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DIFFUSION OF A PLASMA ACROSS A MAGNETIC FIELD

BY

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MAGNETOGASDYNAMICS LABORATORY
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DEPARTMENT OF MECHANICAL ENGINEERING
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OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE
BEDFORD, MASSACHUSETTS

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Abstract

DIFFUSION OF A PLASMA ACROSS A MAGNETIC FIELD

by

Ronald Stephen Finn

An investigation of the problem of plasma diffusion through a magnetic field is made through the macroscopic fluid equations. The problem is set up in a simple one-dimensional geometry of a semi-infinite half-space filled with plasma facing a semi-infinite half-space containing a uniform magnetic field, and the equations are used to describe their subsequent behavior. An attempt is made to solve the equations analytically, but their final solution is obtained numerically. The solutions are also used to determine temperature and entropy increases in the plasma.

DIFFUSION OF A PLASMA ACROSS A MAGNETIC FIELD

by

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I Introduction

There has been very recently a greatly increased interest in scientific and engineering applications of ionized gases and plasmas; this has demanded from known theory a quantitative idea of how plasmas behave under given conditions. An example of these is the problem of plasma containment by a magnetic field; or, stating it more generally, the interaction of a plasma with a magnetic field generated by external sources.

This plasma-field interaction, which is the subject of this discussion, has a number of aspects that complicate its analysis; the complications resulting in non-linear differential equations. As a consequence, the investigation of this interaction is limited to a simple example; this is to make the equations concerned tractable enough to yield quantitative data with a reasonable amount of effort. To illustrate, the processes considered can be crudely described as "slow" motions of plasmas through "strong" magnetic fields, in which the electromagnetic and thermodynamic properties of the plasmas do not change appreciably. What is meant by "slow" motions and "strong" fields will be made clearer further on. In addition to this, the geometry of the problem is reduced to its simplest form. Doing this naturally limits the applicability of the resulting data; still, it produces a good approximation of the plasma-field interaction in a variety of cases.

II General Discussion

The formal statement of this problem most amenable to qualitative investigation is in terms of the so-called macroscopic equations of motion. Essentially these are the equations that result from averaging the motions of many particles; this averaging being done by integration of the Boltzmann equation in velocity space. By means of this integration Spitzer⁽²⁾ defines a current density $\vec{j}$, an average fluid velocity $\vec{u}$, and derives an equation of fluid motion and a generalized form of Ohm's law.

In the case to be considered here, the fluid is assumed to be composed only of singly charged ions of identical masses and of electrons. To generalize to a number of types of ions does not complicate the situation, but it turns out that this does not lead to any additional information. A monatomic gas was chosen simply to make the ratio of specific heats equal to 5/3 in calculations done later. For this case, then, the equation of motion takes the form:

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho (\vec{u} \cdot \nabla) \vec{u} = \vec{j} \times \vec{B} - \nabla p \quad (1)$$

where $\vec{B}$ is the magnetic field strength, p is the scalar pressure, and ρ is the mass density of the ions. This is simply Euler's equation with the addition of a $\vec{j} \times \vec{B}$ force.

There are certain assumptions already implicit in this equation: the mass of the electrons and the gravitational force are both neglected, and the number densities of the electrons and ions are taken to be exactly equal at all points in the fluid. Also, viscosity effects are neglected, allowing what is actually the stress tensor to be taken as a scalar pressure. Accounting for directional pressure differences due to external magnetic fields by using a stress tensor with at least different non-zero diagonal terms might seem actually necessary for this type of problem; however, it will turn out that this does not make any difference in the calculations.

Also for this case, the generalized Ohm's law reduces to the form:

$$\eta \vec{j} = \vec{E} + \vec{u} \times \vec{B} \quad (2)$$

where η is the electrical resistivity and $\vec{E}$ is the electric field. One assumption made here is that the resistivity is a constant scalar; once again, writing it as a diagonal tensor to account for external magnetic fields does not affect the calculations. It must be admitted that neglecting its dependence on the temperature considerably limits the usefulness of the solution; nevertheless, this is the first assumption that will have to be made strictly for simplification. Another assumption is that a $\partial \vec{j} / \partial t$ term is negligible small compared to the others; here the limitation of "slowness" is first applied.

The two equations above resulted from the "plasma" properties of the fluid in this problem. The remaining equations to be introduced are of a more general nature. Firstly, there are the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \quad (3)$$

and the equation of state

$$p = \rho R T \quad (4)$$

where R is the gas constant and T is the temperature. In addition, there are the Maxwell equations

$$\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad (5)$$

and

$$\vec{j} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} \quad (6)$$

The currents due to polarization and the time rate of change of the electric field have been neglected as being small compared to the magnetization current which is included in the $\vec{\nabla} \times \vec{B}$ term in this last equation. The permeability, μ_0 , is taken equal to that of free space.

Finally, there is the energy equation

$$c_v \left(\frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla} \right) \ln(p \rho^{-\gamma}) = \frac{\eta j^2}{\rho T}$$

where c_v is the constant-volume heat capacity and γ is the ratio of

specific heats. This equation implies two very important assumptions; one, that the only source of energy is ohmic heating (viscosity having already been neglected); and the other, that there is no heat transfer in the fluid, i.e., all processes are adiabatic. This last assumption is valid when heat diffusion is much slower than the process to be described.

Now tabulating the equations obtained:

$$\rho \frac{d\vec{u}}{dt} + \rho (\vec{u} \cdot \vec{\nabla}) \vec{u} = \vec{j} \times \vec{B} - \vec{\nabla} p \quad (1)$$

$$\eta \vec{j} = \vec{E} + \vec{u} \times \vec{B} \quad (2)$$

$$\frac{d\rho}{dt} + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \quad (3)$$

$$p = \rho R T \quad (4)$$

$$\vec{\nabla} \times \vec{E} = - \frac{d\vec{B}}{dt} \quad (5)$$

$$\vec{j} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} \quad (6)$$

$$c_v \left(\frac{d}{dt} + \vec{u} \cdot \vec{\nabla} \right) \ln(p \rho^{-\gamma}) = \frac{\eta j^2}{\rho T} \quad (7)$$

it is seen that there are seven equations containing seven variables. It may also be seen that the equations are non-linear, which immediately precludes any straightforward analytical integration of them. Thus, proceeding with their investigation requires some additional simplifications.

Before continuing with this, it is instructive to roughly sketch some properties of plasmas in strong magnetic fields produced by constant external currents. Here "strong magnetic fields" means fields that are much larger than any field set up by currents in the plasma. To the first order, the magnetic force lines tend to be "frozen" in a highly conducting gas. If a motion is imparted to the gas, the force lines are "dragged along" with it. To the next order, current dissipation due to the finite resistivity of the gas allows a relative motion between the gas and the field lines. To see this more clearly, one can imagine a cluster of externally produced parallel magnetic

field lines in a plasma with a finite resistivity. Because the plasma is resistive, the current around an area in it will have to decay if it is not regenerated by externally intensifying the fields. Thus the number of flux lines passing through the area will have to be reduced, and if their total number is constant, some of them must appear outside of the area. Therefore, the cluster spreads out and what the whole process "looks like" is a diffusion of the field lines through the plasma.

III The Infinite One-Dimensional Case

Keeping in mind the limitations of "strong" magnetic fields and "slow" processes, it is now possible to set up a specific example of this problem that will lend itself to further simplification of the seven equations. Consider at time $t = 0$ a perfectly uniform, completely ionized, monatomic plasma of density ρ_0 and resistivity η extending from the yz -plane to infinity along the positive x -axis. The plasma is not allowed to have any motion which is not along the x -axis; i.e., its motion is one-dimensional. Consider also a perfectly uniform magnetic field of intensity B_0 at $t = 0$ extending from the yz -plane to infinity along the negative x -axis. The field lines are all perfectly straight and parallel to the z -axis, pointed in the positive z -direction, and extend to infinity in both z -directions (Figure 1).

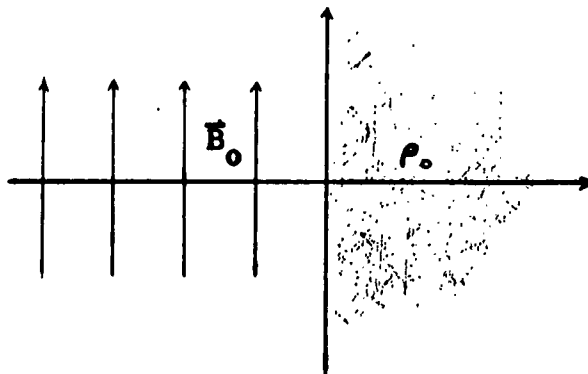


Figure 1.

Because the field is infinite but does not extend past $x = 0$, there is at $t = 0$ an infinite current sheet with the current travelling in the negative y -direction and covering the $x = 0$ surface of the plasma.

To seek a solution for the behavior of this system for times $t > 0$, one first considers the equation of motion

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho (\vec{u} \cdot \nabla) \vec{u} = \vec{j} \times \vec{B} - \nabla p \quad (1)$$

Since the plasma is always uniform in any plane perpendicular to the x -axis, the ∇ -operator reduces to $\partial/\partial x$. Also, if the B -field is always considered in the z -direction, the net currents in the plasma always flow in the y -direction. Then the equation of motion becomes

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = jB - \frac{\partial p}{\partial x} \quad (8)$$

Now it may be seen why it was not necessary to regard the resistivity and the pressure as tensors. Even if the tensor form had been retained, only one term in each would have been used.

At this point the criterion of "slowness" is again applied to the motion of the plasma. This means that the terms on the left are negligibly small, or that the terms on the right nearly cancel. Thus,

$$jB = \frac{\partial p}{\partial x} \quad (9)$$

Since for this configuration the equation

$$\vec{j} = \frac{1}{\mu_0} \nabla \times \vec{B} \quad (6)$$

becomes

$$j = - \frac{1}{\mu_0} \frac{\partial B}{\partial x} \quad (10)$$

because B only has a functional dependence on the x -coordinate, the equation of motion then becomes

$$- \frac{B}{\mu_0} \frac{\partial B}{\partial x} = - \frac{1}{2\mu_0} \frac{\partial B^2}{\partial x} = \frac{\partial p}{\partial x} \quad (11)$$

and this may be integrated to give:

$$p + \frac{B^2}{2\mu_0} = \text{constant} = \frac{B_0^2}{2\mu_0} \quad (12)$$

Thus, $B^2/2\mu_0$ may be considered as a "magnetic pressure", and the total pressure at all points in space remains constant as the magnetic

and plasma pressures vary. Simply from this equation it is possible to see that the plasma leaks slowly into the magnetic field while the magnetic field correspondingly leaks into the plasma. Neither the plasma nor the field has a sharp boundary at $t > 0$; and the current setting up the magnetic field becomes distributed through the plasma rather than being concentrated in a sheet. A plot of plasma density and field strength might then be expected to look like Figure 2.

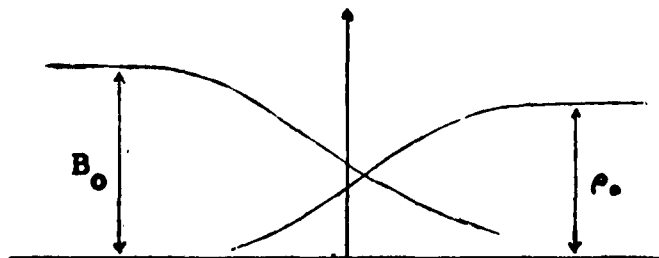


Figure 2.

The next series of steps in seeking a solution to the equations consists of reducing them as far as possible. The calculations are first simplified by using the facts that the magnetic fields are only the z -direction and the currents and therefore electric fields are only in the y -direction (there are no external fields); and that all quantities depend functionally on the x -direction. Ohm's law then becomes

$$\eta j = E - u B \quad (13)$$

and differentiating it with respect to x yields

$$\eta \frac{\partial j}{\partial x} = \frac{\partial E}{\partial x} - B \frac{\partial u}{\partial x} - u \frac{\partial B}{\partial x} \quad (14)$$

The continuity equation becomes

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0 \quad (15)$$

and Faraday's law becomes

$$\frac{\partial E}{\partial x} = - \frac{\partial B}{\partial t} \quad (16)$$

Substituting for $\partial u / \partial x$ from the continuity equation, and for $\partial E / \partial x$ from Faraday's law into the differentiated form of Ohm's law yields

$$\eta \frac{\partial j}{\partial x} = - \frac{\partial B}{\partial t} - u \frac{\partial B}{\partial x} + \frac{B}{\rho} \left(\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} \right) \quad (17)$$

Then substituting for j from the first Maxwell equation yields

$$+ \frac{\eta}{\mu_0} \frac{\partial^2 B}{\partial x^2} = \left(\frac{\partial B}{\partial t} + u \frac{\partial B}{\partial x} \right) - \frac{B}{\rho} \left(\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} \right) \quad (18)$$

Now the process of reduction is carried through starting from the energy equation

$$c_v \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right) \ln(p \rho^{-\gamma}) = \frac{\eta j^2}{\rho T} \quad (19)$$

Substituting for ρT from the equation of state and dividing by c_v yields

$$\frac{1}{p} \left(\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} \right) - \frac{\gamma}{\rho} \left(\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} \right) = \frac{R \eta j^2}{c_v p} \quad (19)$$

Multiplying by p and substituting $R/c_v = \gamma - 1$ and $j = - \frac{1}{\mu_0} \frac{\partial B}{\partial x}$

$$\left(\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} \right) - \gamma \frac{p}{\rho} \left(\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} \right) = \eta \frac{(\gamma - 1)}{\mu_0^2} \left(\frac{\partial B}{\partial x} \right)^2 \quad (20)$$

Now the expression for p in terms of B derived from the equation of motion is substituted, yielding

$$- \frac{B}{\mu_0} \left(\frac{\partial B}{\partial t} + u \frac{\partial B}{\partial x} \right) - \gamma \frac{B_0^2 - B^2}{2\mu_0} \frac{1}{\rho} \left(\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} \right) = \eta \frac{(\gamma - 1)}{\mu_0^2} \left(\frac{\partial B}{\partial x} \right)^2 \quad (21)$$

Simple substitution produces no further reduction after this point, and the process must be continued by the following transformation of coordinates: let

$$X = \int_{-\infty}^x \frac{\rho}{\rho_0} dx \quad ; \quad T = t \quad (22)$$

Then

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial X}{\partial x} \frac{\partial}{\partial X} + \frac{\partial T}{\partial x} \frac{\partial}{\partial T} \\ &= \frac{\rho}{\rho_0} \frac{\partial}{\partial X} \end{aligned} \quad (23)$$

and

$$\frac{\partial}{\partial t} = \frac{\partial X}{\partial t} \frac{\partial}{\partial X} + \frac{\partial T}{\partial t} \frac{\partial}{\partial T} \quad (24)$$

But

$$\frac{\partial X}{\partial t} = \frac{1}{\rho_0} \int_{-\infty}^x \frac{\partial \rho}{\partial t} dx = - \frac{1}{\rho_0} \int_{-\infty}^x \frac{\partial}{\partial x} (\rho u) dx = - \frac{1}{\rho_0} \rho u \quad (25)$$

by the equation of continuity, so that

$$\frac{\partial}{\partial t} = - \frac{\rho u}{\rho_0} \frac{\partial}{\partial X} + \frac{\partial}{\partial T} \quad (26)$$

Thus

$$\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} = - \frac{\rho u}{\rho_0} \frac{\partial}{\partial X} + \frac{\partial}{\partial T} + u \frac{\rho}{\rho_0} \frac{\partial}{\partial X} = \frac{\partial}{\partial T} \quad (27)$$

and substituting appropriately in Eqs. (18) and (21) yields

$$- \frac{\eta}{\mu_0} g \frac{\partial}{\partial X} (g \frac{\partial f}{\partial X}) = \frac{\partial f}{\partial T} - \frac{f}{g} \frac{\partial g}{\partial T} \quad (28)$$

$$- f \frac{\partial f}{\partial T} - \gamma \frac{1-f^2}{2} \frac{1}{g} \frac{\partial g}{\partial T} = \frac{\eta}{\mu_0} (\gamma - 1) g^2 \left(\frac{\partial f}{\partial X} \right)^2 \quad (29)$$

using the dimensionless variables $g = \rho / \rho_0$; $f = B/B_0$.

This transformation has served its intent by eliminating u as a variable; but it has also had other effects in changing the x -axis. The new axis is "squeezed" compared to the real x -axis when the proportionality factor ρ / ρ_0 is small. Also, X is never less than zero because the defining integral is never negative. Thus, large negative values on the x -axis correspond to small positive values on the X -axis, while large positive values on the x -axis correspond to large positive values on the X -axis. However, this transformation has reduced the problem to two equations in two unknowns, and the remaining discussion is involved with trying to solve them.

For further convenience, the two equations will be rewritten in terms of one variable

$$z = \sqrt{\frac{\mu_0}{\eta}} \frac{X}{\sqrt{T}} \quad (30)$$

not to be confused with the z -coordinate, so that

$$\frac{\partial}{\partial X} = \frac{\partial z}{\partial X} \frac{\partial}{\partial z} = \sqrt{\frac{\mu_0}{\eta}} \frac{1}{\sqrt{T}} \frac{\partial}{\partial z} = \frac{z}{X} \frac{\partial}{\partial z} \quad (31)$$

and

$$\frac{\partial}{\partial T} = \frac{\partial z}{\partial T} \frac{\partial}{\partial z} = -\frac{1}{2} \sqrt{\frac{\mu_0}{\eta}} \frac{X}{\sqrt{T}} \frac{\partial}{\partial z} = -\frac{1}{2} \frac{z}{T} \frac{\partial}{\partial z} \quad (32)$$

Eq. (28) then becomes

$$-\frac{\gamma}{\mu_0} g \frac{z}{X} \frac{\partial}{\partial z} \left(g \frac{z}{X} \frac{\partial f}{\partial z} \right) = -\frac{1}{2} \frac{z}{T} \frac{\partial f}{\partial z} + \frac{1}{2} \frac{f}{g} \frac{z}{T} \frac{\partial g}{\partial z} \quad (33)$$

and expanding the term on the left yields

$$\begin{aligned} -\frac{\gamma}{\mu_0} g \frac{z}{X} \left[\frac{z}{X} \frac{\partial g}{\partial z} \frac{\partial f}{\partial z} + \frac{z}{X} g \frac{\partial^2 f}{\partial z^2} + \frac{g}{X} \frac{\partial f}{\partial z} - g \frac{z}{X^2} \frac{\partial f}{\partial z} \frac{\partial X}{\partial z} \right] \\ = -\frac{\gamma}{\mu_0} g \frac{z}{X} \left[\frac{z}{X} \frac{\partial g}{\partial z} \frac{\partial f}{\partial z} + \frac{z}{X} g \frac{\partial^2 f}{\partial z^2} + \frac{g}{X} \frac{\partial f}{\partial z} - \frac{g}{X} \frac{\partial f}{\partial z} \right] \\ = -\frac{\gamma}{\mu_0} g \frac{z^2}{X^2} \left[\frac{\partial g}{\partial z} \frac{\partial f}{\partial z} + g \frac{\partial^2 f}{\partial z^2} \right] \end{aligned} \quad (34)$$

Thus Eq. (33) becomes

$$\frac{2g}{z} \left[\frac{\partial g}{\partial z} \frac{\partial f}{\partial z} + g \frac{\partial^2 f}{\partial z^2} \right] = \frac{\partial f}{\partial z} + \frac{f}{g} \frac{\partial g}{\partial z} \quad (35)$$

Correspondingly, Eq. (29) becomes

$$\frac{f}{2} \frac{z}{T} \frac{\partial f}{\partial z} + \gamma \frac{1-f^2}{4} \frac{1}{g} \frac{z}{T} \frac{\partial g}{\partial z} = \frac{\gamma}{\mu_0} (\gamma - 1) \frac{z^2 g^2}{X^2} \left(\frac{\partial f}{\partial z} \right)^2 \quad (36)$$

or

$$f \frac{\partial f}{\partial z} + \frac{\gamma}{2} (1-f^2) \frac{1}{g} \frac{\partial g}{\partial z} = \frac{2(\gamma - 1)}{z} g^2 \left(\frac{\partial f}{\partial z} \right)^2 \quad (37)$$

Thus, writing primes for $\partial/\partial z$'s

$$\frac{2g}{z} (g'f' + gf'') = -f' + \frac{f}{g} g' \quad (38)$$

$$ff' + \frac{\gamma}{2} (1-f^2) \frac{1}{g} g' = 2(\gamma - 1) \frac{g^2}{z} (f')^2 \quad (39)$$

It may immediately be shown that these equations lead to a process that at least resembles ordinary diffusion. Taking the limiting case of an incompressible plasma; i.e., $g \rightarrow 1$, $g' \rightarrow 0$, and $\gamma \rightarrow \infty$ causes Eq. (39) to yield the value $f=1$ and Eq. (38) to become

$$\frac{2}{z} f'' = -f' \quad (40)$$

Rearranging this as

$$\frac{f''}{f'} = (\ln f')' = -\frac{z}{2} \quad (41)$$

and integrating it yields

$$\ln f' = -\frac{z^2}{4}; \quad f' = e^{-z^2/4} \quad (42)$$

and integrating it again yields

$$f = \int_0^z e^{-z^2/4} dz \quad (43)$$

Thus f in this case is of the form of the error function. What this means is that the plasma sits perfectly still, being incompressible and infinite, and the magnetic field diffuses into it. This is exactly the process undergone by a magnetic field diffusing into a motionless metal conductor.

IV Analytical Integration

Because Eqs. (38) and (39) are non-linear second order differential equations, they cannot be straightforwardly integrated over all values of z . What can be done, however, is to look at the forms of these equations as f and g asymptotically approach values determined from physical reasoning. Remembering that the functions are now in terms of the transformed coordinate X , it may be seen that at large z , g approaches one and f approaches zero. Conversely, very near the origin, f approaches one and g approaches zero. These statements may be made in light of the fact that, at large distances from the origin of the real coordinate, the magnetic field and the plasma should each be relatively undisturbed if only a finite time has elapsed.

With this much knowledge in mind about the behavior of the variables, an attempt may be made to at least partially understand the equations. Taking Eq. (38)

$$f'' = -\frac{g'f'}{g} - \frac{zf'}{2g^2} + \frac{zfg'}{2g^3} \quad (44)$$

$$\frac{f''}{f'} = (\ln f')' = -(\ln g)' - \frac{z}{2g^2} + \frac{zfg'}{2f'g^3} \quad (45)$$

Integrating

$$\ln f' = -\ln g - \int \frac{z dz}{2g^2} + \int \frac{zfg' dz}{2f'g^3} \quad (46)$$

$$f' = \frac{1}{g} \exp \left[-\int \frac{z dz}{2g^2} + \int \frac{zfg' dz}{2f'g^3} \right] \quad (47)$$

Integrating again

$$f = \int \frac{1}{g} \exp \left[- \int \frac{z}{2g^2} dz + \int \frac{zfg' dz}{2f'g^3} \right] dz \quad (48)$$

Both f' and g' go to zero at large z ; thus it is difficult to discuss the behavior of the right-hand side of the exponent at this limit. Nevertheless, it might be expected to go to zero, since the remaining terms give an expression for f similar to that in the case of the incompressible fluid. This is what might be expected at large z where g is nearly undisturbed.

Expressions for the term gf' may be derived from both equations. Eq. (38) may be written

$$(gf')' = - \frac{z}{2g} (f' - \frac{f}{g} g') = - \frac{1}{2} z \left(\frac{f}{g} \right)' \quad (49)$$

which integrates to

$$gf' = - \int \frac{z}{2} \left(\frac{f}{g} \right)' dz \quad (50)$$

Also from Eq. (39)

$$(gf')^2 = \frac{zf f'}{2(\gamma - 1)} + \frac{\gamma}{4(\gamma - 1)} z(1 - f^2) \frac{1}{g} g' \quad (51)$$

$$\frac{2(\gamma - 1)}{zf} g (gf')^2 - gf' - \frac{\gamma}{2} (1 - f^2) \frac{g'}{g} = 0 \quad (52)$$

Solving this quadratic

$$gf' = \frac{1 \pm \sqrt{1 + 4\gamma(\gamma - 1) \frac{(1 - f^2)g g'}{z f^2}}}{\frac{4(\gamma - 1)g}{z f}} \quad (53)$$

Both of these may be integrated to give expressions for f . Unfortunately, none of the above yield any new information about the functional form of either f or g .

It turned out that none of these methods were really helpful in uncoupling the equations to obtain an expression for either f or g . They either led to complicated integral forms or approximations too crude to be useful. It is possible that methods for treating non-linear differential equations that at least lead to further simplifications in this case might exist. Such methods are discussed in texts like Ince⁽¹⁾ and Kaplan⁽³⁾; however, their application is sufficiently difficult that no attempt was made to use them on this problem.

Another possible method of investigation is to substitute various well-known functions for the variables and see if they balance the equations at asymptotic values. A great deal of time and effort was spent trying this, and it proved unequivocally fruitless. The actual functions, though they might be well-behaved, are of sufficient complexity that no closed function resembles either of them. This is illustrated by noting that, in the greatly restrictive case of an incompressible plasma, f still had the form of an open integral.

V Numerical Integration

In order to obtain specific data on the plasma-field interaction, it was necessary to resort to numerical integration. The procedure used is described in Kaplan⁽³⁾, and simply consists of selecting values of the functions and their derivatives at a point, calculating their values at a point close by, and iterating. Thus the method is a numerical first integration; and since f appears to the second order in the equations, it is necessary to integrate twice to obtain values for it. The equations are written in the form:

$$f'' = - \frac{g'f'}{g} - \frac{zf'}{2g^2} + \frac{zfg'}{2g^3} \quad (54)$$

$$f' = (f)' \quad (55)$$

$$g' = \frac{4(\Upsilon - 1)}{\Upsilon} \frac{g^3(f')^2}{z(1-f^2)} - \frac{2}{\Upsilon} \frac{gff'}{(1-f^2)} \quad (56)$$

where the first equation is Eq. (38), and the third is Eq. (39). In doing the calculation, values for f and g and their derivatives were selected at large positive z and the iteration carried toward the origin. These initial values were adjusted until f approached one and g approached zero at the origin and both held their asymptotic values at large z . It would seem more accurate to start at the origin and look for asymptotic values of the functions by iterating away from the origin, but it was extremely difficult to estimate the first derivatives at this point.

The results of the numerical integration are plotted in Figure 3. The variables are left in the dimensionless form of f and g and are plotted as functions of the transformed coordinates.

These transformed coordinates were then recast in the form of the real coordinates by means of the inverse transformations.

$$x = \int_0^x \frac{1}{g} dX ; \quad T = t \quad (57)$$

This was also done numerically. The variables f and g are thus also plotted as functions of x and t in Figure 4. Because the boundary condition $x \rightarrow -\infty$ as $X \rightarrow 0$ is awkward for numerical calculation, the origin of the real axis was determined as follows:

$$X = \int_{-\infty}^x g \cdot dx \quad (58)$$

so that there is an X_0 such that

$$X_0 = \int_{-\infty}^0 g \cdot dx \quad (59)$$

It was then assumed that X_0 and $x = 0$ correspond to $g = \frac{1}{2}$ because the plasma does not move as a mass, due to the balance between the magnetic and gas pressures. This yielded $X_0 = .35$ from Figure 3 and $X_0 = .348$ from a graphical integration of g over x in Figure 4. These figures thus determine the magnetic field strength and the plasma density in both space and time as the diffusion process takes place. By also calculating temperature and entropy changes in the gas, its total behavior may be considered adequately defined.

The temperature of the gas at each point was calculated from the x -axis curves of f and g as follows: since

$$p = \frac{B_0^2 - B^2}{2\mu_0} ; \quad p = \rho RT \quad (60)$$

then

$$T = \frac{p}{\rho R} = \frac{B_0^2 - B^2}{2\mu_0 \rho R} \quad (61)$$

Changing to dimensionless variables

$$T = \frac{B_0^2}{2R\mu_0 \rho_0} \frac{(1 - f^2)}{g} \quad (62)$$

which is plotted in Figure 5 with

$$T_o = \frac{B_o^2}{2R\mu_o \rho_o} \quad (63)$$

Since the process was assumed adiabatic, the specific entropy was calculated from the x-axis curves of g and f by using the expression

$$s = s_o + c_p \ln \frac{\frac{1}{\rho}}{\frac{1}{\rho_o}} + c_v \ln \frac{p}{p_o} \quad (64)$$

Changing to dimensionless variables

$$\begin{aligned} s &= s_o + c_p \ln\left(\frac{1}{g}\right) + c_v \ln\left(\frac{\frac{B_o^2 - B^2}{2\mu_o}}{\frac{B_o^2}{2\mu_o}}\right) \\ &= s_o + c_p \ln\left(\frac{1}{g}\right) + c_v \ln(1 - f^2) \end{aligned} \quad (65)$$

Or, subtracting s_o and dividing by c_v

$$\frac{\Delta s}{c_v} = \ln\left(\frac{1}{g}\right) + \ln(1 - f^2) \quad (66)$$

Then the increase in entropy as a function of x and t is given by $\rho \Delta s = \rho_o g \Delta s$, which was calculated and plotted in Figure 6 using $\gamma = 5/3$. The total increase in entropy as a function of time was obtained by graphically integrating the curve in Figure 6, and was found to be

$$\Delta s = 5.63 c_v \sqrt{\frac{\eta}{\mu_o}} \sqrt{t}$$

All of the calculations for these graphs are only to slide-rule accuracy. Because of this, and because of the error involved in estimating the boundary conditions for the f and g curves in Figure 3, the curve shapes and the derived quantities may only be considered to be correct to somewhat better than an order of magnitude. Greater accuracy would require the services of a computer.

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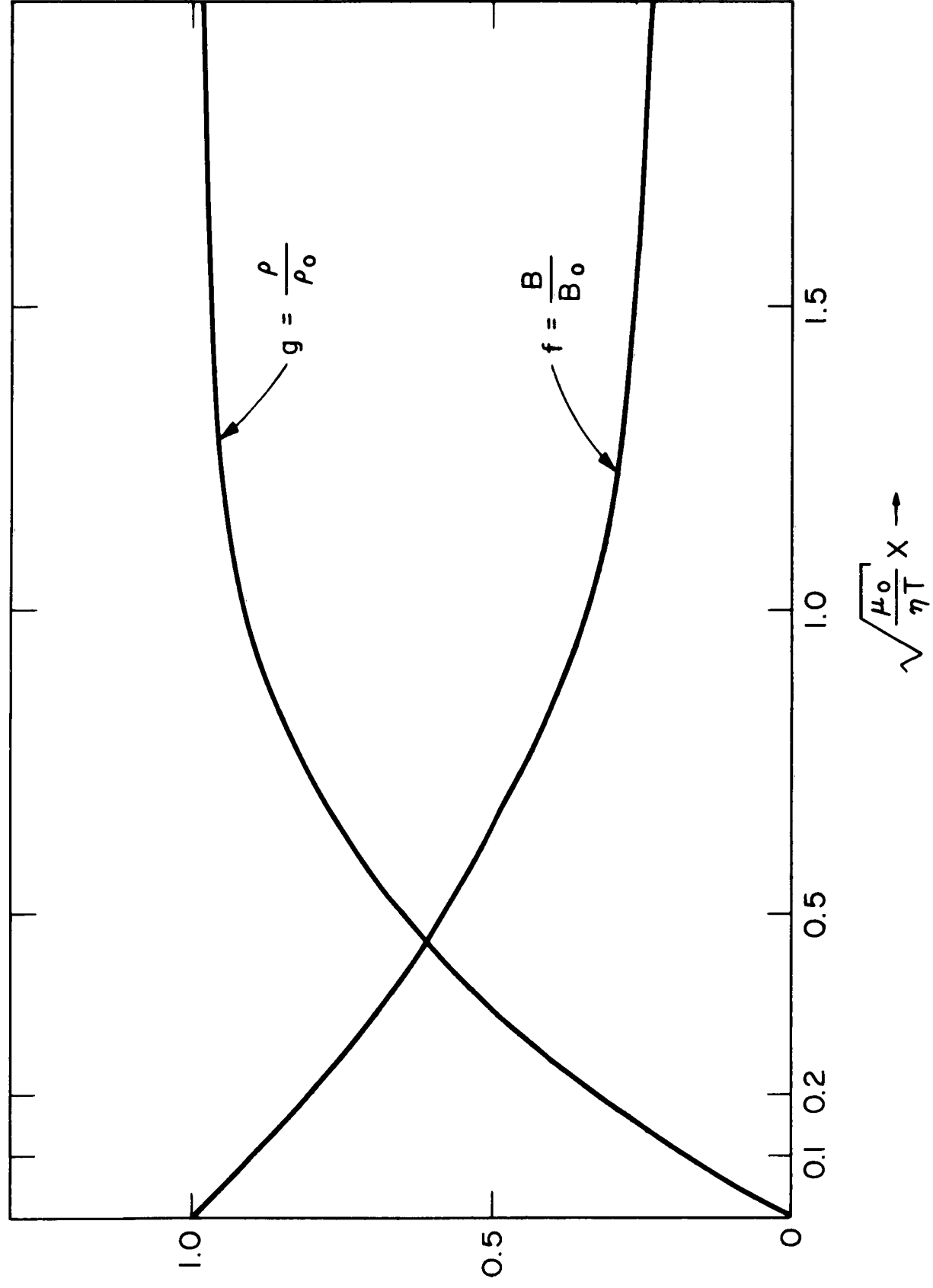


Figure 3. ρ and B as functions of the transformed coordinates.

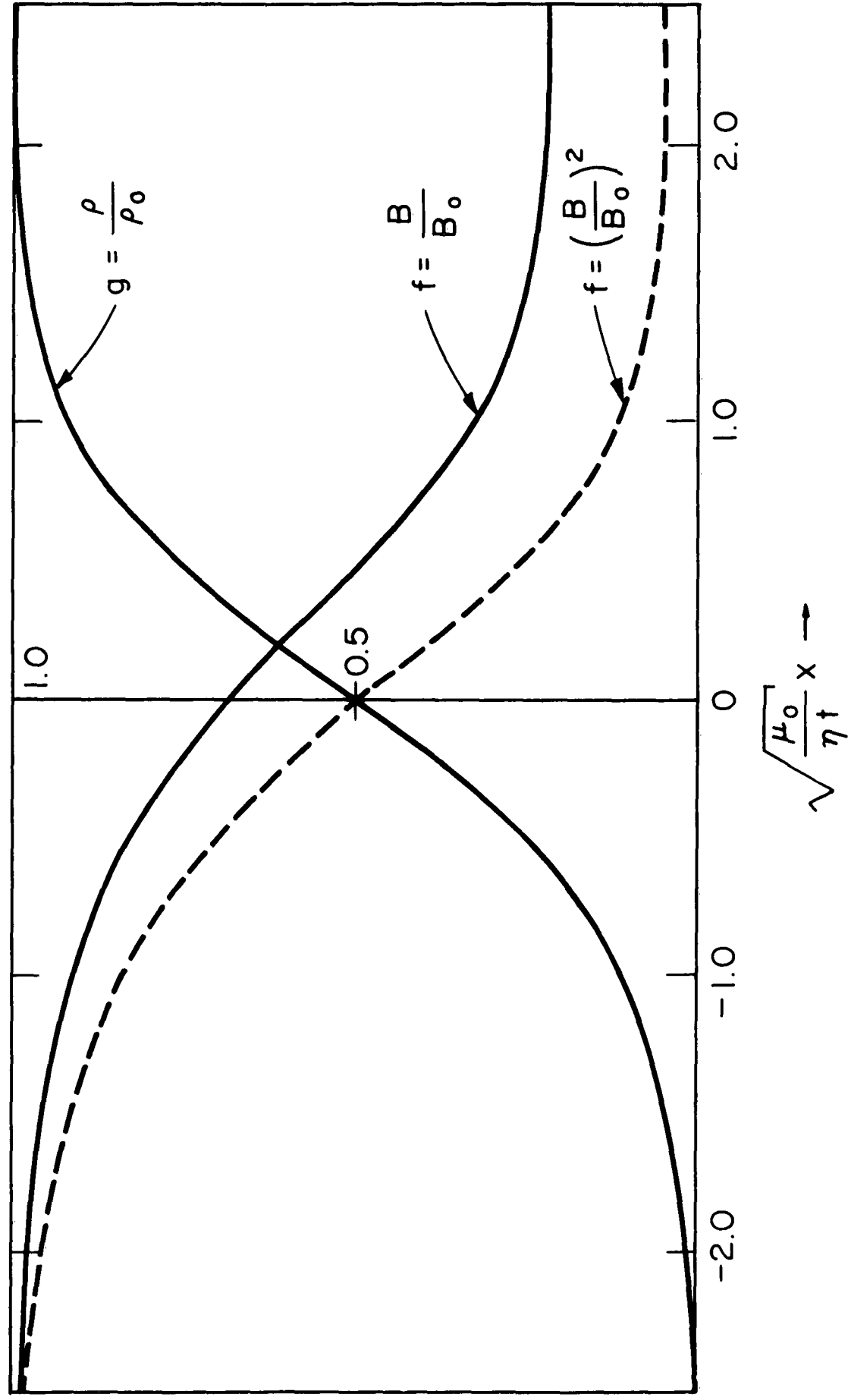


Figure 4. ρ and B as functions of the real coordinates.

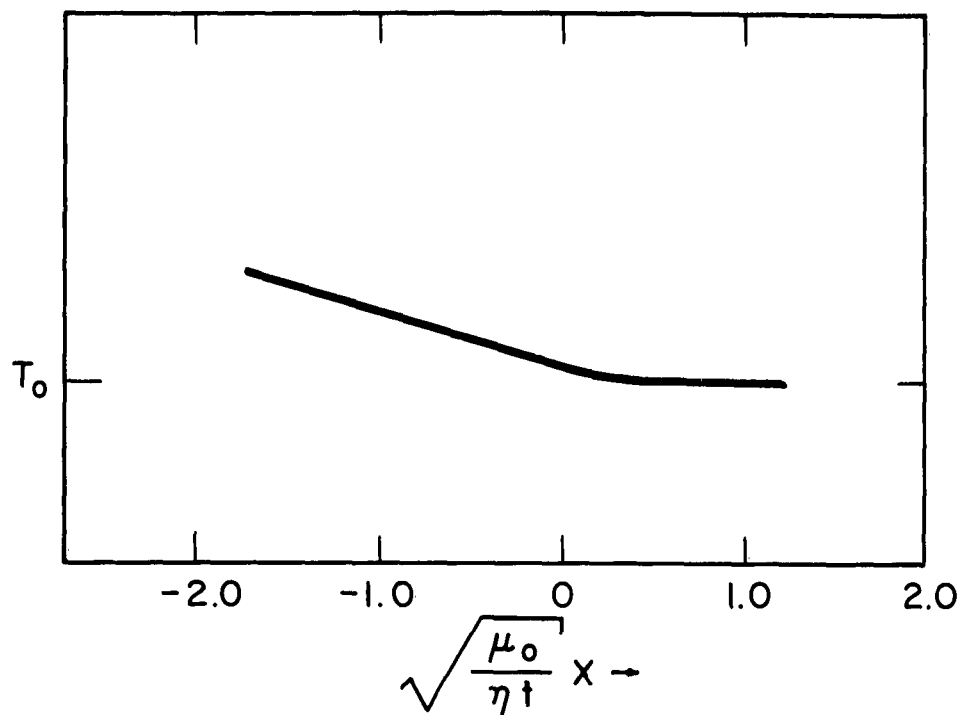


Figure 5. Plasma temperature as a function of the real coordinates.

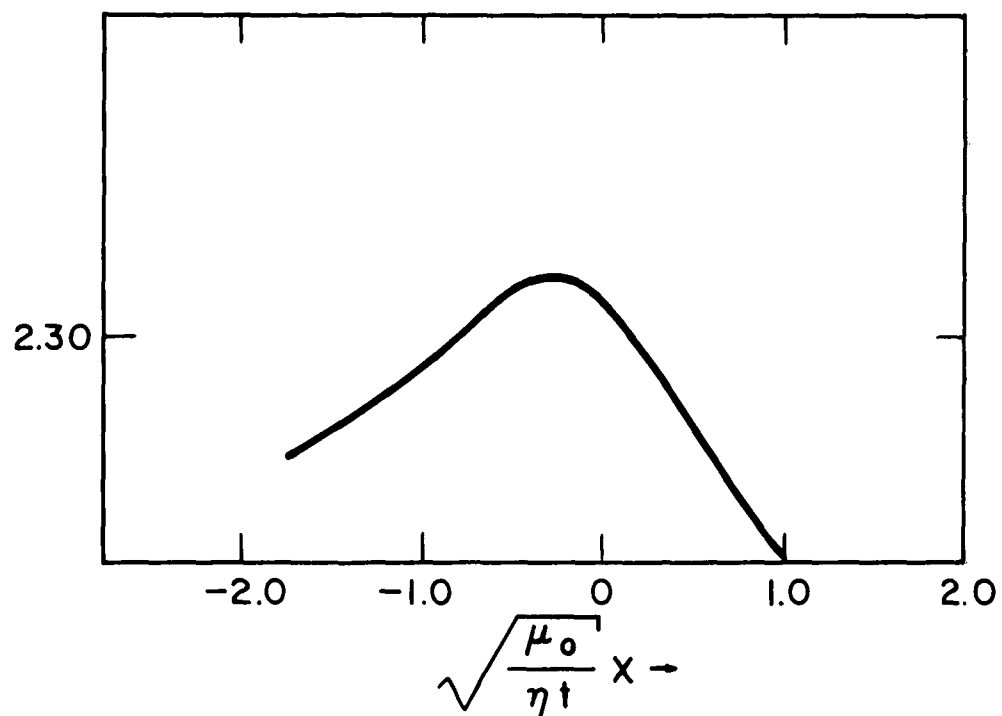


Figure 6. Increase in entropy as a function of the real coordinates.