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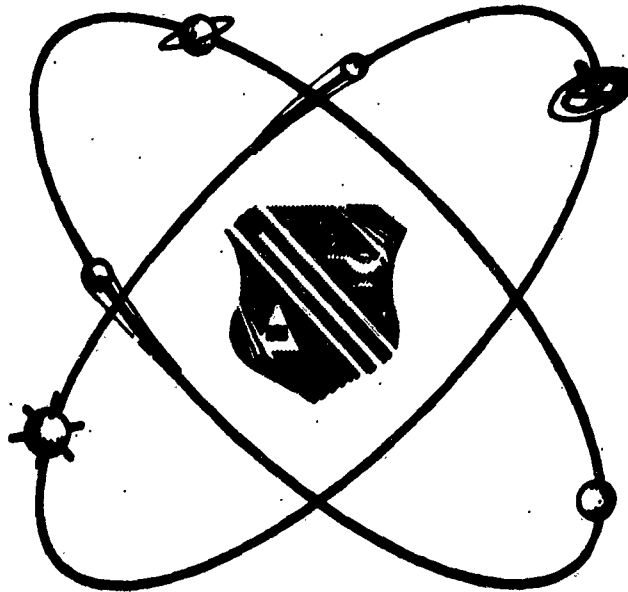
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AIR FORCE OFFICE OF SCIENTIFIC RESEARCH TECHNICAL REPORT

PROBABILITY CONSIDERATIONS ON DESTROYING ICBM'S
WITH INTERCEPTOR SATELLITES

Herbert Knothe



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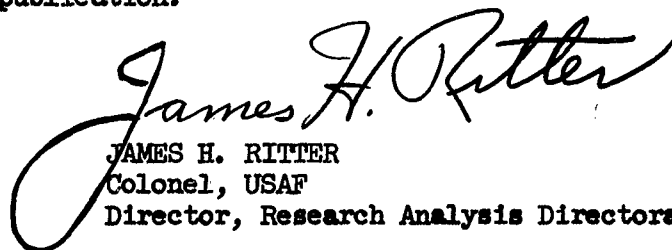
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ABSTRACT

This paper presents a method for calculating the probability that a certain minimum number of a set of ICBM's can be destroyed by a given number of interceptor satellites.

This report is approved for publication.


JAMES H. RITTER
Colonel, USAF
Director, Research Analysis Directorate

PROBABILITY CONSIDERATIONS ON DESTROYING ICBM'S
WITH INTERCEPTOR SATELLITES

The problem, as defined by Dr. Fritz Hoehndorf, may be formulated as follows:

Let m and n be the numbers of interceptors and ICBM's, respectively. Each of the m interceptors is supposed to attack exactly one of the n ICBM's. One ICBM may be attacked by one or more interceptors. The probability that one interceptor destroys the attacked ICBM is p . The probability P_{ik} that the i^{th} interceptor attacks the k^{th} ICBM shall be independent of i, k .

What is the probability that at least $n - \mu$ ICBM's are destroyed or, in other words, at most μ ICBM's escape destruction?

Let us first answer the question: What is the probability of destroying exactly j ICBM's? Let m_1 be the number of interceptors attacking the first ICBM, let m_2 be the number of interceptors attacking the second ICBM, and finally let m_n be the number of interceptors attacking the n^{th} ICBM. It is obvious that the number of different possibilities of attributing m_1, m_2, \dots, m_n interceptors to the first, second, \dots, n^{th} ICBM is

$$\frac{m!}{m_1! m_2! \dots m_n!}$$

Since n^m is the total number of possible distributions of m "elements" among n "boxes," the distribution m_1, m_2, \dots, m_n has the probability

$$\frac{1}{n^m} \cdot \frac{m!}{m_1! m_2! \dots m_n!}$$

The probabilities that the first, second, \dots j^{th} ICBM's escape destruction are $q^{m_1}, q^{m_2}, \dots, q^{m_j}$, respectively, where $q = 1 - p$.

The probabilities that the $(j+1)^{\text{st}}, \dots, n^{\text{th}}$ ICBM's are destroyed are $1 - q^{m_{j+1}}, \dots, 1 - q^{m_n}$, respectively.

Hence the configuration: first j ICBM's not destroyed, remaining $n - j$ ICBM's destroyed, i^{th} ICBM attacked by m_i interceptors ($i = 1, \dots, n$), has the probability

$$\frac{1}{n^m} \cdot \frac{m!}{m_1! m_2! \dots m_n!} q^{m_1} \dots q^{m_j} (1 - q^{m_{j+1}}) \dots (1 - q^{m_n})$$

Since there are $\binom{n}{j}$ possibilities of selecting a set of j escapes out of the n ICBM's we obtain

$$P_j = \binom{n}{j} \frac{1}{n^m} \sum \frac{m!}{m_1! m_2! \dots m_n!} q^{m_1} \dots q^{m_j} (1 - q^{m_{j+1}}) \dots (1 - q^{m_n}) \quad (1)$$

where P_j is the probability that exactly j ICBM's escape destruction

and where the sum has to be extended over all configurations

m_1, m_2, \dots, m_n so that

$$m_1 + m_2 + \dots + m_n = m$$

The sum in formula (1) may be written in a more convenient form.

We substitute in equation (1)

$$m_1 + m_2 + \dots + m_j = v$$

Since each term of the sum in equation (1) for which $m_{j+k} = 0$ ($k \geq 1$) vanishes, we have to consider only those configurations for which

$$m_{j+1} + \dots + m_n = m - (m_1 + \dots + m_j) = m - v \geq n - j$$

or

$$v \leq m - n + j$$

The set of the v elements m_1, m_2, \dots, m_j can be selected in $\binom{m}{v}$ ways and there are j^v possibilities of attributing v interceptors to j ICBM's. The number of possibilities of attributing the remaining $m - v$ interceptors to the remaining $n - j$ ICBM's is

$$\frac{(m - v)!}{m_{j+1}! \dots m_n!}$$

where

$$m_{j+1} + \dots + m_n = m - v$$

Hence it follows

$$P_j = \binom{n}{j} \frac{1}{n^m} \sum_{v=0}^{m-n+j} \binom{m}{v} (jq)^v \sum \frac{(m-v)!}{(m_{j+1})! \dots m_n!} (1-q^{m_{j+1}}) \dots (1-q^{m_n}) \quad (2)$$

where the second sum has to be extended over all possible configurations $m_{j+1} \dots m_n$ satisfying the condition

$$m_{j+1} + \dots + m_n = m - v \quad (3)$$

After determining P_j by means of formula (2) we see that the probability p_μ of killing at least $n - \mu$ ICBM's is

$$p_\mu = \sum_0^\mu P_j$$

In order to estimate the sum

$$S = \sum \frac{(m-v)!}{(m_{j+1})! \dots m_n!} (1-q^{m_{j+1}}) \dots (1-q^{m_n}) \quad (4)$$

the following study is useful. In the sum (4) all terms vanish if

any $m_{j+k} = 0$ ($k \geq 1$). If we now replace $(1 - q^{m_{j+1}}) \dots (1 - q^{m_n})$ by its maximum M_v , under the boundary condition (3) we see that

$$S \leq M_v \cdot \sum \frac{(m - v)!}{(m_{j+1})! \dots m_n!} \quad (5)$$

where only those configurations have to be considered for which

$$m_{j+k} \geq 1$$

This leads to the following problem: How many possibilities are there to distribute ρ "elements" to σ "boxes" ($\rho \geq \sigma$) so that each box contains at least one of the ρ elements?

In order to answer this question we shall derive a recursion formula as follows: Let $\phi_\rho(\sigma, j)$ be the number of possible distributions so that the first j boxes contain at least one element. It is evident that $\phi_\rho(\sigma, j+1)$ equals $\phi_\rho(\sigma, j)$ minus that number of distributions for which the first j boxes contain at least one element and for which the $(j+1)^{\text{st}}$ box is empty. This number is $\phi_\rho(\sigma-1, j)$. Therefore:

$$\phi_\rho(\sigma, j+1) = \phi_\rho(\sigma, j) - \phi_\rho(\sigma-1, j) \quad (6)$$

We now have

$$\phi_\rho(\sigma, 0) = \sigma^\rho \quad (7)$$

From equations (6) and (7) we conclude by induction

$$\phi_p(\sigma, \sigma) = \sigma^p - \sigma(\sigma-1)^p + \binom{\sigma}{2} (\sigma-2)^p - \dots = \sum_{v=0}^{\sigma} (-1)^v \binom{\sigma}{v} (\sigma-v)^p \quad (8)$$

or

$$\phi_p(\sigma, \sigma) = \sum_{v=0}^{\sigma} (-1)^v \binom{\sigma}{v} v^p \quad (9)$$

where $\phi_p(\sigma, \sigma)$ means the number of possibilities to distribute p "elements" to σ "boxes" so that each box contains at least one element. It is desirable to express $\phi_p(\sigma, \sigma)$ in a closed form. For this purpose we write

$$\sum_{v=0}^{\sigma} (-1)^v \binom{\sigma}{v} e^{vx} = (e^x - 1)^\sigma \quad (10)$$

Differentiating (10) p times with respect to x we obtain

$$\phi_p(\sigma, \sigma) = \left. \frac{d^p}{dx^p} (e^x - 1)^\sigma \right|_{x=0} \quad (11)$$

which yields $\phi_p(\sigma, \sigma)$ in the desired closed form.

Equation (11) shows that $\phi_p(\sigma, \sigma)$ may be written as a contour integral in the Gaussian plane, extended over any closed contour encircling the origin once

$$\phi_p(\sigma, \sigma) = \frac{p!}{2\pi i} \oint \frac{(e^z - 1)^\sigma}{z^{p+1}} dz \quad (12)$$

In various ways we can obtain useful estimates for $\phi_p(\sigma, \sigma)$, for instance choosing the unit circle about the origin as contour. However, we shall not dwell upon this method any longer, because the fundamental formula (2) can be transformed into an interesting and much simpler form, handy above all for computations with a digital computer.

With regard to practical calculations the greatest difficulty in evaluating formula (2) is caused by the factorial terms and, above all, by decomposing m into a large number of summands. How this can be avoided shall now be investigated.

The second sum on the right-hand side of equation (2) has the structure

$$\sum \frac{s!}{s_1! \dots s_t!} (1 - q^{s_1}) \dots (1 - q^{s_t}) \quad (13)$$

where

$$s_1 + \dots + s_t = s$$

Eliminating the parentheses in (13) we have $\binom{t}{\alpha}$ products of the kind $q^{s_1} q^{s_2} \dots q^{s_\alpha}$. All these $\binom{t}{\alpha}$ products contribute the same value

$$S = (-1)^\alpha \sum \frac{s!}{s_1! \dots s_t!} q^{s_1} \dots q^{s_\alpha} \quad (14)$$

to the sum in expression (13) if we first keep $s_1, s_2, \dots, s_\alpha$ fixed. We substitute $s_1 + s_2 + \dots + s_\alpha = \tilde{s}$ and write (14) in the form

$$S = (-1)^\alpha \sum \left(\frac{\frac{s!}{(s - \tilde{s})!} \cdot q^{\tilde{s}}}{s_1! s_2! \dots s_\alpha!} \right) \left(\frac{(s - \tilde{s})!}{s_{\alpha+1}! \dots s_t!} \right) \quad (15)$$

Keeping first $s_1, s_2, \dots, s_\alpha$ fixed and summing over $s_{\alpha+1}, \dots, s_t$, we obtain

$$S = \sum_{\tilde{s}=0}^s (-1)^\alpha (t - \alpha)^{s - \tilde{s}} q^{\tilde{s}} \binom{s}{\tilde{s}} \sum \frac{\tilde{s}!}{s_1! \dots s_\alpha!} \quad (16)$$

Summing over s_1, \dots, s_α , and keeping \tilde{s} fixed during this summation, we obtain

$$S = \sum_{\tilde{s}=0}^s (-1)^\alpha (t - \alpha)^{s-\tilde{s}} q^{\tilde{s}} \binom{s}{\tilde{s}} \alpha^{\tilde{s}} \quad (17)$$

or

$$S = (-1)^\alpha (t - \alpha p)^s \quad (18)$$

Equation (18) now leads to the important result

$$\begin{aligned} & \sum \frac{s!}{s_1! \dots s_t!} (1 - q^{s_1}) \dots (1 - q^{s_t}) \\ & = \sum_{\alpha=0}^t (-1)^\alpha \binom{t}{\alpha} (t - \alpha p)^s \end{aligned} \quad (19)$$

Equation (19) finally enables us to give an expression for P_j much more suitable for practical calculations:

$$P_j = \binom{n}{j} \frac{1}{n^m} \sum_{v=0}^{m-n+j} \binom{m}{v} (jq)^v \sum_{\alpha=0}^{n-j} (-1)^\alpha \binom{n-j}{\alpha} (n-j-\alpha p)^{m-v} \quad (20)$$

Let us now represent P_j as a contour integral. For this purpose we apply to the second sum in equation (20) a method analogous to that which led to equation (11). From the relation

$$\sum_{\alpha=0}^{n-j} (-1)^\alpha \binom{n-j}{\alpha} e^{(n-j-\alpha p)x} = e^{(n-j)x} (1 - e^{-px})^{n-j} \quad (21)$$

we conclude by differentiating $(m - \nu)$ times with respect to x , and subsequently substituting $x = 0$

$$\sum_{\alpha=0}^{n-j} (-1)^\alpha \binom{n-j}{\alpha} (n-j-\alpha p)^{m-\nu} = \frac{d^{m-\nu}}{dx^{m-\nu}} \left\{ e^{(n-j)x} (1 - e^{-px})^{n-j} \right\} \Big|_{x=0} \quad (22)$$

or

$$\sum_{\alpha=0}^{n-j} (-1)^\alpha \binom{n-j}{\alpha} (n-j-\alpha p)^{m-\nu} = \frac{(m-\nu)!}{2i\pi} \oint \frac{e^{(n-j)z} (1 - e^{-pz})^{n-j}}{z^{m-\nu+1}} dz \quad (23)$$

where the contour integral has to be extended over a closed contour about the origin in the Gaussian plane of the complex variable z .

Substituting equation (23) in equation (20) we obtain after some elementary operations

$$P_j = \binom{n}{j} \cdot \frac{m!}{n^m} \cdot \frac{1}{2i\pi} \oint \sum_{v=0}^{m-n+j} \frac{(jqz)^v}{v!} \frac{(1 - e^{-pz})^{n-j} e^{(n-j)z}}{z^{m+1}} dz \quad (24)$$

The sum

$$\sum_{v=0}^{m-n+j} \frac{(jqz)^v}{v!} \quad (25)$$

can rigorously be replaced by e^{jqz} because powers of jqz higher than $m-n+j$ would not yield any residue different from zero in the integral of equation (24). Hence it follows that P_j can be represented in the form

$$P_j = \binom{n}{j} \cdot \frac{1}{n^m} \cdot \frac{m!}{2i\pi} \oint \frac{e^{(n-jp)z} (1 - e^{-pz})^{n-j}}{z^{m+1}} dz \quad (26)$$

It is appealing to check equation (26) by proving that indeed

$$\sum_{j=0}^n P_j = 1 \quad (27)$$

Since

$$\sum_{j=0}^n \binom{n}{j} e^{-jpx} (1 - e^{-px})^{n-j} = 1$$

we obtain from equation (26)

$$\sum_{j=0}^n P_j = \frac{m!}{n^m} \cdot \frac{1}{2i\pi} \oint \frac{e^{nz}}{z^{m+1}} dz \quad (28)$$

Since

$$\oint \frac{e^{nz}}{z^{m+1}} dz = 2i\pi \frac{n^m}{m!}$$

equation (27) has been verified.

Equation (26) can be written in the form

$$P_j = \binom{n}{j} \cdot \frac{1}{n^m} \cdot \frac{m!}{2i\pi} \oint \frac{(e^z - e^{qz})^n (e^{pz} - 1)^{-j}}{z^{m+1}} dz \quad (29)$$

It suggests eliminating the parentheses in equation (26) in order to calculate the residues. After some elementary transformations we obtain:

$$P_j = \left(1 - \frac{jp}{n}\right)^m \binom{n}{j} \sum_{\mu=0}^{n-j} (-1)^\mu \binom{n-j}{\mu} \left(1 - \frac{\mu p}{n-jp}\right)^m \quad (30)$$

For practical computations it is always advisable to comprehend two succeeding terms in the sum of equation (30):

$$\begin{aligned} & \binom{n-j}{\mu} \left(1 - \frac{\mu p}{n-jp}\right)^m - \binom{n-j}{\mu+1} \left(1 - \frac{(\mu+1)p}{n-jp}\right)^m \\ &= \binom{n-j}{\mu} \left(1 - \frac{\mu p}{n-jp}\right)^m \left\{1 - \frac{n-j-\mu}{\mu+1} \left(1 - \frac{p}{n-jp-\mu p}\right)^m\right\} \\ &\approx \binom{n-j}{\mu} \left(1 - \frac{\mu p}{n-jp}\right)^m \left(1 - \frac{n-j-\mu}{\mu+1} e^{-\frac{mp}{n-jp-\mu p}}\right) \end{aligned} \quad (31)$$

This approximation is sufficient for all practical purposes. Finally equation (30) assumes the form:

$$P_j \approx \binom{n}{j} \left(1 - \frac{jp}{n}\right)^m \sum_{\sigma=0}^{\frac{n-j-1}{2}} \binom{n-j}{2\sigma} \left(1 - \frac{2\sigma p}{n-jp}\right)^m \left(1 - \frac{n-j-2\sigma}{2\sigma+1} e^{-\frac{mp}{n-jp-2\sigma p}}\right) \quad (32)$$

for an odd $(n-j)$. For an even $(n-j)$ the upper limit of the sum equals

$$\frac{(n-j)}{2}$$

Professor Harry Carver suggested that we calculate the factorial moments of our distribution function, remarking that the general expressions would probably become very simple.

The ν^{th} factorial moment $\mu_{(\nu)}$ is defined as follows:

$$\mu_{(\nu)} = \sum_{j=0}^n j(j-1) \dots (j-\nu+1) P_j \quad (33)$$

or using the notation:

$$j(j-1) \dots (j-\nu+1) = j^{(\nu)}$$

we have:

$$\mu_{(\nu)} = \sum_{j=0}^n j^{(\nu)} P_j \quad (34)$$

In order to calculate these moments we make use of formula (26) which we shall write in a slightly modified form:

$$P_j = \binom{n}{j} \cdot \frac{1}{n^m} \cdot \frac{m!}{2i\pi} \oint \frac{e^{nqz} (e^{pz} - 1)^{n-j}}{z^{m+1}} dz \quad (35)$$

From equation (35) we conclude:

$$\mu(\nu) = \frac{m!}{2i\pi n^m} \oint \frac{e^{nqz} (e^{pz} - 1)^n}{z^{m+1}} \sum_{j=0}^n j^{(\nu)} \binom{n}{j} (e^{pz} - 1)^{-j} dz \quad (36)$$

The sum in the integrand can easily be evaluated as follows: We substitute,

$$e^{pz} - 1 = x^{-1}$$

Since:

$$\sum_{j=0}^n \binom{n}{j} (e^{pz} - 1)^{-j} = \sum_{j=0}^n \binom{n}{j} x^j = (1 + x)^n \quad (37)$$

we obtain by deriving equation (37) ν times with respect to x and multiplying the result by x^ν :

$$\sum_{j=0}^n j^{(\nu)} \binom{n}{j} x^j = n^{(\nu)} x^\nu (1 + x)^{n-\nu} \quad (38)$$

or

$$\sum_{j=0}^n j^{(\nu)} \binom{n}{j} (e^{pz} - 1)^{-j} = n^{(\nu)} e^{pz(n-\nu)} (e^{pz} - 1)^{-n} \quad (39)$$

We now substitute equation (39) in equation (36):

$$\mu(\nu) = \frac{m! \cdot n(\nu)}{2i\pi n^m} \oint \frac{e^{(n - \nu p)z}}{z^{m+1}} dz \quad (40)$$

Calculating the residue of the integrand of equation (40) we finally obtain:

$$\mu(\nu) = n(\nu) \left(1 - \frac{\nu p}{n}\right)^m \quad (41)$$

Our next aim is to give an approximation of our distribution function good to any desired degree of accuracy, taking advantage of the simple expressions for the factorial moments.

We are particularly interested in the case where n is very large i.e., $n \geq 200$. In this case it is justified to replace the summation sign in equation (34) by an integral sign:

$$\mu(\nu) = \int_{j=0}^n j(\nu) P(j) d_j \quad (42)$$

We now perform the similarity transformation:

$$j = \frac{n}{2} (x + 1) \quad (43)$$

which transforms the interval $(0, n)$ into the interval $(-1, 1)$, and obtain:

$$\mu(\nu) = \left(\frac{n}{2}\right)^\nu \int_{-1}^1 \left(1+x\right)\left(1+x-\frac{2}{n}\right)\left(1+x-\frac{4}{n}\right)\dots\left(1+x-\frac{2\nu-2}{n}\right) \tilde{P}(x) dx \quad (44)$$

where $\tilde{P}(x) dx$ is the probability that x lies in the interval $(x, x + dx)$.

Now we take advantage of the simple expressions for the factorial moments in order to expand $\tilde{P}(x)$ into a series of Legendre polynomials. For this purpose we eliminate the parentheses in equation (44):

$$\mu(\nu) = \left(\frac{n}{2}\right)^\nu \left[\int_{-1}^1 (1+x)^\nu \tilde{P}(x) dx - \frac{2}{n} s_1^{(\nu-1)} \int_{-1}^1 (1+x)^{\nu-1} \tilde{P}(x) dx + \frac{2^2}{n^2} s_2^{(\nu-1)} \int_{-1}^1 (1+x)^{\nu-2} \tilde{P}(x) dx \dots + \frac{(-2)^{\nu-1}}{n^{\nu-1}} \right] \quad (45)$$

where $s_1^{(\nu-1)}$ means the i^{th} elementary symmetric function of the first $\nu - 1$ integers.

For practical applications we shall list the first four symmetric functions $s_1(n)$, $s_2(n)$, $s_3(n)$, $s_4(n)$:

$$s_1(n) = \frac{n(n+1)}{2}$$

$$s_2(n) = \frac{n(n+1)(n-1)(3n+2)}{24}$$

(46)

$$s_3(n) = \frac{n^2(n+1)^2(n-2)(n-1)}{48}$$

$$s_4(n) = \frac{n(n+1)(n-1)(n-2)(n-3)(15n^3 + 15n^2 - 10n - 8)}{2^7 \cdot 3^2 \cdot 5}$$

Equation (45) suggests expanding the expression $(1+x)^n$ into a sum of Legendre polynomials. As will be shown in the Appendix, we find:

$$(1+x)^n = \sum_{i=0}^n I_{in} \bar{P}_i \quad (47)$$

where \bar{P}_i is the i^{th} Legendre polynomial normalized so that:

$$\int_{-1}^1 \bar{P}_i^2 dx = 1 \quad (48)$$

and

$$I_{in} = \sqrt{\frac{2i+1}{2}} \quad (49)$$

$$\frac{2^{n+1} (n!)^2}{(2i+1)! \left[(n+1)n - (i+1)i \right] \left[n(n-1) - (i+1)i \right] \dots \left[(i+2)(i+1) - (i+1)i \right]}$$

Let us now expand $\tilde{P}(x)$ into a series of Legendre polynomials:

$$\tilde{P}(x) = \sum_0^{\infty} a_j \bar{P}_j(x) \quad (50)$$

We now substitute equations (41), (47), and (50) in equation (45):

$$n^{(v)} \left(1 - \frac{vp}{n}\right)^m = \left(\frac{n}{2}\right)^v \sum_{i=0}^{v-1} I_{iv} a_i - \frac{2}{n} s_1^{(v-1)} \sum_{i=0}^{v-1} I_{iv-1} a_i$$

$$+ \frac{2^2}{n^2} s_2^{(v-1)} \sum_{i=0}^{v-2} I_{iv-2} a_i \dots \quad (51)$$

Equation (51) enables us to calculate easily the coefficients a_i successively, and is above all suitable for large numbers n and m .

Although equation (51) apparently is complicated, it permits computation of the coefficients a_i without solving a system of linear equations. The numerical evaluation of equation (51) can be performed by means of a desk calculator.

APPENDIX

In order to prove formula (49), page 19, we start from the differential equation for Legendre's polynomials:

$$\frac{d}{dx} \left[(x^2 - 1) P_i' \right] = i(i + 1) P_i \quad (1)$$

We now multiply equation (1) by $(1 + x)^n$ and integrate from -1 to $+1$:

$$\begin{aligned} i(i + 1) \int_{-1}^{+1} (1 + x)^n P_i dx &= \int_{-1}^{+1} \frac{d}{dx} \left[(x^2 - 1) P_i' \right] (1 + x)^n dx \\ &= -n \int_{-1}^{+1} (x^2 - 1) (1 + x)^{n-1} P_i' dx = -n \int_{-1}^{+1} (x - 1) (1 + x)^n P_i' dx \end{aligned} \quad (2)$$

$$\begin{aligned} &= n \int_{-1}^{+1} (1 + x)^n P_i dx + n^2 \int_{-1}^{+1} (x - 1) (1 + x)^{n-1} P_i dx \\ &= (n + n^2) \int_{-1}^{+1} (1 + x)^n P_i dx - 2n^2 \int_{-1}^{+1} (1 + x)^{n-1} P_i dx \end{aligned}$$

From equation (2) we derive the recursion formula:

$$I_{in} = \frac{2n^2}{n^2 + n - i^2 + i} I_{in-1} \quad (3)$$

We apply formula (3) successively until we reach I_{ii} , which is:

$$I_{ii} = \int_{-1}^1 (1+x)^i P_i dx = \int_{-1}^1 x^i P_i dx \quad (4)$$

The integral (4) is well known and may be taken from any textbook*, and from it we then obtain, after some elementary transformations, equation (49), page 19.

*W. Magnus and F. Oberhettinger, Formeln und Sätze für die speziellen Funktionen der mathematischen Physik.

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