UNCLASSIFIED

AD 274 108

Reproduced
by the

ARMED SERVICES TECHNICAL INFORMATION AGENCY
ARLINGTON HALL STATION
ARLINGTON 12, VIRGINIA

UNCLASSIFIED

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

274 108

Report No. 32

ON THE DYNAMICS OF NEAR-EARTH FLIGHT

by B.G. Grunebaum, Ph.D.

ASTIA
1962
508 950

d

CONTRACT N-123(61756)19425 A/PMR

7162 3 /

ABSTRACT

The paper develops equations for near-earth and satellite flights. The use of experimental parameters is avoided as much as possible. Differential equations that defy analytical solution are made a quasi-linear type so numerical integration can be performed with close error bounds. The equations consider both thrust and ballistic phases and enable separate numerical analysis of such effects as oblateness, air drag, earth rotation. All analytical data were assumed to be determinable from doppler radar and/or Transit equipment and the algorithms achieve a reasonable compromise between accuracy and speed.

TABLE OF CONTENTS

ON THE DYNAMICS OF NEAR-EARTH FLIGHT

by Bernard G. Grunebaum, Ph.D.
Chief of Technical Staff
Land-Air, Inc.

The object of this paper is to develop equations for the near-earth flight of missiles and satellites, which permit to obtain accurate numerical results. Any differential equation that defied analytic solution had to be of the quasi-linear first order type so the numerical integration could be performed with close error bounds. The use of experimental parameters was avoided as much as possible since, at the present state of the art, the measurement of such parameters is rather inaccurate in most cases and frequently the error which such parameters pretend to correct is strongly suspected to be smaller than the error derived from the measurements. Both free flight and flight under thrust had to be considered and the equations were required to be such as to enable us to analyze numerically and separately the effects of different physical causes, such as earth oblateness, air drag, and effect of earth rotation on the latter. All analytical data were assumed to be determinable from Doppler Radar and/or Transit equipment and the algorithms had to be such as to achieve a reasonable compromise between accuracy and speed.

These objectives have been met, at least from an analytical standpoint. The paper is nevertheless incomplete, since numerical results still have not been obtained. The conclusion of this report list certain items of programing necessary to complete the study. The work has begun, and when numerical results have been obtained, a concluding report will be made.

The writer wishes to acknowledge valuable help received from Mr. M. R. Claasen in verifying some of the algebraic developments and from Mr. R. Bergman in the form of suggestions regarding apsidal advance.

Basic Principles of Mechanics

For reasons of clearness, we shall recall some of the basic equations of classical dynamics. The following definitions will be used:

x,y,z = orthogonal, right handed, geocentric coordinate system. x,y is assumed to be the equatorial plane and z the polar axis pointing North. The system is assumed inertial, i.e. the earth rotates around the z axis in the x,y direction.

$\bar{r}$ = vector, defining a point P.

$\bar{\rho}$ = projection of $\bar{r}$ on x,y plane.

φ = angle $\bar{\rho}$, $\bar{r}$, positive in the $\bar{\rho}$, z direction.

θ = angle x, $\bar{\rho}$, positive in the x,y direction.

NBT = orthogonal, right handed, unit vector system; N is the unit vector of $\bar{r}$, T is perpendicular to $\bar{r}$ in the $\bar{r}$, z plane in the direction of increasing φ, B is perpendicular to NT in the direction of increasing θ. The plane BT is called the local horizontal by definition.

$\bar{v}$ = velocity vector of P.

$\bar{a}$ = projection of $\bar{v}$ on local horizontal

α = angle between $\bar{v}$ and local horizontal plane; positive if $\bar{v}$ is on the same side as N.

β = angle between $\bar{a}$ and B measured in the B,T direction; i.e. if $\frac{\bar{a}}{|\bar{a}|} = T$, then $\beta = \frac{\pi}{2}$.

Primes indicate differentiation with respect to time. We have the following basic relations:

$$(1) \quad \begin{cases} v \sin \alpha = r' \\ v \cos \alpha \cos \beta = r\theta' \cos \varphi \\ v \cos \alpha \sin \beta = r\varphi' \end{cases}$$

$$(2) \quad a = v \cos \alpha$$

$$(3) \quad \begin{cases} N' = \varphi' T + \cos \varphi \ \theta' B \\ T' = -\varphi' N - \sin \varphi \ \theta' B \\ B' = \ ' \sin \varphi \ T - \theta' \cos \varphi \ N \end{cases}$$

(4) $v^2 = r'^2 + r^2\varphi'^2 + r^2\theta'^2 \cos^2 \varphi$.

Furthermore, a can be easily verified

(5) $\bar{r}' = r'N + r\varphi'T + r\theta' \cos \varphi\, B$

and

(6) $\bar{r}'' = (r'' - r\varphi'^2 - r\theta'^2 \cos^2 \varphi)N + (2r'\varphi' + r\varphi'' + r\theta'^2 \sin \varphi \cos \varphi)T$
$+ (2r'\theta' \cos \varphi + r\theta'' \cos \varphi - 2r\theta'\varphi' \sin \varphi)B$

Proofs for the preceding expressions are elementary and will be omitted.

Choice of Coordinates and Variables

Studying the movement of a point in a field of forces usually yields non-linear differential equations which, except in simple and rather unrealistic cases, defy analytical integration. Moreover, they are usually not of the quasi-linear first order type, and their numerical integration can be lengthy and/or of unknown accuracy. It is well known that this is generally the case when the variables and coordinates of the previous paragraph are used to study the movement of a missile or a satellite. Therefore, our first problem is to choose a set of variables and a system of coordinates that will yield suitable quasi-linear differential equations with all admissible complexities of the field of forces. Such a choice is not necessarily unique or may not be possible at all. The system which we shall recommend has merely been found to be suitable.

The new system of coordinates will be NAR, an orthogonal and right handed set of unit vectors. N is as previously defined, A and R belong to the local horizontal plane. A is the unit vector of $\bar{a}$ so that β is the angle between B and A. Since the system is right handed, β is also the angle between T and R. Thus

(7) $T = A \sin \beta + R \cos \beta$
$B = A \cos \beta - R \sin \beta$

The recommended system of variables is

$v_v = v \sin \alpha$, defined as the vertical component of velocity;
$v_h = v \cos \alpha$, defined as the horizontal component of velocity;
β
r
φ

and, as may be easily verified, in this system the vector acceleration (6) becomes

$$(8) \qquad r'' = (v_v' - \frac{v_h^2}{r})N + (v_h' + \frac{v_h v_v}{r})A + (v_h \beta' + \frac{v_h^2}{r} \cos \beta \tan \varphi)R.$$

Notice that if the sum of all forces F (Per unit mass) acting on the point is decomposed into its components F_N, F_A and F_R, and if these components can be expressed as functions of the cited variables only, and not of their derivatives, as it turns out to be the case in our problem, then we have a first order quasi-linear system as desired, namely

$$(9) \qquad \begin{cases} v_v' = \dfrac{v_h^2}{r} + F_N \\ v_h' = -\dfrac{v_h v_v}{r} + F_A \\ \beta' = -\dfrac{v_h}{r} \cos \beta \tan \varphi + \dfrac{F_R}{v_h} \\ r' = v_v \\ \varphi' = \dfrac{v_h}{r} \sin \beta \end{cases}$$

Although (9) represents a system, generally integrable in closed form, its integral cannot describe completely the movement of a point in terms of position and velocity, because we have only 5 dependent variables. Thus, one additional dependent variable must be introduced, for example θ in the form

$$(10) \qquad \theta' = \frac{v_h}{r} \frac{\cos \beta}{\cos \varphi}$$

Geometrical Relations

Besides the variables involved in (9) and (10), we are usually interested in several other parameters and their rate of variation. We shall now define those parameters and establish expressions that permit calculating them.

Let us denote by Q the plane determined by the vectors $\bar{r}$ and $\bar{v}$. We may call Q the instantaneous plane of movement and it should not be confused with the osculating plane to the trajectory. Q contains the vector A and thus, Q is also the plane NA. We know from classical mechanics that, in a central inverse square force field (Keplerian field) Q is an invariant and later, we shall verify this fact analytically. We are interested in the angles determining the position of Q since their variation may be considered a measure of the departure from Keplerian movement.

Let i, j, k be the unit vectors in the x, y, z directions respectively and let η be the angle between Q and the x, y plane, positive if $|\beta| < \frac{\pi}{2}$. Then

$$\cos \eta = K \cdot R$$

But

$$R = T \cos \beta - B \sin \beta$$

Hence

$$\cos \eta = K \cdot T \cos \beta - K \cdot B \sin \beta$$

Now

$$K \cdot T = \cos \varphi \quad \text{and} \quad K \cdot B = 0$$

Hence

(11) $\cos \eta = \cos \varphi \cos \beta.$

Let $\bar{q}$ be the unit vector in the direction of the intersection between Q and the xy plane, such that

$$\bar{q} = \frac{K \times R}{\sin \eta}.$$

It follows that

$$(12) \qquad \bar{q} \sin \eta = i(\sin \theta \sin \varphi \cos \beta + \cos \theta \sin \beta)$$

$$+ j(-\cos \theta \sin \varphi \cos \beta + \sin \theta \sin \beta).$$

Let μ be the angle between i and $\bar{q}$, positive in the xy direction. Then $\cos \mu = i \cdot \bar{q}$ and thus

$$\cos \mu = \frac{1}{\sin \eta} (\sin \theta \sin \varphi \cos \beta + \cos \theta \sin \beta).$$

Similarly, $\sin \mu = j \cdot \bar{q}$ and thus

$$\sin \mu = \frac{1}{\sin \eta} (- \cos \theta \sin \varphi \cos \beta + \sin \theta \sin \beta).$$

Multiplying these expressions respectively by $\cos \theta$ and $\sin \theta$ and adding, we obtain

$$\sin \eta \cos \mu \cos \theta + \sin \eta \sin \mu \sin \theta = \sin \beta$$

i.e.

$$(13) \qquad \sin \beta = \sin \eta \cos(\mu - \theta).$$

Eliminating β between (11) and (13), we obtain

$$(14) \qquad \tan \varphi = - \tan \eta \sin(\mu - \theta).$$

The minus sign in (14) follows from the condition that at $\beta = 0$, we must have $\mu = \theta - \frac{\pi}{2}$ and $\varphi = \eta$.

Other angles of interest are:

Υ, the angle between $\bar{v}$ and B. Given by

$$(15) \qquad \cos \Upsilon = \cos \alpha \cos \beta$$

ψ, the angle between $\bar{q}$ and $\bar{r}$. Given by

$$(16) \qquad \cos \psi = \cos \varphi \cos(\mu - \theta)$$

positive in the direction of increasing φ.

Precession and Nutation

The rate of variation of μ is uniformly defined as precession through the literature. Authors nevertheless differ in their definition of nutation, but we shall denote by this name the variation of η. In effect and as will be shown, the quantities μ' and η' are not independent.

Differentiating (13) and substituting φ' with (1), we obtain

$$\beta' = \frac{\eta' \sin \eta}{\sin \beta \cos \varphi} - \frac{v_h}{r} \tan \varphi \cos \beta$$

Differentiating (13) and substituting θ' with (1), we obtain

$$\beta' = \eta' \frac{\cos \eta \cos(\mu - \theta)}{\cos \beta} - \mu' \frac{\sin \eta \sin(\mu - \theta)}{\cos \beta} + \\ + \frac{v_h}{r} \frac{\sin \eta \sin(\mu - \theta)}{\cos \varphi}$$

Substracting these expressions and noting that

$$- \tan \varphi \cos \beta = \frac{\sin \eta \sin(\mu - \theta)}{\cos \varphi}$$

by virtue of (11) and (14), we obtain

$$\eta'\left(\frac{\sin\eta}{\sin\beta\cos\varphi} - \frac{\cos\eta\cos(\mu-\theta)}{\cos\beta}\right) + \mu'\,\frac{\sin\eta\sin(\mu-\theta)}{\cos\beta} = 0$$

Using the identities already established, this expression simplifies to

$$\eta'\tan(\mu-\theta) + \mu'\sin\eta\cos\eta = 0 \tag{17}$$

Expression (17) can also be obtained by differentiating (14). An alternate form of (17) is

$$\eta' = \mu'\,\frac{\cos\psi\sin^2\eta}{\sin\varphi} \tag{18}$$

with cos ψ given by (16).

The vector acceleration, as given by (8), can now be expressed as a function of the precession and/or nutation instead of β′. As may be easily verified, the corresponding expressions are

$$(19)\qquad \begin{cases} \bar{r}'' = \left(v_v' - \dfrac{v_h^2}{r}\right)N + \left(v_h' + \dfrac{v_h v_v}{r}\right)A + \\[2ex] \qquad + \text{ either} \begin{cases} \eta'\,\dfrac{v_h}{\cos\psi}\,R \\[2ex] \mu' v_h\,\dfrac{\sin^2\eta}{\sin\varphi}\,R \end{cases} \end{cases}$$

Using expressions (11) and (13), a corresponding quasi-linear system could be written.

<u>Keplerian Movement</u>

This is the simplest situation that can be assumed and it has been studied to exhaustion in the literature. Nevertheless, it is the only case where a simple analytical solution is possible and therefore, it serves at least the purpose of providing us with a check case, i.e. a case where the closeness of the error bounds of our numerical integration procedure can be verified numerically. Therefore, we shall study the Keplerian case with this particular object in mind.

With reference to (9), we have in this case

$F_N = -\frac{MG}{r^2}$ where M is the mass of the earth and G is the universal constant of gravitation.

Besides $F_A = F_R = 0$.

The system (9) becomes therefore

$$(20)\quad \begin{cases} v_v' = \frac{v_h^2}{r} - \frac{MG}{r^2} \\ v_h' = -\frac{v_v v_h}{r} \\ \beta' = -\frac{v_h}{r}\cos\beta\tan\varphi \\ r' = v_v \\ \varphi' = \frac{v_h}{r}\sin\beta \end{cases}$$

where $MG \approx 3.982 \times 10^{20}$ dyne cm²/gr. Integration of (20) yields directly

$$(21)\qquad v_h = \frac{1}{c_1 r}$$

and

$$r'^2 = -\frac{1}{c_1^2 r^2} + \frac{2MG}{r} + c_2$$

where c_1 and c_2 are arbitrary constants. The latter expression can be written in the form

$$(22)\qquad v^2 = \frac{2MG}{r} + c_2$$

Integration of (20) also yields

$$\cos\varphi\cos\beta = \frac{1}{c_3} = \cos\eta$$

where c_3 is another arbitrary constant, thus showing that the angle η is constant. This same result can also be obtained from (19). In effect, since $F_R = 0$, we have $\eta' = \mu' = 0$ showing that not only η but also μ is constant, i.e. the Q plane is invariant. Since quite obviously, $v_h = \psi' r$, we have by virtue of (21)

$$(23) \qquad \psi' = \frac{1}{c_1 r^2}$$

where, by virtue of (11), (13) and (16), ψ can be obtained from

$$(24) \qquad \sin\psi = \frac{\sin\varphi}{\sin\eta}, \qquad \cos\psi = \frac{\tan\beta}{\tan\eta}$$

Let $r' = \frac{dr}{d\psi}\psi' = \frac{1}{c_1 r^2}\frac{dr}{d\psi}$. This transforms (22) into

$$(25) \qquad \left(\frac{dr}{d\psi}\right)^2 = r^2(c_2 c_1^2 r^2 + 2c_1^2 MGr - 1)$$

and the general integral of (25) is

$$(26) \qquad r = \frac{p}{1 - e\cos(\psi + c_4)}$$

where c_4 is another integration constant. (26) is the equation of a conic and for $|e| < 1$, it is the equation of an ellipse of eccentricity e. The values of the parameters e and p are given by

$$(27) \qquad e = \sqrt{1 + \frac{c_2}{(c_1 MG)^2}}$$

and

$$(28) \qquad p = \frac{1}{c_1^2 MG} = a(1 - e^2)$$

where

$$a = \text{semi-major axis of ellipse} = -\frac{MG}{c_2}.$$

It follows that c_2 must be negative for trajectories designed to remain in the gravitational field of the earth. In equation (26), the semi-major axis will be characterized by $\psi = -c_4$.

In order to obtain a relationship with time, we integrate (22) directly, obtaining

$$(29) \qquad \pm t = \frac{1}{c_2}\left[\sqrt{c_2 r^2 + 2MGr - \frac{1}{c_1^2}} + \frac{MG}{\sqrt{-c_2}} \operatorname{argsin} \frac{c_2 r + MG}{\sqrt{(MG)^2 + \frac{c_2}{c_1^2}}}\right] + c_5$$

where c_5 is a constant.

We now have all formulae necessary to verify the numerical integration of (20). Given the initial values of v_v, v_h, β, r, φ and t, the constants of integration can be obtained as follows: c_1 is obtained from (21); c_2 from (22); c_3 and thus also η from the formula following (22). The initial value of ψ can be calculated from (24), and c_4 from (26). Once c_1 and c_2 are known, c_5 can be obtained from (29) and the conventions that must be observed for this determination may be as follows: the positive sign is taken for t and, assuming that the initial data corresponds to a situation prior to impact (missile case), the initial argsin is chosen between $-\frac{\pi}{2}$ and 0. In effect, $c_2 r + MG$ is always negative during normal missile flight. This can be seen by noting that, at the apogee

$$r'^2 = 0 = -\frac{1}{c_1^2 r^2} + \frac{2MG}{r} + c_2$$

following

$$r_a = -\frac{MG}{c_2} + \sqrt{\left(\frac{MG}{c_2}\right)^2 + \frac{1}{c_2 c_1^2}}$$

so that

$$r_a c_2 + MG = c_2\sqrt{\left(\frac{MG}{c_2}\right)^2 + \frac{1}{c_2 c_1^2}} < 0$$

and furthermore, $rc_2 + MG$ is zero at $r = a$, a point usually below the surface of the earth. At impact, argsin will be chosen between $-\pi$ and $-\frac{\pi}{2}$.

Regarding the determination of ψ from (26), notice that prior to apogee, $\psi < -c_4$ and after apogee, $\psi > -c_4$.

Several formulae should be recalled for the Keplerian case. Given the values of r and v at any point, the semi-major axis a can be obtained from

$$(30) \qquad a = \frac{r}{2 - \frac{rv^2}{MG}}$$

If in addition, the angle α is known at that point, the eccentricity e can be obtained from

$$(31) \qquad e^2 = \left(1 - \frac{rv^2}{MG}\right)^2 + \frac{r^2v^2 \sin^2 \alpha}{MGa}$$

Let the index i refer to initial conditions and f to final conditions. Then, the angle $\Delta\psi = \psi_f - \psi_i$ covered during flight, can be obtained from

$$(32) \qquad \sin \Delta\psi = v_{vi}\sqrt{\frac{a(1-e^2)}{MG}}\left[\frac{a}{r_f} - \frac{1}{e^2}\left(\frac{a}{r_f} - 1\right)\right] + \frac{a}{r_f}\left(1 - \frac{r_i v_{hi}^2}{MG}\right)\sqrt{\left(\frac{1}{e^2} - 1\right)\left[1 - \frac{1}{e^2}\left(\frac{r_f}{a} - 1\right)^2\right]}$$

If the apogee occurs at $\psi = \psi_a$, then $\psi_a - \psi_i$ can be obtained from

$$(33) \qquad \begin{cases} \cos(\psi_a - \psi_i) = \frac{1}{e}\left(1 - \frac{r_i v_{hi}^2}{MG}\right) \\ \sin(\psi_a - \psi_i) = \frac{v_{vi}}{e}\sqrt{\frac{a(1-e^2)}{MG}} \end{cases}$$

Let α_{im} be that value of α_i which maximizes $\Delta\psi$ for a given value of v_{vi} and known values of r_i and r_f. This would correspond to the minimum energy trajectory. α_{im} can be obtained from

$$(34) \qquad \sin^2 \alpha_{im} = \frac{MGa}{r_i^2 v_i^2}\left[\frac{3r_f - r_i - \frac{2v_i^2}{MG} r_i r_f}{r_f + r_i} - \left(1 - \frac{r_i v_i^2}{MG}\right)^2\right]$$

If $r_i = r_f$ (implying $\psi_a - \psi_i = \frac{1}{2}\Delta\psi$), (34) reduces to

$$(35)\quad \begin{cases} \sin^2 \alpha_{im} = \dfrac{1 - \dfrac{r_i v_i^2}{MG}}{2 - \dfrac{r_i v_i^2}{MG}} \\[2em] \cos^2 \alpha_{im} = \dfrac{1}{2 - \dfrac{r_i v_i^2}{MG}} = \dfrac{a}{r_i} \end{cases}$$

Let $e = e_m$ if $\alpha_i = \alpha_{im}$. Then, equation (31) can be transformed into

$$(36)\qquad e_m^2 = \frac{2r_i r_f}{a(r_i + r_f)} - 1$$

Define $y = \frac{r_i v_i^2}{MG}$. Then, (35) becomes

$$(37)\quad \begin{cases} \sin^2 \alpha_{im} = \dfrac{1 - y}{2 - y} \\[1em] \cos^2 \alpha_{im} = \dfrac{1}{2 - y} \end{cases}$$

and (31) can be transformed into

$$(38)\qquad e^2 = 1 - y(1 - y) \cos^2 \alpha_i$$

so that (36) becomes

$$(39)\qquad e_m^2 = 1 - y$$

Define $z = \frac{r_f v_f^2}{MG}$. Then, expression (32) can be transformed into

$$(40)\quad \sin \Delta\psi = \frac{1}{e^2}\left[\, y \sin \alpha_i \cos \alpha_i \,(1 - z \cos^2 \alpha_f) + z \sin \alpha_f \cos \alpha_f \,(1 - y \cos^2 \alpha_i)\right]$$

where $z = 2 - \frac{r_f}{r_i}(2 - y)$, and

$$\cos^2 \alpha_f = \frac{y}{z}\frac{r_i}{r_f} \cos^2 \alpha_i$$

In particular, if $r_i = r_f$, then $\alpha_i = \alpha_f$ and (40) becomes

$$(41)\quad \sin \Delta\psi = \frac{2}{e^2}\, y \sin \alpha_i \cos \alpha_i \,(1 - y \cos^2 \alpha_i)$$

and, if in (41) we set $\alpha_i = \alpha_{im}$, we obtain

$$(42)\quad \sin \Delta\psi = \frac{4y\sqrt{1 - y}}{(2 - y)^2}$$

We consider formula (42) quite interesting for estimating the capabilities of a given missile, since it gives the maximum geodetic coverage on a spherical globe. Notice the following table of values

y	$\Delta\psi$
0	0
$\frac{2}{1+\sqrt{2}}$	$\frac{\pi}{2}$
1	π

It should be remembered that these formulae refer to an inertial system, i.e. do not take into consideration the rotation of the earth. This means that the true geodetic coverage will be larger than $\Delta\psi$ if the movement of the missile has a positive West component and smaller than $\Delta\psi$ if there is a positive East component. Thus, in case of war, it is advantagous to have our enemy on the West side. The correction due to the earth's rotation, is not difficult to calculate. For Keplerian movement as well as for any other case, we determine impact latitude and longitude, as well as flight time t in the inertial system and correct longitude only by adding ωt, where ω is the rotation of the earth, positive westward and negative eastward.

Similar considerations apply to the determination of the vector $\bar{v}_i$. As measured by an observer rotating with the earth, a component ωR pointed eastward must be added, which changes the value and the direction of v_i. A further refinement which must not be omitted, is that the local plumb line is not usually perpendicular to the local horizontal, as defined in this paper.

This concludes our discussion and recommended formulae for the Keplerian case. Although far from complete, this paragraph remains within its objective of providing a checking case and approximate initial estimates.

The General Equation of Movement

We designate by this name a particular form of equation (9) or their equivalents in terms of precession and/or nutation for some suitable expression of the force F. To this effect, we decompose F as follows:

F_1 = the force due to the potential of the earth, i.e. gravity;

F_2 = atmospheric drag. This force may be computed either considering or neglecting the effect of earth rotation, the decision depending on some numerical analysis which is pending. By effect of earth rotation, we mean that effect on air drag only, i.e. the latitude correction indicated in the previous paragraph is not being referred to and is always assumed to be necessary after the integration of the equations has been completed;

F_3 = thrust;

F_4 = extraneous forces such as the potential of bodies other than the earth, light pressure, etc. In the case of near-earth missiles, this force is considered negligible and will therefore not be analyzed in this paper. It should be noted that these forces are definitely not negligible in the case of orbiting satellites.

Regarding F_1, there are three cases to be considered, namely:

1. The earth is assumed spherical and the mass distribution is symmetrical with respect to its center O. In this case, F_1 is colinear with N;

II. The earth is a body of revolution masswise around z, which is more restrictive than the same assumption regarding shape only, and thus may be assumed to be an oblate spheriod. In this case, F_1 has components in the N and T directions only;

III. The earth is not a body of revolution, at least masswise. In this case F_1 has components in the N, T and B directions, in probable order of decreasing magnitude.

Case I is actually a particular case of II and, if no other effects are being considered, corresponds to Keplerian movement. Therefore, it is not considered sufficiently important to be studied separately. Although the refinement which case III represents over case II cannot, a priori, be rejected as negligible, this case has not been well studied as to date. Blitzer has recently published a series of formulae for the earth potential assuming axial assymmetry but as to date, their experimental verification is not complete. Thus, we shall confine our study to case II.

For the potential of the earth, we take

$$U = -\frac{MG}{r}\left[1 + \frac{A_2}{r^2}\left(\frac{1}{3} - \sin^2\varphi\right) + \frac{A_3}{2r^3}(5\sin^3\varphi - 3\sin\varphi) + \frac{A_4}{r^4}\left(\frac{3}{35} + \frac{1}{7}\sin^2\varphi - \frac{1}{4}\sin^2 2\varphi\right)\right] \quad (43)$$

where

$A_2 = a_e^2 \times 1.6208 \times 10^{-3}$

$A_3 = a_e^3 \times (2.20 \pm .08) \times 10^{-6}$

$A_4 = a_e^4 \times 2.1 \times 10^{-5}$

a_e = equatorial radius of the earth ≈ 6380 km and MG has been given following formula (20).

Formula (43) represents the first four terms of the familiar expansion of the earth potential in spherical harmonics. This expansion, although convergent, does not converge to the true value because of considerations previously indicated under case III. Four terms is the maximum which may be taken under any circumstance and it is questionable whether the third and fourth terms are reasonable at all.

The sign of U has been taken such that

$$(44) \qquad F_1 = - \text{grad } U$$

so that, in the NBT system, the components of F_1 will be

$$-\frac{\partial U}{\partial r}, \quad -\frac{1}{r}\frac{\partial U}{\partial \varphi}, \quad 0, \quad \text{in that order.}$$

Thus in the NAR system, the components of F_1 will be

$$-\frac{\partial U}{\partial r}, \quad -\frac{\sin\beta}{r}\frac{\partial U}{\partial \varphi}, \quad -\frac{\cos\beta}{r}\frac{\partial U}{\partial \varphi}$$

in that order. Hence, in equation (9), we have

$$(45) \qquad \begin{cases} F_{1_N} = -\dfrac{\partial U}{\partial r} \\ F_{1_A} = -\dfrac{\sin\beta}{r}\dfrac{\partial U}{\partial \varphi} \\ F_{1_R} = -\dfrac{\cos\beta}{r}\dfrac{\partial U}{\partial \varphi} \end{cases}$$

Notice that if <u>only</u> F_1 is being considered, then by virtue of (19) the precession μ can be obtained from

$$(46) \qquad \mu' = \frac{\cos\beta \sin\varphi}{v_h r \sin^2\eta}\frac{\partial U}{\partial \varphi}$$

and, if we consider only the first two terms of (43), then

$$\frac{\partial u}{\partial \varphi} = MG\frac{A_2}{r^3}\sin 2\varphi$$

so that

$$(47) \qquad \mu' = 2MGA_2\frac{\cos\eta \sin^2\varphi}{v_h r^4 \sin^2\eta}$$

We shall study F_2 taking into consideration the rotation of the earth as explained at the beginning of this paragraph. The formula, generally accepted in the literature and in the Transit project, is

$$F_2 = \frac{1}{2} C_D A_s \sigma(h) v_a \bar{v}_a \tag{48}$$

where C_D is a dimensionless coefficient;

A_s = Vehicle's cross section in the direction $\bar{v}_a$;

$\sigma(h)$ = air density as a function of altitude h;

v_a = velocity of the air, at and with respect to P, assuming that the atmosphere rotates rigidly with the earth.

By virtue of the previous definition, the vector $\bar{v}_a$ will be given by

$$\bar{v}_a = \omega r \cos \varphi \cdot \bar{B} - \bar{v} \tag{49}$$

where ω is the angular velocity of rotation of the earth, i.e. assuming one revolution in approximately 24 hours, we have

$$\omega \approx .0041666667 \text{ degrees/sec.}$$

Now, since

$$\bar{v} = v_v N + v_h \cos \beta B + v_h \sin \beta T$$

we have

$$\begin{aligned} \bar{v}_a &= - v_v N + (\omega r \cos \varphi - v_h \cos \beta) B - v_h \sin \beta T \\ &= - v_v N + (\omega r \cos \varphi \cos \beta - v_h) A - \omega r \cos \varphi R \end{aligned} \tag{50}$$

and

$$v_a^2 = v_v^2 + (v_h - \omega r \cos \varphi \cos \beta)^2 + (\omega r \cos \varphi)^2$$

Using (48) and (50), we obtain

$$
(51) \quad \begin{cases} F_{2_N} = -\frac{1}{2} C_D A_s \sigma(h) v_a v_v \\ F_{2_A} = \frac{1}{2} C_D A_s \sigma(h) v_a (\omega r \cos\varphi \cos\beta - v_h) \\ F_{2_R} = -\frac{1}{2} C_D A_s \sigma(h) v_a \omega r \cos\varphi \end{cases}
$$

If we want to neglect the effect of earth rotation, we set $v_a = v$ and $\omega = 0$.

Notice that if F_1 and F_2 are being considered, formula (46) for the precession is not longer valid. Instead, we must use

$$
(52) \qquad \mu' = \frac{\sin\varphi}{v_h \sin^2\eta} \left(\frac{\cos\beta}{r} \frac{\partial U}{\partial\varphi} - \frac{1}{2} C_D A_s \sigma(h) v_a \omega r \cos\varphi \right)
$$

In first approximation, A_s may be considered a constant. Otherwise, it can be expressed as a function of the angle between $\bar{v}$ and $\bar{v}_a$ and in this form, will not upset the quasi-linearity of the equations. $\sigma(h)$ must be known empirically, preferable in analytical form.

Regarding the thrust F_3, it will be considered as follows:

The thrust F_3 will be assumed constant and the mass m of the vehicle will be assumed to vary linearly according to

$$
m = m_o - (m_o - m_1) \frac{t - t_o}{t_1 - t_o}
$$

where m_o = initial mass of vehicle, at time $t = t_o$;

m_1 = mass of vehicle at burnout, $t = t_1$.

Hence

$$
(53) \qquad m = a_1 - a_2 t
$$

where a_1 and a_2 are two positive constants given by

$$(54)\quad \begin{cases} a_1 = \dfrac{m_o t_1 - m_1 t_o}{t_1 - t_o} \\[2ex] a_2 = \dfrac{m_o - m_1}{t_1 - t_o} \end{cases}$$

If in addition, F_s is supposed colinear with $\bar{v}$, then

$$(55)\quad \begin{cases} F_{s_N} = \dfrac{F_s}{m} \sin \alpha \\[1ex] F_{s_A} = \dfrac{F_s}{m} \cos \alpha \\[1ex] F_{s_R} = 0 \end{cases}$$

Using the same technique, additional forces can easily be introduced. For reasons of briefness, we do not carry our analysis beyond this point. Combining (9), (45), (51) and (55), we obtain rather general equations of movement, namely

$$(56)\quad \begin{cases} v_v' = \dfrac{v_h^2}{r} - \dfrac{\partial U}{\partial r} - \dfrac{1}{2} C_D A_s \sigma(h)\ v_a v_v + \dfrac{F_s}{m} \sin \alpha \\[2ex] v_h' = -\dfrac{v_v v_h}{r} - \dfrac{\sin \beta}{r} \dfrac{\partial U}{\partial \varphi} + \dfrac{1}{2} C_D A_s \sigma(h)\ v_a\ (\omega r \cos \varphi \cos \beta - v_h) + \dfrac{F_s}{m} \cos \alpha \\[2ex] \beta' = -\dfrac{v_h}{r} \cos \beta \tan \varphi - \dfrac{\cos \beta}{v_h r} \dfrac{\partial U}{\partial \varphi} - \dfrac{1}{2} C_D A_s \sigma(h)\ v_a\ \omega r \cos \varphi \\[2ex] r' = v_v \\[2ex] \varphi' = \dfrac{v_h}{r} \sin \beta \end{cases}$$

where $\sin \alpha = \dfrac{v_v}{v}$, $\cos \alpha = \dfrac{v_h}{v}$ and m is given by (53) and (54). Since $F_{s_R} = 0$, expression (52) of the precession continues to be valid. Since in general, the angle θ must be known, (10) must be integrated together with (56). In the case of a satellite, the nodal regression can be found very accurately by integrating (52) over a period of revolution.

Nodal Regression

We designate by Nodes the intersection between the orbit of a satellite and the equatorial plane. In each revolution of the satellite, the nodes move on the equatorial plane in a direction contrary to the projection of the satellite's movement on the equatorial plane, by an amount $\Delta\mu$ called nodal regression.

In order to obtain $\Delta\mu$ under consideration of the earth's oblateness, air drag and effect of earth rotation on the latter, expression (52) must be integrated numerically over a period of revolution. If air drag and associated effects may be considered negligible, a second approximation of $\Delta\mu$ can be obtained from the numerical integration of (46). A third approximation can be obtained by numerical integration of (47) and we must remember that as we increase the degree of approximation, we are in this case, decreasing accuracy. A fourth approximation is obtainable analytically by assuming the hypothesis conducive to (47) and in addition, assuming that during a particular revolution, the departure from Keplerian movement is negligible. This calculation was performed for the first time by Leon Blitzer and published in the Journal of Applied Physics, Vol. 28, No. 11, Nov. 1957. Blitzer's results have been widely used, in many cases with apparent indiscrimination as to expectable accuracy.

In order to emphasize this point, we present our own proof of Blitzer's formula. From (47) and (24), we obtain

$$\mu' = 2MG\, A_2 \frac{\cos\eta \sin^2\psi}{v_h\, r^4}$$

and since $v_h = \psi' r$

$$\mu' = 2MG\, A_2 \cos\eta \frac{\sin^2\psi}{\psi'\, r^5}$$

Let T_0 be the time between consecutive nodal passages. As t varies from 0 to T_0, ψ varies between 0 and approximately 2π, actually to a somewhat smaller value. Hence

$$\Delta\mu = \int_0^{T_0} \mu' \, dt = 2MG \, A_2 \cos \eta \int_0^{T_0} \frac{\sin^2 \psi}{\psi' \cdot r^5} \, dt$$

$$= 2MG \, A_2 \cos \eta \int_0^{2\pi} \frac{\sin^2 \psi}{\psi'^2 \, r^5} \, d\psi$$

where $\cos \eta$ is held constant by virtue of the assumption that we have Keplerian movement.

In order to eliminate ψ', we use expression (23), obtaining

$$(57) \qquad \Delta\mu = 2MG \, A_2 \, C_1^2 \cos \eta \int_0^{2\pi} \frac{\sin^2 \psi}{r} \, d\psi$$

c_1 can be caluclated by considering the situation at apogee. Here $r = a(1 + e)$ and $v_h = v_a$ where e is the eccentricity of the particular orbit and v_a the velocity at apogee. (Do not confuse with velocity relative to air). Hence, since $\psi' = \frac{v_h}{r}$, we have

$$c_1 = \frac{1}{v_a \, a(1 - e)}$$

v_a is obtained from (30) by setting $r = a(1 + e)$. We obtain

$$v_a = \sqrt{\frac{MG}{a} \frac{1 - e}{1 + e}}$$

so that

$$(58) \qquad c_1^2 = \frac{1}{MGa(1 - e^2)}$$

Furthermore, by virtue of (26) and (28)

$$r = \frac{a(1 - e^2)}{1 - e \cos(\psi + c_4)}$$

Substituting this expres ion and (58) in (57), we obtain

$$\Delta\mu = \frac{2A_2 \cos \eta}{a^2(1 - e^2)^2} \int_0^{2\pi} \sin^2 \psi \, [1 - e \cos (\psi + c_4)] \, d\psi$$

Now $\int_0^{2\pi} \sin^2 \psi \cos (\psi + c_4) \, d\psi = 0$ for all constant c_4 (another approximation) and $\int_0^{2\pi} \sin^2 \psi \, d\psi = \pi$.

Hence

$$\Delta\mu = \frac{2A_2 \pi \cos \eta}{a^2(1 - e^2)^2} \tag{59}$$

(59) is Blitzer's formula.

One of the objects of this paper is to calculate the departure of (59) from the numerical integrals of (52), (46) and (47).

Advance of the Line of Apsides

The line of apsides is defined as the radius vector corresponding to apogee. Let ψ_{a1} and ψ_{a2} be the values of ψ_a, the value of ψ corresponding to apogee - in two consecutive orbital revolutions of a satellite and in that order. The advance of the line of apsides is defined as

$$\Delta\psi_a = \psi_{a2} - \psi_{a1} \tag{60}$$

and this movement is called retrograde if $\Delta\psi_a$ is negative.

Again, we list approximations in order of decreasing accuracy. A first approximation of $\Delta\psi_a$ can be obtained from the numerical integration of (56). Let φ_a and β_a be the values of φ and β at r = maximum. Then by virtue of (11),

$$\cos \eta_a = \cos \varphi_a \cos \beta_a$$

so that ψ_a can be calculated from

$$\sin \psi_a = \frac{\sin \varphi_a}{\sin \eta_a} \qquad \text{and/or} \qquad \cos \psi_a = \frac{\tan \beta_a}{\tan \eta_a}$$

by virtue of (24). A similar value of ψ_a is obtained at the next maximum of r and substitution in (60) yields $\Delta\psi_a$.

A second approximation can be obtained by neglecting in (56), all forces not due to the potential of the earth. A third and important approximation corresponds to the previous case where we consider for the earth's potential, only the first two terms of its expansion in spherical harmonics as given by (43). The system (56) becomes, in this case

$$(61)\quad \begin{cases} v_v' = \dfrac{v_h^2}{r} - \dfrac{MG}{r^2}\left[1 + \dfrac{3A_2}{r^2}\left(\dfrac{1}{3} - \sin^2\varphi\right)\right] \\ v_h' = -\dfrac{v_v v_h}{r} - MG\dfrac{A_2}{r^4}\sin 2\varphi \sin\beta \\ \beta' = -\dfrac{v_h}{r}\cos\beta\tan\varphi - MG\dfrac{A_2}{v_h r^4}\sin 2\varphi\cos\beta \\ r' = v_v \\ \varphi' = \dfrac{v_h}{r}\sin\beta \end{cases}$$

By using the angles ψ and η instead of φ and β, the system (61) can be replaced by the equivalent system

$$(62)\quad \begin{cases} v_v' = \dfrac{v_h^2}{r} - \dfrac{MG}{r^2}\left[1 + \dfrac{3A_2}{r^2}\left(\dfrac{1}{3} - \sin^2\psi\sin^2\eta\right)\right] \\ v_h' = -\dfrac{v_v v_h}{r} - MG\dfrac{A_2}{r^4}\sin^2\eta\sin 2\psi \\ \eta' = -MG\dfrac{A_2}{2v_h r^4}\sin 2\eta\sin 2\psi \\ r' = v_v \\ \psi' = \dfrac{v_h}{r} \end{cases}$$

Details of the transformation are rather obvious and are being omitted for reasons of briefness. For the calculation of the line of apsides, the system

(62) should be preferred over (61), since its numerical integration is somewhat faster and thus, saves machine time.

A fourth approximation can be obtained from the analytical integration of (62), a process to which we shall refer again with somewhat more detail, in a later paragraph of this paper. It yields a widely known formula namely

$$(63) \quad \Delta\psi_a = \frac{2\pi A_2}{a^2 (1 - e^2)^2} \left(2 - \frac{5}{2} \sin^2 \eta\right)$$

generally also attributed to Blitzer and published in the same reference of the previous paragraph. The approximations introduced by (63) and not involved in (62) are of two kinds: first a and e are respectively the semi-major axis and eccentricity at the first apogee where the inclination of the Q plane is η and must be calculated using the assumption that the movement is Keplerian at that time; second, $\Delta\psi_a$ as given by (63) represents some average value of the advance of the line of apside, obtained by neglecting the terms which indicate its dependence on ψ_{a_1}. This point will be clarified later.

In general, the same comments apply to formula (63) and (59). Nevertheless, (63) illustrates a fact which can be easily evidenced by physical reasoning, namely that the line of apsides advances if the orbit is sufficiently close to the equator, has retrograde movement if the orbit is sufficiently near polar and is stationary at some intermediate value of η. This value, according to (63), is approximately 63.5° and is not in good agreement with experimental observation.

The Analytical Integral of a Special Non-Keplerian Case

The case that we are referring to, corresponds to the systems (61) or (62) of the previous paragraph. The results are only partial, that is, these systems were not completely integrated and only the values of certain orbital parameters were obtained. Originality for these calculations is usually attributed to L. E. Cunningham, although to the best of the writers knowledge, the derivations have not been published yet as this is being written. The results, in somewhat different form, can be found in the book by T. E. Sterne, An Introduction to Celestial Mechanics, pp. 122-124. For reasons of briefness, we must omit the derivations.

It is being assumed that the trajectory can be decomposed into infinitesimal segments, each one representing a piece of a Keplerian orbit. This assumption, obviously does not represent an approximation, but it requires that we sharpen the definitions of our parameters; namely, let a, e, η and ψ_{a_1} represent respectively the semi-major axis, eccentricity, inclination of Q plane and apogee angle for some particular position along the trajectory. In addition, K_a, K_e, K_η, K_μ, K_ψ, K_m is a set of constants, depending on the orbital parameters at that particular point. The determination of these constants will be clarified later. The following formulae will represent a, e, η, μ, ψ_a and the mean anomaly m as a function of ψ. The mean anomaly is given by m = nt where t is time since apogee passage and n is the angular frequency given by $n = \sqrt{MG/a^3}$.

$$
\begin{aligned}
a(\psi) = K_a &+ \frac{2A_2}{a(1-e^2)^2}\Big\{e\left(1+\frac{e^2}{4}\right)\left(1-\frac{3}{2}\sin^2\eta\right)\cos(\psi-\psi_{a_1}) \\
&+ \frac{3}{4}e\left(1+\frac{e^2}{4}\right)\sin^2\eta\cos(\psi+\psi_{a_1}) + \frac{e^3}{16}\sin^2\eta\cos(3\psi_{a_1}-\psi) \\
&+ \frac{e^2}{2}\left(1-\frac{3}{2}\sin^2\eta\right)\cos 2(\psi-\psi_{a_1}) + \frac{1}{2}\left(1+\frac{3}{2}e^2\right)\sin^2\eta\cos 2\psi \\
&+ \frac{e^3}{12}\left(1-\frac{3}{2}\sin^2\eta\right)\cos 3(\psi-\psi_{a_1}) + \frac{3}{4}e\left(1+\frac{e^2}{4}\right)\sin^2\eta\cos(3\psi-\psi_{a_1}) \\
&+ \frac{3}{8}e^2\sin^2\eta\cos 2(2\psi-\psi_{a_1}) + \frac{e^3}{16}\sin^2\eta\cos(5\psi-3\psi_{a_1})\Big\}
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
e(\psi) = K_e &+ \frac{A_e}{a^2(1-e^2)^2}\Big\{\left(1+\frac{e^2}{4}\right)\left(1-\frac{3}{2}\sin^2\eta\right)\cos(\psi-\psi_{a_1}) \\
&+ \frac{1}{4}\left(1+\frac{11}{4}e^2\right)\sin^2\eta\cos(\psi+\psi_{a_1}) + \frac{e^2}{16}\sin^2\eta\cos(3\psi_{a_1}-\psi) \\
&+ \frac{e}{2}\left(1-\frac{3}{2}\sin^2\eta\right)\cos 2(\psi-\psi_{a_1}) + \frac{5}{4}e\sin^2\eta\cos 2\psi \\
&+ \frac{e^2}{12}\left(1-\frac{3}{2}\sin^2\eta\right)\cos 3(\psi-\psi_{a_1}) + \frac{1}{12}\left(7+\frac{17}{4}e^2\right)\sin^2\eta\cos(3\psi-\psi_{a_1}) \\
&+ \frac{3}{8}e\sin^2\eta\cos 2(2\psi-\psi_{a_1}) + \frac{e^2}{16}\sin^2\eta\cos(5\psi-3\psi_{a_1})\Big\}
\end{aligned}
\tag{65}
$$

(66) $$\eta(\psi) = K_\eta + \frac{A_2 \sin^2 \eta}{4a^2(1 - e^2)^2} [e \cos(\psi + \psi_{a1}) + \cos 2\psi + \frac{e}{3} \cos(3\psi - \psi_{a1})]$$

(67) $$\mu(\psi) = K_\mu - \frac{A_2 \cos \eta}{2a^2(1 - e^2)^2} [2(\psi - \psi_{a1}) + 2e \sin(\psi - \psi_{a1}) - e \sin(\psi + \psi_{a1})$$

$$- \sin 2\psi - \frac{e}{3} \sin(3\psi - \psi_{a1})]$$

(68)

$$\psi_a(\psi) = K_\psi + \frac{A_2}{2a^2(1 - e^2)^2} \{(4 - 5 \sin^2 \eta)(\psi - \psi_{a1}) + \frac{1}{e}[(2 - 3 \sin^2 \eta)$$

$$+ e^2(\frac{7}{2} - \frac{17}{4} \sin^2 \eta) + \frac{e^2}{4} \sin^2 \eta \cos 2\psi_{a1}] \sin(\psi - \psi_{a1})$$

$$- \frac{1}{e}[e^2 + \frac{1}{2}(1 - \frac{7}{2} e^2) \sin^2 \eta] \sin(\psi + \psi_{a1}) + (1 - \frac{3}{2} \sin^2 \eta) \sin 2(\psi - \psi_{a1})$$

$$- (1 - \frac{5}{2} \sin^2 \eta) \sin 2\psi + \frac{e}{6}(1 - \frac{3}{2} \sin^2 \eta) \sin 3(\psi - \psi_{a1})$$

$$- \frac{1}{3e}[e^2 - \frac{1}{2}(7 + \frac{19}{4} e^2) \sin^2 \eta] \sin(3\psi - \psi_{a1})$$

$$+ \frac{3}{4} \sin^2 \eta \sin 2(2\psi - \psi_{a1}) + \frac{e}{8} \sin^2 \eta \sin(5\psi - 3\psi_{a1})\}$$

(69)

$$m(\psi) = K_m + n_r t + \frac{A_2}{ea^2(1 - e^2)^{3/2}} \{-[(1 - \frac{e^2}{4})(1 - \frac{3}{2} \sin^2 \eta)$$

$$+ \frac{e^2}{8} \sin^2 \eta \cos 2\psi_{a1}] \sin(\psi - \psi_{a1}) + \frac{1}{4}(1 + \frac{3}{2} e^2) \sin^2 \eta \sin(\psi - \psi_{a1})$$

$$- \frac{e}{2}(1 - \frac{3}{2} \sin^2 \eta) \sin 2(\psi - \psi_{a1}) - \frac{e^2}{12} (1 - \frac{3}{2} \sin^2 \eta) \sin 3(\psi - \psi_{a1})$$

$$- \frac{1}{12}(7 - \frac{e^2}{4}) \sin^2 \eta \sin(3\psi - \psi_{a1}) - \frac{3}{8} e \sin^2 \eta \sin 2(2\psi - \psi_{a_1})$$

$$- \frac{e^2}{16} \sin^2 \eta \sin(5\psi - 3\psi_{a1})\}$$

where, in (69), t is the time since apogee passage at $m = m(\psi)$ and n_r is the mean anomalistic motion, namely

$$(70) \qquad n_r = n\left[1 + \frac{A_2\, a}{r^3}\,(1 - 3 \sin^2 \varphi)\right]$$

n being the angular frequency at the reference point, namely $\sqrt{\frac{MG}{a^3}}$, φ and r corresponding also to the reference point.

The constants K must be determined so that, when ψ takes the value corresponding to the reference point, $a(\psi) = a$, $e(\psi) = e$, etc.

We are particularly interested in the variation of the parameters μ and ψ_a and, from (67) and (68), it appears that this variation is not periodic. In order to facilitate calculations, we set the reference at the first apogee. Then, if $\psi - \psi_{a1}$ is approximately equal to 2π, we will be at the second apogee. Hence

$$(71) \quad \mu(\psi_{a2}) = K_\mu - \frac{A_2 \cos \eta}{2a^2(1 - e^2)^2}\left[4\pi - \left(1 + \frac{4}{3} e\right) \sin 2\psi_{a1}\right]$$

$$(72) \quad \psi_{a2} = K_\psi + \frac{A_2}{2a^2(1 - e^2)^2}\Big\{2\pi(4 - 5 \sin^2 \eta)$$

$$- \left\{\frac{1}{3e}\left[4e^2 - \left(2 + \frac{61}{8}e^2\right) \sin^2 \eta\right] + 1 - \frac{1}{4}\left(13 + \frac{e}{2}\right) \sin^2 \eta\right\} \sin 2\psi_{a1}\Big\}$$

Neglecting the periodic terms, in order to obtain an average, we have approximately

$$\mu(\psi_{a2}) = K_\mu - \frac{2\pi\, A_2 \cos \eta}{a^2 (1 - e^2)^2}$$

and

$$\psi_{a2} = K_\psi + \frac{A_2\, \pi}{a^2 (1 - e^2)^2}\,(4 - 5 \sin^2 \eta)$$

Now, if we neglect the periodic terms in (67) and (68), use as reference point the first apogee and set $\psi = \psi_{a1}$, we see readily that $K_\mu = \mu(\psi_{a1})$ and $K_\psi = \psi_{a1}$. Hence

$$(73) \qquad \Delta\mu = -\frac{2\pi A_2 \cos \eta}{a^2(1 - e^2)^2}$$

and

$$(74) \qquad \Delta\psi_a = \frac{\pi A_2}{a^2(1 - e^2)^2} (4 - 5 \sin^2 \eta)$$

(74) is identical to (63); the negative sign in (73) indicates that the nodal precession is retrograde, so that (73) is equivalent to (59) which was derived by a direct procedure. The formulae (71) and (72) should be considered superior to (59) and (63) respectively, if the departures from the averages are desired.

<u>On the Integration of Quasi-linear Systems</u>

Consider a vector variable

$$\bar{x}(x_1, x_2, x_3, \ldots x_n)$$

and its derivative

$$\bar{x}'(x_1', x_2', x_3', \ldots x_n')$$

where primes indicate differentiation with respect to an independent variable t. We are concerned with presenting, without proof for reasons of briefness, a numerical integration algorithm involving error bounds, for a class of equations reducible to the form

$$(75) \qquad \bar{x}' = f(\bar{x}, t)$$

where f is a vector function.

It follows from (75) that

$$d\bar{x} = f(\bar{x}, t)\, dt$$

an expression which, for the discrete case, will be given the following alternate interpretations:

$$\Delta\bar{x} \approx f(\bar{x}, t)\,\Delta t$$

and

$$\Delta\bar{x} \approx f(\bar{x} + \Delta\bar{x}, t + \Delta t)\,\Delta t$$

and the function f will be required to satisfy the quite general condition

$$(76) \qquad \Delta\bar{x} = [\alpha\, f(\bar{x}, t) + (1 - \alpha)\, f(\bar{x} + \Delta\bar{x}, t + \Delta t)]\,\Delta t$$

where $0 \leq \alpha \leq 1$ for all points $\bar{x}$ of the integration domain. Then, the integration can take the form

$$(77) \qquad \Delta\bar{x} = \frac{1}{2}[f(\bar{x}, t) + f(\bar{x} + \Delta\bar{x}, t + \Delta t)]\,\Delta t$$

and the corresponding error $\Delta\varepsilon$ will satisfy

$$(78) \qquad \Delta\varepsilon \leq \frac{1}{2}|[f(\bar{x}, t) - f(\bar{x} + \Delta\bar{x}, t + \Delta t)]|\Delta t$$

Since the expressions (77) and (78) do not represent an explicit algorithm, a second order error, assumed negligible, will be introducted by using, in the right member of (77), the value $\Delta\bar{x} = f(\bar{x}, t)\,\Delta t$. $\Delta\varepsilon$ is obviously a vector increment and expression (78) should be intepreted in the sense that the components of $\Delta\varepsilon$ are half of the absolute values of the differences between the components of the two functions involved.

An integration routine, for arbitrary f, should have the following structure. Given

$$\bar{x}_i(x_{1i}, x_{2i}, \ldots x_{ni}),\ t_i,\ \Delta t$$

and the functions

$$x'_m = f_m(x_1, x_2, \ldots x_n, t), \quad m = 1, (1), n$$

calculate

$$\bar{x}_f(x_{1f}, x_{2f}, \ldots x_{nf}), \quad t_f, \ \bar{\varepsilon}(\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n)$$

subject to either

$$x_{rf} = K$$

or

$$t_f = K$$

where r and K are known. All data and results should be printed out and suitable entries should be provided for the functions f_m.

Procedure.

1. Set $x_1, x_2, \ldots x_n$, t. If this step is executed for the first time, the values $\bar{x}_i$ and t_i should be used; at every further execution, use the results from the previous calculation.
2. Calculate

$$(\Delta x_1)_o = f_1(\bar{x}, t)\ \Delta t$$

$$(\Delta x_2)_o = f_2(\bar{x}, t)\ \Delta t$$

$$(\Delta x_n)_o = f_n(\bar{x}, t)\ \Delta t$$

using the values from step 1. for $\bar{x}$ and t

3. Calculate

$$\Delta x_1 = \frac{1}{2}\{f_1(\bar{x}, t) + f_1[\bar{x} + (\Delta\bar{x})_o, t + \Delta t]\} \Delta t$$

$$\Delta x_2 = \frac{1}{2}\{f_2(\bar{x}, t) + f_2[\bar{x} + (\Delta\bar{x})_o, t + \Delta t]\} \Delta t$$

$$\Delta x_n = \frac{1}{2}\{f_n(\bar{x}, t) + f_n[\bar{x} + (\Delta\bar{x})_o, t + \Delta t]\} \Delta t$$

4. Obtain the components of $\Delta\varepsilon$ as follows

$$\Delta\varepsilon_1 = \frac{1}{2}|f_1(\bar{x}, t) - f_1(\bar{x} + \Delta\bar{x}, t + \Delta t)| \Delta t$$

$$\Delta\varepsilon_2 = \frac{1}{2}|f_2(\bar{x}, t) - f_2(\bar{x} + \Delta\bar{x}, t + \Delta t)| \Delta t$$

$$\Delta\varepsilon_n = \frac{1}{2}|f_n(\bar{x}, t) - f_n(\bar{x} + \Delta\bar{x}, t + \Delta t)| \Delta t$$

5. Calculate the cumulative error $\varepsilon(\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n)$ by adding the results of step 4. into the corresponding locations.
6. Calculate new values of the variables, in the form

$$x_1 + \Delta x_1$$

$$x_2 + \Delta x_2$$

$$x_n + \Delta x_n$$

$$t + \Delta t$$

and return to 1. until x_{rf} or r_f takes the specified value. If this occurs, print out results.

Coordinate Transformations

All our formulae refer to a geocentric inertial system as described in the first paragraph. The six scalars necessary to describe position and velocity of a point, have been substituted in most cases by the set of variables

(79) $r,\ v_v,\ v_h,\ \theta\ \varphi,\ \beta$

and if other variables are being used, the necessary transformations are indicated. Nevertheless, data given and requested by military organizations are usually referred to a different coordinate system. The latter will be described in this paragraph and in addition, we shall indicate the formulae to transform from this military system to our inertial system and vice-versa.

The military system is right handed, orthogonal, usually denoted by XYZ, with the origin fixed at some point on the surface of the earth and rotating rigidly with it. Let $P_1(\bar{R})$ be this origin, where $\bar{R}$ is the corresponding radius vector from the origin of our geocentric system. Then Z is pointing in the $\bar{R}$ direction, i.e. is the local vertical by definition and XY define the local horizontal plane, perpendicular to $\bar{R}$. Y points south, i.e. belongs to the Zz plane and X points west.

Let $\bar{R}o$ be the projection of $\bar{R}$ on the xy plane, φ_o the angle $\bar{R}o\bar{R}$ in that direction and θ_o the value of θ corresponding to $\bar{\rho} = \bar{R}o$. As before let t denote time and assume that at t = 0, $\bar{R}o$ coincides with the positive direction of the x axis. Thus

$$(80) \qquad \theta_o = \omega t \qquad (\omega = .0041666667 \text{ deg/sec})$$

We may assume in addition that at time $t = \frac{\theta_o}{\omega}$, the x axis has longitude zero. Then θ_o is its longitude east of P_1, i.e. $-\theta_o$ is its longitude west.

Let $\bar{X}_1$, $\bar{Y}_1$, $\bar{Z}_1$ denote the unit vectors corresponding to XYZ and as before, denote by i, j, K the unit vectors in the xyz directions. As usual, primes will indicate derivation with respect to time. Then

$$(81) \quad \begin{cases} \bar{X}_1 = i \sin\theta_o - j\cos\theta_o \\ \bar{Y}_1 = i\cos\theta_o \sin\varphi_o + j\sin\theta_o\sin\varphi_o - K\cos\varphi_o \\ \bar{Z}_1 = i\cos\theta_o\cos\varphi_o + j\sin\theta_o\cos\varphi_o + K\sin\varphi_o \end{cases}$$

and

$$
(82)\quad \begin{cases} \bar{X}_1' = \omega(i \cos \theta_o + j \sin \theta_o) \\ \bar{Y}_1' = \omega \sin \varphi_o(- i \sin \theta_o + j \cos \theta_o) \\ \bar{Z}_1' = \omega \cos \varphi_o(- i \sin \theta_o + j \cos \theta_o) \end{cases}
$$

With reference to our inertial system, if the point P, the movement of which is being studied, is defined in the XYZ system by the vector $\bar{r}_o$ and if R is the magnitude of $\bar{R}$, we have

$$
\bar{r} = R\bar{Z}_1 + \bar{r}_o
$$

$$
(83)\quad \begin{cases} = R(i \cos \theta_o \cos \varphi_o + j \sin \theta_o \cos \varphi_o + K \sin \varphi_o) + X\bar{X}_1 + Y\bar{Y}_1 + Z\bar{Z}_1 \\ = i(R \cos \theta_o \cos \varphi_o + X \sin \theta_o + Y \cos \theta_o \sin \varphi_o + Z \cos \theta_o \cos \varphi_o) \\ + j(R \sin \theta_o \cos \varphi_o - X \cos \theta_o + Y \sin \theta_o \sin \varphi_o + Z \sin \theta_o \cos \varphi_o) \\ + K(R \sin \varphi_o - Y \cos \varphi_o + Z \sin \varphi_o) \\ = xi + yj + zK \end{cases}
$$

Since R must be known and θ_o is given by (80), (83) permits the calculation of the inertial coordinates xyz as a function of XYZ and the corresponding time.

Furthermore

$$
\begin{aligned}
\bar{r}' &= R\bar{Z}_1' + \bar{r}_o' \\
&= R\bar{Z}_1' + X'\bar{X}_1 + Y'\bar{Y}_1 + Z'\bar{Z}_1 + X\bar{X}_1' + Y\bar{Y}_1' + Z\bar{Z}_1' \\
&= R\,\omega \cos \varphi_o(- i \sin \theta_o + j \cos \theta_o) + X'(i \sin \theta_o - j \cos \theta_o) \\
&+ Y'(i \cos \theta_o \sin \varphi_o + j \sin \theta_o \sin \varphi_o - K \cos \varphi_o) \\
&+ Z'(i \cos \theta_o \cos \varphi_o + j \sin \theta_o \cos \varphi_o + K \sin \varphi_o) \\
&+ X\omega(i \cos \theta_o + j \sin \theta_o) + Y\omega \sin \varphi_o(- i \sin \theta_o + j \cos \theta_o) \\
&+ Z\omega \cos \varphi_o(- i \sin \theta_o + j \cos \theta_o)
\end{aligned}
$$

$$(84)\quad \begin{cases} \bar{r}' = i(-R\omega \sin\theta_o \cos\varphi_o + X' \sin\theta_o + Y' \cos\theta_o \sin\varphi_o + Z' \cos\theta_o \cos\varphi_o \\ \qquad + X\omega \cos\theta_o - Y\omega \sin\theta_o \sin\varphi_o - Z\omega \sin\theta_o \cos\varphi_o) \\ \quad + j(R\omega \cos\theta_o \cos\varphi_o - X' \cos\theta_o + Y' \sin\theta_o \sin\varphi_o + Z' \sin\theta_o \cos\varphi_o \\ \qquad + X\omega \sin\theta_o + Y\omega \cos\theta_o \sin\varphi_o + Z\omega \cos\theta_o \cos\varphi_o) \\ \quad + K(-Y' \cos\varphi_o + Z' \sin\varphi_o) \\ \quad = X'i + y'j + z'K = \bar{v} \end{cases}$$

So that (84) gives the inertial components of the velocity. The angles θ and φ can be obtained from

$$\tan\theta = \frac{y}{x} \qquad \text{and} \qquad \tan\varphi = \frac{z}{\sqrt{x^2 + y^2}}$$

where

$$x^2 + y^2 = X^2 + R^2 \cos^2\varphi_o + (Y \sin\varphi_o + Z \cos\varphi_o)$$

$$(Y \sin\varphi_o + Z \cos\varphi_o + 2R \cos\varphi_o)$$

v_v follows from

$$v_v = \bar{N} \cdot \bar{v}$$
$$= (i \cos\theta \cos\varphi + j \sin\theta \cos\varphi + K \sin\varphi) \cdot \bar{v}$$

and v_h can be computed from $v_h^2 = v^2 - v_v^2$.

The angle β can be obtained from

$$(85) \qquad \cos\beta = \frac{r \cos\varphi}{x^2\, v_h} \, (y'x - yx') \cos^2\theta$$

Notice that the angles φ and φ_o correspond to <u>geocentric</u> latitude. The corresponding <u>geodetic</u> latitude is Φ and Φ_o and given approximately by

(86) $\tan \bar{\Phi} = \frac{\tan \varphi}{1 - \varepsilon^2}$

where ε^2 = .0067226701 = oblateness of the earth. The corresponding radius of the earth is approximately

(87) $R_e = \frac{a_e}{\sqrt{1 - \varepsilon^2 \sin^2 \Phi}} \quad \frac{\cos \Phi}{\cos \varphi}$

where a_e = equatorial radius of the earth = 2.093×10^7 ft.

In the determination of the angles θ, φ and β, the following considerations apply. For θ, we have

$$\tan \theta = \frac{y}{x}$$

as noted. If $\tan \theta$ is positive, then $0 < \theta < \frac{\pi}{2}$ if $x, y > 0$, and $\pi < \theta < \frac{3\pi}{2}$ if $x, y < 0$. If $\tan \theta$ is negative, then $\frac{\pi}{2} < \theta < \pi$ if $x < 0$, and $\frac{3\pi}{2} < 0 < 2\pi$ if $x > 0$. In other words, a standard argtan routine can be used.

Regarding φ, we have

$$\tan \varphi = \frac{z}{\sqrt{x^2 + y^2}} \qquad \text{and} \qquad |\varphi| \leq \frac{\pi}{2}$$

Thus, φ has the same sign as z.

Regarding β as given by (85), we must compute the sign of

(88) $v_T = - x' \cos \theta \sin \varphi - y' \sin \theta \sin \varphi + z' \cos \varphi$

where v_T is the projection of $\bar{v}$ on $\bar{T}$. If v_T is positive, then $0 < \beta \leq \frac{\pi}{2}$ if $\cos \beta$ is positive, and $\frac{\pi}{2} < \beta < \pi$ if $\cos \beta$ is negative. If v_T is negative, then $\pi < \beta \leq \frac{3\pi}{2}$ if $\cos \beta$ is negative and $\frac{3\pi}{2} < \beta < 2\pi$ if $\cos \beta$ is positive. Schematically:

Signum v_T	Signum $\cos \beta$	Bounds of β
+	+	$0 \to \frac{\pi}{2}$
+	-	$\frac{\pi}{2} \to \pi$
-	-	$\pi \to \frac{3\pi}{2}$
-	+	$\frac{3\pi}{2} \to 2\pi$

The preceding theory permits thus the transformation from the system

$$\Phi_o,\ t,\ h,\ X.\ Y,\ Z,\ X',\ Y',\ Z'$$

(h = altitude of P_1 above sea level) into the system

$$(89)\qquad r,\ v_v,\ v_h,\ \theta,\ \varphi,\ \beta$$

The inverse problem would be the determination of

$$X,\ Y,\ Z,\ X',\ Y',\ Z'$$

given (89) and Φ_o, t, h. From these three scalars, we can determine

$$R = h + \frac{a_e}{\sqrt{1 - \varepsilon^2 \sin^2 \Phi}} \quad \frac{\cos \Phi_0}{\cos \varphi_o}$$

where $\tan \varphi_o = (1 - \varepsilon^2) \tan \Phi_o$

and $\theta_o = \omega t$

Next, we compute

$$x = r \cos \varphi \cos \theta$$
$$y = r \cos \varphi \sin \theta$$
$$z = r \sin \varphi.$$

The substitution of these values in (83) permits the determination of X, Y, Z. Next, we comoute

$$(90)\qquad \begin{cases} x' = v_v \cos \theta \cos \varphi - v_h(\sin \theta \cos \beta + \cos \theta \sin \varphi \sin \beta) \\ y' = v_v \sin \theta \cos \varphi + v_h(\cos \theta \cos \beta - \sin \theta \sin \varphi \sin \beta) \\ z' = v_v \sin \varphi + v_h \cos \varphi \sin \beta \end{cases}$$

and find X′,Y′,Z′ by substitution in (84). Thus, the inverse problem can also be considered solved.

If the geodetic distance, measured on the surface of the earth, between two points p_1 and p_2 is desired, i.e.

$$\Lambda = \text{dist. } p_1(\theta_1, \varphi_1), \quad p_2(\theta_2, \varphi_2)$$

it can be calculated with

$$\Lambda = \frac{1}{2}(R_1 + R_2) \text{ arg cos } [\cos(\theta_2 - \theta_1 - \omega\Delta t) \cos(\varphi_2 - \varphi_1)] \tag{91}$$

where

$$R_{1,2} = \frac{a_e}{\sqrt{1 - \varepsilon^2 \sin^2 \Phi_{1,2}}} \quad \frac{\cos \Phi_{1,2}}{\cos \varphi_{1,2}}, \quad \text{and} \quad \tan \Phi_{1,2} = \frac{\tan \varphi_{1,2}}{1 - \varepsilon^2},$$

and Δt is the time interval for moving from P_1 to P_2.

Atmospheric Refraction

In order to measure the parameters involved in the preceding analysis, either with Transit equipment or solely ground equipment, it is necessary to have some criterion for the refraction effect of the atmosphere. Due to both periodic and secular variations, no great degree of reliability can be expected from any such criterion and for all practical purposes, we recommend to use the "Model Atmosphere", derived by the Central Radio Propagation Laboratories of the National Bureau of Standards in 1959.

Let

n = refractive index
N = refractivity

related by

$$N = (n - 1) \cdot 10^6$$

and let

N_o = refractivity at ground level.

Then, it has been established that

$$N_o = \frac{77.6}{T}\left[p + \frac{4810\, e_s\, (RH)}{T}\right] \tag{92}$$

where

T = temperature, degrees Kelvin to $\frac{1}{10}$ degree;

p = atmospheric pressure to nearest millibar;

e_s = saturation water pressure, to $\frac{1}{10}$ millibar. May be read from tables of temperature versus water pressure;

RH = relative humidity to 1%

Furthermore, for a height H above tracking equipment, it has been established that

(93)

$$N = N_o e^{-c_e H}$$

where

$$c_e = \ln \frac{N_o}{N_o + \Delta N}$$

and

$$\Delta N = -7.32 e^{.005577 N_o}$$

It follows, for the average refractivity $\bar{N}$ that

(94)
$$\bar{N} = \frac{1}{H}\int_o^H N_o dH = \frac{N_o}{c_e H} - \frac{N_o}{c_e H e^{c_e H}}$$

The second term of the last member being usually negligible. The refraction correction should not be confused or superseded by the squint angle correction, which considers the angle between the RF axis and measuring (digital) axis of a radar antenna.

Conclusions

Excluding interior ballistics, the preceding theory will permit the accurate calculation of the trajectory of near-earth missiles and satellites. At this point, we cannot indicate which effects should be taken into consideration in each particular case. Such a decision would require the following programs, the preparation of which is pending:

Two programs to perform the coordinate transformations indicated in the text;

A subroutine for the integration of quasi-linear first order systems. Such a subroutine should not be specific on the functions involved;

A subroutine for the analytical integration of the Keplerian case. The main object of this program would be to check the previous one.

Programs for the different cases presented in the text, in order to evaluate each perturbing effect separately;

Programs for the evaluation of nodal regression and apsidal advance, for different cases.

An accurate connection with Transit equipment can be easily made, using the expressions of the previous paragraph. Lately, extensive studies seem to be in progress, which consider the departure of the earth's potential from axial symmetry. It should be noticed that any new expression for the potential which may result from these studies, may easily be treated using the same techniques of this paper.

AD-

Land-Air, Inc., Point Mugu, Calif.
ON THE DYNAMICS OF NEAR-EARTH FLIGHT by Bernard G. Grunebaum, Derivation of orbit equations. 15 February 1962, 41p.
Contract N-123(61756)19425A/PMR Unclassified Report

The paper develops equations for near-earth and satellite flights. The use of experimental parameters is avoided as much as possible. Differential equations that defy analytical solution are made a quasi-linear type so numerical integration can be performed with close error bounds. The equations consider both thrust and ballistic phases and enable separate numerical analysis of such effects as oblateness, air drag, earth rotation. All analytical data were assumed to be determinable from doppler radar and/or Transit equipment and the algorithms achieve a reasonable compromise between accuracy and speed.

UNCLASSIFIED

1. Radar range computers
2. Radar programing
3. Computers
4. Digital computers
I. Grunebaum, Bernard G.
II. Pacific Missile Range
III. Contract N-123(61756) 19425A/PMR

UNCLASSIFIED

AD-

Land-Air, Inc., Point Mugu, Calif.
ON THE DYNAMICS OF NEAR-EARTH FLIGHT by Bernard G. Grunebaum, Derivation of orbit equations. 15 February 1962, 41p.
Contract N-123(61756)19425A/PMR Unclassified Report

The paper develops equations for near-earth and satellite flights. The use of experimental parameters is avoided as much as possible. Differential equations that defy analytical solution are made a quasi-linear type so numerical integration can be performed with close error bounds. The equations consider both thrust and ballistic phases and enable separate numerical analysis of such effects as oblateness, air drag, earth rotation. All analytical data were assumed to be determinable from doppler radar and/or Transit equipment and the algorithms achieve a reasonable compromise between accuracy and speed.

UNCLASSIFIED

1. Radar range computers
2. Radar programing
3. Computers
4. Digital computers
I. Grunebaum, Bernard G.
II. Pacific Missile Range
III. Contract N-123(61756) 19425A/PMR

UNCLASSIFIED

AD-

Land-Air, Inc., Point Mugu, Calif.
ON THE DYNAMICS OF NEAR-EARTH FLIGHT by Bernard G. Grunebaum, Derivation of orbit equations. 15 February 1962, 41p.
Contract N-123(61756)19425A/PMR Unclassified Report

The paper develops equations for near-earth and satellite flights. The use of experimental parameters is avoided as much as possible. Differential equations that defy analytical solution are made a quasi-linear type so numerical integration can be performed with close error bounds. The equations consider both thrust and ballistic phases and enable separate numerical analysis of such effects as oblateness, air drag, earth rotation. All analytical data were assumed to be determinable from doppler radar and/or Transit equipment and the algorithms achieve a reasonable compromise between accuracy and speed.

UNCLASSIFIED

1. Radar range computers
2. Radar programing
3. Computers
4. Digital computers
I. Grunebaum, Bernard G.
II. Pacific Missile Range
III. Contract N-123(61756) 19425A/PMR

UNCLASSIFIED

AD-

Land-Air, Inc., Point Mugu, Calif.
ON THE DYNAMICS OF NEAR-EARTH FLIGHT by Bernard G. Grunebaum, Derivation of orbit equations. 15 February 1962, 41p.
Contract N-123(61756)19425A/PMR Unclassified Report

The paper develops equations for near-earth and satellite flights. The use of experimental parameters is avoided as much as possible. Differential equations that defy analytical solution are made a quasi-linear type so numerical integration can be performed with close error bounds. The equations consider both thrust and ballistic phases and enable separate numerical analysis of such effects as oblateness, air drag, earth rotation. All analytical data were assumed to be determinable from doppler radar and/or Transit equipment and the algorithms achieve a reasonable compromise between accuracy and speed.

UNCLASSIFIED

1. Radar range computers
2. Radar programing
3. Computers
4. Digital computers
I. Grunebaum, Bernard G.
II. Pacific Missile Range
III. Contract N-123(61756) 19425A/PMR

UNCLASSIFIED