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Stability Study of Pulse-Width Modulated and Nonlinear Sampled-Data Systems

by

T. Nishimura

Series No. 60, Issue No. 353

Contract No. AF 18(600)-1521

April 4, 1961

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**Electronics Research Laboratory
University of California
Berkeley, California**

**STABILITY STUDY OF PULSE-WIDTH MODULATED AND NONLINEAR
SAMPLED-DATA SYSTEMS**

by

T. Nishimura

**Institute of Engineering Research
Series No. 60, Issue No. 353**

**Physics Division
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ABSTRACT

In the previous report⁽¹⁾, the fundamental equation that describes limit cycles in nonlinear sampled-data systems has been derived. In that case, the equivalence of limit cycles with finite pulsed systems having a periodically varying sampling-rate is observed, and the methods of analysis applied to the latter are extended to obtain these limit cycles with the aid of final value theorem.

This fundamental equation is modified and is simplified to some extent under certain assumptions as it can be applied to systems both with and without integrators. The limitation on the longest period of saturated and unsaturated oscillation is investigated and the critical gain for their existence is derived, starting from the modified fundamental equation. Also, the stability of limit cycles as well as the equilibrium point is considered, based on Neace's method and its modification.

Through this study, various kinds of non-linearities, namely, pulse-width modulation, relay saturating amplifier with linear zone and quantized level amplifier are discussed, and examples are presented for each of these cases. Self-excited oscillations are mainly examined, as well as the possible existence and stability of limit cycles, however, the method can be extended to forced oscillations.

Finally, experimental works are performed in order to verify the theoretical results by means of digital computers.

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CHAPTER 1

Introduction

Nonlinear sampled-data feedback control systems have been extensively investigated in recent years (Fig. 1). One of the most important topics of these studies has been the stability problem of such nonlinear sampled-data systems. Lyapunov's second method has been introduced as a powerful tool for the analysis of the stability problems.^{4, 5, 6} At the same time, the classical describing-function method has been used in analogy with the case of nonlinear continuous systems.⁷

Among the various types of nonlinearities the pulse-width modulation has been studied by such people as R. F. Nease⁸, R. E. Andeen,^{9, 10} E. Polak,¹¹ T. T. Kadota,⁶ W. L. Nelson,¹² Shao Da Chuan,¹³ I. V. Pyshkin,¹⁴ and E. I. Jury and T. Nishimura^{1, 2, 3}. For example, R. E. Andeen^{9, 10} presented an approximate method for the analysis by replacing the pulse-width modulator by the equivalent pulse-amplitude modulator. Hence, his method is restricted by the small-signal condition because of that approximation. On the other hand, Shao Da Chuan¹³ gave an exact analysis of the limit cycle in the PWM system using the canonical-form expression. However, his analysis is limited to the limit cycle of two sampling periods. I. V. Pyshkin¹⁴ presented an extensive method to prove the existence of limit cycles in PWM systems, but still he had to resort to certain approximation using the low-pass characteristics of the linear plant and the stability problem was not substantially attacked.

T. T. Kadota⁶ applied the second method of Lyapunov for the stability discussion and succeeded in deriving a sufficient condition for asymptotic stability in the large of PWM systems, although his condition was rather conservative compared with the actual stability condition obtained experimentally.

E. I. Jury and T. Nishimura^{1, 2, 3} derived a method to find the exact behavior of limit cycles in PWM systems extending the theory of the periodically-varying finite-pulse-width systems. Also, they discussed certain features of the stability of such systems in connection

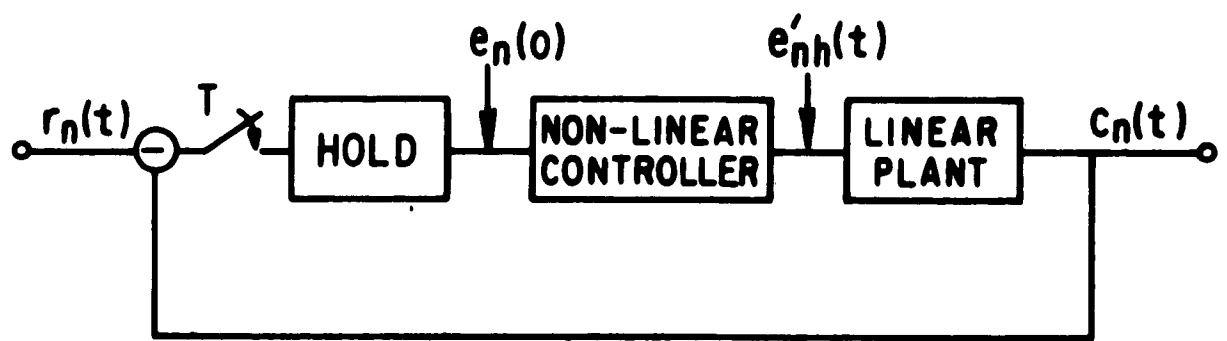


FIG. 1 NON-LINEAR SAMPLED DATA CONTROL SYSTEM

with the stability of limit cycles when they are in the simplest form, namely, of the two sampling periods.

In this report, the method introduced in the above references is extended and simplified in its form. Hence, this report is in direct continuation with the previous one ⁽¹⁾ and further detail of that report will be summarized in the last part of this introduction.

The main subject of this report is the stability problem of PWM sampled-data systems. By the Lyapunov-function method one attempts to find the Lyapunov function which is positive definite and is decreasing for every sampling instant. In order to find such a general function one has to expect the worst case might happen, without paying attention to the various features of disturbances within the system. This fact leads to the result that the sufficient condition derived by this method is often excessively conservative, as is seen in the case of Ref. 6.

In contrast to such a comprehensive, macroscopic method, we adopt the microscopic method in which we start by separating the possible modes of oscillation into the adequate categories, then analyze the stability of oscillations of each category from its simplest mode and forward the analysis until we cover all the possible modes of oscillations.

This laborious work may appear almost prohibitive in its beginning. However, by finding the certain regularities which govern such oscillations, we can achieve the precise analysis of the stability problem. This is the basic attitude of this report towards the specified problem.

In Chapter II the simplified form which yields the feature of limit cycles will be derived and this equation will be extended to include the system which contains pure integrators.

In Chapter III we will derive the method to find the longest period of limit cycle of relay mode oscillation. Then we will discuss the stability boundary for the relay mode oscillation. The same technique will be extended to unsaturated oscillation as well as to other nonlinear systems.

In Chapter IV the stability of the equilibrium point will be investigated and final conclusions will be derived on the stability of the PWM sampled-data systems.

Review of the Previous Report¹

We present a short review about the content of the previous report entitled "On the Periodic Modes of Oscillations in Pulse-Width Modulated Systems"¹ and reproduce those equations which will be used in this report. The scheme of the PWM system is shown in Fig. 2. The PWM controller (lead type) has such characteristics that its output is a unit pulse (positive or negative), the sign of the pulse is identical with the sign of the control error at the sampling instant ($e_n(0)$) and the pulse width is proportional to the magnitude of $e_n(0)$.

Since the sampling period T is fixed, the maximum pulse width is T and the saturation will occur beyond this point. Then the output of the PWM controller during the $(n+1)$ th sampling period will be given by the product of the sign function $\gamma(n)$ and the unit pulse function $u_h(t)$. The origin of the time axis t is placed at the n th sampling instant.

$$e'_{nh}(t) = \gamma(n)u_h(t) \quad (1.1)$$

where

$$\gamma(n) = \frac{e_n(0)}{|e_n(0)|} \quad (1.2)$$

and this takes only +1 or -1.

$$\begin{aligned} u_h(t) &= 1 && \text{for } 0 \leq t \leq h_n \\ &= 0 && \text{elsewhere} \end{aligned} \quad (1.3)$$

Also the pulse width is given as follows:

$$\begin{aligned} h_n &= \alpha |e_n(0)| && \text{for } 0 \leq \alpha |e_n(0)| \leq T \\ &= T && \text{for } \alpha |e_n(0)| > T \end{aligned} \quad (1.4)$$

where α is a gain of the PWM controller. The incremental response $\Delta C_n(s)$ is defined as the product of the transfer function $KG(s)$ and the Laplace transform form of the pulsed output $e'_{nh}(t)$ from the PWM controller

$$\Delta C_n(s) = KG(s)E'_{nh}(s) \quad (1.5)$$

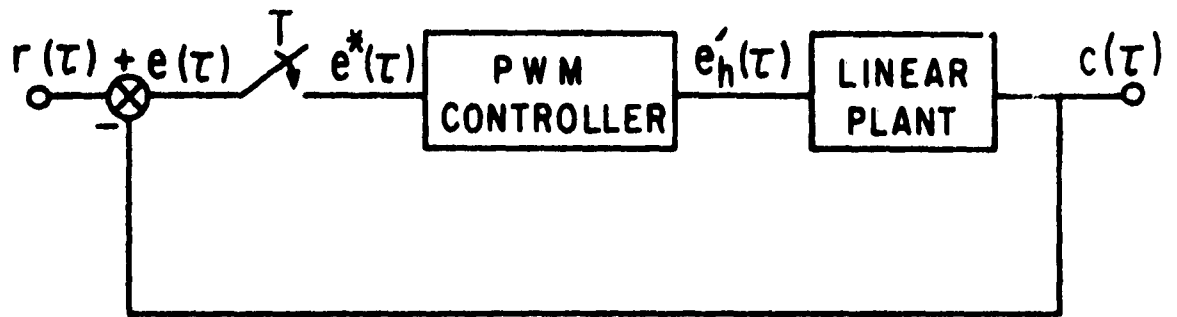
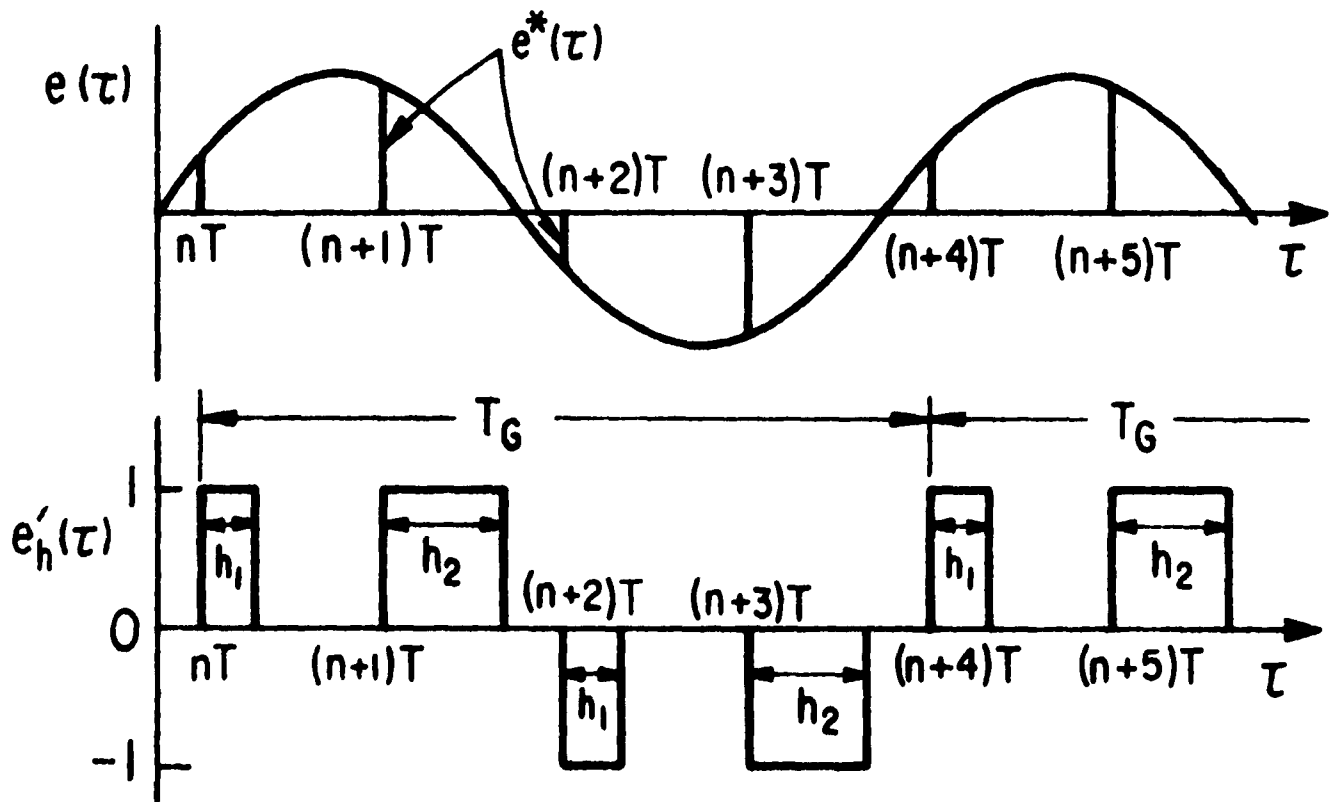


FIG. 2a PULSE WIDTH MODULATED SYSTEM



$e(\tau)$, $e^*(\tau)$; INPUT & SAMPLED INPUT
TO CONTROLLER

$e'_h(\tau)$; OUTPUT OF CONTROLLER

FIG. 2b INPUT & OUTPUT OF PWM CONTROLLER

where

$$E'_{nh}(s) = \mathcal{L}[e'_{nh}(t)] = \gamma(n) \frac{1 - e^{-h_n s}}{s} \quad (1.6)$$

Then the incremental response of each limit cycle is given by the summation of $\Delta C_n(s)$ during one limit cycle, multiplied by the relative delay factor

$$\Delta C_s(s) = \sum_{l=0}^{M-1} \Delta C_l(s) e^{-l s T} \quad (1.7)$$

The z-transform of this becomes

$$\Delta C_s^*(z) = \sum_{l=0}^{M-1} z^{-l} \Delta C_l^*(z) \quad (1.8)$$

Applying the skip-sampling¹⁶ operation to $\Delta C_s^*(z)$ yields the incremental response at the instants 0, T_G , $2T_G$, ..., where $T_G = MT$, the period of the limit cycle:

$$\Delta C_{0s}^*(Z) = Z_s[\Delta C_s^*(z)] \quad (1.9)$$

Then it is shown that the response at the beginning of the limit cycle is obtained by adding $\Delta C_s^*(Z)$ for all the periods of limit cycles and by applying the final value theorem¹⁷ to the summation. In this case, it is assumed that $KG(s)$ has no integrator in order to insure the convergence of limiting process.

Thus the final result is derived as follows:

$$\begin{aligned} c_{0s} &= \lim_{Z \rightarrow 1} \Delta C_{0s}^*(Z) \\ &= \lim_{Z \rightarrow 1} Z \left\{ \sum_{l=0}^{M-1} z^{-l} Z[KG(s)E'_{lh}(s)] \right\} \end{aligned} \quad (1.10)$$

This equation can be applied for any type of nonlinearities. In the case of the PWM system, $E'_{lh}(s)$ in the above equation is replaced by Eq. (1.6).

Ex. (1.10) can be extended in general form to give the response at the j th sampling instant of the limit cycle.

$$\begin{aligned} c_{js} &= \lim_{Z \rightarrow 1} \Delta C_{js}^*(Z) \\ &= \lim_{Z \rightarrow 1} Z_s \cdot z^j \sum_{l=0}^{M-1} z^{-l} [KG(s) E'_{lh}(s)] \end{aligned} \quad (1.11)$$

$$\text{since } \Delta C_{js}^*(Z) = Z_s [z^j \Delta C_s^*(z)] \quad (1.12)$$

When $KG(s)$ has an integrator, the problem is solved with the aid of difference equations.

Using these equations, the limiting cycle of $M = 2$ and $M = 4$ are analyzed in the examples of the previous report¹ and the existence of limit cycles of the PWM mode and the relay mode as well as the stable region is indicated. Also the critical gains for each region are specified for the limit cycle of $M = 2$.

In the appendix, the stability of the limit cycle is discussed, which is another important problem of the limit cycle as much as its existence. The discussion is based on the important theorems given by Nease.⁸ The definition of the stable limit cycle is given as follows.

DEFINITION: A limit cycle is said to be absolutely stable if any small perturbation about the limit cycle approaches 0 as $n \rightarrow \infty$.

Then two theorems are referred to which present the method for testing the stability of limit cycles.

THEOREM 3. Assume that the nonlinear difference equations

$$X_{n+1} = F(X_n) \quad (1.13)$$

have a periodic solution S_n of period M , and that the functions $F(X_n)$ are single valued and possess continuous first partial derivatives. The first approximation of the difference equation for small perturbations about this periodic solution S_n is

$$Y_{n+1} = [A_n] Y_n \quad (1.14)$$

and the solution of this equation determines the stability of the periodic solutions S_n if $[A_n]$ is nonsingular at all of the solution points. In the above equation Y_n is the perturbation about S_n and the components of $[A_n]$ are

$$a_{ij, n} = \left. \frac{\partial F_i(X_n)}{\partial X_j} \right|_{X_n = S_n} \quad (1.15)$$

THEOREM 4. The system of Eq. (1.12) is stable if all the eigenvalues of the matrix

$$[A_G] = [A_{n+M-1}][A_{n+M-2}] \cdots [A_n] \quad (1.16)$$

lie inside the unit circle. Then all the solutions tend to 0 as n becomes large.

Theorem 4 of Ref. 8 is partially stated in the above, considering only the necessary part pertinent to this discussion. Combining these two theorems, the condition for the stability of limit cycles will be reduced to the following statement that "all the eigenvalues of the matrix $[A_G]$ which consist of the first partial derivatives of $F(X_n)$ lie inside the unit circle at all the periodic solution points S_n ." With this condition satisfied, the small perturbation will tend to 0 as $n \rightarrow \infty$, hence the limit cycle is said to be stable according to the definition of previously defined stability of limit cycles.

CHAPTER 11

Modification of Fundamental Equation for Limit Cycles

The general form of the equations which give the response at the j th sampling instant of the limit cycle in the nonlinear sampled-data system has been derived in Eq. (1.11). It is assumed that $KG(s)$ has no poles at the origin.

Certain modification of Eq. (1.11) is attempted in the following part. Since the content of the skip-sampling operator $Z_s []$ is a function of s , i. e., a sampled function with the period T , the operator Z_s samples the sampled function of T with the period $T_G = MT$. Hence this sampling process is redundant and the first sampling process with period T may be removed. For that purpose, the content of the Z_s operator, i. e., $\Delta C_{js}^*(z)$ shall be modified in such a way that $\Delta C_{js}^*(z)$ contains the incremental response of one period of limit cycle preceding the j th sampling instant. Hence, the new incremental response $\Delta C_{is}^*(z)$ is described as follows.

$$\Delta C_{is}^*(z) = \sum_{l=0}^{M-1} z^{-l} \mathcal{Z} [KG(s)E'_{l+1,h}(s)] \quad (2.1)$$

$i = 0, 1, \dots, M-1.$

It is understood that the system has already been on the limit cycle, hence $E'_{l+1,h}(s)$ has the periodical feature with the period T_G .

$$E'_{l+1,h}(s) = E'_{l+1-M,h}(s) \quad \text{when } l+1 \geq M \quad (2.2)$$

Substituting $\Delta C_{is}^*(z)$ for $\Delta C_{js}^*(z)$ in Eq. (1.11) yields the desired response at the i th sampling instant.

$$\begin{aligned} c_{is} &= \lim_{Z \rightarrow 1} Z_s [\Delta C_{is}^*(z)] \\ &= \lim_{Z \rightarrow 1} Z_s \left[\sum_{l=0}^{M-1} z^{-l} \mathcal{Z} [KG(s)E'_{l+1,h}(s)] \right] \quad (2.3) \\ &\quad i = 0, 1, \dots, M-1 \end{aligned}$$

Inverting the content of the Z_s operator to the function of s , and replacing the skip sampling operator Z_s by the ordinary $\mathcal{Z}$ operator with the period T , the above equation becomes as Eq. (2.4), given below, where $Z = e^{MsT_G}$.

is multiplied to the content of the bracket while Z^{-1} is multiplied to the outside of the bracket.

$$\begin{aligned}
 c_{is} &= \lim_{Z \rightarrow 1} \mathcal{Z}_{TG} \left(e^{MsT} \left[\sum_{l=0}^{M-1} e^{-lsT} KG(s) E'_{l+1, h}(s) \right] \right) Z^{-1} \\
 &= \lim_{\substack{Z \rightarrow 1 \\ s \rightarrow 0}} \frac{Z^{-1}}{Zj} \int_{\Gamma} \frac{e^{MpT} KG(p) \sum_{l=0}^{M-1} e^{-lpT} E'_{l+1, h}(p)}{1 - e^{-MT(s-p)}} dp \\
 &\quad i = 0, 1, \dots, M-1 \quad (2.4)
 \end{aligned}$$

The multiplication of e^{MsT} to the content of the bracket is done to ensure the convergence of the integrand to zero along the infinite semicircle on the right-half plane, and this does not cause any change in the final result under the condition that $Z \rightarrow 1$.

The limiting process $s \rightarrow 0$ may be performed before the integration if the path of integration Γ along the $j\omega$ axis is taken sufficiently close to the imaginary axis of the p -plane. Hence, Eq. (2.4) is reduced to

$$c_{is} = \frac{1}{Zj} \int_{\Gamma} \frac{e^{MpT} KG(p) \sum_{l=0}^{M-1} e^{-lpT} E'_{l+1, h}(p)}{1 - e^{-MPt}} dp \quad (2.5)$$

This is the modified form of Eq. (2.1).

Further simplification is possible when the oscillation is symmetrical and monotonic. The symmetrical oscillation is such that the same shape of oscillation is repeated for every half period of the limit cycle with an opposite sign. The word monotonic in this case implies that the input $e'_{l0}(t)$ to the nonlinear component is positive for the first half period of the limit cycle and is negative for the other half period. These conditions are expressed mathematically as follows:

Letting $M = 2\mu$ (μ = number of samples in half period of limit cycle)

$$e'_{l,s} = -e'_{l+\mu,s} > 0 \quad \text{for } l = 0, 1, \dots, \mu-1 \quad (2.6)$$

and

$$\begin{aligned} \gamma(l) &= +1 & \text{for } l = 0, 1, \dots, \mu-1 \\ \gamma(l) &= -1 & \text{for } l = \mu, \mu+1, \dots, M-1 \end{aligned} \quad (2.7)$$

The above assumption is justified in most of the cases in practice because the linear plant usually possesses the low-pass characteristics, which render the oscillations smooth and monotonic.

Also it is emphasized that the assumption of symmetry is placed on the input of the linear plant, as in Eq. (2.6), not on its output. Hence, the asymmetrical oscillation with respect to the plant output, as observed in the second-order relay system, (Fig. 9 of reference 1) is still the symmetrical oscillation with respect to the plant input and satisfies the condition of Eq. (2.6)

Since

$$E'_{l+\mu+1,h}(s) = -E'_{l+1,h}(s) \quad (2.8)$$

Eq. (2.5) is rewritten as follows

$$\begin{aligned} c_{is} &= \frac{1}{2\pi j} \int_{\Gamma} \frac{e^{2\mu pT} KG(p) \sum_{l=0}^{\mu-1} e^{-lpT} E'_{l+1,h}(p) (1 - e^{-\mu pT})}{1 - e^{2\mu pT}} dp \\ &= \frac{1}{2\pi j} \int_{\Gamma} \frac{e^{\mu pT} KG(p) \sum_{l=0}^{\mu-1} e^{-lpT} E'_{l+1,h}(p)}{1 + e^{\mu pT}} dp \end{aligned} \quad (2.9)$$

Thus, half of the zeros of the denominator of the integrand, which includes the zero at the origin, are cancelled by the zeros of $(1 - e^{-\mu pT})$ in the numerator by introducing the condition of symmetry and monotonicity of

oscillations.[†] This elimination of the pole at the origin yields a great contribution for the analysis of systems with integrators which shall be developed in the following part.

Equation (2.10) will be rewritten for the case when the function of the nonlinear component is the pulse-width modulation. Then,

$$E'_{lh}(s) = \gamma(l) \frac{1 - e^{-sh_l}}{s} \quad (2.11)$$

where $\gamma(l)$ is specified by Eq. (1.2), and

$$\begin{aligned} h_l &= a |e_l(0)| \\ &= a |r_l(0) - c_l(0)| \end{aligned} \quad (2.12)$$

Substituting Eq. (2.11) into Eq. (2.10) yields

$$c_{ls} = -\frac{1}{2\pi j} \int_{\Gamma} \frac{e^{\mu p T} KG(p)}{1 + e^{\mu p T}} \sum_{l=0}^{\mu-1} \frac{\gamma(l+1) e^{-l p T} (1 - e^{-p h_{l+1}})}{p} dp \quad (2.13)$$

So far, it is assumed that $KG(s)$ has no integrator. The problem of $KG(s)$

[†] The conditions of symmetry and monotonicity are the sufficient conditions for cancelling the pole at the origin of the integrand of Eq. (2.06), but are not the necessary conditions. Actually, the condition of monotonicity is not required at all in order to cancel the pole at the origin. Only the condition that every $e'_{lh}(0)$ in one limit cycle have its pair of the same magnitude, but of opposite sign, $e'_{lh}(0)$ in the same limit cycle, is required for that purpose. In special cases, this condition is further reduced to that the summation of $e'_{lh}(0)$ is equal to zero, as in the case of the quantized level amplifier.

For example, the four-period limit cycle of 1, k, -k, -1, $|k| < 1$, which is shown in the example of Ref. 19 for the saturating amplifier system, satisfied the above condition.

However, this does not assure the existence of such limit cycle as shown in the same reference in which the existence of the above limit cycle is denied. Moreover, the stability of such asymmetrical, non-monotonic limit cycle is quite dubious, as is noticed in the experimental observation. Therefore, the introduction of the assumption of symmetry and monotonicity is practical when the linear plant has the low-pass characteristics, as in most cases. Also, we emphasize that this assumption does not introduce any approximation for the analysis at all.

with integrators will be discussed in the following part.

An example of second-order system with a single integrator is solved in the previous section with the aid of difference equations. The reason for the difference equations being used in addition to our fundamental equation is that the pole at $Z = 1$ which originates from the single integrator of the plant makes the application of the final value theorem impossible, since when Z approaches unity the term $\Delta C_g^*(Z)$ would diverge because of the pole at $Z = 1$. This fact can also be observed when we take a look at the modified equation (2.4).

When the integral is evaluated by the residue method for all the poles of $KG(p)$, the term $1 - e^{-MT(s-p)}$ causes the integrand to diverge when the residue at $p = 0$ is evaluated, and when $s \rightarrow 0$ and this does not give any finite value for the residue.

However, if the condition of symmetry and monotonicity is introduced to such a system, this troublesome pole at the origin is eliminated as $s \rightarrow 0$ and the evaluation of the residue at the origin becomes possible. This is seen in Eq. (2.10) in which the pole at the origin of the integrand has already been removed.

Therefore $KG(p)$ may contain not only a single integrator, but also a double or triple or any higher order of integrator at the origin. And it is proved that the simplified equation (2.10) is valid for any shape of the plant $KG(s)$ if it satisfies the physically realizable condition under the assumption of symmetrical and monotonic oscillation. Also, Eq. (2.10) is applicable for any type of nonlinearities if the output $e'_{nh}(t)$ of nonlinear element can be specified as a function of the input and output of the over-all system.

We have derived three fundamental equations for limit cycles in this chapter, as shown in Eqs. (1.11), (2.5), and (2.9). We will explain briefly the advantage and disadvantage of using each of these equations.

The first equation of Eq. (1.11) has the disadvantage of having two kinds of z -transformation, namely, the s -transformation with respect to T and the skip-sampling operation with respect to T_G . However, in actual calculation, we can perform the skip sampling operation by picking up the necessary terms out of the expansion in powers of z^{-1} , without carrying over the integration of sampling operation. Moreover, we are forced to use Eq. (1.11) when the system contains digital processing units which are given as a function of z , not of s .

The second equation of Eq. (2.5) may be used in most cases except when the system has digital processing units. However, we must be careful in carrying over the integration because the integrand contains the delay factors in the form of e^{-pT} and more labor is required for the computation.

The third equation of Eq. (2.9) is useful when the oscillation is symmetric and monotonic, and is the most convenient form among these fundamental equations. When these assumptions are violated, as observed in the examples of the quantized level amplifier of Section 2.2, (3), we have to use either Eq. (1.11) or Eq. (2.5).

When the plant has an integrator, Eq. (2.5) or Eq. (2.9) can be used, although they give only the oscillations of zero D. C. component.⁺ In case the oscillation of non-zero D. C. component is desired or the values of derivatives of responses are required, the difference equations are very helpful as shown in section V of reference 1.

Example 2.1

The equations which yield the solution for this symmetrical PWM mode oscillation are illustrated in this example.

The plant is chosen as second order and the period of the limit cycle is $M = 2\mu$ sampling periods.

$$KG(s) = \frac{K}{s(s+b)} \quad (2.14)$$

In Eq. (2.13), let $i = 0$ and take $\gamma(l+i) = \gamma(l) = +1$ for all l , referring to Eq. (2.7). Hence Eq. (2.13) becomes as follows.

⁺ D. C. component implies the average value of responses at the sampling instants during one period of limit cycle.

$$\begin{aligned}
c_{0s} &= -\frac{1}{2\pi j} \int_{\Gamma} \frac{e^{\mu p T}}{1 + e^{\mu p T}} \frac{K}{p(p+b)} \sum_{l=0}^{\mu-1} \frac{(1 - e^{-h_l p}) e^{-l p T}}{p} dp \\
&= -\frac{K}{b} \left\{ \frac{h_0}{2} - \frac{e^{-\mu b T} (e^{bh_0} - 1)}{b(1 + e^{-\mu b T})} + \frac{h_1}{2} - \frac{e^{-(\mu-1)b T} (e^{bh_1} - 1)}{b(1 + e^{-\mu b T})} + \dots \right. \\
&\quad \left. + \frac{h_{\mu-1}}{2} - \frac{e^{-b T} (e^{bh_{\mu-1}} - 1)}{b(1 + e^{-\mu b T})} \right\} \quad (2.15)
\end{aligned}$$

Multiplying $-a$ to both sides of the above equation and knowing $h_0 = -ac_{0s}$, the following equation is derived:

$$h_0 = \frac{aK}{b} \sum_{l=0}^{\mu-1} \left\{ \frac{h_l}{2} - \frac{e^{-(\mu-l)b T} (e^{bh_l} - 1)}{b(1 + e^{-\mu b T})} \right\} \quad (2.16)$$

For $i = 1$, it is observed from Eq. (2.7) that

$$\begin{aligned}
\gamma(l+1) &= +1 & \text{for } l = 0, 1, \dots, \mu-2 \\
&= -1 & \text{for } l = \mu-1
\end{aligned} \quad (2.17)$$

and

$$h_{\mu} = h_0$$

Hence, $h_1 (= -ac_{1s})$ is obtained in a similar manner

$$\begin{aligned}
h_1 &= \frac{aK}{b} \left\{ \sum_{l=0}^{\mu-2} \left[\frac{h_{l+1}}{2} - \frac{e^{-(\mu-l)b T} (e^{bh_{l+1}} - 1)}{b(1 + e^{-\mu b T})} \right] \right. \\
&\quad \left. - \left[\frac{h_0}{2} - \frac{e^{-b T} (e^{bh_0} - 1)}{b(1 + e^{-\mu b T})} \right] \right\} \quad (2.18)
\end{aligned}$$

In general,

$$\begin{aligned}
\gamma(l+i) &= +1 & \text{for } l = 0, 1, \dots, \mu-1-i \\
&= -1 & \text{for } l = \mu-i, \mu-i+1, \dots, \mu-1
\end{aligned} \quad (2.19)$$

and

$$h_{\mu-1+i} = h_{i-1}$$

Then, using these results

$$h_i = \frac{eK}{b} \left\{ \sum_{l=0}^{\mu-i-1} \left[\frac{h_{l+i}}{2} - \frac{e^{-(\mu-l)bT} (e^{bh_{l+i-1}})}{b(1 + e^{-\mu bT})} \right] - \sum_{l=\mu-i}^{\mu-1} \left[\frac{h_{l+i-\mu}}{2} - \frac{e^{-(\mu-l)bT} (e^{bh_{l+i-\mu-1}})}{b(1 + e^{-\mu bT})} \right] \right\} \quad (2.20)$$

$$i = 0, 1, \dots, \mu-1$$

The solutions of the equations which are given by Eq. (2.20) for $i = 0, 1, \dots, \mu-1$ yield the exact feature of the PWM mode oscillation that will be sustained within the closed-loop PWM sampled-data system. Although the numerical solutions of these transcendental equations are not easily obtained when $\mu \geq 2$, certain approximations for the exponential terms are possible, as discussed in the previous section. Also the programming on the high-speed digital computer will enable us to solve these equations without approximation when the system constant, the sampling period, and the system gains are specified.

Example 2.2 Quantized Level Amplifier

In this example, the nonlinear gain amplifier has a characteristic of quantized level (2 levels) as shown in Fig. (3).

This characteristic is represented as follows.

$$\begin{aligned}
 e'_n(t) &= 1 && \text{if } d < e_n(0) \\
 &= 0.5 && \text{if } d/2 < e_n(0) < d \\
 &= 0 && \text{if } -d/2 < e_n(0) < d/2 \quad (2.21) \\
 &= -0.5 && \text{if } -d < e_n(0) < -d/2 \\
 &= -1 && \text{if } e_n(0) < -d
 \end{aligned}$$

where $ad = 1$

The linear plant is again second order given by Eq. (2.14).

We will demonstrate the existence of asymmetrical oscillation with the period of three sampling periods.

For such an asymmetrical oscillation, Eq. (2.5) must be used.

Letting $M = 3$, and it is assumed that the output of the nonlinear component has a sequence of +1, 0, -1, hence $y(0) = 1$, $y(2) = -1$. Substituting these values into Eq. (2.5), together with Eq. (2.21), yields

$$c_{0s} = -\frac{K}{b} \left[\frac{2T}{3} - \frac{1 - e^{-2bT}}{b(1 + e^{-bT} + e^{-2bT})} \right] \quad (2.22)$$

$$c_{1s} = \frac{K}{b} \left[\frac{T}{3} - \frac{1 - e^{-bT}}{b(1 + e^{-bT} + e^{-2bT})} \right] \quad (2.23)$$

$$c_{2s} = \frac{K}{b} \left[\frac{T}{3} - \frac{e^{-bT} - e^{-2bT}}{b(1 + e^{-bT} + e^{-2bT})} \right] \quad (2.24)$$

[†] The stability of this limit cycle is tested by the method which is described in Chapter I[†] as well as by the digital computer experiment. It is found that this limit cycle is unstable and will move to the relay mode oscillation of $M = 4$.

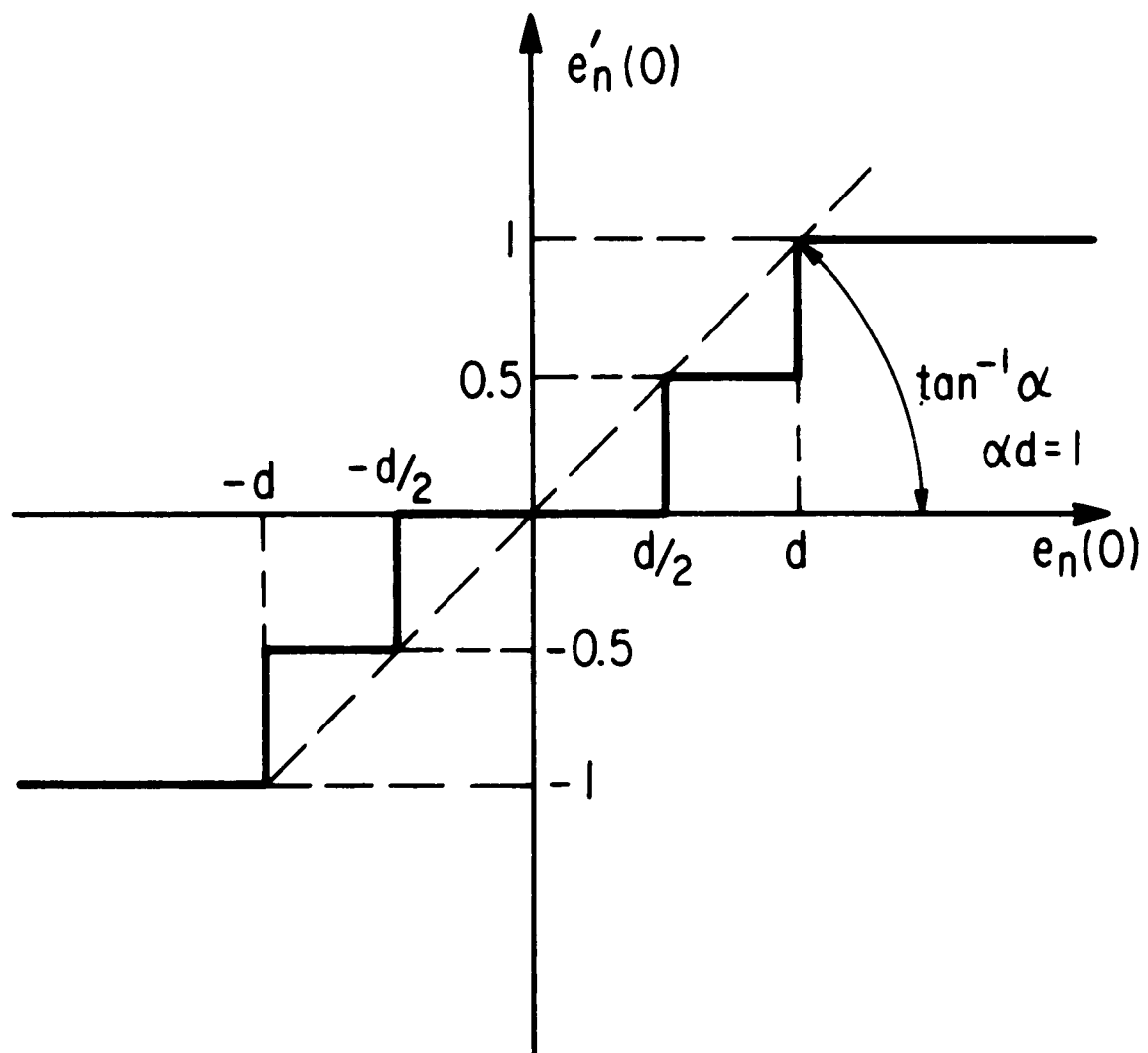


FIG. 3 CHARACTERISTIC OF QUANTIZED LEVEL AMPLIFIER

When $T = 2$, $b = 1$, $K = 1$ and $d = 1/3$ ($a=3$), those values are calculated as follows.

$$c_{0s} = -0.482$$

$$c_{1s} = -0.083$$

$$c_{2s} = 0.565$$

The output of the quantised level amplifier corresponding to c_{0s} is +1, 0 for c_{1s} , -1 for c_{2s} , which satisfy the original assumption of the sequence. This asymmetrical limit cycle is shown in Fig. (4).

Another example of an asymmetrical limit cycle will be demonstrated for the same system as the previous example, except that $d = 1/2$ ($a=2$) in this case. The sequence of the output from the nonlinear component is assumed as +1, 0, -1/2, 0, 1/2, 0, -1, having seven samples in one period of limit cycle.

Again Eq. (2.5) is used for the analysis of such asymmetrical oscillation. For example, the response at the first sampling instant of the limit cycle is given by

$$c_{0s} = -\frac{K}{b} \left[\frac{5}{7} T - \frac{(1 - e^{-bT})(2 - e^{-2bT} + e^{-4bT} - 2e^{-6bT})}{2b(1 - e^{-7bT})} \right] \quad (2.25)$$

$$= -0.571$$

Similarly, the responses at the other sampling instants of the limit cycle are obtained as follows.

$$c_{1s} = -0.178$$

$$c_{2s} = 0.470$$

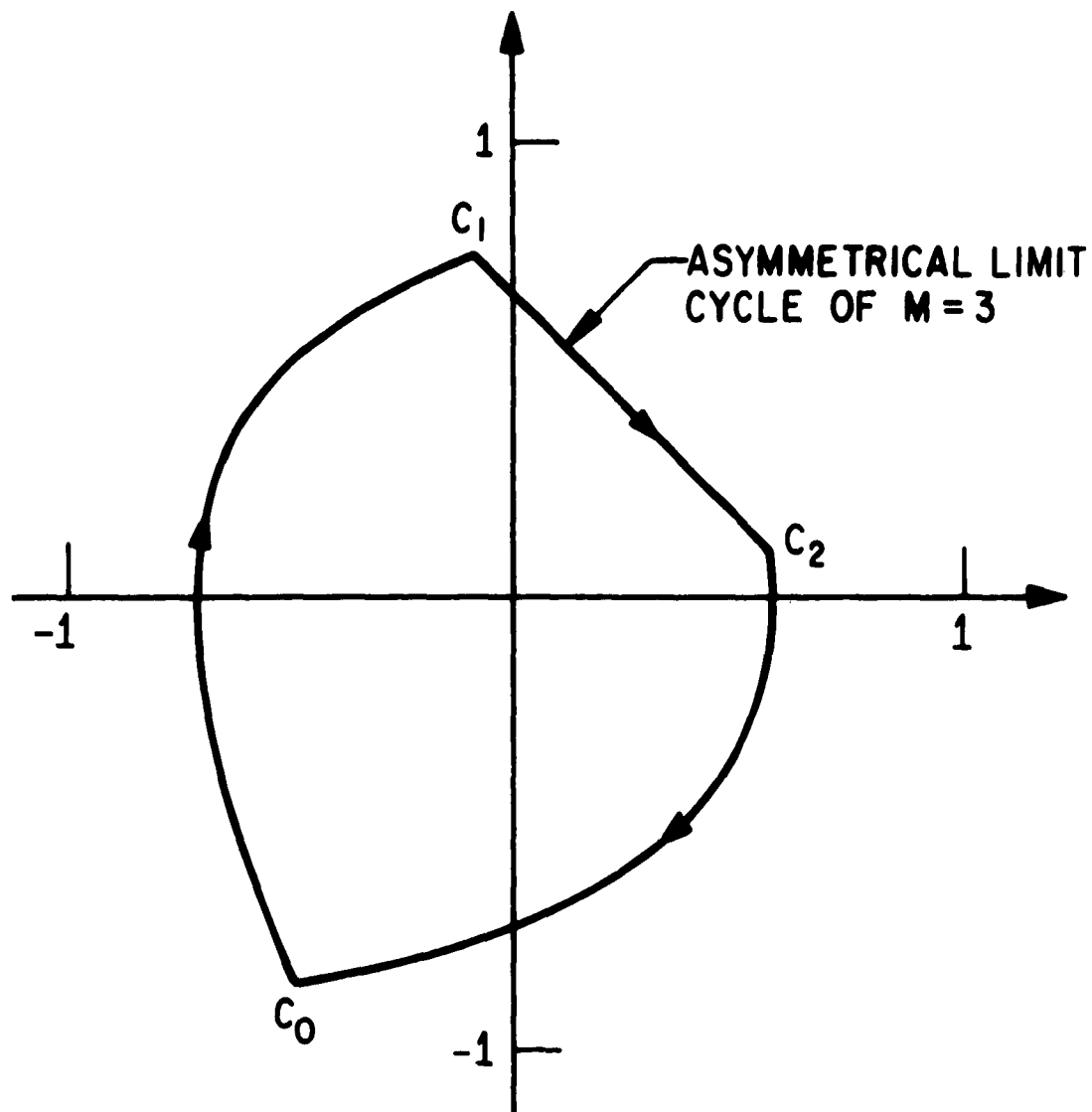
$$c_{3s} = -0.011$$

$$c_{4s} = -0.371$$

$$c_{5s} = 0.147$$

$$c_{6s} = 0.514$$

Applying these values of responses to Eq. (2.21) (letting $e_n(0) = -c_n(0)$), we find that the assumed sequence will be reproduced from the nonlinear gain amplifier and such a limit cycle will be maintained. The feature of this limit cycle is shown in Fig. (5). These two limit cycles are proved to be stable limit cycles by the digital computer experiment.



$$KG(s) = \frac{1}{s(s+1)} \quad T=2 \quad \alpha=3$$

$$e'_n : +1, 0, -1$$

FIG. 4 ASYMMETRICAL LIMIT CYCLE OF QUANTIZED
LEVEL AMPLIFIER SYSTEM

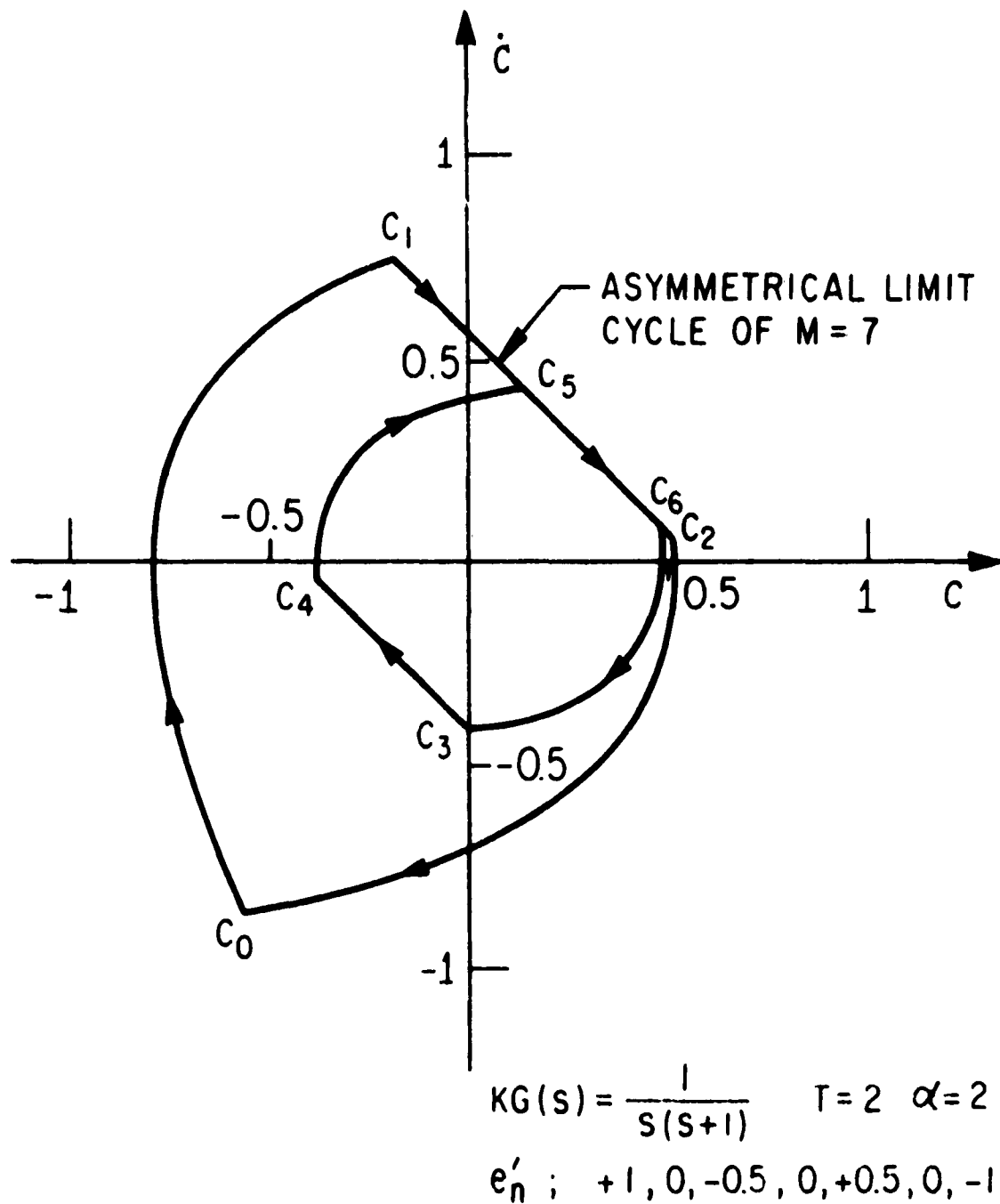


FIG. 5 ASYMMETRICAL LIMIT CYCLE OF QUANTIZED LEVEL AMPLIFIER SYSTEM

CHAPTER III

ANALYSIS OF SATURATED AND UNSATURATED OSCILLATIONS AND THEIR STABILITY BOUNDARIES

In the previous chapter, we have derived a fundamental equation to trace the exact behavior of limit cycles which are sustained within the nonlinear sampled-data systems. We have also demonstrated that such oscillations can be eliminated by reducing the gain in PWM systems when they are in the simplest mode. However, the problem of finding the gain boundary to eliminate all the possible limit cycles requires an enormous amount of work.

But we will proceed to solve the stability problem in this chapter by the localized approach to saturated and unsaturated oscillations in PWM systems.

3.1 Limitation on the Period of Limit Cycles of Relay Mode Oscillations

It has been observed that there exists a certain limitation on the longest period for the limit cycle which is sustained within the autonomous relay sampled-data system. Isawa and Weaver²⁰ discussed this problem on the second-order system and derived an equation which gives the maximum half period of limit cycle as a function of the sampling period, introducing a fictitious delay to the sampler. Also Pyshkin¹⁴ treated the same problem using the describing-function method.

We will show that this problem can be solved for any order of the linear plant without any approximation. Under the assumption of symmetric and monotonic oscillation at the output of the nonlinear component, which is an ideal relay in this case, Eq. (2.13) is valid for the relay mode oscillation if all h_{l+1} are replaced by the sampling period T . Hence

$$c_{1s} = - \frac{1}{2\pi j} \int_{\Gamma} \frac{e^{\mu PT} K G(p)}{1 + e^{\mu PT}} \sum_{l=0}^{N-1} \frac{\gamma(l+1) e^{-lPT} (1 - e^{-PT})}{p} dp \quad (3.1)$$

It is to be recalled that the output of the nonlinear element $e'_{fs}(0)$ is positive for the first half period and negative for the other half period. This implies that the response $c_{fs}(0)$ is negative for the first half period and positive for the other half if the nonlinear component such as relay, PWM, or saturating amplifier has no hysteresis in it (no input is assumed). Therefore,

$$\begin{aligned} \gamma(l) &= +1 && \text{for } l = 0, 1, \dots, \mu-1 \\ &= -1 && \text{for } l = \mu, \mu+1, \dots, M-1 \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} c_{fs} &< 0 && \text{for } l = 0, 1, \dots, \mu-1 \\ c_{fs} &> 0 && \text{for } l = \mu, \mu+1, \dots, M-1 \end{aligned} \quad (3.3)$$

Then the maximum number $\mu_{\max}$ of sampling periods in the half period of limit cycle is easily obtained by investigating the polarity of $c_{\mu-1,s}$ which is the response at the last sampling instant contained in the first half period of the limit cycle, and is given by Eq. (3.1), letting $l = \mu-1$ in it,

$$c_{\mu-1,s} = -\frac{1}{2\pi j} \int_{\Gamma} \frac{e^{\mu p T} K G(p)}{1 + e^{\mu p T}} \frac{(1 - e^{-p T})}{p} \sum_{l=0}^{\mu-1} \gamma(l+\mu-1) e^{-l p T} dp \quad (3.4)$$

Since c_{0s} is negative by the original assumption, the polarity of $c_{\mu-1,s}$ is tested starting from $\mu = 2$, i. e., from c_{1s} . $c_{\mu-1,s}$ may be negative up to a certain $\mu = \mu'$, but may become positive for all $\mu \geq \mu'+1$. In that case this critical μ' is taken as $\mu_{\max}$ since $c_{\mu-1,s}$ must be negative by the original assumption, and whenever this assumption is violated we may conclude that such limit cycle with the half period μT cannot be sustained in that system.

This conclusion will be illustrated for first-order and second-order plants in Examples 3.1 and 3.2.

Example 3.1

When the plant transfer function is first order, given by

$$KG(s) = \frac{K}{s+b} \quad (3.5)$$

The response $c_{\mu-1, s}$ at the margin of the first half wave is similarly obtained. Letting $i = \mu-1$ in Eq. (2.19)

$$\begin{aligned} \gamma(l+\mu) &= +1 & \text{for } l = 0 \\ &= -1 & \text{for } l = 1, 2, \dots, \mu-1 \end{aligned} \quad (3.6)$$

and

$$c_0, c_1, \dots, c_{\mu-1} < 0$$

Hence Eq. (3.4) becomes

$$\begin{aligned} c_{\mu-1, s} &= -\frac{1}{2\pi j} \int_{\Gamma} \frac{e^{\mu p T} KG(p)}{1 + e^{\mu p T}} \frac{(1 - e^{-pT})}{p} \left(1 - \sum_{l=1}^{\mu-1} e^{-l p T} \right) dp \\ &= \frac{K}{b} \frac{(1 - e^{-bT})}{1 + e^{-\mu b T}} [1 + e^{-bT} + \dots + e^{-(\mu-2)bT} - e^{-(\mu-1)bT}] \quad (3.8) \end{aligned}$$

It is clearly observed that $c_{\mu-1, s}$ is positive for all $\mu \geq 2$, violating the original assumption of Eq. (3.7). Therefore it is concluded that the first-order relay sampled-data system can maintain only the relay mode oscillation of two sampling periods ($\mu = 1$) and that the longer period oscillation cannot be sustained in that system. The above statement is true for any plant gain K , plant time-constant b , and sampling period T .

Example 3.2

Next the case of the second-order system with an integrator will be discussed. The plant transfer-function is given by Eq. (2.19).

Then, letting all $h_{l+1} \rightarrow T$ in Eq. (2.15) of Example 2.1 yields a set of solutions $c_{0s}, c_{1s}, \dots, c_{\mu-1, s}$ for the responses at all sampling instants.

$$c_{is} = - \frac{K}{b} \left\{ \sum_{l=0}^{\mu-i-1} \left[\frac{T}{2} - \frac{e^{-(\mu-l-1)bT}(1 - e^{-bT})}{b(1 + e^{-\mu bT})} \right] \right. \\ \left. - \sum_{l=\mu-i}^{\mu-1} \left[\frac{T}{2} - \frac{e^{-(\mu-l-1)bT}(1 - e^{-bT})}{b(1 + e^{-\mu bT})} \right] \right\} \quad (3.9)$$

For $\mu = 1$ and $i = \mu - 1 = 0$

$$c_{0s} = - \frac{K}{b} \left[\frac{T}{2} - \frac{1 - e^{-bT}}{b(1 + e^{-bT})} \right] = - \frac{K}{2b^2} \cdot \frac{bT(1 + e^{-bT}) - 2(1 - e^{-bT})}{1 + e^{-bT}} \quad (3.10)$$

It can be easily proved that the numerator inside the bracket is positive for all $bT \rightarrow 0$. Hence it has been shown that c_{0s} is negative for all T .

Then for $\mu = 2$ and $i = \mu - 1 = 1$

$$c_{1s} = - \frac{K}{b^2} \frac{(1 - e^{-bT})^2}{1 + e^{-2bT}} < 0 \quad (3.11)$$

Therefore, c_{1s} is negative for any sampling period T .

Next for $\mu = 3$ and $i = \mu - 1 = 2$ in Eq. (3.9)

$$c_{2s} = \frac{K}{b} \left[\frac{T}{2} - \frac{(1 - e^{-bT})}{b(1 + e^{-3bT})} (1 + e^{-bT} - e^{-2bT}) \right] \quad (3.12)$$

Simple trial will show that c_{2s} is negative for small T , but when T becomes larger c_{2s} will become positive, violating the original assumption. Thus we can find a certain critical T_c by solving Eq. (3.12) for T when c_{2s} is equated to zero. Then the oscillation of $\mu = 3$ or $M = 6$ can exist for $T < T_c$ but it cannot be sustained for $T > T_c$.

A similar phenomenon is observed for larger μ . Here we may derive an equation which yields the relation between the maximum μ and the sampling period T . Letting $i = \mu - 1$ in Eq. (3.9),

$$\begin{aligned}
c_{\mu-1, s} &= -\frac{K}{b} \left\{ \frac{T}{2} - \frac{e^{-(\mu-1)bT}(1 - e^{-bT})}{b(1 + e^{-\mu bT})} \right. \\
&\quad \left. - \sum_{l=1}^{\mu-1} \left[\frac{T}{2} - \frac{e^{-(\mu-l-1)bT}(1 - e^{-bT})}{b(1 + e^{-\mu bT})} \right] \right\} \\
&= \frac{K}{b} \left\{ \frac{bT}{2} (\mu - 2) - e^{bT} \cdot \frac{1 - e^{-\mu bT}}{1 + e^{-\mu bT}} + e^{bT} - 1 \right\} \quad (3.13)
\end{aligned}$$

By the original assumption, $c_{\mu-1, s} < 0$. Hence $\mu_{\max}$ is obtained by finding the maximum μ which satisfies the following inequality

$$\frac{bT}{2} (\mu - 2) - e^{bT} \frac{1 - e^{-\mu bT}}{1 + e^{-\mu bT}} + e^{bT} - 1 < 0 \quad (3.14)$$

Thus we are led to the following interesting conclusion, that the limit cycle of two sampling periods and four sampling periods can exist in the relay sampled-data system for any sampling period T , but that for $\mu \geq 3$ the maximum number of sampling periods which can be contained in one limit cycle is restricted by certain conditions that are specified by the system constant b and sampling period T , as shown in Eq. (3.14). $\mu_{\max}$ as a function of the sampling period T is plotted on Fig. 6.

The discussion and conclusion so far can directly be applied to the relay mode (saturated) oscillation in PWM systems. The fact that there exists a limit on the period of the limit cycle for the relay mode oscillation must be remembered and this will be very useful in stability discussion of PWM systems that will be given later.

3.2 Gain Boundary of Relay Mode Oscillations in PWM Systems

It is observed in Section IV of reference 1 that the oscillation will be shifted from the PWM mode oscillation to the relay mode oscillation when the over-all gain aK is increased, because of the saturating characteristics of pulse width modulator. The critical gain between these two modes is denoted as aK_c and is obtained for the limit cycle of two sampling periods in Section IV of reference 1. It is not a difficult problem to obtain the critical gain for longer periods,

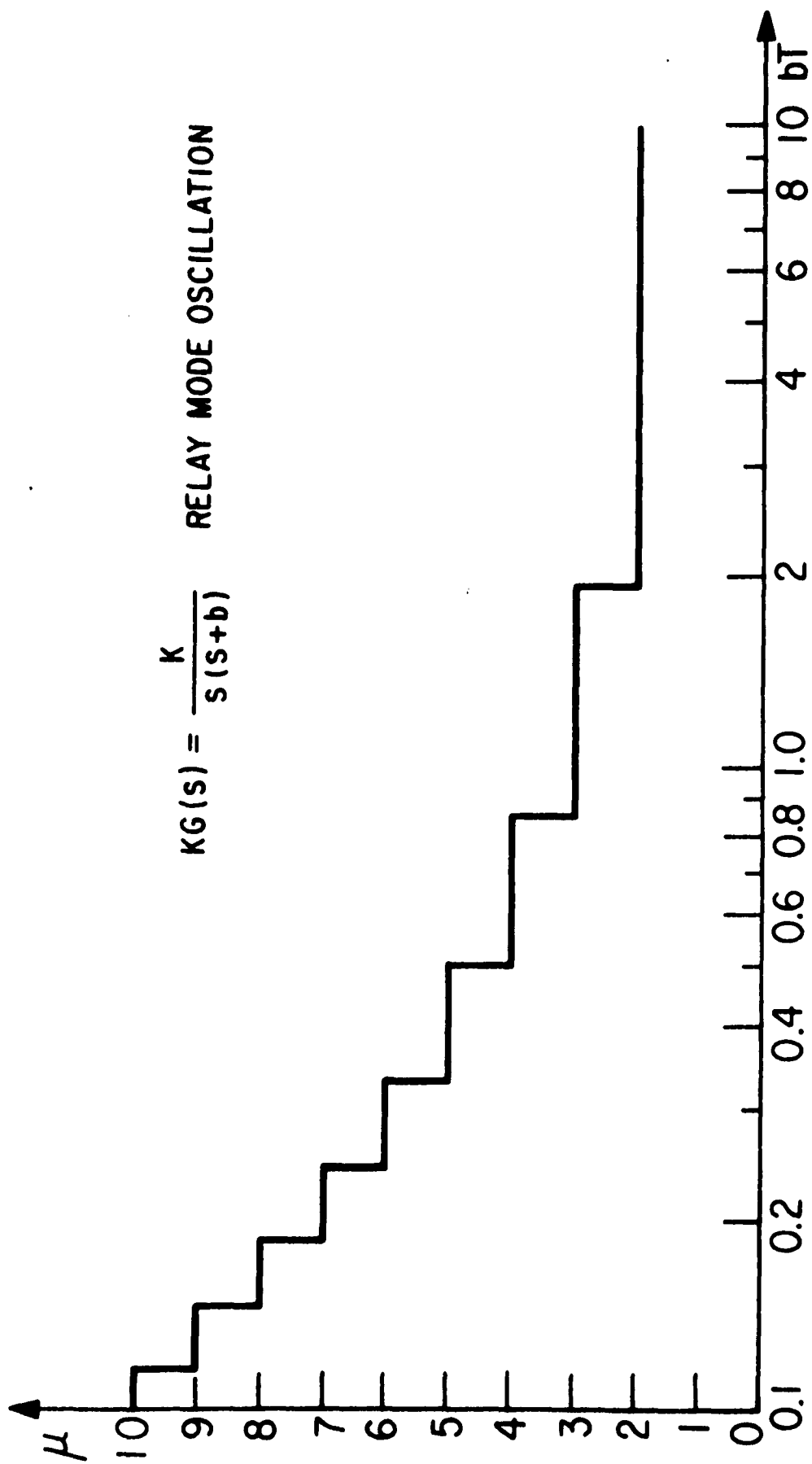


FIG. 6. MAXIMUM NUMBER OF SAMPLING PERIODS CONTAINED IN
HALF PERIOD OF LIMIT CYCLE

since we have already derived the general equation for the PWM mode oscillation in the previous section. The transition will occur when all pulses h_i of each sampling period reach saturation and fill every sampling period of limit cycle. However, we have derived the equation of c_{is} as a function of aK in Eq. (2.13) and we know for the autonomous system that

$$h_i = -a c_{is} \quad i = 0, 1, \dots, \mu-1 \quad (3.15)$$

when the period of the limit cycle is chosen as $2\mu T$. Then the critical gain aK_c is given as such a gain that every h_i becomes simultaneously equal to T on both sides of Eq. (2.13); thus the sum of h_i for the half period becomes equal to the half period μT .

Combining Eqs. (2.13) and (3.15) and letting $h_i \rightarrow T$ yields

$$\begin{aligned} \mu T &= \lim_{h_i \rightarrow T} \sum_{i=0}^{\mu-1} h_i = \lim_{h_i \rightarrow T} \left(-a \sum_{i=0}^{\mu-1} c_{is} \right) \\ &= \frac{aK}{2\pi j} \sum_{i=0}^{\mu-1} \int_{\Gamma} \frac{e^{\mu p T} G(p) \sum_{l=0}^{\mu-1} \gamma(l+1) e^{-l p T} (1 - e^{-p T})}{(1 + e^{\mu p T}) p} dp \\ &= \frac{aK}{2\pi j} \int_{\Gamma} \frac{G(p)(1 - e^{p T}) \left(- \sum_{i=0}^{\mu-1} \sum_{l=0}^{\mu-1} \gamma(l+1) e^{(\mu-l-1)p T} \right)}{(1 + e^{\mu p T}) p} dp \end{aligned} \quad (3.16)$$

Taking out the terms under the double summation inside the bracket of the integrand and denoting this as $f(\mu, pT)$, we have

$$\begin{aligned} f(\mu, pT) &= - \sum_{i=0}^{\mu-1} \sum_{l=0}^{\mu-1} \gamma(l+1) e^{(\mu-l-1)p T} \\ &= - \sum_{l=0}^{\mu-1} e^{(\mu-l-1)p T} \sum_{i=0}^{\mu-1} \gamma(l+1) \end{aligned} \quad (3.17)$$

Actual values of $\gamma(l+1)$ are given by Eq. (2.19). Substituting their values into the above yields the general form of $f(\mu, pT)$ as follows

$$f(\mu, pT) = \mu - 2 + (\mu - 4)e^{pT} + (\mu - 6)e^{2pT} + \dots - (\mu - 2)e^{(\mu-2)pT} - \mu e^{(\mu-1)pT} \quad (3.18)$$

Denoting the gain which satisfies Eq. (3.15) as $\alpha K_g(\mu)$ it is given by

$$\alpha K_g(\mu) = \frac{\mu T}{\frac{1}{2\pi j} \int_{\Gamma} \frac{G(p)(1 - e^{pT})f(\mu, pT)}{p(1 + e^{\mu pT})} dp} \quad (3.19)$$

It must be pointed out that the assumption of the simultaneous saturation of the pulse width h_1 is not practical except when $\mu = 1$, since the wave shape of the response may become somewhat like a sine wave, but may not appear like a square wave. Therefore most of $|c_{10}|$ becomes larger than T when all the pulse widths are saturated, so the left hand side of Eq. (3.16) may be larger than μT .

Hence, $\alpha K_g(\mu)$ which is derived from Eq. (3.16) must be regarded as the lowest gain that might allow the existence of the specified saturated oscillation; it gives the sufficient condition for the non-existence of the specified saturated oscillation. Also it may be regarded as the critical gain when the square wave assumption is introduced as explained later, then it gives approximately the gain when the average of $|c_{10}|$ is equal to T . The above equation, Eq. (3.19), shall be applied for the cases of the first- and second-order systems in Examples 3.3 and 3.4.

Example 3.3

In the first place, when the transfer function of the plant is the first order, given by Eq. (3.5), and when $\mu = 1$, αK_g is easily calculated as follows:

$$f(\mu, pT) = -1 \quad \text{for } \mu = 1 \quad (3.20)$$

Hence, by Eq. (3.19)

$$\alpha K_g = \frac{T}{\frac{1}{2\pi j} \int_{\Gamma} \frac{1 - e^{pT}}{(1 + e^{pT})^p} \frac{-1}{p+b} dp} = bT \frac{1 + e^{-bT}}{1 - e^{-bT}} \quad (3.21)$$

Example 3.4

When the plant is second order, given by Eq. (2.14), the integral of Eq. (3.19) is first evaluated as follows

$$\begin{aligned} \frac{1}{2\pi j} \int_{\Gamma} \frac{(1 - e^{pT}) f(\mu, pT)}{1 + e^{\mu pT}} \frac{dp}{p^2(p+b)} \\ = \frac{1}{2b^2} \left\{ b_{\mu} T + \frac{2(1 - e^{-bT}) f(\mu, -bT)}{1 + e^{-\mu bT}} \right\} \end{aligned} \quad (3.22)$$

where $f(\mu, -bT)$ is obtained by substituting $-bT$ for p in Eq. (3.18). Substituting this into Eq. (3.19) yields αK_g as a function of μ :

$$\alpha K_g(\mu) = \frac{2b^2_{\mu} T(1 + e^{-\mu bT})}{b_{\mu} T(1 + e^{-\mu bT}) + 2(1 - e^{-bT}) f(\mu, -bT)} \quad (3.23)$$

αK_g is actually calculated for $\mu = 1, 2, 3, 4$ as follows

$$\alpha K_g(1) = \frac{2b^2 T(1 + e^{-bT})}{bT(1 + e^{-bT}) - 2(1 - e^{-bT})} \quad (3.24)$$

$$\alpha K_g(2) = \frac{b^2 T(1 + e^{-2bT})}{bT(1 + e^{-2bT}) - e^{-bT}(1 - e^{-bT})} \quad (3.25)$$

$$\alpha K_g(3) = \frac{b^2 T(1 + e^{-3bT})}{3bT(1 + e^{-3bT}) + 2(1 - e^{-bT})(1 - e^{-bT} - 3e^{-2bT})} \quad (3.26)$$

$$\alpha K_g(4) = \frac{4b^2 T(1 + e^{-4bT})}{2bT(1 + e^{-4bT}) + 2(1 - e^{-bT})(1 - e^{-2bT} - 2e^{-3bT})} \quad (3.26a)$$

It was found in the previous section that a limit exists for the longest period of the relay mode oscillation in the PWM system as well as in the relay system.

When this restriction is combined with the results about the critical gain for the relay mode oscillation we reach an important conclusion concerning the stability boundary of the relay mode oscillation.

Since we know how to find $\mu_{\max}$ for the relay mode oscillation and also the boundary gain corresponding to each μ , we can find out the lowest gain for any relay mode oscillation by inspecting every αK_g as a function of μ , where μ ranges from 1 to $\mu_{\max}$. We may conclude that no relay mode oscillation can exist below that gain, since we have covered all the possible modes of relay oscillation.

This lowest gain $\min(\alpha K_g)$ is the stability boundary with respect to the relay mode oscillation, and it is clearly a sufficient condition for the non-existence of that type of oscillation.

This derivation of the stability boundary for the relay mode oscillation shall be illustrated on the first-order and second-order plants in the following part.

In case of the first-order plant, it has been shown that only the limit cycle of two sampling periods, i. e., $\mu = 1$, can exist, and we derived the critical gain αK_g for this mode in Eq. (3.21). Hence this αK_g is the stability boundary for the relay mode oscillation. When the gain is lower than this αK_g the system is stable as far as the relay mode oscillation is concerned. However, it is pointed out in Chapter II that this αK_g exactly coincides with the absolute stability boundary derived by Kadota⁹ using Lyapunov's second method.

Therefore, the question will arise whether we may take the gain boundary for relay mode oscillation as the absolute stability boundary or not, and if the answer is affirmative, how we can justify it. This question will be answered in the following part.

Next, the second-order plant will be discussed. In this case, the problem is not so simple as in the first-order system, since we found that

the limit cycles of two and four sampling periods always exist for any T and the longer period oscillation can exist when T becomes shorter. However, we can prove fortunately that $\alpha K_g(\mu)$ is the monotonically decreasing function of μ , hence the lowest gain boundary can be obtained only by calculating αK_g for $\mu_{\max}$ which is specified by Eq. (3.14) for a given T . The proof of the decreasing characteristic of αK_g is given in Appendix B. Thus, the steps to follow in finding the stability boundary are first, find $\mu_{\max}$ by Eq. (3.14) for a specified T , then find αK_g corresponding to this $\mu_{\max}$ by Eq. (3.24); ultimately this $\alpha K_g(\mu_{\max})$ is the stability boundary for the relay mode oscillation for that T . The curve of this boundary is plotted in Fig. 7.

3.3 Gain Boundary of Unsaturated Oscillations

In the previous sections, we have derived the sufficient condition for non-existence of relay (saturated) mode oscillation. Extension of the same technique to the unsaturated oscillation is attempted in this section. The problem is not so simple in this case because the pulse width may take any value between zero and T and is not fixed at T as in the saturated oscillation. The derivation of the equation which gives the longest period for the unsaturated oscillation as well as the equation which yields the boundary gain for such oscillation may become prohibitively complicated, since each pulse width at each sampling instant differs from the other and they cannot be easily calculated.

However, by introducing the sine wave approximation or square wave approximation we can eliminate such complexity to some extent and can follow almost the same steps as in the case of saturated oscillations to reach the gain boundary of unsaturated oscillation.

We consider the square wave approximation in which we assume that the pulse widths are constant and are equal to h , which may be considered as the average value of pulse widths during one period of limit cycle.

In that case, letting

$$h_{l+i} = h$$

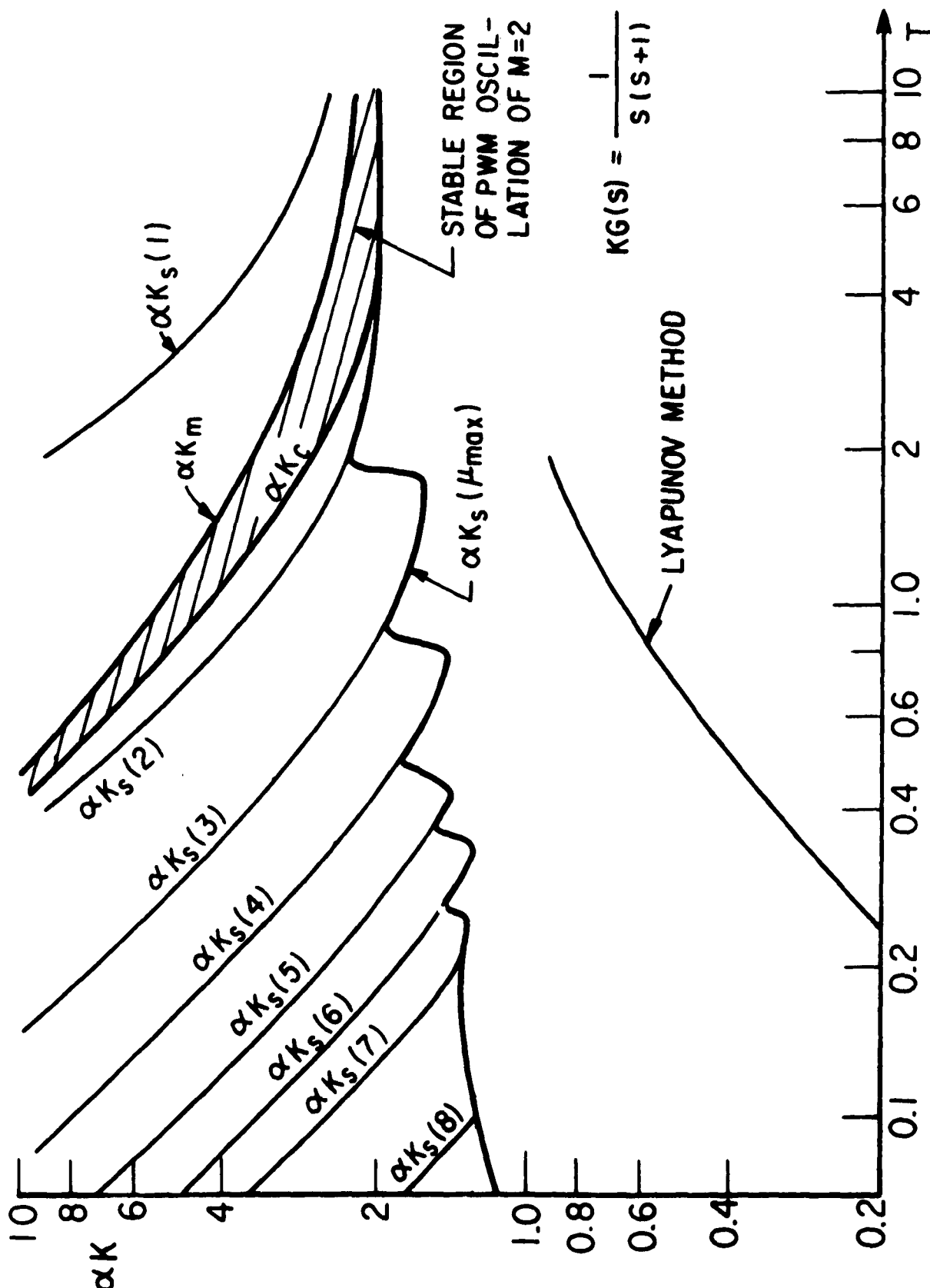


FIG 7 GAIN BOUNDARY FOR SATURATED OSCILLATION AND STABLE REGION OF PWM OSCILLATION OF $M = 2$

in Eq. (2.13) yields

$$c_{is} = \frac{1}{2\pi j} \int_{\Gamma} \frac{e^{\mu p T} K G(p)}{1 + e^{\mu p T}} \frac{1 - e^{-h p}}{p} \sum_{l=0}^{\mu-1} \gamma(l+\mu-1) e^{-l p T} dp \quad (3.27)$$

Then, we can find $\mu_{\max}$ which gives the longest period admitted for the unsaturated oscillation by testing the polarity of $c_{\mu-1, s}$ and using $\mu_{\max}$.

The gain boundary is given as follows by modifying Eq. (3.19).

$$a K_g(\mu, h) = \frac{\mu h}{\frac{1}{2\pi j} \int_p \frac{G(p) e^{p(T-h)} (1 - e^{-bh}) \gamma(\mu, pT)}{p(1 + e^{\mu p T})} dp} \quad (3.28)$$

The applications of these two equations shall be illustrated on the first order and second order systems.

Example 3.5

In case of the first order system, the equation to give $c_{\mu-1, s}$ becomes,

$$c_{\mu-1, s} = \frac{K}{b} \cdot \frac{e^{-b(T-h)} (1 - e^{-bh})}{1 + e^{-\mu b T}} [1 + e^{-bT} + \dots + e^{-(\mu-2)bT} \cdot e^{-(\mu-1)bT}] \quad (3.29)$$

Observing Eq. (3.29), the same conclusion is derived as in the saturated oscillation that the longest period of the unsaturated oscillation is $2T$, and the oscillation of $\mu \geq 2$ cannot be sustained in the first order system with the square wave approximation.

Hence we find, for unsaturated oscillation, that

$$\mu_{\max} = 1$$

The corresponding gain can be easily calculated from Eq. (3.28) as follows

$$a K_g(h) = bh \frac{1 + e^{bT}}{e^{bh} - 1} = a K_c \quad (3.30)$$

This coincides with the critical gain of the PWM oscillation as derived in the Section IV of reference 1

Example 3.6

The second order system is discussed. Under the square wave assumption, the equation for $c_{\mu-1, s}$ is easily derived from Eq. (3.27) as follows

$$c_{\mu-1, s} = \frac{K}{b^2} \left[\frac{bh}{2} (\mu-2) - \frac{e^{bh} - 1}{1 - e^{-bT}} \cdot \frac{1 - e^{-\mu bT}}{1 + e^{-\mu bT}} + e^{bh} - 1 \right]. \quad (3.31)$$

Then the maximum number of samples which can be contained in one half of the period of limit cycle of unsaturated oscillation is obtained by finding the maximum μ which satisfies the following inequality

$$\frac{bh}{2} (\mu-2) - \frac{e^{bh} - 1}{1 - e^{-bT}} \cdot \frac{1 - e^{-\mu bT}}{1 + e^{-\mu bT}} + e^{bh} - 1 < 0 \quad (3.32)$$

Then $\mu_{\max}(h)$ are calculated as a function of T and h . In Appendix C, the proof is given to show that $\mu_{\max}$ for an unsaturated oscillation is always equal to or smaller than $\mu_{\max}$ of saturated oscillation.

$$\mu_{\max}(h) \leq \mu_{\max}(T) \quad (3.33)$$

This relation will be very useful for the analysis of unsaturated oscillation as well as for the derivation of the gain boundary of such oscillations. The gain boundary of the unsaturated oscillation is given by $aK_g(h)$ of Eq. (3.28) when μ takes the value of $\mu_{\max}(h)$ obtained above. Thus

$$aK_g(\mu, h) = \frac{2b^2 \mu h (1 + e^{-\mu bT})}{b \mu h (1 + e^{-\mu bT}) + 2e^{-bT} (e^{bh} - 1) K(\mu, -bT)} \quad (3.34)$$

where $\mu = \mu_{\max}(h)$ obtained from Eq. (3.32).

The values of $aK_g(h)$ are calculated for various h and T and shown in Fig. (8). It is noticed that $aK_g(h)$ is very close to $aK_g(T)$ or slightly above $aK_g(T)$, as seen from the same figure.

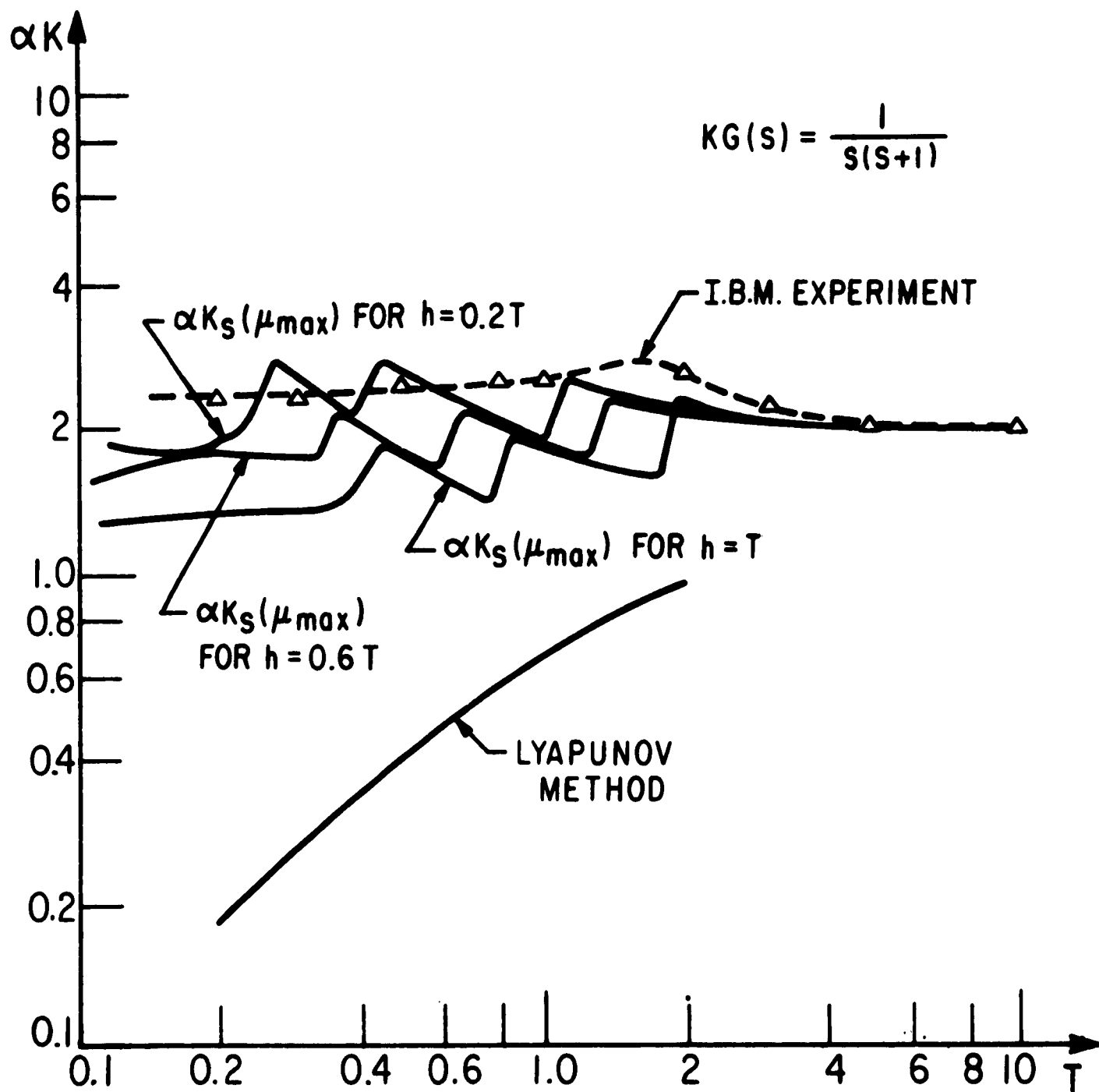


FIG.8 $\alpha K_S(\mu_{\max})$ FOR SATURATED AND UNSATURATED OSCILLATION AND EXPERIMENTAL STABILITY BY I.B.M. 704

It can be observed by comparing Eq. (B.1) and (B.7) of Appendix B that $a K_g(\mu, T)$ is smaller than $a K_g(\mu, h)$ as long as $D(\mu)$ is positive.

$$a K_g(\mu, T) \leq a K_g(\mu, h) \quad (3.35)$$

if $D(\mu) \geq 0$

Then using the relation of Eq. (C.1) that has been proved in Appendix C, the following relation can be derived.

$$a K_g(\mu_{\max}(T), T) \leq a K_g(\mu_{\max}(h), T) \leq a K_g(\mu_{\max}(h), h) \quad (3.36)$$

for $D(\mu) \geq 0$

$D(\mu)$ is positive for $bT \leq 1.9$.

Therefore $a K_g(\mu_{\max}(T), T)$ is the lowest gain boundary for the existence of saturated and unsaturated oscillation.

When $bT > 1.9$, $D(\mu) < 0$

hence,

$$a K_g(\mu, T) \geq a K_g(\mu, h)$$

And the smallest of $a K_g(\mu, h)$ is obtained by letting $h = 0$ in Eq. (3.34). Eventually it is reduced to $2b$. Therefore $a K = 2b$ yields the lowest gain boundary for $bT > 1.9$. Combination of these two boundary curves presents the sufficient condition for non-existence of saturated and unsaturated oscillation of PWM systems. This is plotted in Fig. (8) and we can observe that it is very close to the experimental data obtained by IBM 704. We mentioned that the pulse width of square wave approximation can be regarded as the average value of the pulse width during one period of limit cycle. However, another approach to this problem is possible and it is explained in the following. When we find the maximum pulse-width $h_{\max}$ and minimum pulse-width $h_{\min}$ during one period of limit cycle, we can calculate the corresponding $a K_g(h_{\max})$ and $a K_g(h_{\min})$.

The actual gain boundary corresponding to the actual wave shape must lie in-between these two gains. But, we know that $\mu_{\max}$ is an increasing function of h and $a K_g(\mu, h)$ is an decreasing function of μ as well as of h . Therefore

$$\mu_{\max}(h_{\max}) > \mu_{\max}(h_{\min})$$

and

$$\alpha K_g(\mu, h_{\max}) < \alpha K_g(\mu, h_{\min})$$

$$\alpha K_g(\mu_1, h) < \alpha K_g(\mu_2, h)$$

$$\text{for } \mu_1 > \mu_2 \text{ and } D(\mu) > 0$$

Combining these properties we may conclude that αK_g corresponding to $h_{\max}$ is smaller than the one corresponding to $h_{\min}$ for $bT < 1.9$. Hence we take $\alpha K_g(h_{\max})$ as the lowest gain for the existence of the unsaturated oscillation of which maximum pulse-width is $h_{\max}$. Actual gain must be above this value. When $bT > 1.9$, we take $\alpha K = 2b$ as the lowest gain.

3.4 Extension of Theories to Other Types of Nonlinearities.

The noticeable result which has been derived in this chapter is that the method to obtain $\mu_{\max}$ and the corresponding αK_g can be extended to other types of nonlinearities.

Whether the nonlinearity is a saturating amplifier with linear region or a quantised level amplifier does not matter, provided that they have a complete saturation as shown in Fig. 3. We can extend our theories to the analysis of such systems

The limitation on the longest period of limit cycles for such nonlinear systems is exactly identical with the one for the PWM system, because they will behave just like a relay system when the oscillation remains in the completely saturated region. Therefore, the equation to find $\mu_{\max}$ given by Eq. (3.8) and Eq. (3.14), is valid without any change for the nonlinearities which have the completely saturated region.

We need a slight modification on the equation that gives the gain boundary of saturated mode oscillation.

Let α be the tangent of the inclination of the line which combines the origin and the edge of the saturated region. Then, as far as the saturated region is concerned, we have the following input-output relations of nonlinear component,

$$\begin{aligned}
e'_n(t) &= 1 & 1 < a e_n(0) \\
e'_n(t) &= N[e_n(0)] & \text{if } -1 < a e_n(0) < 1 \\
&= -1 & \text{if } a e_n(0) < -1
\end{aligned} \tag{3.37}$$

$N [\]$ in the above equation indicates the particular nonlinear amplitude-depending-function which can be specified for each case of nonlinearities.

In case of such amplitude-dependent nonlinear functions, Eq. (2.5) or (2.9) can still be used to obtain the responses, and the output of the nonlinear component $E'_{nh}(s)$ is given as follows.

$$E'_{nh}(s) = N(e_n(0)) \frac{1 - e^{-Ts}}{s} \tag{3.38}$$

Substituting this into Eq. (2.68) yields

$$c_{is} = - \frac{1}{2\pi j} \int_p \frac{e^{\mu p T} K G(s)}{1 + e^{\mu p T}} \sum_{l=0}^{\mu-1} \frac{N(e_{l+i}(0)) e^{-l p T} (1 - e^{-p T})}{p} dp \tag{3.39}$$

When the oscillation is in the completely saturated mode, the nonlinear gain factor given by $N(\)$ will become equal to 1 or -1 as shown in Eq. (3.37). Hence, the equation which yields the responses at the sampling instants becomes identical with the one for the PWM system given by Eq. (3.1). However, because of the saturation, the input to the nonlinear component during the first half period of the limit cycle becomes as follows.

$$-a c_{is} \geq 1 \quad \text{for } i = 0, 1, \dots, \mu-1 \tag{3.40}$$

Substituting Eq. (3.39) into the above equation and obtaining its sum for it from 0 to $\mu-1$ yields

$$\begin{aligned}
\mu &\leq - \left(\sum_{i=0}^{\mu-1} a c_{is} \right) \\
&= \frac{a K}{2\pi j} \int_{\Gamma} \frac{G(p)(1 - e^{p T}) \mu(\mu, p T)}{p(1 + e^{\mu p T})} dp \tag{3.41}
\end{aligned}$$

Hence the gain boundary for saturated mode oscillation becomes

$$aK_g = \frac{1}{2\pi j} \int_{\Gamma} \frac{G(p)(1 - e^{pT})f(\mu, pT)}{p(1 + e^{\mu pT})} dp \quad (3.42)$$

It is easily observed by comparing the above equation with Eq. (3.19) for PWM systems that aK_g for amplitude-dependent nonlinearities can be obtained by multiplying $1/T$ to aK_g for the PWM system.

This multiplied factor $1/T$ gives an inclining characteristic to aK_g as shown in Fig. (40), different from the flat characteristic of aK_g for the PWM system which is shown in Fig. (7). $aK_g(\mu_{\max})$ for the saturating gain amplifier as well as for the quantized level amplifier are plotted in Fig. (10) and (11). They give sufficiently close gain boundaries to the experimental data on IBM 704.

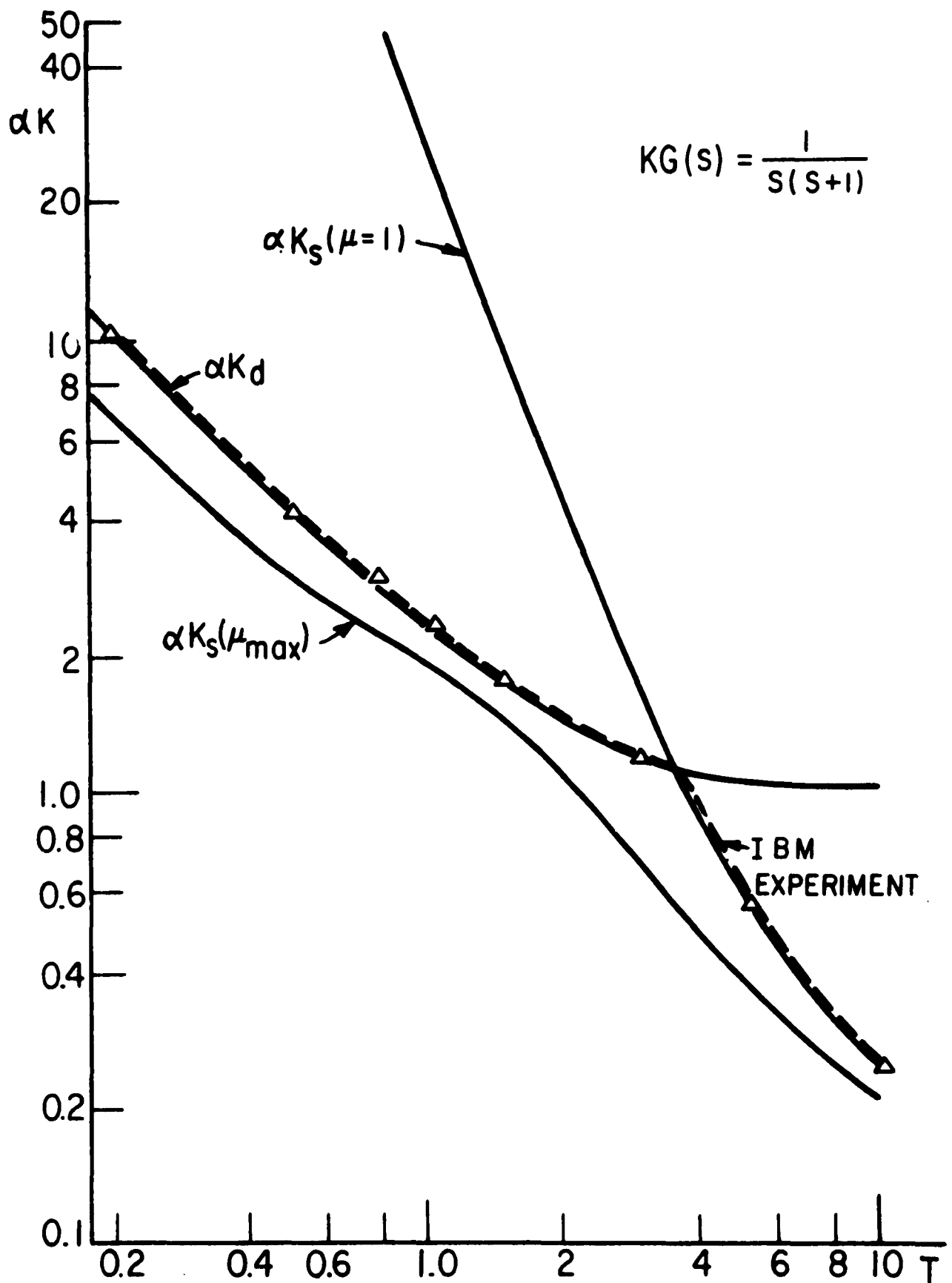


FIG.10 $\alpha K_S(\mu=1)$, αK_d , AND $\alpha K_S(\mu_{max})$ FOR SATURATING AMPLIFIER WITH LINEAR ZONE AND STABILITY BOUNDARY BY IBM 704

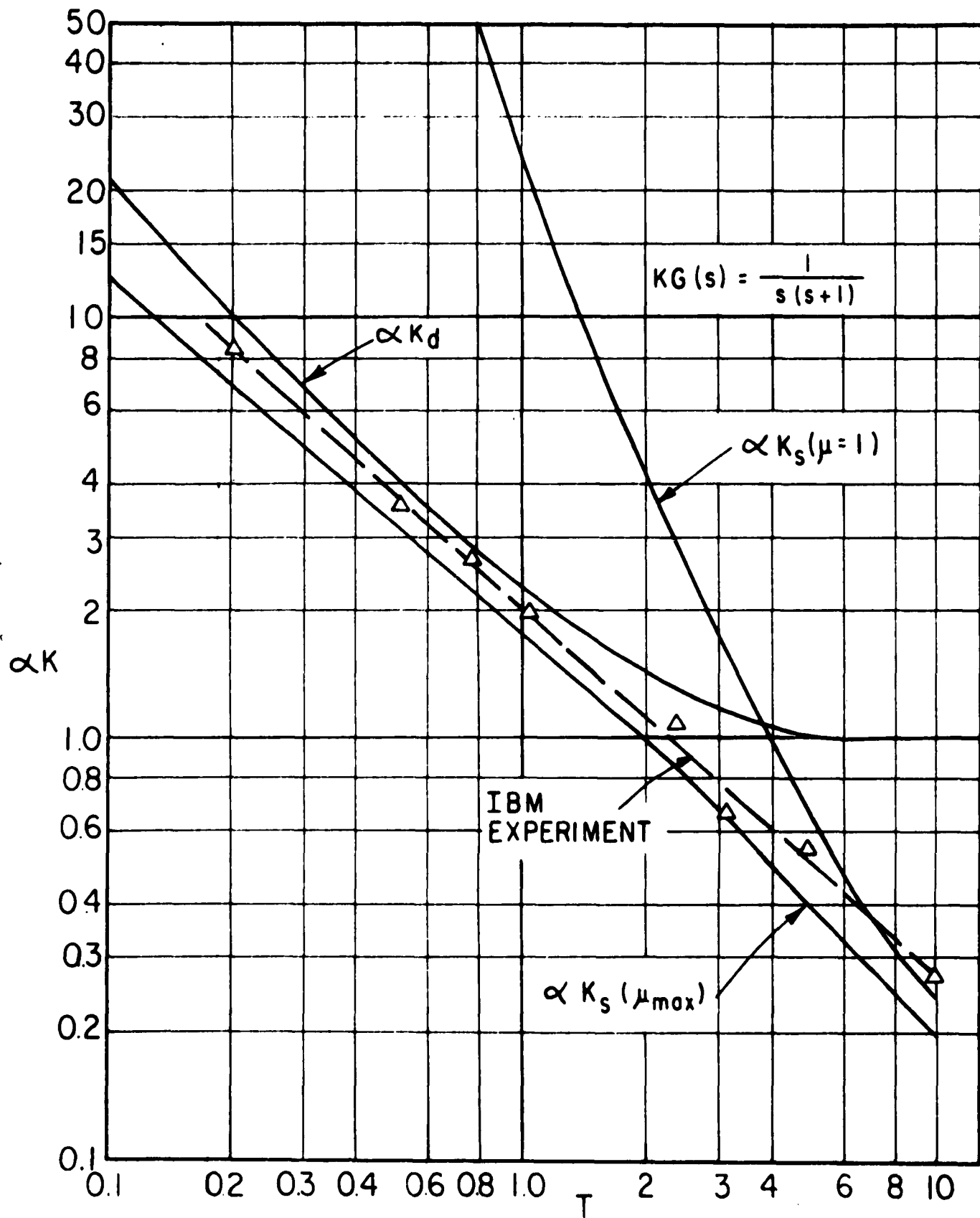


FIG. 11 $\alpha K_s(\mu=1)$, αK_d , AND $\alpha K_s(\mu_{max})$ FOR
 QUANTIZED LEVEL AMPLIFIER AND
 STABILITY BOUNDARY BY I B M 704

CHAPTER IV

Stability of the Equilibrium Point

It was shown in reference 8 that the stability of the limit cycle is tested by examining the eigenvalues of the characteristic matrix, which consists of the first partial derivatives of the variables. Also the equilibrium point can be regarded as the limit cycle of one sampling period.

The basic steps to test the stability of limit cycles which are given in reference 8 were presented in Chapter I. The basic assumption of this linearisation of the nonlinear difference equation is that the perturbation around the limit cycle or the equilibrium point is sufficiently small. However, the actual limitation on the magnitude of the perturbation which permits such approximation is not specified explicitly. We will indicate that such approximation becomes valid by evaluating the error which is brought about by linear approximation.

We will derive the total linearisation technique for that purpose in general form. It is known that the difference equations of the response and its derivatives for linear sampled-data systems are written in the following matrix form. ²¹

$$c_{n+1}^{(j)} = [Q_i^{(j)}] c_n^{(i)} + Q_{rn}^{(j)} \quad j = 0, 1, \dots, q-1 \quad (4.1)$$

where $Q_i^{(j)}$ consists of system constants and $Q_{rn}^{(j)}$ represents the effect of the input, and q is the order of the linear plant. Similar representation is possible for nonlinear sampled-data systems, using a vector, X_n , for the system variables and a nonlinear vector $N(X_n)$ for the output from the nonlinear component.

$$X_{n+1} = [Q] X_n + N(X_n) \quad (4.2)$$

and it is assumed that X_n and X_{n+1} is on the equilibrium point, S_0 , which is the origin in most cases. Then if we let the components of the vector, X_n , be $x_{1n}, x_{2n}, \dots, x_{qn}$, these x_{jn} corresponds to $c_n^{(j)}$ in Eq. (4.1). Also the characteristic matrix $[Q]$ of the linear plant corresponds to $[Q_i^{(j)}]$ in Eq. (4.1).

Letting Y_n with its component $y_{1n}, y_{2n}, \dots, y_{qn}$ be the small

disturbance of X_n . Eq. (4.2) is rewritten when such disturbance Y_n is added to X_n .

$$(X_{n+1} + Y_{n+1})] = [Q](X_n + Y_n)] + N(X_n + Y_n)] \quad (4.3)$$

or

$$X_{n+1}] + Y_{n+1}] = [Q] X_n] + [Q] Y_n] + N(X_n + Y_n)] \quad (4.4)$$

However in most nonlinearities, the output of the nonlinear component is only the function of input itself, and is not the function of its derivatives. Therefore,

$$N(X_n) = N(x_{1n}) = N(-c_n) \quad (4.5)$$

In that case, all the partial derivatives of $N(X_n)$ vanish when it is differentiated by x_{jn} for $j \neq 1$. Hence $N(X_n + Y_n)$ can be expanded into powers of x_{1n} by Taylor expansion

$$N(X_n + Y_n) = N(X_n) + \frac{\partial N}{\partial x_{1n}} y_{1n} + \frac{1}{2!} \frac{\partial^2 N}{\partial x_{1n}^2} y_{1n}^2 + \dots \quad (4.6)$$

where all the partial derivatives are evaluated at $X_n = S_0$. Denoting the summation of all the terms as $N'(y_{1n})$ except for the first term in the above equation, it becomes

$$N(X_n + Y_n) = N(X_n) + N'(y_{1n}) \quad (4.7)$$

Substituting this into Eq. (4.4) and using Eq. (4.2) yields the difference equation of the perturbation around the equilibrium point.

$$Y_{n+1}] = [Q] Y_n] + N'(y_{1n})] \quad (4.8)$$

Then denoting the summation of all the terms as $N''(y_{1n})$ except for the first two terms in Eq. (4.6), we have

$$N'(y_{1n}) = \frac{\partial N}{\partial x_{1n}} y_{1n} + N''(y_{1n}) \quad (4.9)$$

Substituting this into Eq. (4.8) yields

$$\begin{aligned} Y_{n+1} &= [Q] Y_n + \left[\frac{\partial N}{\partial x_{1n}} y_{1n} \right] + N''(y_{1n}) \\ &= [A_n] Y_n + N''(y_{1n}) \end{aligned} \quad (4.10)$$

where the component in the first column of A_n is the sum of $\partial N / \partial x_{1n}$ and the corresponding term in Q , and the other components of two matrices are all identical.

$$\begin{aligned} a_{11} &= Q_{11} + \frac{\partial N_1}{\partial x_{1n}} \\ a_{ij} &= Q_{ij} \quad j = 2, \dots, q \end{aligned} \quad (4.11)$$

It is observed that $[A_n]$ given above is identical with $[A_n]$ of Eq. (1.14) in the previous section. However, Eq. (4.10) has no approximation whereas Eq. (1.14) is the first approximation of Eq. (4.10), neglecting nonlinear vector, $N''(y_{1n})$. For certain types of nonlinearities it can be proved that $N''(Y_n)$ is finite inside certain region S , which is a vector space of Y_n .

In other words, we can find such R_n and R (with their components R_{1n} and R_1 , respectively) that

$$N''_i(y_{1n}) = k_i R_{1n} y_{1n} \quad \text{for all } i \quad (4.12)$$

and

$$0 \leq R_{1n} < R_1 \quad \text{for all } n \text{ and } i \text{ if } Y_n \in S \quad (4.13)$$

R_1 is a positive finite number and so is R_{1n} . The number k with its component k_i is introduced for mathematical convenience to represent the sign and gain factor and also is a finite number. Then substituting Eq. (4.12) into Eq. (4.10) yields

$$\begin{aligned} Y_{n+1} &= [A_n] Y_n + k R_n y_{1n} \\ &= [B_n] Y_n \end{aligned} \quad (4.14)$$

where the component in the first column of B_n is the sum of the corresponding component of A_n and the component of $k R_n$, and the other components of

two matrices are all identical.

$$\begin{aligned}
 b_{i1} &= a_{i1} + k_i R_{in} \\
 &= Q_{i1} + \frac{\partial N}{\partial x_{in}} + k_i R_{in} \\
 b_{ij} &= a_{ij} = Q_{ij} \quad \text{for } j = 2, \dots, q
 \end{aligned} \tag{4.15}$$

This linearized difference equation in its form shown in Eq. (4.14') is essentially a nonlinear difference equation because the matrix $[B_n]$ is a nonlinear matrix containing the nonlinear factor, R_n . However, Eq. (4.14) is derived without introducing any approximation and is valid inside the vector space S for all n . And we know that R_n is bounded as seen in Eq. (4.13) for all n , hence, applying the stability criterion to the matrix $[B_n]$ in the same manner for the linear matrix, we may derive certain conditions for the stability of the difference equation of Eq. (4.14), using the boundedness of R_n . When the condition obtained above is satisfied, all the eigenvalues of $[B_n]$ lie inside the unit circle for all n .

Although this condition does not necessarily guarantee that any disturbance $Y_n \rightarrow 0$ as $n \rightarrow \infty$, it gives stricter restriction on the stable region than the condition on the incrementally linearized matrix $[A_n]$ which is derived in the previous section. Also, by evaluating explicitly the magnitude of the error factor kR , which is caused by the linearisation, we can estimate the range of S in which the totally linearised matrix $[B_n]$ can be regarded as a time-invariant matrix $[B]$. Within that region, the test on the single matrix $[B]$ ensures the stability of the equilibrium point. Then, if $[B_n]$ is a stable matrix for all n , i. e., if the eigenvalues of $[B_n]$ lie inside the unit circle, the equilibrium point is stable within the region that the matrix $[B_n]$ can be regarded as a constant matrix $[B]$. Such a region can be found by evaluating the magnitude of the error vector kR and by comparing it with other time invariant components of the matrix B_n .

Example 4.1

The total linearization method is illustrated on the PWM system and

its usefulness will be proved. The linear plant chosen are again first- and second-order systems.

(1) First-Order System. For the first-order system, the difference equation is

$$c_{n+1} = c_n e^{-bT} + \frac{K}{b} \gamma(n) e^{-bT} (e^{bh_{n-1}}) \quad (4.16)$$

where

$$h_n = -a \gamma(n) c_n \quad (4.17)$$

Obviously $X_n = x_{1n} = c_n$ and $Q = e^{-bT}$ and

$$N(X_n) = N(c_n) = \frac{K}{b} \gamma(n) e^{-bT} (e^{bh_{n-1}}) \quad (4.18)$$

in Eq. (4.12). The equilibrium point $S_0 = 0$. The difference equation of the perturbation y_n around the origin becomes, from Eq. (4.10) and (4.11)

$$y_{n+1} = A_n y_n + N''(y_n) \quad (4.19)$$

where

$$A_n = Q + \left. \frac{\partial N}{\partial c_n} \right|_{c_n=0} = e^{-bT} - a K e^{-bT} \quad (4.20)$$

and

$$\begin{aligned} N''(y_n) &= \sum_{i=2}^{\infty} \frac{1}{i!} \left. \frac{\partial^{(i)} N}{\partial c_n^{(i)}} \right|_{c_n=0} y_n^i \\ &= -\frac{K}{b} \gamma(n) e^{-bT} \sum_{i=2}^{\infty} \frac{1}{i!} (-a b \gamma(n) y_n)^i \end{aligned} \quad (4.21)$$

By Eq. (4.12)

$$N''(y_n) = k R_n y_n \quad (4.22)$$

where k is chosen as follows

$$k = -a K$$

Then equating Eqs. (4.21) and (4.22), we find R_n

$$R_n = e^{-bT} \sum_{i=2}^{\infty} \frac{1}{i!} (-a b \gamma(n) y_n)^{i-1} \quad (4.23)$$

It is noticed that $-\gamma(n) y_n$ is always positive because $\gamma(n) = -1$ when $y_n > 0$ and $\gamma(n) = +1$ when $y_n < 0$. Also $|a y_n|$ will not exceed the sampling period T because of saturation. Thus

$$0 \leq -a \gamma(n) y_n b \leq bT \quad (4.24)$$

Then, using Eq. (4.23) and the above result, R_n is bounded as follows:

$$\begin{aligned} 0 \leq R_n &\leq e^{-bT} \sum_{i=2}^{\infty} \frac{1}{i!} (bT)^{i-1} \\ &= e^{-bT} \frac{1}{bT} [e^{bT} - 1 - bT] \\ &= R \end{aligned} \quad (4.25)$$

The totally linearised difference equation of the perturbation around the origin is from Eq. (4.14)

$$y_{n+1} = B_n y_n \quad (4.26)$$

where B_n is obtained from Eqs. (4.15) and (4.20)

$$B_n = A_n + kR_n + (1 - aK)e^{-bT} - aKR_n \quad (4.27)$$

Eq. (4.26) is stable if

$$|B_n| < 1 \quad (4.28)$$

Substituting Eq. (4.27) into the above

$$|e^{-bT}(1 - aK) - aKR_n| < 1 \quad (4.29)$$

This is rewritten as follows:

$$(1 - e^{-bT} + aKe^{-bT} + aKR_n)(1 + e^{-bT} - aK(e^{-bT} + R_n)) > 0 \quad (4.30)$$

The content of the first bracket is positive since aK and R_n are non-negative quantities. Hence, the stability condition is reduced to

$$aK < \frac{1 + e^{-bT}}{R_n + e^{-bT}} \quad (4.31)$$

R_n is not a constant, but is restricted between 0 and R for all n by Eq. (4.25). Therefore, the maximum value of R_n is substituted into Eq. (4.31) in order to obtain the lowest boundary of aK . Thus

$$aK < \frac{1 + e^{-bT}}{R + e^{-bT}} = bT \frac{1 + e^{-bT}}{1 - e^{-bT}} \quad (4.32)$$

Various important results have been obtained concerning the stability of the first-order PWM system. Reviewing and combining these results, a conclusion of the stability of such system can be derived.

In the first place, it is proved that the longest period of relay mode oscillation is two sampling periods, $\mu=1$. In the second place, the gain boundary corresponding to $\mu = 1$ is obtained.

$$aK_g(\mu = 1) = bT \frac{1 + e^{-bT}}{1 - e^{-bT}}$$

Thus, we may state that no relay mode oscillation can exist below $aK_g(\mu = 1)$. In the third place, it has been proved that the origin is stable for any perturbation below the gain which is given above. This fact eliminates the possibility of the existence of PWM oscillations or any other irregular oscillations below $aK_g(\mu = 1)$.

By these three steps, we could have successfully eliminated the existence of all the types of oscillations below $aK_g(\mu = 1)$. On the other hand, it has been shown in the previous section that the relay mode oscillation can exist above $aK_g(\mu = 1)$ and is stable. Hence, we have shown that $aK < aK_g(\mu = 1)$ is the necessary and sufficient condition for the absolute stability of the first-order PWM system.

(2) Second-Order System

Next the case of the second-order system will be discussed. The transfer function is given by Eq. (2.14) as

$$KG(s) = \frac{K}{s(s+b)}$$

and the difference equations will be derived as follows

$$\begin{aligned} c_{n+1} &= \gamma(n) \frac{K}{b^2} (bh_n - e^{-b(T-h_n)} + e^{-bT}) + c_n + \frac{1 - e^{-bT}}{b} c_n \\ &= f(c_n, c_n) \end{aligned} \quad (4.33)$$

$$\begin{aligned} \dot{c}_{n+1} &= \gamma(n) \frac{K}{b} (e^{-b(T-h_n)} - e^{-bT}) + e^{-bT} \dot{c}_n \\ &= g(c_n, \dot{c}_n) \end{aligned} \quad (4.34)$$

Then the components matrix $[A_n]$ of Eq. (2.14) is given by Eq. (1.15).

$$[A_0] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad (4.35)$$

where

$$a_{11} = \frac{\partial f}{\partial c_n} = -\frac{aK}{b} (1 - e^{-b(T-h_n)}) + 1 \quad (4.36)$$

$$a_{12} = \frac{\partial f}{\partial \dot{c}_n} = \frac{1 - e^{-bT}}{b} \quad (4.37)$$

$$a_{21} = \frac{\partial g}{\partial c_n} = -aKe^{-b(T-h_n)} \quad (4.38)$$

$$a_{22} = \frac{\partial g}{\partial \dot{c}_n} = e^{-bT} \quad (4.39)$$

Let $x_{1n} = c_n$, $x_{2n} = \dot{c}_n$, and y_{1n} , y_{2n} be the corresponding perturbations, and the equilibrium point, S_0 , be the origin. Then $[Q] X_n$ of Eq. (4.2) represents first two terms in the right hand side of Eqs. (4.33) and (4.34) and $N(X_n)$ represents the last terms of two equations. Obviously N is solely a function of c_n , since $h_n = -a \gamma(n) c_n$. The components A_n of Eq. (4.10) are given by Eqs. (4.35) to (4.38). Choosing $k_1 = aK$, $k_2 = -aK$ in Eq. (4.12) we can find R_{1n} and R_{2n} .

$$N_1''(y_{1n}) = aKR_{1n}y_{1n} = \sum_{i=2}^{\infty} \frac{1}{i!} \left. \frac{\partial^{(i)} N_1}{\partial c_n^{(i)}} y_{1n}^i \right|_{c_n=0} \quad (4.40)$$

and

$$N_2''(y_{1n}) = -aKR_{2n}y_{1n} = \sum_{i=2}^{\infty} \frac{1}{i!} \left. \frac{\partial^{(i)} N_2}{\partial c_n^{(i)}} y_{1n}^i \right|_{c_n=0} \quad (4.41)$$

Then the limiting value of R_{1n} and R_{2n} are obtained.

From Eqs. (4.33) and (4.40) we have

$$aKR_{1n} = \frac{aKe^{-bT}}{b} \sum_{i=2}^{\infty} (-ab\gamma(n)y_{1n})^{i-1} \quad (4.42)$$

Because of the saturation,

$$0 \leq -ab\gamma(n)y_{1n} \leq bT \quad (4.43)$$

Hence

$$\begin{aligned} 0 \leq R_{1n} &\leq \frac{1}{b} e^{-bT} \sum_{i=2}^{\infty} \frac{1}{i!} (bT)^{i-1} \\ &= \frac{e^{-bT}}{b^2 T} (e^{bT} - bT - 1) \\ &= R_1 = R/b \end{aligned} \quad (4.44)$$

where R is given by Eq. (4.25). Similarly from Eqs. (4.34) and (4.41)

$$-aKR_{2n} = -aKe^{-bT} \sum_{i=2}^{\infty} \frac{1}{i!} (bT)^{i-1} \quad (4.45)$$

Hence, referring to Eq. (4.44)

$$0 \leq R_{2n} = bR_{1n} \leq R \quad (4.46)$$

The totally linearised form is given by Eq. (4.14')

$$Y_{n+1} = [B_n] Y_n \quad (4.47)$$

where the components of B_n are given by Eq. (4.15) and Eqs. (4.35) to (4.38)

$$[B_n] = \begin{bmatrix} a_{11} + aKR_{1n} & a_{12} \\ a_{21} - aKR_{2n} & a_{22} \end{bmatrix}_{c_n=0} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad (4.48)$$

$[B_n]$ is stable if the following two conditions are satisfied by its components.²²

$$I. (1 + b_{11}b_{22} - b_{12}b_{21})(1 - b_{11}b_{22} + b_{12}b_{21}) > 0 \quad (4.49)$$

$$II. (1 + b_{11}b_{22} - b_{12}b_{21} + b_{11} + b_{22})(1 + b_{11}b_{22} - b_{12}b_{21} - b_{11} - b_{22}) > 0 \quad (4.50)$$

Condition I is tested first. by using Eqs. (4.35) to (4.38) and (4.48)

$$b_{11}b_{22} - b_{12}b_{21} = e^{-bT} + aKR_{1n} \quad (4.51)$$

The content of the first bracket is clearly positive since R_{1n} is a non-negative quantity by Eq. (4.44). Hence, the content of the second bracket must be positive to satisfy Condition I.

$$1 > e^{-bT} + aKR_{1n}$$

or

$$aK < \frac{1 - e^{-bT}}{R_{1n}} \quad (4.52)$$

The above inequality is satisfied for all n , if R_{1n} is replaced by its maximum value which is given by Eq. (4.44)

$$\begin{aligned} aK < \frac{1 - e^{-bT}}{R_1} &= \frac{(1 - e^{-bT})b^2T}{1 - bTe^{-bT} - e^{-bT}} \\ &= aK_d \end{aligned} \quad (4.53)$$

Thus Condition I is satisfied for all $aK < aK_d$.

Next, Condition II is investigated. Using Eqs. (4.35) to (4.38) and (4.48)

$$\begin{aligned} 1 + b_{11}b_{22} - b_{12}b_{21} - (b_{11} + b_{22}) \\ = \frac{aK}{b} (1 - e^{-bT}) > 0 \end{aligned} \quad (4.54')$$

and, also

$$\begin{aligned} 1 + b_{11}b_{22} - b_{12}b_{21} + b_{11} + b_{22} \\ = 2(1 + e^{-bT}) - \frac{aK}{b} (1 - e^{-bT}) + 2aKR_{2n} \end{aligned} \quad (4.55)$$

and this must be positive to satisfy Condition II. Hence,

$$aK \left(\frac{1 - e^{-bT}}{b} - 2R_{2n} \right) < 2(1 + e^{-bT}) \quad (4.56)$$

This inequality is satisfied for all n when R_{2n} is replaced by its minimum value which is equal to zero by Eq. (4.46). Thus,

$$aK < 2b \frac{(1 + e^{-bT})}{1 - e^{-bT}} = aK_c \quad (4.57)$$

It is observed that this aK_c is identically equal to the lowest gain for the PWM oscillation of $M = 2$ which is obtained in reference 1. When these two conditions are combined we may state that the origin is stable if

$$\alpha K < \min(\alpha K_c, \alpha K_d)$$

and if the system remains within the region that the matrix $[B_n]$ can be approximated by a time invariant matrix $[B]$.

Thus we could have succeeded in eliminating all the regular or irregular oscillations around the origin.

In the first order system, the test on $[B_n]$ gave the absolute stability boundary αK_c .

In the second order system, we could have derived αK_d from Condition I which could not have been obtained from the incrementally linearized matrix $[A_n]$.

We have mainly discussed the saturated oscillations in Chapter III, and the small oscillation has been studied in this chapter.

Combining the results obtained from these two different approaches will give a good means of solution for the stability problem.

$\alpha K(\mu_{\max})$ and the experimental stability curve obtained by IBM 704 are shown in Fig. (8). Also $\min(\alpha K_c, \alpha K_d)$ are plotted in Fig. (19) together with the experimental curve. It can be observed that both curves give sufficiently close values to the experimental data.

The stability condition obtained by Lyapunov's method⁶ is added to these figures for the sake of comparison.

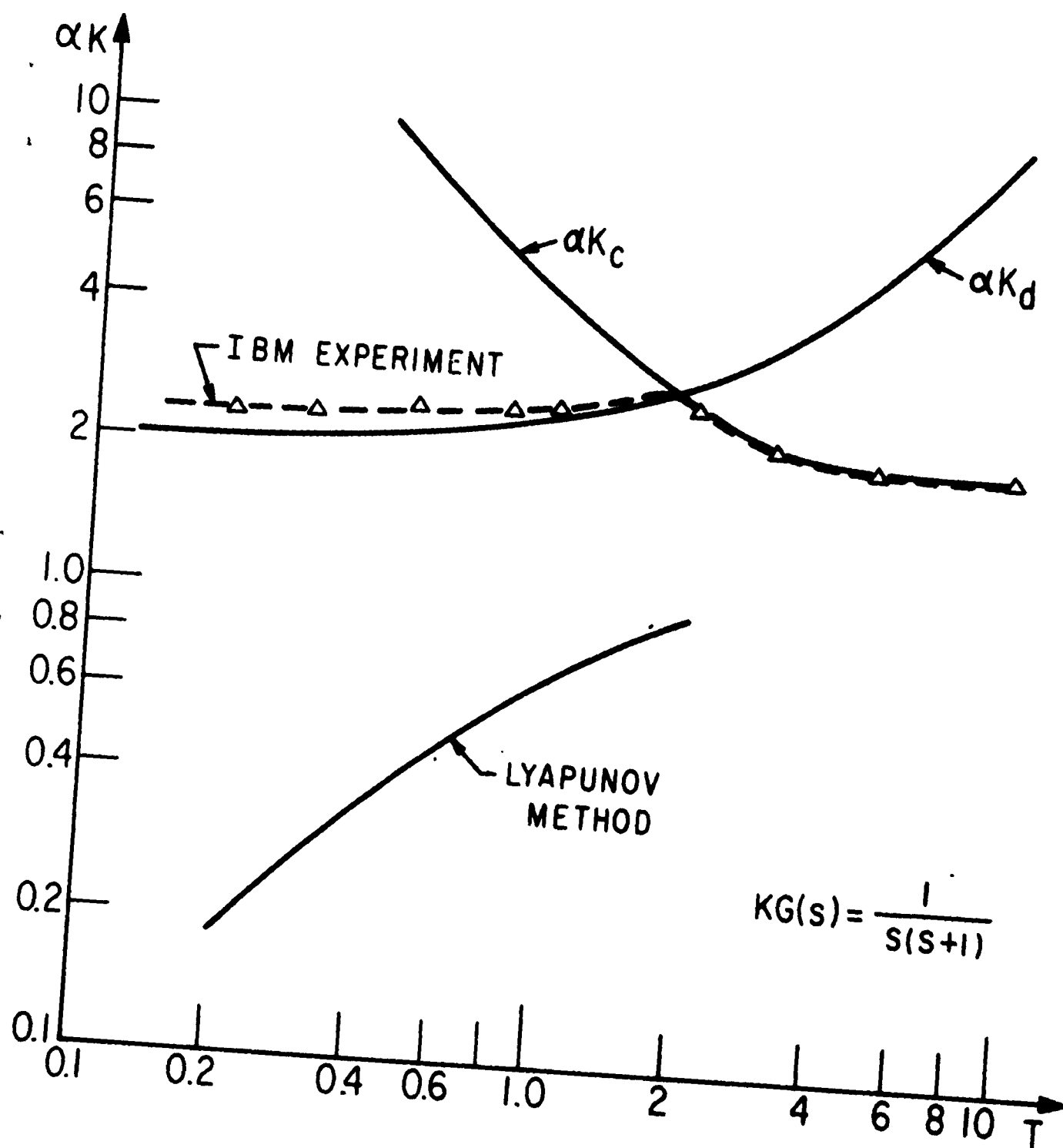


FIG. 9 αK_c , αK_d AND EXPERIMENTAL STABILITY BOUNDARY BY IBM 704

CHAPTER V

CONCLUSION

5.1 Conclusion

Periodical oscillations within nonlinear sampled-data systems, as a whole, has been investigated in this report.

The fundamental equation that gives the exact feature of limit cycles will be very useful for the precise analysis of such systems. Oscillations which are almost periodical can also be treated by this equation for an approximate description of their behavior.

Thus the fundamental equation and its modified equation can be very powerful tools in investigating the steady state of nonlinear sampled-data systems. When the periodical input is applied, the same equation yields the desired responses, and the results are superior in their precision to the one obtained by the describing-function method. They are based on the transform method, and can be applied for any order of the plant transfer-function. Moreover, they can be used for any shape of nonlinearities, not necessarily PWM, but also relay, saturating amplifier and quantised level amplifier.

Similar approaches to the problem of limit cycles are possible as shown by Shao Da Chuan¹³ by means of the canonical-form representation or by H. C. Torng^{18,19} by means of the discrete-function method. However, in these studies, setting up of state equations or difference equations for each case is always required and the size of system equations will become larger and larger when the order of the plant increases.

On the other hand, in our transform method, every equation that gives the exact feature of responses can be derived from one fundamental equation which is common to all the plants and to all the nonlinearities.

The steps that must be followed are just substituting the actual form of the plant transfer-function into the fundamental equation and giving consideration to the particular type of nonlinearities.

The stability problem was another important topic of this thesis. As was mentioned in Chapter I, our attitude towards this problem is microscopic in contrast to the macroscopic approach such as Lyapunov's second method.

Eliminating the possible existence of all saturated and unsaturated oscillations, the sufficient condition for the non-existence of such oscillations in the PWM system has been derived and is compared with the experimental results, as well as with the sufficient condition for the asymptotic stability in the large as derived by Lyapunov's second method.

The stability boundary derived by our method gives a closer criterion for the experimental results than that obtained by the Lyapunov method. The reason for this can be attributed, in the first place, to the difficulty in finding the best Lyapunov function for the specified types of nonlinearity; in the second place, to the fundamental property of such a macroscopic method wherein one must expect that the worst case might happen, without paying attention to the limitation on oscillations inside the system imposed by the operation of the nonlinear function.

We would never deprecate the approach from the macroscopic point of view. We might even expect that the straightforward method to find the best Lyapunov function to give the necessary and sufficient condition for the stability could be established. However, we believe that our localized approach is also useful in designing nonlinear sampled-data systems.

It is frequent that the system designer wishes to eliminate the particular modes of oscillations, especially of the fundamental frequency (half of the sampling frequency) or of a few of its subharmonics. Or he may attempt to eliminate only the oscillations of large amplitude which probably remain in the saturated regions. In such cases, the method that we have established is directly applicable; and the designer will obtain the sufficient and satisfactory information on his problem without wasting time in the struggle to find a comprehensive stability condition that may lead to an excessively conservative result.

The modification of Neace's method to test the stability of an equilibrium point is attempted. It is worthwhile to notice that the stability boundary obtained by this method yields the closed criterion to the experimental result as shown in Fig. 9.

Emphasis is placed on the PWM system in this thesis, although the analysis has always been extended to other types of nonlinearities.

As far as the analysis is concerned, no approximation is introduced, such as small-signal condition or sine-wave approximation. The PWM controller is a very sensitive device while it is operating in the unsaturated region; also by means of its saturating property it prevents the plant from

receiving an excessive influence from the input.

We observe in Fig. 9 that the operating region (stable region) of the PWM systems is rather uniform, due to the flat characteristic of the stability boundary. Therefore, the range of the choice of sampling frequencies is very wide. On the other hand, in case of other nonlinearities, the choice of sampling frequency is rather limited because of the inclined characteristics of the stability boundary (Figs. 10, 11). The PWM system is superior to the relay system in the sense that the former has a stable region while the latter can never get rid of the oscillations of the fundamental frequency and its first subharmonics. Also it is observed experimentally that the PWM system reaches the equilibrium state considerably faster than the other types of nonlinearities such as the saturating amplifier with linear region, starting from the same initial conditions. From this fact, we may state that the PWM controller is a more sensitive device than other nonlinear, amplitude-dependent controlling devices. And this will offer an advantage to the optimal control by means of the PWM system.

5.2 Suggestions for Future Work

When the nonlinear sampled-data system is represented by a set of nonlinear difference equations, it may be reduced to a totally linearized form as follows:

$$Y(n+1) = [B_n] Y(n)$$

where $Y(n)$ is a vector representing the state at the n th sampling instant and $[B_n]$ is a totally linearized matrix and is a nonlinear function of $Y(n)$.

We may find a certain region of B_n where B_n is stable, i.e., the eigenvalue of $[B_n]$ lies inside the unit circle if $Y(n)$ belongs to the certain vector space S .

It was pointed out by Kodama²³ that it is false to say that the equilibrium point is asymptotically stable in the large if $[B_n]$ is stable for all the points of state space. The above fact is demonstrated by his counter examples, by shaping the nonlinear gain curve for that purpose.

However, we believe that under certain conditions the above will hold true. And it is worth while to look forward to such conditions. This possibility is supported by the following fact. When the matrices $[B_1]$

and $[B_2]$ are stable, $[B_1]^2$ and $[B_2]^2$ are also stable by the Frobenius theorem. However, nothing is assured on the stability of the product matrix $[B_1] \cdot [B_2]$. But these two matrices are closely related to each other by the nonlinear system equation. If this relationship is precisely investigated, we may find the conditions under which $[B_1] \cdot [B_2]$ becomes also stable. And it may be extended to the case when the number of multiplied matrices is increased, possibly to infinity.

Also we noticed that various conclusions that are derived for individual matrices cannot be applied to their product at all. In general, we have to carry over the tedious multiplication of matrix elements and we usually find a very different conclusion from what we expect from the individual matrix. It will be very helpful if, for example, the stability criteria on the individual matrix can be extended to the product of matrices without carrying over the actual matrix multiplication process.

Another suggestion on the study of nonlinear sampled-data systems is the appropriate use of high-speed digital computers. Nowadays, the digital computer is extensively adopted as a controlling device of sampled-data systems. However, it can be used for simulating the sampled-data system in its programs. We can perform any type of experiment on sampled-data systems that are incorporated in programs of digital computers in the form of difference equations.

We believe that a complicated conclusion obtained by certain theoretical investigations can be accepted if it is formulated for suitable usage of digital computers, since the digital computer can give the desired data instantly when it is wanted. It will be advisable for researchers in sampled-data systems always to keep the possible utilization of digital computers in mind.

APPENDIX A

EXPERIMENTAL WORKS BY DIGITAL COMPUTORS

In order to verify the theoretical works developed in this report, high-speed digital computers are used extensively.

Because of the basic property of the sampled-data systems that has been the main subject of this thesis, the digital computer is well fitted for synthesizing the system and performing the experimental works.

The function of the closed loop nonlinear sampled-data system as shown in Fig. (1) is completely represented by a set of nonlinear difference equations as Eq. (4.16) or Eqs. (4.33) and (4.34). These nonlinear difference equations can be easily incorporated into programs of digital computers because of their iterative properties. Responses at the end of every sampling period can be used as the initial conditions for the following sampling instants; this operation is conveniently performed by using the transfer command and the index register in digital computers.

Setting of initial conditions is quite arbitrary and the accuracy of the computation is incompatible with that of analogue computers.

A series of experiments is performed using Bendix 15 to verify the existence and behavior of limit cycles as derived in Chapter II. Typical oscillations of PWM mode or relay mode are observed when suitable gain is given to the system.

aK_c or aK_g are accepted as boundary gains and a conspicuous difference of behavior of responses is observed on both sides of these critical gains.

Finally, a series of stability tests is performed in order to endorse the various stability boundaries obtained in Chapter III and Chapter IV. A wide range of initial conditions (40-60 points distributed on the phase plane) is selected, and for every initial condition the responses are calculated up to 100-200 sampling instants in order to examine the convergence of responses.

Varying the gain a gradually over the critical regions, the border line between stable and unstable regions is traced with good accuracy. These tests are repeated varying the sampling period T and the results are plotted as a function of the sampling frequency as shown in Fig. (8)

and (9).

As an example, the flow chart of the program for the second order PWM system is shown in Fig. (12).

This is programmed for the purpose of obtaining the responses of the second order system at every sampling instant when the initial conditions are specified.

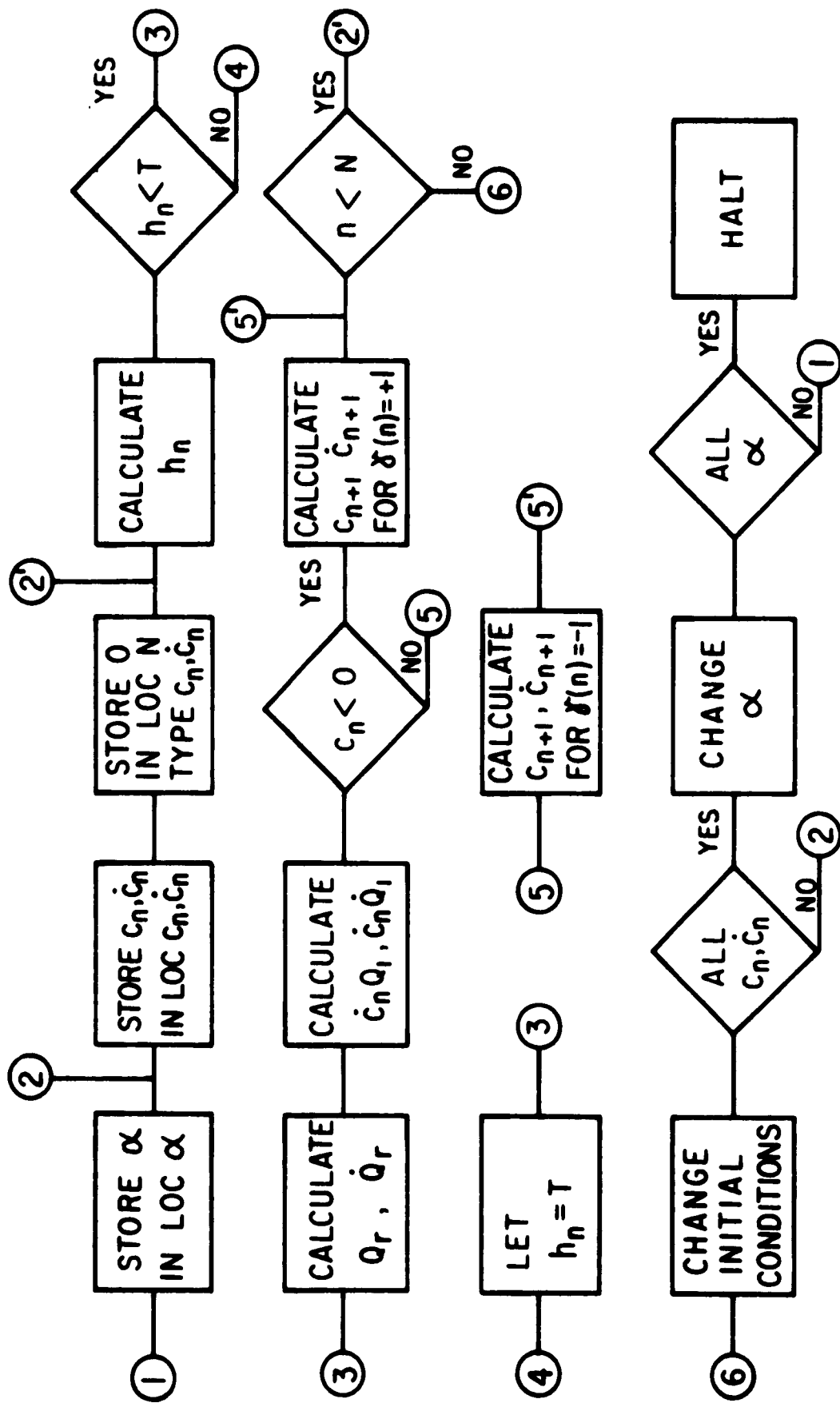


FIG. 12 FLOW CHART OF PROGRAMMING ON BENDIX 15 (2nd ORDER PWM SYSTEM)

APPENDIX B

PROOF OF DECREASING CHARACTERISTIC OF $\alpha K_g(\mu)$

The objective of this appendix is to prove the gain boundary αK_g for saturated oscillation is a monotonically decreasing function of μ . When $\alpha K_g(\mu)$ is given by Eq. (3.23) for the second-order system, it can be rewritten as follows.

$$\alpha K_g(\mu) = \frac{2b}{1 + 2(1 - e^{-bT}) \left[\frac{f(\mu, -bT)}{b\mu T(1 + e^{-\mu bT})} \right]} \quad (\text{B. 1})$$

In order to prove the decreasing characteristic of $\alpha K_g(\mu)$, it is enough to prove the increasing characteristic of the function inside the large bracket of the denominator of the above equation.

Let

$$D(\mu) = \frac{f(\mu, -bT)}{b\mu T(1 + e^{-\mu bT})} \quad (\text{B. 2})$$

then, the incremental difference $\Delta D(\mu)$ is given by

$$\Delta D(\mu) = D(\mu+1) - D(\mu) \quad (\text{B. 3})$$

If $\Delta D(\mu)$ is positive for all μ , $D(\mu)$ has a monotonically increasing characteristic. Substituting Eq. (B. 2) into (B. 3) yields $\Delta D(\mu)$

$$\Delta D(\mu) = \frac{\mu(1 + e^{-\mu bT})f(\mu+1, -bT) - (\mu+1)(1 + e^{-(\mu+1)bT})f(\mu, -bT)}{b\mu T(1 + e^{-\mu bT}) \cdot b(\mu+1)T(1 + e^{-(\mu+1)bT})} \quad (\text{B. 4})$$

$f(\mu, -bT)$ is obtained by substituting $-bT$ into the place of pT in Eq. (3.17) or in Eq. (3.18). When this is substituted into Eq. (B. 4) the numerator of $\Delta D(\mu)$ can be reduced to the following form.

$$\begin{aligned} \text{Numerator of } \Delta D(\mu) &= 2 \sum_{l=0}^{\mu-1} (l+1)(e^{-lbT} - e^{-(2\mu-l-1)bT}) \\ &= 2 \sum_{l=0}^{\mu-1} (l+1)e^{-lbT} (1 - e^{-(2\mu-l-1)bT}) \end{aligned} \quad (\text{B. 5})$$

This is obviously positive for all μ which is a positive integer.

Therefore

$$\Delta D(\mu) > 0 \quad \text{for } \mu = 1, 2, 3, \dots \quad (\text{B. 6})$$

Thus we could prove the monotonic increasing property of $D(\mu)$, hence the monotonic decreasing property of $\alpha K_g(\mu)$. A similar property can be proved for $\alpha K_g(\mu)$ of unsaturated oscillations. It is given by Eq. (3.34) and is rewritten as follows

$$\alpha K_g(\mu, h) = \frac{2b}{1 + 2(1 - e^{-bT}) \left[\frac{e^{bh} - 1}{e^{bT} - 1} \cdot \frac{f(\mu, -bT)}{\mu bT(1 + e^{-\mu bT})} \right]} \quad (\text{B. 7})$$

The function inside of the large bracket of the denominator of the above equation is related to $D(\mu)$ of Eq. (B. 2) as follows

$$\text{Function inside bracket} = \frac{e^{bh} - 1}{e^{bT} - 1} \quad (\text{B. 8})$$

Therefore, when $D(\mu)$ has an increasing property, naturally the above function possesses the same property, which makes $\alpha K_g(\mu, h)$ have a decreasing property.

APPENDIX C

PROOF OF THE RELATION $\mu_{\max}(h) \leq \mu_{\max}(T)$

In order to prove the relation that

$$\mu_{\max}(h) \leq \mu_{\max}(T) \quad \text{for } 0 \leq h \leq T \quad (\text{C.1})$$

where $\mu_{\max}(T)$ is the maximum positive integer that satisfies the inequality of Eq. (3.14) and $\mu_{\max}(h)$ is the maximum positive integer that satisfies the inequality of Eq. (3.33). We denote the left hand side of Eq. (3.32) as $g(h)$

$$g(h) = g_1(h) - g_2(h) \quad (\text{C.2})$$

where

$$g_1(h) = \frac{bh}{2} (\mu - 2) \quad (\text{C.3})$$

$$g_2(h) = (e^{bh} - 1) \left[\frac{1}{1 - e^{-bT}} \cdot \frac{1 - e^{-\mu bT}}{1 + e^{-\mu bT}} - 1 \right] \quad (\text{C.4})$$

If the following relation is proved,

$$g(h) \geq g(T) \quad \text{for } h \leq T \quad (\text{C.5})$$

then it implies the relation of Eq. (C.1) since $g(h)$ is a monotonically increasing function of μ .

From the following relation

$$\frac{g_2(h)}{g_2(T)} = \frac{e^{bh} - 1}{e^{bT} - 1} = \frac{h}{T} \cdot \frac{1 + \frac{b^2 h^2}{2} + \dots}{1 + \frac{b^2 T^2}{2} + \dots} \leq \frac{h}{T} \quad (\text{C.6})$$

we obtain

$$1 \geq \frac{g_1(h)}{g_1(T)} \geq \frac{g_2(h)}{g_2(T)} \quad (\text{C.7})$$

Let

$$\frac{g_1(h)}{g_1(T)} = 1 - \alpha \quad \alpha \geq 0 \quad (\text{C.8})$$

$$\frac{g_2(h)}{g_2(T)} = 1 - \beta \quad \beta \geq 0 \quad (C. 9)$$

Then

$$0 \leq \alpha \leq \beta$$

Therefore

$$g_1(h) - g_1(T) - (g_2(h) - g_2(T)) = -\alpha(g_1(T) - \frac{\beta}{\alpha}g_2(T)) \geq -\alpha(g_1(T) - g_2(T)) \geq 0 \quad (C. 10)$$

since

$$g_1(T) - g_2(T) = g(T) \leq 0 \quad \text{for } \mu \leq \mu_{\max}$$

Eq. (C. 10) implies Eq. (C. 5), thus the proof has been completed.

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