \ll 1$, Eq. (2.1) can be approximated by

$$L(N) \doteq \exp(-NP_f) \quad \text{or} \quad \ln L(N) \doteq -NP_f \tag{2.2}$$

which is the well-known survivorship or "reliability" - function for chance failures; $(1/P_f) = T_R$ represents the "return number" of failures based on the condition $R < S$.

When the resistance of the structure decreases with increasing number of load-applications ("fatigue") or with the duration of loading ("creep-rupture"), $F_R(R)$ is a function of N or of time. (Similarly, $F_S(S)$ may be a function of N if a definite trend of change in service conditions with age must be expected.) Therefore P_f becomes a function of N (or of time) and Eq. (2.1) is replaced by

$$L(N) = \prod_{n=1}^{N} \left[1 - P_f(N) \right] \quad , \tag{2.3}$$

or, since $P_f(n) \ll 1$

$$L(N) = \exp\left[- \sum_{n=1}^{N} P_f(n) \right] . \qquad (2.4)$$

Thus, the reliability function $L(N)$ is related to the change of $F_R(R)$ with N through its effect on $F_\nu(\nu)$ and $F_\nu(1) = P_f$, both of which according to Eqs. (1.3) and (1.4) are functions of N. $L(N)$ is the stochastic limit of the proportion of structures not failing under sequences of N randomly applied loads S each of which contributes to the gradually accumulating damage.

The relation between $L(N)$ and $P_f(N)$ is obtained from Eq. (2.4):

$$- d\ln L(N)/dN = P_f(N) \qquad (2.5)$$

or

$$L(N) = \exp\left[- \int_0^N P_f(n)dn \right] . \qquad (2.6)$$

In general,

$$r(N) = - d\ln L(N)/dN \qquad (2.7)$$

is called risk function (or intensity function [5]).

II. SAFETY ANALYSIS FOR FAILURE UNDER ULTIMATE LOAD.

Methods of safety analysis for failure under ultimate load are developed by assuming independence of $F_R(R)$ and $F_S(S)$ of each other as well as of N, and by introducing the two forms of the functions $F_R(R)$ and $F_S(S)$ that have been found to provide the best representation of observed statistical distributions of relevant material properties and of random loading: the logarithmic-normal distribution and the asymptotic distributions of extreme values.

3. Logarithmic-Normal Distributions. Introducing the following functions

$$F_R(R) = \Phi\left(\frac{\log R/\check{R}}{\delta_R}\right) \qquad (3.1)$$

and

$$F_S(S) = \Phi\left(\frac{\log S/\check{S}}{\delta_S}\right) \qquad (3.2)$$

where

$$\Phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp(-u^2/2)du , \qquad (3.3)$$

$\check{R}$ and $\check{S}$ denote medians of R and S and $\delta_R = \sigma(\log R)$ and

$\delta_S = \sigma(\log S)$ are the standard deviations of log R and log S respectively, it is obvious that the distribution of $\log \nu = \log (R/S) = \log R - \log S$ is normal (since the distributions of log R and of log S are normal) with mean $\log (\check{R}/\check{S})$ and standard deviation $\delta = \sqrt{\delta_R^2 + \delta_S^2}$. Hence,

$$F_\nu(\nu) = \Phi[(\log \nu/\nu_0)/\delta] \tag{3.4}$$

where $\nu_0 = \check{R}/\check{S}$ is the "central" value of the safety factor, and therefore

$$P_f = F_\nu(1) = \Phi[-(\log \nu_0)/\delta]. \tag{3.5}$$

It is expedient to establish the relations between the measures of dispersion of the logarithmic-normal distribution δ_R and δ_S and the coefficients of variation $v_R = \sigma_R(R)/\bar{R}$ and $v_S = \sigma_S(S)/\bar{S}$ which are usually known for actual observations where $\bar{R}$ and $\bar{S}$ denote the means of R and S.

It can be shown that

$$v_R^2 = \exp (\bar{a}\,\delta_R)^2 - 1, \quad v_S^2 = \exp (\bar{a}\,\delta_S)^2 - 1 \tag{3.6}$$

where $\bar{a} = \ln 10 = 2.3026$. The relations

$$(\sigma_R/\check{R})^2 = v_R^2 \exp (\bar{a}\,\delta_R)^2, \quad (\sigma_S/\check{S})^2 = v_S^2 \exp (\bar{a}\,\delta_S)^2 \tag{3.7}$$

permit the conversion of the coefficients of variation into the ratios $(\sigma_R/\check{R})$ and $(\sigma_S/\check{S})$ based on the medians.

In order to illustrate the safety analysis, ratios $(\sigma_R/\check{R}) = 0.05$, 0.10 and 0.15 as well as $(\sigma_S/\check{S}) = 0.10$, 0.20 and 0.30 have been assumed. Table 1 presents the associated values of $\delta_R(\delta_S)$ and $v_R(v_S)$, Table 2 the resulting values δ for various combinations of $(\sigma_R/\check{R})$ and $(\sigma_S/\check{S})$.

Using Eq. (3.4), the functions $F_\nu(\nu)$ for various combinations of $(\sigma_R/\check{R})$ and $(\sigma_S/\check{S})$ have been evaluated with ν_0 as parameter and presented in Figs. 1(a) to 1(i) where $P(\nu) = F_\nu(\nu)$ on logarithmic scale have been plotted against ν. The probability density of $\log \nu$ for $\sigma_R/\check{R} = \sigma_S/\check{S} = 0.10$ is shown in Fig. 2 illustrating the fact that the conventional assumption that some constant value of the safety factor can be associated with a structure is meaningless: any value of $0 < \nu < \infty$ is possible, although values in the central range are much more likely. The range $0 < \nu < 1$ defines failure, the range $\nu \geq 1$ survival. Therefore the probability of failure $P_f = F_\nu(1)$ can be read directly from the diagrams of $P(\nu)$ for a specific ratio of ν_0 at the intersection $\nu = 1$ (see Fig. 1). Conversely, for specified values of $(\sigma_R/\check{R})$, $(\sigma_S/\check{S})$ and P_f, the required ratio ν_0 ("central" safety factor) is obtained.

Obviously Eq. (3.5) can also be used directly for this purpose without recourse to graphic representation. Table 3 presents values ν_0 associated with various ratios $(\sigma_R/\tilde{R})$ and $(\sigma_S/\tilde{S})$ for different levels of P_f obtained by using Eq. (3.5).

For a nonstatistical load $S = S^*$ and $\sigma_S = 0$. Eq. (3.5) reduces to

$$P_f = \Phi\left[-(\log \nu_0)/\delta_R\right] \tag{3.8}$$

where $\nu_0 = \check{R}/S^*$.

4. <u>Extreme Value Distributions</u>. Introducing the so-called third asymptotic (or Weibull) distribution of smallest values [6] for R and the so-called second asymptotic (or Frechet) distribution of largest values [7] for S, or

$$F_R(R) = 1 - \exp\left[-\left(\frac{R}{\tilde{R}}\right)^{\alpha}\right] \tag{4.1}$$

and

$$F_S(S) = \exp\left[-\left(\frac{S}{\tilde{S}}\right)^{-\beta}\right] \tag{4.2}$$

where $\tilde{R}$ and $\tilde{S}$ are the "characteristic" values (close to modes) of the distributions and α and β are scale factors that are inverse functions of the standard deviations $\sigma(\log R)$ and $\sigma(\log S)$, the distribution $F_\nu(\nu)$ is obtained from Eq. (1.3) in the form

$$F_\nu(\nu) = 1 - \int_0^\infty \exp\left[-t - \left(\frac{\nu}{\nu_0}\right)^{\alpha} t^{-\frac{\alpha}{\beta}}\right] dt \tag{4.3}$$

where $t = (S/\tilde{S})^{-\beta}$ and $\nu_0 = \tilde{R}/\tilde{S}$ or, alternatively,

$$F_\nu(\nu) = 1 - \int_0^\infty \exp\left[-t - \left(\frac{\nu}{\nu_0}\right)^{\beta} t^{-\frac{\beta}{\alpha}}\right] dt\,. \tag{4.4}$$

By the substitution $u = \exp(-t)$, Eq. (4.3) is transformed into

$$F_\nu(\nu) = 1 - \int_0^1 \exp\left[-\left(\frac{\nu}{\nu_0}\right)^{\alpha} (-\ln u)^{-\frac{\alpha}{\beta}}\right] du \tag{4.5}$$

which is a form convenient for numerical evaluation provided ν is not too small compared to unity.

The probability of failure is therefore

$$P_f = F_\nu(1) = 1 - \int_0^\infty \exp\left[-t - \left(\frac{1}{\nu_0}\right)^\alpha t^{-\frac{\alpha}{\beta}}\right]dt =$$

$$1 - \int_0^\infty \exp\left[-t - \left(\frac{1}{\nu_0}\right)^\beta t^{-\frac{\beta}{\alpha}}\right]dt \qquad (4.6)$$

or, in terms of u,

$$P_f = F_\nu(1) = 1 - \int_0^1 \exp\left[-\left(\frac{1}{\nu_0}\right)^\alpha (-\ln u)^{-\alpha/\beta}\right]du$$

$$= 1 - \int_0^1 \exp\left[-\left(\frac{1}{\nu_0}\right)^\beta (-\ln u)^{\beta/\alpha}\right]du . \qquad (4.7)$$

The relations between the measures of dispersion α and β and the coefficients of variations $v_R = \sigma_R/\tilde{R}$ and $v_S = \sigma_S/\tilde{S}$ or the ratios $\sigma_R/\tilde{R}$ and $\sigma_S/\tilde{S}$ are obtained in the following form.

$$v_R^2 = \left[\Gamma\left(1+\frac{2}{\alpha}\right) - \Gamma\left(1+\frac{1}{\alpha}\right)^2\right] / \Gamma\left(1+\frac{1}{\alpha}\right)^2 \qquad (4.8)$$

and

$$v_S^2 = \left[\Gamma\left(1-\frac{2}{\beta}\right) - \Gamma\left(1-\frac{1}{\beta}\right)^2\right] / \Gamma\left(1-\frac{1}{\beta}\right)^2 \qquad (4.9)$$

as well as

$$(\sigma_R/\tilde{R})^2 = \Gamma\left(1+\frac{2}{\alpha}\right) - \Gamma\left(1+\frac{1}{\alpha}\right)^2 \qquad (4.10)$$

and

$$(\sigma_S/\tilde{S})^2 = \Gamma\left(1-\frac{2}{\beta}\right) - \Gamma\left(1-\frac{1}{\beta}\right)^2 \qquad (4.11)$$

Using the above equations the values of Table 4 (a) and (b) have been computed.

By numerical evaluation of Eq. (4.5) the functions $F_\nu(\nu) = P(\nu)$ for various combinations of $(\sigma_R/\tilde{R})$ and $(\sigma_S/\tilde{S})$ have been constructed with $\nu_0 = \tilde{R}/\tilde{S}$ as parameter and are presented in Figs. 3(a) to 3(i). For this evaluation use has been made of the approximation of Eqs. (4.3) and (4.4) by

$$F_\nu(\nu) = a\Gamma(1-b) \qquad (4.12)$$

obtained with the substitution $b = \alpha/\beta$ or β/α $(b < 1)$ and $a = (\nu/\nu_0)^\alpha$ or $(\nu/\nu_0)^\beta$ provided $a \ll 1$. The probability density of ν for

$\sigma_R/\tilde{R} = \sigma_S/\tilde{S} = 0.1$ has been illustrated in Fig. 4.

The probability of failure $P_f = F_\nu(1)$ can again be read directly from the diagrams $P(\nu)$ at $\nu = 1$ for any specified "central" safety factor ν_0. Table 5 presents values of ν_0 associated with various ratios of $(\sigma_R/\tilde{R})$ and $(\sigma_S/\tilde{S})$ for different levels of P_f, obtained from Figs. 3(a) to 3(i).

In the particular case of a non-statistical maximum load intensity $S = S^*$ and therefore $\sigma_S = 0$ and $\sigma_S/\tilde{S} = 0$, the probability of failure is obtained directly from the reduced Eq. (4.5) introducing $\nu=1$:

$$F_\nu(\nu) = 1 - \exp\left[-\left(\frac{\nu}{\nu_0}\right)^\alpha\right] \tag{4.13}$$

where $\nu = R/S^*$ and $\nu_0 = \tilde{R}/S^*$, and

$$P_f = 1 - \exp\left[-\nu_0^{-\alpha}\right] \tag{4.14}$$

which is easily evaluated.

Comparison of the values of ν_0 in Tables 3 and 5 at the same levels of P_f associated with the same ratios of $(\sigma_R/\tilde{R})$, $(\sigma_S/\tilde{S})$ and $(\sigma_R/\tilde{R})$, $(\sigma_S/\tilde{S})$ respectively, shows the differences resulting from the assumptions of logarithmic-normal and extremal distribution of the variables R and S: for extremal distributions much higher "central" safety factors are required to ensure the same probabaility of failure as for logarithmic-normal distributions. It should, however, be considered that the load-populations considered are not the same. The extremal distributions represent partial populations obtained by selecting the largest values in samples of the whole population (gusts exceeding a certain intensity, flood levels, etc.), while logarithmic-normal populations usually represent the entity of loads $S > 0$. Thus the number of applications of extremal loads is much smaller than the total number N, so that in order to ensure a specific value of L(N) a much higher value of P_f can be accepted.

With respect to the use of extremal distributions for the representation of resistance properties it has been shown [8] that superior production-control is likely to result in logarithmic-normal distributions, while poor control leads to extremal distributions.

5. Safety Factors for "Maximum Load" and "Minimum Strength". In conventional design it is usually assumed that the safety factor can be based on a "maximum load" and a "minimum resistance". However, with the exception of non-statistical loads, such as the fluid or bulk-pressure in storage containers or floor-loads in warehouses, no absolute maximum can be specified; similarly, no absolute minimum of the resistance values can be known, only values representing the smallest observation in samples of finite size. Thus a "minimum" resistance R_{min} will always be associated with a finite probability p of not being attained and a "maximum" load S_{max} with a finite probability q of being exceeded, however small these probabilities are selected. Thus $R_{min} = R_p = r_p\tilde{R}$ (for logarithmic-normal distributions) or $r_p\tilde{R}$ (for extremal distributions) and

$S_{max} = S_q = s_q \check{S}$ (for logarithmic-normal distribution) or $s_q \tilde{S}$ (for extremal distributions). The ratio r_p is related to p in the case of logarithmic-normal distributions by the equation

$$p = \Phi[(\log r_p)/\delta_R] \quad (5.1)$$

and in the case of extremal distributions of smallest values by the equation

$$p = 1 - \exp(-r_p^{\alpha}) . \quad (5.2)$$

Similarly the ratio s_q is related to q by

$$q = \bar{\Phi}[(\log s_q)/\delta_S] \quad (5.3)$$

where $\bar{\Phi} = 1 - \Phi$ in the case of logarithmic-normal distributions, and by

$$q = 1 - \exp(- s_q^{-\beta}) \quad (5.4)$$

in the case of extremal distributions of largest values.

The "conventional" safety factor $\bar{\nu}$ is now defined by

$$\bar{\nu} = R_p/S_q = \nu_o r_p/s_q \quad (5.5)$$

and is thus related to the "central" factor ν_o and the selected ratios r_p and s_q. For a non-statistical maximum load S^*,

$$\bar{\nu} = R_p/S^* = \nu_o r_p . \quad (5.6)$$

It is, in general, unlikely that the "minimum" value R_p can be specified with better approximation than $p = 0.1$: with samples consisting of 9 specimens the probability of values smaller than the smallest value of the sample is 0.1. Analysis of actual acceptance tests of structural steel has shown that about 10 percent of the observed yield-stress values fall below the specified minimum 9.

The specification of the "maximum" load S_q depends on the length of the time series of load observations from which it is to be determined. With a recurrence number of $(1/q)$ of loads exceeding S_q, the value of q can be selected as small as desired if the series of observations is long enough.

For $p = 0.1$ and $q = 0.1, 0.01, 0.001$ and 0.0001, the ratios r_p and s_q have been computed using Eqs. (5.1) to (5.4) and listed in Tables 6 for logarithmic-normal distributions and in Tables 7 for extremal distributions. Plotting the relations $\nu_L(P_f)$ with the aid of Tables 3 (logarithmic-normal distribution, $p = q = 0.5$) and 5 (extremal distributions, $p = q = 1 - 1/e$) the relations $\bar{\nu}(P_f)$ for various combinations of (p,q) can be obtained by the use of Eqs. (5.5) and (5.6). These relations are presented in Figs. 5(a) to 5(ℓ) for logarithmic-normal distributions and in Figs. 6(a) to 6(ℓ) for extremal distributions. These diagrams permit the selection of "conventional" safety factors $\bar{\nu}$ if the probability level q associated with the "maximum" load can be estimated, and thus provide a means of interpreting the probability of failure associated with current safety factors.

6. Multiple-Member Structures. (a) Structures without redundancy. Considering a structure consisting of n members each of which is subject to the same load (total load divided by n) and assuming that the structure fails if any one of its members fails, the probability of failure P_{f_n} of the structure under a single load application is $P_f = 1 - \prod_{k=1}^{n} (1 - p_k) \doteq \sum_{k=1}^{n} p_k$ where p_k is the probability of failure of the k-th member.

If probabilities of failure $p_k = P_f/n$ are assumed equal for all members and if the over-all probability of failure P_f is to be retained, it is necessary to reduce the probability of failure from P_f to P_f/n. Assuming logarithmic-normal distributions for R and S, this reduction requires:

1) decrease of $\sigma_R/\breve{R}$ keeping $\nu_0 = \breve{R}/\breve{S}$ and $\sigma_S/\breve{S}$ constant;
2) increase of ν_0 keeping $\sigma_S/\breve{S}$ and $\sigma_R/\breve{R}$ constant;
3) combination of (1) and (2).

Operations (1) and (2) are discussed using Eqs. (6.1) and (6.2) for the conditions of statistical and non-statistical load $\sigma_S \neq 0$ and $\sigma_S = 0$ respectively.

$$p_k = \Phi\left[-(\log \nu_0)/\delta\right] \tag{6.1}$$

$$p_k = \Phi\left[-(\log \nu_0)/\delta_R\right] \tag{6.2}$$

ad (1): values of $\sigma_R/\breve{R}$ are computed which would guarantee the specified probability of failure $p_k = P_f/n$ of each member and P_f of the whole structure for given values of $\sigma_S/\breve{S}$ and ν_0. Examples have been computed for the assumptions $P_f = 10^{-6}$, $\nu_0 = 5.0$ and $\sigma_S/S = 0$, 0.10, 0.20 and 0.30. Values of σ_R/R thus obtained are listed in Table [illegible]; Fig. 7 presents the relationship between $\sigma_R/\breve{R}$ and n, illustrating the necessity of reducing $\sigma_R/\breve{R}$ as n increases.

ad (2): Values of ν_0 are computed which would guarantee the specified probability of failure under given $\sigma_R/\breve{R}$ and $\sigma_S/\breve{S}$. Examples have been computed for $P_f = 10^{-6}$, $\sigma_R/\breve{R} = 0.05$, 0.10, 0.15 and $\sigma_S/\breve{S} = 0$, 0.10, 0.20, 0.30. The results are shown in Table 9 and in Fig. 8 which illustrates the necessity of increasing ν_0 as n increases. Figs. 7 and 8 illustrate the relative effects of improved control or of increased safety factors on the design of non-redundant multiple-member structures to a specified probability of failure.

(b) Probability of Survival of Redundant Structures.
The survivorship function of a non-redundant structure of n members subject to N independent load applications is

$$L(N) = [\exp(-Np_k)]^n = \exp(-NP_f)$$

being identical with that of a single member structure of probability of failure $P_f = n\, p_k$.

Failure of an n-fold redundant structure will result from the failure of the j-th member, after it has survived the failure of $(j-1)$ members. In other words its probability of survival will drop so rapidly between the $(j-1)$-th and the j-th failure that further survival will become very unlikely, though not impossible.

Considering a structure of n-fold redundancy under the assumptions that (a) the total load is always equally distributed among the existing members; (b) the resistances of all members belong to one and the same statistical population, and defining by N_j the interval (in terms of number of load application) between the $(j-1)$-th and j-th failures, and by p_j the probability of failure of each member during this interval, when $(n-j+1)$ members still exist, the probability that such a member will survive N_j load applications is $(1-p_j)^{N_j}$ or (since $p_j \ll 1$) $\exp(-N_j p_j)$.

The probability that $n-j+1$ members will survive N_j load applications is $\exp[-(n-j+1)\, N_j p_j]$, from which the probability function Eq. (6.3) and the density function Eq. (6.4) of N_j are obtained.

$$F_{N_j}(N_j) = 1 - \exp[-(n-j+1)N_j p_j] \tag{6.3}$$

$$f_{N_j}(N_j) = (n-j+1)p_j \exp[-(n-j+1)N_j p_j] \tag{6.4}$$

The corresponding characteristic functions are

$$Q_j(\omega) = \int_0^\infty \exp(i\omega N_j) f_{N_j}(N_j)\, dN_j = \frac{1}{1 - \frac{i\omega}{(n-j+1)p_j}} \tag{6.5}$$

$$[\mathit{Im}\,\omega > -(n-j+1)\,p_j].$$

Then by the virtue of law of convolution the characteristic function of $N = \sum_{k=1}^{n} N_k$ is

$$Q(\omega) = \prod_{K=1}^{n} \left[\frac{1}{1 - \frac{i\omega}{(n-k+1)p_K}} \right] . \tag{6.6}$$

Making use of the Fourier inversion formula, the probability density function of N is given by

$$f_N(N) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Q(\omega) \exp(-i\omega N)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 - \frac{i\omega}{np_1}} \cdot \frac{1}{1 - \frac{i\omega}{(n-1)p_2}} \cdots \frac{1}{1 - \frac{i\omega}{p_n}} \exp(i\omega N)\, dN \tag{6.7}$$

which is easily evaluated by the method of partial fractions:

$$f_N(N) = np_1 C_1 \exp[-np_1 N] + (n-1)p_2 C_2 \exp[-(n-1)p_2 N] + \cdots + p_n C_n \exp[-p_n N] \tag{6.8}$$

or the survivorship function is

$$L(N) = C_1 \exp[-np_1 N] + C_2 \exp[-(n-1)p_2 N] + \cdots + C_n \exp[-p_n N] \tag{6.9}$$

where

$$C_j = \frac{1}{1 - \frac{(n-j+1)p_j}{np_1}} \cdot \frac{1}{1 - \frac{(n-j+1)p_j}{(n-1)p_2}} \cdots \frac{1}{1 - \frac{(n-j+1)p_j}{(n-j+2)p_{j-1}}}$$

$$\cdot \frac{1}{1 - \frac{(n-j+1)p_j}{(n-j)p_{j+1}}} \cdots \frac{1}{1 - \frac{(n-j+1)p_j}{p_n}} .$$

Considering as an example the case of a structure of three redundancies in which R (total resistance) and S (total load) are logarithmic-normally distributed with $\sigma_R/\breve{R} = 0.05$ ($\delta_R = 0.02166$) and $\sigma_S/\breve{S} = 0.10$ ($\delta_S = 0.04311$) respectively, $\nu_0 = R/S$ is assumed to be 1.773 in order that $p_1 = 10^{-7}$, $p_2 = 6.03 \times 10^{-2}$ and $p_3 = 1 - 1.26 \times 10^{-6}$. Since further computation shows that $C_1 = (1 + 2.79 \times 10^{-6})$, $C_2 = -2.82 \times 10^{-6}$ and $C_3 = 4.00 \times 10^{-8}$,

$$L(N) = (1 + 2.79 \times 10^{-6}) \exp[-3 \times 10^{-7} N] - 2.82 \times 10^{-6} \exp[-1.206 \times 10^{-1} N] + 4.00 \times 10^{-8} \exp[-(1 - 1.26 \times 10^{-6}) N]. \qquad (6.10)$$

There is practically no difference between the above equation and the survivorship function $L(N) = e^{-3 \times 10^{-7} N}$ associated with the structure of no redundancy. Important, however, is the fact that the structure has not failed but survives with a reduced probability of survival even after one member fails: in the given example, the survivorship function in terms of the time (N') measured from the instant when one of the members fails is approximately $L_1(N') = e^{-1.206 \times 10^{-1} N'}$ since the failure of the second member causes almost immediate collapse of the entire structure in view of the large value of p_3. From the engineering point of view, therefore, and for structures such as aircraft design should be based on the consideration that the structure after failure of one member should have a probability of survival $L_1(N^*) = e^{-1.206 \times 10^{-1} N^*}$ sufficiently close to unity for the reduced operational time N^* (flight time between two airports of maximum on the flight course) so that the structure can be expected to safely reach the point where the failed member can be replaced. If the probability of survival for which the structure is initially designed, considering its expected total operational life, is matched by the probability of survival of the structure after failure of its first member, considering as "reduced operational life" the longest time that might be necessary to reach a point of repair, the redundancy of the structure is as fully utilized as practically possible. Assuming a ratio of total and "reduced" operational life of the order of 10^3 to 10^4, the probability of failure of the structure may be increased in the same proportion by the failure of its first redundant member without reducing its "operational" safety (principle of "fail-safe" design). Finally it may be worth noting that by the virtue of the central limit theorem the distribution of $N = \sum_{k=1}^{n} N_k$ approaches asymptotically a normal distribution as n increases.

7. Probability of Failure Under Combined Loading. The expression for the probability of failure under combined loading is illustrated by considering the example of a short column (free at the top and fixed at the bottom) subject to vertical load (live load + dead load) and the load due to earthquake.

The manner in which the expression for P_f is established is quite general, although a particular load and structure are assumed.

The principal load $\vec{W}_p$ is the sum of live load $\vec{W}_\ell$ at the top acting downward, and of dead load $\vec{W}_d$ acting at the mass center, while the secondary load due to earthquake is $\vec{\zeta} W_d$ acting also at the mass center, where $\vec{\zeta}$ denotes the acceleration due to earthquake and W_d is the weight of the column.

The total load is $\vec{W} = \vec{W}_\ell + \vec{W}_d + \vec{\zeta} W_d$. Since it is assumed that the magnitude W_ℓ of $\vec{W}_\ell$ and the magnitude and direction of $\vec{\zeta}$ are statistical variables, $\vec{W}$ is also a statistical variable having a three-dimensional probability density function $f(W, \theta, \phi)$ in the domain $0 \leq \theta < 2\pi$, $0 \leq \phi < 2\pi$ and $0 \leq W$ where W is the absolute value of $\vec{W}$. (See Fig. 9).

If it can be assumed that the critical load $W^*(\theta, \phi)$ in the direction (θ, ϕ) is known, so that any load larger than W^* produces failure of the structure, the probability of failure p_f is

$$p_f = \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \int_{W^*(\theta,\phi)}^{\infty} f(W, \theta, \phi)\, dW \,. \tag{7.1}$$

If $W^*(\theta, \phi)$ is also a statistical variable,

$$p_f = \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \int_0^{\infty} g(W^*, \theta, \phi)\, dW^* \int_{W^*(\theta,\phi)}^{\infty} f(W, \theta, \phi)\, dW \tag{7.2}$$

where $g(W^*, \theta, \phi)$ is the probability density of W^*, θ and ϕ.

On the basis of Eqs. (7.1) and (7.2) a theoretical justification of the concept of allowable increase of design stress for combined loads can be given. Design specification of bridges as well as buildings, for instance, permits increased design stresses when secondary loads such as wind and loads due to earthquakes are simultaneously applied with the principal loads such as live load and dead load.

If, for simplicity, it is assumed that $W^*(\theta, \phi)$ is independent of ϕ and not a statistical variable and that the occurrence of $\vec{\zeta}$ is symmetric with respect to the z axis the problem is reduced to a two-dimensional one (see Fig. 10) with replacement of $f(W, \theta, \phi)d\phi$ in Eq. (7.1) by $f(W, \theta)d\phi/(2\pi)$:

$$p_f = \int_0^{2\pi} d\theta \int_{W^*(\theta)}^{\infty} f(W, \theta)\, dW \tag{7.3}$$

In spite of its apparently simple form, Eq. (7.3) is hardly usable because it is difficult to evaluate $f(W, \theta)$. Writing $\vec{\zeta}_H$ and $\vec{\zeta}_V$ for horizontal and vertical components of $\vec{\zeta}$ the probability of failure is computed by considering the stresses due to $\vec{W}_p$, $\vec{\zeta}_H W_d$ and $\vec{\zeta}_V W_d$ separately under the assumptions that ζ_H and ζ_V are mutually independent statistical variables with normal joint probability density Eq. (7.4) and that $W_p = |\vec{W}_p| = |\vec{W}_\ell + \vec{W}_d|$ has an exponential probability density Eq. (7.5), where $\zeta_H = \pm\ |\vec{\zeta}_H|$ and $\zeta_V = \pm\ |\vec{\zeta}_V|$ (positive signs are taken when the directions of $\vec{\zeta}_H$ and $\vec{\zeta}_V$ are identical with the positive directions of x and z axis respectively).

$$f_{\zeta_H, \zeta_V}(\zeta_H, \zeta_V) = \frac{1}{2\pi\sigma_H\sigma_V} \exp\left[-\frac{1}{2}\left(\zeta_H^2/\sigma_H^2 + \zeta_V^2/\sigma_V^2\right)\right] \tag{7.4}$$

$$(-\infty < \zeta_H, \zeta_V < \infty)$$

$$f_{W_p}(W_p) = h \cdot \exp[-h(W_p - W_d)] \qquad (W \geq W_d) \qquad (7.5)$$

The stress $\tau_1 = W/A$ due to $\vec{W}$ is uniformly distributed on the bottom section and its probability density is

$$f_{\tau_1}(\tau_1) = Ah \cdot \exp[-Ah(\tau_1 - \tau_d)] \qquad (7.6)$$

where $\tau_d = W_d/A$, A being the cross-sectional area of the column. The stress τ_2 due to $\xi_V W_d$ is also uniformly distributed.

$$\tau_2 = \xi_V W_d / A = \xi_V \tau_d \qquad (7.7)$$

τ_2 has a normal probability density with zero mean and standard deviation $\sigma_2 = \sigma_V \tau_d$. The distribution of the stress τ_3 due to $\xi_H W_d$ is linear over the bottom section. Denoting by τ_{3E} and τ_{3F} the stresses at points E and F respectively (Fig. 10),

$$\tau_{3E} = -\tau_{3F} = \xi_H W_d l/(2Z) = \xi_H \tau_d' \qquad (7.8)$$

where Z is the section modulus and $\tau_d' = 2l\tau_d/\rho$, l and ρ being height and radius of the circular column. The statistical distributions of τ_{3E} and τ_{3F} are both normal with zero mean and standard deviation $\sigma_3 = \sigma_H \tau_d'$. Hence, the total stresses at points E and F are

$$\tau_E = \tau_1 + \tau_2 + \tau_{3E}$$

$$\tau_F = \tau_1 + \tau_2 + \tau_{3F} = \tau_1 + \tau_2 - \tau_{3E}.$$

It can be shown that the probability densities of τ_E and τ_F are identical and so are the distribution functions of τ_E and τ_F.

Letting τ represent τ_E and τ_F,

$$f_\tau(\tau) = Ah \cdot \exp[-Ah(\tau - \tau_d) + A^2h^2\sigma_{23}^2/2]\, \Phi[(\tau - \tau_d - Ah\sigma_{23}^2)/\sigma_{23}]$$

and

(7.9)

$$F_\tau(\tau) = \Phi[(\tau - \tau_d)/\sigma_{23}] - \exp[-Ah(\tau - \tau_d) + A^2h^2\sigma_2^2/2]\, \Phi[(\tau - \tau_d - Ah\sigma_{23}^2)/\sigma_{23}]$$

(7.10)

where $\sigma_{23} = \sqrt{\sigma_2^2 + \sigma_3^2}$.

Finally, the probability of failure of the column is approximately

$$P_f = 2[1 - F_\tau(\tau_C^*) + F(\tau_T^*)] \tag{7.11}$$

where τ_C^* (>0) and τ_T^* (<0) are the compressive and tensile strength of the material.

All parameters needed for computation of the probability of failure are functions of the distributions of loads and the cross-sectional area. In order to ensure the same probability of failure for a given loading condition, the cross-sectional area must be independent of whether the design is based on the primary load only or whether secondary loads are also considered.

When the design of the column is based only on the primary load $\vec{W}_p$ with design load W_{des} and the allowable (compressive) stress τ_{a0}, the area A is obviously $A = W_{des}/\tau_{a0}$. If the horizontal component $\zeta_{H_{des}}$ is taken as the design value of the secondary load, and if the same area is used the maximum stress τ_a is

$$\tau_a = \frac{W_{des}}{A} + \frac{2\sqrt{\pi}\, W_d \ell}{A^{3/2}} \zeta_{H_{des}} \tag{7.12}$$

Eq. (7.12) indicates that the combined loading requires τ_a as allowable stress for the same probability of failure. With $\tau_{a0} = W_{des}/A$ and $\tau_d = W_d/A$, Eq. (7.12) becomes

$$\tau_a = \tau_{a0} + 2\sqrt{\frac{\pi\tau_{a0}}{W_{des}}}\, \tau_d \ell \zeta_{H_{des}}. \tag{7.13}$$

Since W_{des} and τ_{a0} are specified quantities, a straight-line relationship is obtained between τ_a and $\zeta_{H_{des}}$ which assures a constant probability of failure. The relationship is shown schematically in Fig. 11. This is the theoretical basis on which the increase of the allowable stress is justified. Although a simple structure and a simple load combination are used for illustration, the concept underlying the above discussion is quite general.

8. <u>The Use of Separate Load Factors</u>. In some recent specifications[10] the use of separate load factors is recommended instead of a single safety factor to be applied to the total load. Thus, if $S_{d,q}$ and $S_{\ell,q}$ are introduced as design dead and live loads it is recommended to use separate factors α_q associated with $S_{d,q}$ and β_q with $S_{\ell,q}$ for the evaluation of the total design load to be compared with R_p, the design carrying capacity, instead of using $\bar{\nu}(S_{d,q} + S_{\ell,q})$.

Clearly, separate load factors cannot be related to the concept of probability of failure unless they can be related to $\bar{\nu}$. Thus the problem that arises is the following: what are the representative values of two statistical variables x and y with probability densities $f_x(x)$ and $f_y(y)$ respectively when only $z = x + y$ is given? It is obvious that there is an infinite number of combinations of x and y which add up to z. Out of those there are, however, the most probable values x^* and y^* which might be regarded as representative.

Mathematically, these are the values which make their joint probability density function maximum. Since the joint distribution function is

$$f_x(x)f_y(y) = f_x(x)f_y(z-x) = f_x(z-y)f_y(y), \tag{8.1}$$

x^* and y^* are the roots of the following equations

$$d[f_x(x)f_y(z-x)]/dx = 0, \quad d[f_x(z-y)f_y(y)]/dy = 0 \tag{8.2}$$

which are also respectively equivalent to

$$\frac{f_x'(x)}{f_x(x)} = \frac{f_y'(z-x)}{f_y(z-x)} \tag{8.3}$$

$$\frac{f_y'(y)}{f_y(y)} = \frac{f_x'(z-x)}{f_x(z-x)} . \tag{8.4}$$

Returning to the problem of load factors, x and y are replaced by S_d and S_ℓ respectively in Eqs. (8.3) and (8.4). Furthermore, by setting $z = S_q$, the roots $S_{d,q}$ and $S_{\ell,q}$ which are considered as design loads are obtained. Evidently,

$$S_q = S_{d,q} + S_{\ell,q} . \tag{8.5}$$

By setting $z = R_p$, roots S_d^* and S_ℓ^* are obtained such that

$$R_p = S_d^* + S_\ell^*. \tag{8.6}$$

R_p and S_q in this discussion are related by Eq. (5.5): $R_p = \bar{\nu} S_q$. In order to clarify the problem, these quantities are schematically illustrated in Fig. 12.

Expressing S_d^* and S_ℓ^* in terms of $S_{d,q}$ and $S_{\ell,q}$ in Eq. (8.6):

$$R_p = \alpha_q S_{d,q} + \beta_q S_{\ell,q} \tag{8.7}$$

When both S_d and S_ℓ are normally distributed with means $\bar{S}_d$ and $\bar{S}_\ell$ and standard deviations σ_d and σ_ℓ, and v_d, v_ℓ, γ, k' and k are defined by

$$v_d = \sigma_d/\bar{S}_d, \quad v_\ell = \sigma_\ell/\bar{S}_\ell, \quad \gamma = \bar{S}_\ell/\bar{S}_d, \quad S_q = k'\bar{S}_d, \quad R_p = k\bar{S}_d, \tag{8.8}$$

the probability density functions of S_d and S_ℓ are

$$
\left.
\begin{aligned}
f_{S_d}(S_d) &= \frac{1}{\sqrt{2\pi}\, v_d \bar{S}_d} \exp\left[-(S_d - \bar{S}_d)^2/(2 v_d^2 \bar{S}_d^2)\right] \\
f_{S_\ell}(S_\ell) &= \frac{1}{\sqrt{2\pi}\, \gamma v_\ell \bar{S}_d} \exp\left[-(S_\ell - \gamma \bar{S}_d)^2/(2\gamma^2 v_\ell^2 \bar{S}_d^2)\right].
\end{aligned}
\right\} \qquad (8.9)
$$

From Eqs. (8.3), (8.4) and (8.9), $S_{d,q}$ and $S_{\ell,q}$ are obtained in terms of $\bar{S}_d$ and S_ℓ respectively.

$$
S_{d,q} = \frac{v_d^2(k'-\gamma) + \gamma^2 v_\ell^2}{v_d^2 + \gamma^2 v_\ell^2}\, \bar{S}_d \qquad (8.10)
$$

$$
S_{\ell,q} = \frac{v_d^2 + (k'-1)\gamma v_\ell^2}{v_d^2 + \gamma^2 v_\ell^2}\, \bar{S}_\ell \qquad (8.11)
$$

Since S_d and S_ℓ are normally distributed, $S = S_d + S_\ell$ is also normally distributed with mean $\bar{S} = \bar{S}_d + \bar{S}_\ell$ and standard deviation $\sigma_S = \sqrt{\sigma_d^2 + \sigma_\ell^2}$; the design value S_q is

$$
S_q = \bar{S} + f(q)\, \sigma_S
$$

where $f(q)$ is a function of q obtainable from a table of the error distribution. Then, with the aid of Eq. (8.8),

$$
k' = S_q/\bar{S}_d = 1 + \gamma + f(q)\sqrt{v_d^2 + \gamma^2 v_\ell^2}\,. \qquad (8.12)
$$

In the present example, $v_d = 0.05$ and $v_\ell = 0.20$ are used; for γ two assumptions are made: $\gamma = 3.0$ for a railway bridge and $\gamma = 0.5$ for a highway bridge. If q is assumed to be 0.01, $f(q) = 2.33$. With these values, $S_{d,q}$ and $S_{\ell,q}$ are obtained from Eqs. (8.10) and (8.11) after computing k' making use of Eq. (8.12):

$S_{d,0.01} = 1.01\, \bar{S}_d$ and $S_{\ell,0.01} = 1.46\, \bar{S}_\ell$ for $\gamma = 3.0$ and

$S_{d,0.01} = 1.05\, \bar{S}_d$ and $S_{\ell,0.01} = 1.42\, \bar{S}_\ell$ for $\gamma = 0.50$.

The expressions for S_d^* and S_ℓ^* are obtained by replacing k' by k in Eqs. (8.10) and (8.11) respectively:

$$S_d^* = \frac{v_d^2(k-\gamma) + \gamma^2 v_\ell^2}{v_d^2 + \gamma^2 v_\ell^2} \bar{S}_d \tag{8.13}$$

$$S_\ell^* = \frac{v_d^2 + (k-1)\gamma v_\ell^2}{v_d^2 + \gamma^2 v_\ell^2} \bar{S}_\ell \tag{8.14}$$

Substituting S_d^* and S_ℓ^* into Eq. (8.6) after expressing them in terms of $S_{d,q}$ and $S_{\ell,q}$ with the aid of Eqs. (8.10) and (8.11),

$$R_p = \frac{v_d^2(k-\gamma)+\gamma^2 v_\ell^2}{v_d^2(k'-\gamma)+\gamma^2 v_\ell^2} S_{d,q} + \frac{v_d^2+(k-1)\gamma v_\ell^2}{v_d^2+(k'-1)\gamma v_\ell^2} S_{\ell,q} . \tag{8.15}$$

Hence by comparison with Eq. (8.7),

$$\alpha_q = \frac{v_d^2(k-\gamma)+\gamma^2 v_\ell^2}{v_d^2(k'-\gamma)+\gamma^2 v_\ell^2}, \quad \beta_q = \frac{v_d^2+(k-1)\gamma v_\ell^2}{v_d^2+(k'-1)\gamma v_\ell^2} . \tag{8.16}$$

Therefore, with $k' = 5.403$ ($\gamma = 3.0$),

$$\alpha_{0.01} = \frac{0.0025(k-3)+0.36}{0.366} \quad \beta_{0.01} = \frac{0.12(k-1)+0.0025}{0.531} \tag{8.17}$$

and with $k' = 1.761$ ($\gamma = 0.5$),

$$\alpha_{0.01} = \frac{0.0025(k-0.5)+0.01}{0.0132}, \quad \beta_{0.01} = \frac{0.02(k-1)+0.0025}{0.0177} . \tag{8.18}$$

If the distribution of load R is also normal with mean $\bar{R}$ and standard deviation $\sigma_R = 0.05\,\bar{R}$, and assuming $p = 0.1$ for ultimate strength R_p: $R_{0.1} = \bar{R} - f(0.1)\ \sigma_R = 0.936\ \bar{R}$. Since both R and $S = S_d + S_\ell$ are distributed normally,

$$P_f = P(R-S < 0) = \Phi\left[-(\bar{R}-\bar{S})/\sigma\right] \tag{8.19}$$

where $\sigma = \sqrt{\sigma_R^2 + \sigma_d^2 + \sigma_\ell^2}$.

For $\gamma = 3.0$, $v_d = 0.05$, and $v_\ell = 0.20$, $\bar{S} = 4.0\ \bar{S}_d$, $\bar{R} = 1.07k\ \bar{S}_d$ and $\sigma = \sqrt{0.00288k^2 + 0.3625}\ \bar{S}_d$ and for $\gamma = 0.5$, $v_d = 0.05$ and $v_\ell = 0.20$,

$\bar{S} = 1.5\,\bar{S}_d$, $\bar{R} = 1.07k\,\bar{S}_d$ and $\sigma = \sqrt{0.00288k^2 + 0.0125}\,\bar{S}_d$.

Thus, assuming k, $\alpha_{0.01}$ and $\beta_{0.01}$ are obtained from Eq. (8.17) or (8.18) and P_f from Eq. (8.19). Tables 10(a) and (b) list $\alpha_{0.01}$, $\beta_{0.01}$ and P_f, and Fig. 13 shows the factors $\alpha_{0.01}$ and $\beta_{0.01}$ as functions of P_f. In this way the arbitrariness of the selection of separate load factors can be eliminated and the factors associated with definite probabilities of failure. However, the analysis for any but normal distribution functions of loads and carrying capacity is prohibitive. It appears therefore that no practical purpose is served by the use of separate load factors instead of a single factor of safety.

III. STATISTICAL ANALYSIS OF FATIGUE UNDER CONSTANT AND RANDOM LOAD.

While ultimate load design is based on the assumption that the probability of failure that arises from the criterion $S > R$ or $\nu < 1$ is independent of the $(N-1)$ loads in the sequence preceding its occurrence at the N-th load application, design for fatigue considers the fact that some damage is produced in each load application and that therefore the accumulation of such damage in the course of repeated load applications will finally produce failure. Thus the probability of survival will decrease with increasing number of load applications; the form of the survivorship or "reliability" function characterizing the process of damage accumulation as "fatigue" differs therefore significantly from the "reliability" function $L(N) = \exp(-NP_f)$ established for ultimate load design.

9. Fatigue Mechanism. Denoting by Δ the accumulated fatigue damage, by $f(S)$ the effect of the momentary stress intensity S and by $g(\Delta)$ the effect of the accumulated damage (history) on the damage rate $d\Delta/dN$, the assumption is made that

$$d\Delta/dN = f(S)g(\Delta). \tag{9.1}$$

The applied stress sequence is assumed to be made up of stress intensities $S_i (i = 1, \ldots, n)$ appearing at random, with frequencies determined by the a priori probabilities p_i of their occurrence.

According to Eq. (9.1) for the sequence of N loads ,

$$\int_0^N d\Delta/g(\Delta) = G(\Delta) = \int_0^N f(S)dN = \sum_{k=1}^{N} f(S_{i_k}) \tag{9.2}$$

where S_{i_k} indicates the stress intensity S_i which is the k-th in the sequence of N load applications. Introducing a critical value Δ^* of the accumulated damage at which fatigue failure occurs, the fatigue life N can be defined as the minimum value N which, in Eq. (9.2), produces $G(\Delta^*) = G^*$.

If $f(S)$ is considered to be a statistical function, $G(\Delta)$ becomes a statistical variable for a given N; conversely N is statistically distributed for a given value of $G(\Delta)$. Be virtue of the central limit theorem the distribution of $G(\Delta)$ will tend toward normality. Under constant stress intensity S_i, this asymptotic normal distribution has mean Nm_i and standard deviation $\sigma_i\sqrt{N}$, where m_i and σ_i denote the mean and standard deviation of $f(S_i)$. Therefore, considering the above failure criterion, the probability function of fatigue lives $N(>0)$ is given by

$$F_N(N) = \bar{\Phi}[(G^*-Nm_i)/(\sigma_i\sqrt{N})] = \bar{\Phi}[(1-Nm_{i0})/\sigma_{i0}\sqrt{N})] \quad (9.3)$$

with the following expressions for the median $\breve{N}_i$, mean $\bar{N}_i$ and variance $\sigma^2_{N_i}$.

$$\breve{N}_i = 1/m_{i0} \quad (9.4)$$

$$\bar{N}_i = \breve{N}_i \left[1 + \tfrac{1}{2}(\sigma^2_{i0}/m_{i0})\right] \quad (9.5)$$

$$\sigma^2_{N_i} = N_i^2 \left[\sigma^2_{i0}/m_{i0} + \tfrac{5}{4}(\sigma^2_{i0}/m_{i0})^2\right] \quad (9.6)$$

where $m_{i0} = m_i/G^*$ and $\sigma_{i0} = \sigma_i/G^*$ are evaluated from the results of fatigue tests, $\Phi(t)$ is given by Eq. (3.3) and $\bar{\Phi}(t) = 1 - \Phi(t)$.

The survivorship function

$$L(N) = \Phi\left[(G^*-Nm_i)/(\sigma_i\sqrt{N})\right] = \Phi\left[(1-Nm_{i0})/(\sigma_{i0}\sqrt{N})\right] \quad (9.7)$$

can be approximated by a series of m observations of the fatigue life N_k $(k=1, 2, ---, m)$ arranged in ascending order, since the expected value 11 of $L(N_k)$ is

$$\overline{L(N_k)} = 1 - k/(m+1) \quad (9.8)$$

Introducing the variable

$$t = (G^*-Nm_i)/\sigma_i\sqrt{N} = (1-Nm_{i0})/\sigma_{i0}\sqrt{N} \quad (9.9)$$

linear relations are obtained

$$t\sqrt{N} = G^*/\sigma_i-(m_i/\sigma_i)N = 1/\sigma_{i0} -(m_{i0}/\sigma_{i0})N \quad (9.10)$$

between $t\sqrt{N}$ and N. If t_k corresponds to the observation N_k so that

$$\overline{L(N_k)} = \overline{\Phi(t_k)} = 1 - k/(m+1) \ , \qquad (9.11)$$

the values t_k can be obtained from a table of error functions. The set of points $(t_k\sqrt{N_k}, N_k)$ must be reproducible by a straight line in the coordinate system $(t\sqrt{N}, N)$ if the hypotheses concerning the fatigue mechanism which leads to Eq. (9.3) is to be reasonably valid. In Fig. 14(a)-(f) the evaluation of test results obtained on 7075 aluminum alloy is presented and shows fair agreement with the proposed theory. The values m_{i0} and σ_{i0} obtained by graphically fitted straight lines to the data are listed in Table 11.

It is worth noting that the following asymptotic equalities have been proved in the renewal theory [12] of mathematical statistics.

$$\bar{N}_i = G^*/m_i$$

$$\sigma_{N_i} = \frac{\sigma_i^2}{m_i^3} G^* + \left[\frac{5(\sigma_i^2 + m_i^2)}{4m_i^4} - \frac{2\mu_3'}{3m_i^3} - \frac{(\sigma_i^2 + m_i^2)}{2m_i^2}\right]$$

where μ_3' is the third moment of $f(S_i)$ about the origin. The first equation can be written in the form

$$\bar{N}_i = 1/m_{i0} \qquad (9.12)$$

while the knowledge of the values of σ_{i0} and m_{i0} given in Table 11, together with the assumption that the coefficient of skewness is of the order of unity leads to the following approximation for σ_{N_i}.

$$\sigma_{N_i} = \bar{N}_i^2 \left[\sigma_{i0}^2/m_{i0} + \frac{5}{4}(\sigma_{i0}^2/m_{i0})^2\right] \qquad (9.13)$$

The last assumption may be justified by the fact that the coefficient of skewness of the exponential distribution is 2.

Comparison of Eqs. (9.12) and (9.13) with Eqs. (9.4) and (9.6) shows that there is a difference because there m_{i0} is estimated as an inverse of median $\breve{N}$ of N according to the distribution function Eq. (9.3), while here it is obtained as the inverse of mean $\bar{N}$ of N according to the renewal theory.

However, since the difference between $\bar{N}$ and $\breve{N}$ is as small as shown in Eq. (9.5), it is concluded that the estimations of m_{i0} and σ_{i0} by the two alternative methods give practically identical values.

The damage accumulation due to fatigue was treated first as a renewal problem by E. Parzen. [13]

A demonstration of the possibility of obtaining the distribution function $F_z(z)$ of $z = f(S_i)$ is another interesting point suggested by the renewal theory which relates the Laplace trnasforms $\mathcal{L}[F_z(z)]$ of $F_z(z)$ and $\mathcal{L}[\bar{N}(G)]$ of the renewal function $\bar{N}(G)$ in the form

$$\mathcal{L}[F_z(z)] = \frac{\mathcal{L}[\bar{N}(G)]}{1 + s\mathcal{L}[\bar{N}(G)]} \tag{9.14}$$

where

$$\mathcal{L}[F_z(z)] = \int_0^\infty e^{-sz} F_z(z)\, dz$$

and

$$\mathcal{L}[\bar{N}(G)] = \int_0^\infty e^{-sG} \bar{N}(G) dG .$$

The renewal function is defined as the mean value of the distribution of the maximum number of load applications which porduces a given $G = G(\Delta)$ in Eq. (9.2).

Eq. (9.14) suggests, therefore, that the distributions $F_z(z)$ of $z = f(S_i)$ might be obtained by the Laplace inversion theorm if the mean value of the maximum number of load applications associated with $G = G(\Delta)$ as a function of G could be observed by experiment.

10. Stress Interaction and Cumulative Damage Rule. Under conditions of fatigue under random loading stress-interaction effects have to be considered [14] which at the stress-amplitudes S_i arise from the intermittent application of stress amplitudes $S_j > S_i$, changing the average of $f(S_i)$ from m_i to m_i^*. A stress-interaction factor ω_i is introduced in the form

$$\omega_i = N_i/N_i' \tag{10.1}$$

where N_i and N_i' denote respectively the fatigue life at stress level S_i without and with intermittent applications of amplitudes $S_j > S_i$. The linear cumulative damage rule the validity of which has been disproved [15]

$$\sum_{i=1}^{n} p_i\, N_R/N_i = 1 \tag{10.2}$$

where N_R is the random life is therefore modified in the form

$$\sum_{i=1}^{n} p_i N_R/N_i' = \sum_{i=1}^{n} p_i\, \omega_i\, N_R/N_i \tag{10.3}$$

which reproduces results of random fatigue tests fairly well.

From Eq. (9.2) considering the failure condition $G(\Delta^*) = G^*$, the following expressions are obtained:

(a) without stress-interaction

$$G^* = \sum_{k_1=1}^{p_1 N_R} f(S_{1_{k_1}}) + \cdots\cdots\cdots + \sum_{k_n=1}^{p_n N_R} f(S_{n_{k_n}}) \qquad (10.4)$$

(b) with stress-interaction

$$G^* = \sum_{k_1=1}^{p_1 N_R} f^*(S_{1_{k_1}}) + \cdots\cdots\cdots + \sum_{k_n=1}^{p_n N_R} f^*(S_{n_{k_n}}) \qquad (10.5)$$

where $f^*(S_i)$ is a statistical stress-effect function modified for stress interaction, with mean m_i^* and standard deviation σ_i^*. If only the average trend is considered Eqs. (10.4) and (10.5) can be replaced by

$$G^* = p_1 m_1 \breve{N}_R + \cdots\cdots\cdots + p_n m_n \breve{N}_R \qquad (10.6)$$

and

$$G^* = p_1 m_1^* \breve{N}_R + \cdots\cdots\cdots + p_n m_n^* \breve{N}_R \,. \qquad (10.7)$$

The stress-interaction factor is now defined by

$$\omega = m^*/m \,. \qquad (10.8)$$

Since $p_i m_i = p_i\, m_{i0}\, G^* = p_i G^*/\breve{N}$ and $p_i m_i^* = p_i m_i \omega_i = p_i m_{i0} \omega_i G^* = p_i \omega_i G^*/\breve{N}_i$, Eqs. (10.6) and (10.7) are identical with the linear and the modified (quasi-linear) rules of damage accumulation Eqs. (10.2) and (10.3) if $\breve{N}_R$ and $\breve{N}_i$ are used instead of N_R and N_i.

The concept of stress-interaction factors leads to that of a "fictitious" S-N diagram embodying the stress-interaction effects at all stress-amplitudes $S_i = S$ [14]. Since $m_{i0} = 1/\breve{N}_i$, the diagram (log S- log m_0) has a positive slope of the same magnitude as the (negative) slope of the (log S - log $\breve{N}$) diagram, as shown in Fig. 15. To each load spectrum corresponds a straight-line fictitious (log S - log m_0^*) diagram intersecting the (log S - log m_0) diagram at the maximum stress-amplitude of the spectrum $S_{i,max}$. The equations of these relations are

$$m_0/m_{i,max} = (S/S_{i,max})^{\xi} \qquad (10.9)$$

and

$$m_0^*/m_{i,max} = (S/S_{i,max})^{\eta} \,. \qquad (10.10)$$

Therefore, the interaction factor

$$\omega = (S/S_{i,max})^{\eta - \xi} \tag{10.11}$$

depends on $S_{i,max}$ and $(\eta - \xi)$ where η in turn depends on the load spectrum.

The observed log S - log m_0 relation for 7075-aluminum (small specimens in rotating bending) [16] is of the form of Eq. (10.9) with $\xi = 8.24$ and $m_{i,max} = 1.66 \times 10^{-4}$ for $S_{i,max} = 6.97 \times 10^4$ or $m_{i,max} = 4.12 \times 10^{-4}$ for $S_{i,max} = 7.79 \times 10^4$, as shown in Fig. 16.

Since Eq. (9.2) with stress interaction effect can be written in the following form

$$G(\Delta) = \sum_{k_1=1}^{p_1 N_R} f^*(S_{1k_1}) + \cdots\cdots\cdots + \sum_{k_n=1}^{p_n N_R} f^*(S_{nk_n}) , \tag{9.2'}$$

$G_{-}(\Delta)$ is normally distributed by virtue of central limit theorem with mean $N_R \bar{m}_R$ and standard derivation $\sqrt{N_R}\ \bar{\sigma}_R$ where

$$\bar{m}_R = \sum_{i=1}^{n} p_i m_i^* \quad \text{and} \quad \bar{\sigma}_R = \sqrt{\sum_{i=1}^{n} p_i \sigma_i^{*2}} \ .$$

Therefore, considering the failure criterion Eq. (10.5), the distribution function of fatigue life N_R under random loading is

$$L(N_R) = \Phi\left[\frac{G^* - N_R \bar{m}_R}{\sqrt{N_R}\ \bar{\sigma}_R}\right] = \Phi\left[\frac{1 - N_R \bar{m}_{R_0}}{\sqrt{N_R}\ \bar{\sigma}_{R_0}}\right] \tag{10.12}$$

where

$$\bar{m}_{R0} = \bar{m}_R/G^* = \sum_{i=1}^{n} p_i m_{i0}^* = \sum_{i=1}^{n} p_i m_i \omega_i \tag{10.13}$$

and

$$\bar{\sigma}_{R0} = \bar{\sigma}_R/G^* = \sqrt{\sum_{i=1}^{n} p_i \sigma_{i0}^{*\,2}} \ . \tag{10.14}$$

With the aid of Eq. (10.12) the values of $\bar{m}_{R0} = \bar{m}_R/G^*$ and $\bar{\sigma}_{R0} = \bar{\sigma}_R/G^*$ can be determined in the same way as m_{i0} and σ_{i0}. In Fig. 17(a)-(i) results of nine sets of random fatigue tests of 20 specimens each of 7075 aluminum alloy [16] are plotted in the $t\sqrt{N}$ versus N cordinate system. The resulting values $\bar{m}_{R0}$ and $\bar{\sigma}_{R0}$ have been obtained from the fitted straight lines and are listed in Table 11. The exponential load spectra used in these experiments are of the form

$$f_s(s_i) = h \cdot \exp\left[-h(s_i - s_{i,min})\right] \qquad (10.15)$$

where $s_i = S_i/S_u$ and $s_{i,min} = S_{i,min}/S_u$ are stress-ratios, $S_u = 82{,}000$ psi being the ultimate tensile strength. The values of s_i, p_i and h are listed in Table 12. The maximum stresses are either $S_{i,max} = 77{,}900$ psi or $S_{i,max} = 69{,}700$ psi. Assuming straight line fictitious fatigue diagrams the use of Eqs. (10.11) and (10.13) with known values p_i, m_{i0} and $\bar{m}_{R0}$ permits the evaluation of η and ω_i by trial and error. The results are presented in Tables 13 and 14 including $m^*_{i0} = m_{i0}\,\omega_i$.

11. Scatter in Fatigue Life under Random Loading. While the variance σ_{NR}^2 of fatigue life under random loading can be directly evaluated using Eq. (9.6) in which σ_{Ni}, m_{i0} and σ_{i0} are replaced by σ_{NR}, $\bar{m}_{R0}$ and $\bar{\sigma}_{R0}$ respectively, where $\bar{m}_{R0}$ and $\bar{\sigma}_{R0}$ are known quantities, the real problem is to establish the theoretical basis on which σ_{NR} can be predicted from the results of fatigue tests at constant stress-amplitudes. In these tests σ_i is a function of S_i and therefore of m_i, m_i being related to S_i as shown in Fig. 16. The assumption is now introduced that the relation of σ_0 and m_0 is of the form

$$\sigma_0 = a' m_0^{b'} \qquad (11.1)$$

as indicated by the results on 7075 aluminum alloy presented in Fig. 18 that can be fitted by Eq. (11.1) with $\log a' = -1.11$ and $b' = 0.397$ or

$$\log \sigma_0 = 0.397 \log m_0 - 1.11\,. \qquad (11.2)$$

It is interesting to note that the points $(\bar{m}_{R0}, \bar{\sigma}_{R0})$ obtained from random fatigue tests and plotted on the same diagram (open circles) suggest that Eq. (11.2) adequately represents the relation of $\bar{\sigma}_{R0}$ and $\bar{m}_{R0}$:

$$\log \bar{\sigma}_{R0} = 0.397 \log \bar{m}_{R0} - 1.11 \qquad (11.2')$$

Thus the assumption seems justified that the relation between $\sigma_0^* = \sigma/G^*$ and $m_0^* = m/G^* = m_0\,\omega$ is of the same form

$$\log \sigma_0^* = 0.397 \log m_0^* - 1.11\,. \qquad (11.2'')$$

Comparison of Eq. (11.1) with Eq. (11.2") in the form $\sigma_0^* = a' m_0^{*b'}$ leads therefore to the relation where

$$\sigma_0^* = \sigma_0 \omega' \qquad (11.3)$$

$$\omega' = \omega^{b'} \qquad (11.4)$$

is the stress-interaction factor associated with the standard deviation of the stress-effect function. Using Eqs. (10.14) and (11.3),

$$\bar{\sigma}_{R0} = \sqrt{\sum_{i=1}^{n} p_i \sigma_{i0}^{*2}} = \sqrt{\sum_{i=1}^{n} p_i \sigma_{i0}^2 \omega_i'^2} \qquad (11.5)$$

where the quantities p_i, σ_{i0} and $\omega_i' = \omega_i^{b'}$ are known.

If the values $\bar{\sigma}_{R0}$ obtained from Eq. (11.5) are found to be in good agreement with directly observed values, the theoretical approach based on integration of the assumed damage mechanism Eq. (9.1) and the application of statistical rules appears to be justified. In the three rows of Table 15, observed values, values based on Eq. (11.5) and values obtained with the aid of the purely empirical Eq. (11.2") are presented. The comparison shows as fair an agreement between theory and observation as can in general be expected in fatigue; it also shows that Eq. (11.2") provides a reasonable practical approximation. ($\sigma_{i0}^* = \sigma_{i0} \omega' = \sigma_{i0} \omega^{b'}$ is also listed in Table 14).

In the representation and interpretation of fatigue under constant and random loading the third asymptotic distribution of extreme (smallest) values has been extensively used [17]. The survivorship function is of the form

$$L(N) = \exp\left[-\left(\frac{N-N_0}{V-N_0}\right)^{\alpha}\right] \qquad (11.6)$$

where V is the "characteristic" life at $L(V) = e^{-1}$, N_0 is the minimum life and α is a scale parameter inversely related to the standard deviation of $\log (N-N_0)$. While statistical superposition of Eq. (11.6), written for various constant stress amplitudes S_i and considering the effect of stress-interaction, results in a damage accumulation rule for the evaluation of the characteristic random life V_R [14], this rule could only be rigorously applied if the relation between $1/\alpha_i$ and $1/\alpha_R$ were known. Such a relation is, however, not known nor can it be derived on a theoretical basis similar to that underlying Eq. (11.5).

As a first estimate it might be assumed that the relation between $1/\alpha_i$ and V_i is the same as that between $1/\alpha_R$ and V_R. To test this assumption observed values $1/\alpha_i'$ $(= 1/2.303\alpha_i)$ and $1/\alpha_R'$ $(= 1/2.303\alpha_R)$ have in Fig. 19 been plotted against $\log V_i$ and $\log V_R$ respectively. Since both sets of points seem to belong to the same population, a rough estimate of $1/\alpha_R$ can be obtained in this manner.

12. <u>Comparison Between Extremal, Logarithmic Normal and Proposed Distribution of Fatigue Lives</u>. In Fig. 20(a)-(c) several series of fatigue test results on 7075 aluminum alloy [16] have been plotted on extremal probability paper together with the above three distributions fitted to the test results. It is quite obvious that within the range $0.05 < L(N) < 0.95$ the test results can be equally well fitted by any of the three distributions and no valid distinction can be made between them. Such a distinction appears only when the rate or "risk" of damage associated with these distributions are compared.

The risk function $r(N) = -\, d\ln L(N)/dN$ is related to $L(N)$ according to Eq. (2.7). Considering, for valid comparison, the existence of a minimum life N_0 not only in the extremal function Eq. (11.6) but also in the logarithmic-normal and in the proposed distribution, the following survivorship functions must be considered:

$$L(N) = \bar{\Phi}\left[\frac{1}{\delta_N} \log \frac{N-N_0}{\breve{N}-N_0}\right] \tag{12.1}$$

for the logarithmic-normal distribution, with $\delta_N = \sigma\,[\log (N-N_0)]$ and $\bar{\Phi} = 1 - \Phi$, and

$$L(N) = \Phi\left[\frac{1-m_0(N-N_0)}{\sigma_0\sqrt{N-N_0}}\right] = \Phi\left[\left(1 - \frac{N-N_0}{\breve{N}-N_0}\right)/\left(\varepsilon\sqrt{\frac{N-N_0}{\breve{N}-N_0}}\right)\right] \tag{12.2}$$

for the proposed distribution, where $\varepsilon = \sigma_0/\sqrt{m_0}$. Introducing the variable

$$t' = (N-N_0)/(\breve{N}-N_0) \quad \text{or} \quad t' = (N-N_0)/(V-N_0), \tag{12.3}$$

the following results are obtained with the aid of Eq. (2.7).

For the extremal function

$$L(t') = \exp\,[-t'^{\alpha}],$$

the risk function is

$$r(t') = \alpha t'^{\alpha-1}, \tag{12.4}$$

for the logarithmic normal distribution

$$L(t') = \bar{\Phi}\left(\frac{1}{\delta_N} \log t\right),$$

the risk function is

$$r(t') = \frac{0.434}{\sqrt{2\pi}\, b_N t' \, \Phi\left(\frac{1}{\delta_N} \log t'\right)} \exp\left[-(\log t')^2/(2\delta_N^{\,2})\right] \quad (12.5)$$,

and for the proposed distribution

$$L(t') = \Phi\left[(1-t')/(\varepsilon\sqrt{t'})\right] ,$$

the risk function is

$$r(t') = \frac{1}{2\varepsilon\sqrt{2\pi}\, \Phi\left(\frac{1-t'}{\varepsilon\sqrt{t'}}\right)} \left(t'^{-3/2} + t'^{-1/2}\right) \exp\left[-\frac{1}{2}\left(\frac{1-t'}{\varepsilon\sqrt{t'}}\right)^2\right] \quad (12.6)$$.

The form of the functions (12.4), (12.5) and (12.6) differs significantly. As t' goes to infinity, $r(t')$ increases monotonically for the extremal distribution with $\alpha > 1$, it decreases to zero for the logarithmic-normal distribution and it tends towards a constant value $1/2\varepsilon^2$ for the proposed distribution.

The density-functions, survivorship-functions and risk functions for the three distributions have been plotted in Figs. 21-23.

It is worth noting, however, that the rate of failure associated with the proposed distribution tends towards a power function of N asymptotically if the assumption is made that the stress effect function is not only a function of stress level but also of number of load applications such that the average is $m_o(N) = a'' m_o N^{a''-1}$ and the standard deviation is $\sigma_o(N) = \sqrt{2}\sigma_o b''^{1/2} N^{(2b''-1)/2}$ leading to the survivorship function $L(N) = \Phi[(1-m_o N^{a''})/(\sigma_o N^{b''})]$ which is identical with Eq. (9.7) when $a'' = 1$ and $b'' = 1/2$.

IV. SAFETY ANALYSIS FOR COMBINED FATIGUE AND ULTIMATE LOAD FAILURE.

13. Survivorship Function for Combination of Failures. The final failure of a structure, the carrying capacity of which is progressively reduced by fatigue so that its probability of survival diminishes according to the reliability function characterizing the fatigue process, may occur prematurely under the single application of an "ultimate" load, if the momentarily existing carrying capacity, whether still virtually unaffected by fatigue or already somewhat reduced by it, cannot sustain this load. In fact the difference between final fatigue failure and ultimate load failure is only one of the load-intensity producing such failure: under conditions of developed extensive fatigue damage due to operational loads, an operational load intensity of relatively high probability of occurrence (short recurrence period) may cause final failure, while ultimate load failure is, by definition, a failure occurring under an exceptionally severe, exceptionally rare load intensity that happens to exceed the "static" carrying capacity of the structure, unaffected by fatigue. Thus, in the early part of the life of a structure, when fatigue damage is non-existent or insignificant the failure criterion is that of ultimate load failure. As fatigue damage accumulates and

reduces the "static" carrying capacity of the structure, a joint failure criterion must be formulated combining fatigue with ultimate load failure in such a way as to provide for the alternative possibilities of fatigue failure under essentially operational load conditions and ultimate load failure under exceptional loads. To this end the total load spectrum is divided, somewhat arbitrarily, into two parts; one, containing the operational loads up to a limiting load intensity (S_{max}) of a recurrence period about equal to the expected operational life, and considered to produce fatigue damage; the other made up of the load-intensities above this limit (S_{max}) which, because of their rare occurrence during the operational life, do not significantly contribute to fatigue.

The random application of N loads belonging to the first part will produce a survivorship function $L'(N)$ characteristic of fatigue and obtainable from random fatigue tests based on this part of the spectrum. The probability of survival $L'(N)$ is reduced by the danger that the occurrence of a load belonging to the second (high) part of the spectrum may cause "static" failure before the fatigue damage accumulation is extensive enough to make the probability of an actual fatigue failure under a load belonging to the first part of the spectrum significant. Obviously the probability of such "static" failure increases as the carrying capacity is gradually reduced by fatigue. It should, however, be kept in mind that, according to test-results, the reduction of the initial resisting area by only a few percent due to a spreading fatigue crack does not noticeably reduce the "static" carrying capacity.

The survivorship function for the combination of fatigue and ultimate load is obtained by adding the "risks" or rates of failure due to fatigue and the ultimate load, determining the associated survivorship function with the aid of Eq. (2.7). Hence

$$r(N) = -\, d\ln L(N)/dN = r'(N) + P_f(N) \tag{13.1}$$

where $r'(N)$ is the rate of fatigue failures and $P_f(N)$ the rate of ultimate failures. Since

$$L'(N) = \exp\left[-\int_0^N r'(N)\, dN\right], \tag{13.2}$$

Eq. (13.1) becomes by integration

$$L(N) = L'(N) \exp\left[-\int_0^N P_f(N)\, dN\right] \tag{13.3}$$

which is the basic equation for the determination of $L(N)$.

14. Distribution of Damage. The advantage of assuming a combined physical-statistical mechanism of fatigue damage such as Eq. (9.1) or Eq. (9.2) is not only that a survivorship function $L'(N)$ can be derived from it but also a distribution function of Δ can be established that is compatible with $L'(N)$.

Since the right hand side of Eq. (9.2) is governed by a normal distribution with mean $N\bar{m}_R$ and standard deviation $\bar{\sigma}_R \sqrt{N}$, it is possible to derive the distribution function $F_\Delta(\Delta)$ of Δ for a fixed value of N in the form

$$F_\Delta(\Delta) = \Phi\left[\frac{G(\Delta) - N\bar{m}_R}{\sqrt{N}\bar{\sigma}_R}\right] \tag{14.1}$$

where $G(\Delta)$ is assumed to be a monotonically increasing function of Δ. On the other hand, for a fixed value of Δ, the distribution function $F_N(N)$ is obtained in the form

$$F_N(N) = \bar{\Phi}\left[\frac{G(\Delta) - N\bar{m}_R}{\sqrt{N}\bar{\sigma}_R}\right] \tag{14.2}$$

which turns out to be Eq. (9.3) if Δ is replaced by Δ^*.

It is also feasible to construct a distribution function $F_\Delta(\Delta)$ which is compatible with the extremal distribution Eq. (11.6). One possible form is

$$F_\Delta(\Delta) = \exp\left\{-\left[\frac{(N-N_0)\bar{m}'_R}{G'(\Delta)}\right]^\alpha\right\} \tag{14.3}$$

where $\bar{m}'_R$ is a constant and $G_e(\Delta)$ is a monotonically increasing function of Δ.

For constant value of Δ, the distribution of N

$$F_N(N) = 1 - \exp\left\{-\left[\frac{(N-N_0)\bar{m}'_R}{G_e(\Delta)}\right]^\alpha\right\} \tag{14.4}$$

which turns out to be Eq. (11.6) when $\frac{\bar{m}'_R}{G_e(\Delta^*)} = \frac{1}{V-N_0}$ is introduced.

It seems however difficult to relate either Eq. (14.3) or Eq. (14.4) to a fatigue mechanism from which the dispersion of fatigue lives under random load would follow automatically.

In the following, Eq. (14.1) is used as the expression of the distribution of Δ unless otherwise indicated.

15. Mathematical Formulation of Risk of Failure. Assuming the form $c\Delta^K$ for the history function $g(\Delta)$ and for zero initial conditions, $G(\Delta) = \Delta^{K'}/K'c$ where $K' = 1 - K$ and $0 \le K \le 1$. Eq. (9.2) becomes, therefore,

$$\Delta^{K'} = c' \sum_{k=1}^{N} f(S_{i_k}) \tag{15.1}$$

which implies that the trend of the relationship

$$\Delta^{\kappa'} = c' m_i N \qquad \text{or} \qquad \Delta^{\kappa'} = c' \bar{m}_R N \tag{15.2}$$

is to be observed by experiment for constant or random loading, where $c' = \kappa' c$. For the present discussion, Δ is defined such that

$$\Delta = (R_1 - R_N)/R_1 \tag{15.3}$$

where R_1 and R_N are the initial and the momentary static strength of the structure or specimen respectively. Combining Eqs. (15.2) and (15.3),

$$\left(\frac{R_1 - R_N}{R_1}\right)^{\kappa'} = c' m_i N \quad (\text{ or } = c' \bar{m}_R N). \tag{15.4}$$

In the case of uniaxial loading $R_N = \tau_0 A_N$ and $R_1 = \tau_0 A_1$, where τ_0 is the material strength parameter and A_N and A_1 are the momentary and the initial cross-sectional areas of the structure or specimen. Inserting these into Eqs. (15.3) and (15.4),

$$\Delta = (A_1 - A_N)/A_1 \tag{15.3'}$$

and

$$\left(\frac{A_1 - A_N}{A_1}\right)^{\kappa'} = c' m_i N . \tag{15.4'}$$

The latter relation is fairly well realized in experiments, for example those performed by W. Weibull [18] and by J.L. Kepert and A.O. Payne. [19] Using logarithmic scale for both Δ and N, Fig. 24 shows the experimental results, to which straight lines can be fitted. This suggests the validity of Eq. (15.4') at least as a first approximation. On the basis of this consideration, the mathematical formulation of the risk of failure $r(N) = r'(N) + P_f(N)$ can be established, emphasizing the determination of the risk of failure $P_f(N)$ due to the load from extreme portion of the spectrum since $r'(N)$ is derivable from the known function $L'(N)$.

Writing $R_N = R_1 \phi_N(\Delta)$ and $\nu'_N = R_N/S = \phi_N(\Delta)\, R_1/S$ where $S > S_{max}$ and $\phi_N(\Delta)$ is a monotonically decreasing function of Δ, and also writing $\nu' = R_1/S$, the risk of failure $P_f(N)$ (the momentary probability of failure) is given by either Eq. (15.5) or Eq. (15.6).

$$P_f(N) = P(\nu'_N < 1) = P\left[\nu' \, \phi_N(\Delta) < 1\right] \tag{15.5}$$

$$P_f(N) = P(R_N - S < 0) = P\left[R_1 \phi_N(\Delta) - S < 0\right] \tag{15.6}$$

It is possible to obtain the distribution of Δ from the postulate that $G(\Delta)$ is normally distributed: the knowledge of this distribution in turn permits the determination of the distribution of $\phi_N(\Delta)$.

For the evaluation of $P_f(N)$, the approach implied in Eq. (15.5) is used; denoting the conditional probability density function of S relative to the hypothesis that $S > S_{max}$ by $f_{S_2}(S) = f_s(S)/p_t$ $(S > S_{max})$ where $p_t = 1-F_S(S_{max})$, the conditional distribution $F_{\nu'}(\nu')$ of $\nu' = R_1/S$ relative to the same hypothesis is obtained making use of $f_{S_2}(S)$ and $f_{R_1}(R_1)$ in the same manner as in Part I. The conditional distribution $F_{\nu'_N}(\nu'_N)$ is obtained as the distribution of the product of ν' and $\phi(\Delta)$. $P_f(N)$ then turns out to be $p_t F_{\nu'_N}(1)$.

In the following discussion, the statistical variation of R_1 is neglected. This can be justified by the fact that the statistical variation of R_1 is much narrower than that of $\phi_N(\Delta)$. However, even with this simplifying assumption the distributions of both ν'_N and $R_N - S$ turn out to be of a form which makes actual analysis extremely difficult except for $\kappa' = 1$. Since

$$\Delta = (R_1 - R_N)/R_1,$$

$$\phi_N(\Delta) = 1 - \Delta \tag{15.7}$$

and

$$\nu'_N = \nu'(1 - \Delta). \tag{15.8}$$

The distribution of $\nu' = R_1/S$ under the assumption that R_1 is constant is directly obtained from the distribution of S.

Assuming an exponential probability density

$$f_S(S) = h' \exp\left[-h'(S-S_{min})\right] \quad (S \geqq S_{min}) \tag{15.9}$$

which represents, for instance, the gust-load distribution for airplane wings fairly well, the truncated distribution $f_{S_1}(S)$, obtained by considering that values $S > S_{max}$ do not contribute to fatigue damage but may cause ultimate load failure, can be written in the form

$$f_{S_1}(S) = h' \exp\left[-h'(S-S_{min})\right] \Big/ \left\{1-\exp\left[-h'(S_{max}-S_{min})\right]\right\} \quad (S_{max} \geqq S \geqq S_{min}). \tag{15.10}$$

Use of $f_{S_2}(S) = h' \exp\left[-h'(S-S_{max})\right]$ that is the distribution of S larger than S_{max} in Eq. (15.9) as the extreme portion of the load spectrum produces

$$F_{\nu'}(\nu') = \exp\left[-h'(R_1/\nu' - S_{max})\right] \tag{15.11}$$

which gives the density function,

$$f_{\nu'}(\nu') = \frac{h'R_1}{\nu'^2} \exp\left[-h'(R_1/\nu' - S_{max})\right] . \qquad (15.12)$$

By virtue of Eq. (15.1), Δ itself is normally distributed with mean $\bar{m}_R N$ and standard deviation $\bar{\sigma}_R \sqrt{N}$, assuming $\kappa' = 1$ and $c = 1$. Therefore the distribution of $\bar{\Delta} = 1 - \Delta$ is also normal with probability density

$$f_{\bar{\Delta}}(\bar{\Delta}) = \frac{1}{\sqrt{2\pi N}\,\bar{\sigma}_R} \exp\left\{-\left[\bar{\Delta} - (1-\bar{m}_R N)\right]^2/(2N\bar{\sigma}^2_R)\right\} . \qquad (15.13)$$

Making use of Eq. (15.12) and (15.13) it can be shown that the distribution function $F_{\nu'_N}(\nu'_N)$ of $\nu'_N = \nu'(1-\Delta) = \nu'\bar{\Delta}$ is given by

$$F_{\nu'_N}(\nu'_N) = \bar{\Phi}\left[(S_{max}\nu'_N - \bar{R}_N + h'\sigma^2_{R_N}/\nu'_N)/\sigma_{R_N}\right] \exp\left[h'(S_{max} - \bar{R}_N/\nu'_N) + \right.$$

$$\left. + \frac{1}{2} h'^2 \sigma^2_{R_N}/\nu'^2_N\right] + \Phi\left[(S_{max}\nu'_N - \bar{R}_N)/\sigma_{R_N}\right] \qquad (15.14)$$

where $\bar{R}_N = R_1(1 - \bar{m}_R N)$ and $\sigma_{R_N} = R_1 \sqrt{N}\,\bar{\sigma}_R$. Then $P_f(N)$ is

$$P_f(N) = p_t \times \left\{\bar{\Phi}\left[(S_{max} - \bar{R}_N + h'\sigma^2_{R_N})/\sigma_{R_N}\right] \exp\left[h'(S_{max} - \bar{R}_N) + \right.\right.$$

$$\left.\left. + h'^2\sigma^2_{R_N}/2\right] + \left[(S_{max} - \bar{R}_N)/\sigma_{R_N}\right]\right\} \qquad (15.15)$$

where $p_t \equiv P(S > S_{max}) = \exp\left[-h'(S_{max} - S_{min})\right]$.

When $\kappa' \neq 1$, the best way is to apply Eq. (1.2) directly in conjunction with Eq. (15.6):

$$P_f(N) = \int_{S_{max}}^{\infty} h' \exp\left[-h'(S - S_{min})\right] F_{R_N}(S)\,dS \qquad (15.16)$$

Since $G(\Delta) = \frac{1}{c'}\left[(R_1 - R_N)/R_1\right]^{\kappa'}$ is normally distributed with mean $N\bar{m}_R$ and standard deviation $\sqrt{N}\bar{\sigma}_R$,

$$F_{R_N}(R_N) = \bar{\Phi}\left[\frac{(R_1 - R_N)^{\kappa'} - c' N\bar{m}_R R_1^{\kappa'}}{c' \sqrt{N}\bar{\sigma}_R R_1^{\kappa'}}\right] . \qquad (15.17)$$

Substituting Eq. (15.17) into Eq. (15.16), the desired expression for $P_f(N)$ is obtained:

$$P_f(N) = \int_{S_{max}}^{\infty} h' \exp\left[-h'(S - S_{min})\right] \bar{\Phi}\left[\frac{(R_1 - S)^{\kappa'} - c' N\bar{m}_R R_1^{\kappa'}}{c' \sqrt{N}\bar{\sigma}_R R_1^{\kappa'}}\right] dS \qquad (15.18)$$

It is also possible to establish the expression when the distribution of Δ is given by Eq. (14.3). Assuming $G_e(\Delta)$ has the form $G_e(\Delta) = \Delta^{\lambda}/d$ (λ,d = constants), the trend of the relation between Δ and N is

$$\Delta^{\lambda} = d\bar{m}'_R(N - N_0) \tag{15.19}$$

which is essentially the same as Eq. (15.2). The distribution of R_N is

$$F_{R_N}(R_N) = 1\text{-exp}\left[-\left\{\frac{[d(N-N_0)\bar{m}'_R]^{1/\lambda} R_1}{R_1 - R_N}\right\}^{\alpha\lambda}\right]. \tag{15.20}$$

Finally,

$$P_f(N) = \int_{S_{max}}^{\infty} h'\exp\left[-h'(S-S_{min})\right]\left\{1\text{-exp}\left[-\left\{\frac{[d(N-N_0)\bar{m}'_R]^{1/\lambda} R_1}{R_1 - S}\right\}^{\alpha\lambda}\right]\right\} dS. \tag{15.21}$$

The survivorship function L(N) can now be obtained with the aid of Eq. (13.3).

16. Numerical Example. A numerical example is based on tests on 2024 aluminum which has an initial strength R_1 = 64,000 psi [14]. A set of 20 specimens of this material was subjected to random fatigue tests with six stress levels applied to the specimen in a randomized order with the cycle ratios drawn from an exponential distribution. Table 16 shows these stress levels and their cycle ratios. The exponential distribution of the load is shown schematically in Fig. 25. Using Eq. (15.9) for the form of this distribution, $h' = 2.7 \times 10^{-4}$ and S_{min} = 19,200 psi.

$$\left.\begin{aligned} f_S(S) &= 2.70\times10^{-4}\exp\left[-2.70\times10^{-4}(S - 19{,}200)\right] \\ f_{S_2}(S) &= 2.70\times10^{-4}\exp\left[-2.70\times10^{-4}(S - 57{,}600)\right] \end{aligned}\right\} \tag{16.1}$$

This experiment is understood in such a way that the part of the distribution for which S is greater than 57,600 psi was truncated: in other words, $f_{S_1}(S)$ is obtained by setting in Eq. (15.10) S_{min} = 19,200 psi and S_{max} = 57,600 psi. Then $p_t = P(S > 57{,}600) = 3.16\times10^{-5}$ and $P(S > 64{,}000) = 5.62\times10^{-6}$. It is worth noting that the latter probability is $P_f(0)$. Furthermore $\bar{m}_{R0} = 2.98\times10^{-7}$ and $\bar{\sigma}_{R0} = 2.53\times10^{-4}$ are obtained by plotting the result of the above mentioned random fatigue test on Fig. 26.

For expediency it is assumed in the following numerical example that $K' = 1$ and c = 1 so that Eq. (15.15) can be used. For the example considered the quantities in Eq. (15.15) are

$p_t = 3.16 \times 10^{-5}$, $\quad S_{max}$ = 57,600 psi

$h' = 2.70 \times 10^{-4}$, $\quad \bar{R}_N = 64{,}000\,(1 - \bar{m}_R N)$

and $\quad \sigma_{R_N} = 64{,}000\,\sqrt{N}\,\bar{\sigma}_R$

where $\bar{m}_R$ and $\bar{\sigma}_R$ are evaluated after the determination of G^* [$= G(\Delta^*)$].

For $\kappa' = 1$,

$$G^* = \Delta^* = (R_1 - R_N^*)/R_1$$

where R_N^* is the static strength such that the specimen fails when its static strength is reduced to this value. Strictly speaking, this definition does not determine R_N^* uniquely because of the statistical character of the loading.

The assumption that the extreme part of the load spectrum does not contribute to fatigue damage is equivalent to the statement that $\bar{m}_{R0}$ and $\bar{\sigma}_{R0}$ are not affected by this part of the spectrum. Considering this assumption together with the fact that there is no stress in the spectrum $f_{S1}(S)$ exceeding 54,000 psi and therefore $R_N^* < 54,400$ psi, and denoting by T_R^* the return number of loads higher than R_N^*, the trend of the decrease of the static strength R_N from $R = 54,400$ psi caused by T_R^* load applications can be expressed by Eq. (16.2) making use of Eq. (15.2) with $\kappa' = 1$ and $c' = 1$ since $\Delta = (64,000 - 54,400)/64,000 = 0.15$ when $R = 54,400$ psi.

$$\Delta^* - 0.15 = \bar{m}_{R0}\,\Delta^* T_R^* \tag{16.2}$$

Conversely, Eq. (16.2) is used for the determination of R_N^* as a first approximation because both Δ^* and T_R^* are functions of R_N^*.

Assuming R_N^* to be very close to 54,400 psi, the return number $T_R^* = 1/0.00018 = 5.55 \times 10^3$ from which R_N^* is obtained with the aid of Eq. (16.2).

$$R_N^* = 54,384 \doteq 54,400 \text{ psi}$$

Hence, the value 54,400 psi can be used as R_N^* or 0.15 as Δ^*. $\bar{m}_R$ and $\bar{\sigma}_R$ are, therefore,

$$\bar{m}_R = 0.15\,\bar{m}_{R0} = 4.47 \times 10^{-8}$$

$$\bar{\sigma}_R = 0.15\,\bar{\sigma}_{R0} = 3.80 \times 10^{-5}.$$

All information to make use of Eq. (15.15) for the determination of $p_f(N)$ has now been obtained. The result is listed in Table 17 and shown in Fig. 27. The value of $L'(N)$ is also tabulated in Table 18 and shown in Fig. 28. $\exp[-\int_0^N p_f(N)dN]$ is evaluated by the method of numerical integration and listed in Table 19. Since $L'(N)$ is nearly equal to unity for $N < 10^6$, Table 19 gives $L(N) = L'(N)\exp[-\int_0^N p_f(N)dN]$ which is also presented in Fig. 28. From this result it can be concluded that the fatigue life would be reduced by approximately one order of magnitude due to the application of loads pertaining to the extreme portion of the spectrum.

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TABLE 1

Relation between Standard Deviation $\delta_R(\delta_S)$ of $\log R(\log S)$ and Coefficient of Variation $v_R(v_S)$ based on mean $\bar{R}(\bar{S})$, or $\sigma_R/\breve{R}(\sigma_S/\breve{S})$ based on Median $\breve{R}(\breve{S})$ of R(S) for Logarithmic Normal Distribution

$\frac{\sigma_R}{\breve{R}}$ $(\frac{\sigma_S}{\breve{S}})$	0.05	0.10	0.15	0.20	0.30
$\delta_R(\delta_S)$	0.0217	0.0431	0.0641	0.0844	0.123
$v_R(v_S)$	0.0499	0.0995	0.148	0.196	0.288

TABLE 2

Standard Deviation of $\log R \pm \log S$ as a Function of $\sigma_S/\breve{S}$ and $\sigma_R/\breve{R}$ for Logarithmic Normal Distribution of R and S.

$\sigma_S/\breve{S}$	0.10	0.10	0.10	0.20	0.20	0.20	0.30	0.30	0.30
$\sigma_R/\breve{R}$	0.05	0.10	0.15	0.05	0.10	0.15	0.05	0.10	0.15
δ	0.04824	0.06096	0.07724	0.08717	0.09480	0.10601	0.12462	0.13008	0.13846

TABLE 3

Relation between Probability of Failure P_f and Central Safety Factor ν_0 for Logarithmic Normal Distributions of R and S

$\frac{\sigma_S}{\bar{S}}$	$\frac{\sigma_R}{\bar{R}}$	P_f					
		10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
0	0.05	1.06	1.12	1.17	1.20	1.24	1.27
0	0.10	1.14	1.26	1.36	1.45	1.53	1.60
0	0.15	1.21	1.41	1.58	1.73	1.88	2.02
0.10	0.05	1.15	1.30	1.41	1.51	1.61	1.70
0.10	0.10	1.20	1.39	1.54	1.68	1.82	1.95
0.10	0.15	1.26	1.51	1.74	1.93	2.14	2.33
0.20	0.05	1.29	1.60	1.86	2.11	2.35	2.60
0.20	0.10	1.32	1.66	1.96	2.25	2.53	2.82
0.20	0.15	1.37	1.77	2.13	2.47	2.83	3.19
0.30	0.05	1.45	1.95	2.44	2.90	3.40	3.91
0.30	0.10	1.47	2.01	2.57	3.04	3.58	4.15
0.30	0.15	1.50	2.10	2.68	3.26	3.89	4.55

TABLE 4

Relation between Parameter α (β) and Coefficient of Variation v_R (v_S) based on Mean $\bar{R}$ ($\bar{S}$) or $\sigma_R/\tilde{R}$ ($\sigma_S/\tilde{S}$) based on Characteristic Values $\tilde{R}$ ($\tilde{S}$) for Extremal Distribution of R(S)

(a)

$\sigma_R/\tilde{R}$	0.05	0.10	0.15
v_R	0.0511	0.104	0.160
α	24.4	11.8	7.41

(b)

$\sigma_S/\tilde{S}$	0.10	0.20	0.30
v_S	0.956	0.183	0.264
β	14.3	7.84	5.75

TABLE 5

Relation between Probability of Failure P_f and Central Safety Factor ν_0 for Extremal Distributions of R and S

$\frac{\sigma_S}{\tilde{S}}$	$\frac{\sigma_R}{\tilde{R}}$	P_f					
		10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
0	0.05	1.10	1.20	1.33	1.45	1.60	1.77
0	0.10	1.20	1.50	1.80	2.17	2.60	3.20
0	0.15	1.30	1.85	2.60	3.55	4.80	6.45
0.10	0.05	1.20	1.50	1.75	2.00	2.30	2.75
0.10	0.10	1.30	1.70	2.05	2.50	3.00	3.70
0.10	0.15	1.45	2.00	2.75	3.80	5.20	6.90
0.20	0.05	1.40	1.80	2.50	3.35	4.40	6.00
0.20	0.10	1.45	2.00	2.70	3.60	4.90	6.90
0.20	0.15	1.60	2.30	3.40	4.70	6.40	8.85
0.30	0.05	1.50	2.30	3.50	5.20	7.70	11.30
0.30	0.10	1.60	2.45	3.70	5.50	8.20	11.80
0.30	0.15	1.70	2.70	4.25	6.35	9.50	14.00

TABLE 6

Ratios r_p and s_q for Logarithmic Normal Distributions

(a) r_p

$\sigma_R/\breve{R}$	0.05	0.10	0.15
δ_R	0.0217	0.0431	0.0641
$r_{0.1}$	0.938	0.881	0.828

(b) s_q

$\sigma_S/\breve{S}$	0.10	0.20	0.30
δ_S	0.0431	0.0844	0.123
$s_{0.1}$	1.14	1.28	1.44
$s_{0.01}$	1.26	1.57	1.93
$s_{0.001}$	1.36	1.82	2.39
$s_{0.0001}$	1.43	2.06	2.85

TABLE 7

Ratios r_p and s_q for Extremal Distributions

(a) r_p

$\sigma_R/\tilde{R}$	0.05	0.10	0.15
α	24.4	11.8	7.41
$r_{0.1}$	0.912	0.826	0.738

(b) s_q

$\sigma_S/\tilde{S}$	0.10	0.20	0.30
β	14.3	7.84	5.75
$s_{0.1}$	1.17	1.33	1.48
$s_{0.01}$	1.38	1.80	2.23
$s_{0.001}$	1.62	2.41	3.32
$s_{0.0001}$	1.90	3.24	4.96

TABLE 8

Improvement of Material Control (in Terms of Decrease of $\sigma_R/\bar{R}$) as a Function of n to Ensure Constant Probability of Failure $P_f = 10^{-6}$ of a Non-Redundant Structure of n Members ($\nu_0 = 5.0$)

n	1	2	5	10	20	50	100	200	500	1000	10,000
P_k	10^{-6}	5×10^{-7}	2×10^{-7}	10^{-7}	5×10^{-8}	2×10^{-8}	10^{-8}	5×10^{-9}	2×10^{-9}	10^{-9}	10^{-10}
$\sigma_S/\bar{S} = 0$	0.372	0.358	0.343	0.333	0.322	0.313	0.306	0.298	0.290	0.284	0.265
$\sigma_S/\bar{S} = 0.10$	0.351	0.340	0.324	0.311	0.303	0.292	0.284	0.276	0.269	0.261	0.243
$\sigma_S/\bar{S} = 0.20$	0.296	0.282	0.263	0.252	0.238	0.226	0.219	0.209	0.198	0.190	0.166
$\sigma_S/\bar{S} = 0.30$	0.193	0.174	0.148	0.127	0.105	0.0777	0.0447				

TABLE 9

Increase of Central Safety Factor ν_0 as Function of n to Ensure Constant Probability of Failure $P_f = 10^{-6}$ of a Non-Redundant Structure of n Members

$\sigma_S / \bar{S}$	$\sigma_R / \bar{R}$	n	1	2	5	10	20	50	100	200	500	1000	10,000
		p_k	10^{-6}	$5x10^{-7}$	$2x10^{-7}$	10^{-7}	$5x10^{-8}$	$2x10^{-8}$	10^{-8}	$5x10^{-9}$	$2x10^{-9}$	10^{-9}	10^{-10}
0	0.05		1.27	1.28	1.29	1.30	1.31	1.32	1.32	1.33	1.34	1.35	1.37
0	0.10		1.60	1.62	1.65	1.68	1.70	1.72	1.74	1.76	1.79	1.81	1.88
0	0.15		2.01	2.06	2.11	2.15	2.20	2.25	2.29	2.33	2.38	2.42	2.55
0.10	0.05		1.69	1.72	1.76	1.78	1.81	1.84	1.86	1.89	1.92	1.95	2.01
0.10	0.10		1.95	1.99	2.04	2.08	2.12	2.16	2.20	2.23	2.28	2.32	2.44
0.10	0.15		2.32	2.38	2.46	2.52	2.58	2.66	2.71	2.77	2.85	2.90	3.10
0.20	0.05		2.59	2.66	2.77	2.84	2.92	3.01	3.08	3.15	3.26	3.33	3.58
0.20	0.10		2.82	2.90	3.03	3.11	3.20	3.32	3.40	3.49	3.61	3.70	4.01
0.20	0.15		3.18	3.29	3.45	3.56	3.68	3.82	3.92	4.04	4.20	4.32	4.73
0.30	0.05		3.90	4.06	4.38	4.45	4.61	4.83	4.98	5.16	5.40	5.58	6.20
0.30	0.10		4.14	4.32	4.57	4.75	4.94	5.18	5.36	5.55	5.82	6.02	6.72
0.30	0.15		4.53	4.75	5.04	5.25	5.48	5.76	5.97	6.20	6.52	6.75	7.61

TABLE 10

Separte Load Factors α_q and β_q as Functions of Probability of Failure

(a) $\alpha_{0.01}$ and $\beta_{0.01}$ (γ = 3.0)

k	4.0	5.0	6.0	6.5	7.0
ν_0	1.07	1.34	1.60	1.74	1.87
P_f	3.30×10^{-1}	2.02×10^{-2}	1.90×10^{-4}	1.00×10^{-5}	4.07×10^{-7}
$\alpha_{0.01}$	0.991	0.997	1.004	1.008	1.011
$\beta_{0.01}$	0.685	0.912	1.138	1.251	1.365

(b) $\alpha_{0.01}$ and $\beta_{0.01}$ (γ = 0.5)

k	1.6	1.8	1.9	2.0	2.1
ν_0	1.14	1.29	1.35	1.43	1.50
P_f	6.81×10^{-2}	1.90×10^{-3}	2.24×10^{-4}	1.78×10^{-5}	1.15×10^{-6}
$\alpha_{0.01}$	0.965	1.005	1.021	1.041	1.060
$\beta_{0.01}$	0.820	1.045	1.160	1.270	1.385

TABLE 11

Means m_{i0}, $\bar{m}_{R0}$ (in Terms of $\bar{G}^{*}$) and Standard Deviations σ_{i0}, $\bar{\sigma}_{R0}$ of Stress Effect Function

Test No.	1	2	3	4	5
Stress* Level	const. 29,500	const. 37,300	const. 44,900	const. 52,600	const. 60,300
m_{i0}	1.48×10^{-7}	7.87×10^{-7}	4.31×10^{-6}	1.64×10^{-5}	4.65×10^{-5}
σ_{i0}	1.61×10^{-4}	3.04×10^{-4}	6.20×10^{-4}	1.16×10^{-3}	1.00×10^{-3}

Test No.	6	20	21	22	23
Stress Level	const. 68,000	random	random	random	random
m_{i0}, $\bar{m}_{R0}$	1.38×10^{-4}	1.11×10^{-6}	5.49×10^{-6}	2.21×10^{-5}	8.34×10^{-7}
σ_{i0}, $\bar{\sigma}_{R0}$	2.71×10^{-3}	3.99×10^{-4}	5.05×10^{-4}	8.64×10^{-4}	4.06×10^{-4}

Test No.	24	25	26	27	28
Stress Level	random	random	random	random	random
$\bar{m}_{R0}$	4.97×10^{-6}	2.03×10^{-5}	1.35×10^{-7}	3.39×10^{-6}	1.25×10^{-5}
$\bar{\sigma}_{R0}$	6.10×10^{-4}	8.68×10^{-4}	2.03×10^{-4}	6.71×10^{-4}	9.47×10^{-4}

* Stresses are given in psi.

TABLE 12

Load Spectra for Random Fatigue Tests of AA 7075 Al.

Test Series No.	s_i / h	.35	.45	.55	.65	.75	.85	.95
20	17.3	.822	.1456	.02664	.00458	.00100	.000182	
21	17.3		.822	.1456	.02664	.00458	.00100	.000182
22	17.3			.822	.1456	.02664	.00458	.00100
23	22.9	.900	.0900	.00900	.000900	.0000900	.0000100	
24	22.9		.900	.0900	.00900	.000900	.0000900	.0000100
25	22.9			.900	.0900	.00900	.000900	.0000900
26	34.3	.9684	.0306	.00100	.000030	.000001	.00000003	
27	34.3		.9684	.0306	.00100	.000030	.00000100	.00000003
28	34.3			.9684	.030600	.00100	.0000300	.00000100

TABLE 13

Inverse of the Slope, η, of log S - log m_0 Diagram

Test No.	20	21	22	23	24	25	26	27	28
η	6.74	6.68	6.14	6.48	6.32	5.88	8.24	6.56	6.54

TABLE 14

Stress Interaction Factor ω_i, Mean m^*_{i0} and Standard Deviation σ^*_{i0} (in Terms of G^*) of Modified Stress Effect Function.

Test No.	Stress level S_i	S_1	S_2	S_3	S_4	S_5	S_6	S_7
	(lb/in²)	2.87×10^{4}	3.69×10^{4}	4.51×10^{4}	5.33×10^{4}	6.15×10^{4}	6.97×10^{4}	7.79×10^{4}
	m_{i0}	1.12×10^{-7}	8.83×10^{-7}	4.60×10^{-6}	1.91×10^{-5}	5.96×10^{-5}	1.66×10^{-4}	4.12×10^{-4}
	σ_{i0}	1.35×10^{-4}	3.07×10^{-4}	5.90×10^{-4}	1.04×10^{-3}	1.63×10^{-3}	2.45×10^{-3}	3.52×10^{-3}
20	p_i	0.822	0.1456	0.02664	0.00458	0.001	0.000182	
	ω_i	3.79	2.59	1.92	1.50	1.20	1.00	
	m^*_{i0}	4.23×10^{-7}	2.29×10^{-6}	8.79×10^{-6}	2.86×10^{-5}	7.17×10^{-5}	1.66×10^{-4}	
	σ^*_{i0}	2.29×10^{-4}	4.48×10^{-4}	7.64×10^{-4}	1.22×10^{-3}	1.75×10^{-3}	2.46×10^{-3}	
21	p_i		0.822	0.1456	0.02664	0.00453	0.001	0.000182
	ω_i		3.20	2.35	1.85	1.45	1.19	1.00
	m^*_{i0}		2.82×10^{-6}	1.08×10^{-5}	3.44×10^{-5}	8.65×10^{-5}	1.98×10^{-4}	4.12×10^{-4}
	σ^*_{i0}		4.87×10^{-4}	8.28×10^{-4}	1.31×10^{-3}	1.89×10^{-3}	2.63×10^{-3}	3.52×10^{-3}
22	p_i			0.822	0.1456	0.02664	0.00453	0.001
	ω_i			3.16	2.21	1.65	1.27	1.00
	m^*_{i0}			1.45×10^{-5}	4.21×10^{-5}	9.84×10^{-5}	2.10×10^{-4}	4.12×10^{-4}
	σ^*_{i0}			9.31×10^{-4}	1.42×10^{-3}	1.99×10^{-3}	2.69×10^{-3}	3.52×10^{-3}
23	p_i	0.9	0.09	0.009	0.0009	0.00009	0.000009	
	ω_i	4.78	3.06	2.14	1.61	1.24	1.00	
	m^*_{i0}	5.32×10^{-7}	2.71×10^{-6}	9.84×10^{-6}	3.06×10^{-5}	7.38×10^{-5}	1.66×10^{-4}	
	σ^*_{i0}	2.51×10^{-4}	4.79×10^{-4}	7.98×10^{-4}	1.25×10^{-3}	1.78×10^{-3}	2.45×10^{-3}	

Table 14 (Cont'd)

Test No.		S_1	S_2	S_3	S_4	S_5	S_6	S_7
24	p_i		0.9	0.09	0.009	0.0009	0.00009	0.000009
	ω_i		4.19	2.86	2.06	1.59	1.24	1.00
	m^*_{i0}		3.70×10^{-6}	1.32×10^{-5}	3.94×10^{-5}	9.45×10^{-5}	2.06×10^{-4}	4.12×10^{-4}
	σ^*_{i0}		5.41×10^{-4}	8.96×10^{-4}	1.38×10^{-3}	1.96×10^{-3}	2.67×10^{-3}	3.52×10^{-3}
25	p_i			0.9	0.09	0.009	0.0009	0.00009
	ω_i			3.65	2.44	1.76	1.31	1.00
	m^*_{i0}			1.67×10^{-5}	4.64×10^{-5}	1.05×10^{-4}	2.17×10^{-4}	4.12×10^{-4}
	σ^*_{i0}			9.87×10^{-4}	1.48×10^{-3}	2.04×10^{-3}	2.72×10^{-3}	3.52×10^{-3}
26	p_i	0.9684	0.0306	0.001	0.00003	0.000001	0.00000003	
	ω_i	0.955	0.969	0.977	0.986	0.993	1.00	
	m^*_{i0}	1.06×10^{-7}	8.55×10^{-7}	4.49×10^{-6}	1.88×10^{-5}	5.92×10^{-5}	1.66×10^{-4}	
	σ^*_{i0}	1.32×10^{-4}	3.03×10^{-4}	5.85×10^{-4}	1.03×10^{-3}	1.63×10^{-3}	2.45×10^{-3}	
27	p_i		0.9684	0.0306	0.001	0.00003	0.000001	0.00000003
	ω_i		3.50	2.51	1.89	1.50	1.21	1.00
	m^*_{i0}		3.09×10^{-6}	1.15×10^{-5}	3.60×10^{-5}	8.91×10^{-5}	2.01×10^{-4}	4.12×10^{-4}
	σ^*_{i0}		5.05×10^{-4}	8.52×10^{-4}	1.34×10^{-3}	1.92×10^{-3}	2.64×10^{-3}	3.52×10^{-3}
28	p_i			0.9684	0.0306	0.001	0.00003	0.000001
	ω_i			2.54	1.90	1.50	1.21	1.00
	m^*_{i0}			1.17×10^{-5}	3.62×10^{-5}	8.95×10^{-5}	2.01×10^{-4}	4.12×10^{-4}
	σ^*_{i0}			8.55×10^{-4}	1.34×10^{-3}	1.92×10^{-3}	2.64×10^{-3}	3.52×10^{-3}

TABLE 15

Comparison of Standard Deviation $\bar{\sigma}_{R0}$ Estimated by Various Methods

Test No.	20	21	22	23	24
Experiment	3.99×10^{-4}	5.05×10^{-4}	8.64×10^{-4}	4.06×10^{-4}	6.10×10^{-4}
Theory	3.14×10^{-4}	6.05×10^{-4}	1.08×10^{-3}	2.84×10^{-4}	5.97×10^{-4}
Eq.(11-2")	3.35×10^{-4}	6.32×10^{-4}	1.10×10^{-3}	3.08×10^{-4}	6.08×10^{-4}

Test No.	25	26	27	28
Experiment	8.68×10^{-4}	2.03×10^{-4}	6.71×10^{-4}	9.47×10^{-4}
Theory	1.06×10^{-3}	1.44×10^{-4}	5.20×10^{-4}	8.75×10^{-4}
Eq.(11-2")	1.07×10^{-3}	1.46×10^{-4}	5.23×10^{-4}	8.85×10^{-4}

TABLE 16

Load Spectrum for Random Fatigue Test of AA 2024 Al.

Stress Level	S_1	S_2	S_3	S_4	S_5	S_6	S_7
S in S_u	0.35	0.45	0.55	0.65	0.75	0.85	0.95
S in psi	22,400	28,800	35,200	41,600	48,000	54,400	60,800
Cycle Ratio	.822	.1456	.02664	.00458	.00100	.000182	0

TABLE 17

Probability of Failure $P_f(N)$ as a Function of N

N	10^3	10^4	10^5	$2x10^5$	$4x10^5$	$6x10^5$	$8x10^5$
$P_f(N)$	$5.62x10^{-6}$	$5.63x10^{-6}$	$6.05x10^{-6}$	$7.05x10^{-6}$	$8.47x10^{-6}$	$1.05x10^{-5}$	$1.25x10^{-}$
log P_f	$\bar{6}.750$	$\bar{6}.750$	$\bar{6}.781$	$\bar{6}.848$	$\bar{6}.928$	$\bar{5}.021$	$\bar{5}.097$

N	10^6	$2x10^6$	$4x10^6$	$6x10^6$	$8x10^6$	10^7
$P_f(N)$	$1.43x10^{-5}$	$2.26x10^{-5}$	$2.96x10^{-5}$	$3.11x10^{-5}$	$3.15x10^{-5}$	$3.16x10^{-5}$
log P_f	$\bar{5}.155$	$\bar{5}.354$	$\bar{5}.471$	$\bar{5}.493$	$\bar{5}.498$	$\bar{5}.500$

TABLE 18

Survivorship Function associated with Fatigue

$L'(N)$

N	10^3	10^4	10^5	$2x10^5$	$4x10^5$	$6x10^5$	$8x10^5$
$L'(N)$	1	1	1	1	1	$1-1.51x10^{-5}$	0.9996
$\log L'(N)$	0	0	0	0	0	0	$\doteq 0$

N	10^6	$2x10^6$	$4x10^6$	$6x10^6$	$8x10^6$	10^7	$2x10^7$
$L'(N)$	0.9972	0.8686	0.352	0.102	0.0274	0.00678	$5.37x10^{-6}$
$\log L'(N)$	$\bar{1}.998$	$\bar{1}.939$	$\bar{1}.546$	$\bar{1}.009$	$\bar{2}.438$	$\bar{3}.831$	$\bar{6}.730$

TABLE 19

Survivorship Function for the Combination of Fatigue and Ultimate Load

$$\exp\left[-\int_0^N P_f(N)\,dN\right]$$

N	10^4	$2x10^4$	$4x10^4$	$6x10^4$	$8x10^4$	10^5
$\exp\left[-\int_0^N P_f(N)dN\right]$	0.944	0.893	0.798	0.700	0.615	0.549

N	$2x10^5$	$4x10^5$	$6x10^5$	$8x10^5$	10^6
$\exp\left[-\int_0^N P_f(N)dN\right]$	0.283	0.0603	0.00912	0.000912	0.0000631

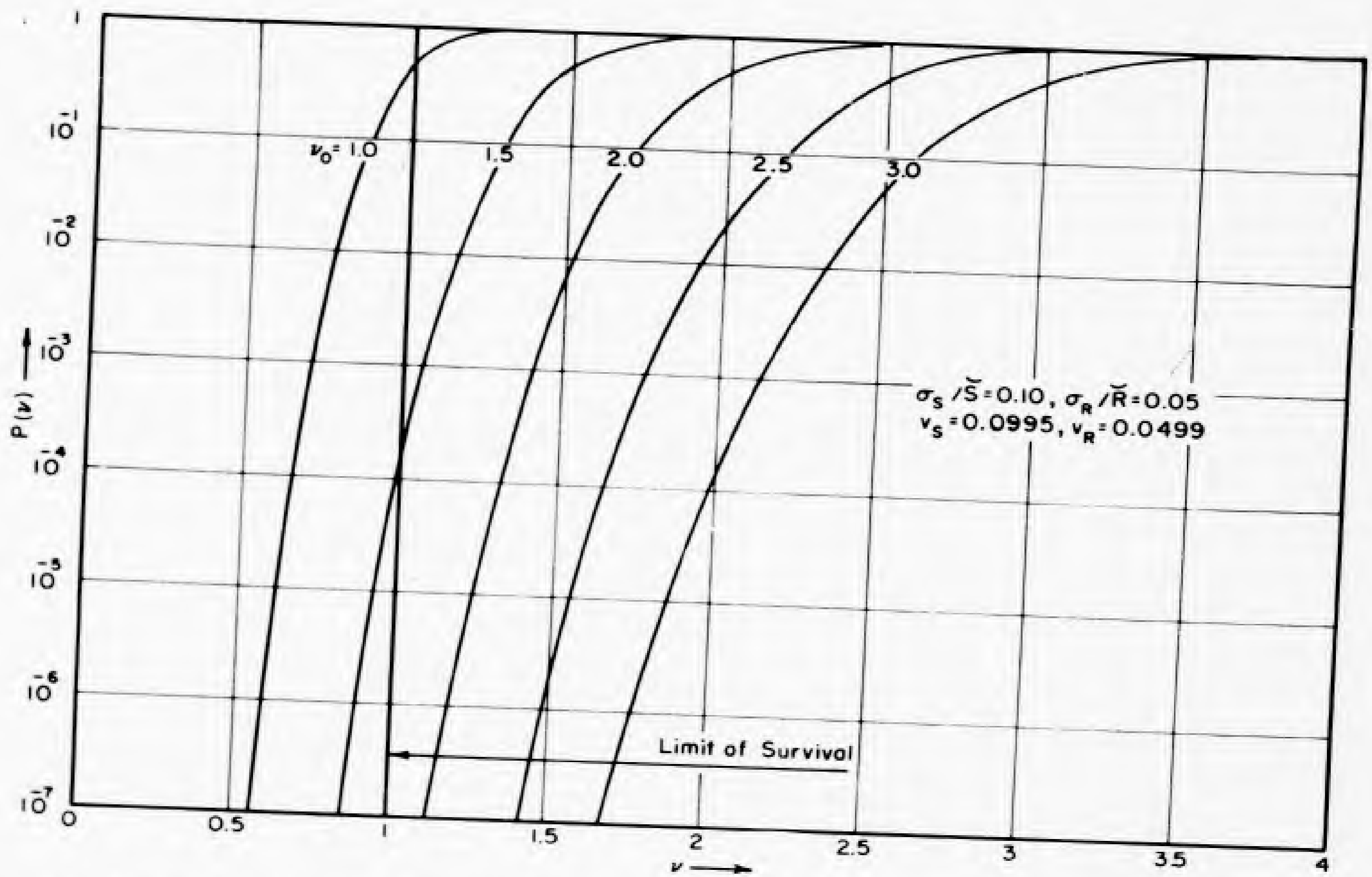

Fig. 1 (a)

Figure 1. Distribution Function of ν with Logarithmic-Normal Distribution of R and S

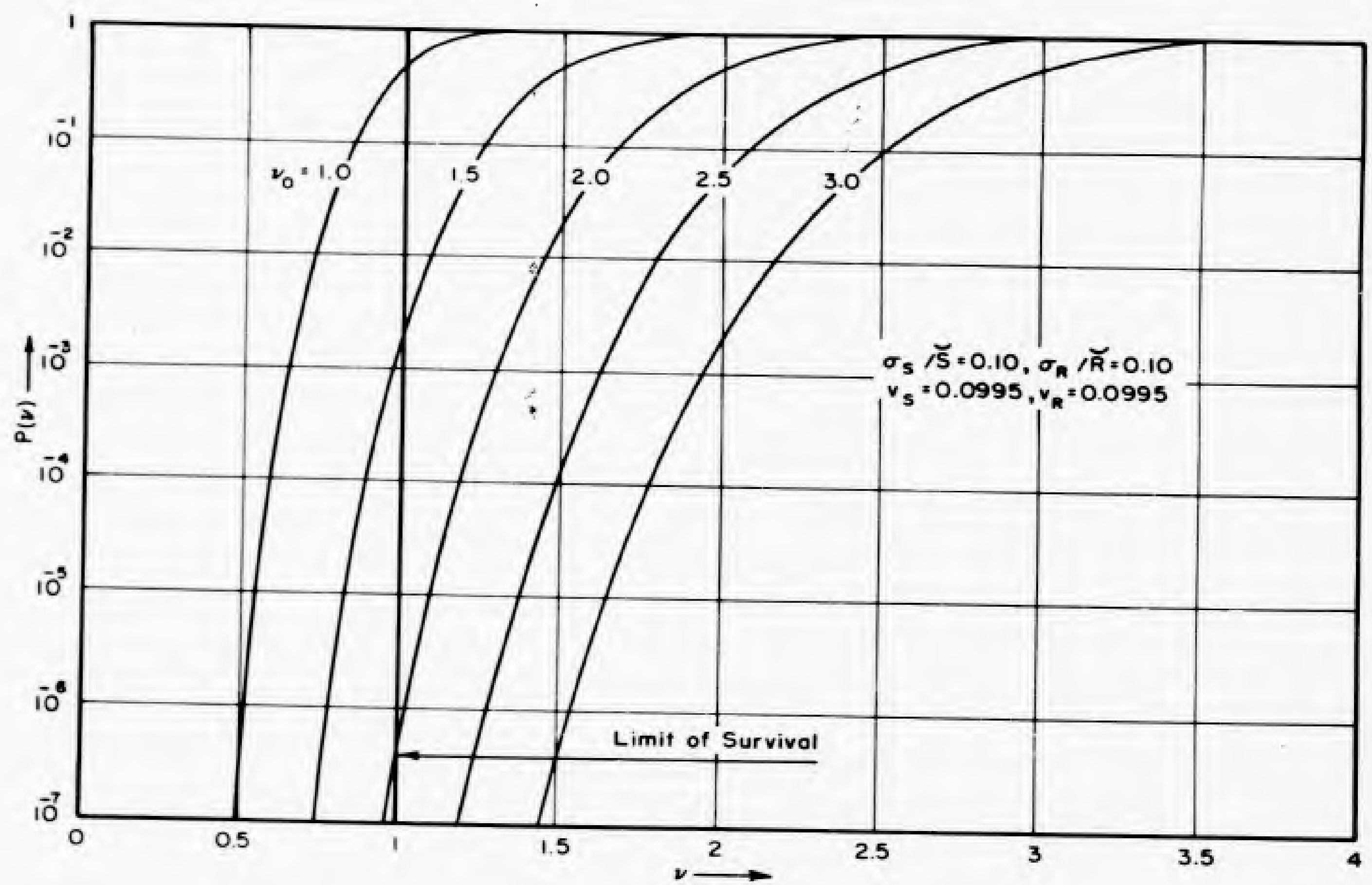

Fig. 1 (b)

Figure 1. Distribution Function of with Logarithmic-Normal Distribution of R and S

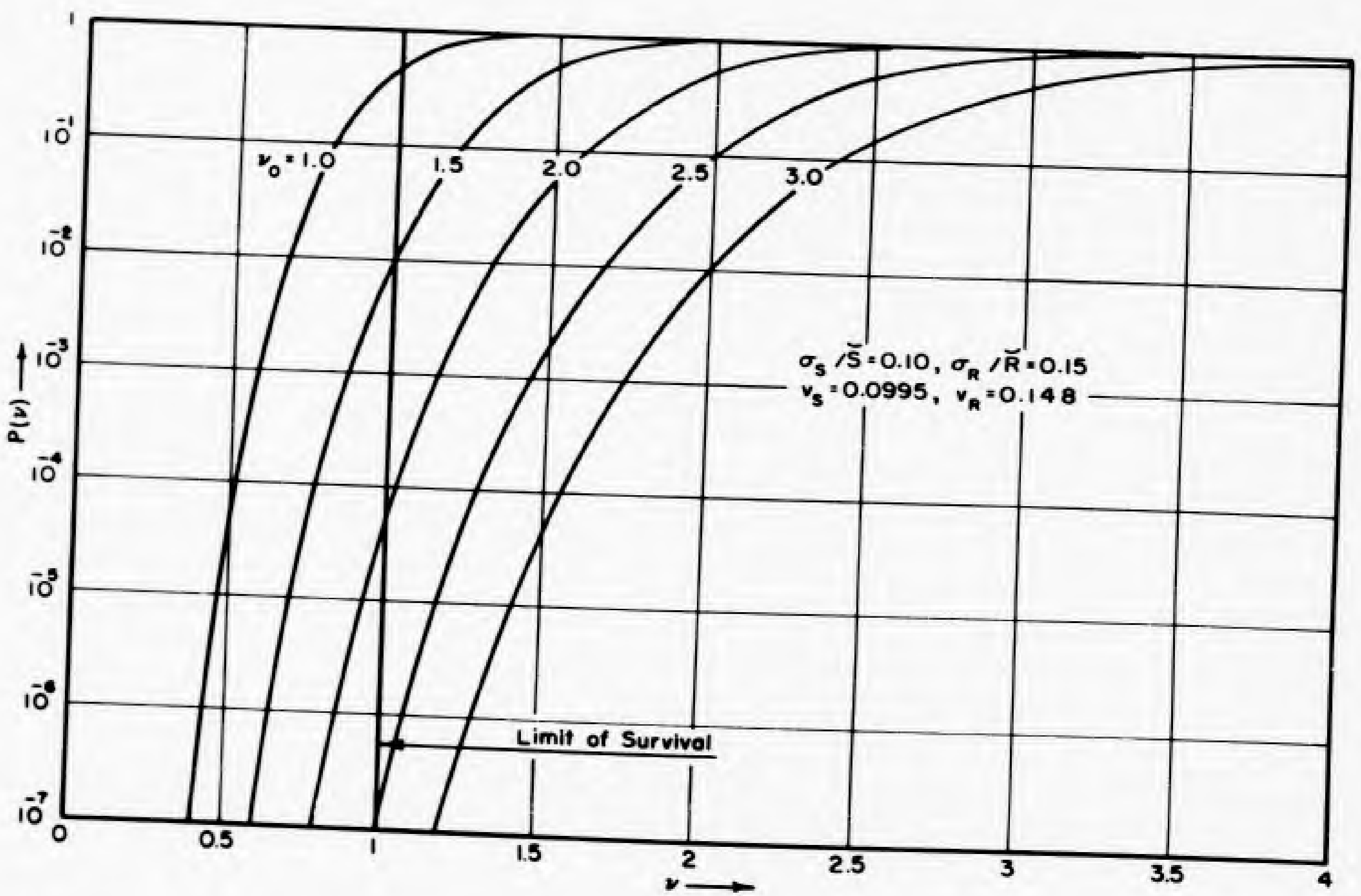

Fig. 1 (c)

Figure 1. Distribution Function of with Logarithmic-Normal Distribution of R and S

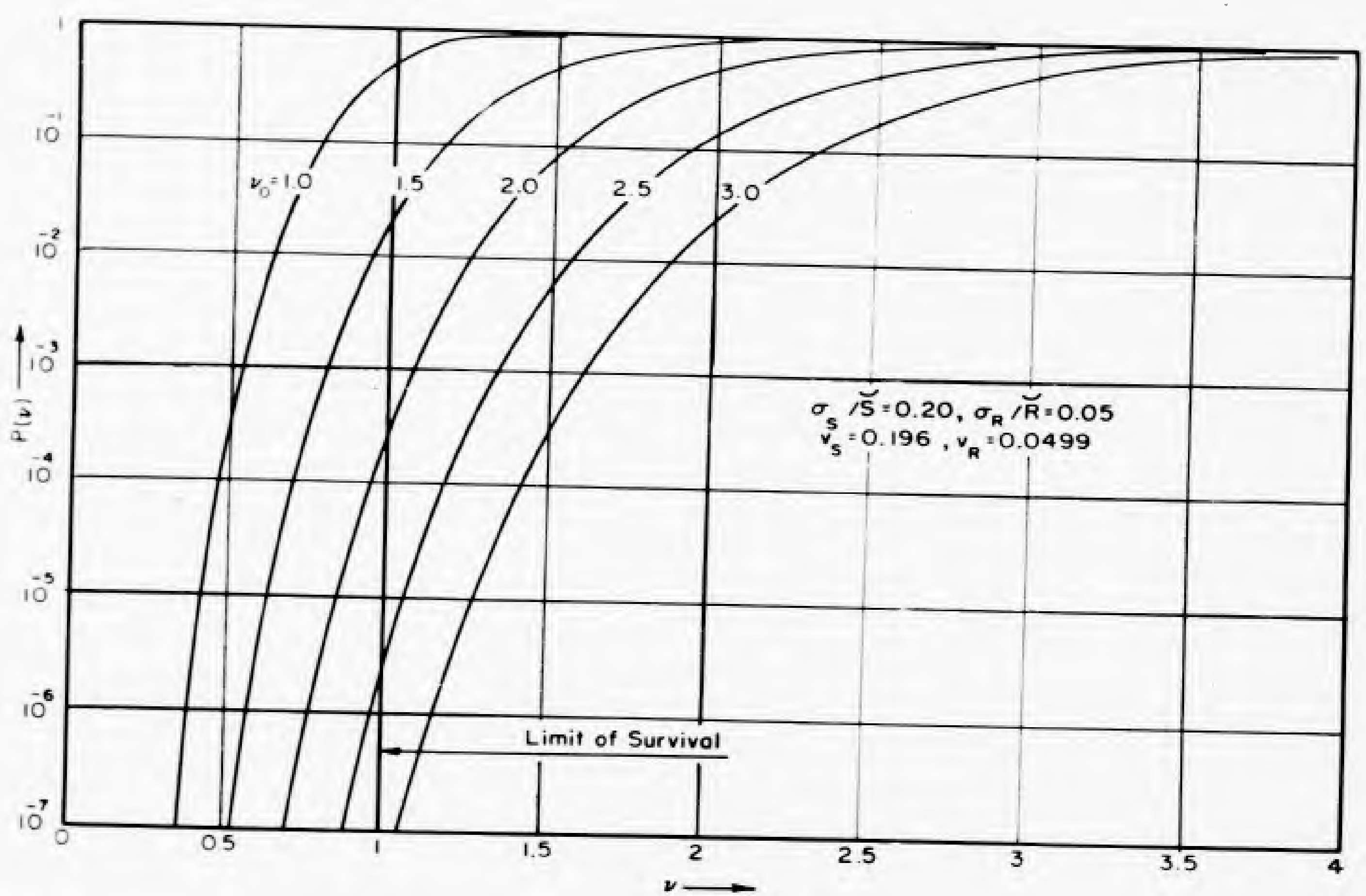

Fig. 1 (d)

Figure 1. Distribution Function of with Logarithmic-Normal Distribution of R and S

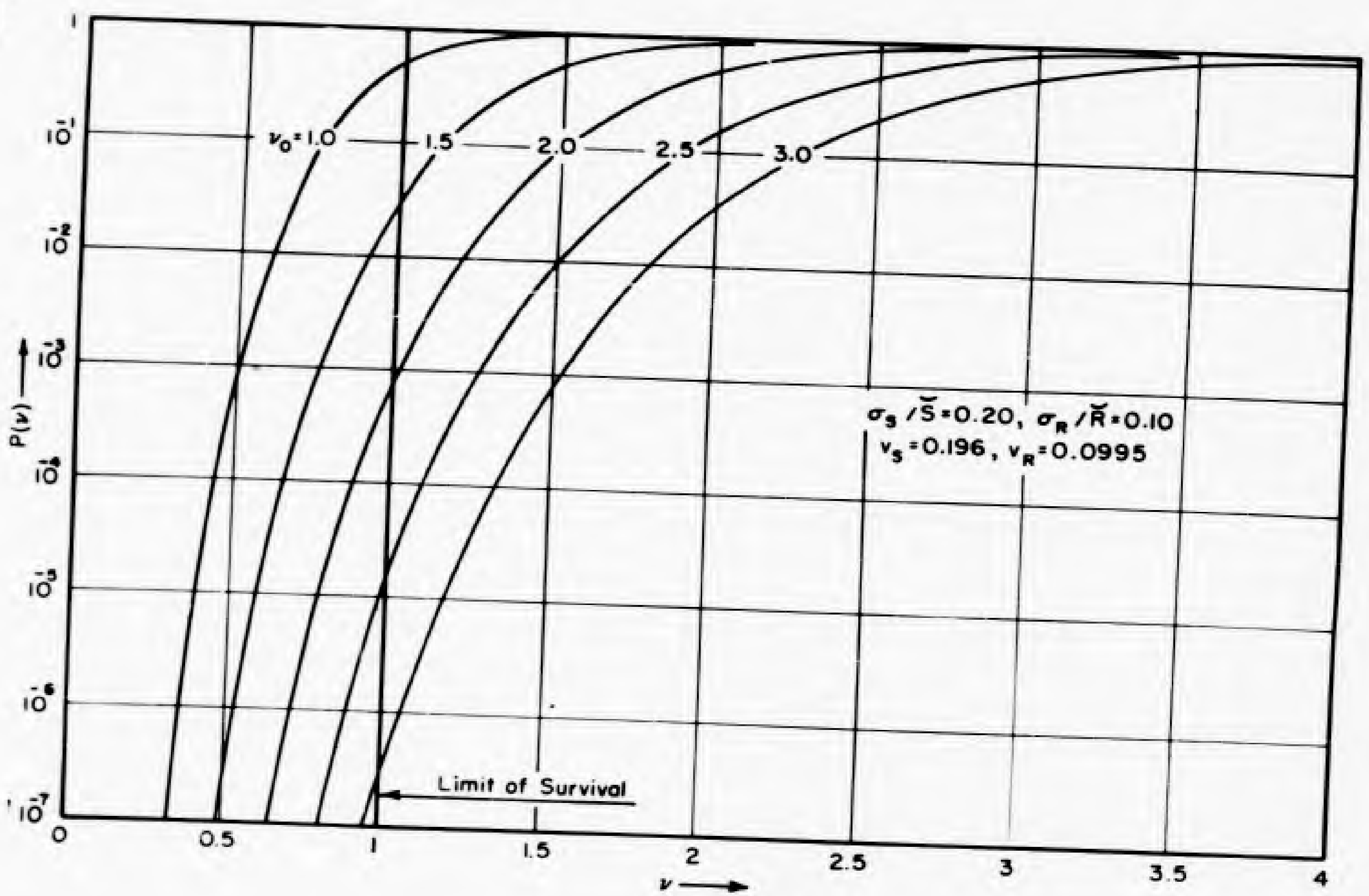

Fig. 1 (e)

Figure 1. Distribution Function of with Logarithmic-Normal Distribution of R and S

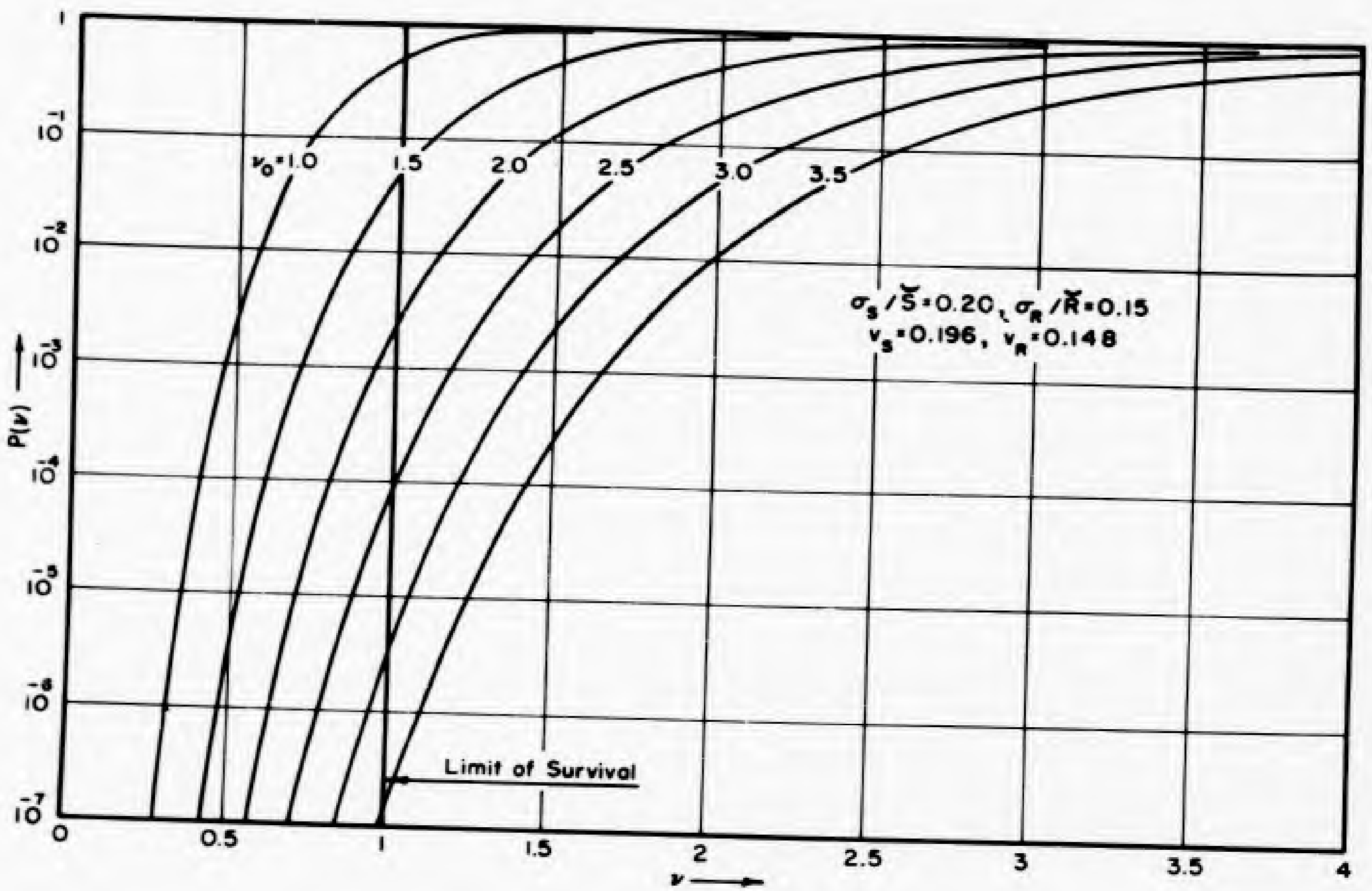

Fig. 1 (f)

Figure 1. Distribution Function of with Logarithmic-Normal Distribution of R and S

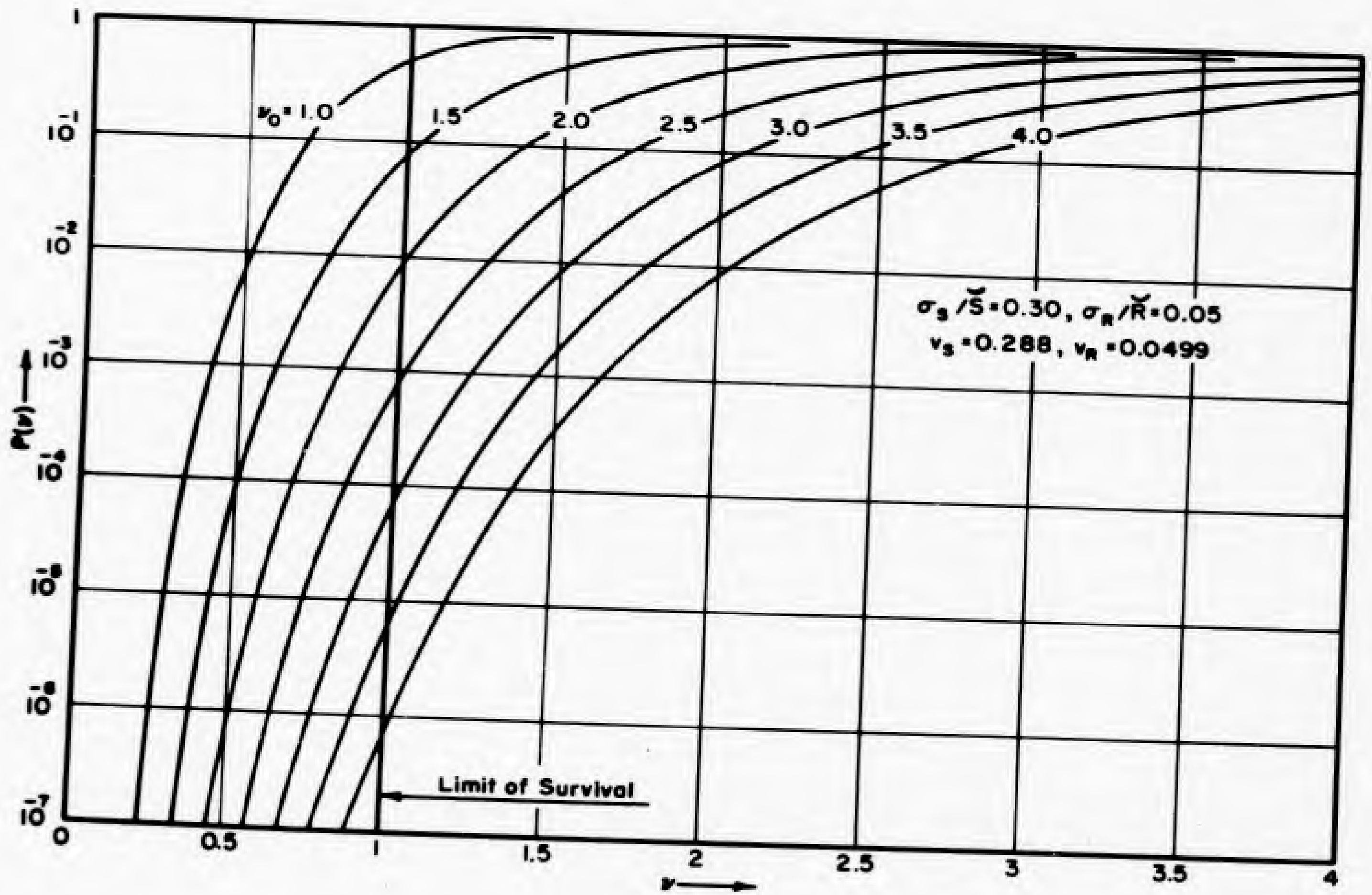

Fig. 1 (g)

Figure 1. Dist ribution Function of with Logarithmic-Normal Distribution of R and S

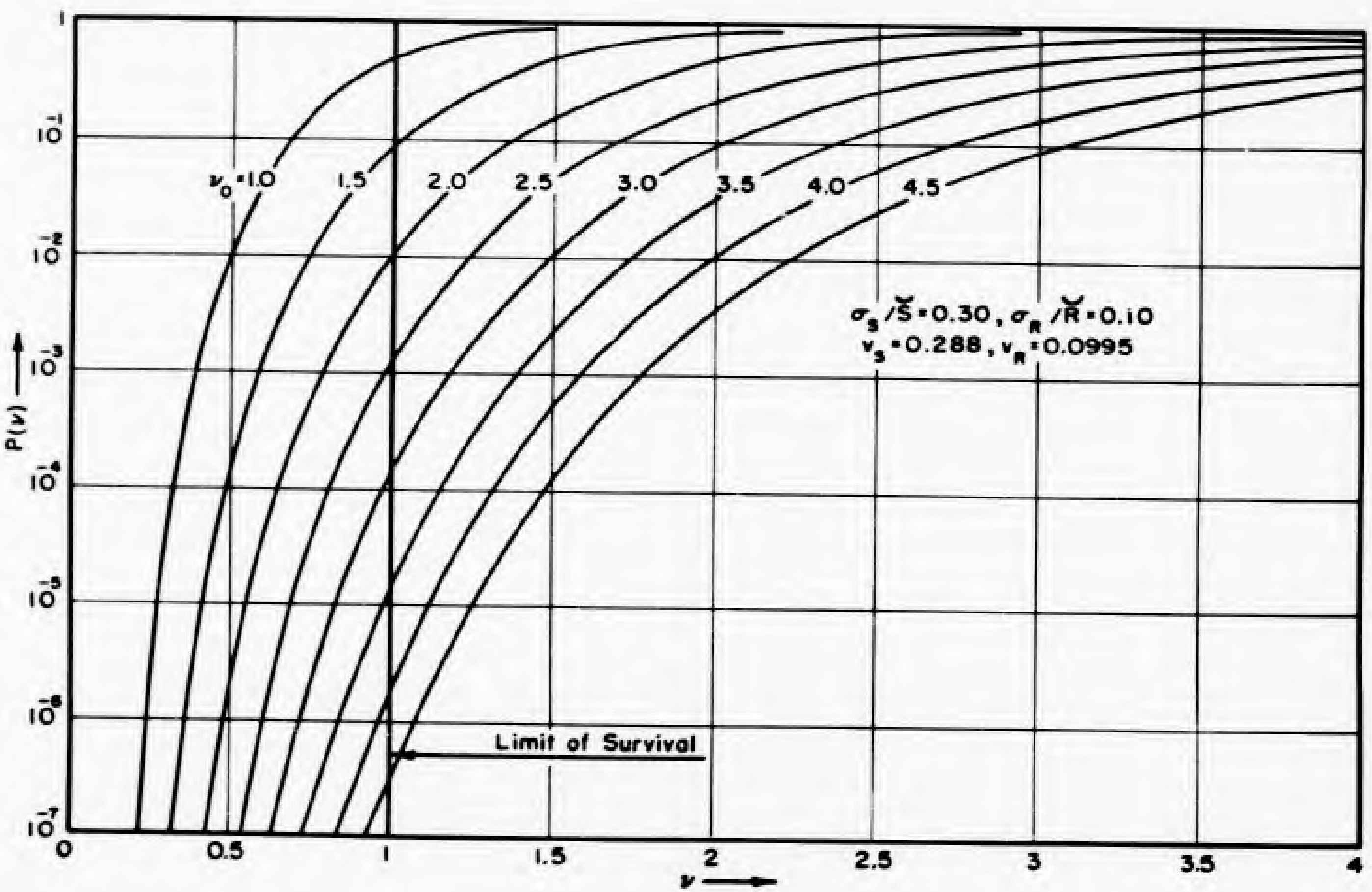

Fig. 1 (h)

Figure 1. Distribution Function of ν with Logarithmic-Normal Distribution of R and S

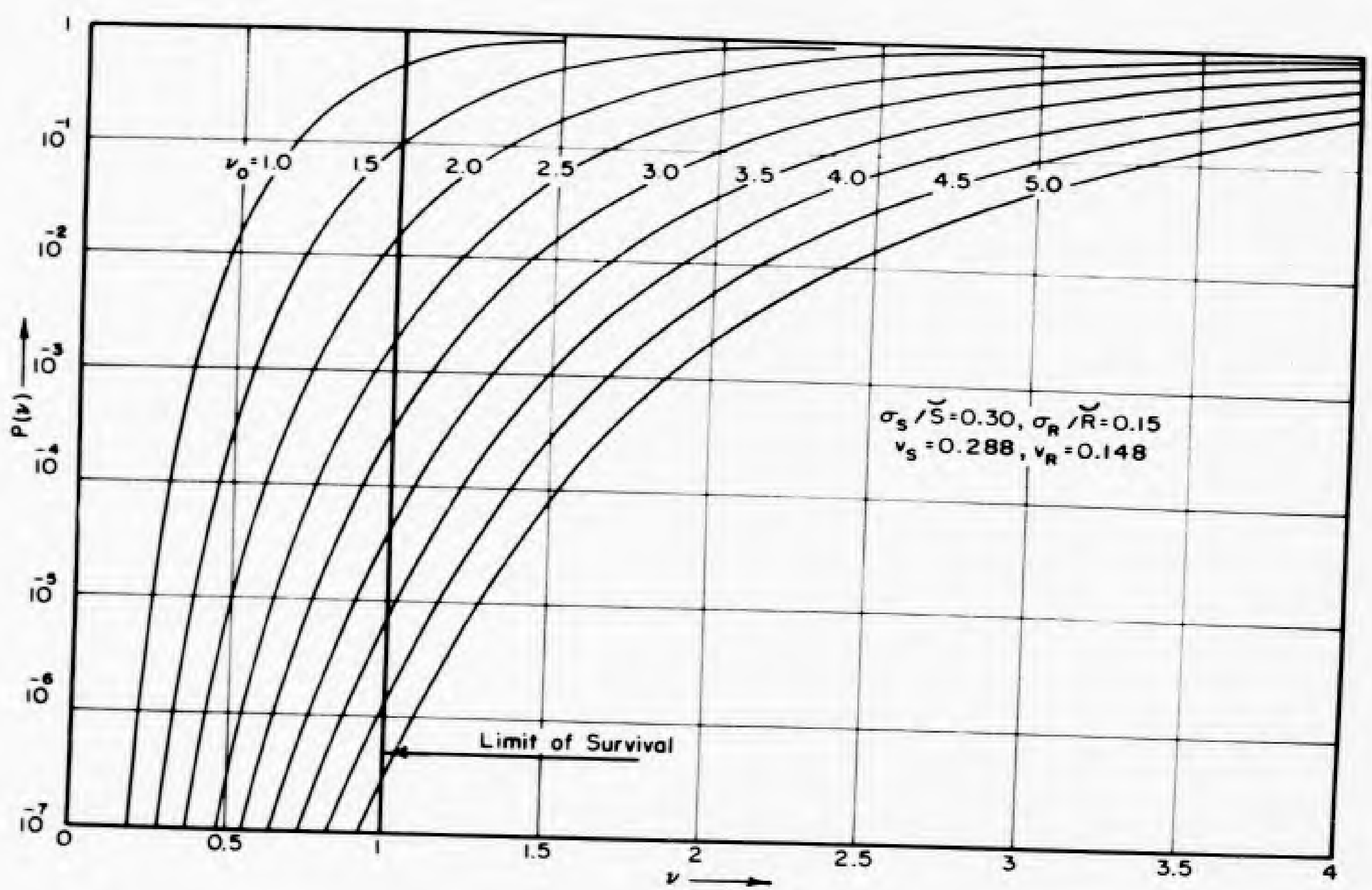

Figure 1. Distribution Function of **Fig. 1 (i)** with Logarithmic-Normal Distribution of R and S

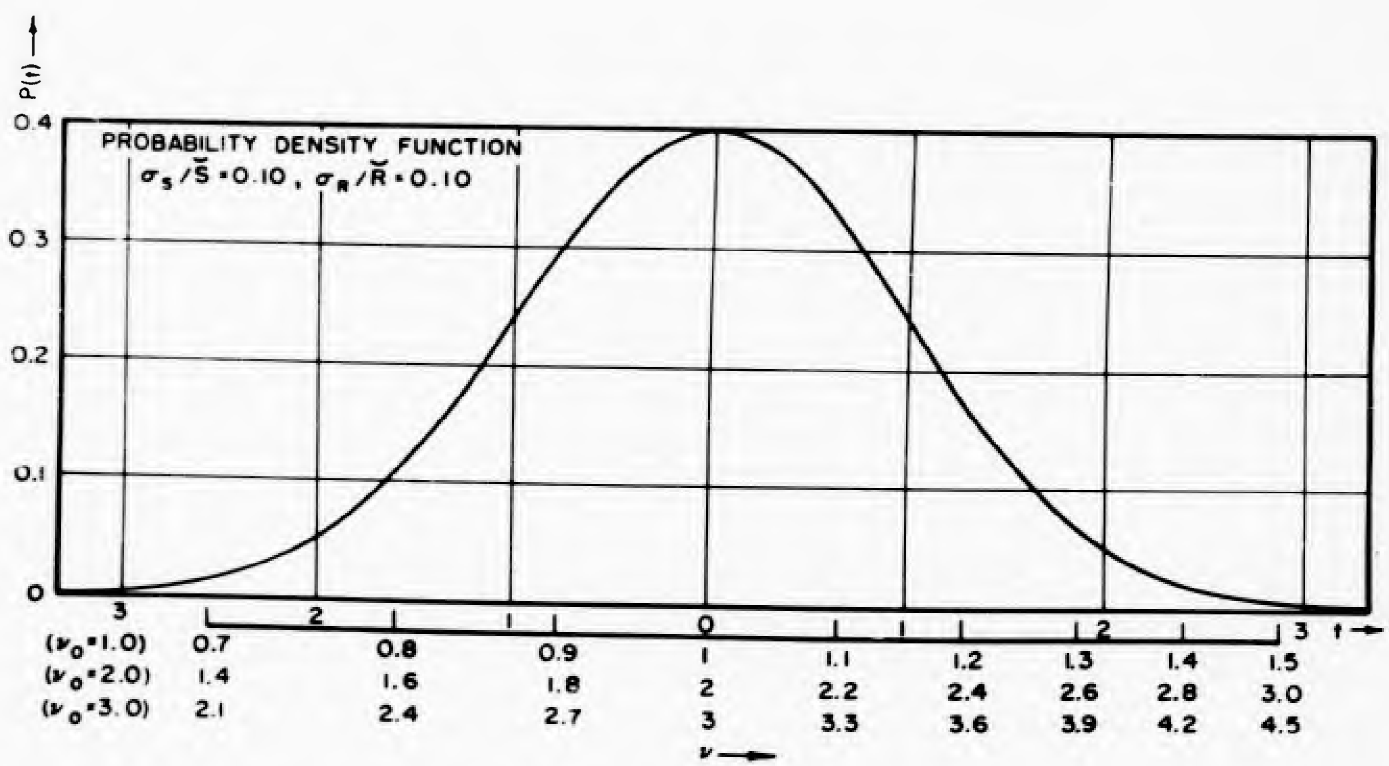

Figure 2. Probability Density of ν with Logarithmic-Normal Distributions of R and S

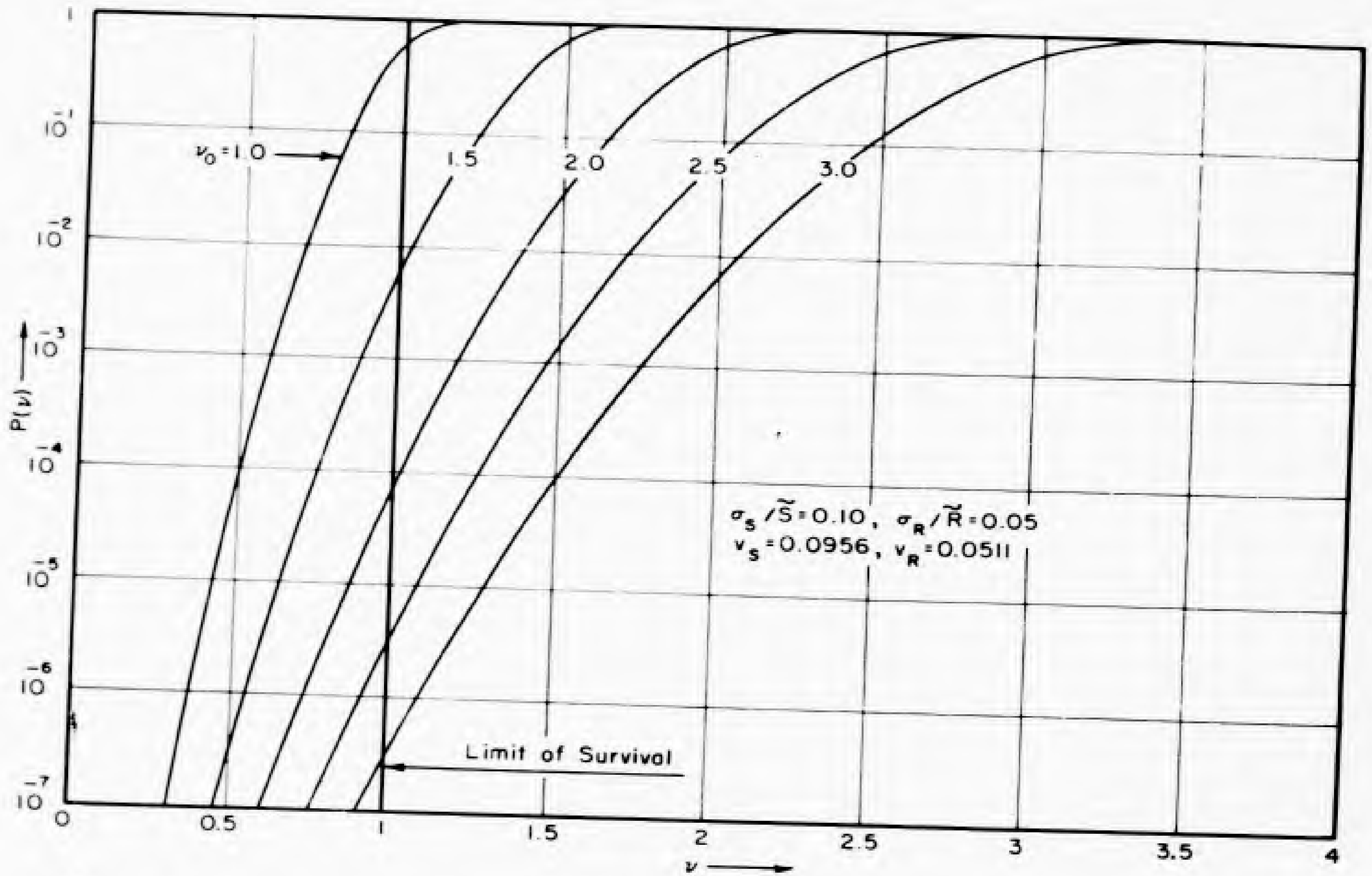

Fig. 3 (a)

Figure 3. Distribution Function of ν with Extremal Distribution of R and S

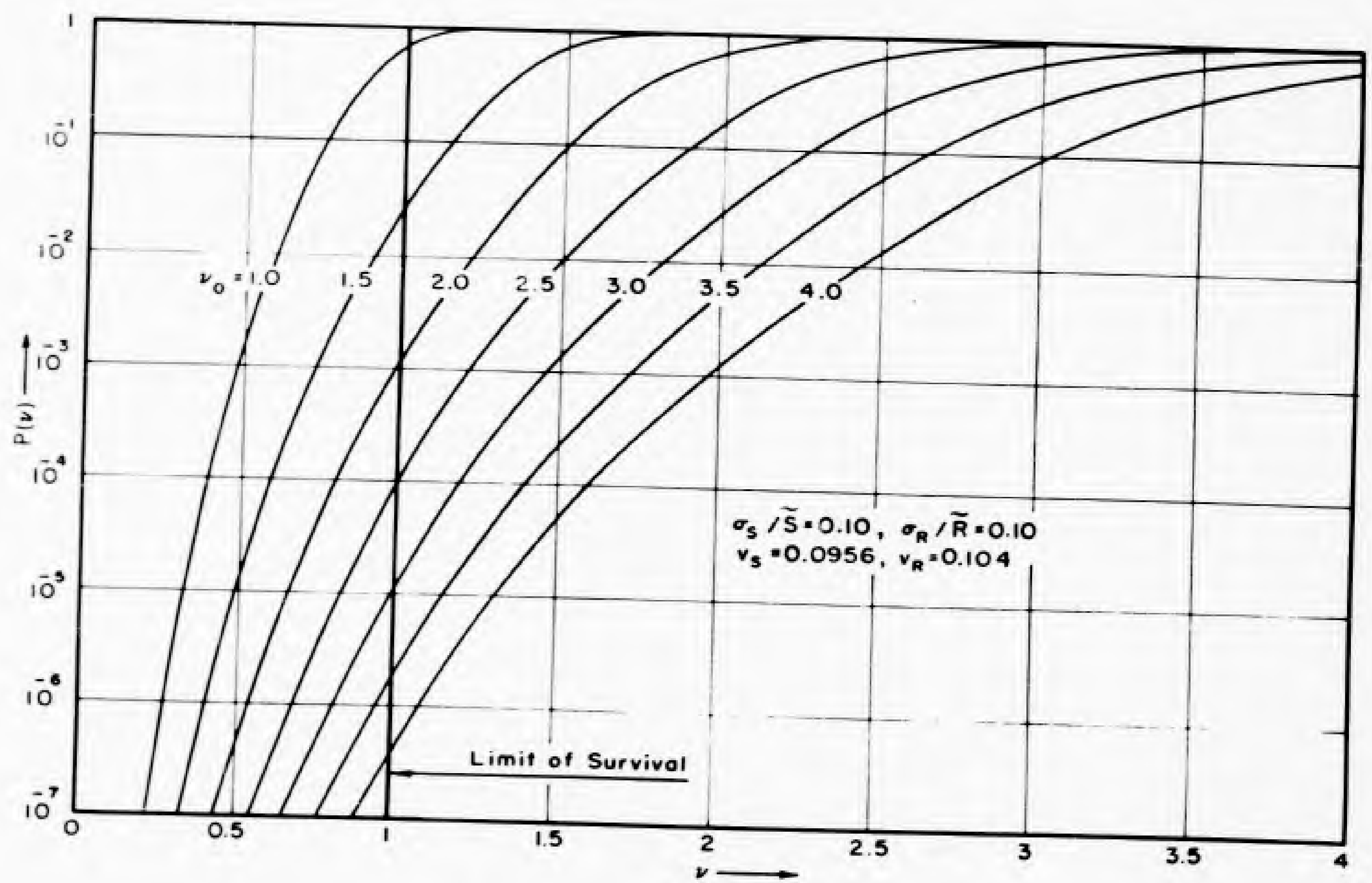

Fig. 3 (b)

Figure 3. Distribution Function of with Extremal Distribution of R and S

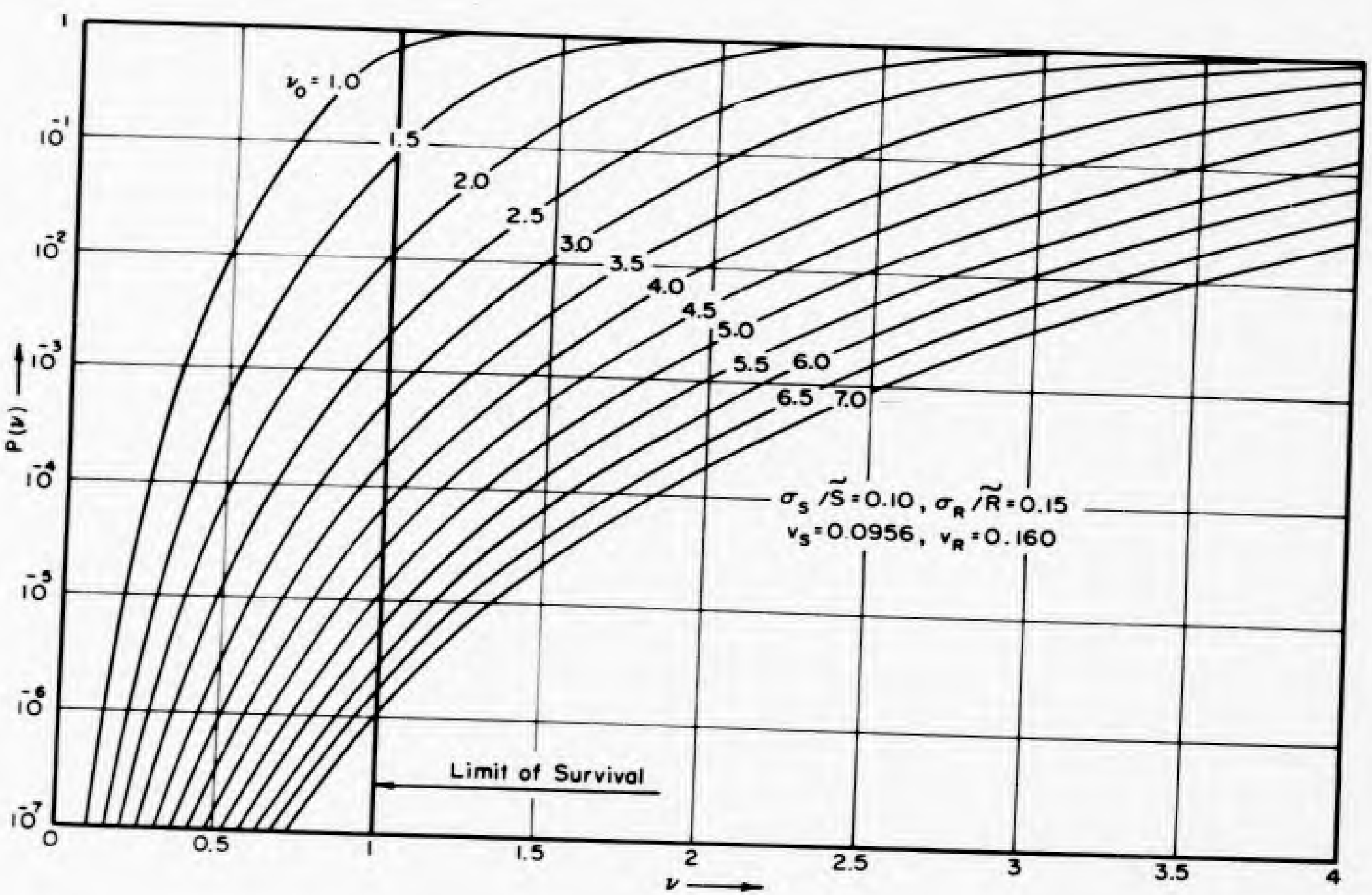

Figure 3. Distribution Function of **Fig. 3 (c)** with Extremal Distribution of R and S

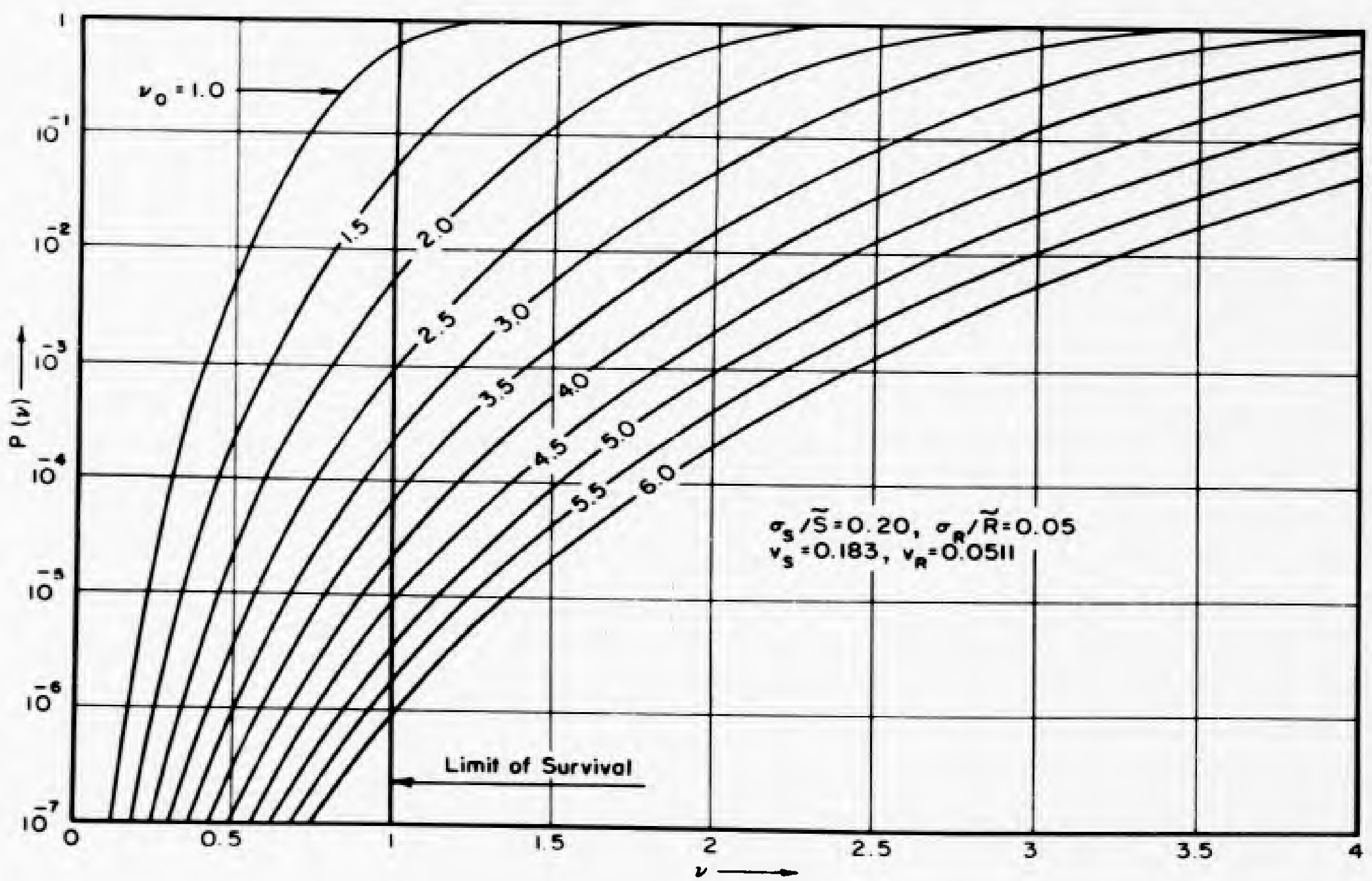

Fig. 3 (d)
Figure 3. Distribution Function of with Extremal Distribution of R and S

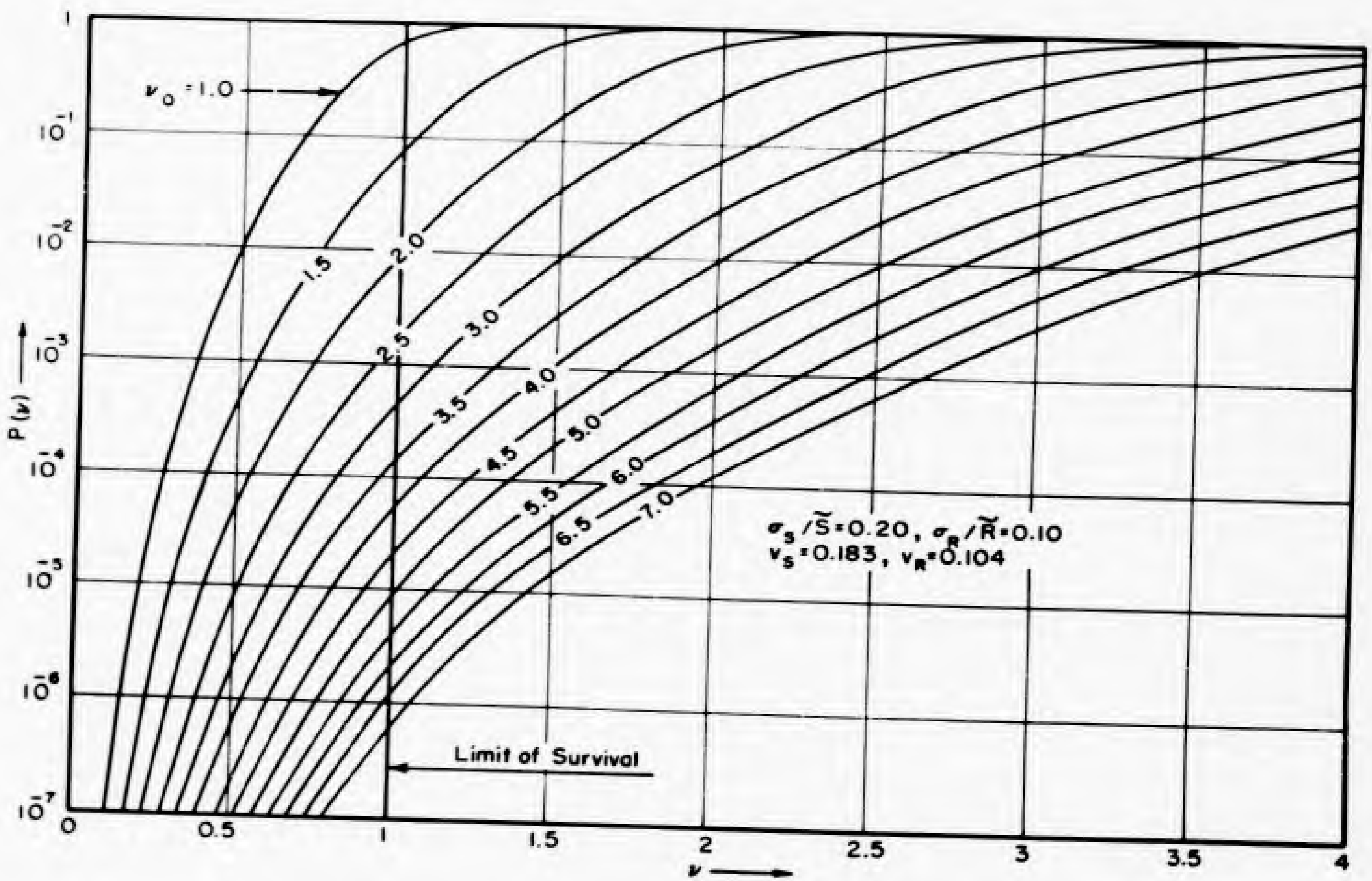

Fig. 3 (e)

Figure 3. Distribution Function of with Extremal Distribution of R and S

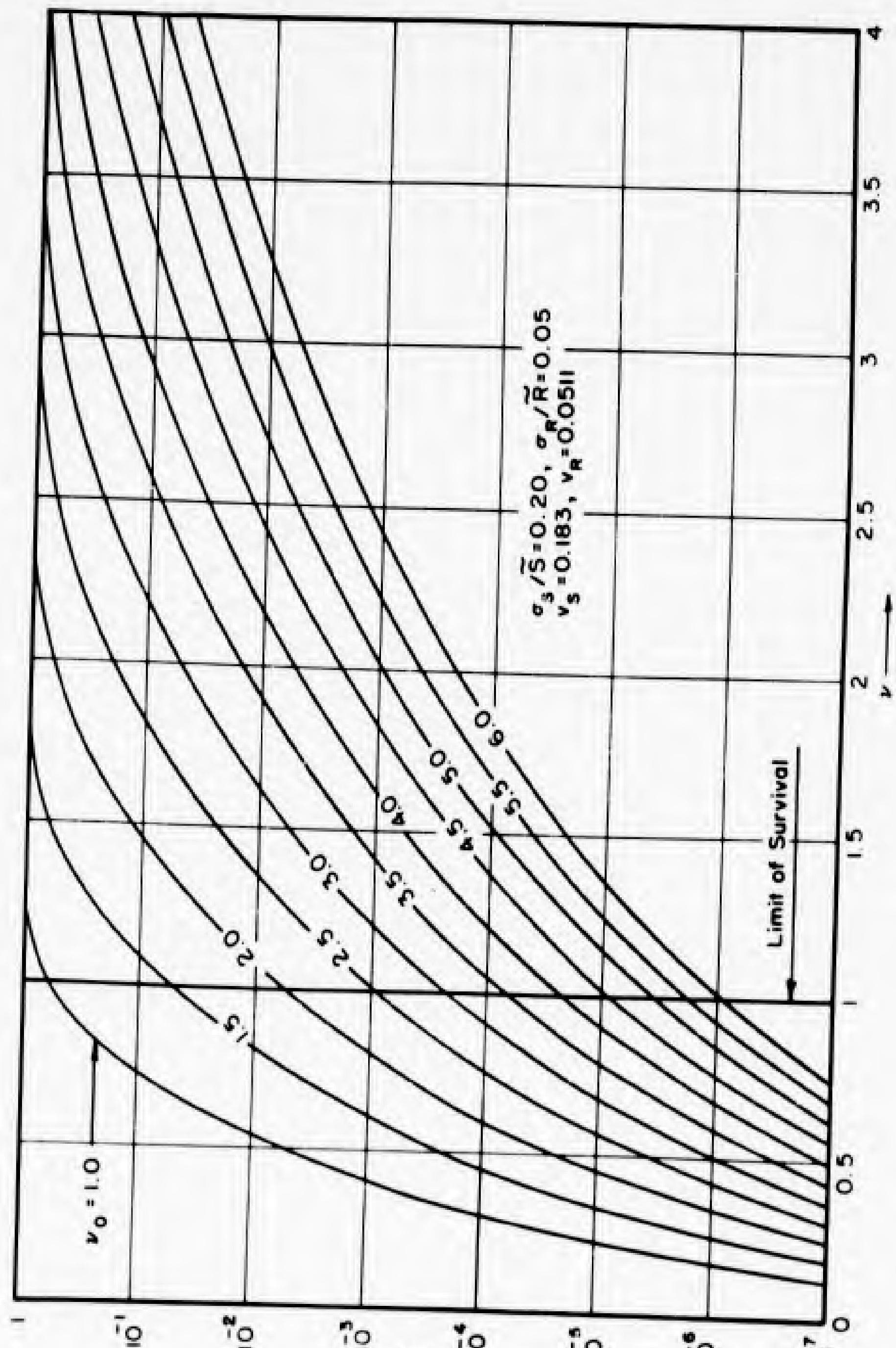

Fig. 3 (d)

Figure 3. Distribution Function of ν with Extremal Distribution of R and S

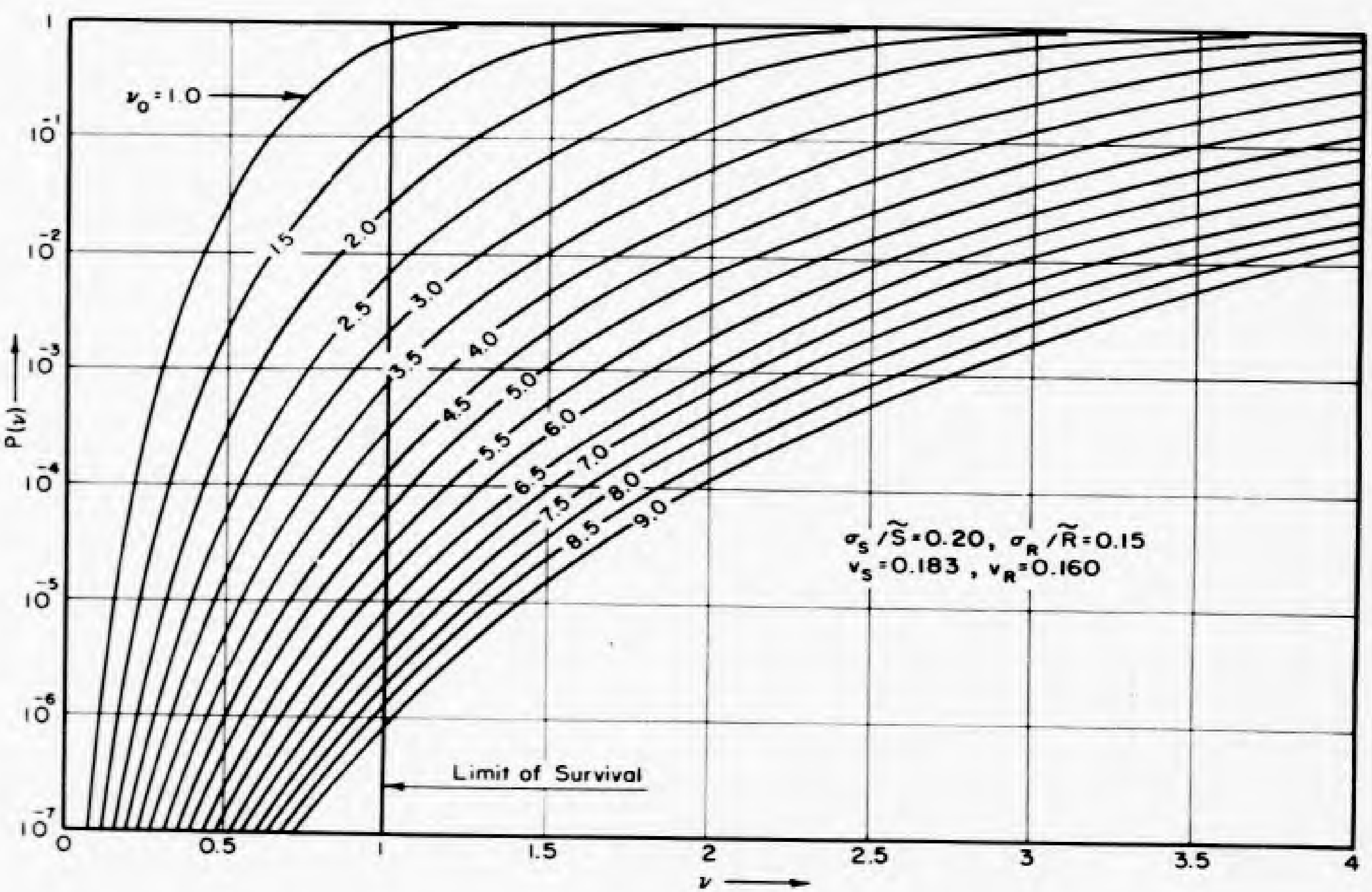

Fig. 3 (f)

Figure 3. Distribution Function of with Extremal Distribution of R and S

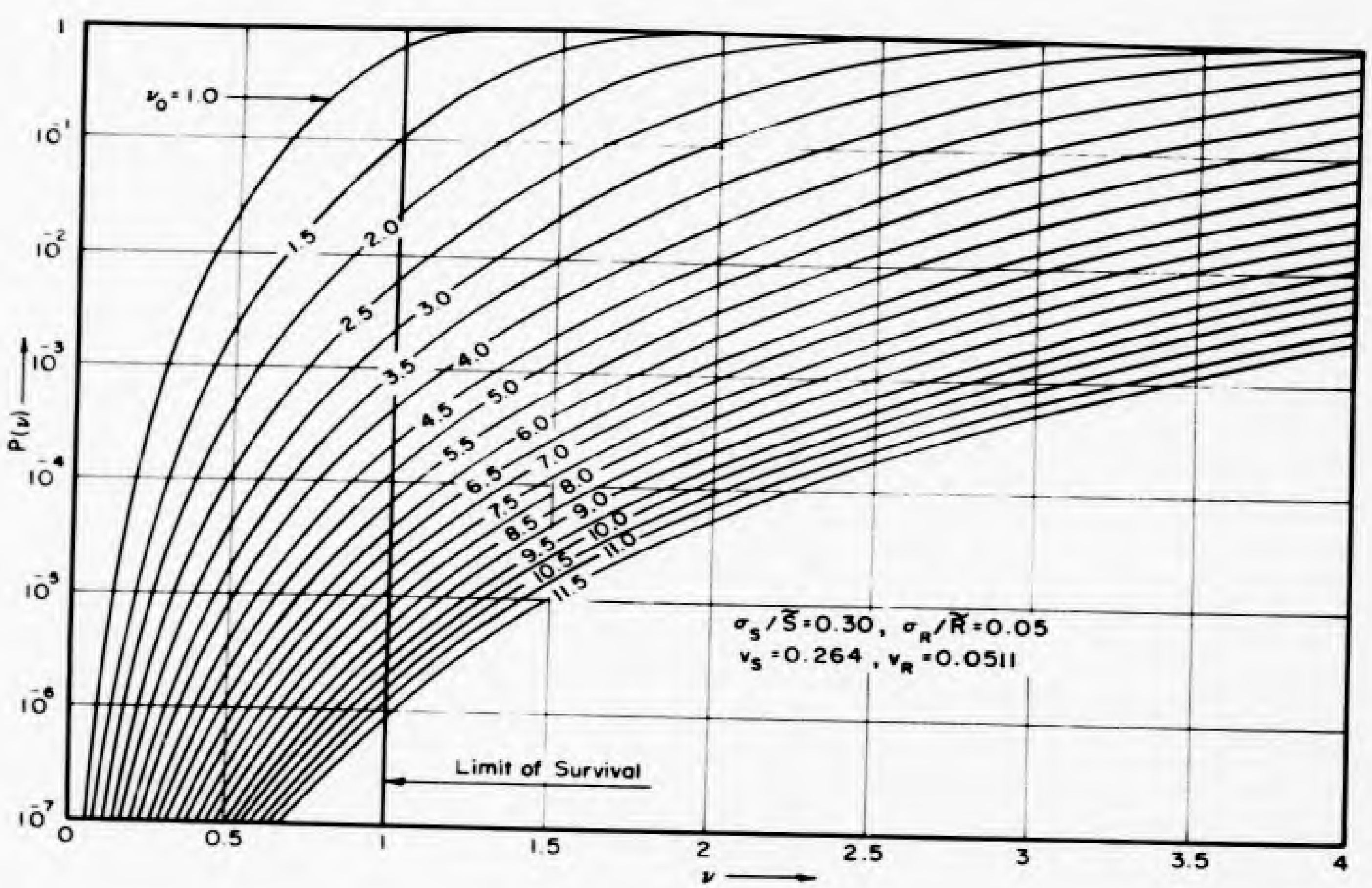

Figure 3. Distribution Function of **Fig. 3 (g)** with Extremal Distribution of R and S

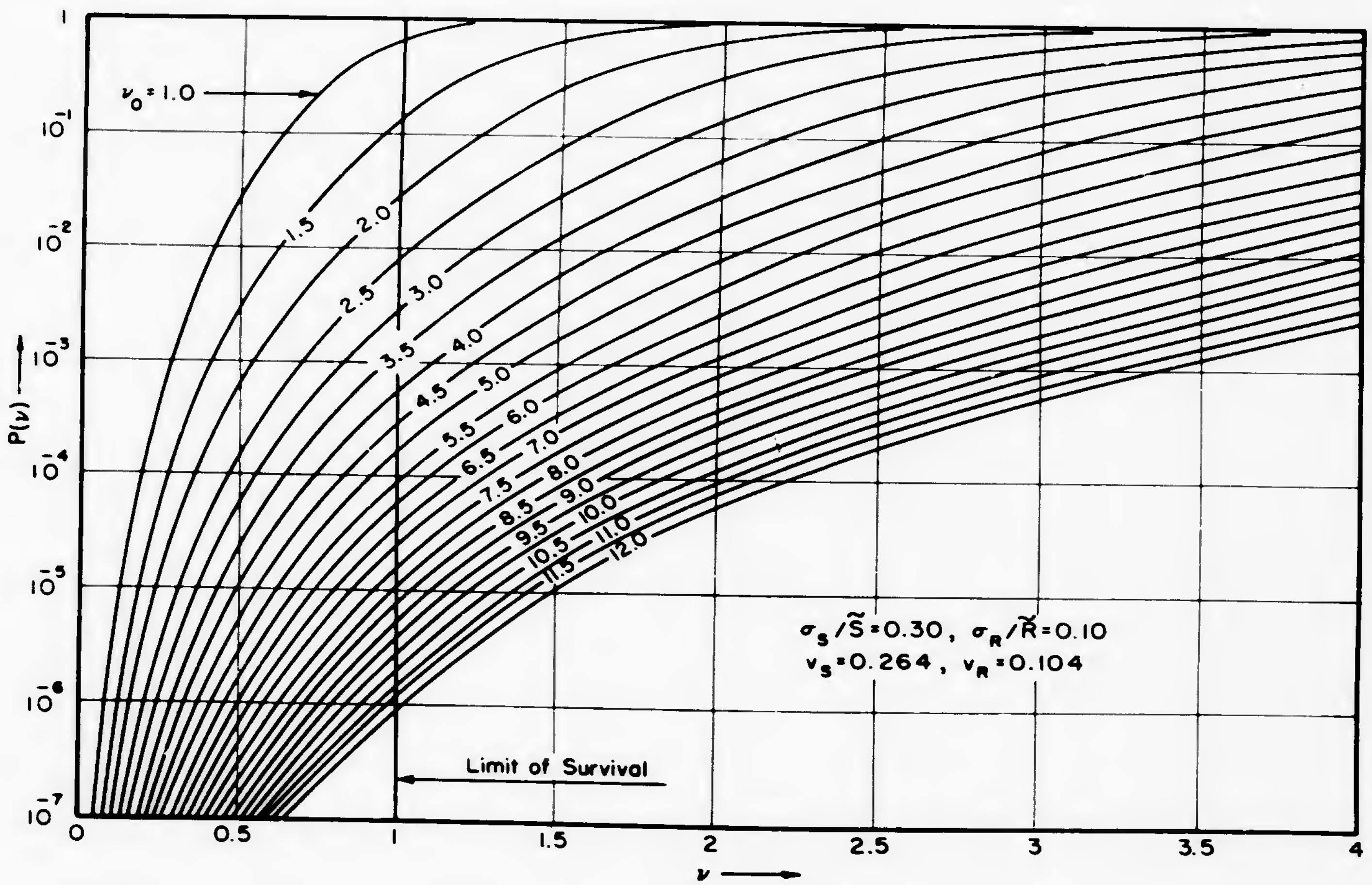

Fig. 3 (h)

Figure 3. Distribution Function of ν with Extremal Distribution of R and S

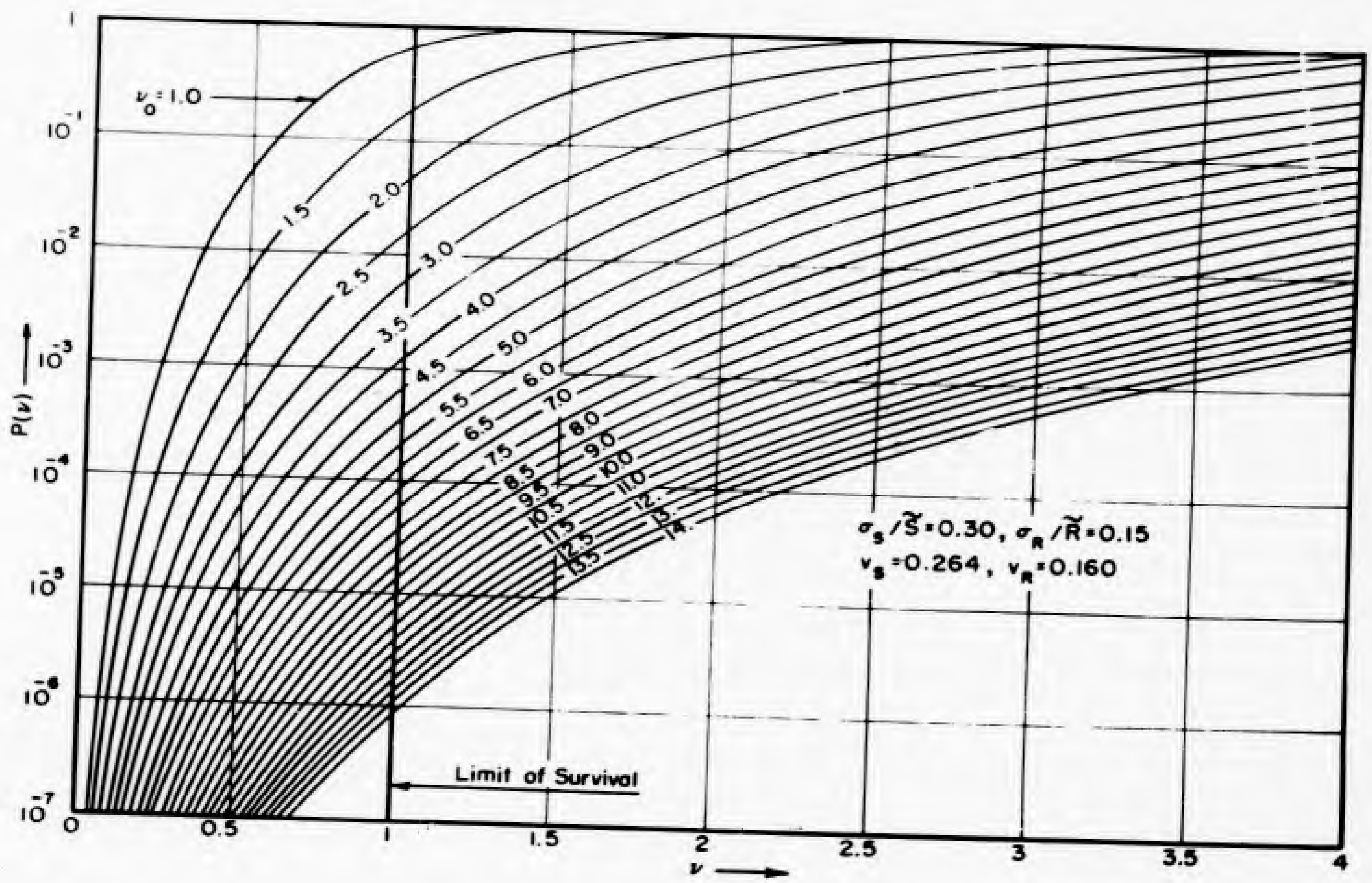

Figure 3. Distribution Function of Fig. 3 (i) with Extremal Distribution of R and S

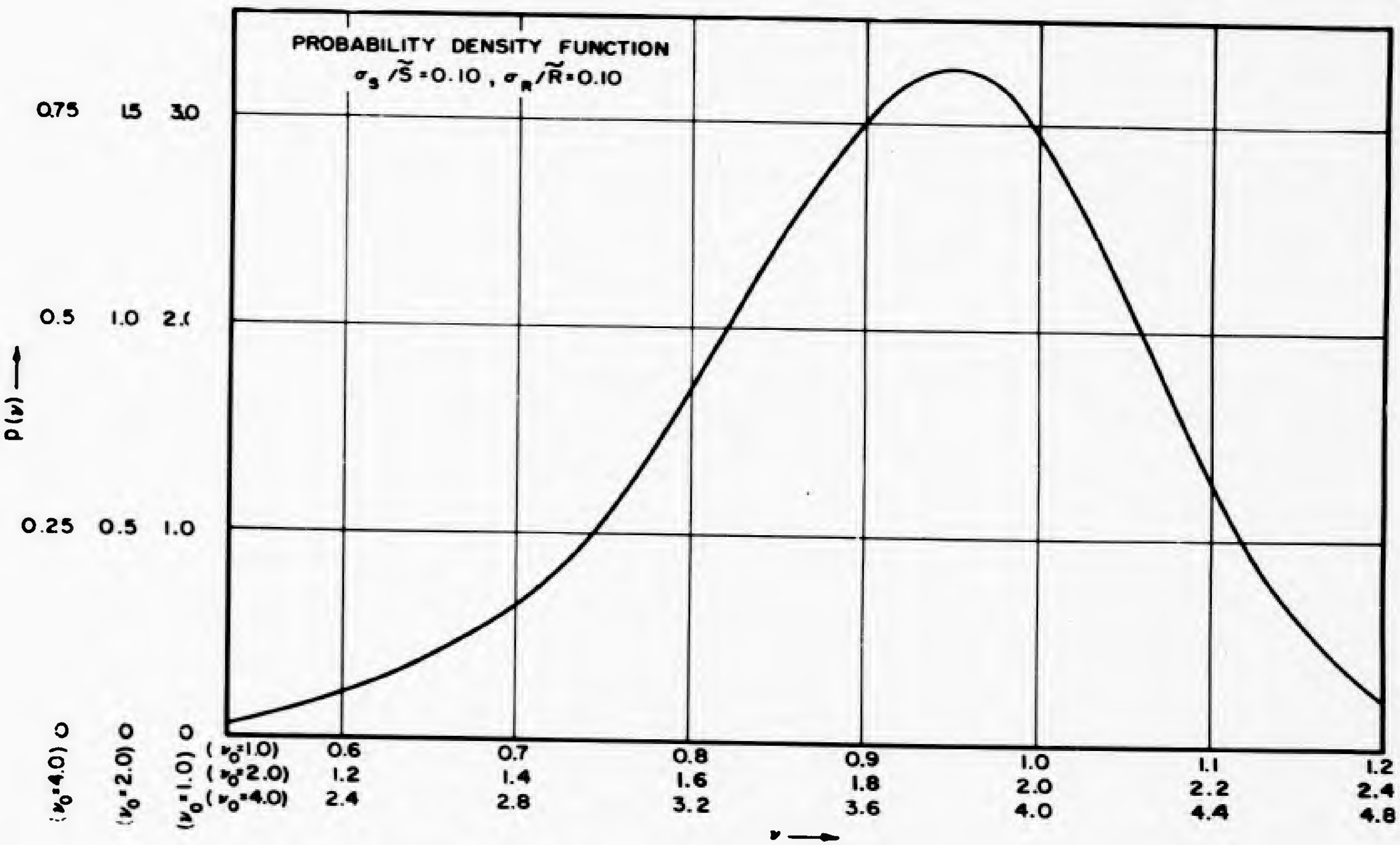

Figure 4. Probability Density of ν with Extremal Distributions of R and S

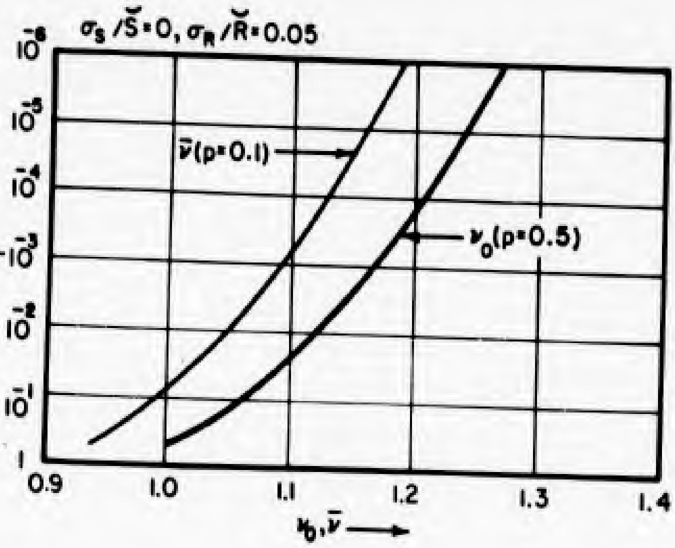

Fig. 5 (a)

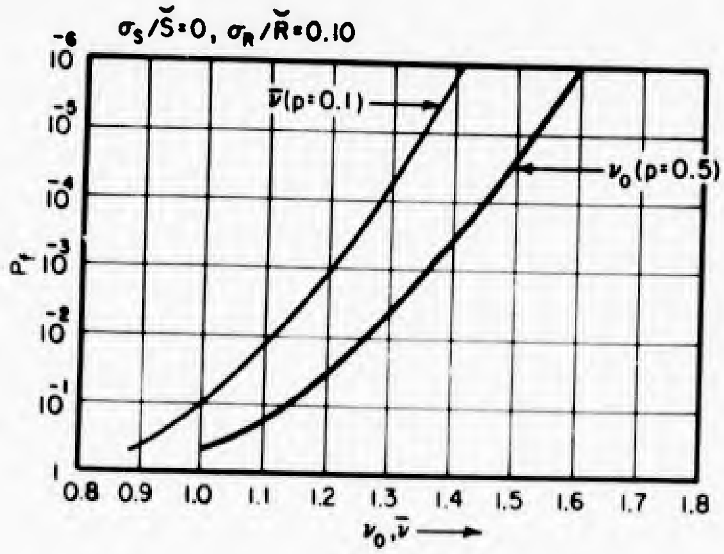

Fig. 5 (b)

Figure 5. Relation between Probability of Failure P_f and Central Safety Factor ν_0 and "Conventional" Safety Factor $\bar{\nu}$ with Logarithmic-Normal Distributions of R and S

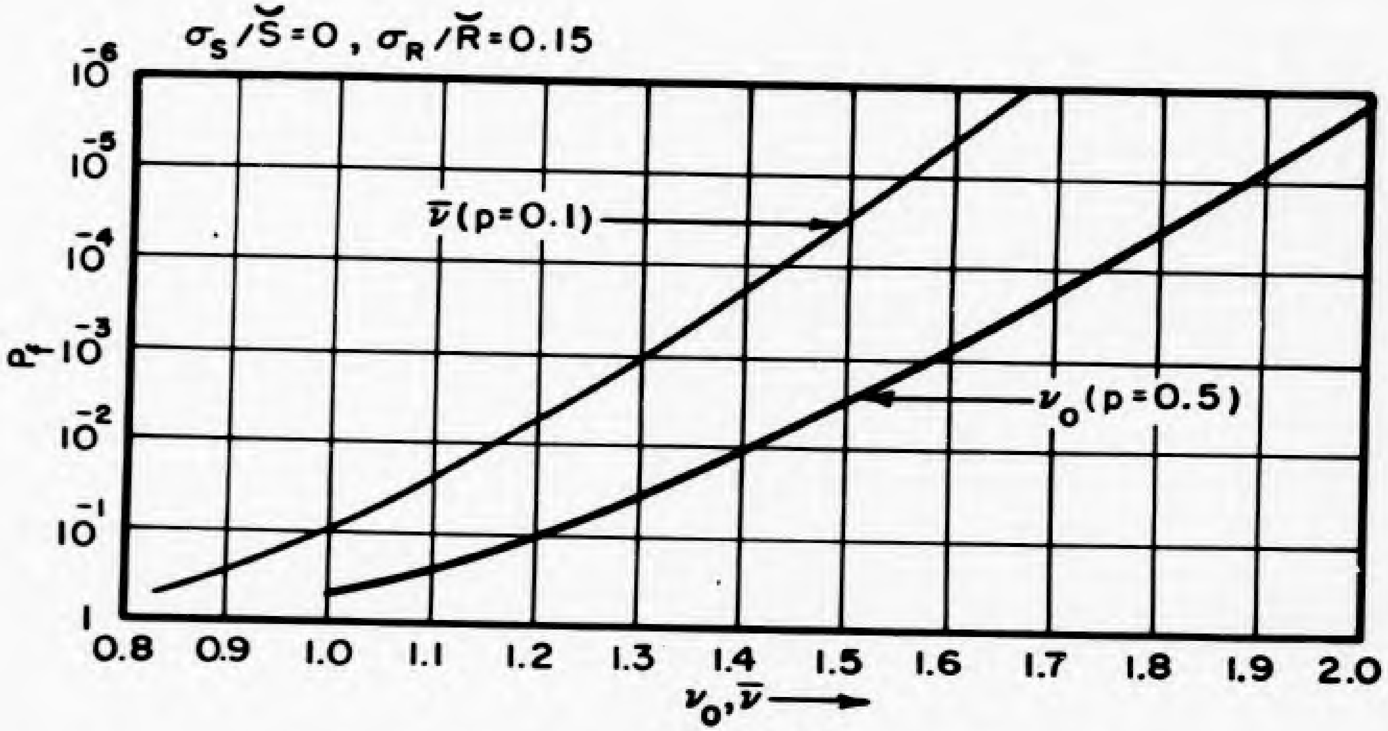

Fig. 5 (c)

Figure 5. Relation between Probability of Failure P_f and Central Safety Factor $_0$ and "Conventional" Safety Factor with Logarithmic-Normal Distributions of R and S

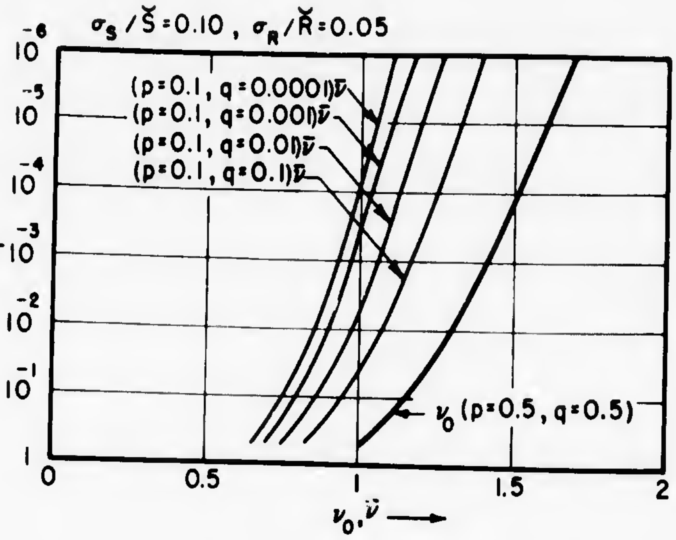

Fig. 5 (d)

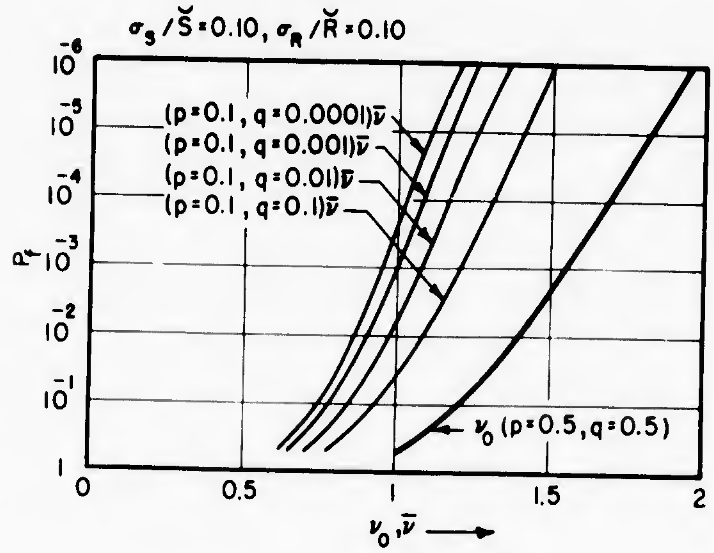

Fig. 5 (e)

Figure 5. Relation between Probability of Failure P_f and Central Safety Factor $_0$ and "Conventional" Safety Factor with Logarithmic-Normal Distributions of R and S

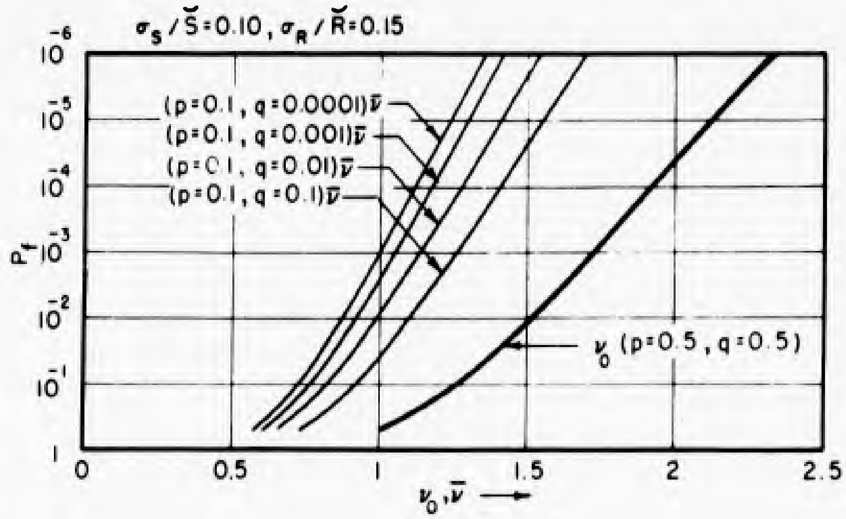

Fig. 5 (f)

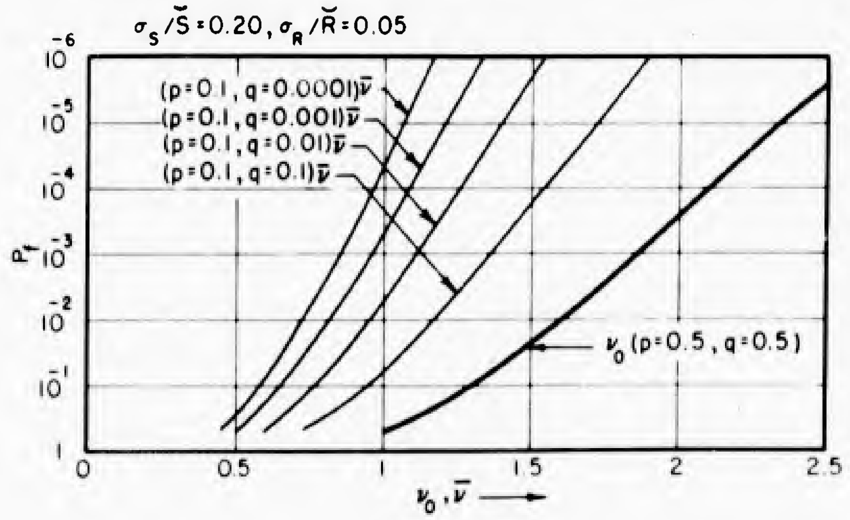

Fig. 5 (g)

Figure 5. Relation between Probability of Failure P_f and Central Safety Factor $_0$ and "Conventional" Safety Factor with Logarithmic-Normal Distributions of R and S

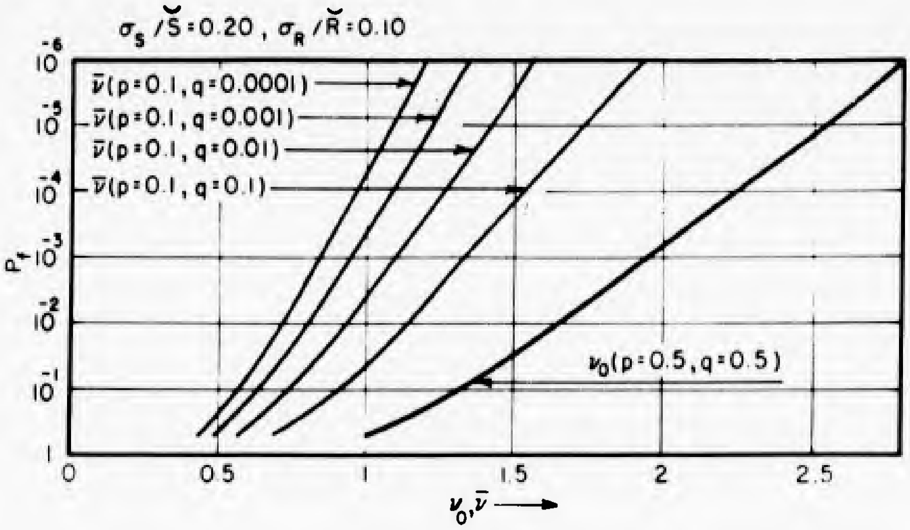

Fig. 5 (h)

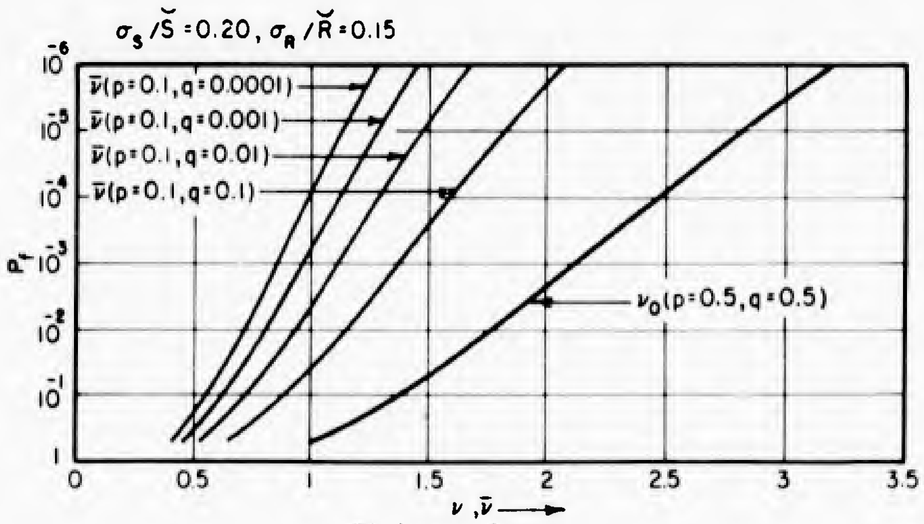

Fig. 5 (i)

Figure 5. Relation between Probability of Failure P_f and Central Safety Factor $_0$ and "Conventional" Safety Factor with Logarithmic-Normal Distributions of R and S

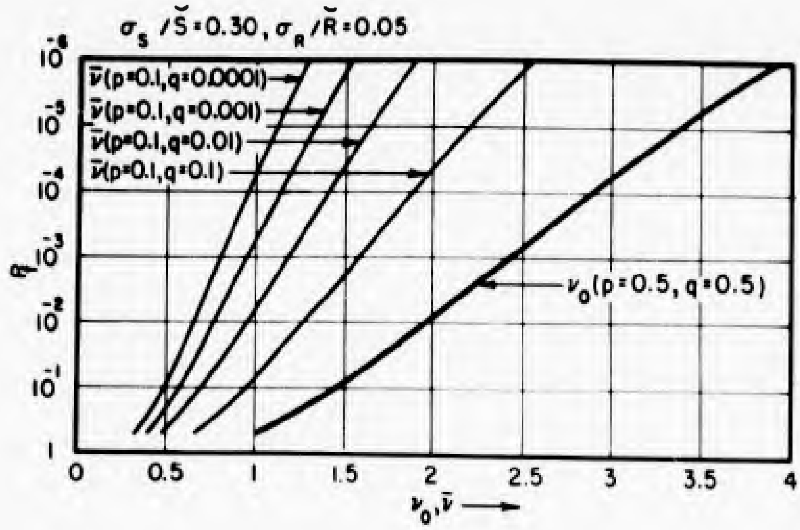

Fig. 5 (j)

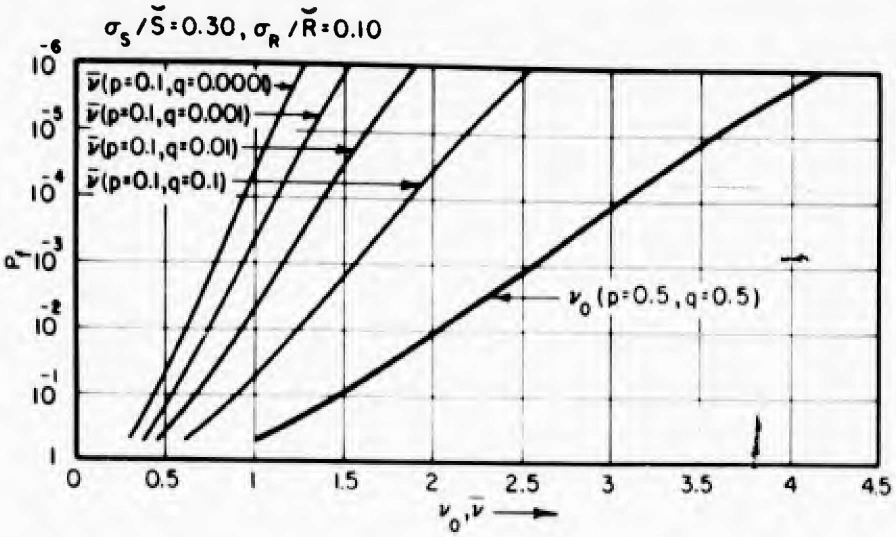

Fig. 5 (k)

Figure 5. Relation between Probability of Failure P_f and Central Safety Factor $_0$ and "Conventional" Safety Factor with Logarithmic-Normal Distributions of R and S

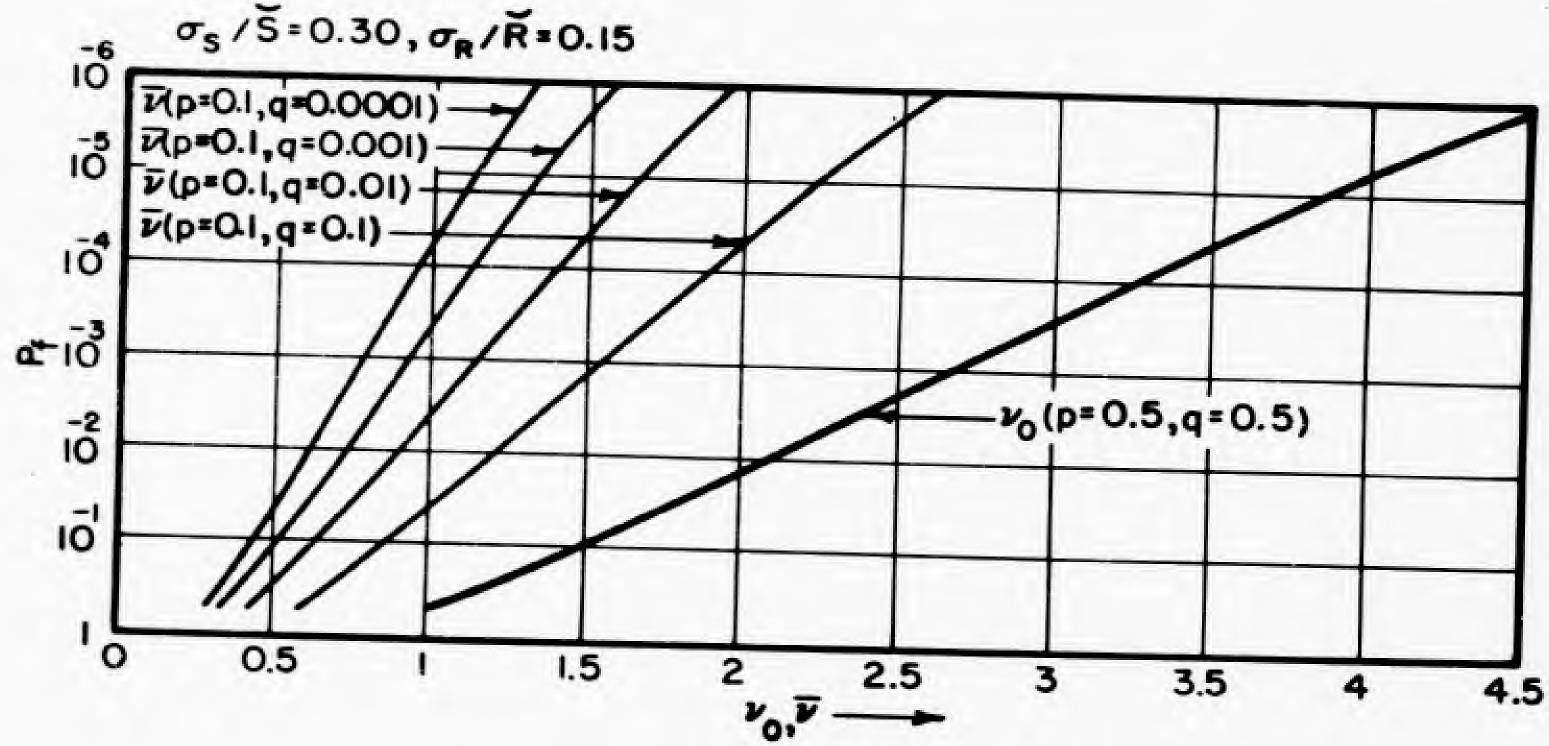

Fig. 5 (ℓ)

Figure 5. Relation between Probability of Failure P_f and Central Safety Factor $_0$ and "Conventional" Safety Factor with Logarithmic-Normal Distributions of R and S

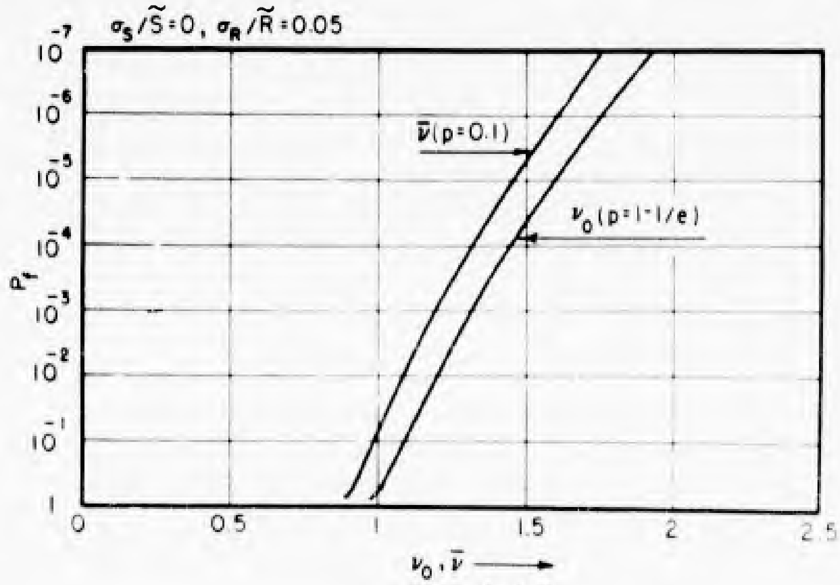

Fig. 6 (a)

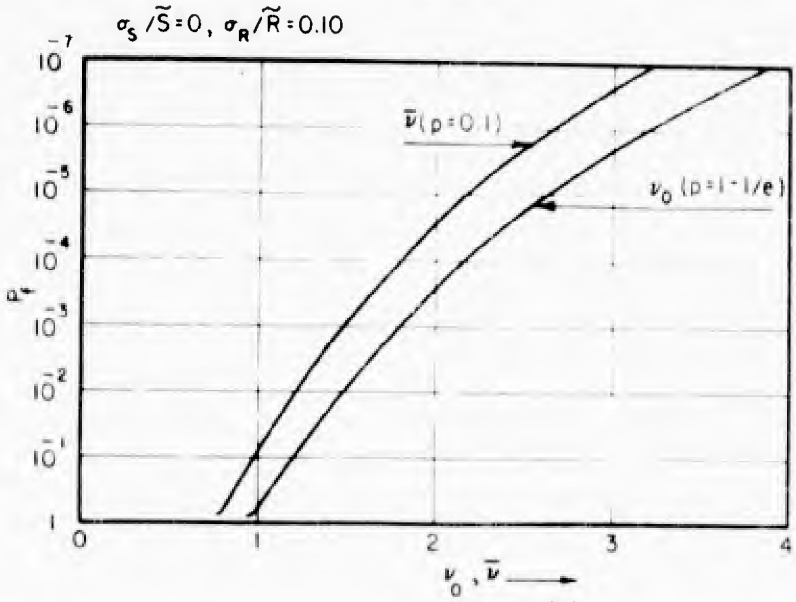

Fig. 6 (b)

Figure 6. Relation between Probability of Failure P_f and Central Safety Factor ν_0 and "Conventional" Safety Factor $\bar{\nu}$ with Extremal Distributions of R and S

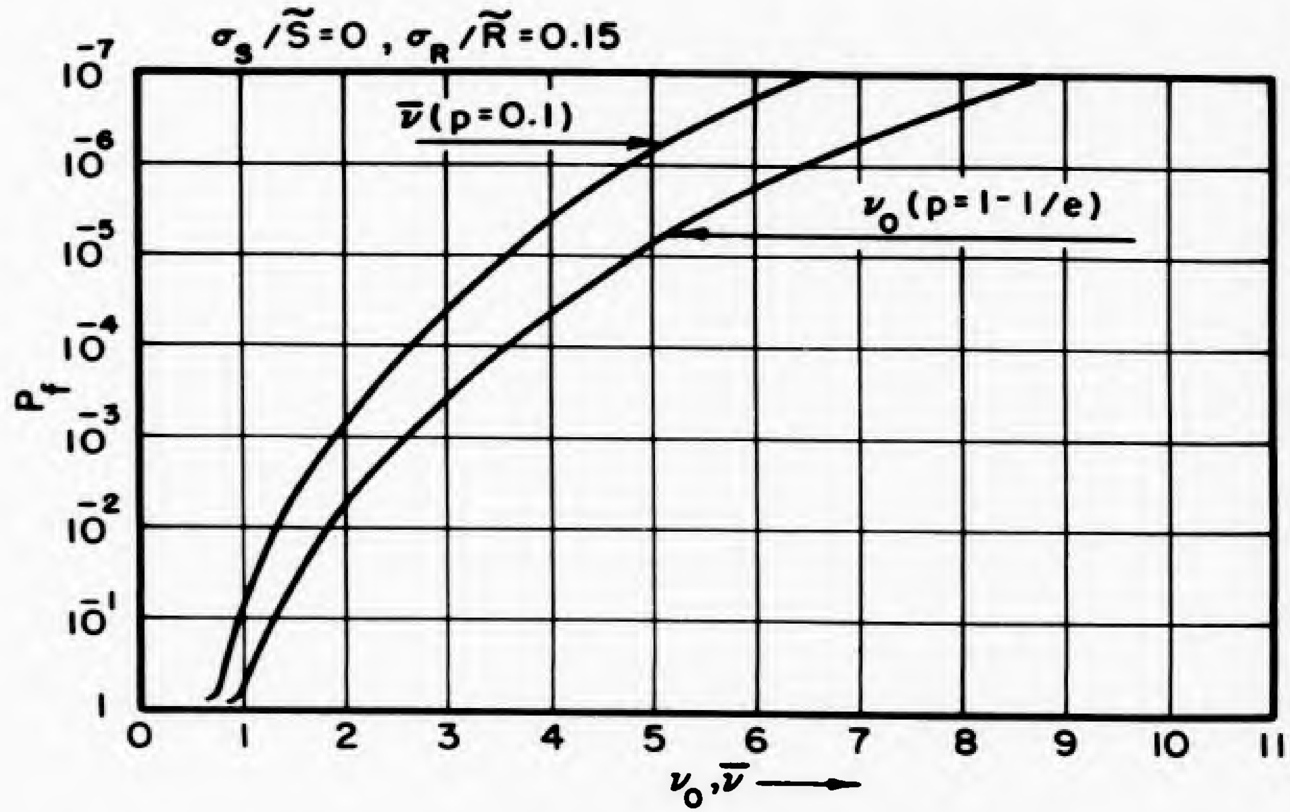

Fig. 6 (c)

Figure 6. Relation between Probability of Failure P_f and Central Safety Factor ν_0 and "Conventional" Safety Factor $\bar{\nu}$ with Extremal Distributions of R and S

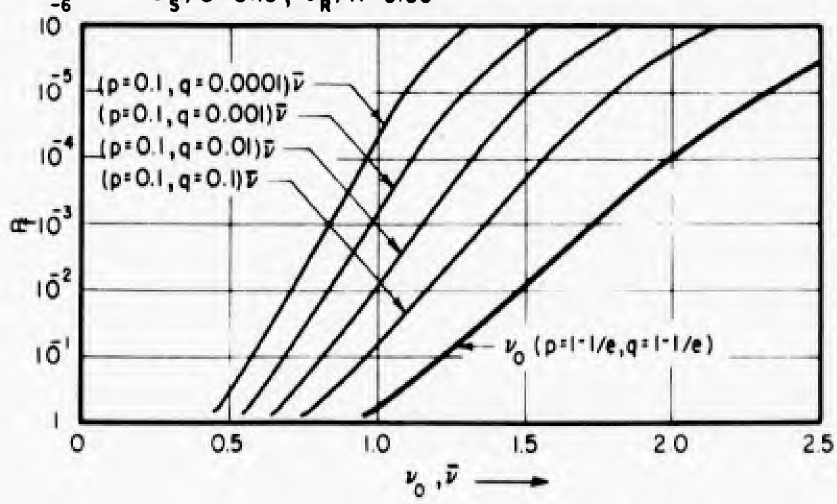

Fig. 6 (d)

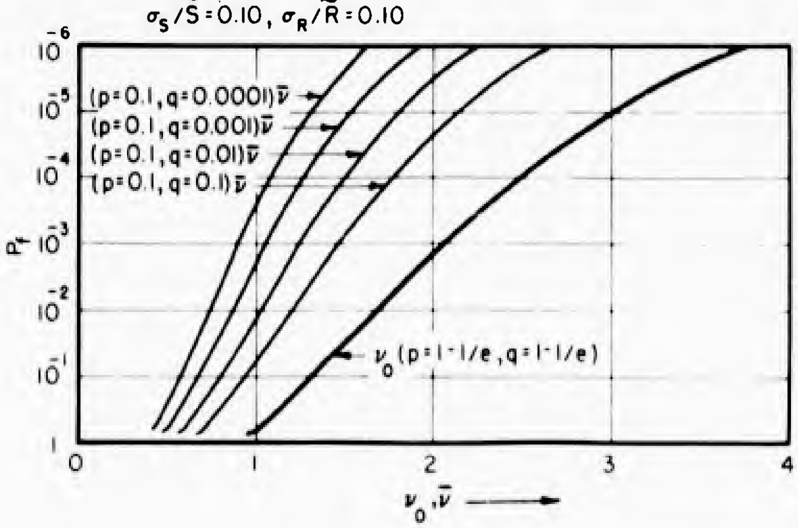

Fig. 6 (e)

Figure 6. Relation between Probability of Failure P_f and Central Safety Factor ν_0 and "Conventional" Safety Factor $\bar{\nu}$ with Extremal Distributions of R and S

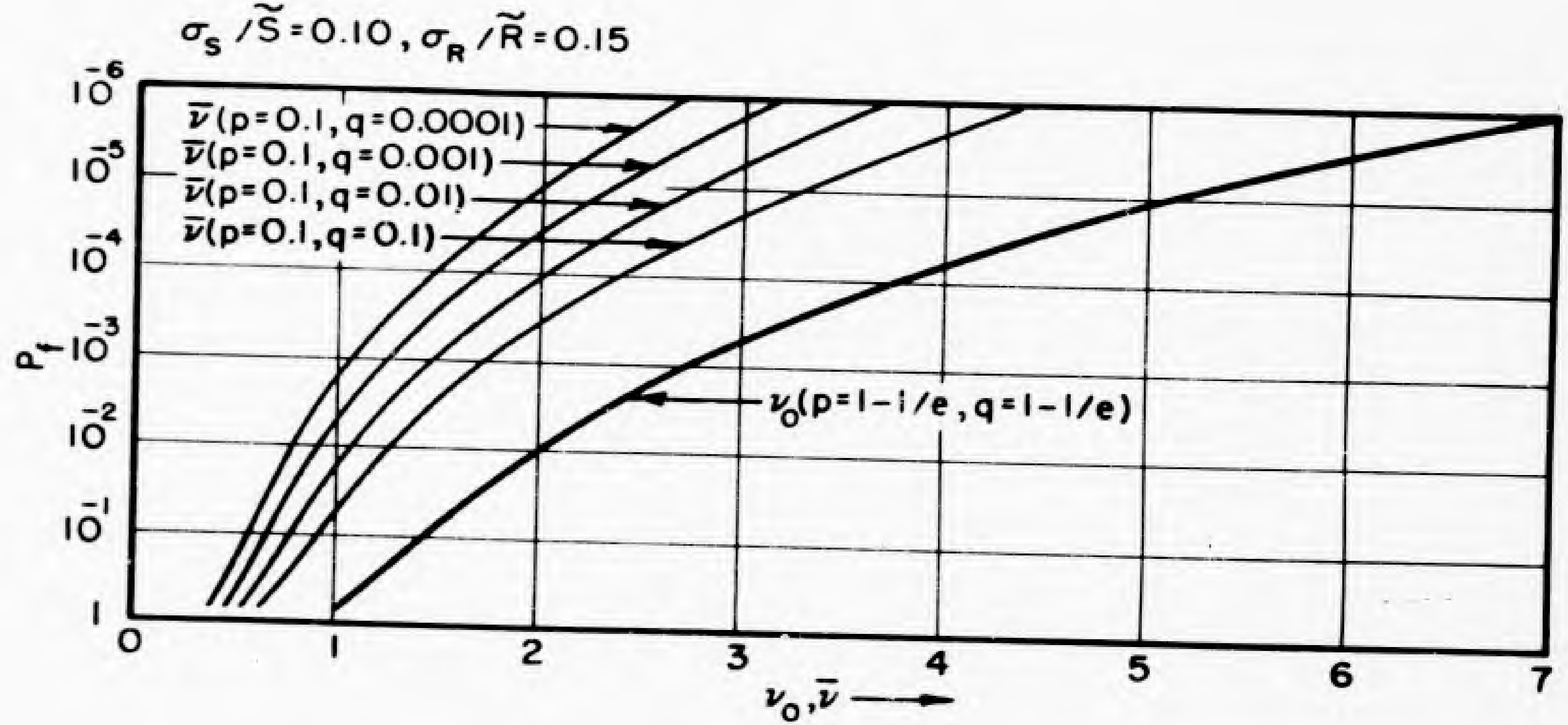

Fig. 6 (f)

Figure 6. Relation between Probability of Failure P_f and Central Safety Factor $_0$ and "Conventional" Safety Factor with Extremal Distributions of R and S

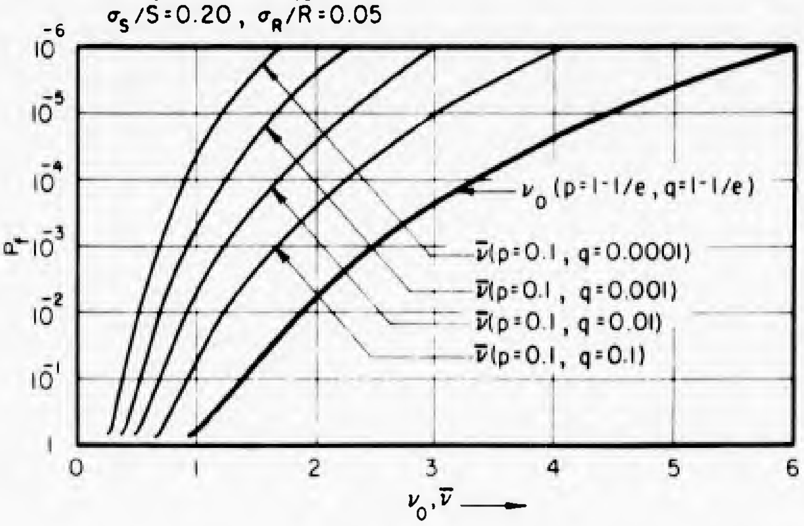

Fig. 6 (g)

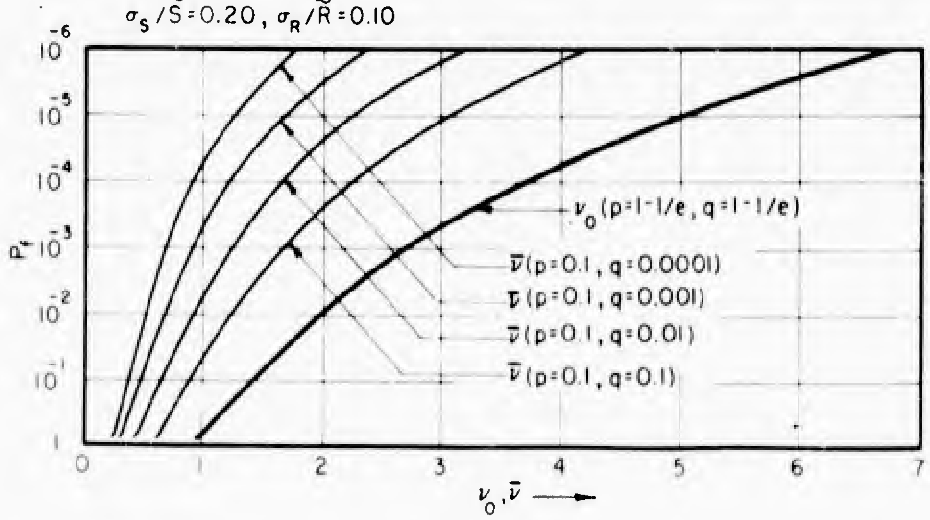

Fig. 6 (h)

Figure 6. Relation between Probability of Failure P_f and Central Safety Factor ν_0 and "Conventional" Safety Factor $\bar{\nu}$ with Extremal Distributions of R and S

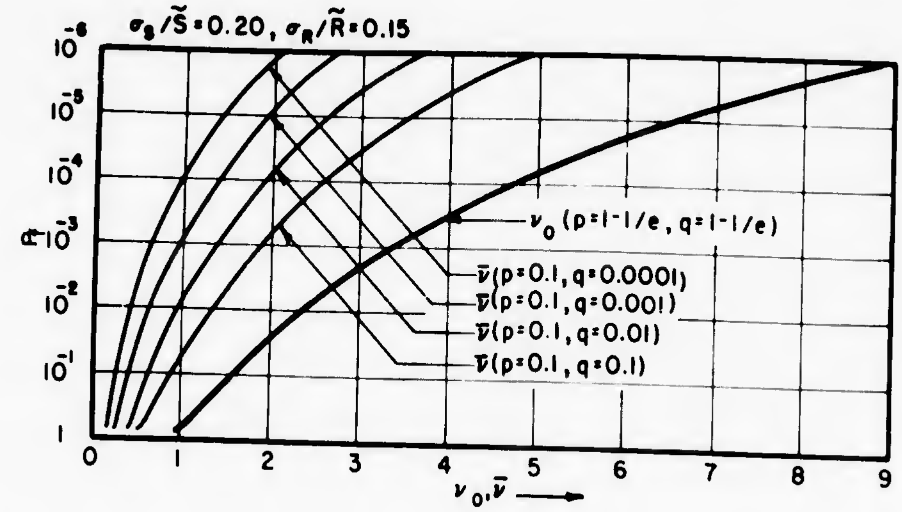

Fig. 6 (i)

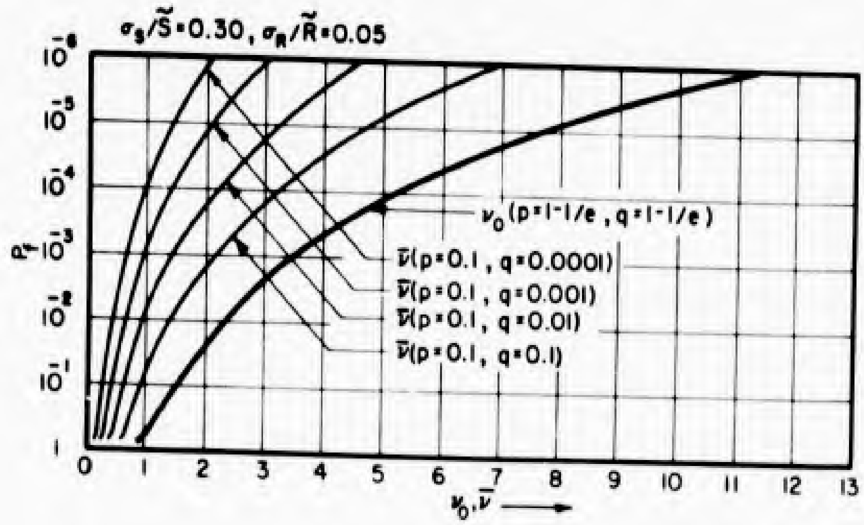

Fig. 6 (j)

Figure 6. Relation between Probability of Failure P_f and Central Safety Factor $_0$ and "Conventional" Safety Factor with Extremal Distributions of R and S

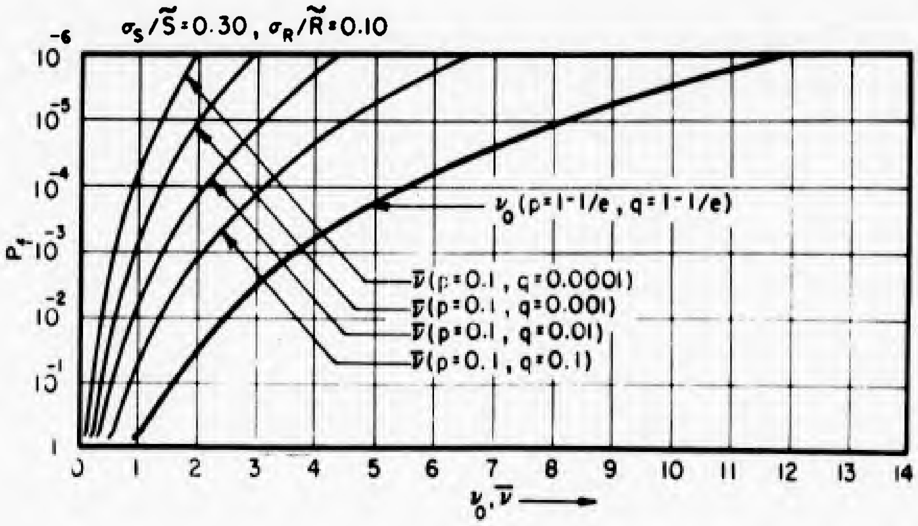

Fig. 6 (k)

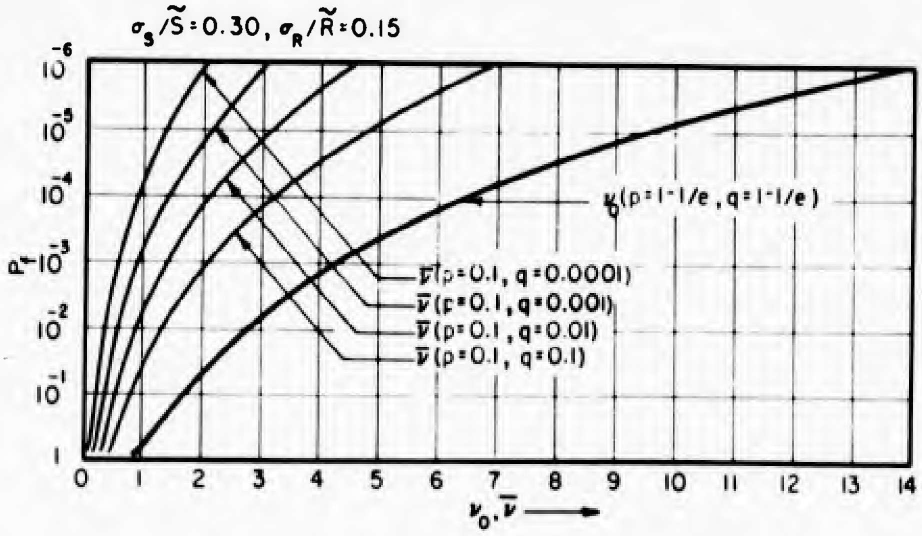

Fig. 6 (ℓ)

Figure 6. Relation between Probability of Failure P_f and Central Safety Factor ν_0 and "Conventional" Safety Factor $\bar{\nu}$ with Extremal Distributions of R and S

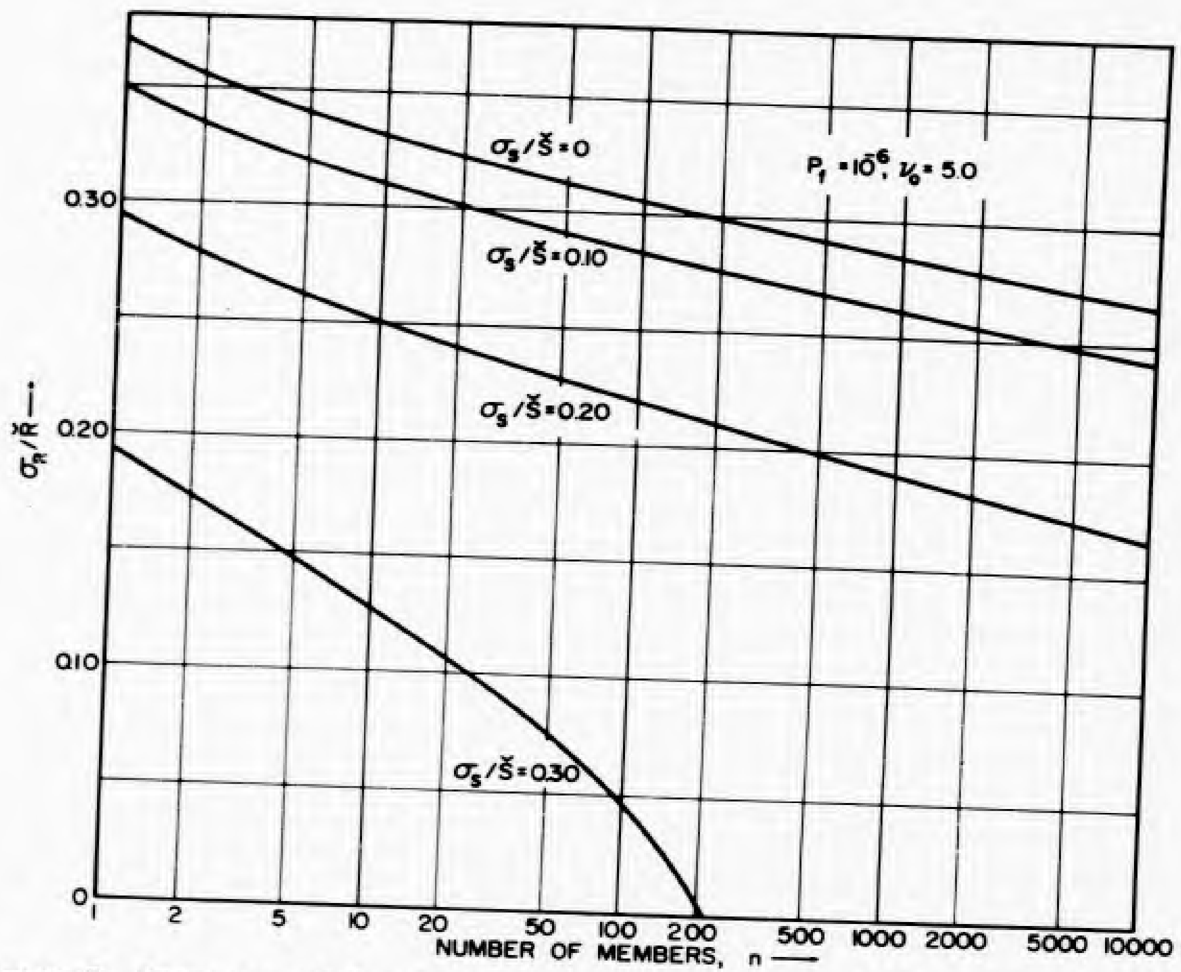

Figure 7. Increase of Improvement of Material Control (in Terms of Decrease of $\sigma_R/\bar{R}$) as a Function of n to Ensure Constant Probability of Failure $P_f=10^{-6}$ of a Non-Redundant Structure of n Members ($\nu_0=5.0$)

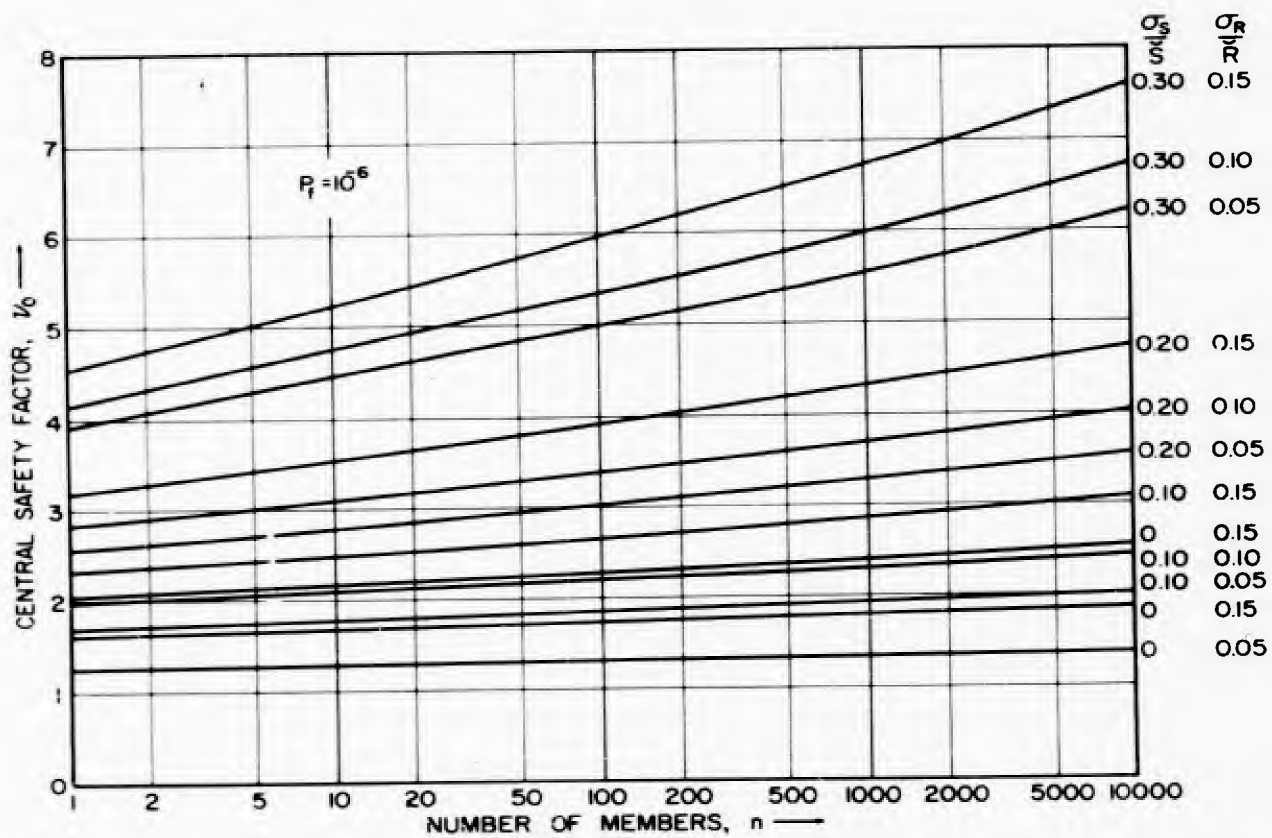

Figure 8. Increase of Central Safety Factor ν_0 as Function of n to Ensure Constant Probability of Failure $P_f = 10^{-6}$ of a Non-Redundant Structure of n Members

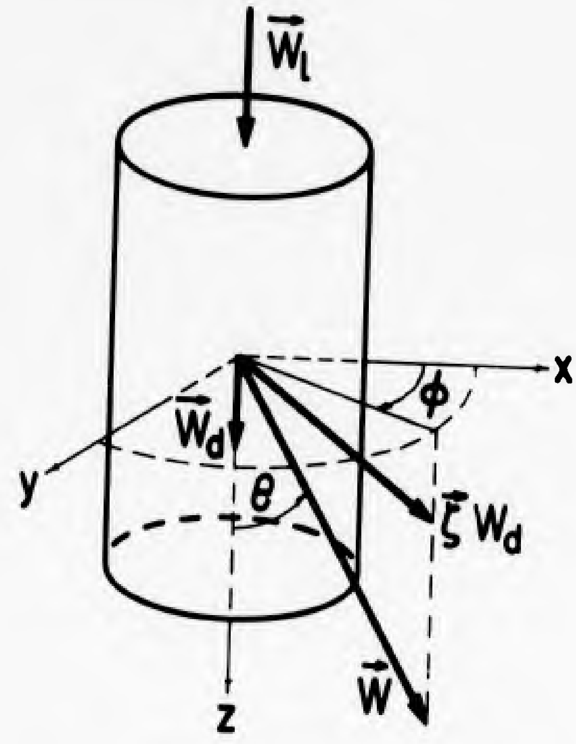

Figure 9. Short Column under Combined Three-Dimensional Loading

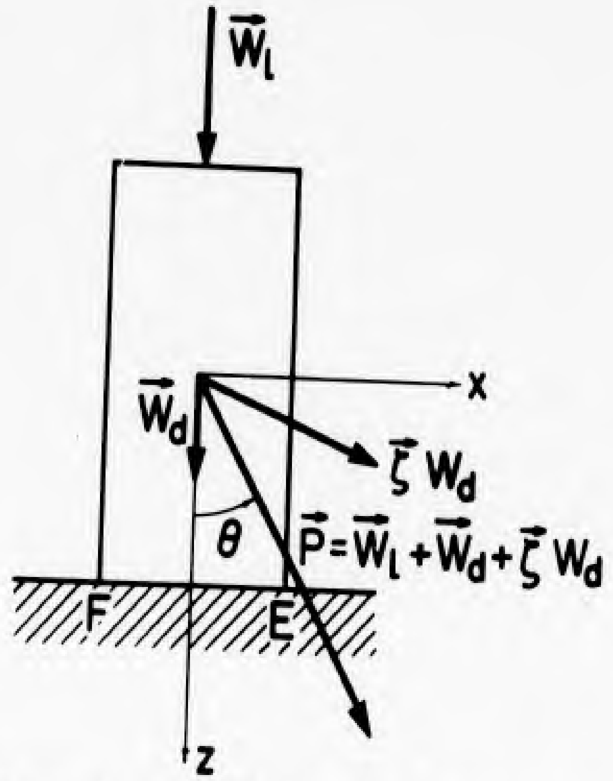

Figure 10. Short Column under Combined Two-Dimensional Loading

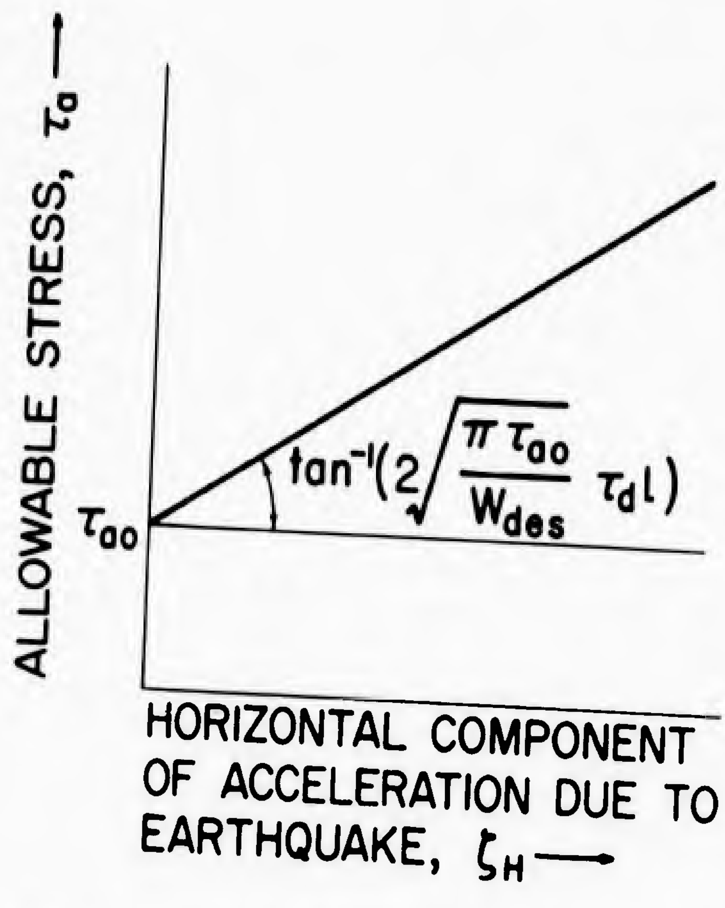

Figure 11. Allowable Increase of Design Load for Horizontal Acceleration due to Earthquake as Secondary Load

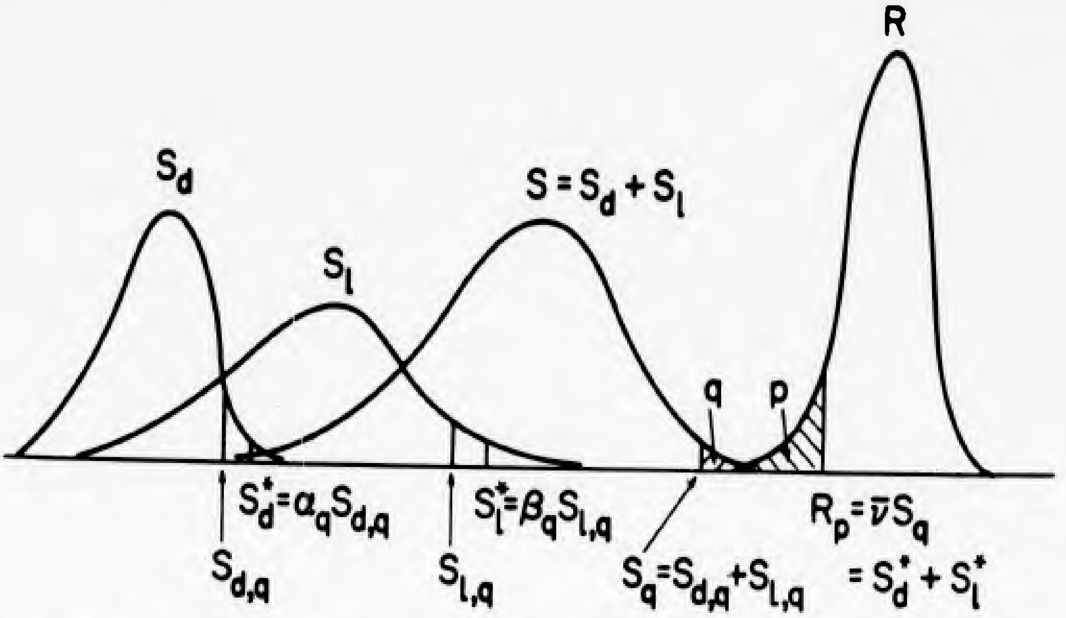

Figure 12. Schematical Illustration for R_p, S_q, S_d^*, S_ℓ^*, $S_{d,q}$, $S_{\ell,q}$, α_q and β_q

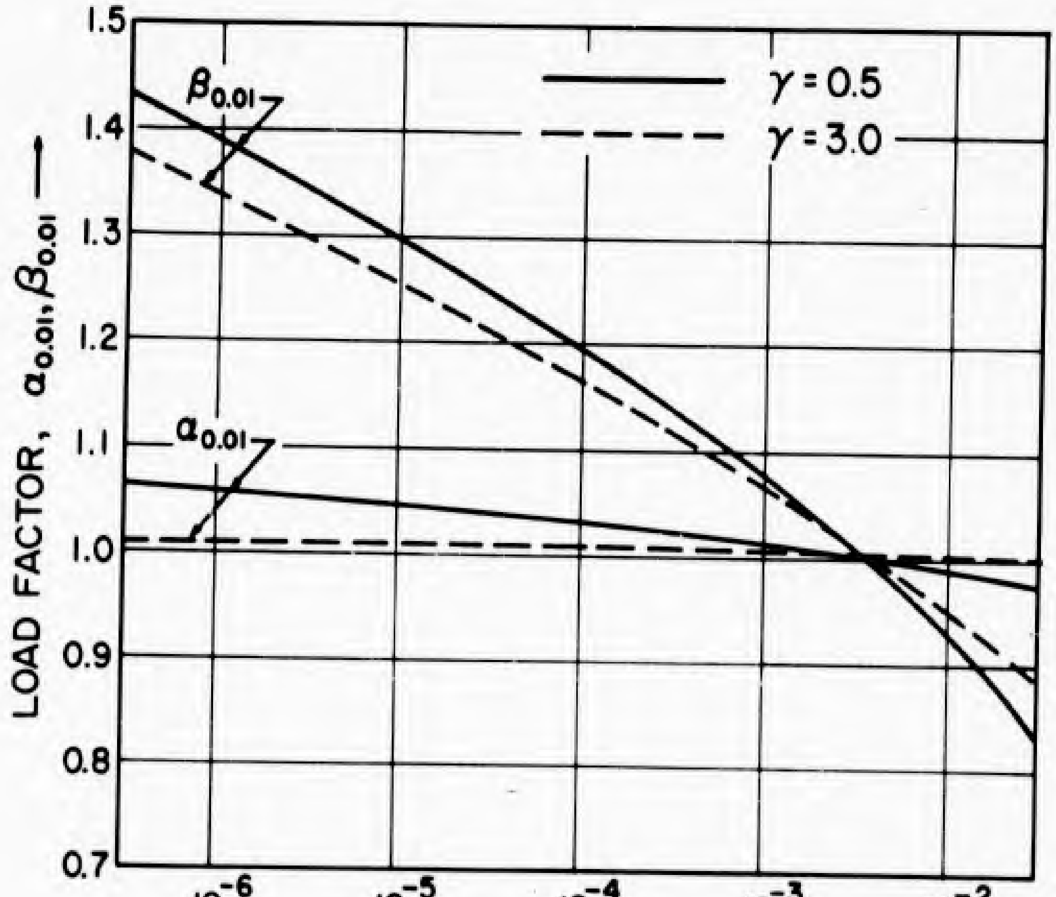

Figure 13. Separate Load Factors α_q and β_q as Functions of Probability of Failure

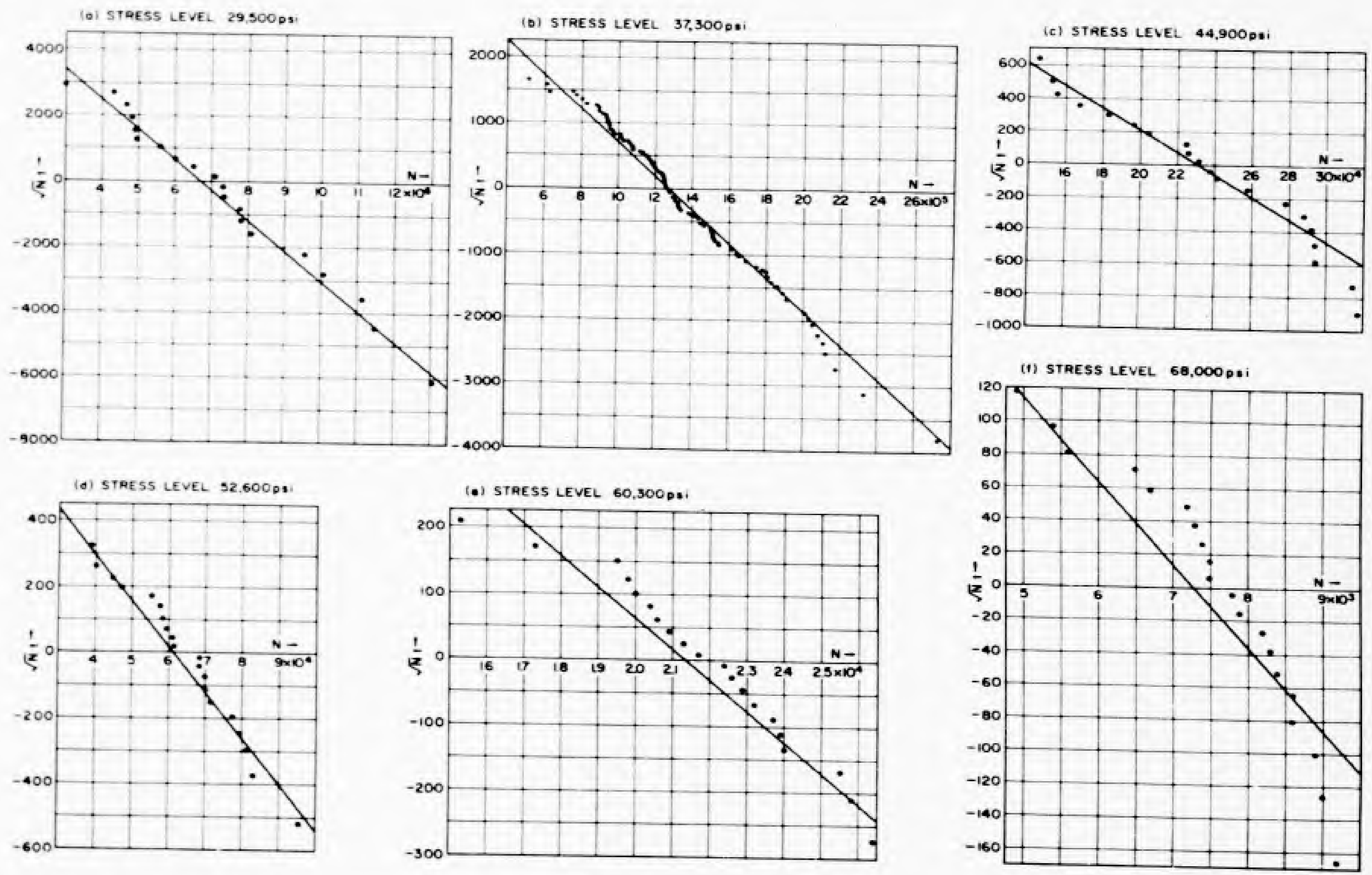

Figure 14. Representation of Constant Amplitude Fatigue Test Results on AA 7075 Aluminum in $\sqrt{N}t$ - N Coordinate System

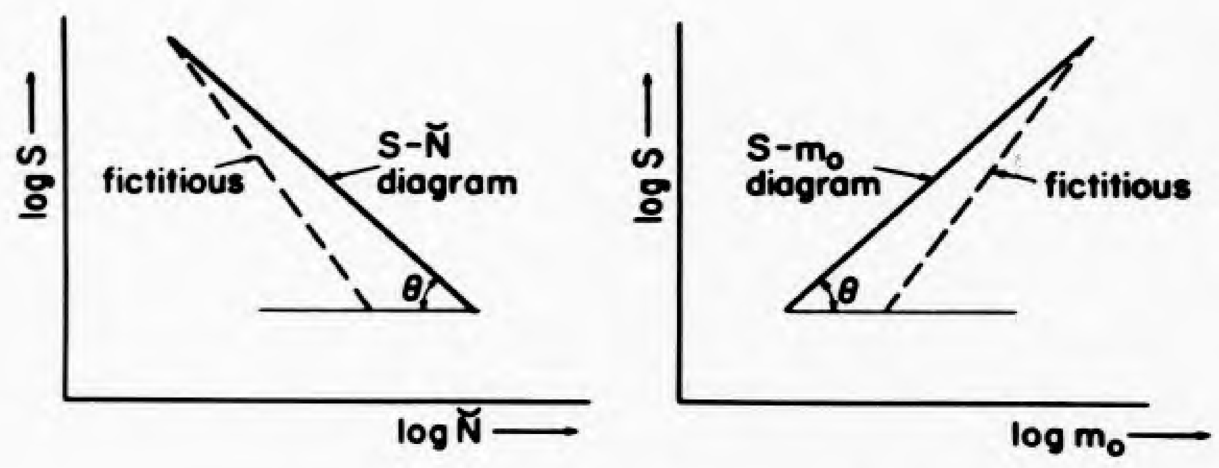

Figure 15. Schematic Illustration of S - $\breve{N}$ and S - m_0 Diagrams

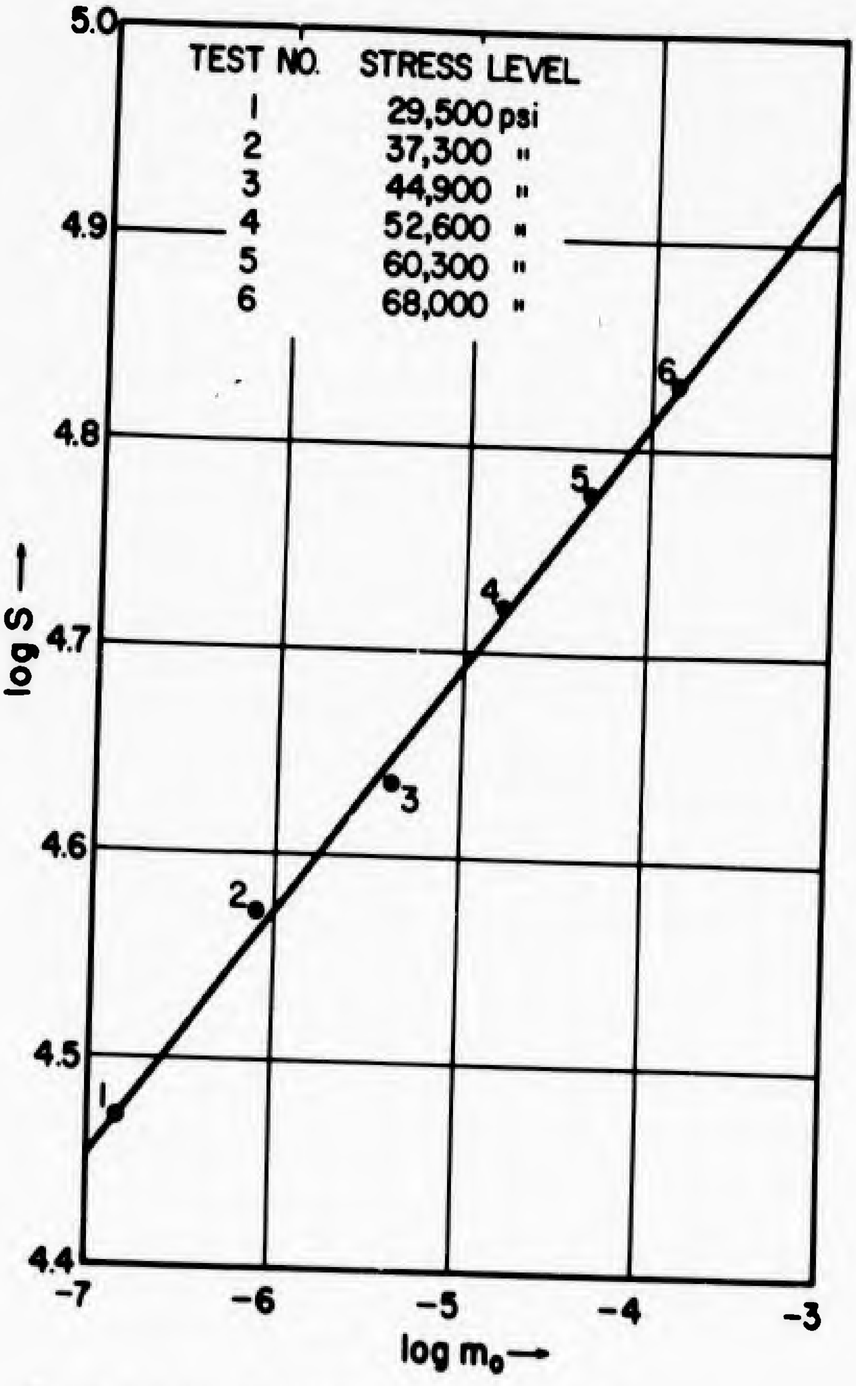

Figure 16. S - m_0 Relation for AA 7075 Al.

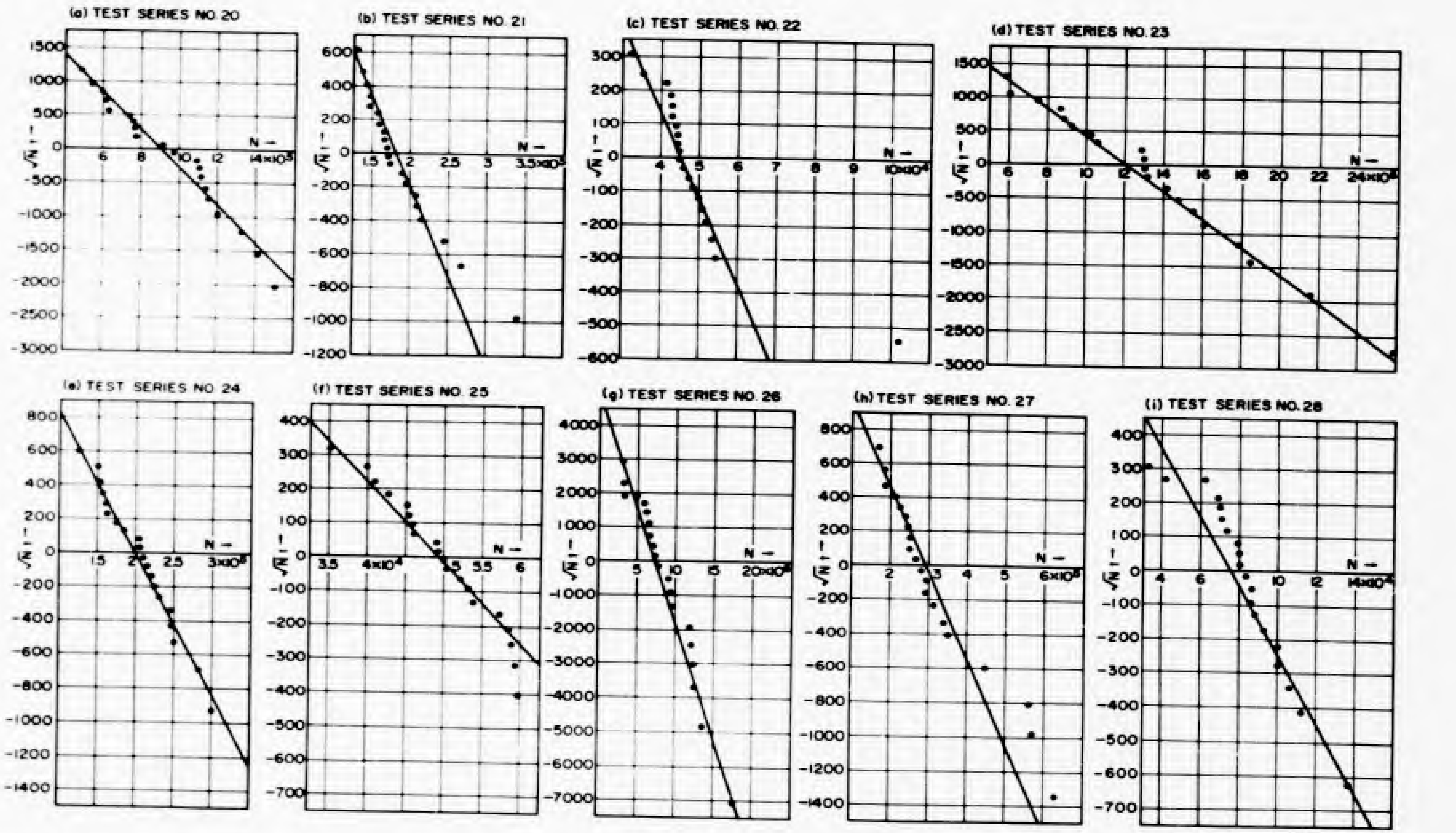

Figure 17. Representation of Random Fatigue Test Results on **AA 7075 Aluminum** in $\sqrt{N}t$ - N Coordinate System

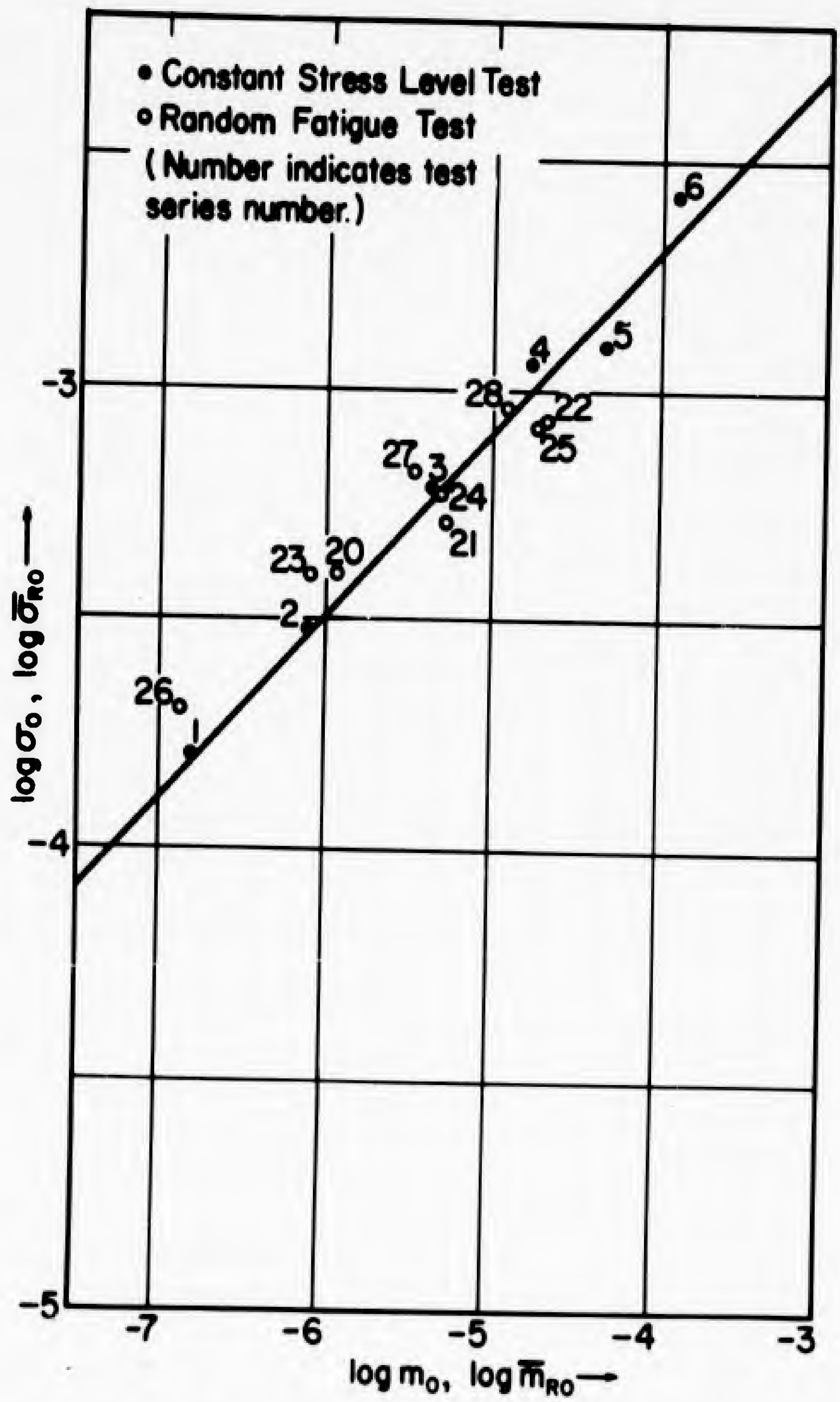

Figure 18. m_0 - σ_0 Relation for AA 7075 Al.

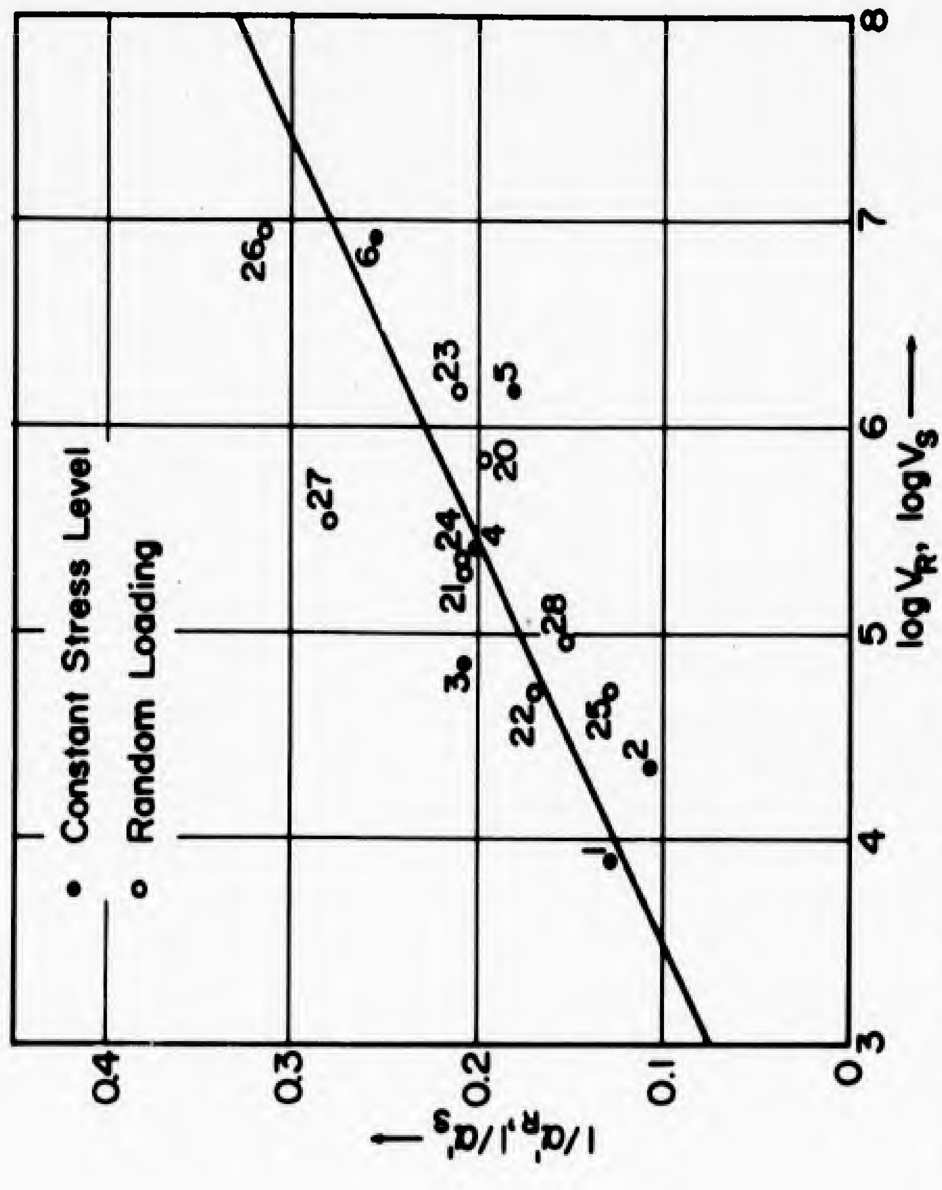

Figure 19. $1/\alpha'_R$, $1/\alpha'_S$ - V_R, V_S Relation for AA 7075 Aluminum

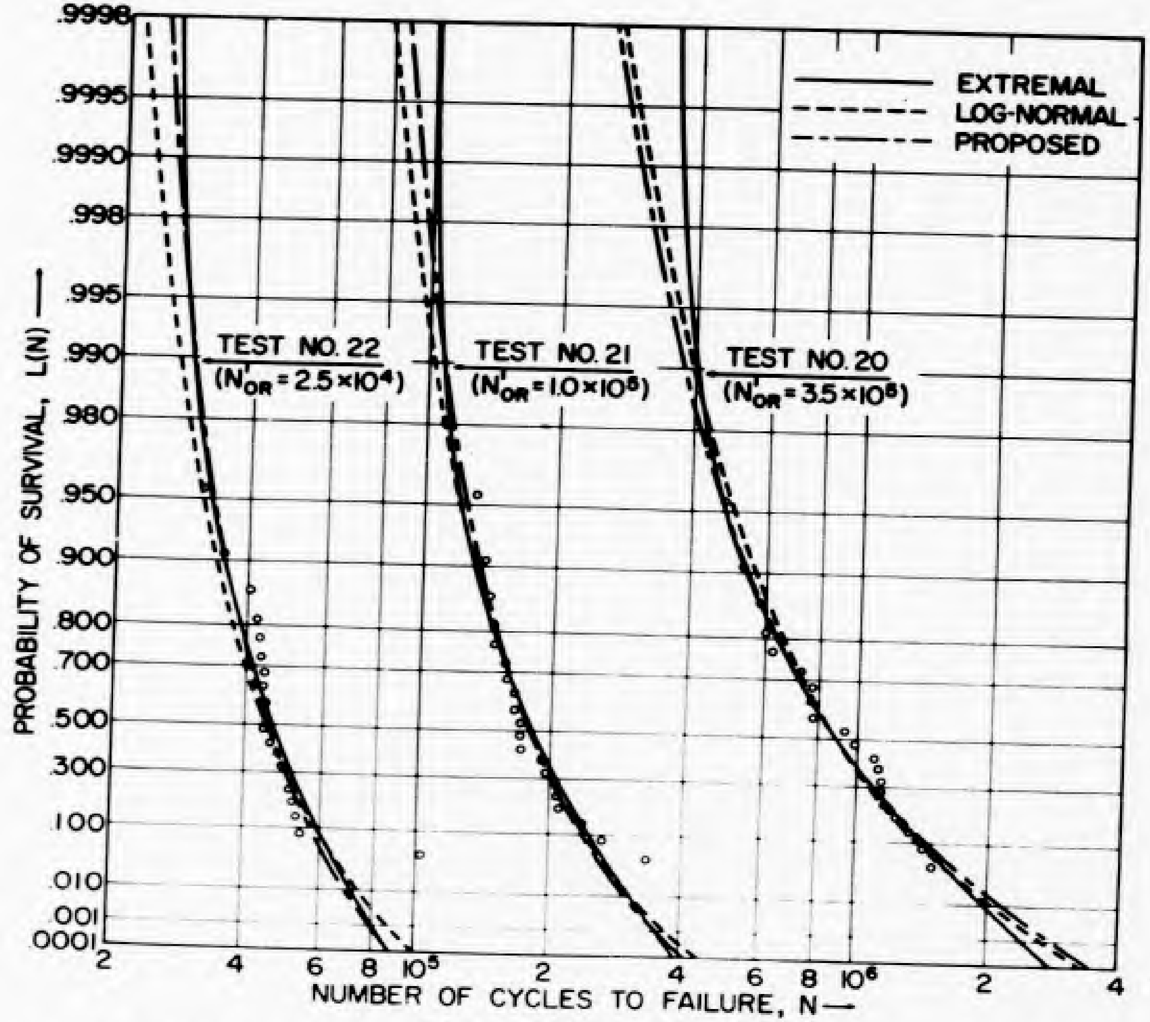

Fig. 20 (a)

Figure 20. Extremal, Logarithmic-Normal and Proposed Survivorship Functions for AA 7075 Aluminum under Random Loading

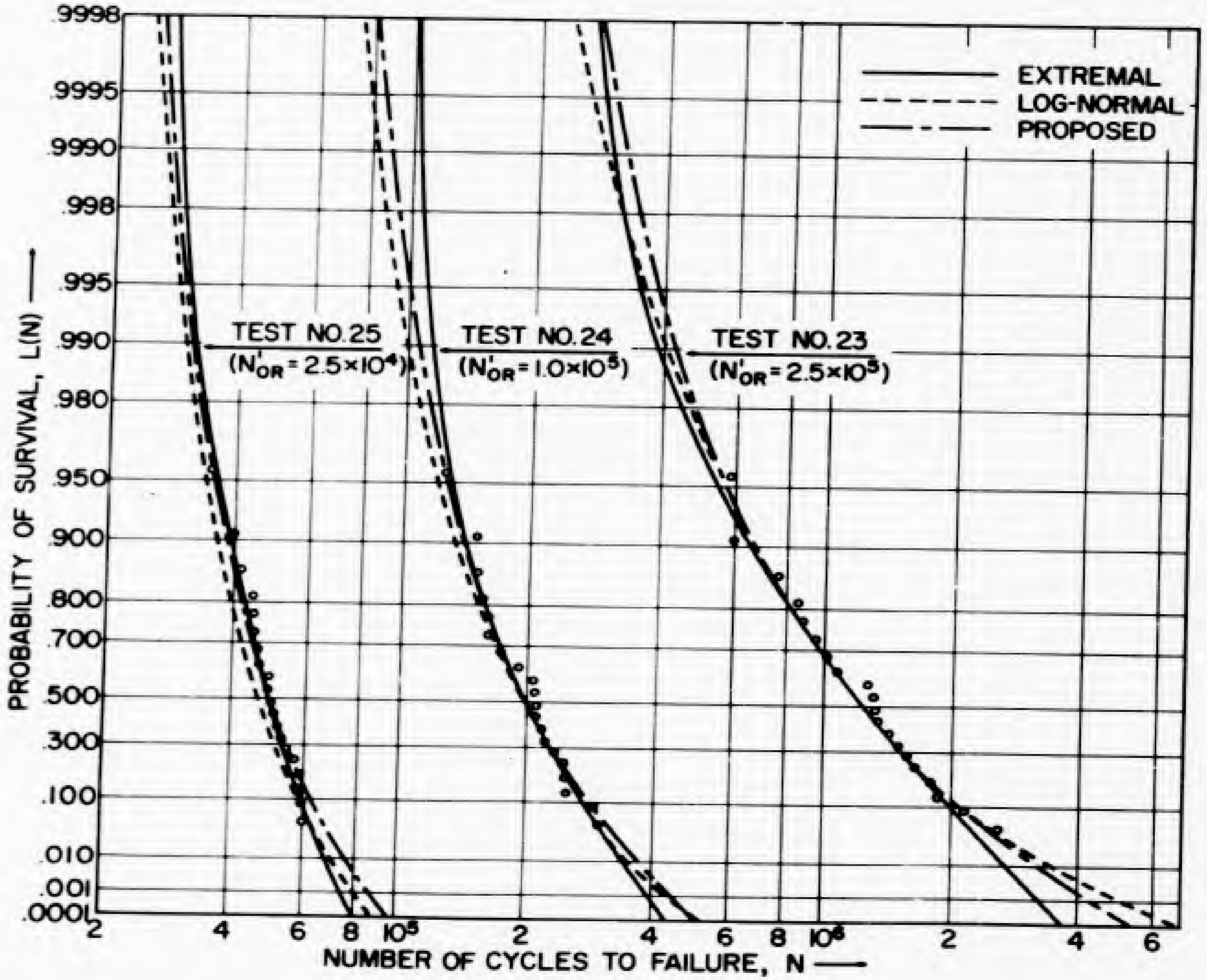

Fig. 20 (b)

Figure 20. Extremal, Logarithmic-Normal and Proposed Survivorship Functions for AA 7075 Aluminum under Random Loading

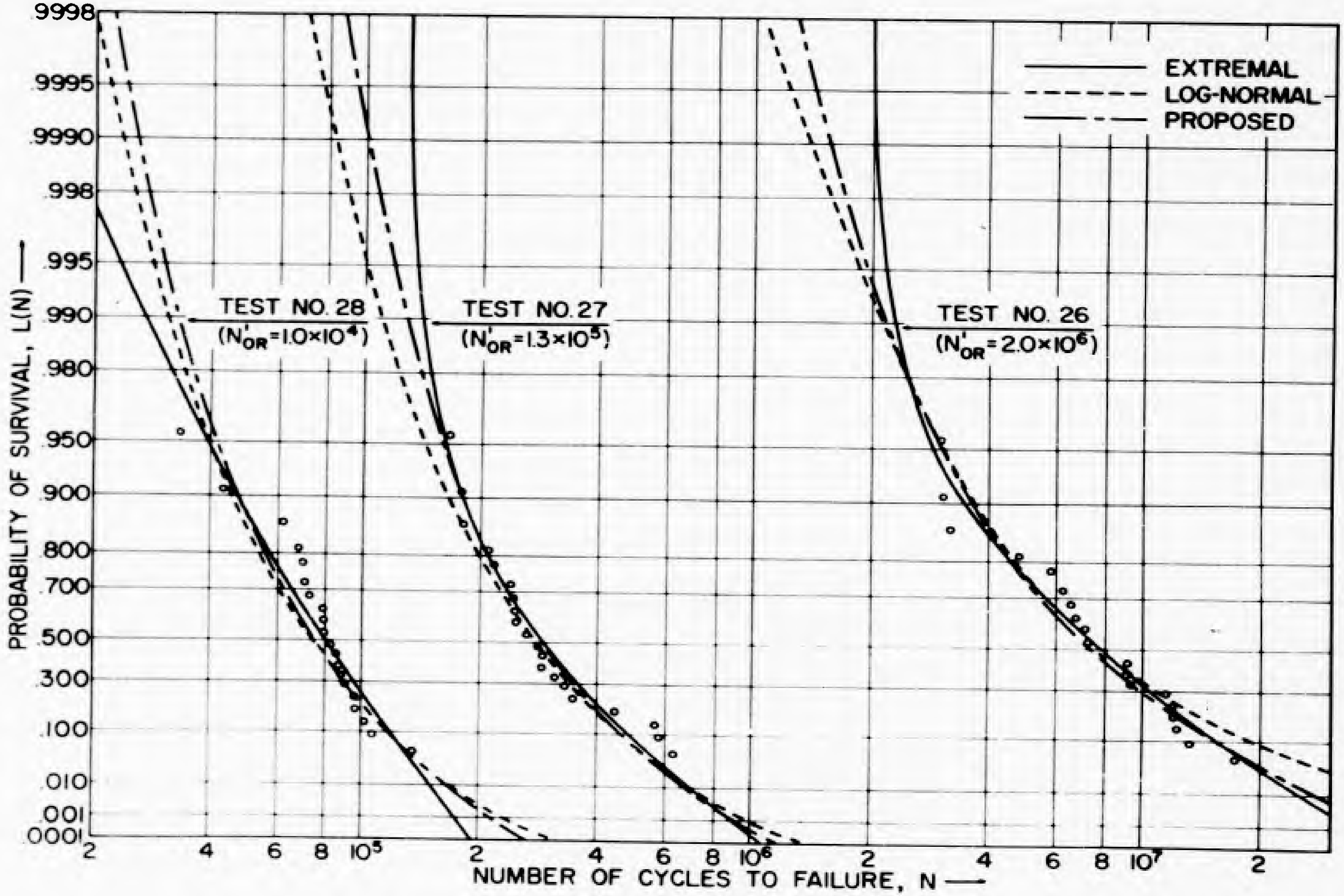

Fig. 20 (c)

Figure 20. Extremal, Logarithmic-Normal and Proposed Survivorship Functions for AA 7075 Aluminum under Random Loading

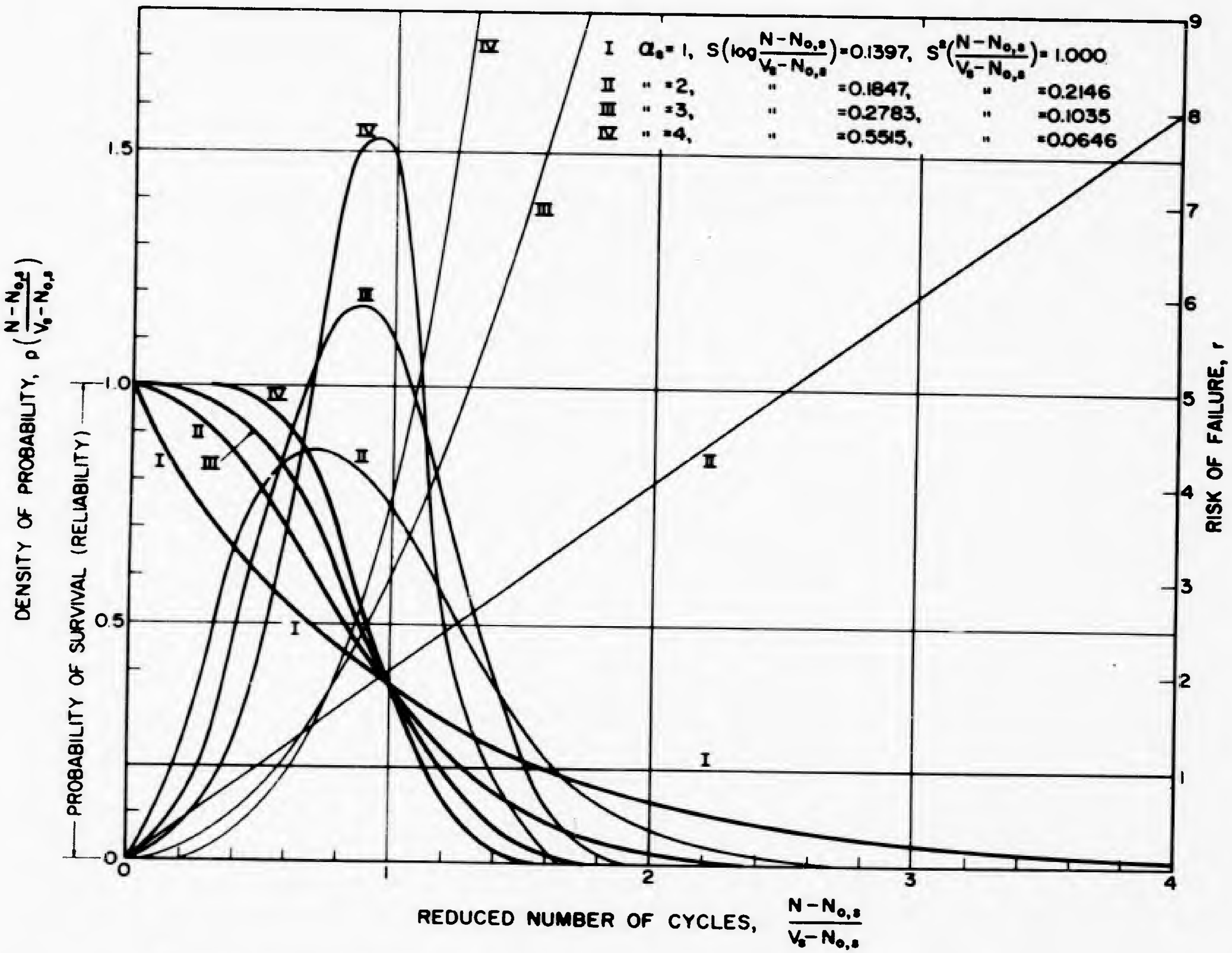

Figure 21. Distribution Function, Probability Density, and Risk Function of Extremal Distribution

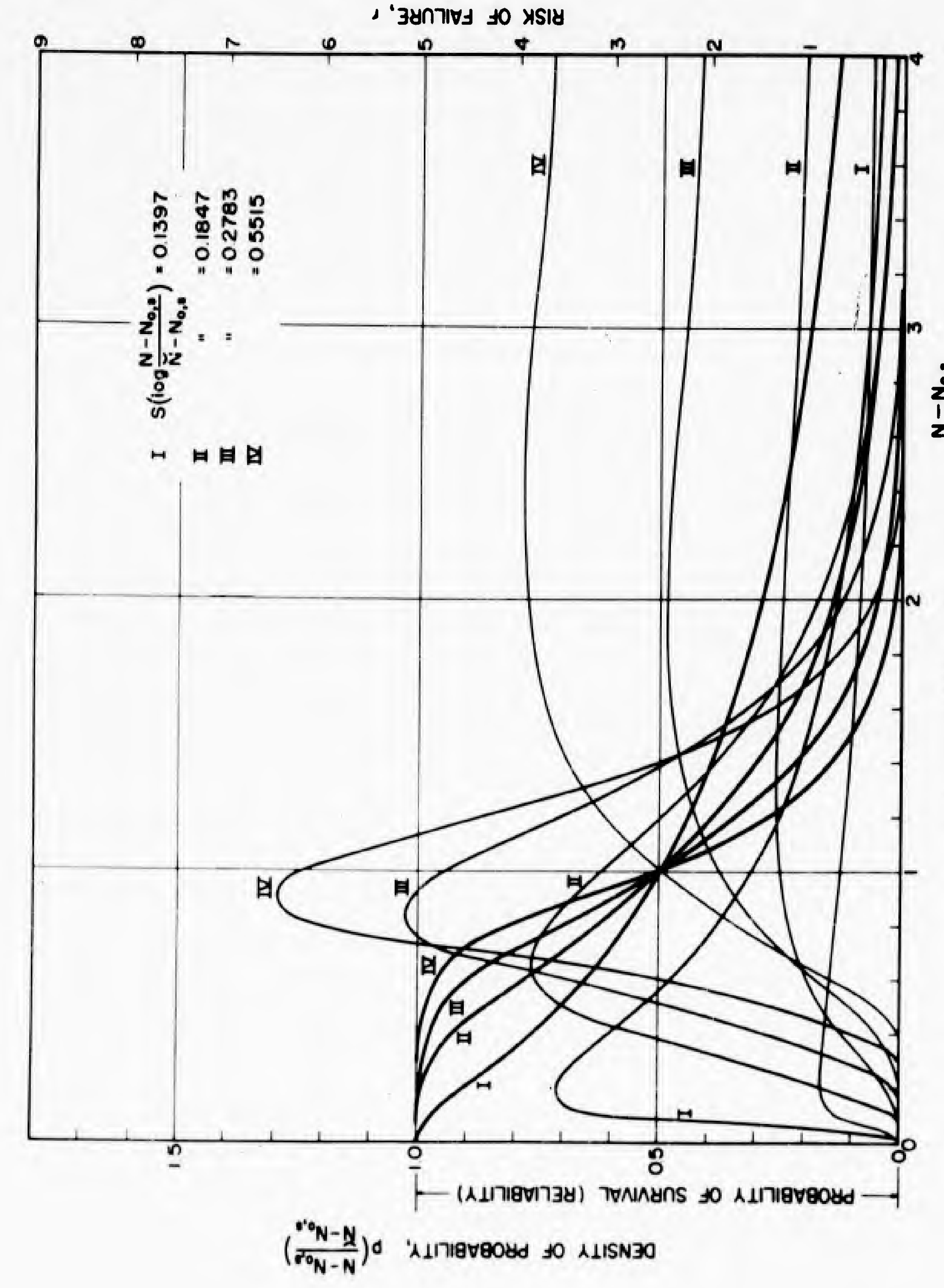

Figure 22. Distribution Function, Probability Density and Risk Function of Logarithmic-Normal Distribution

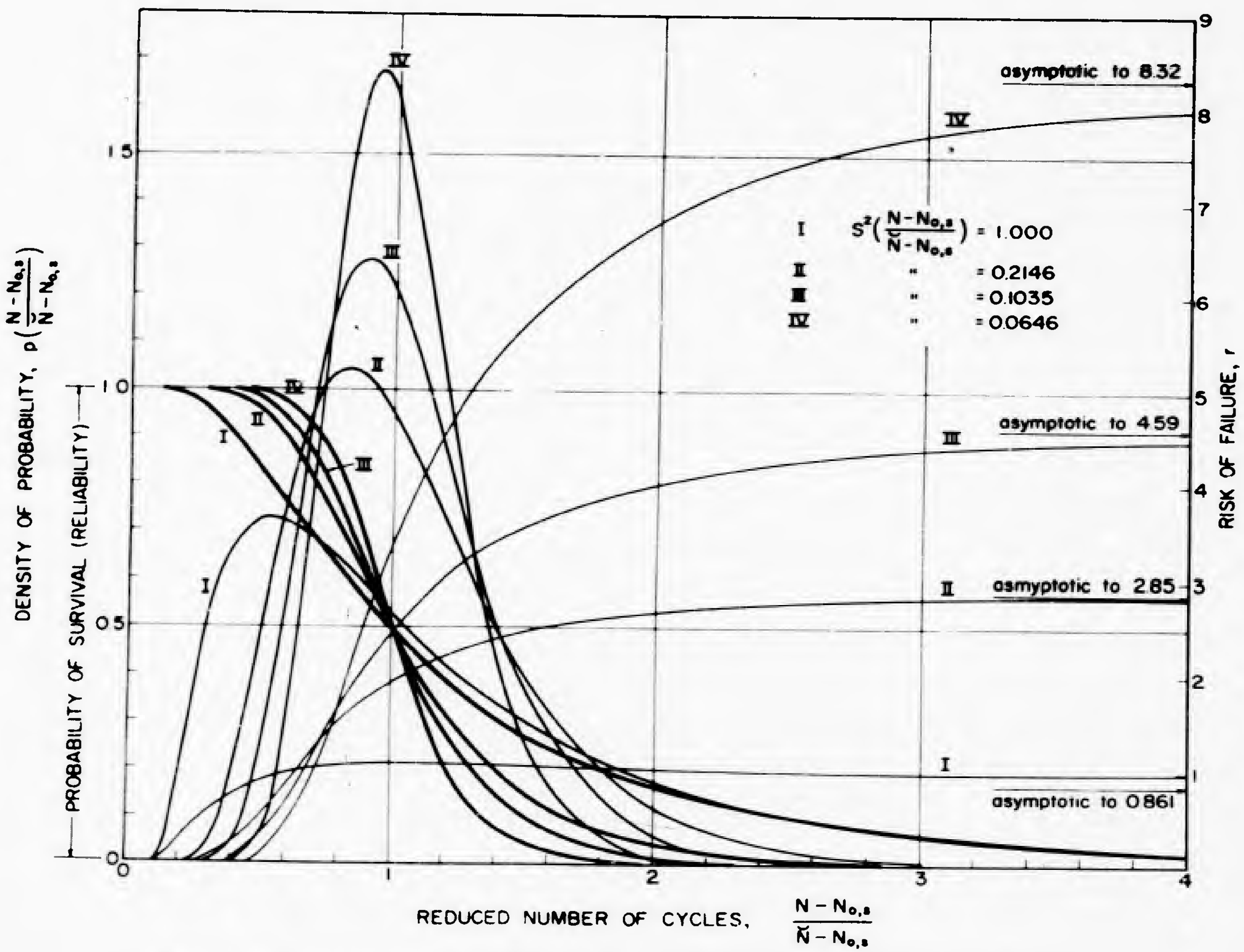

Figure 23. Distribution Function, Probability Density and Risk Function of Proposed Distribution

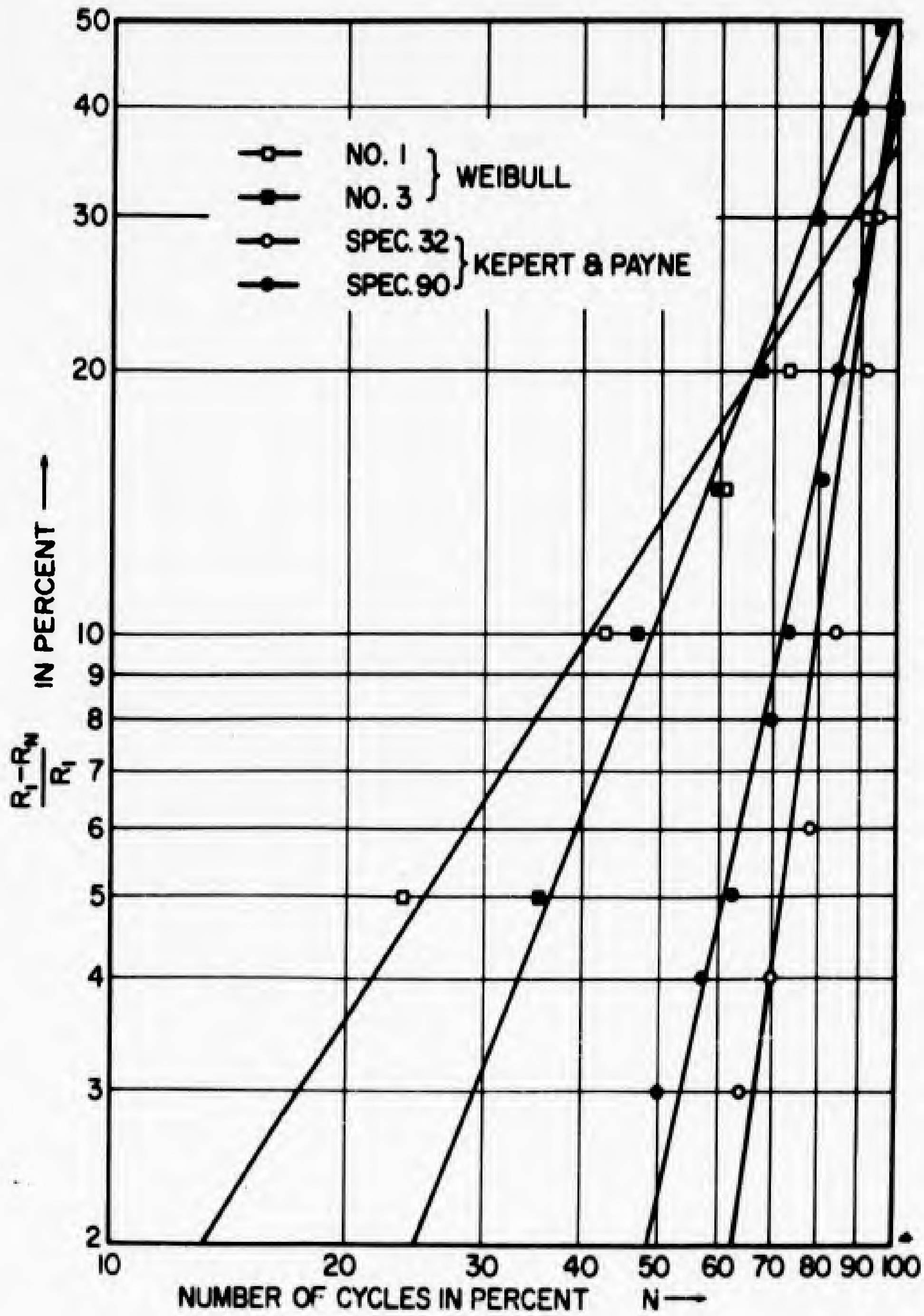

Figure 24. Empirical Relation between Reduction of Static Strength and Cycle Ratio (W. Weibull[18] and J. L. Kepert and A. O. Payne[19])

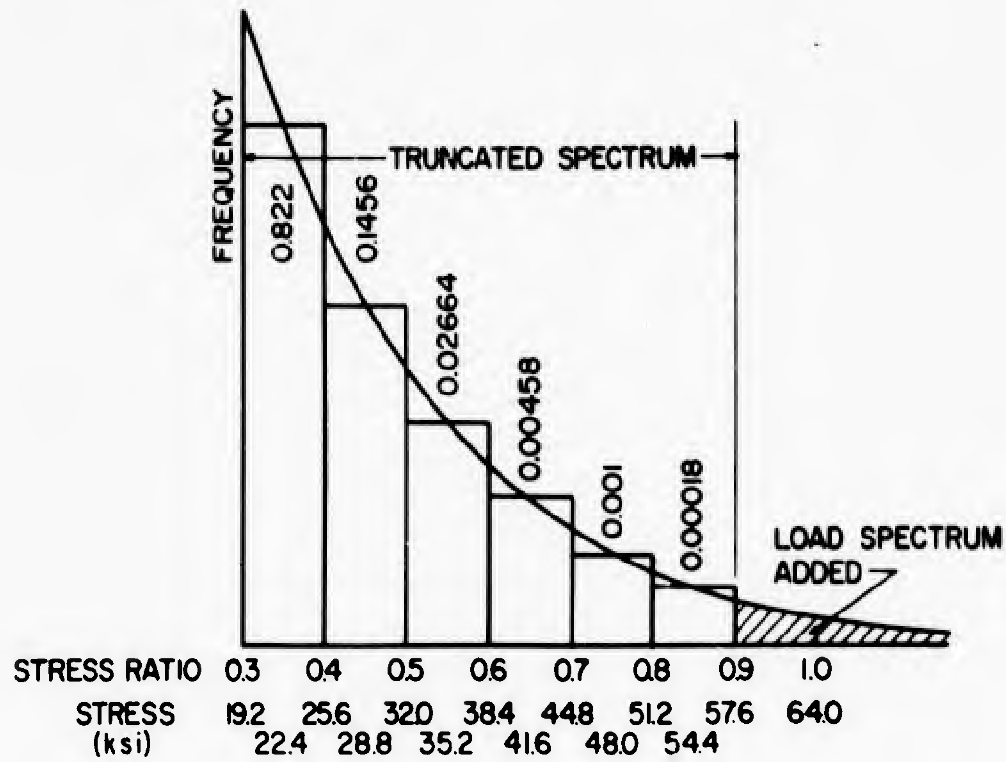

Figure 25. Schematical Representation of Exponential Truncated and Full Load Spectrum

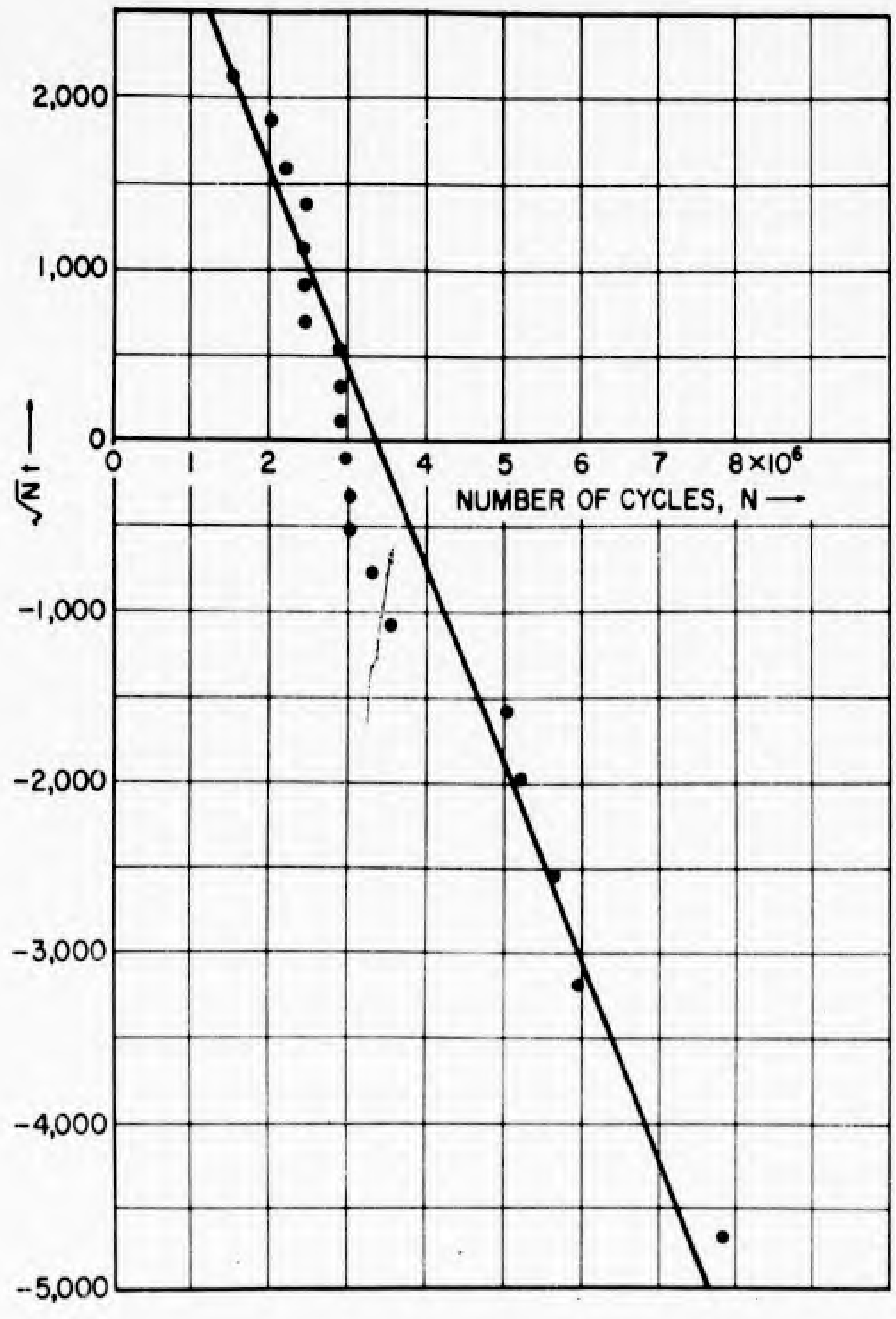

Figure 26. Representation of a Random Fatigue Test Result on AA 2024 Aluminum in $\sqrt{N}t$ - N Coordinate System

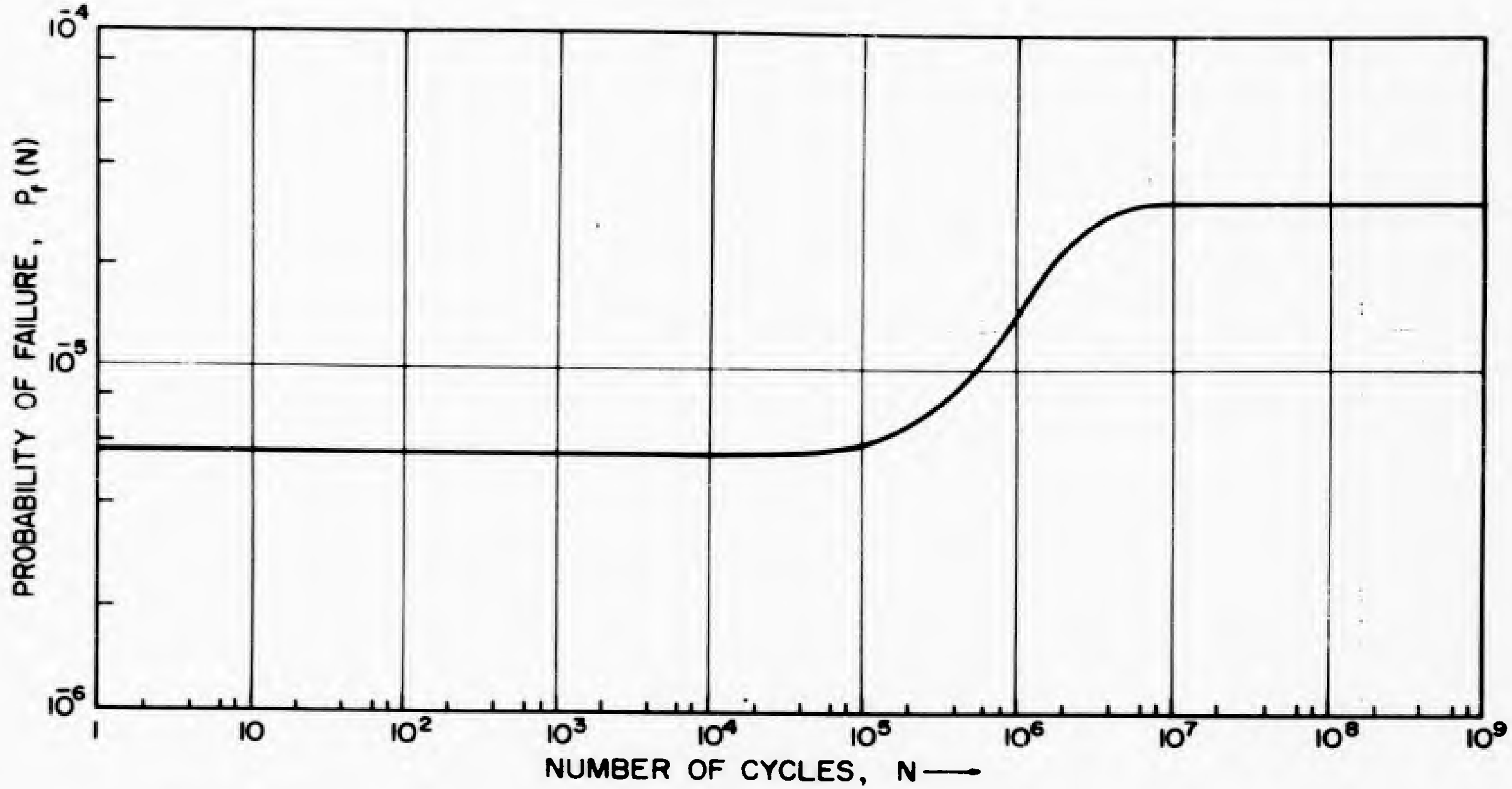

Figure 27. Probability of Failure $P_f(N)$ due to Extremes of Load Spectrum as a Function of Number $\bar{N}$ of Load Application

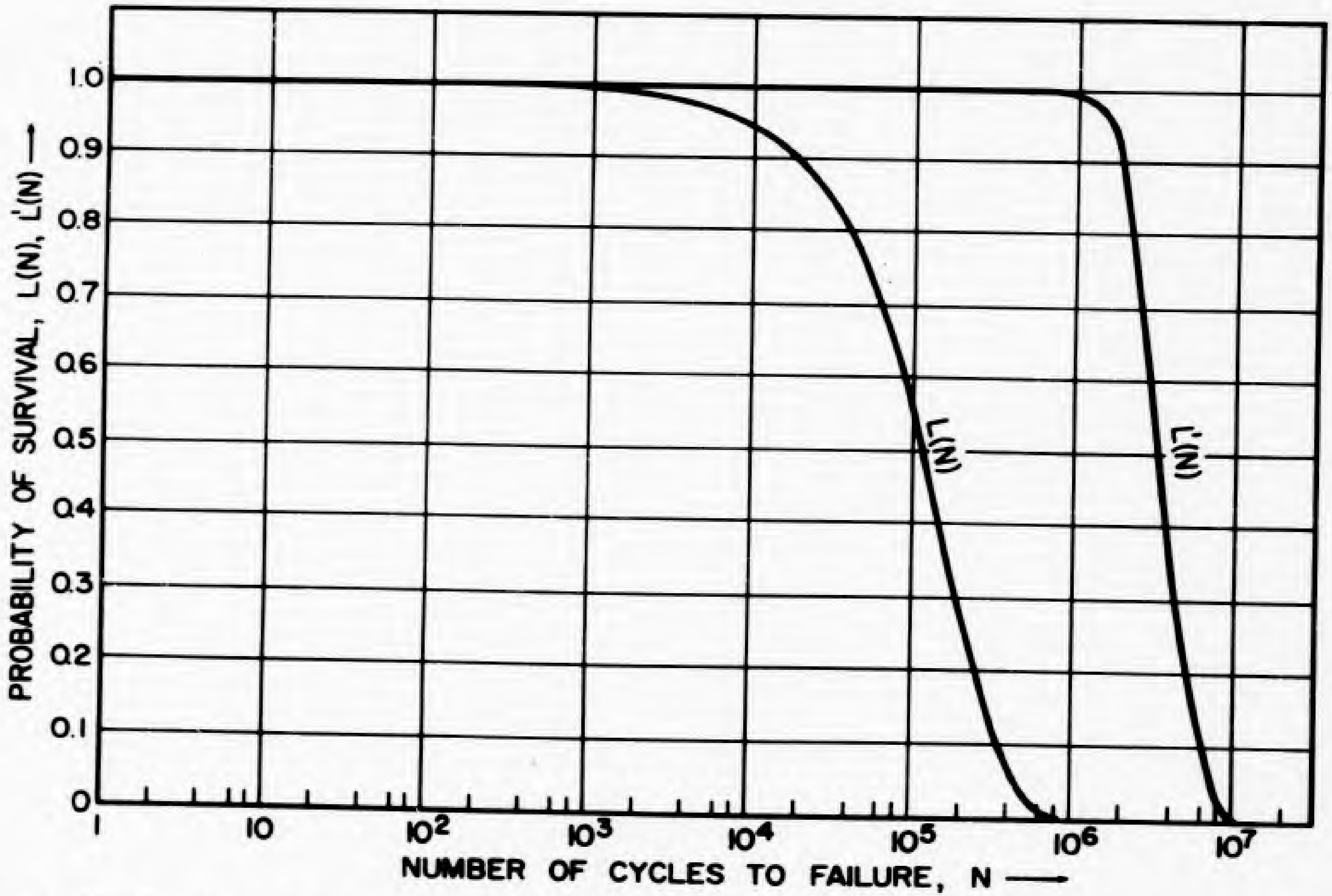

Figure 28. Survivorship Functions L'(N) and L(N) under Truncated and Full Load Spectrum

COLUMBIA UNIVERSITY, New York, N. Y. STRUCTURAL SAFETY UNDER CONDITIONS OF ULTIMATE LOAD FAILURE AND FATIGUE, by A. M. Freudenthal and M. Shinozuka, Oct 1961. 108p. incl. figs, tables and refs. (Project 7351; Task 73521)(WADD TR 61-177)(Contract AF 33(616)-7042)

Unclassified report

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