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MARCH 1962

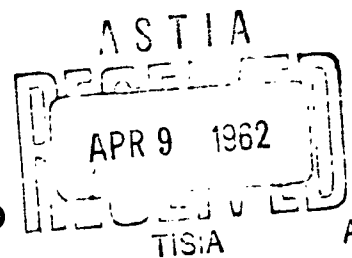
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THE ANALYTIC SIGNAL REPRESENTATION OF MODULATED WAVEFORMS

Edward Bedrosian

PREPARED FOR:

UNITED STATES AIR FORCE PROJECT RAND



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The **RAND** Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

This report is a result of RAND's continuing study in communications and propagation. It is prompted by the need to develop new and efficient modulation techniques which will make better use of existing equipment and available bandwidth. This report reveals a new type of frequency modulation (FM) that retains many of the advantages of existing FM while reducing the bandwidth needed. A patent disclosure is contemplated. The report should be of interest to agencies and contractors concerned with the design and application of FM in communications systems.

SUMMARY

This report offers a rather general and mathematically convenient formulation of analog-modulated signals which makes use of the analytic signal concept. Known types of modulation are readily identified as special cases. As a result of examining the various cases which the model embraces, a new type of modulation has been discovered--single sideband frequency modulation (SSB FM)--which can be derived from a conventional phase-modulated signal by an additional amplitude modulation, using the exponential function of the modulating signal's Hilbert transform. The resulting modulated signal will have a one-sided spectrum about the carrier frequency, will be compatible with existing FM receivers, and will cause a decrease in signal bandwidth.

ACKNOWLEDGMENT

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I. INTRODUCTION

Recent studies of compatible single sideband (SSB) modulation systems have stimulated interest in the theoretical aspects of simultaneous amplitude and phase modulation.^{(1-4)*} These studies are notable for at least two reasons which are not connected with the specific application. First, they underscore the fact that although amplitude and phase modulation are distinct techniques, they have an intimate connection. Second, theoretical studies invariably introduce the Hilbert transform^(5,6) sooner or later and become, in fact, applications of the "analytic signal."^(7,8)

Other studies dealing with various types of signal and noise waveforms⁽⁹⁻¹²⁾ further attest to the growing utility of the analytic signal representation as a theoretical device. This paper considers the application of the analytic signal to Baghdady's representation of analog modulation⁽¹³⁾ in order to obtain a more general formulation which embraces known types of analog modulation and in which signals having one-sided spectra are implicit.

*This list of references is by no means complete. The first three are merely the most recent which have come to the author's attention. Lyannoy (Ref. 4) reviewed the previous literature giving numerous references and pointed out that the notion of simultaneous amplitude and phase modulation originated with Tetel'baum in 1937. Additional references are given by Kahn (Ref. 3).

II. THE ANALYTIC SIGNAL

Basically, the analytic signal is a complex function of a real variable whose real and imaginary parts are a Hilbert pair. In practice, the actual waveform under consideration is identified with either the real or the imaginary part of the analytic signal; the analytic signal is then simply substituted for the actual signal for the purpose of analysis.

Frequently the result is a more convenient and compact notation which can be used in convolution integrals or in conjunction with transfer functions in the same fashion as the real signal and which, in addition, has several interesting and useful properties:

- 1) The Fourier transform of the analytic signal vanishes for negative frequencies (also, the real and imaginary parts have identical power spectra).
- 2) The magnitude of the analytic signal is the envelope of the actual waveform. Though defined purely mathematically, the envelope has physical significance for narrow-band signals.
- 3) The phase of the analytic signal is equal to the phase of the actual waveform.

It is clear from the foregoing that the analytic signal is simply a formalized version of the "rotating vector" or "phasor" that is frequently used in circuit analysis and studies of communication systems.* The theoretical justification of this representation is gratifying because it brings to light some of the otherwise unsuspected properties listed above as well as eliminating the need for its intuitive introduction. Details of these properties are given in Refs. 5, 7-12, 15, and 16.

*The date at which the rotating vector was introduced is difficult to determine precisely. Many of the modern concepts associated with analysis of FM and SSB systems are given by Wheeler (Ref. 14).

III. THE MODULATED SIGNAL

While the actual processes employed in generating an analog-modulated signal may differ considerably for various types of modulation, the resulting signal appears representable as the product of two time functions, viz.,

$$s(t) = c(t) m(f(t)) \quad (1)$$

where c is a function representing the carrier and m is the modulation functional representing an operation on the modulating signal f . The form of Eq. (1) has the advantage of mathematically disassociating the operation of providing a carrier from that of characterizing the actual modulation technique.

THE CARRIER FUNCTION

The purpose of the carrier function in Eq. (1) is to transfer the intelligence spectrum to a frequency region which is more suitable for propagation. The effect on the resulting spectrum is seen from the Fourier transform of the product⁽¹⁷⁾ in Eq. (1),

$$S(\omega) = \int_{-\infty}^{\infty} C(\mu) M(\omega - \mu) d\mu \quad (2)$$

If the carrier is a narrow-band waveform as might be employed for noise modulation, then the convolution expressed in Eq. (2) results in a spreading of the signal band as well as its translation to the vicinity of the carrier frequency. If the carrier is a pure sinusoid, then a simple frequency translation is effected and Eq. (2) becomes

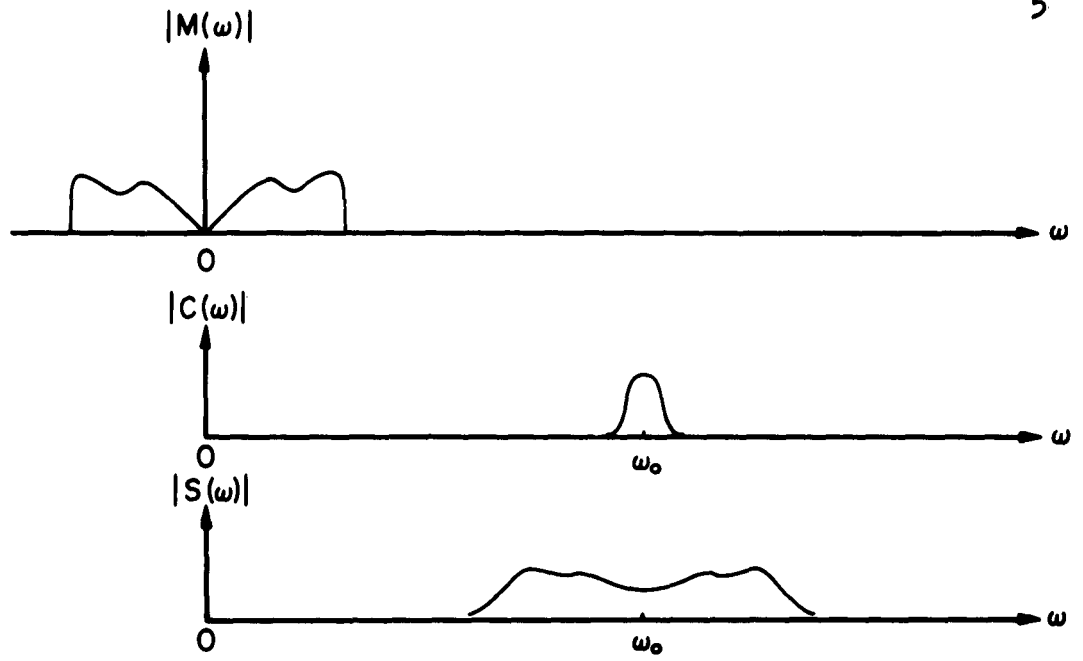
$$S(\omega) = \frac{1}{2} [M(\omega - \omega_0) + M(\omega + \omega_0)] \quad (3)$$

If the carrier is written as an analytic signal whose spectrum, therefore, exists only for positive frequencies, then the frequency transfer is only toward the positive frequencies as illustrated in Figs. 1a and 1b for $\omega_0 > 0$. Whether or not $s(t)$ is in turn analytic depends on the spectral extent of both $c(t)$ and $m(t)$, the test being whether or not $S(\omega)$ vanishes for ω negative.

THE MODULATION FUNCTION

Usable modulation functions must be limited, of course, to those which yield modulated signals amenable to subsequent demodulation. The techniques currently available are: coherent (amplitude) detection in which a replica of the carrier function is used to render the amplitude of the modulation function, phase or frequency detection in which the oscillatory character of a narrow-band modulated signal permits an approximate measurement of its phase or frequency, and envelope detection in which a suitable rectifying and filtering network can yield an approximation to the envelope of a narrow-band modulated signal. Thus, the modulation function must cause the modulating signal to appear (though not necessarily linearly) in either the amplitude, phase/frequency, or envelope of the modulated signal. The linear and exponential functionals are unique in their ability to produce just these effects and are, therefore, the only ones considered here.

As might be suspected, the linear functional corresponds to amplitude modulation. The exponential functional, however, corresponds not only to



a) General (band limited) analytic carrier

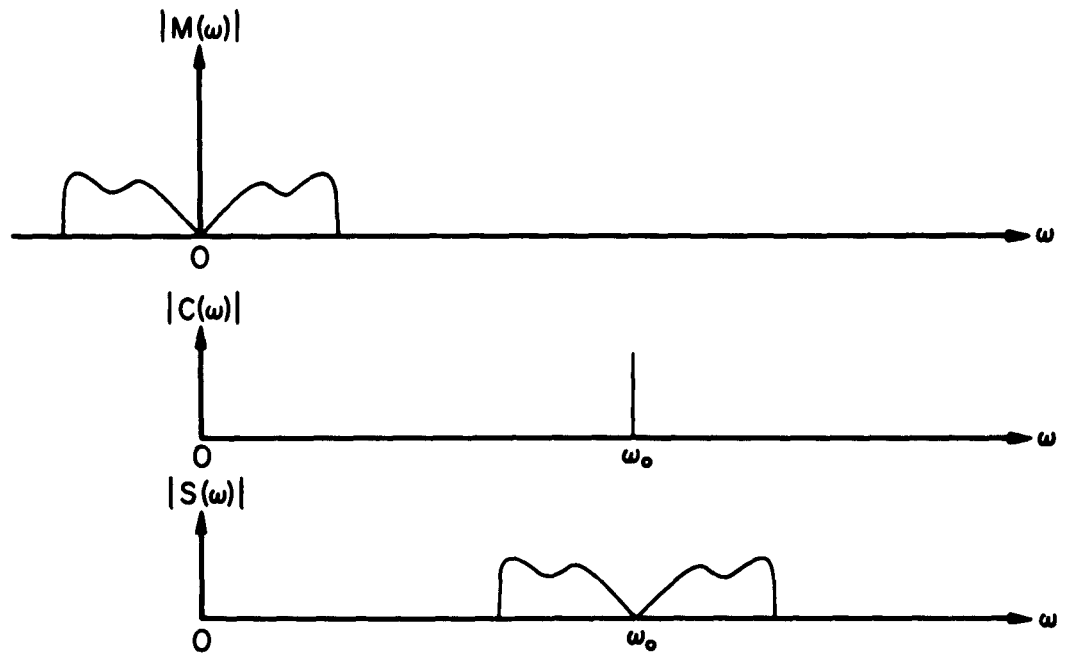
b) Harmonic analytic carrier, $e^{i\omega_0 t}$

Fig. 1 — Spectral translations due to introduction of analytic carrier

the familiar phase modulation^{*} but also to the emergent technique of simultaneous amplitude and phase modulation which may be better characterized as envelope modulation. The latter will be discussed in greater detail in a subsequent section.

If the modulation function is written for the analytic as well as the purely real modulating signals, then both the SSB and double sideband (DSB) forms of modulated signal appear. The SSB form results, of course, from the fact that an analytic function m of an analytic function f is itself analytic and therefore has a spectrum which vanishes for negative frequencies. Frequency translation by the carrier function then results in a modulated signal which has only an upper sideband. A one-sided spectrum having only a lower sideband can be generated by using the complex conjugate of the analytic function f , since its spectrum vanishes for positive frequencies.

^{*} No generic differentiation need be made between phase and frequency modulation since they differ only by a linear operation (differentiation or integration) on the modulating signal. Frequency modulation can be obtained by phase modulating with the integral of the modulating signal and conversely using differentiation.

IV. HARMONIC-CARRIER SYSTEMS

The modulated signals resulting from the foregoing modulation functions are best illustrated by using the harmonic analytic carrier $e^{i\omega_0 t}$ in Eq. (1). For each modulation function $m(t)$ a complex modulated signal $s(t)$ results. When analytic, both its real and imaginary parts are valid representations of the actual modulated signal, and its magnitude gives the envelope. The modulating signal is denoted by $f(t)$ and its Hilbert transform by $\hat{f}(t)$.

LINEAR (AMPLITUDE) MODULATION

Double Sideband

$$m(t) = f(t)$$

$$s(t) = f(t) \cos \omega_0 t + i f(t) \sin \omega_0 t \quad (4)$$

$$|s(t)| = f(t), \quad \omega_0 \geq \omega_{\max}$$

The representation of a conventional amplitude-modulated signal* is immediately apparent in Eq. (4). It is analytic for $\omega_0 > \omega_{\max}$, where $\omega_{\max}$ is the highest angular frequency component in $f(t)$, as noted both by Oswald⁽⁹⁾ and by Wozencraft.⁽¹⁶⁾ Coherent detection can recover $f(t)$ exactly for all ω_0 . Theoretically, the signal must be analytic to permit envelope detection; practically, it must also be narrow-band.

*The carrier is suppressed in this representation, but the distinction is trivial since it is easily inserted by adding a constant to $m(t)$ in Eqs. (4) and (5).

Single Sideband

$$\begin{aligned}
m(t) &= f(t) + i \hat{f}(t) \\
s(t) &= \left[f(t) \cos \omega_0 t - \hat{f}(t) \sin \omega_0 t \right] \\
&\quad + i \left[f(t) \sin \omega_0 t + \hat{f}(t) \cos \omega_0 t \right] \\
|s(t)| &= \left[f^2(t) + \hat{f}^2(t) \right]^{1/2}, \quad \omega_0 > 0
\end{aligned} \tag{5}$$

That Eq. (5) represents a SSB amplitude-modulated signal is easily shown by noting that $\cos at$ and $\sin at$ are a Hilbert pair, so the Fourier series representations of $f(t)$ and $\hat{f}(t)$ are

$$f(t) = \sum_{n=0}^{\infty} c_n \cos(\omega_n t + \varphi_n); \quad \hat{f}(t) = \sum_{n=0}^{\infty} c_n \sin(\omega_n t + \varphi_n)$$

Thus, Eq. (5) can be written

$$s(t) = \sum_{n=0}^{\infty} c_n \cos[(\omega_n + \omega_0) t + \varphi_n] + i \sum_{n=0}^{\infty} c_n \sin[(\omega_n + \omega_0) t + \varphi_n]$$

where the simple translation of all frequency components by an amount ω_0 is apparent. The representation is analytic for all values of $\omega_0 > 0$ and only coherent detection can recover $f(t)$ exactly.

EXPONENTIAL (PHASE) MODULATION

Double Sideband

$$\begin{aligned}
 m(t) &= e^{i f(t)} \\
 s(t) &= \cos [\omega_0 t + f(t)] + i \sin [\omega_0 t + f(t)] \\
 |s(t)| &\approx 1, \quad \omega_0 \gg \Omega_{\max}
 \end{aligned} \tag{6}$$

The conventional representation of phase modulation is apparent in Eq. (6); it is designated as DSB to emphasize that the spectrum is two-sided about ω_0 . Even if the modulating signal $f(t)$ is band-limited, the exponential operation used in generating $m(t)$ assures that it will have a spectrum of infinite extent in both positive and negative directions. Shifting the center frequency to ω_0 cannot, therefore, produce a spectrum containing only positive frequencies so, as noted also by Dugundji,⁽¹¹⁾ the representation is not analytic. In a practical sense, of course, it is known that the spectrum falls off rapidly beyond some frequency. Thus, the representation can be taken approximately to be analytic if the center frequency is high compared with the highest significant frequency component $\Omega_{\max}$ in $m(t)$.

Single Sideband

$$\begin{aligned}
 m(t) &= e^{i[f(t) + i \hat{f}(t)]} \\
 s(t) &= e^{-\hat{f}(t)} \cos [\omega_0 t + f(t)] + i e^{-\hat{f}(t)} \sin [\omega_0 t + f(t)] \\
 |s(t)| &= e^{-\hat{f}(t)}, \quad \omega_0 > 0
 \end{aligned} \tag{7}$$

The modulated signal given by Eq. (7) appears to be novel. Since the modulation function is now analytic, it contains no negative frequencies (though the spectrum still extends infinitely in the positive direction). Thus, $s(t)$ is analytic for all values of $\omega_0 > 0$.

By analogy with Eq. (5), the modulated signal given by Eq. (7) represents a SSB version of phase modulation since it contains no frequency components below ω_0 . The implication is that, though the instantaneous frequency

$$\omega_{\text{inst}} = \frac{d\phi}{dt} = \omega_0 + \dot{f}(t)$$

has the same excursions for both Eqs. (6) and (7), the multiplicative factor $e^{-\hat{f}(t)}$ causes the lower sideband to disappear in Eq. (7). To demonstrate that this is indeed the case and, also, to gain insight as to the effect on the upper sideband, consider frequency modulation by the single sinusoid

$$\dot{f}(t) = \Omega \cos \omega t$$

where Ω is the peak frequency deviation. The phase function and its Hilbert transform become

$$f(t) = \frac{\Omega}{\omega} \sin \omega t, \quad \hat{f}(t) = -\frac{\Omega}{\omega} \cos \omega t$$

where Ω/ω is the deviation ratio or modulation index.

The modulation function in Eq. (7) becomes

$$m(t) = e^{i[f(t) + i \hat{f}(t)]} = \exp\left[\frac{\Omega}{\omega} e^{i\omega t}\right] = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\Omega}{\omega}\right)^k e^{ik\omega t}$$

yielding the analytic modulated signal

$$m(t) e^{i\omega_0 t} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\Omega}{\omega}\right)^k e^{i(\omega_0 + k\omega)t}$$

The actual modulated signal may be taken as the real part or

$$e^{-\hat{f}(t)} \cos[\omega_0 t + f(t)] = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\Omega}{\omega}\right)^k \cos(\omega_0 + k\omega)t \quad (8)$$

from which the one-sided nature of the spectrum is clear.* The mean-square value of the amplitude factor is $I_0(\frac{2\Omega}{\omega})$ so the ratio of peak-to-average power over the modulation cycle becomes

$$e^{\frac{2\Omega}{\omega}} / I_0(\frac{2\Omega}{\omega})$$

A conventional frequency-modulated (FM) signal using the same modulating signal has the familiar expansion

$$\cos[\omega_0 t + f(t)] = \sum_{k=-\infty}^{\infty} J_k\left(\frac{\Omega}{\omega}\right) \cos(\omega_0 + k\omega)t \quad (9)$$

* It would be difficult to find a more striking example of the mathematical simplicity afforded by the use of the analytic signal than that demonstrated by the development leading to the Fourier series expansion given by Eq. (8). The conventional approach is to multiply the

Fourier series expansion $e^{-\hat{f}(t)} = \sum_{n=-\infty}^{\infty} I_n\left(\frac{\Omega}{\omega}\right) \cos n\omega t$ by that for

$\cos[\omega_0 t + f(t)]$ given by Eq. (9) and then reduce the double summation; needless to say, the procedure is quite tedious.

The line spectra of Eqs. (8) and (9) are plotted in Fig. 2 for several deviation ratios. The component magnitudes have been adjusted so that the spectra represent signals of equal average power. Generally speaking, the cancellation of the lower sideband is accompanied by an extension of the upper sideband. Nevertheless, the "width" of the one-sided spectra appear reduced by roughly one-third.

The one-sided signal can be generated from a conventional phase-modulated signal by forming the Hilbert transform (by means, say, of a $\pi/2$ phase shifter) and amplitude modulating with the exponential; limiting in the receiver will then restore the lower sideband and permit the use of a conventional discriminator. However, the new threshold behavior is not clear.

EXPONENTIAL (ENVELOPE) MODULATION

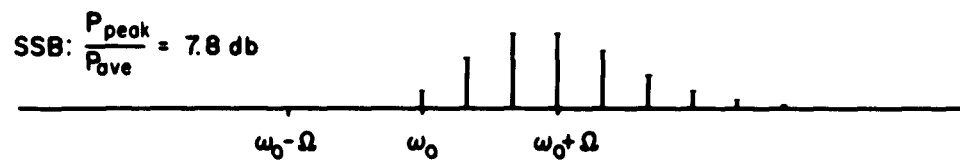
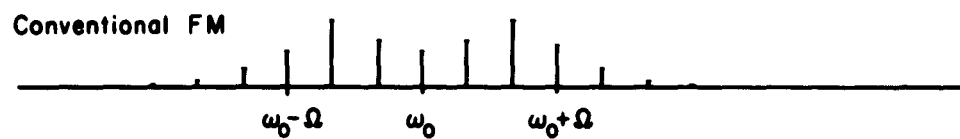
Double Sideband

$$\begin{aligned}
 m(t) &= e^{g(t)}, \quad g(t) = \alpha \log f(t), \quad f(t) > 0 \\
 s(t) &= f^\alpha(t) \cos \omega_0 t + i f^\alpha(t) \sin \omega_0 t \\
 |s(t)| &\approx f^\alpha(t), \quad \omega_0 \gg \omega_{\max}
 \end{aligned} \tag{10}$$

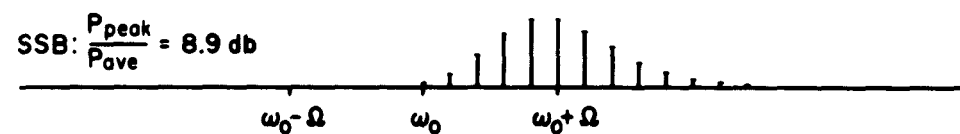
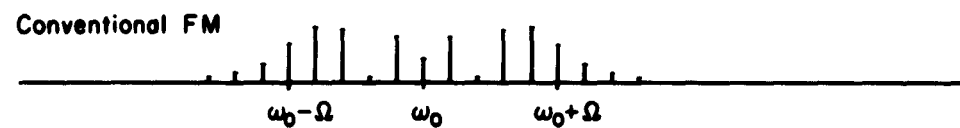
The intermediate function $g(t)$ is introduced in order to produce the desired modulated effects in Eq. (11). The difference between Eq. (4) and Eq. (10) is obviously trivial and the latter has been included mainly for completeness; they are, of course, identical for $\alpha = 1$.



a) Modulation index $\frac{\Omega}{\omega} = 1$



b) Modulation index $\frac{\Omega}{\omega} = 3$



c) Modulation index $\frac{\Omega}{\omega} = 5$

Fig.2 —Magnitudes of spectral components for conventional and SSB forms of FM with sinusoidal modulation

Single Sideband

$$\begin{aligned}
m(t) &= e^{[g(t) + i \hat{g}(t)]}, \quad g(t) = \alpha \log f(t), \quad f(t) > 0 \\
s(t) &= r^\alpha(t) \cos \left[\omega_0 t + \alpha \widehat{\log f(t)} \right] \\
&\quad + i r^\alpha(t) \sin \left[\omega_0 t + \alpha \widehat{\log f(t)} \right] \\
|s(t)| &= r^\alpha(t), \quad \omega_0 > 0
\end{aligned} \tag{11}$$

The SSB version given by Eq. (11) is related to Eq. (7), from which it can be derived by considering the logarithm of the modulating signal. The reason these two classes of exponential modulation have been made distinct here is that Eqs. (6) and (7) contain the modulating signal in the phase of the modulated signal, while Eqs. (10) and (11) contain the modulating signal in the envelope; hence, the designations.

The exponent α is important because of the relationship which the choice of its value as either 1 or 1/2 bears on the question of compatible SSB modulation. Powers⁽²⁾ considers the case $\alpha = 1/2$ so that the modulating signal is contained in the square of the envelope, thus requiring a square-law envelope detector for distortionless reproduction. Since it is this case which produces a modulated signal having the same spectral width as a conventional SSB signal, he concludes that true compatibility (i.e., a system usable with a linear envelope detector) is impossible.

Iyanncv⁽⁴⁾ reaches the same conclusion and, incidentally, labels the case Optimal Amplitude and Phase Modulation (OAPM) for Square-Law Detection. He also considers $\alpha = 1$ or OAPM for Linear Detection and points

out that this modulated signal occupies a bandwidth just twice the maximum modulating frequency and that it is compatible with a linear envelope detector.

Naturally, there would be no advantage to a compatible SSB system which has the same spectral width as a conventional DSB system, but Lyannoy adds that the bandwidth can be halved in practice by suitable filtering because of the characteristics of speech. This derives from the fact that "a real signal has components whose modulation depth approaches 100 per cent only in the range up to 1500 or 2000 c/s, the modulation depth decreasing rapidly thereafter." Hence, removal of the upper half of the signal spectrum permits the large-modulation-index, low-frequency components to be received without distortion; while the small-modulation-index, high-frequency components are received with a small but tolerable distortion.

Kahn⁽³⁾ makes virtually identical observations with regard to his Compatible SSB (CSSB) system, but it is outside the scope of this study to make a comparative analysis. It is apparent, however, that the two systems have a close fundamental relationship.

V. CONCLUSION

It has been shown that analog-modulated signals are representable in general as the product of a carrier function and a modulation function. When an analytic signal is chosen for the carrier, its effect spectrally becomes one of translating the signal spectrum to the vicinity of the carrier frequency. The process is one of pure translation when the carrier is harmonic, $\exp(i\omega_0 t)$, or one of translation plus spreading when the carrier has a finite spectral extent (i.e., when noiselike).

The useful forms of the modulation function are the linear and exponential functionals from which three classes of modulated signals can be derived, viz., amplitude, phase, and envelope modulation. Within these classes, the use of the purely real or analytic signal forms of the modulating signal leads naturally to the DSB and SSB (i.e., two-sided and one-sided) spectral forms, respectively, of the modulated signal.

The existing familiar forms of modulation are readily identified within the above structure. In addition, a new form of modulation which may be called SSB FM is revealed. It can be derived from a conventional phase-modulated signal by an additional amplitude modulation using the exponential function of the modulating signal's Hilbert transform. The resulting modulated signal will have a one-sided spectrum about the carrier frequency and be compatible with existing FM receivers. The advantage is a decrease in signal bandwidth; the disadvantage, the loss of a constant transmitter output level.

From a practical viewpoint, this form of FM may find little acceptance in wide-band systems, not only because of the unfavorable peak-to-

average power ratio required of the transmitter, but also because of the lack of any particular pressure to conserve the radio spectrum at present in services, such as broadcast FM or specialized military and civilian communication systems.

On the other hand, some of the bands assigned to public use are quite crowded and users are therefore disposed toward modulation techniques which make more efficient use of the available spectrum. A compatible narrow-band SSB FM system of the type described here may prove useful in such applications by narrowing the signal bandwidth for a given modulation index without sacrificing the "improvement" or the immunity to impulse noise associated with FM. The peak-to-average power ratio of a narrow-band SSB FM signal may prove an acceptable burden in that case.

Observation of the comparable (i.e., SSB) form of envelope modulation commends Powers' suggestion of the application of the non-compatible, square-law type of envelope modulation ($n = 1/2$ in Eq. (9)) as a means of SSB data transmission.⁽²⁾ Conventional SSB is adequate for speech or music because of the well-known tolerance of the human ear for phase or even frequency errors in reproduction. However, data waveforms must be reproduced without such errors, so exact carrier re-insertion is required (a procedure which is frequently difficult). The fact that the modulating signal is contained in the envelope of Eq. (9) indicates that square-law envelope detection should result in distortionless reception, thus obviating carrier re-insertion without sacrificing spectral economy.

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