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PERTURBATIONS OF A SYNCHRONOUS SATELLITE DUE TO THE TRIAXIALITY OF THE EARTH

R. H. Frick and T. B. Garber

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The RAND Corporation
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PREFACE

The work presented in this report was done under NASA Contract NASr-21(02), monitored by the Director of Communication Studies, in the Office of Applications. The general study area of the contract comprises technical studies relating to communication satellites. As part of this work, RAND has paid particular attention to guidance and orbit stabilization problems of 24-hour synchronous satellites. The present study shows that the problem of maintaining such satellites in an exactly stationary position will be more complex than had been previously expected. The results should be of concern to all agencies and contractors involved in the development of a synchronous satellite.

A summary of the findings of this study was presented to NASA on October 20, 1961.

SUMMARY

This paper presents the results of an investigation of the behavior of a synchronous (24-hour) satellite as affected by the triaxiality of the earth. This includes not only the effect of the equatorial bulge but also the ellipticity of the earth's equatorial section. The results indicate that there are only two positions on the equator (123°9' West Longitude and 56°51' East Longitude) at which a truly synchronous satellite can exist in a stable condition. In order to establish a synchronous satellite at any other longitude, station-keeping propulsion of the order of 51 ft/sec for each year of operation would be required. If no station-keeping propulsion is provided, the satellite will execute long period (greater than 1.3 years) oscillations about the closest of the two stable positions mentioned above.

CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

a_r	radial acceleration component
a_θ	easterly acceleration component
a_ϕ	northerly acceleration component
F_r	radial force component
F_θ	easterly force component
F_ϕ	northerly force component
g_o	gravitational acceleration at earth's surface
I	specific impulse of fuel
J_2	earth oblateness coefficient
$J_2^{(2)}$	equatorial ellipticity coefficient
$K(\gamma_o)$	complete elliptic integral of the first kind
k_1	modified oblateness coefficient
k_2	modified ellipticity coefficient
m	mass of satellite
m_p	mass of station-keeping fuel
n	number of station-keeping corrections in a time T
$\bar{n}_1$	unit vector in an easterly direction
$\bar{n}_2$	unit vector in a northerly direction
R_o	radius of the earth
r	radial distance from the center of the earth
r_c	synchronous satellite orbital radius
r_o	synchronous satellite orbital radius for a spherical earth
$\bar{r}_1$	unit vector in the radial direction

s	operation $\frac{d}{d\tau}$
s_i	roots of the characteristic equation ($i = 1...4$)
T	operating time of the satellite
T_o	oscillation period
t	time
t_o	time of drift before correction
U	earth's gravitational potential
V_Θ	total corrective velocity required in time T
ΔV_Θ	drift velocity at time t_o
γ	satellite angular position relative to minor axis of earth's equatorial section
γ_o	initial value of γ
δ	limit of integration
ϵ_1	phase angle
Θ	spherical coordinate of satellite in the equatorial plane
Θ_E	angular position of earth's minor axis
$\dot{\Theta}_E$	earth's angular rate
τ	non-dimensional time
ϕ	spherical coordinate of satellite in the meridian plane

I. INTRODUCTION

In recent years the idea of establishing an artificial satellite in a synchronous equatorial orbit about the earth has become increasingly attractive. Since such a satellite by definition would remain above the same point on the earth's equator, it could be used as a communication relay station between any two points on the earth which are within its field of view.

If it is assumed that the mass distribution of the earth is spherically symmetric, then the resulting gravitational potential varies inversely as the radial distance from the earth's center and is independent of either latitude or longitude. Under this assumption, it is a simple matter to demonstrate that a satellite on a circular orbit at an altitude of 22,236 st mi would have an orbital period of one sidereal day. Thus, such a satellite in the earth's equatorial plane would appear to be fixed relative to the earth.

It has long been recognized that in order to take into account the actual non-spherical nature of the earth's mass distribution, it is necessary to include additional terms in its potential function in the form of higher order spherical harmonics. These additional terms depend not only on the radial distance but also on latitude and/or longitude. As a result of observations of some of the artificial satellites launched thus far,[(1)] it has been possible to refine the estimates of the magnitude of these additional spherical harmonic terms. The two additional terms considered in the above reference are (1) the zonal solid spherical harmonic corresponding to the usual oblateness of the earth such that the equatorial radius exceeds the polar radius, and (2) the sectorial solid spherical harmonic which results from the ellipticity of the earth's equatorial section. The first of these terms is latitude dependent while the second is a function of longitude as well.

The question now arises as to what effect these modifications in the earth's potential might have on the synchronous nature of a 24-hour satellite. The present paper addresses this problem by setting up the general equations of motion of a satellite under the influence of the expanded potential function. These equations are then linearized in terms of small deviations from a circular synchronous orbit. On the assumption that the orbital injection procedure is perfect and that the initial deviations from the synchronous condition are all zero, the above equations can be solved for the forced solution resulting from the added terms in the potential function which act as driving functions in the linearized equations.

On the basis of these results it is possible to determine the amount of propulsion required to maintain a synchronous satellite at a given longitude on the equator.

While the solution of the linearized equations is only valid for small variations in orbital radius and for variations in epoch angle up to 10° relative to the synchronous position, an approximate equation is also developed which allows large variations in epoch angle but is still restricted to small radial displacements. By means of this equation the behavior of the satellite can be described in the event no propulsion is provided to maintain position.

II. METHOD OF ANALYSIS

STATEMENT OF THE PROBLEM

The problem which is solved in the succeeding sections of this report is the determination of the motion of a presumably synchronous satellite relative to the earth when the effects of earth's oblateness and the ellipticity of its equatorial section are included in the gravitational potential function. For the purposes of this analysis, the gravitational attraction of the sun and moon, the sun's radiation pressure, and residual drag effects are neglected.

EQUATIONS OF MOTION

Coordinate System

The reference system adopted is shown in Fig. 1 where the XYZ system is an inertial reference with origin at the earth's center and OZ along the polar axis. In this system the position of the satellite P is specified by the spherical coordinates θ, ϕ, and r. Also, the line OA represents the instantaneous position of the minor axis of the earth's equatorial section so that the angle γ represents the difference in longitude between the satellite and this minor axis reference line. For a truly synchronous satellite γ should remain constant. In addition, three unit vectors, $\bar{r}_1$, $\bar{n}_1$ and $\bar{n}_2$ are specified at P corresponding to displacements due to small changes in r, θ, and ϕ, respectively.

Acceleration Components

The components of acceleration in the direction of the three unit vectors specified above can be expressed in terms of r, θ, and ϕ as follows

$$a_r = \ddot{r} - r\dot{\theta}^2 \cos^2\phi - r\dot{\phi}^2 \qquad (1)$$

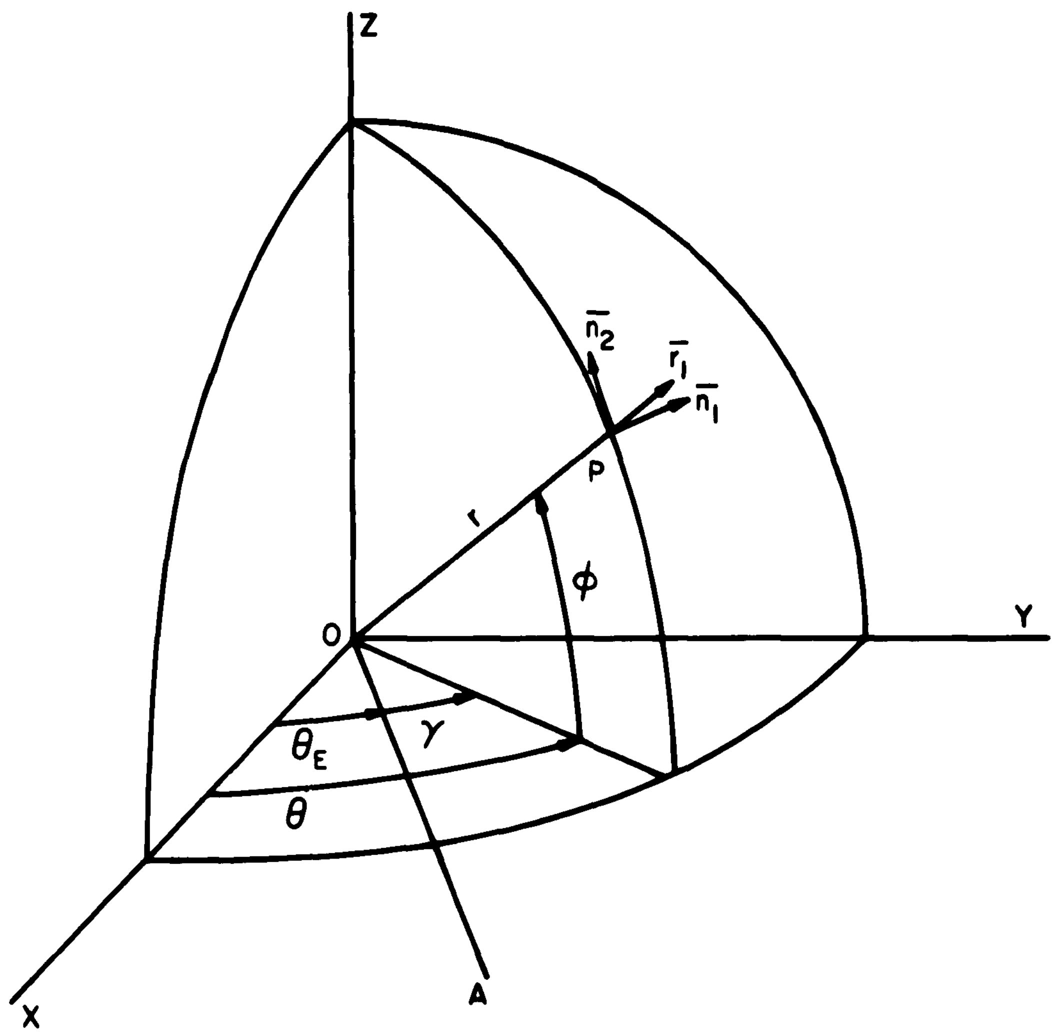

Fig. 1 — Coordinate system

$$a_{\theta} = \frac{1}{r \cos \phi} \frac{d}{dt} \left[r^2 \dot{\theta} \cos^2 \phi \right] \tag{2}$$

$$a_{\phi} = \frac{1}{r} \frac{d}{dt} \left[r^2 \dot{\phi} \right] + r\dot{\theta}^2 \cos \phi \sin \phi \tag{3}$$

Force Components

The formulation of the earth's gravitational potential presented in Reference 1 is used to determine the corresponding force components.

$$U = \frac{g_o R_o^2}{r} \left[1 - J_2 \frac{R_o^2}{r^2} \left(\frac{3 \sin^2 \phi - 1}{2} \right) + 3 J_2^{(2)} \frac{R_o^2}{r^2} \cos^2 \phi \cos 2\gamma \right] \tag{4}$$

where

R_o = earth's radius

g_o = gravitational acceleration at the earth's surface

J_2 = non-dimensional multiplier

$J_2^{(2)}$ = non-dimensional multiplier

In Eq. (4) the first term results from the spherical part of the earth's mass distribution, the second or J_2 term from the usual oblateness about the polar axis and the third or $J_2^{(2)}$ term from the ellipticity of the earth's equatorial section. It is seen that the J_2 term depends only on the latitude, ϕ, while the $J_2^{(2)}$ term depends not only on the latitude but also on the relative longitude, γ. See Fig. 2.

The numerical values of J_2 and $J_2^{(2)}$ have been determined in Ref. 1 on the basis of observations of Vanguards II and III as follows

$$J_2 = + 1.08219 \times 10^{-3}$$

$$J_2^{(2)} = - 5.35 \times 10^{-6}$$

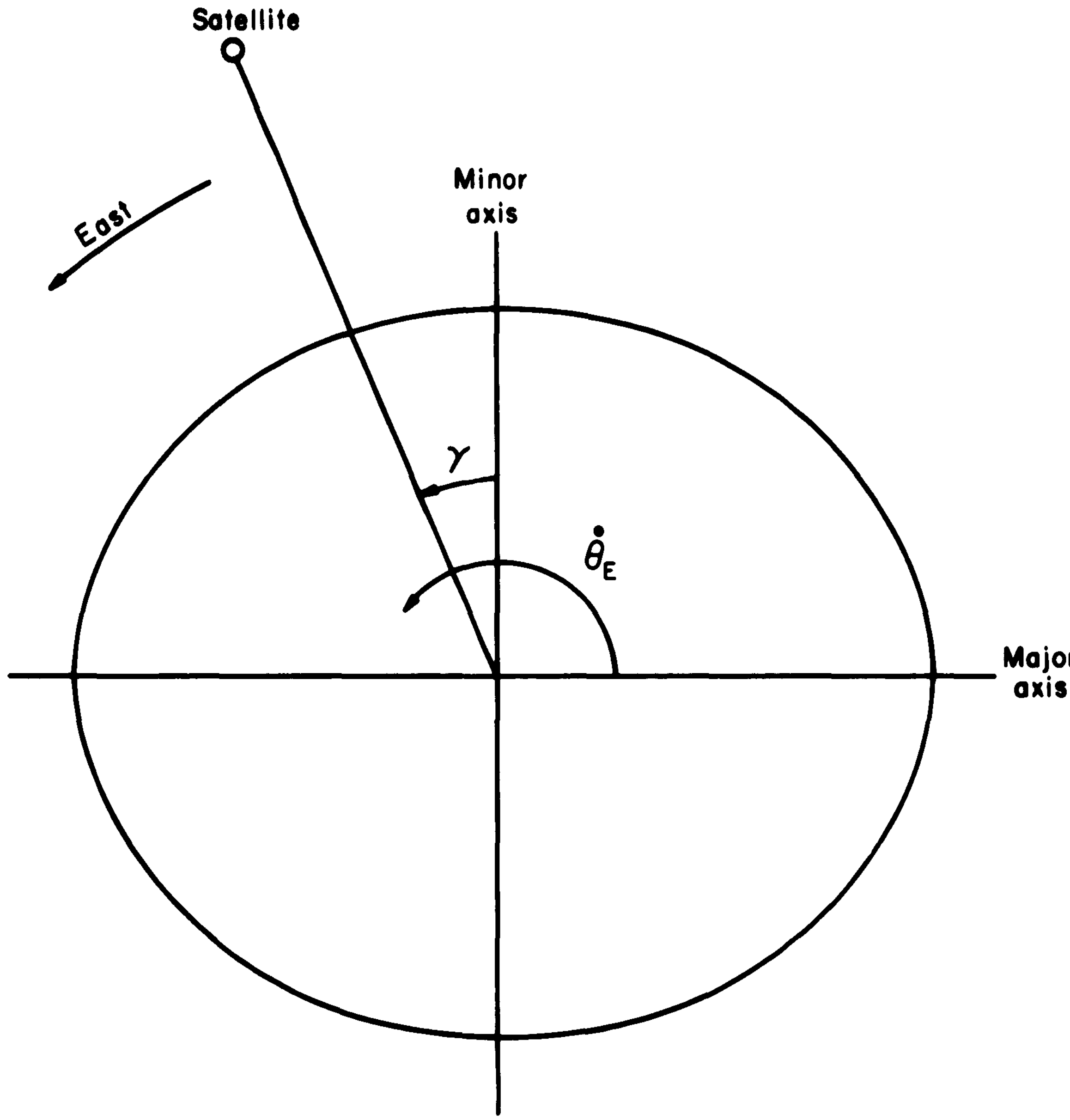

Fig. 2 — Earth's equatorial section (ellipticity exaggerated)

A value of $J_2^{(2)}$ of this magnitude is equivalent to a difference of 671 ft between the semi-major and semi-minor axes of the equatorial section.

The desired force components can be obtained from Eq. (4) as follows

$$\frac{F_r}{m} = \frac{\partial U}{\partial r} = -\frac{g_o R_o^2}{r^2} + \frac{3J_2\, g_o R_o^4}{2r^4}(3\sin^2\phi - 1)$$

$$-\frac{9J_2^{(2)}\, g_o R_o^4}{r^4}\cos^2\phi \cos 2\gamma \qquad (5)$$

$$\frac{F_\theta}{m} = \frac{1}{r}\frac{\partial U}{\partial \theta} = -\frac{6J_2^{(2)}\, g_o R_o^4}{r^4}\cos^2\phi \sin 2\gamma \qquad (6)$$

$$\frac{F_\phi}{m} = \frac{1}{r}\frac{\partial U}{\partial \phi} = -\frac{3J_2\, g_o R_o^4}{r^4}\sin\phi\cos\phi$$

$$-\frac{6J_2^{(2)}\, g_o R_o^4}{r^4}\cos\phi\sin\phi\cos 2\gamma \qquad (7)$$

where

m = mass of the satellite

$\gamma = \theta - \theta_E$

Complete Equations

The general form of the equations of motion is obtained by equating the corresponding acceleration and force expressions from Eqs. (1), (2), (3), (5), (6) and (7) to give the following

$$\ddot{r} - r\dot{\theta}^2\cos^2\phi - r\dot{\phi}^2 = -\frac{g_o R_o^2}{r^2} + \frac{3J_2\, g_o R_o^4}{2r^4}(3\sin^2\phi - 1)$$

$$-\frac{9J_2^{(2)}\, g_o R_o^{\,4}}{r^4}\cos^2\phi\,\cos 2\gamma \tag{8}$$

$$\frac{1}{r\cos\phi}\,\frac{d}{dt}\left[r^2\dot{\theta}\,\cos^2\phi\right] = -\frac{6J_2^{(2)}\, g_o R_o^{\,4}}{r^4}\cos^2\phi\,\sin 2\gamma \tag{9}$$

$$\frac{1}{r}\frac{d}{dt}\left[r^2\dot{\phi}\right] + r\dot{\theta}^2\cos\phi\,\sin\phi = -\frac{3J_2\, g_o R_o^{\,4}}{r^4}\sin\phi\,\cos\phi$$

$$-\frac{6J_2^{(2)}\, g_o R_o^{\,4}}{r^4}\sin\phi\,\cos\phi\,\cos 2\gamma \tag{10}$$

Since the present analysis is concerned with an equatorial satellite, the above equations can be simplified by assuming that ϕ is a small angle, in which case Eqs. (8), (9) and (10) reduce to

$$\ddot{r} - r\dot{\theta}^2 = -\frac{g_o R_o^{\,2}}{r^2} - \frac{3J_2\, g_o R_o^{\,4}}{2r^4} - \frac{9J_2^{(2)}\, g_o R_o^{\,4}}{r^4}\cos 2\gamma \tag{11}$$

$$\frac{1}{r}\,\frac{d}{dt}\left[r^2\dot{\theta}\right] = -\frac{6J_2^{(2)}\, g_o R_o^{\,4}}{r^4}\sin 2\gamma \tag{12}$$

$$\frac{1}{r}\,\frac{d}{dt}\left[r^2\dot{\phi}\right] + r\dot{\theta}^2\,\phi = 0 \tag{13}$$

where terms of the order of $J_2\phi$ and $J_2^{(2)}\phi$ have been neglected in Eq. (13).

Since the J_2 and $J_2^{(2)}$ terms do not have any significant effect on the ϕ equation, the rest of the analysis will be concerned only with motion in the orbital plane, as described by Eqs. (11) and (12).

LINEARIZED EQUATIONS

Since this paper is primarily concerned with deviations of the satellite from a nominally synchronous orbit, it is convenient to linearize

Eqs. (11) and (12) in the vicinity of the conditions for such an orbit. This is accomplished by defining the following set of variables

$$r = r_c + \Delta r \tag{14}$$

$$\theta = \theta_E + \gamma_o + \Delta\gamma \tag{15}$$

$$\gamma = \gamma_o + \Delta\gamma \tag{16}$$

where

r_c = orbital radius of a synchronous orbit

γ_o = longitude difference between the minor axis of earth's equatorial section and the desired synchronous position of the satellite.

Substitution of Eqs. (14), (15) and (16) into Eq. (11) gives

$$\ddot{\Delta r} - r_c\dot{\theta}_E^2 - \dot{\theta}_E^2\,\Delta r - 2r_c\,\dot{\theta}_E\,\Delta\dot{\gamma} = -\frac{g_oR_o^{\,2}}{r_c^{\,2}}\left(1 - \frac{2\Delta r}{r_c}\right)$$

$$-\frac{3J_2\,g_oR_o^{\,4}}{2\,r_c^{\,4}}\left(1 - \frac{4\Delta r}{r_c}\right)$$

$$-\frac{9J_2^{(2)}\,g_oR_o^{\,4}}{r_c^{\,4}}\left(1 - \frac{4\Delta r}{r_c}\right)\left[\cos 2\gamma_o - 2\Delta\gamma\sin 2\gamma_o\right] \tag{17}$$

By equating the steady state terms to zero as follows

$$r_c\dot{\theta}_E^2 - \frac{g_oR_o^{\,2}}{r_c^{\,2}} - \frac{3J_2\,g_oR_o^{\,4}}{2\,r_c^{\,4}} = 0 \tag{18}$$

a relationship is obtained which defines the value of the synchronous orbital radius, r_c. Thus it is seen that earth oblateness modifies the value of

r_c from that obtained for a spherical earth.

By combining Eqs. (17) and (18) and neglecting higher order terms the following equation results

$$\ddot{\Delta r} - \left[\dot{\theta}_E^2 + \frac{2\,g_o R_o^{\,2}}{r_c^{\,3}}\right] \Delta r - 2\,r_c\,\dot{\theta}_E\,\dot{\Delta\gamma}$$

$$- \frac{18 J_2^{(2)}\,g_o R_o^{\,4}\,\sin 2\gamma_o}{r_c^{\,4}}\,\Delta\gamma = - \frac{9 J_2^{(2)}\,g_o R_o^{\,4}}{r_c^{\,4}}\,\cos 2\gamma_o \qquad (19)$$

Equation (19) can be further simplified by introducing the following lumped constants

$$k_1 = J_2 \frac{R_o^{\,2}}{r_c^{\,2}} \qquad (20)$$

$$k_2 = - J_2^{(2)} \frac{R_o^{\,2}}{r_c^{\,2}} \qquad (21)$$

and by making use of the approximate form of Eq. (18) in which the J_2 term is neglected to give

$$\dot{\theta}_E^2 = \frac{g_o R_o^{\,2}}{r_c^{\,3}} \qquad (22)$$

Thus Eq. (19) becomes

$$\frac{\ddot{\Delta r}}{r_c} - 3\dot{\theta}_E^2\,\frac{\Delta r}{r_c} - 2\dot{\theta}_E\,\dot{\Delta\gamma} + 18 k_2\,\dot{\theta}_E^2\,\sin 2\gamma_o\,\Delta\gamma =$$

$$+ 9 k_2\,\dot{\theta}_E^2\,\cos 2\gamma_o \qquad (23)$$

Finally, if a non-dimensional time τ is defined by the relation

$$\tau = \dot{\theta}_E t \qquad (24)$$

Eq. (23) can be rewritten as

$$\frac{d^2}{d\tau^2}\left(\frac{\Delta r}{r_c}\right) - 3\left(\frac{\Delta r}{r_c}\right) - 2\,\frac{d\Delta\gamma}{d\tau} + (18k_2 \sin 2\gamma_o)\,\Delta\gamma = +\,9k_2 \cos 2\gamma_o \tag{25}$$

In a completely analogous manner Eq. (12) can also be linearized and put into non-dimensional form as follows

$$2\,\frac{d}{d\tau}\left(\frac{\Delta r}{r_c}\right) + (30k_2 \sin 2\gamma_o)\,\frac{\Delta r}{r_c} + \frac{d^2\Delta\gamma}{d\tau^2} - (12k_2 \cos 2\gamma_o)\,\Delta\gamma = +\,6k_2 \sin 2\gamma_o \tag{26}$$

The preceding analysis has reduced the two non-linear differential equations of motion (11) and (12) to two coupled linear differential equations (25) and (26) with constant coefficients relating the variables $\Delta r/r_c$ and $\Delta\gamma$.

By combining Eqs. (25) and (26) separate equations for $\Delta r/r_c$ and $\Delta\gamma$ are obtained in operator form as follows

$$\left[s^4 + s^2 + (24k_2 \sin 2\gamma_o)\,s + (36k_2 \cos 2\gamma_o)\right]\frac{\Delta r}{r_c} = -\,108k_2^2 \tag{27}$$

$$\left[s^4 + s^2 + (24k_2 \sin 2\gamma_o)\,s + (36k_2 \cos 2\gamma_o)\right]\Delta\gamma = -\,18k_2 \sin 2\gamma_o \tag{28}$$

where $s = \frac{d}{d\tau}$

It is interesting to note at this point that the above equations depend only upon k_2, which is related to the ellipticity of the equatorial section. The only significant effect of the other oblateness term is to modify the value of r_c through Eq. (18).

In order to solve Eqs. (27) and (28) the roots of the characteristic operator equation are obtained in the form

$$s_1 = +\,12k_2 \sin 2\gamma_o + j \tag{29}$$

$$s_2 = +\,12k_2 \sin 2\gamma_o - j \tag{30}$$

$$s_3 = -\,12k_2 \sin 2\gamma_o + 6\sqrt{-k_2 \cos 2\gamma_o} \tag{31}$$

$$s_4 = -\,12k_2 \sin 2\gamma_o - 6\sqrt{-k_2 \cos 2\gamma_o} \tag{32}$$

The form of the solution depends on the sign of $\cos 2\gamma_o$ since this determines whether s_3 and s_4 are real or conjugate complex. If $\cos 2\gamma_o < 0$ then the solution is of the form

$$\begin{aligned}\frac{\Delta r}{r_c} = {} & 3k_2\left[1 + 7\sin^2 2\gamma_o\right]^{1/2} e^{+(12k_2 \sin 2\gamma_o)\tau} \cos(\tau - \epsilon_1) \\ & - 2\tan 2\gamma_o \sqrt{-k_2 \cos 2\gamma_o}\, e^{-(12k_2 \sin 2\gamma_o)\tau} \sinh\left(6\sqrt{-k_2 \cos 2\gamma_o}\right)\tau \\ & - \frac{3k_2}{\cos 2\gamma_o}\end{aligned} \tag{33}$$

$$\begin{aligned}\Delta\gamma = {} & -\,6k_2\left[1 + 7\sin^2 2\gamma_o\right]^{1/2} e^{+(12k_2 \sin 2\gamma_o)\tau} \sin(\tau - \epsilon_1) \\ & + \frac{1}{2}\tan 2\gamma_o\, e^{-(12k_2 \sin 2\gamma_o)\tau} \cosh\left(6\sqrt{-k_2 \cos 2\gamma_o}\right)\tau \\ & - \frac{1}{2}\tan 2\gamma_o\end{aligned} \tag{34}$$

where

$$\tan \epsilon_1 = \frac{4}{3} \tan 2\gamma_o \tag{35}$$

In obtaining the above solution, it has been assumed that the initial orbital injection errors have been reduced to zero so that ($\Delta r_o = \Delta\dot{r}_o = \Delta\gamma_o = \Delta\dot{\gamma}_o = 0$) and, as a result, Eqs. (33) and (34) represent the driven solution resulting from the $J_2^{(2)}$ term in the earth's potential function.

An examination of the above solution shows that a number of simplifications can be made in view of the magnitude of the constant k_2 (of the order of 10^{-7}). The exponential terms have time constants in excess of 300 yr and can be replaced by unity. Likewise, terms involving k_2 as a multiplier can be neglected compared to those involving $k_2^{1/2}$. Thus, a simplified version of the solution can be written as

$$\frac{\Delta r}{r_c} = -2 \tan 2\gamma_0 \sqrt{-k_2 \cos 2\gamma_0}\, \sinh (\sigma \sqrt{-k_2 \cos 2\gamma_0})\tau \tag{36}$$

$$\Delta\gamma = -\frac{1}{2} \tan 2\gamma_0 \left[1 - \cosh (\sigma \sqrt{-k_2 \cos 2\gamma_0})\tau\right] \tag{37}$$

This form of the solution applies for $45^\circ < |\gamma_0| < 90^\circ$.

If the value of $\cos 2\gamma_0 > 0$, then the roots s_3 and s_4 become complex and it can be shown that the form of the solution analogous to Eqs. (36) and (37) is given by

$$\frac{\Delta r}{r_c} = 2 \tan 2\gamma_0 \sqrt{k_2 \cos 2\gamma_0}\, \sin (\sigma \sqrt{k_2 \cos 2\gamma_0})\tau \tag{38}$$

$$\Delta\gamma = -\frac{1}{2} \tan 2\gamma_0 \left[1 - \cos (\sigma \sqrt{k_2 \cos 2\gamma_0})\tau\right] \tag{39}$$

This form of the solution applies for $|\gamma_0| < 45^\circ$.

Due to the linearization procedure used in obtaining the above solutions, they are only valid as long as

$$\Delta r \ll r_c$$

$$\Delta\gamma \doteq \sin \Delta\gamma$$

An examination of Eqs. (36) and (38) indicates that the first condition will be satisfied for values of τ equivalent to a three year duration, at which time Δr would be about one per cent of r_c.

However, the buildup of $\Delta\gamma$ is more rapid and may reach a value of 10° in about two months. Thus it is seen that the behavior of $\Delta\gamma$ is the determining factor with regard to the time interval over which the solutions remain valid.

LARGE ANGLE EQUATIONS

In view of the fact that $\Delta\gamma$ exceeds the linear range in the above solution after a reasonable length of time, it was felt that a solution which was valid for large values of $\Delta\gamma$ would be desirable. This can be obtained from Eqs. (11) and (12) by substituting $\theta_E + \gamma$ for θ to give

$$\ddot{r} - r\,(\dot{\theta}_E + \dot{\gamma})^2 = -\frac{g_o R_o^{\,2}}{r^2} - \frac{3J_2\, g_o R_o^{\,4}}{2r^4} - \frac{9J_2^{(2)} g_o R_o^{\,4}}{r^4}\cos 2\gamma \tag{40}$$

$$\ddot{\gamma} + \frac{2\dot{r}}{r}\,(\dot{\theta}_E + \dot{\gamma}) = -\frac{6J_2^{(2)} g_o R_o^{\,4}}{r^5}\sin 2\gamma \tag{41}$$

By successive differentiation these equations can be combined to give a fourth order equation in γ as follows

$$\ddot{\ddot{\gamma}} + \dot{\theta}_E^2\,\ddot{\gamma} + 24k_2\,\dot{\theta}_E^2 \sin 2\gamma\,\dot{\gamma} + 18k_2\,\dot{\theta}_E^4 \sin 2\gamma = 0 \tag{42}$$

While this equation still requires that both $\dot{r}$ and $\dot{\gamma}$ remain small, it does not restrict the value of γ. Since it has already been demonstrated that the variation in γ is a low frequency solution with negligible damping, Eq. (42) can be simplified by dropping the first and fourth derivative terms. In terms of the non-dimensional time τ the equation becomes

$$\frac{d^2\gamma}{d\tau^2} + 18k_2 \sin 2\gamma = 0 \tag{43}$$

from which a first integral can be obtained in the form

$$\frac{d\gamma}{d\tau} = -6\sqrt{k_2}\,\sin\gamma_o\left[1-\frac{\sin^2\gamma}{\sin^2\gamma_o}\right]^{1/2} \tag{44}$$

Equation (44) can then be expressed in the form of an elliptic integral as follows

$$\tau = \frac{1}{6\sqrt{k_2}}\int_{\frac{\pi}{2}}^{\delta}\frac{d\psi}{\left[1-\sin^2\gamma_o\,\sin^2\psi\right]^{1/2}} \tag{45}$$

where

$$\delta = \sin^{-1}\left(\frac{\sin\gamma}{\sin\gamma_o}\right) \tag{46}$$

From Eq. (45) it is seen that the resulting angular motion of the satellite will be a large amplitude oscillation about the position of the minor axis of the earth's equatorial section with an amplitude of γ_o. The period of this oscillation is given by

$$T_o = \frac{2}{3\sqrt{k_2}\,\dot{\phi}_E}\int_{o}^{\frac{\pi}{2}}\frac{d\psi}{\left[1-\sin^2\gamma_o\,\sin^2\psi\right]^{1/2}} \tag{47}$$

where the integral is $K(\gamma_o)$, the complete elliptic integral of the first kind.

The variations in the orbital radius r during this oscillation can also be determined by substituting Eq. (44) into Eq. (41) and integrating.

In this integration r^5 on the right side of Eq. (41) is assumed to be equal to $r_c^{\,5}$. The resulting variation of r is given by

$$\frac{r}{r_c} = \left[1 - 6\sqrt{k_2}\ \sin\gamma_o \left[1 - \frac{\sin^2\gamma}{\sin^2\gamma_o}\right]^{1/2}\right]^{-2/3} \tag{48}$$

which can be simplified to

$$\frac{\Delta r}{r_c} = 4\sqrt{k_2}\ \sin\gamma_o \left[1 - \frac{\sin^2\gamma}{\sin^2\gamma_o}\right]^{1/2} \tag{49}$$

A comparison of Eqs. (44) and (49) shows that

$$\frac{\Delta r}{r_c} = -\frac{2}{3}\frac{d\gamma}{d\tau} \tag{50}$$

Thus, when γ is increasing in magnitude, Δr is negative and the orbital radius is slightly less than r_c; conversely, when γ is decreasing, Δr is positive and the orbital radius is slightly larger than r_c. The maximum excursion in Δr occurs when γ equals zero at the minor axis and is given by

$$\frac{\Delta r_{max}}{r_c} = \pm\ 4\sqrt{k_2}\ \sin\gamma_o \tag{51}$$

III. RESULTS AND DISCUSSION

SYNCHRONOUS SATELLITE

As a result of the foregoing analysis, the effects of the non-spherical mass distribution of the earth on a nominally synchronous satellite can be evaluated. The effect of the usual earth's oblateness, or J_2 term in the earth's potential, is simply to increase the radius of the synchronous orbit by a small amount, as indicated in Eq. (18). The solution to Eq. (18) is to a good approximation

$$r_c = r_o\left(1 + \frac{1}{2} J_2 \frac{R_o^{\,2}}{r_o^{\,2}}\right) \tag{52}$$

where r_o is the solution of Eq. (18) when J_2 is zero. For $\dot{\Theta}_E$ corresponding to a period of one sidereal day (86164 sec), the value of r_o is given by

$$r_o = 26194.9 \text{ st mi}$$

From Eq. (52) it is found that the J_2 term increases the orbital radius by .32 st mi to a value of

$$r_c = 26195.2 \text{ st mi}$$

This effect would not be difficult to eliminate as part of the orbital injection corrections.

On the other hand, the effect of the ellipticity of the earth's equatorial section is a more serious problem. An examination of Eqs. (37) and (39) shows that for values of $\Delta\gamma$ up to about 10° both equations can be approximated by the relation

$$\Delta\gamma = (-\ 9k_2\ \dot{\Theta}_E^{\,2} \sin 2\gamma_o)t^2 \tag{53}$$

In a similar manner Eqs. (36) and (38) can both be approximated by the relation

$$\frac{\Delta r}{r_c} = + (12k_2 \dot{\theta}_E \sin 2\gamma_o)t \tag{54}$$

From Eq. (53) it is possible to compute the time required for a satellite to drift from an initial epoch angle γ_o through an angle $\Delta\gamma$. Figure 3 is a plot of this time interval for $\Delta\gamma$ equal to 10° as a function of the initial epoch angle γ_o. It is seen that the drift time has a minimum of .1738 yr or about 2 months for γ_o equal to $\pm$ 45° with the drift being toward the minor axis ($\gamma = 0^\circ$). Also, the drift time becomes infinite for γ_o equal to 0°, $\pm$ 90°, 180°. However, $\pm$ 90° represent positions of unstable equilibrium as demonstrated by the large amplitude solution, while 0° and 180° are positions of stable equilibrium. <u>Thus, unless station-keeping propulsion is incorporated into the satellite design these two stable positions are the only locations at which a truly synchronous satellite can exist.</u>

It is also possible to determine by means of Eq. (54) the change in Δr which occurs during the 10° drift which was assumed in Fig. 3. In Fig. 4, Δr is plotted as a function of γ_o and it is seen that the maximum value is about 15 miles. Thus the major effect of the $J_2^{(2)}$ term is to produce a change in the relative angular position, γ, of the satellite as seen from the earth.

STATION-KEEPING PROPULSION

If it is desirable to maintain a satellite in a synchronous position other than the two stable positions described above, the amount of propulsion required for station-keeping becomes an important consideration. To determine this, Eqs. (53) and (54) are differentiated to give

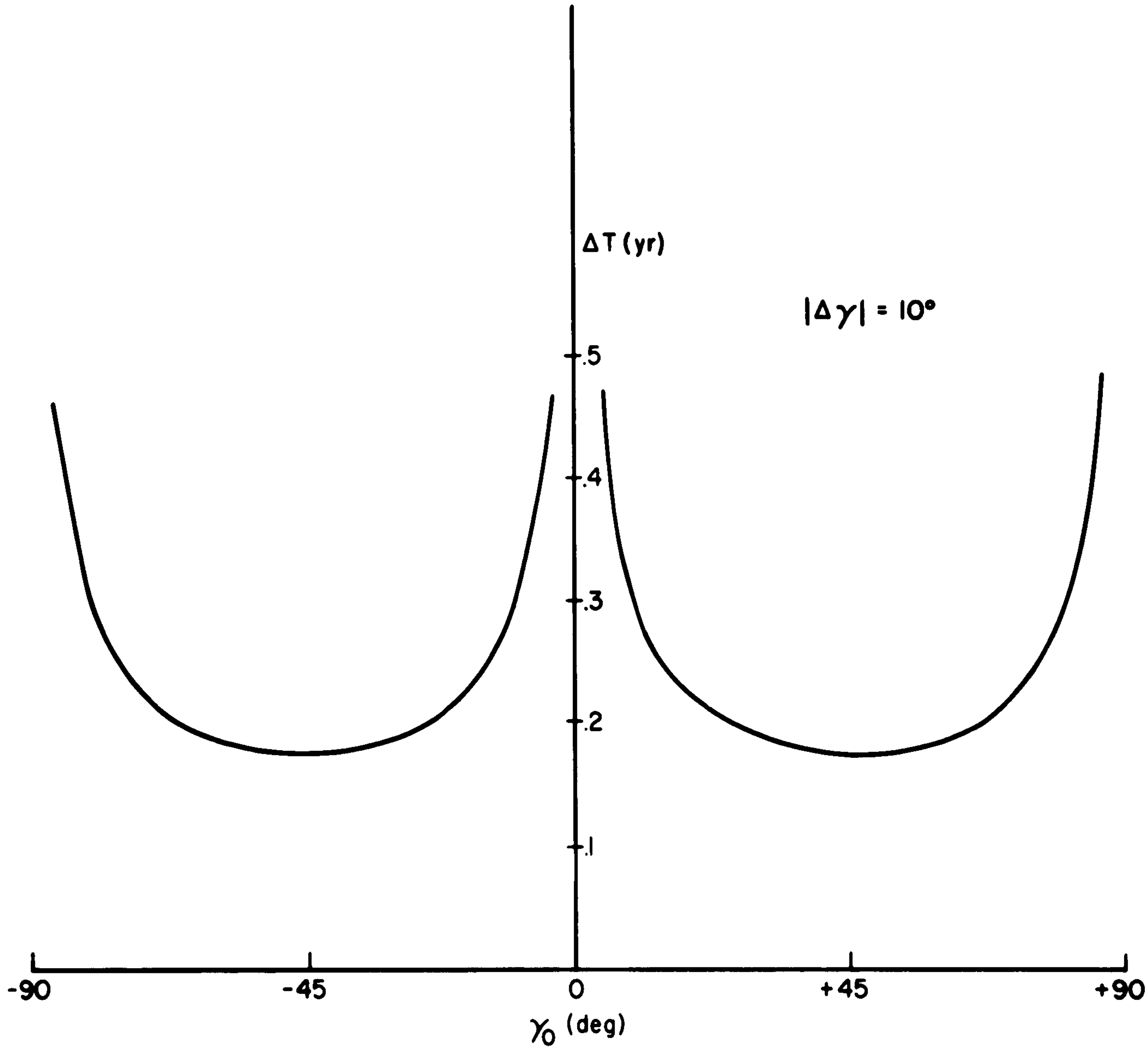

Fig. 3 — Drift time as a function of γ_0

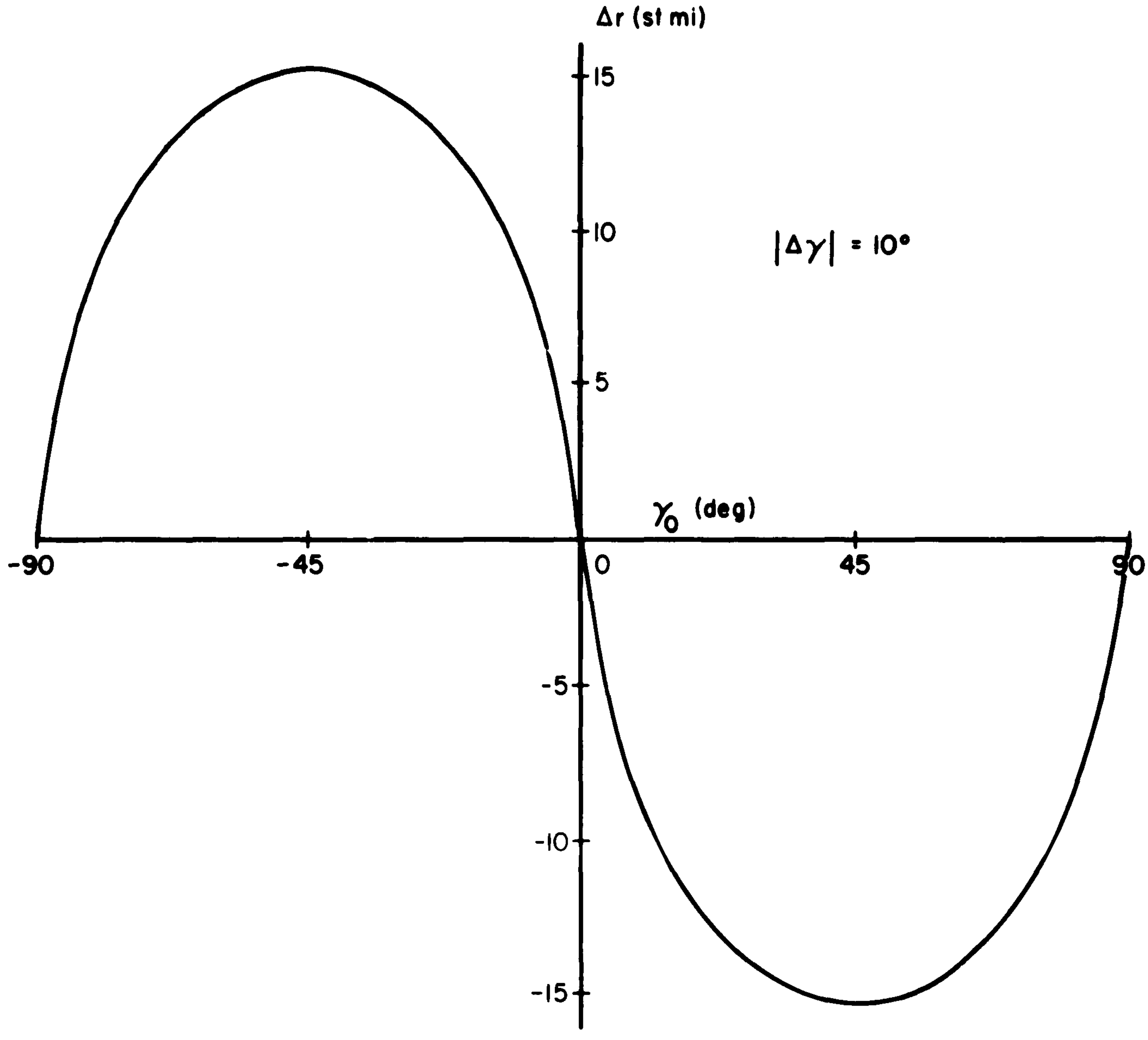

Fig. 4 — Radial error as a function of γ_0

$$\Delta\dot{\gamma} = -(18k_2\,\dot{\theta}_E^2 \sin 2\gamma_o)t \qquad (55)$$

and

$$\Delta\dot{r} = -12k_2\,\dot{\theta}_E\, r_c \sin 2\gamma_o \qquad (56)$$

Equation (56) gives the radial velocity resulting from the perturbation of the orbit, while the tangential velocity component is obtained by multiplying Eq. (55) by r_c to give

$$\Delta V_\theta = -18k_2\, r_c\, \dot{\theta}_E^2\, t \sin 2\gamma_o \qquad (57)$$

From Eq. (56) it can be shown that the constant radial component of velocity has a maximum value of .0148 ft/sec for γ_o equal to 45°. On the other hand, from Eq. (57) it can be shown that ΔV_θ builds up linearly at the rate of .1395 ft/sec/day. Thus after one day, the tangential component of velocity ΔV_θ is the dominant perturbation in the satellite motion. To restore the satellite to its original position at an angle γ_o after it has drifted toward the minor axis for time t_o it is necessary to apply sufficient propulsion to change its velocity by an amount $2\Delta V_\theta$ in a direction opposite to the drift. Since the system is conservative, this reversal of the drift velocity will cause the satellite to move back to its original value of γ_o in an additional time interval, t_o. The satellite then reverses direction and begins to drift again toward the minor axis and the above procedure is repeated. Thus, at intervals of $2t_o$, it is necessary to make a velocity change of $-2\Delta V_\theta$. In a total time of T equal to $2nt_o$ the total velocity change V_θ is given by

$$V_\theta = -2\sum \Delta V_\theta$$

$$= + 36k_2 \, r_c \, \dot{\theta}_E^2 \, nt_o \sin 2\gamma_o$$

$$= + 18k_2 \, r_c \, \dot{\theta}_E^2 \, T \sin 2\gamma_o \qquad (58)$$

where n is the number of corrective impulses in time T. Thus the velocity change per unit time is given by

$$\frac{V_\theta}{T} = + 18k_2 \, r_c \, \dot{\theta}_E^2 \sin 2\gamma_o$$

$$= 50.9 \sin 2\gamma_o \left(\frac{ft/sec}{yr}\right) \qquad (59)$$

It is seen that the propulsion requirement might be as much as 50.9 ft/sec for each year and this figure is independent of the time interval between corrections. However, the time interval would be determined by assigning a permissible value of the drift angle $\Delta\gamma$ and using Eq. (53) to determine the corresponding value of t_o.

It should be emphasized that the above propulsion requirement is only for station-keeping and does not include the propulsion required for the elimination of initial condition errors during orbital injection.

A plot of the yearly velocity requirement superposed on a Mercators Projection of the earth is shown in Fig. 5. Thus, for a given position of the satellite on the equator, the yearly velocity requirement can be read directly. Based on Ref. 1 this curve has been plotted with the minor axis of the earth's equatorial section located at 123° 9' West Longitude, and 56° 51' East Longitude.

The mass of fuel for the above velocity requirement can be determined from the relation

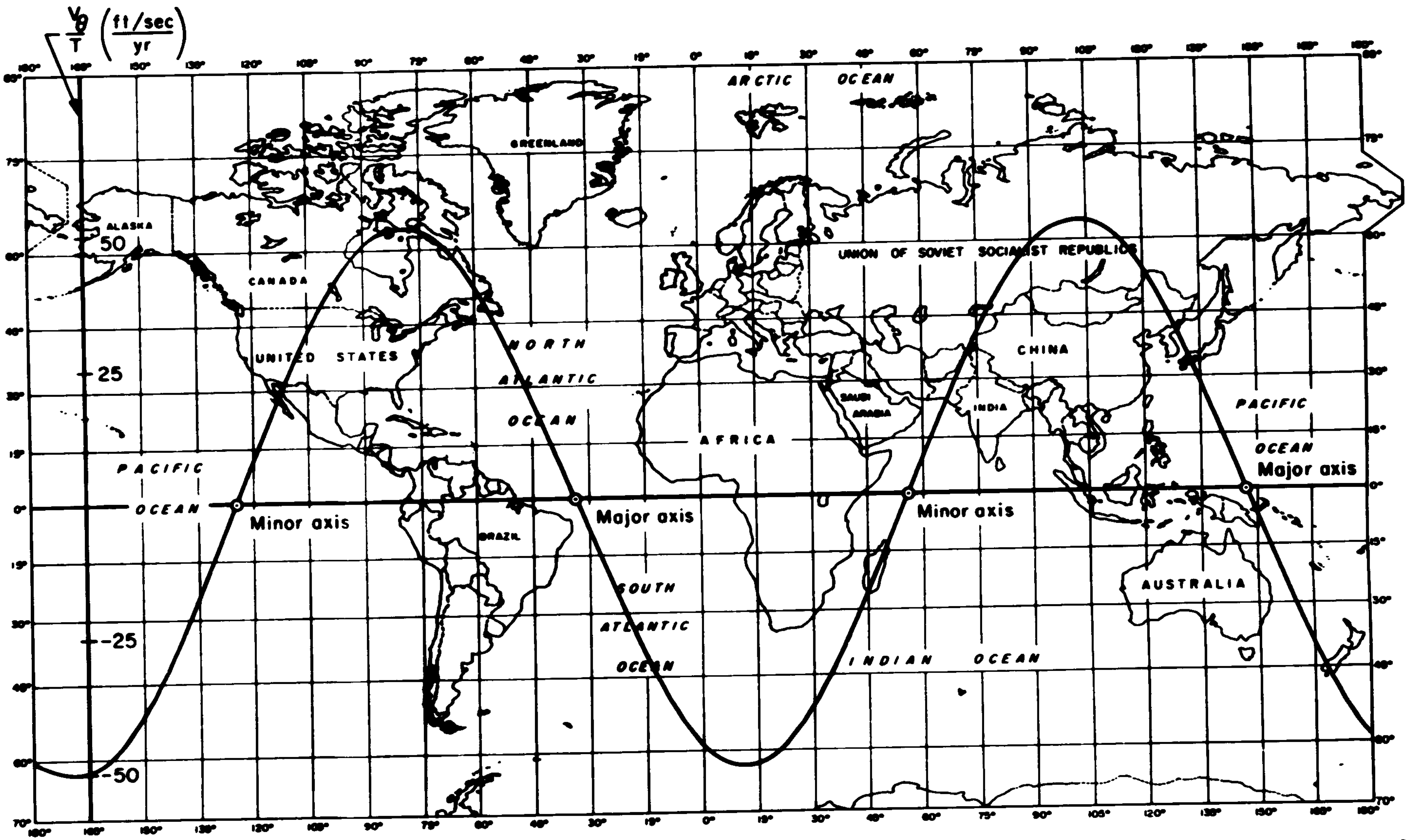

Fig. 5 — Velocity requirement per year as a function of synchronous position

$$\frac{m_p}{m} \doteq \frac{V_\theta}{g_o I} \tag{60}$$

where

m_p = mass of fuel per year

m = total mass on orbit

I = specific impulse of the fuel

Thus for a cold gas system ($I \doteq 75$ sec) as much as 2.1 per cent of the mass on orbit would be required per year for station-keeping propulsion.

It should be noted that in addition to the fuel weight, an attitude sensing and control system would be required for this type of operation, resulting in an additional weight requirement.

LARGE ANGLE OSCILLATIONS

In the absence of any station-keeping propulsion, the satellite will execute large angle oscillations in the equatorial plane about the position of the minor axis ($\gamma = 0^\circ$ or 180°). The period of these oscillations is given by Eq. (47) and has been plotted as a function of the amplitude, γ_o, in Fig. 6.

During these oscillations the values of γ and Δr as a function of time can be determined from Eqs. (45) and (49). Figure 7 is a plot of Δr as a function of γ for an oscillation with an initial amplitude of 45°, with time points at intervals of one tenth of the total period (T_o = 1.541 yr). It is seen that starting at γ_o equal to + 45° the satellite initially moves toward smaller values of γ and toward positive increments in the orbital

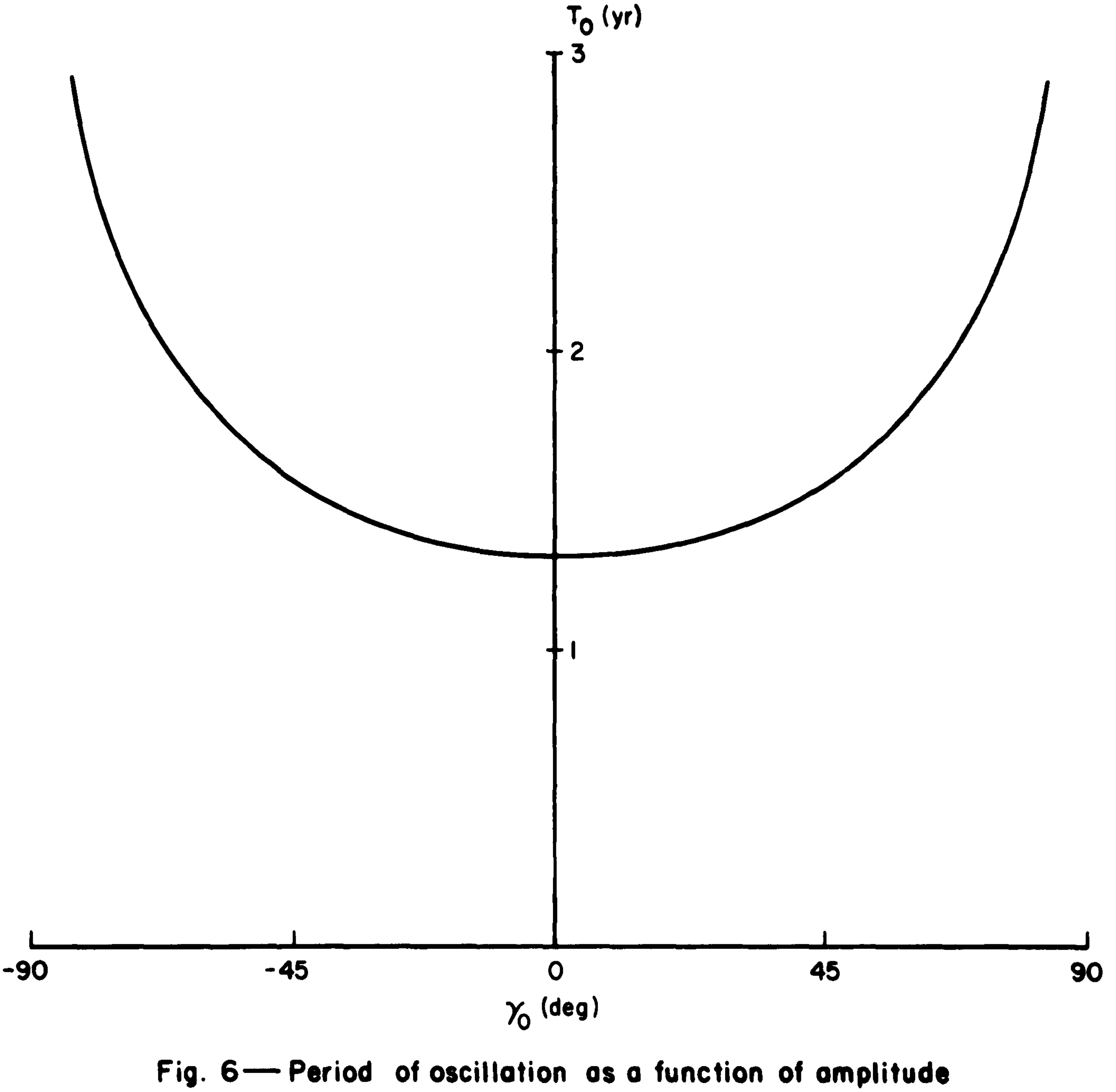

Fig. 6— Period of oscillation as a function of amplitude

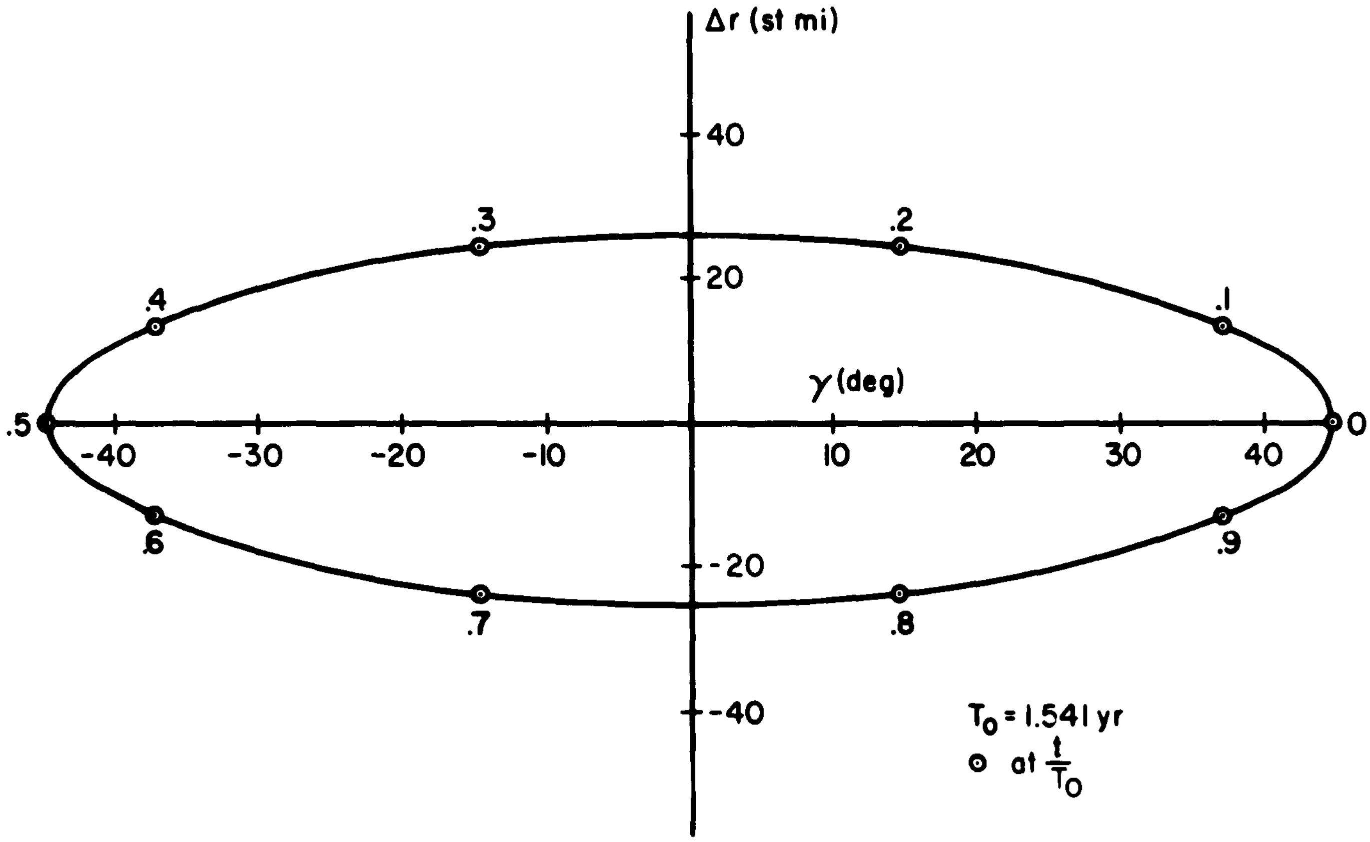

Fig. 7 — Variation in radius during oscillation

radius. The increment Δr reaches a maximum of + 25.9 mi at γ equal to zero, after which it decreases to zero at - 45°. As γ begins to increase, Δr becomes negative, reaching a minimum of - 25.9 miles at γ equal to 0 and then increasing to zero as the oscillation completes its cycle at + 45°. Figure 8 shows the magnitude of the maximum excursion of Δr as a function of the amplitude γ_o, as determined by Eq. (51).

From the above discussion it is seen that a satellite in the immediate vicinity of the minor axis ($\gamma = 0^\circ$ or 180°) will execute small angular oscillations about either of these two stable positions. On the other hand, a satellite in the vicinity of the major axis ($\gamma = \pm 90^\circ$) is in unstable equilibrium and the slightest disturbance will start a long period oscillation with a double amplitude of 180° about the position of the minor axis.

Reference 2 indicates that a synchronous satellite would be a very effective tool for the determination of the ellipticity of the earth's equator al plane. It is evident from the above analysis and discussion that if a satellite of this type were established in orbit and allowed to execute large angle oscillations, the midpoint of oscillation would be at the position of the minor axis of the equatorial section. A measurement of the oscillatory period and the amplitude, γ_o, could be used in Eq. (47) to determine the value of k_2 and thereby the parameter $J_2^{(2)}$ in the potential function of the earth.

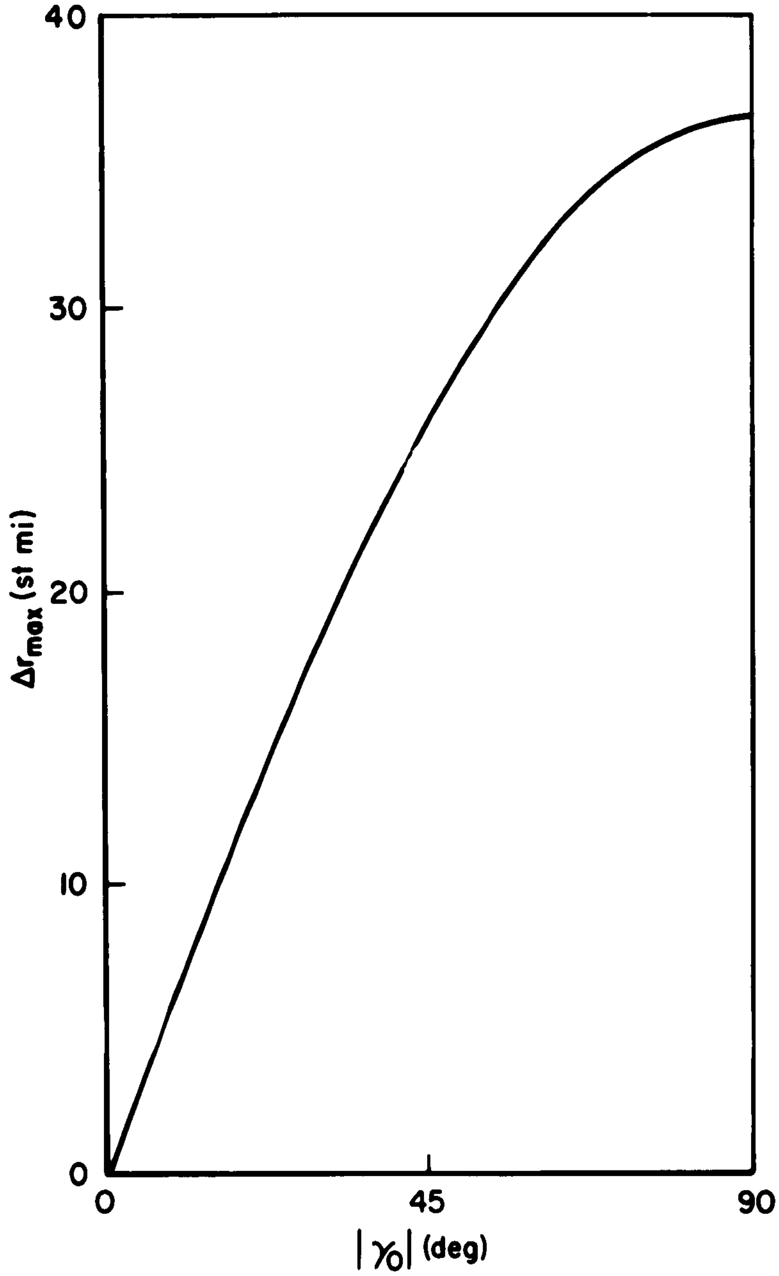

Fig. 8 — Maximum radial variation as a function of amplitude of oscillation

IV. CONCLUSIONS

As a result of the analysis presented in this paper, the following conclusions can be stated.

- The effect of the earth's equatorial bulge (J_2 term in the potential) is to increase the radius of a synchronous satellite orbit by .32 mile.

- The effect of the ellipticity of the earth's equatorial section ($J_2^{(2)}$ term in the potential) is to produce large angle oscillations of the satellite about the position of the minor axis of the equatorial section with periods in excess of 1.3 years.

- Associated with these large angle oscillations are variations of the orbital radius of the same period but with amplitudes less than 37 miles.

- Two stable positions exist for a synchronous satellite. These are at the longitudes of the extremities of the minor axis of the equatorial section, namely, 123° 9' W and 56° 51' E.

- The longitudes of the two extremities of the major axis of the equatorial section are positions of unstable equilibrium at which any small disturbance will set up an angular oscillation with a double amplitude of 180° about the position of the minor axis.

- To establish a synchronous satellite at any longitude other than the two stable positions, it is necessary to provide station-keeping propulsion which may amount to as much as 51 ft/sec/yr. This requirement is in addition to that necessary for the initial orbital injection corrections.

- A synchronous satellite would be a useful tool for determining a more exact value of the ellipticity of the earth's equatorial section both in magnitude and orientation of the axes.

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2. O'Keefe, J. A., A. Eckels and R. K. Squires, "Vanguard Measurements Give Pear-Shaped Component of Earth's Figure," Science, Vol. 129, 1959, pp. 565-566; J. Geophys. Res., Vol. 64, 1959, p. 2389.